SUFFICIENT CONDITIONS FOR THE EXISTENCE OF MINIMIZING HARMONIC MAPS WITH AXIAL SYMMETRY IN THE SMALL-AVERAGE REGIME

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Abstract. The paper concerns the analysis of global minimizers of a Dirichlet-type energy functional defined on the space of vector fields $H^1(S,T)$, where S and T are surfaces of revolution. The energy functional we consider is closely related to a reduced model in the variational theory of micromagnetism for the analysis of observable magnetization states in curved thin films. We show that axially symmetric minimizers always exist, and if the target surface T is never flat, then any coexisting minimizer must have line symmetry. Thus, the minimization problem reduces to the computation of an optimal one-dimensional profile. We also provide a necessary and sufficient condition for energy minimizers to be axially symmetric.

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1. Introduction and motivation

For given surfaces of revolution $S, T \subseteq \mathbb{R}^3$ around the $\boldsymbol{e}_3$ axis, we consider the Dirichlet-type energy defined for every $\boldsymbol{m} \in H^1(S,T)$ by

$$\mathcal{E}_\omega(\boldsymbol{m}) := \int_S |\nabla \boldsymbol{m}(\xi)|^2 \mathrm{d}\xi + \int_S g(\boldsymbol{m}(\xi)\cdot \boldsymbol{a}(\xi))\mathrm{d}\xi + \int_S |\langle \boldsymbol{m}\rangle_{S_\xi} \times \boldsymbol{e}_3|^2 \omega^2(\xi)\mathrm{d}\xi, \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}_+$ is an anisotropic potential, $\boldsymbol{a} : S \to \mathbb{R}^3$ is a prescribed vector field that can be either axially symmetric or axially antisymmetric (see section 2.1.3), $\langle \boldsymbol{m}\rangle_{S_\xi}$ is the average of $\boldsymbol{m}$ along the circle of latitude $S_\xi := (\xi \cdot \boldsymbol{e}_3)\boldsymbol{e}_3 + \mathbb{S}^1(\xi \cdot \boldsymbol{e}_1)$, i.e.,

$$\langle \boldsymbol{m}\rangle_{S_\xi} = \frac{1}{|S_\xi|}\int_{S_\xi} \boldsymbol{m}(\xi)\mathrm{d}\xi, \tag{1.2}$$

and $\omega : S \to \mathbb{R}_+$ is a generic measurable function that weights the strength of the tendency of $\langle \boldsymbol{m}\rangle_{S_\xi}$ to be aligned along the unit vector $\boldsymbol{e}_3$. Detailed hypotheses on the regularity of the involved quantities, as well as precise definitions of the terms employed, will be given in section 2. Here, we only want to point out that the term vector field is used interchangeably with map, i.e., no tangential requirement is implicit in its use.

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The main aim of this paper is to show that under mild conditions on the weight function ω, every global minimizer of $\mathcal{E}_\omega$ has axial symmetry. When $\boldsymbol{a}$ is axially symmetric, a particular case of our findings gives a necessary and sufficient condition for energy minimizers to be axially symmetric: the existence of axially symmetric energy minimizers is equivalent to the existence of *axially null-average* minimizers, i.e., to the existence of minimizers $\boldsymbol{m} \in H^1(S,T)$ such that $\langle \boldsymbol{m} \rangle_{S_\xi} \times \boldsymbol{e}_3 = 0$ for every $\xi \in S$ (see section 4 and Theorem 3 in there). This characterization also holds when the last term in (1.1) is absent. Note that while it is always the case that axially symmetric configurations are axially null average, minimality allows for the converse implication.

In what follows, to shorten notation and enhance readability, we set

$$\mathcal{D}_S(\boldsymbol{m}) := \int_S |\nabla \boldsymbol{m}(\xi)|^2 \mathrm{d}\xi, \quad \mathcal{A}_S(\boldsymbol{m}) := \int_S g(\boldsymbol{m}(\xi) \cdot \boldsymbol{a}(\xi)) \mathrm{d}\xi, \tag{1.3}$$

and

$$\mathcal{P}_S(\boldsymbol{m}) := \int_S |\langle \boldsymbol{m} \rangle_{S_\xi} \times \boldsymbol{e}_3|^2 \omega^2(\xi) \mathrm{d}\xi, \tag{1.4}$$

so that the energy functional we are interested in takes the form

$$\mathcal{E}_\omega(\boldsymbol{m}) = \mathcal{D}_S(\boldsymbol{m}) + \mathcal{A}_S(\boldsymbol{m}) + \mathcal{P}_S(\boldsymbol{m}). \tag{1.5}$$

Our investigations fit with that thread of results concerned with the study of harmonic maps with symmetries. Indeed, when S is a surface with boundary, our analysis applies to the existence of axially symmetric solutions of the Euler-Lagrange equations of $\mathcal{E}_\omega$, which are harmonic-type equations between surfaces of revolution.

Also, when $T := \mathbb{S}^2$, the energy functional $\mathcal{E}_\omega$ is closely related to a reduced model in the variational theory of micromagnetism, a model relevant for the description of observable magnetization states in curved thin films. As explained in section 1.2, the functional $\mathcal{E}_\omega$ can be considered a model of the micromagnetic energy functional in the asymptotic regime of curved thin films where, in addition to magnetocrystalline and shape anisotropies, higher-order magnetostatic effects are taken into account through the term $\mathcal{P}_S$, which favors configurations that are S_ξ-null-average in the perpendicular component. According to our findings, micromagnetic ground states always have axial symmetry in the curved thin-film regime and under the influence of nonnegligible higher-order magnetostatic effects. The conclusions we reach can be applied to other significant physical systems, e.g., to understand the existence of symmetric textures in the Oseen-Frank theory of nematic liquid crystals [23, 34].

1.1. Outline. In the next section 1.2, we briefly present the physical context that led us to the investigation of (1.1), so to give the reader a broader perspective on the relevance of the energy functional $\mathcal{E}_\omega$. In section 1.3, after a brief review of earlier research on the symmetry of harmonic maps, we discuss previous studies on the symmetry properties of minimizers of the micromagnetic energy functional in the curved thin film regime. In section 2, we describe the rigorous setting of the problem and detail the contributions of the present work. Proofs are given in section 3. In section 4, we present further results and some applications to the existence of axially symmetric solutions of elliptic PDEs.

1.2. Physical context. Over the last decade, considerable experimental and theoretical research has been done on the physics of ferromagnetic systems with curved shapes. One of the main reasons is that the curved geometry can lead to effective antisymmetric interactions and, as a result, to the consequent formation of magnetic skyrmions, i.e., chiral spin textures with a non-trivial topological degree, even in the absence of spin-orbit coupling mechanism, responsible for Dzyaloshinskii–Moriya interactions (DMI). The evidence of these states sheds light on the role of geometry in magnetism: magnetic skyrmions can be stabilized by curvature effects only, in contrast to the planar case where DMI is required [31].

Also, recent advances in the nanofabrication of magnetic hollow particles have sparked interest in these geometries, which lead to artificial materials with unexpected characteristics and diverse applications spanning from logic devices to biomedicine [40].

From a mathematical point of view, the analysis of mesoscale magnetism in curved geometry can be carried out within the framework of the variational theory of micromagnetism [8, 16, 27], where the order parameter is the magnetization $\boldsymbol{m}$ subject to the *saturation* constraint of being $\mathbb{S}^2$-valued. In this framework, the energy term $\mathcal{A}_S$ accounts for the so-called *crystal* and *shape-anisotropy effects*. Indeed, it is well-known that when the ferromagnet occupies a thin shell whose thickness is significantly smaller than the size of the system, the dominant energy contribution is encapsulated in the energy functional [9, 13, 17, 22]

$$\mathcal{F}_\kappa : \boldsymbol{m} \in H^1(S, \mathbb{S}^2) \mapsto \int_S |\nabla \boldsymbol{m}(\xi)|^2 \mathrm{d}\xi + \kappa \int_S (\boldsymbol{m}(\xi) \cdot \boldsymbol{n}(\xi))^2 \mathrm{d}\xi, \tag{1.6}$$

where S is the surface generating the hollow nanoparticle by extrusion along the normal direction $\boldsymbol{n}$, and $\kappa \in \mathbb{R}$ is an effective anisotropy parameter accounting for both *shape* and *crystal* anisotropy. For large $\kappa > 0$, tangential vector fields are energetically favored; for large $\kappa < 0$, i.e., when perpendicular crystal anisotropy prevails over shape anisotropy, energy minimization prefers normal vector fields. Note that, from the variational point of view, the expression of $\mathcal{F}_\kappa$ is nothing but a particular case of the energy functional $\mathcal{E}_\omega$ when $T := \mathbb{S}^2$, $\boldsymbol{a}(\xi) := \boldsymbol{n}_S(\xi)$ and $g(s) := \kappa s^2$ if $\kappa > 0$ and $g(s) := |\kappa|(1 - s^2)$ if $\kappa < 0$. In other words, our energy term $\mathcal{A}_S$ in (1.3) accounts for possible generalizations of the second term in (1.6). Other typical expressions of $\boldsymbol{a}$ and g issuing from the variational theories of micromagnetics and nematic liquid crystals are $\boldsymbol{a}(\xi) := \boldsymbol{n}_S(\xi)$, $\boldsymbol{a}(\xi) := \boldsymbol{n}_T(\xi)$, $\boldsymbol{a}(\xi) = \boldsymbol{e}_3$, or $g(s) = \lambda(1 - s^2)^2$ for some $\lambda \in \mathbb{R}_+$.

In the language of the variational theory of micromagnetism, we can think of the energy term $\mathcal{P}_S$ as accounting for higher-order effects in the long-range magnetic dipole-dipole interactions. Indeed, it is well-known that in the classical three-dimensional setting, the magnetostatic self-energy associated with a distribution of magnetization $\boldsymbol{m} \in H^1(\Omega, \mathbb{R}^3)$ in a domain $\Omega \subseteq \mathbb{R}^3$, favors solenoidal vector fields, i.e., divergence-free vector fields that are tangential to the boundary [7, 8]. From the variational point of view, this means that the magnetostatic self-energy is minimized when

$$\operatorname{div} \boldsymbol{m} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{m} \cdot \boldsymbol{n}_{\partial\Omega} = 0 \quad \text{on } \partial\Omega. \tag{1.7}$$

Now, as shown in [9, 13, 22], but also apparent from (1.6), at the leading order in the energy reduction from $3d$ to $2d$, only the tendency of being tangential to the boundary is preserved. At the same time, any aspect of the divergence-free conditions is lost. To explain this last point better, we recall that any vector-field $\boldsymbol{m} \in H^1(\Omega, \mathbb{R}^3)$ satisfying (1.7) is necessarily null-average in Ω. Indeed, for $i \in \{1, 2, 3\}$, we have $\operatorname{div}(\boldsymbol{m} x_i) = x_i \operatorname{div} \boldsymbol{m} + \boldsymbol{m} \cdot \boldsymbol{e}_i$; therefore, if $\boldsymbol{m}$ satisfies (1.7), then $\langle \boldsymbol{m} \cdot \boldsymbol{e}_i \rangle_\Omega = \langle \operatorname{div}(\boldsymbol{m} x_i) \rangle_\Omega = 0$, because of the divergence theorem and the tangential boundary condition on $\boldsymbol{m}$. Despite being a simple mathematical consequence of (1.7), the favoring of null-average configurations is often the main qualitative property used to describe the physical effect of the demagnetizing field (see, e.g., [5]).

Maintaining some of these structural consequences of (1.7) in the passage from $3d$ to $2d$ would be desirable, and this is the main motivation for considering the term $\mathcal{P}_S$ in our analysis: to keep track of some of the qualitative features of the demagnetizing field that get lost at the leading order in the energy expansion. Indeed, under suitable symmetry assumptions, we can infer something stronger than $\langle \boldsymbol{m} \rangle_\Omega = 0$. Specifically, let Ω_δ be an open tubular neighborhood, of sufficiently small thickness $\delta > 0$, of a smooth surface of revolution S. For simplicity, we assume that $\boldsymbol{n}_S(\xi) \neq \pm \boldsymbol{e}_3$ except possibly for a finite set of points in S (compare it with *never flat* condition in Definition 2). Let J_δ be the projection of Ω_δ onto the $\boldsymbol{e}_3$-axis and, for every $h \in J_\delta$, let $\Sigma_{\delta,h}$ be the *horizontal* section of Ω_δ at height h. We claim that if $\boldsymbol{m}$ satisfies (1.7), and $\boldsymbol{m} \cdot \boldsymbol{e}_3$ depends only on the vertical coordinate corresponding to $\boldsymbol{e}_3$, then for every $h \in J_\delta$ there holds $\langle \boldsymbol{m} \times \boldsymbol{e}_3 \rangle_{\Sigma_{\delta,h}} = 0$. In fact, using $\operatorname{div}(\boldsymbol{m} x_i) = \boldsymbol{m} \cdot \boldsymbol{e}_i$ with $i \in \{1,2\}$, by the divergence theorem we have

$$\int_{\Sigma_{\delta,h}} \boldsymbol{m}(x) \cdot \boldsymbol{e}_i \mathrm{d}x = \partial_3(\boldsymbol{m} \cdot \boldsymbol{e}_3)(h) \int_{\Sigma_{\delta,h}} x \cdot \boldsymbol{e}_i \mathrm{d}x + \int_{\partial \Sigma_{\delta,h}} \sigma_i \boldsymbol{m}(\sigma) \cdot \boldsymbol{\nu}_{\partial \Sigma_{\delta,h}}(\sigma) \mathrm{d}\sigma = 0$$

because, as we are going to show next, both integrals on the right-hand side vanish. By limiting arguments it follows that $\langle \boldsymbol{m} \times \boldsymbol{e}_3 \rangle_{S_\xi} = 0$ for every $\xi \in S$, and this is the hard contraint version of the $\mathcal{P}_S$ term in (1.1).

The first term in the above formula vanishes because, by construction, $\Sigma_{\delta,h}$ is an annulus. To show that also the second integral vanishes, we begin observing that $\boldsymbol{n}_{\partial\Omega_\delta}(\sigma)$ is in the plane spanned by $\boldsymbol{\nu}_{\partial\Sigma_{\delta,h}}(\sigma)$ and $\boldsymbol{e}_3$ and, therefore, given that $\boldsymbol{m}$ satisfies (1.7), we have

$$0 = \boldsymbol{m} \cdot \boldsymbol{n}_{\partial\Omega_\delta} = (\boldsymbol{n}_{\partial\Omega_\delta} \cdot \boldsymbol{\nu}_{\partial\Sigma_{\delta,h}})(\boldsymbol{m} \cdot \boldsymbol{\nu}_{\partial\Sigma_{\delta,h}}) + (\boldsymbol{n}_{\partial\Omega_\delta} \cdot \boldsymbol{e}_3)(\boldsymbol{m} \cdot \boldsymbol{e}_3).$$

Since $\boldsymbol{n}_S(\xi) \neq \pm \boldsymbol{e}_3$ except possibly for a finite set of points in S, the previous relation implies that on $\partial\Sigma_{\delta,h}$ we have

$$\boldsymbol{m} \cdot \boldsymbol{\nu}_{\partial\Sigma_{\delta,h}} = -\frac{(\boldsymbol{n}_{\partial\Omega_\delta} \cdot \boldsymbol{e}_3)(\boldsymbol{m} \cdot \boldsymbol{e}_3)}{\boldsymbol{n}_{\partial\Omega_\delta} \cdot \boldsymbol{\nu}_{\partial\Sigma_{\delta,h}}}.$$

However, the right-hand side of the previous relation depends only on h and not on the specific point of $\partial\Sigma_{\delta,h}$. It follows that

$$\int_{\partial\Sigma_{\delta,h}} \sigma_i \boldsymbol{m}(\sigma) \cdot \boldsymbol{\nu}_{\partial\Sigma_{\delta,h}}(\sigma) \mathrm{d}\sigma = \boldsymbol{m} \cdot \boldsymbol{\nu}_{\partial\Sigma_{\delta,h}} \int_{\partial\Sigma_{\delta,h}} \sigma_i \mathrm{d}\sigma = 0.$$

Although the argument fails when one drops the assumption that $\boldsymbol{m} \cdot \boldsymbol{e}_3$ depends only on the vertical direction $\boldsymbol{e}_3$, what we pictured motivates us enough to explore the impact that a soft penalization term like $\mathcal{P}_S$ in (1.4) has on the energetic landscape.

1.3. State of the art. The minimization problem for Dirichlet-type energy functionals between manifolds naturally arises in differential geometry in the study of unit-speed geodesics and minimal submanifolds [19, 39]. It also occurs in nonlinear field theories every time local interactions are brought into the realm of elastic deformations. Prominent examples are the elastic free energy in the Oseen-Frank theory of nematic liquid crystals [3, 41], symmetric and antisymmetric exchange interactions in the variational theory of micromagnetism [8, 27], and M-theory [1, 4]. The systematic treatment gained impetus after the seminal work of Eells and Sampson [20], who showed that under certain technical conditions on the base and target manifolds, every continuous map is homotopic to a harmonic map.

The existence of minimizing harmonic maps with symmetry has been the topic of several studies, typically set around specific geometries. In [36, 37], it is shown that for any symmetrical domain in $\mathbb{R}^2$ and any symmetrical boundary datum that takes values in a closed hemisphere, minimizing harmonic maps must be radially symmetric. In [12], using projection and averaging procedures introduced in [11], Coron and Helein study harmonic diffeomorphisms between the Euclidean n-ball (minus a finite number of points) and a

Riemannian manifold. They show that under some conditions on the involved parameters, there always exists a smooth SO(3)-equivariant map defined on B^3 with values in $E^3(a) := \{(x,y) \in \mathbb{R}^3 \times \mathbb{R} : |x|^2 + y^2/a^2 = 1\}$ improving some earlier results of Baldes [2]. In [24], Hardt, Lin, and Poon investigated the existence of $\mathbb{S}^2$-valued axially symmetric harmonic maps with prescribed singularities (see also [24, 25] and [32]).

The literature on the subject is vast, and only a systematic review could give due recognition to the results obtained over the years. Some references are available in [6, 35]. Here, we limit ourselves to the literature intimately related to our investigations on the energy functional $\mathcal{E}_\omega$, and we refer the reader to [19, 26, 30, 39] and the references therein for further results on the topic.

Magnetic hollow nanoparticles usually are of the shape of a surface of revolution, and symmetry properties of the ground-states are usually derived in specific geometries. For spherical thin films, i.e., when $S = \mathbb{S}^2$, we showed in [18] that when $\kappa \leqslant -4$, the normal vector fields $\pm\boldsymbol{n}$ are the *only global* minimizers of the energy functional $\mathcal{F}_\kappa$ defined in (1.6). The interest in results of this kind is in the topological remark that $\pm\boldsymbol{n}$ carry different *skyrmion numbers* because of $\deg(\pm\boldsymbol{n}) = \pm 1$ —see also [33] for stationary states topologically distinct from the ground state. In physical terms, this translates into their robustness against thermal fluctuations and external perturbations with far-reaching consequences for modern magnetic storage technologies [21]. The vector fields $\pm\boldsymbol{n}$ have full rotational symmetry, and their local stability persists up to $\kappa < 0$. They lose stability for $\kappa > 0$ when, therefore, new ground states have to appear. No more than this is currently known. The reason is that when $\kappa > -4$, the energy landscape of $\mathcal{F}_\kappa$ is very hard to describe analytically, and it is unclear which aspects of the symmetry of the problem are retained in the shape of minimizers. Numerics suggest that when $\kappa > 0$, the energy $\mathcal{F}_\kappa$ exhibits magnetic states with skyrmion numbers 0, ± 1, all having axial symmetry [28, 29]. However, no mathematical evidence of this has been established. Our paper aims to gain some insight into the question.

In [14], it is shown that when $S = \mathbb{S}^1 \times [0,1]$, the $\mathbb{S}^1$-valued normal vector fields $\pm\boldsymbol{n}$ are the unique ground state when provided that κ is sufficiently negative ($\kappa \leqslant -3$), but almost nothing is currently known when $\kappa > -3$. The only thing one can say is that if the ground states are, as we expect, in-plane, then global minimizers lose axial symmetry because they are no more null average (see [14, Figure 5] and [10]).

However, the axial symmetry of the minimizers certainly fails, for example, when $S = \mathbb{D}$ is the unit disk in $\mathbb{R}^2$. Indeed, on the one hand, as we show in [15], the absence of curvature favors radial ground states rather than axially symmetric, i.e., minimizers satisfy the relation $\boldsymbol{m}(x) = \boldsymbol{m}(|x|)$ for every $x \in \mathbb{D}$. On the other hand, our proofs still work if one adds to the expression of $\mathcal{E}_\omega$ a penalization on the boundary as in [15]. Therefore, we conclude that in the generality we are treating the problem, it is essential to consider the term $\mathcal{P}_S$ to obtain the existence of axially symmetric minimizers regardless of the specific choices of S and T in the class of surfaces of revolution.

2. Contributions of the present work

2.1. Notation and setup. The main results of the present work concern the symmetry properties of energy minimizing maps. To state our results precisely, we need to set up the framework, the mathematical notation, and the terminology used throughout the paper.

2.1.1. *Sobolev spaces on surfaces.* For a given C^1-surface $S \subset \mathbb{R}^3$, we denote by $H^1(S,\mathbb{R}^3)$ the Sobolev space of vector-valued functions defined on S endowed with the norm

$$\|\boldsymbol{m}\|^2_{H^1(S,\mathbb{R}^3)} := \int_S |\boldsymbol{m}(\xi)|^2 \mathrm{d}\xi + \int_S |\nabla \boldsymbol{m}(\xi)|^2 \mathrm{d}\xi. \tag{2.1}$$

Here, ∇ is the tangential gradient of $\boldsymbol{m}$ at $\xi \in S$, and $|\nabla \boldsymbol{m}(\xi)|^2 = \sum_{i=1}^2 |\partial_{\boldsymbol{\mu}_i(\xi)} \boldsymbol{m}(\xi)|^2$ if $(\boldsymbol{\mu}_1(\xi), \boldsymbol{\mu}_2(\xi))$ is an orthonormal basis of $T_\xi S$. Also, given two surfaces $S, T \subseteq \mathbb{R}^3$, we write $H^1(S,T)$ for the metric subspace of $H^1(S,\mathbb{R}^3)$ made by vector-valued functions with values in T.

2.1.2. *Surfaces of revolution.* In what follows we denote by $I \subseteq \mathbb{R}$ a closed interval, by $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard ordered basis of $\mathbb{R}^3$, and by $A^\top(\phi)$ the rotation matrix about the $\boldsymbol{e}_3$-axis given by

$$A^\mathsf{T}(\phi) := \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

By a regular simple curve, we mean the image of a C^1-map $\gamma : I \mapsto \mathbb{R}^3$ such that $\dot\gamma(t) \neq 0$ for every $t \in I$, and with no self-intersections, i.e., such that the only possible loss of injectivity in γ arises at the endpoints of I, case in which the curve closes into a loop. As customary, we often refer to γ as a curve rather than just its image.

Given a regular simple curve $\gamma : t \in I \mapsto \gamma(t) = (x(t), 0, z(t)) \in \mathbb{R}^3$, with $x(t) \geqslant 0$, the surface of revolution $S \subseteq \mathbb{R}^3$ generated by γ is the image of the parameterization defined for $0 \leqslant \phi \leqslant 2\pi$ and $t \in I$ by

$$\boldsymbol{\xi}(\phi, t) = A^\top(\phi)\gamma(t). \tag{2.3}$$

In more intrinsic terms, given a regular simple curve γ in the x,z-plane, which lies at a nonnegative distance from the $\boldsymbol{e}_3$-axis, the surface of revolution S generated by γ is the set $S := \cup_{\xi\in\gamma}((\xi\cdot\boldsymbol{e}_3)\boldsymbol{e}_3 + \mathbb{S}^1(\xi\cdot\boldsymbol{e}_1))$, where $\mathbb{S}^1(r)$ is the circle in $\mathbb{R}^2 \times \{0\}$ centered at the origin and of radius $r > 0$. For our purposes, it is convenient to denote by $S_\xi := (\xi\cdot\boldsymbol{e}_3)\boldsymbol{e}_3 + \mathbb{S}^1(\xi\cdot\boldsymbol{e}_1)$ the circle of latitude at $\xi \in S$. After that, we have that

$$S := \cup_{\xi\in\gamma} S_\xi. \tag{2.4}$$

Given that γ is defined on a closed interval, the resulting surface of revolution is always *topologically* closed and possibly with a boundary (see Figure 1).

The tangent space to S at $\boldsymbol{\xi}(\phi,t)$ is generated by the two vectors

$$\boldsymbol{\tau}_\phi(\phi,t) := \partial_\phi \boldsymbol{\xi}(\phi,t) = \partial_\phi A^\top(\phi)\gamma(t), \tag{2.5}$$

$$\boldsymbol{\tau}_t(\phi,t) := \partial_t \boldsymbol{\xi}(\phi,t) = A^\top(\phi)\dot\gamma(t). \tag{2.6}$$

Given that $A(\phi)\partial_\phi A^\top(\phi)v = \boldsymbol{e}_3 \times v$ for every $v \in \mathbb{R}^3$, one easily gets that

$$\boldsymbol{\tau}_\phi(\phi,t)\cdot\boldsymbol{\tau}_t(\phi,t) = A(\phi)\partial_\phi A^\top(\phi)\gamma(t)\cdot\dot\gamma(t) = (\boldsymbol{e}_3\times\gamma(t))\cdot\dot\gamma(t) = (\gamma(t)\times\dot\gamma(t))\cdot\boldsymbol{e}_3 = 0, \tag{2.7}$$

because $\gamma(t)\times\dot\gamma(t)$ is directed along $\boldsymbol{e}_2$. Thus, the coordinate system is *orthogonal* but, in general, *not* orthonormal. Indeed, the metric coefficients are given by

$$\mathfrak{h}_1(t) := |\boldsymbol{\tau}_\phi| = |x(t)|, \quad \mathfrak{h}_2(t) := |\boldsymbol{\tau}_t| = |\dot\gamma(t)|. \tag{2.8}$$

Since $\boldsymbol{\tau}_\phi(\phi,t)$ and $\boldsymbol{\tau}_t(\phi,t)$ are orthogonal, the area element on S assumes the form

$$\sqrt{\mathfrak{g}(t)} = |\partial_\phi\boldsymbol{\xi}\times\partial_t\boldsymbol{\xi}| = |\boldsymbol{\tau}_\phi(\phi,t)|\cdot|\boldsymbol{\tau}_t(\phi,t)| = \mathfrak{h}_1(t)\cdot\mathfrak{h}_2(t). \tag{2.9}$$

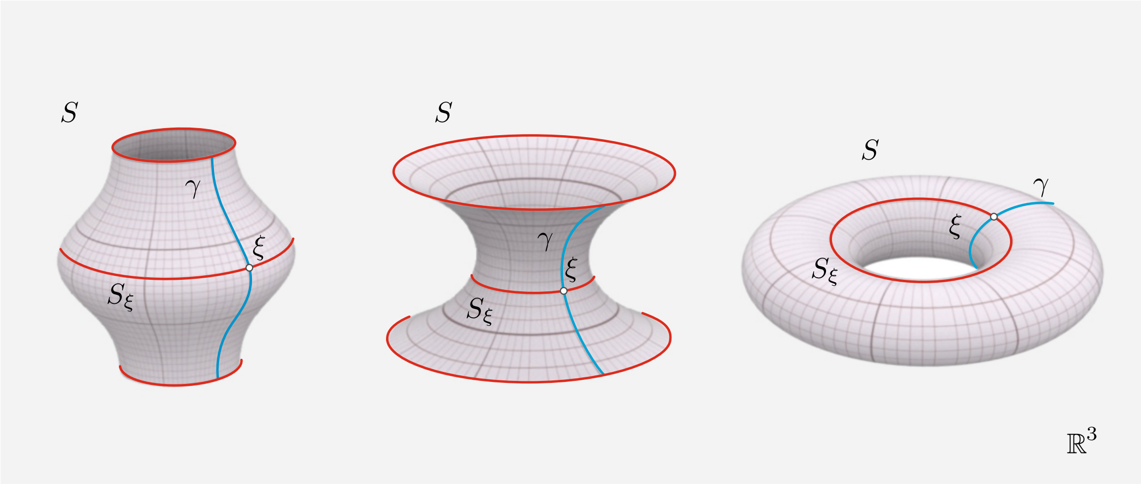

FIGURE 1. Given a regular curve γ in the x, z-plane, which lies at a nonnegative distance from the $\boldsymbol{e}_3$-axis, the surface of revolution S generated by γ is the set $S := \cup_{\xi\in\gamma} S_\xi$, where $S_\xi := (\xi \cdot \boldsymbol{e}_3)\boldsymbol{e}_3 + \mathbb{S}^1(\xi \cdot \boldsymbol{e}_1)$ is the circle of latitude at $\xi \in S$. The resulting surface can have a boundary or not.

Note that the area element in (2.9) depends only on the t variable and, by the regularity hypotheses in Remark 1, we have that $\sqrt{\mathfrak{g}(t)} > 0$ for every $t \in I$ with the possible exception of the two boundary points of γ if γ touches the $\boldsymbol{e}_3$-axis.

Remark 1. (*On the regularity of surfaces of revolution*) Throughout the paper, we always assume that the generated surface of revolution S is *regular*, i.e., that the parameterization (2.3) is of class C^1, and that the induced area element (2.9) never vanishes. Now, since the generating curve γ is regular, the surface of revolution S induced by γ is certainly regular whenever $\varrho_\gamma := \inf_{t\in I} x(t) > 0$. Instead, if $\varrho_\gamma = 0$, we assume, as in the case of the meridian arc generating $\mathbb{S}^2$, that γ touches the $\boldsymbol{e}_3$-axis perpendicularly and (at most) at two distinct points so that a smooth surface of revolution results. In what follows, whenever we name a surface of revolution, it is always understood that the previous regularity assumptions are met.

2.1.3. *Symmetric vector fields.* The results of our paper guarantee the existence of minimizers with specific symmetry properties. The notion of an axially symmetric vector field is standard.

Definition 1. *With $S, T \subseteq \mathbb{R}^3$ being two surfaces of revolution, we say that a vector field $\boldsymbol{u} : S \to T$ is* axially symmetric *when the following property holds:*

$$\boldsymbol{u}(A^\top(\phi)\xi) = A^\top(\phi)\boldsymbol{u}(\xi) \quad \forall\phi \in \mathbb{R}, \forall\xi \in S. \tag{2.10}$$

We say that $\boldsymbol{u}$ is axially antisymmetric *if*

$$\boldsymbol{u}(A^\top(\phi)\xi) = A(\phi)\boldsymbol{u}(\xi) \quad \forall\phi \in \mathbb{R}, \forall\xi \in S. \tag{2.11}$$

We say that a vector field $\boldsymbol{u}$ has line symmetry *if for every fixed $\xi \in S$ one has either $\boldsymbol{u}(A^\top(\phi)\xi) = A^\top(\phi)\boldsymbol{u}(\xi)$ for every $\phi \in \mathbb{R}$ or $\boldsymbol{u}(A^\top(\phi)\xi) = A(\phi)\boldsymbol{u}(\xi)$ for every $\phi \in \mathbb{R}$.*

Remark 2. Surfaces of revolution are G-sets with respect to the action of the group G of rotations around the $\boldsymbol{e}_3$-axis. In the language of transformation groups, an axially symmetric vector field can be equivalently defined as an equivariant map with respect to the G-sets S and T, where G is the group of rotations around the $\boldsymbol{e}_3$-axis. Similarly, one could define an axially *anti*symmetric vector field as an equi*contra*variant map with respect to the G-sets S and T.

In local coordinates, and with the convenient abuse of notation $\boldsymbol{u}(\phi,t) := (\boldsymbol{u}\circ\boldsymbol{\xi})(\phi,t)$, the vector field $\boldsymbol{u} : S \to T$ is axially symmetric if, and only if, $\boldsymbol{u}(\phi,t) = A^\top(\phi)\boldsymbol{u}(0,t)$ for every $(\phi,t)\in\mathbb{R}\times I$, and is axially antisymmetric if, and only if, $\boldsymbol{u}(\phi,t) = A(\phi)\boldsymbol{u}(0,t)$ for every $(\phi,t)\in\mathbb{R}\times I$. Instead, in general, if $\boldsymbol{u}$ has line symmetry, then we are still allowing for the existence of $t_1 \neq t_2 \in I$ such that $\boldsymbol{u}(\phi,t_1) = A^\mathsf{T}(\phi)\boldsymbol{u}(0,t_1)$ and $\boldsymbol{u}(\phi,t_2) = A(\phi)\boldsymbol{u}(0,t_2)$ for every $\phi\in\mathbb{R}$.

Note that the class of axially symmetric vector-field is essentially disjoint from the class of axially antisymmetric vector fields. Indeed, if $A^\mathsf{T}(\phi)\boldsymbol{\alpha}(t) = A(\phi)\boldsymbol{\beta}(t)$ for T-valued profiles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, then necessarily $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ is directed along the $\boldsymbol{e}_3$-axis. Also, note that the vector field $\boldsymbol{v}(\phi,t) = A(\phi)\boldsymbol{\alpha}(t)$ is axially antisymmetric if, and only if, $\boldsymbol{u}(\phi,t) = \boldsymbol{v}(-\phi,t)$ is axially symmetric.

We provide some examples to help clarify the notion of line symmetry. Suppose $\gamma : I \to \mathbb{R}^3$ is the image of the curve generating the surface of revolution S and $\boldsymbol{\alpha} : I \mapsto T$ is a smooth vector field along γ. The vector field $\boldsymbol{u} : S \to T$ defined in local coordinates by $\boldsymbol{u}(\phi,t) = A^\top(\phi)\boldsymbol{\alpha}(t)$ is axially symmetric. The vector field $\boldsymbol{v} : S \to T$ defined in local coordinates by $\boldsymbol{v}(\phi,t) = A(\phi)\boldsymbol{\alpha}(t)$ is axially antisymmetric. Axially symmetric and antisymmetric vector fields belong to the class of vector fields with line symmetries. However, there are examples of vector fields with line symmetry that are neither axially symmetric nor axially antisymmetric. For that, consider the surface of revolution $S = 2\mathbb{D}$ given by the disk of radius two centered at the origin. If $\boldsymbol{v} : \mathbb{D} \to T$ is any axially antisymmetric vector field such that $\boldsymbol{v}_{|\partial\mathbb{D}} \equiv \boldsymbol{e}_3$, and $\boldsymbol{u} : 2\mathbb{D}\backslash\mathbb{D} \to T$ is any axially symmetric vector field such that $\boldsymbol{u}_{|\partial\mathbb{D}} \equiv \boldsymbol{e}_3$, then the vector field $\boldsymbol{w} : 2\mathbb{D} \to T$ obtained by gluing $\boldsymbol{u}$ and $\boldsymbol{v}$ along $\partial\mathbb{D}$ is a continuous vector field with line symmetry, which is neither axially symmetric nor axially antisymmetric; the example is prototypical because it can be easily generalized to build smooth line-symmetric vector fields defined on arbitrary surfaces of revolution that are not axially symmetric nor axially antisymmetric.

We stress that for a given profile $\boldsymbol{\alpha} : I \to T$, the axially symmetric vector field $A^\top(\phi)\boldsymbol{\alpha}(t)$ and the axially antisymmetric vector field $A(\phi)\boldsymbol{\alpha}(t)$ generated by $\boldsymbol{\alpha}$ are, in general, different from the global point of view. For example (cf. Figure 2), for $\boldsymbol{\alpha} : t \in I \mapsto (\sin t, 0, \cos t) \in \mathbb{S}^2$, $I = [-1,1]$, the axially symmetric vector field $A^\mathsf{T}(\phi)\boldsymbol{\alpha}(t)$ has mirror symmetry with respect to every plane orthogonal to $\mathbb{S}^1 \times \{0\}$, while the generated axially antisymmetric vector field has only a finite number of mirror symmetries. However, note that both share the same Dirichlet energy $\mathcal{D}_S$ and the same energy $\mathcal{P}_S$. In our setting, the anisotropy term $\mathcal{A}_S$ is the only contribution that, from the variational point of view, can ultimately favor one over the other.

2.2. Statement of main results. Even though our primary motivation comes from curved thin-film structures, we formulate and prove our results for the case where the target space is a generic surface of revolution rather than just $\mathbb{S}^2$. With the usual abuse of notation, we set $\omega(\phi,t) := \omega(\boldsymbol{\xi}(\phi,t))$, $(\phi,t)\in[0,2\pi]\times I$, and, for every $t\in I$ we introduce the *circular integral weight*

$$W^2(t) := \int_0^{2\pi} \omega^2(\phi,t)\mathrm{d}\phi. \tag{2.12}$$

We recall that the metric coefficient $\mathfrak{h}_1(t) := |x(t)|$ measures the distance of the curve γ generating S from the $\boldsymbol{e}_3$-axis or, equivalently, the radius of the circle of latitude $S_{\gamma(t)}$. The first result of our paper is stated in the following theorem.

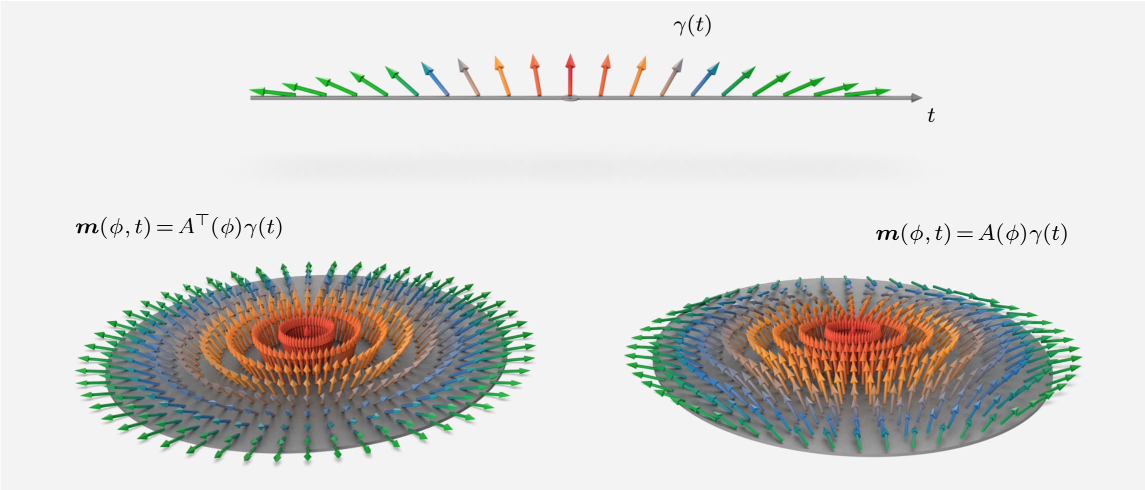

Figure 2. Given the profile $\gamma : t \in I \mapsto (\sin t, 0, \cos t) \in \mathbb{S}^2$, $I = [-1, 1]$, the axially symmetric vector field $\boldsymbol{m}(\phi,t) := A^{\mathsf{T}}(\phi)\gamma(t)$ (depicted on the left) and the axially antisymmetric vector field $\boldsymbol{m}(\phi,t) := A(\phi)\gamma(t)$ (depicted on the right) can look pretty different. Nevertheless, they have the same Dirichlet energy.

Theorem 1. *For given surfaces of revolution $S, T \subseteq \mathbb{R}^3$, we consider the energy functional defined for every $\boldsymbol{m} \in H^1(S,T)$ by (cf. (1.1))*

$$\mathcal{E}_\omega(\boldsymbol{m}) := \int_S |\nabla \boldsymbol{m}(\xi)|^2 \mathrm{d}\xi + \int_S g(\boldsymbol{m}(\xi) \cdot \boldsymbol{a}(\xi)) \mathrm{d}\xi + \int_S |\langle \boldsymbol{m} \rangle_{S_\xi} \times \boldsymbol{e}_3|^2 \omega^2(\xi) \mathrm{d}\xi$$

with $\omega : S \to \mathbb{R}_+$ a measurable weight such that $\sup_{t\in I}(\mathfrak{h}_1(t)W(t)) < +\infty$, $g : \mathbb{R} \to \mathbb{R}_+$ a Lipschitz function and $\boldsymbol{a} : S \to \mathbb{R}^3$ a Lipschitz vector field.

If $\mathfrak{h}_1(t)W(t) > \sqrt{2\pi}$ for every $t \in I$, the following assertions hold:

i. If the vector field $\boldsymbol{a}$ is axially symmetric, then any minimizer $\boldsymbol{m} \in H^1(S,T)$ of $\mathcal{E}_\omega$ has an axially symmetric representative, in the sense that if $\boldsymbol{m}$ is a minimizer of $\mathcal{E}_\omega$ then there exists an axially symmetric vector field $\boldsymbol{u} \in H^1(S,T)$, built from $\boldsymbol{m}$, such that $\mathcal{E}_\omega(\boldsymbol{u}) = \mathcal{E}_\omega(\boldsymbol{m})$.

ii. If the vector field $\boldsymbol{a}$ is axially antisymmetric, then any minimizer $\boldsymbol{m} \in H^1(S,T)$ of $\mathcal{E}_\omega$ has an axially antisymmetric representative, in the sense that if $\boldsymbol{m}$ is a minimizer of $\mathcal{E}_\omega$ then there exists an axially antisymmetric vector field $\boldsymbol{u} \in H^1(S,T)$, built from $\boldsymbol{m}$, such that $\mathcal{E}_\omega(\boldsymbol{u}) = \mathcal{E}_\omega(\boldsymbol{m})$.

In any case, if $\boldsymbol{a}$ is axially symmetric or antisymmetric, when $\mathfrak{h}_1(t)W(t) > \sqrt{2\pi}$ any minimizer $\boldsymbol{m} \in H^1(S,T)$ of $\mathcal{E}_\omega$ is of the form

$$\boldsymbol{m}(\phi,t) = \boldsymbol{\alpha}_\perp(t)\cos\phi + \boldsymbol{\beta}_\perp(t)\sin\phi + \eta(t)\boldsymbol{e}_3. \tag{2.13}$$

for some $\boldsymbol{\alpha}_\perp, \boldsymbol{\beta}_\perp \in H^1(I,\mathbb{R}^2)$, $\eta \in H^1(I,\mathbb{R})$, subject to the constraint of the resulting $\boldsymbol{m}$ being T-valued.

Remark 3. Configurations of the form (2.13) belong to the class of vector fields satisfying the property that $\langle \boldsymbol{m}_\perp \rangle_{S_\xi} = 0$ for every $\xi \in S$, and later on referred to as *axially null-average* (see Definition 3). Thus, in particular, Theorem 1 says that any minimizer is axially null-average provided that $\mathfrak{h}_1(t)W(t) > \sqrt{2\pi}$ for every $t \in I$. When $\inf_{t\in I}(\mathfrak{h}_1(t)W(t)) = \sqrt{2\pi}$ we cannot conclude that minimizers are still axially null-average. However, it will be evident from the proof (cf. (3.34)) that any minimizer must be of the (more general) form $\boldsymbol{m}(\phi,t) =$

$\boldsymbol{\zeta}(t) + \boldsymbol{\alpha}_\perp(t)\cos\phi + \boldsymbol{\beta}_\perp(t)\sin\phi$ for suitable $\boldsymbol{\alpha}_\perp, \boldsymbol{\beta}_\perp \in H^1(I, \mathbb{R}^2)$, $\boldsymbol{\zeta} \in H^1(I, \mathbb{R}^3)$ subject to the constraint of the resulting $\boldsymbol{m}$ being T-valued.

Remark 4. The assumption of T being a surface of revolution (around the $\boldsymbol{e}_3$-axis) cannot be removed. Indeed, the entire argument is based on the closure property that if $\boldsymbol{m}$ is T-valued, then $A^\top(\phi)\boldsymbol{m}(\xi) \in T$ for every $\phi \in \mathbb{R}$ and every $\xi \in S$. However, note that it is not necessary to (and we do not) assume $\boldsymbol{a}$ to be T-valued. When we say that $\boldsymbol{a} : S \to \mathbb{R}^3$ is, e.g., axially symmetric, what we rigorously mean, that matches with Definition 1, is that the image of $\boldsymbol{a}$ is included in a surface of revolution which can be different from T.

Remark 5. As it will be apparent from the proofs, our arguments still work if one considers a multivariable potential of the form $g(\boldsymbol{m}(\xi)\cdot\boldsymbol{a}_1(\xi), \boldsymbol{m}(\xi)\cdot\boldsymbol{a}_2(\xi), \ldots, \boldsymbol{m}(\xi)\cdot\boldsymbol{a}_n(\xi))$ where the vector fields $(\boldsymbol{a}_i)_{i=1}^n$ are either all axially symmetric or all axially antisymmetric. Also, as we explain in Remark 6, our arguments extend to the case in which the surface of revolution S has a boundary, and we look for minimizers in $H^1(S, T)$ satisfying prescribed axially symmetric (or axially antisymmetric) Dirichlet boundary conditions. More generally, our results still hold when the anisotropy energy $\mathcal{A}_S$ assumes the form

$$\mathcal{A}_S(\boldsymbol{m}) := \int_S g(\boldsymbol{m}(\xi)\cdot\boldsymbol{a}_1(\xi))\mathrm{d}\xi + \int_{\partial S} b(\boldsymbol{m}(\sigma)\cdot\boldsymbol{a}_2(\sigma))\mathrm{d}\sigma,$$

i.e., when, as in [15], boundary penalizations are involved. To control the complexity of the formulas, we do not treat these generalizations. However, it should be evident from the arguments that our results cover these more general settings with no technical changes at all.

The expression of the minimizer $\boldsymbol{m}$ in (2.13) allows for both axially symmetric and antisymmetric vector fields. Indeed, if we set $\boldsymbol{\alpha}(t) := \boldsymbol{\alpha}_\perp(t) + \eta(t)\boldsymbol{e}_3$ then we can express the vector field $\boldsymbol{m}$ in (2.13) under the equivalent forms (cf. (3.40)-(3.41))

$$\boldsymbol{m}(\phi, t) = A^\mathsf{T}(\phi)\boldsymbol{\alpha}(t) + (\sin\phi)(\boldsymbol{\beta}_\perp(t) - \boldsymbol{e}_3 \times \boldsymbol{\alpha}_\perp(t)), \tag{2.14}$$

and

$$\boldsymbol{m}(\phi, t) = A(\phi)\boldsymbol{\alpha}(t) + (\sin\phi)(\boldsymbol{\beta}_\perp(t) + \boldsymbol{e}_3 \times \boldsymbol{\alpha}_\perp(t)). \tag{2.15}$$

Thus, $\boldsymbol{m}$ is axially symmetric when $\boldsymbol{\beta}_\perp(t) = \boldsymbol{e}_3 \times \boldsymbol{\alpha}_\perp(t)$, and it is axially antisymmetric when $\boldsymbol{\beta}_\perp(t) = -\boldsymbol{e}_3 \times \boldsymbol{\alpha}_\perp(t)$. Theorem 1 guarantees the existence of axially symmetric or axially antisymmetric representatives, but, in general, it cannot rule out the coexistence of minimizers with a broken symmetry term of the type $(\sin\phi)(\boldsymbol{\beta}_\perp(t) \pm \boldsymbol{e}_3 \times \boldsymbol{\alpha}_\perp(t))$. But things go better if one requires an additional condition on the target manifold T, which prevents the existence of flat zones in T.

Definition 2. *Let $\boldsymbol{\pi}_\perp$ be the projection of $\mathbb{R}^3$ onto $\Sigma_0 := \mathbb{R}^2 \times \{0\}$ and, for every $z \in \mathbb{R}$, let $\Sigma_z := z\boldsymbol{e}_3 + \Sigma_0$ be the plane parallel to Σ_0 and passing through $z\boldsymbol{e}_3$. Note that if T is a surface of revolution, then $\boldsymbol{\pi}_\perp(T \cap \Sigma_z)$ is either empty or the union of circles in Σ_0. We say that T is* never flat *if for any $z \in \mathbb{R}$ the following property holds: either $\boldsymbol{\pi}_\perp(T \cap \Sigma_z)$ is empty, or $\boldsymbol{\pi}_\perp(T \cap \Sigma_z)$ consists of a unique circle, or it consists of a finite family of circles at a positive distance from each other.*

If the profile γ generating the surface of revolution T is the graph of a smooth function, i.e., of the form $\gamma(t) = (t, 0, z(t))$, then the never-flat condition amounts to asking that the function z does not have intervals where it is constant. The never-flat condition is quite general and satisfied, e.g., by all surfaces of revolution represented in Figure 1. Instead, the unit disk in Σ_0 does not satisfy the never-flat condition.

Theorem 2. *For given surfaces of revolution $S, T \subseteq \mathbb{R}^3$, we consider the energy functional defined for every $\boldsymbol{m} \in H^1(S,T)$ by (cf. (1.1))*

$$\mathcal{E}_\omega(\boldsymbol{m}) := \int_S |\nabla \boldsymbol{m}(\xi)|^2 \mathrm{d}\xi + \int_S g(\boldsymbol{m}(\xi)\cdot \boldsymbol{a}(\xi))\mathrm{d}\xi + \int_S |\langle \boldsymbol{m}\rangle_{S_\xi} \times \boldsymbol{e}_3|^2 \omega^2(\xi)\mathrm{d}\xi$$

with $\omega : S \to \mathbb{R}_+$ a measurable weight such that $\sup_{t\in I}(\mathfrak{h}_1(t)W(t)) < +\infty$, $g : \mathbb{R} \to \mathbb{R}_+$ a Lipschitz function and $\boldsymbol{a} : S \to \mathbb{R}^3$ a Lipschitz vector field.

If $\mathfrak{h}_1(t)W(t) > \sqrt{2\pi}$ for every $t \in I$ and T is never flat, then the following assertions hold:

i. If the vector field $\boldsymbol{a}$ is axially symmetric, then either every *minimizer is axially symmetric, or there also coexist minimizers with line symmetry. An axially symmetric minimizer is given by $\boldsymbol{m}(\phi,t) := A^{\mathsf{T}}(\phi)\boldsymbol{\gamma}_s(t)$ where $\boldsymbol{\gamma}_s$ is a solution to the one-dimensional minimization problem*

$$\mathcal{F}_s(\boldsymbol{\gamma}) := \mathcal{E}_\omega(A^{\mathsf{T}}(\phi)\boldsymbol{\gamma}(t))$$

among all possible profiles $\boldsymbol{\gamma} \in H^1(I,T)$.

ii. If the vector field $\boldsymbol{a}$ is axially antisymmetric, then either every *minimizer is axially antisymmetric, or there also coexist minimizers with line symmetry. An axially antisymmetric minimizer is given by $\boldsymbol{m}(\phi,t) := A(\phi)\boldsymbol{\gamma}_a(t)$ where $\boldsymbol{\gamma}_a$ is a solution to the one-dimensional minimization problem*

$$\mathcal{F}_a(\boldsymbol{\gamma}) := \mathcal{E}_\omega(A(\phi)\boldsymbol{\gamma}(t))$$

among all possible profiles $\boldsymbol{\gamma} \in H^1(I,T)$.

In any case, if the vector field $\boldsymbol{a}$ is axially symmetric or antisymmetric, then any minimizer $\boldsymbol{m} \in H^1(S,T)$ of $\mathcal{E}_\omega$ has line symmetry and is of the form

$$\boldsymbol{m}(\phi,t) = \boldsymbol{\alpha}_\perp(t)\cos\phi + \boldsymbol{\beta}_\perp(t)\sin\phi + \eta(t)\boldsymbol{e}_3. \tag{2.16}$$

for some $\boldsymbol{\alpha}_\perp, \boldsymbol{\beta}_\perp \in H^1(I,\mathbb{R}^2)$, $\eta \in H^1(I,\mathbb{R})$, subject to the orthogonality conditions $\boldsymbol{\beta}_\perp(t)\cdot \boldsymbol{\alpha}_\perp(t) \equiv 0$ and $|\boldsymbol{\beta}_\perp(t)| \equiv |\boldsymbol{\alpha}_\perp(t)|$, and to the conditions of the resulting $\boldsymbol{m}$ being T-valued.

3. Axially (anti)symmetric minimizers: Proofs of Theorems 1 and 2

We prove Theorems 1 and 2, assuming that the vector field $\boldsymbol{a} : S \to \mathbb{R}^3$ is axially symmetric. The proof works the same when $\boldsymbol{a} : S \to \mathbb{R}^3$ is axially antisymmetric, provided that the words *symmetric* and *antisymmetric* are exchanged in the proper obvious places and formula (3.23) is replaced with the assignment $\boldsymbol{u}(\phi,t) := A(\phi)A^{\mathsf{T}}(\phi_*)\boldsymbol{m}(\phi_*,t)$.

For clarity, we subdivide the proof in several steps. In order to improve the readability of the argument, we summarize here the steps. In Step 1, we reformulate the energy functional in local coordinates through the parameterization $\boldsymbol{\xi}$ of S defined by (2.3). In Step 2, we derive an auxiliary estimate which, in particular, assures that the construction in Step 3 is well-posed. In Step 3, we show that if $\mathfrak{h}_1(t)W(t) \geqslant \sqrt{2\pi}$, then given any minimizer $\boldsymbol{m}$ of $\mathcal{E}_\omega$, there exists an axially symmetric vector field $\boldsymbol{u} \in H^1(S,T)$, built from $\boldsymbol{m}$, with the same minimal energy. In working out the details in Step 3 we assume that the minimal vector field $\boldsymbol{m}$ is smooth, and we show in Step 5 how to avoid this assumption. In Step 4, we finalize the proofs of Theorems 1 and 2.

$\boxed{\text{Step 1}}$ First, we rephrase the problem in local coordinates by using the parameterization $\boldsymbol{\xi}$ of S defined in (2.3). With the usual abuse of notation, we set $\boldsymbol{m}(\phi,t) := \boldsymbol{m}(\boldsymbol{\xi}(\phi,t))$, being aware that the context will always clarify such an overload of the symbol $\boldsymbol{m}$. In local coordinates, the shape anisotropy term $\mathcal{A}_S$ in (1.3) reads as

$$\mathcal{A}_S(\boldsymbol{m}) = \int_{t\in I}\int_0^{2\pi} g(\boldsymbol{m}(\phi,t)\cdot\boldsymbol{a}(\phi,t))\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t. \tag{3.1}$$

Also, with $\boldsymbol{m}_\perp := (\boldsymbol{m}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1+(\boldsymbol{m}\cdot\boldsymbol{e}_2)\boldsymbol{e}_2$, and the assignment $\langle\boldsymbol{m}_\perp\rangle(t) := \frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{m}_\perp(\theta,t)\mathrm{d}\theta$, the term $\mathcal{P}_S$ in (1.4) can be written as follows:

$$\begin{aligned}
\mathcal{P}_S(\boldsymbol{m}) &= \int_S |\omega(\xi)\langle\boldsymbol{m}_\perp\rangle_{S_\xi}|^2\mathrm{d}\xi && (3.2)\\
&= \int_{t\in I}\int_0^{2\pi}\left|\frac{\omega(\boldsymbol{\xi}(\phi,t))}{|S_{\boldsymbol{\xi}(\phi,t)}|}\int_{S_{\boldsymbol{\xi}(\phi,t)}}\boldsymbol{m}_\perp(\sigma)\mathrm{d}\sigma\right|^2\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t && (3.3)\\
&= \int_{t\in I}\int_0^{2\pi}\omega^2(\boldsymbol{\xi}(\phi,t))\left|\frac{1}{|2\pi\mathfrak{h}_1(t)|}\int_0^{2\pi}\boldsymbol{m}_\perp(\theta,t)|\partial_\theta\boldsymbol{\xi}(\theta,t)|\mathrm{d}\theta\right|^2\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t && (3.4)\\
&= \int_{t\in I}\left(\int_0^{2\pi}\omega^2(\boldsymbol{\xi}(\phi,t))\mathrm{d}\phi\right)|\langle\boldsymbol{m}_\perp\rangle(t)|^2\sqrt{\mathfrak{g}(t)}\mathrm{d}t && (3.5)\\
&= \int_{t\in I} W^2(t)|\langle\boldsymbol{m}_\perp\rangle(t)|^2\sqrt{\mathfrak{g}(t)}\mathrm{d}t. && (3.6)
\end{aligned}$$

In computing the previous expressions we took into account that $|S_{\boldsymbol{\xi}(\phi,t)}| = 2\pi\mathfrak{h}_1(t)$ and $|\partial_\theta\boldsymbol{\xi}(\theta,t)| = \mathfrak{h}_1(t)$.

Finally, we focus on the expression in local coordinates of the Dirichlet energy term $\mathcal{D}_S$ in (1.3). To that end, we observe that with the notation introduced in Section 2, in particular (2.5) and (2.6), and with the usual abuse of notation $\boldsymbol{m}(\phi,t) := (\boldsymbol{m}\circ\boldsymbol{\xi})(\phi,t)$, there holds that

$$(\partial_{\boldsymbol{\tau}_\phi}\boldsymbol{m}\circ\boldsymbol{\xi})(\phi,t) = \partial_\phi\boldsymbol{m}(\phi,t), \qquad (\partial_{\boldsymbol{\tau}_t}\boldsymbol{m}\circ\boldsymbol{\xi})(\phi,t) = \partial_t\boldsymbol{m}(\phi,t), \tag{3.7}$$

because, e.g., $\partial_\phi\boldsymbol{m}(\phi,t) = [D\boldsymbol{m}\circ\boldsymbol{\xi}(\phi,t)]\boldsymbol{\tau}_\phi(\phi,t)$. Hence, we find that

$$|(\nabla\boldsymbol{m}\circ\boldsymbol{\xi})(\phi,t)|^2 = \frac{|\partial_\phi\boldsymbol{m}(\phi,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t\boldsymbol{m}(\phi,t)|^2}{\mathfrak{h}_2^2(t)} \tag{3.8}$$

with $\mathfrak{h}_1$ and $\mathfrak{h}_2$ the metric coefficients defined in (2.8). It follows that the Dirichlet energy reads under the form

$$\int_S|\nabla\boldsymbol{m}(\xi)|^2\mathrm{d}\xi = \int_{t\in I}\int_0^{2\pi}\frac{|\partial_\phi\boldsymbol{m}(\phi,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t\boldsymbol{m}(\phi,t)|^2}{\mathfrak{h}_2^2(t)}\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t. \tag{3.9}$$

Overall, in the coordinate chart induced by $\boldsymbol{\xi}$, the energy functional on $H^1(S,T)$ assumes the form

$$\begin{aligned}
\mathcal{E}_\omega(\boldsymbol{m}) = \int_{t\in I}\int_0^{2\pi}&\frac{|\partial_\phi\boldsymbol{m}(\phi,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t\boldsymbol{m}(\phi,t)|^2}{\mathfrak{h}_2^2(t)}\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t\\
&+\int_{t\in I}W^2(t)|\langle\boldsymbol{m}_\perp\rangle(t)|^2\sqrt{\mathfrak{g}(t)}\mathrm{d}t\\
&\qquad+\int_{t\in I}\int_0^{2\pi}g(\boldsymbol{m}(\phi,t)\cdot\boldsymbol{a}(\phi,t))\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t.
\end{aligned} \tag{3.10}$$

From now on, we are going to work with the local coordinates expression (3.10).

$\boxed{\text{Step 2}}$ The main aim of this step is to guarantee that the construction in the next Step 3 is well-posed. In what follows, we denoted by $x(\xi)$ the distance of $\xi\in S$ from the $\boldsymbol{e}_3$-axis, i.e., the radius of the circle of latitude S_ξ. Note that, in local coordinates, $x(\boldsymbol{\xi}(\phi,t)) = \mathfrak{h}_1(t)$. We want to show that if $\boldsymbol{m} \in H^1(S,T)$ has finite $\mathcal{E}_\omega$-energy (in particular, if it is a minimizer) and $\mathfrak{h}_1^2(t)W^2(t) \geqslant 2\pi$ then

$$\int_S |\boldsymbol{m}_\perp(\xi)|^2 \frac{\mathrm{d}\xi}{x^2(\xi)} < +\infty. \tag{3.11}$$

In order to establish (3.11), it is sufficient to show that in local coordinates one has

$$\int_{t\in I}\int_0^{2\pi} \frac{|\boldsymbol{m}_\perp(\phi,t)|^2}{\mathfrak{h}_1^2(t)}\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t < +\infty. \tag{3.12}$$

For that, we recall that for periodic functions there holds the following Poincaré-Wirtinger inequality

$$\int_0^{2\pi} |\boldsymbol{u}(\phi) - \langle \boldsymbol{u}\rangle|^2 \mathrm{d}\phi \leqslant \int_0^{2\pi} |\partial_\phi \boldsymbol{u}(\phi)|^2\mathrm{d}\phi, \quad \langle \boldsymbol{u}\rangle := (2\pi)^{-1}\int_0^{2\pi} \boldsymbol{u}(\phi)\mathrm{d}\phi. \tag{3.13}$$

Moreover, given the quadratic setting, the left-hand side of the previous relation can be written as

$$\int_0^{2\pi} |\boldsymbol{u}(\phi) - \langle \boldsymbol{u}\rangle|^2 \mathrm{d}\phi = \int_0^{2\pi} |\boldsymbol{u}(\phi)|^2 - |\langle \boldsymbol{u}\rangle|^2\mathrm{d}\phi. \tag{3.14}$$

Therefore, if we write $\boldsymbol{m} = \boldsymbol{m}_\perp + (\boldsymbol{m}\cdot\boldsymbol{e}_3)\boldsymbol{e}_3$ then for every $t \in I$, there holds

$$\begin{aligned} 0 &\leqslant \int_{t\in I}\int_0^{2\pi} \frac{|\boldsymbol{m}_\perp(\phi,t)|^2 - |\langle \boldsymbol{m}_\perp\rangle(t)|^2}{\mathfrak{h}_1^2(t)}\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t \\ &\leqslant \int_{t\in I}\int_0^{2\pi} \frac{|\partial_\phi\boldsymbol{m}_\perp(\phi,t)|^2}{\mathfrak{h}_1^2(t)}\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t. \end{aligned} \tag{3.15}$$

Hence, whenever $\mathfrak{h}_1^2(t)W^2(t) \geqslant 2\pi$ we get

$$\begin{aligned} 0 &\leqslant \int_{t\in I}\int_0^{2\pi} \frac{|\boldsymbol{m}_\perp(\phi,t)|^2}{\mathfrak{h}_1^2(t)}\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t && (3.16)\\ &\leqslant \int_{t\in I}\int_0^{2\pi} \frac{|\partial_\phi\boldsymbol{m}_\perp(\phi,t)|^2}{\mathfrak{h}_1^2(t)}\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t + 2\pi\int_{t\in I}\frac{|\langle \boldsymbol{m}_\perp\rangle(t)|^2}{\mathfrak{h}_1^2(t)}\sqrt{\mathfrak{g}(t)}\mathrm{d}t && (3.17)\\ &\leqslant \int_{t\in I}\int_0^{2\pi} \frac{|\partial_\phi\boldsymbol{m}_\perp(\phi,t)|^2}{\mathfrak{h}_1^2(t)}\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t + \int_{t\in I} W^2(t)|\langle \boldsymbol{m}_\perp\rangle(t)|^2\sqrt{\mathfrak{g}(t)}\mathrm{d}t, && (3.18) \end{aligned}$$

and this shows that the integral in (3.11) is finite provided that $\boldsymbol{m} \in H^1(S,T)$ has finite $\mathcal{E}_\omega$-energy and $\mathfrak{h}_1^2(t)W^2(t) \geqslant 2\pi$.

$\boxed{\text{Step 3}}$ *Given any minimizer $\boldsymbol{m}$ of $\mathcal{E}_\omega$, there exists an axially symmetric vector field $\boldsymbol{u} \in H^1(S,T)$, built from $\boldsymbol{m}$, with the same minimal energy.*

By direct methods in the calculus of variations, we know that there exists a (global) minimizer $\boldsymbol{m} \in H^1(S,T)$ of the energy functional $\mathcal{E}_\omega$, but, in general, more than one. We want to show that given any (global) minimizer $\boldsymbol{m}$ of $\mathcal{E}_\omega$, there exists an axially symmetric texture $\boldsymbol{u} \in H^1(S,T)$, built from $\boldsymbol{m}$, with the same minimal energy, i.e., such that $\mathcal{E}_\omega(\boldsymbol{u}) = \mathcal{E}_\omega(\boldsymbol{m})$. We first assume that $\boldsymbol{m} \in H^1(S,T)$ is a smooth minimizer of the energy $\mathcal{E}_\omega$. Afterward, in Step 5, we show how to remove this assumption through a density argument.

The idea is to introduce the real variable function

$$\Phi_{\mathcal{E}} : \phi \in [0,2\pi] \mapsto \ \Phi_{\mathcal{D}}(\phi) + \Phi_{\mathcal{A}}(\phi) \in \mathbb{R}_+, \tag{3.19}$$

where

$$\Phi_{\mathcal{D}}(\phi) := \int_{t\in I}\Big(\frac{|\boldsymbol{m}_{\perp}(\phi,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t \boldsymbol{m}(\phi,t)|^2}{\mathfrak{h}_2^2(t)}\Big)\sqrt{\mathfrak{g}(t)}\mathrm{d}t, \tag{3.20}$$

$$\Phi_{\mathcal{A}}(\phi) := \int_{t\in I} g(\boldsymbol{m}(\phi,t)\cdot \boldsymbol{a}(\phi,t))\sqrt{\mathfrak{g}(t)}\mathrm{d}t. \tag{3.21}$$

For $\phi_* \in \operatorname{argmin}_{\phi\in[0,2\pi]} \Phi_{\mathcal{E}}(\phi)$ we have that

$$\begin{aligned}\Phi_{\mathcal{E}}(\phi_*) = \int_{t\in I}\Big(\frac{|\boldsymbol{m}_{\perp}(\phi_*,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t \boldsymbol{m}(\phi_*,t)|^2}{\mathfrak{h}_2^2(t)}\Big)\sqrt{\mathfrak{g}(t)}\mathrm{d}t \\ + \int_{t\in I} g(\boldsymbol{m}(\phi_*,t)\cdot \boldsymbol{a}(\phi_*,t))\sqrt{\mathfrak{g}(t)}\mathrm{d}t.\end{aligned} \tag{3.22}$$

Next, we define a new vector field $\boldsymbol{u} \in H^1(S,T)$ via the relation

$$\boldsymbol{u}(\phi,t) := A^{\top}(\phi)A(\phi_*)\boldsymbol{m}(\phi_*,t). \tag{3.23}$$

We then have $\partial_t \boldsymbol{u}(\phi,t) = A^{\top}(\phi)A(\phi_*)\partial_t \boldsymbol{m}(\phi_*,t)$ and $\partial_\phi \boldsymbol{u}(\phi,t) = \partial_\phi A^{\top}(\phi)A(\phi_*)\boldsymbol{m}(\phi_*,t)$, from which, observing that $A(\phi)\partial_\phi A^{\mathsf{T}}(\phi)\boldsymbol{v} = \boldsymbol{e}_3 \times \boldsymbol{v}$ for every $\boldsymbol{v} \in \mathbb{R}^3$, we infer that

$$|\partial_t \boldsymbol{u}(\phi,t)|^2 = |\partial_t \boldsymbol{m}(\phi_*,t)|^2, \tag{3.24}$$

and

$$\begin{aligned}|\partial_\phi \boldsymbol{u}(\phi,t)|^2 &= |A(\phi)\partial_\phi A^{\top}(\phi)A(\phi_*)\boldsymbol{m}(\phi_*,t)|^2 \\ &= |\boldsymbol{e}_3 \times A(\phi_*)\boldsymbol{m}(\phi_*,t)|^2 \\ &= |\boldsymbol{e}_3 \times \boldsymbol{m}(\phi_*,t)|^2 \\ &= |\boldsymbol{m}_{\perp}(\phi_*,t)|^2.\end{aligned} \tag{3.25}$$

Remark 6. (*The construction preserves boundary conditions*) The construction that leads to (3.23) is compatible with eventually prescribed boundary conditions: if S is a surface of revolution with boundary, then $\boldsymbol{m}_{\partial S} \equiv \boldsymbol{u}_{\partial S}$. Indeed, suppose to fix the ideas that $I = [0,1]$ and that, e.g., $\boldsymbol{m}(\phi,1) = A^{\top}(\phi)\boldsymbol{e}$ for some $\boldsymbol{e} \in T$, then we have $\boldsymbol{u}(\phi,1) := A^{\top}(\phi)A(\phi_*)\boldsymbol{m}(\phi_*,1) = A^{\top}(\phi)A(\phi_*)A^{\top}(\phi_*)\boldsymbol{e} = A^{\top}(\phi)\boldsymbol{e}$ and, therefore, the boundary value is preserved.

Also, using the local coordinate representation of the characterization of axially symmetric vector fields expressed by (2.10), we obtain that

$$\begin{aligned}g(\boldsymbol{u}(\phi,t)\cdot \boldsymbol{a}(\phi,t)) &= g(A^{\top}(\phi)A(\phi_*)\boldsymbol{m}(\phi_*,t)\cdot A^{\top}(\phi)\boldsymbol{a}(0,t)) \\ &= g(A(\phi_*)\boldsymbol{m}(\phi_*,t)\cdot \boldsymbol{a}(0,t)) \\ &= g(\boldsymbol{m}(\phi_*,t)\cdot A^{\top}(\phi_*)\boldsymbol{a}(0,t)) \\ &= g(\boldsymbol{m}(\phi_*,t)\cdot \boldsymbol{a}(\phi_*,t)).\end{aligned} \tag{3.26}$$

From the previous computations, using (3.22) and (3.18), we infer that when $\mathfrak{h}_1^2(t)W^2(t) \geqslant 2\pi$, the following estimates hold

$$\begin{aligned}
\mathcal{E}_\omega(\boldsymbol{u}) &= \int_0^{2\pi}\int_{t\in I} \frac{|\boldsymbol{m}_\perp(\phi_*,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t \boldsymbol{m}(\phi_*,t)|^2}{\mathfrak{h}_2^2(t)}\sqrt{\mathfrak{g}(t)}\mathrm{d}t\mathrm{d}\phi \\
&\qquad + \int_0^{2\pi}\int_{t\in I} g(\boldsymbol{m}(\phi_*,t)\cdot\boldsymbol{a}(\phi_*,t))\sqrt{\mathfrak{g}(t)}\mathrm{d}t\mathrm{d}\phi && (3.27)\\
&= \int_0^{2\pi}\Phi_{\mathcal{E}}(\phi_*)\mathrm{d}\phi && (3.28)\\
&\leqslant \int_0^{2\pi}\Phi_{\mathcal{E}}(\phi)\mathrm{d}\phi && (3.29)\\
&= \int_{t\in I}\int_0^{2\pi} \frac{|\boldsymbol{m}_\perp(\phi,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t \boldsymbol{m}(\phi,t)|^2}{\mathfrak{h}_2^2(t)}\sqrt{\mathfrak{g}(t)}\mathrm{d}\phi\mathrm{d}t \\
&\qquad + \int_{t\in I}\int_0^{2\pi} g(\boldsymbol{m}(\phi,t)\cdot\boldsymbol{a}(\phi,t))\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t && (3.30)\\
&\leqslant \int_{t\in I}\int_0^{2\pi} \frac{|\partial_\phi\boldsymbol{m}_\perp(\phi,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t \boldsymbol{m}(\phi,t)|^2}{\mathfrak{h}_2^2(t)}\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t \\
&\qquad + \int_{t\in I} W^2(t)|\langle\boldsymbol{m}_\perp\rangle(t)|^2\sqrt{\mathfrak{g}(t)}\mathrm{d}t \\
&\qquad\qquad + \int_{t\in I}\int_0^{2\pi} g(\boldsymbol{m}(\phi,t)\cdot\boldsymbol{a}(\phi,t))\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t && (3.31)\\
&\leqslant \mathcal{E}_\omega(\boldsymbol{m}). && (3.32)
\end{aligned}$$

The minimality of $\boldsymbol{m}$ entails the equality $\mathcal{E}_\omega(\boldsymbol{m}) = \mathcal{E}_\omega(\boldsymbol{u})$ from which the conclusions *i.* and *ii* of Theorem 1 follow.

$\boxed{\text{Step 4}}$ Up to now, we know that if $\boldsymbol{m}$ is a smooth minimizer of $\mathcal{E}_\omega$ then there exists an axially symmetric vector field $\boldsymbol{u} \in H^1(S,T)$, built from $\boldsymbol{m}$, such that $\mathcal{E}_\omega(\boldsymbol{u}) = \mathcal{E}_\omega(\boldsymbol{m})$. We now show that, as a consequence, if $\mathfrak{h}_1(t)W(t) > \sqrt{2\pi}$, then the conclusions of Theorem 2 hold, i.e., that any minimizer of $\mathcal{E}_\omega$ is necessarily axially symmetric. Indeed, given that $\mathcal{E}_\omega(\boldsymbol{u}) = \mathcal{E}_\omega(\boldsymbol{m})$, one gets that (3.30)=(3.31)=(3.32), and these equalities entail that

$$\begin{aligned}
\int_{t\in I}\int_0^{2\pi} \frac{|\partial_\phi\boldsymbol{m}(\phi,t)|^2}{\mathfrak{h}_1^2(t)} + \int_{t\in I}\left[W^2(t) - \frac{2\pi}{\mathfrak{h}_1^2(t)}\right]|\langle\boldsymbol{m}_\perp\rangle(t)|^2\sqrt{\mathfrak{g}(t)}\mathrm{d}t \\
= \int_{t\in I}\int_0^{2\pi} \frac{|\boldsymbol{m}_\perp(\phi,t) - \langle\boldsymbol{m}_\perp\rangle(t)|^2}{\mathfrak{h}_1^2(t)}\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t. \qquad (3.33)
\end{aligned}$$

But then, Poincaré-Wirtinger inequality (3.13), together with the previous equality (3.33), implies that any minimizer for which (3.30)=(3.31)=(3.32), must necessarily satisfy the relations

$$\left[W^2(t) - \frac{2\pi}{\mathfrak{h}_1^2(t)}\right]|\langle\boldsymbol{m}_\perp\rangle(t)|^2 = 0, \qquad (3.34)$$

$$|\partial_\phi\boldsymbol{m}(\phi,t)\cdot\boldsymbol{e}_3|^2 = 0. \qquad (3.35)$$

In writing the previous two relations, we took into account that by hypotheses (see Remark 1), we have $\sqrt{\mathfrak{g}(t)} > 0$ for every $t \in I$ with the possible exception of the two boundary points of γ if γ touches the $\boldsymbol{e}_3$-axis. Also, from (3.34), (3.35), and (3.33) we get that any minimizer of $\mathcal{E}_\omega$ satisfies the relation

$$\int_{t\in I}\left(\int_0^{2\pi}(|\partial_\phi \boldsymbol{m}_\perp(\phi,t)|^2 - |\boldsymbol{m}_\perp(\phi,t) - \langle \boldsymbol{m}_\perp\rangle(t)|^2)\mathrm{d}\phi\right)\frac{\sqrt{\mathfrak{g}(t)}}{\mathfrak{h}_1^2(t)}\mathrm{d}t = 0, \tag{3.36}$$

and the integrand is nonnegative by Poincaré–Wirtinger inequality. It follows that for a.e. $t\in I$ there holds

$$\int_0^{2\pi}(|\partial_\phi \boldsymbol{m}_\perp(\phi,t)|^2 - |\boldsymbol{m}_\perp(\phi,t) - \langle \boldsymbol{m}_\perp\rangle(t)|^2)\mathrm{d}\phi = 0. \tag{3.37}$$

But this means that the equality sign is reached in the Poincaré-Wirtinger inequality (3.13), and this is known to happen if, and only if, $\boldsymbol{m}_\perp(\phi,t) = \langle \boldsymbol{m}_\perp\rangle(t) + \boldsymbol{\alpha}_\perp(t)\cos\phi + \boldsymbol{\beta}_\perp(t)\sin\phi$ for suitable functions $\boldsymbol{\alpha}_\perp, \boldsymbol{\beta}_\perp : t\in I \mapsto \mathbb{R}^2\times\{0\}$. Also, we know from (3.35) that $\boldsymbol{m}(\phi,t)\cdot\boldsymbol{e}_3$ depends only on the t-variable. Therefore, every minimizer is of the form

$$\boldsymbol{m}(\phi,t) = \langle \boldsymbol{m}_\perp\rangle(t) + \boldsymbol{\alpha}_\perp(t)\cos\phi + \boldsymbol{\beta}_\perp(t)\sin\phi + \eta(t)\boldsymbol{e}_3 \tag{3.38}$$

for a suitable scalar function $f : I \to \mathbb{R}$ which is nothing but $\boldsymbol{m}(\phi,t)\cdot\boldsymbol{e}_3$. Moreover, if $\mathfrak{h}_1(t)W(t) > \sqrt{2\pi}$, then $\langle \boldsymbol{m}_\perp\rangle(t) \equiv 0$ and, therefore, the minimizer $\boldsymbol{m}$ is necessarily of the form

$$\boldsymbol{m}(\phi,t) = \boldsymbol{\alpha}_\perp(t)\cos\phi + \boldsymbol{\beta}_\perp(t)\sin\phi + \eta(t)\boldsymbol{e}_3\,. \tag{3.39}$$

This proves (2.13) and completes the proof of Theorem 1.

From now on, we focus on the claims made in Theorem 2, and therefore, we assume that the target surface T is never flat.

Note that, in general, the previous expression includes both axially symmetric and antisymmetric vector fields. Indeed, if we set $\boldsymbol{\alpha}(t) := (\boldsymbol{\alpha}_\perp(t), \eta(t))$ then we can express the vector field $\boldsymbol{m}$ in (3.39) both as a perturbation of an axially symmetric vector field,

$$\begin{aligned}\boldsymbol{m}(\phi,t) &= (\cos\phi)\boldsymbol{\alpha}_\perp(t) + (\sin\phi)\boldsymbol{e}_3\times\boldsymbol{\alpha}_\perp(t) + (\boldsymbol{e}_3\otimes\boldsymbol{e}_3)\boldsymbol{\alpha}(t)\\ &\qquad + (\sin\phi)(\boldsymbol{\beta}_\perp(t) - \boldsymbol{e}_3\times\boldsymbol{\alpha}_\perp(t))\\ &= A^{\mathsf{T}}(\phi)\boldsymbol{\alpha}(t) + (\sin\phi)(\boldsymbol{\beta}_\perp(t) - \boldsymbol{e}_3\times\boldsymbol{\alpha}_\perp(t))\end{aligned} \tag{3.40}$$

and as a perturbation of an axially antisymmetric vector field,

$$\begin{aligned}\boldsymbol{m}(\phi,t) &= (\cos\phi)\boldsymbol{\alpha}_\perp(t) - (\sin\phi)\boldsymbol{e}_3\times\boldsymbol{\alpha}_\perp(t) + (\boldsymbol{e}_3\otimes\boldsymbol{e}_3)\boldsymbol{\alpha}(t)\\ &\qquad + (\sin\phi)(\boldsymbol{\beta}_\perp(t) + \boldsymbol{e}_3\times\boldsymbol{\alpha}_\perp(t))\\ &= A(\phi)\boldsymbol{\alpha}(t) + (\sin\phi)(\boldsymbol{\beta}_\perp(t) + \boldsymbol{e}_3\times\boldsymbol{\alpha}_\perp(t)).\end{aligned} \tag{3.41}$$

In other words, $\boldsymbol{m}$ is axially symmetric when $\boldsymbol{\beta}_\perp(t) = \boldsymbol{e}_3\times\boldsymbol{\alpha}_\perp(t)$ and axially antisymmetric when $\boldsymbol{\beta}_\perp(t) = -\boldsymbol{e}_3\times\boldsymbol{\alpha}_\perp(t)$. As a consequence, for a minimal configuration $\boldsymbol{m}$ of the form (3.39) to have line symmetry, it is sufficient to satisfy the orthogonality conditions

$$|\boldsymbol{\beta}_\perp(t)| \equiv |\boldsymbol{\alpha}_\perp(t)|, \quad \boldsymbol{\beta}_\perp(t)\cdot\boldsymbol{\alpha}_\perp(t) \equiv 0. \tag{3.42}$$

Indeed, if these conditions are met, then for every $t\in I$ there holds $\boldsymbol{\beta}_\perp(t) = \pm\boldsymbol{e}_3\times\boldsymbol{\alpha}_\perp(t)$, and therefore, from (3.40)-(3.41), we get that for every $(\phi,t)\in\mathbb{R}\times I$ either $\boldsymbol{m}(\phi,t) = A^{\mathsf{T}}(\phi)\boldsymbol{\alpha}(t)$ or $\boldsymbol{m}(\phi,t) = A(\phi)\boldsymbol{\alpha}(t)$.

It remains to show that any minimizer satisfies the orthogonality conditions (3.42). But this is a consequence of the assumption on the target manifold T of being never flat (cf. Definition 2). Indeed, for any $t\in I$ the map $\boldsymbol{m}_\perp(\cdot,t)$ takes values in $\boldsymbol{\pi}_\perp(T\cap\Sigma_{\eta(t)})$. This means that $|\boldsymbol{m}_\perp(\cdot,t)|$ is a Sobolev function taking values into a finite set, i.e., in the set of radii associated with the circles in $\boldsymbol{\pi}_\perp(T\cap\Sigma_{\eta(t)})$. It follows that $\phi\in I\mapsto|\boldsymbol{m}_\perp(\cdot,t)|$ is constant and, therefore, that $|\boldsymbol{m}_\perp(\cdot,t)|$ depends only on the t-variable. Therefore we get

$$|\boldsymbol{\alpha}_\perp(t)|^2=|\boldsymbol{m}_\perp(0,t)|^2=|\boldsymbol{m}_\perp(\pi/2,t)|^2=|\boldsymbol{\beta}_\perp(t)|^2, \tag{3.43}$$

and this proves the first condition in (3.42). But now, from (3.39) and (3.43), we obtain

$$|\boldsymbol{m}_\perp(\phi,t)|^2=|\boldsymbol{\alpha}_\perp(t)|^2+\boldsymbol{\beta}_\perp(t)\cdot\boldsymbol{\alpha}_\perp(t)\sin(2\phi) \tag{3.44}$$

and the only way for $|\boldsymbol{m}_\perp(\phi,t)|$ to be constant in ϕ is that also the second orthogonality condition in (3.42) is satisfied.

If the vector field $\boldsymbol{a}$ is axially symmetric, given that we know from Step 3 the existence of axially symmetric minimizers of $\mathcal{E}_\omega$, we conclude an axially symmetric minimizer is given by $\boldsymbol{m}(\phi,t):=A^{\mathsf{T}}(\phi)\boldsymbol{\gamma}_s(t)$ where $\boldsymbol{\gamma}_s$ is a solution to the one-dimensional minimization problem

$$\begin{aligned}\mathcal{F}_s(\boldsymbol{\gamma}):=\mathcal{E}_\omega(A^{\mathsf{T}}(\phi)\boldsymbol{\gamma}(t))=2\pi\int_{t\in I}\frac{|\boldsymbol{\gamma}_\perp(t)|^2}{\mathfrak{h}_1^2(t)}+\frac{|\dot{\boldsymbol{\gamma}}_\perp(t)|^2}{\mathfrak{h}_2^2(t)}\sqrt{\mathfrak{g}(t)}\mathrm{d}t\\ +\int_{t\in I}\int_0^{2\pi}g(\boldsymbol{\gamma}(t)\cdot A(\phi)\boldsymbol{a}(\phi,t))\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t\end{aligned} \tag{3.45}$$

among all possible profiles $\boldsymbol{\gamma}\in H^1(I,T)$.

$\boxed{\text{Step 5}}$ In this final step, we show how to remove the smoothness assumption on the minimizer. For that, we improve the ideas in [14]. Let $\boldsymbol{m}$ be a minimizer of $\mathcal{E}_\omega$ and let $\boldsymbol{m}^\varepsilon\in C^\infty(S,T)$ be a family such that $\boldsymbol{m}^\varepsilon\to\boldsymbol{m}$ strongly in $H^1(S,T)$. Such a family certainly exists because of a well-known result of Schoen and Uhlenbeck (see [38, p.267]). For every $\varepsilon>0$, we consider the continuous function

$$\begin{aligned}\Phi^\varepsilon_{\mathcal{E}}(\phi)=\int_{t\in I}\left(\frac{|\boldsymbol{m}^\varepsilon_\perp(\phi,t)|^2}{\mathfrak{h}_1^2(t)}+\frac{|\partial_t\boldsymbol{m}^\varepsilon(\phi,t)|^2}{\mathfrak{h}_2^2(t)}\right)\sqrt{\mathfrak{g}(t)}\mathrm{d}t\\ +\int_{t\in I}g(\boldsymbol{m}^\varepsilon(\phi,t)\cdot\boldsymbol{a}(\phi_*,t))\sqrt{\mathfrak{g}(t)}\mathrm{d}t,\end{aligned} \tag{3.46}$$

and we choose a point $\phi_\varepsilon\in\operatorname{argmin}_{\phi\in[0,2\pi]}\Phi^\varepsilon_{\mathcal{E}}(\phi)$. We then define the axially symmetric vector field $\boldsymbol{u}_\varepsilon\in H^1(S,T)$ via the relation

$$\boldsymbol{u}_\varepsilon(\phi,t):=A^\top(\phi)A(\phi_\varepsilon)\boldsymbol{m}^\varepsilon(\phi_\varepsilon,t). \tag{3.47}$$

Computing as in STEP 3, we have that

$$\mathcal{E}_\omega(\boldsymbol{u}_\varepsilon) = \int_0^{2\pi}\int_{t\in I} \frac{|\boldsymbol{m}_\perp^\varepsilon(\phi_\varepsilon,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t \boldsymbol{m}^\varepsilon(\phi_\varepsilon,t)|^2}{\mathfrak{h}_2^2(t)} \mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t \tag{3.48}$$

$$+ \int_0^{2\pi}\int_{t\in I} g(\boldsymbol{m}^\varepsilon(\phi_\varepsilon,t)\cdot \boldsymbol{a}(\phi_\varepsilon,t))\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t \tag{3.49}$$

$$= \int_0^{2\pi} \Phi_{\mathcal{E}}^\varepsilon(\phi_\varepsilon)\mathrm{d}\phi \tag{3.50}$$

$$\leqslant \int_0^{2\pi} \Phi_{\mathcal{E}}^\varepsilon(\phi)\mathrm{d}\phi \tag{3.51}$$

$$= \int_{t\in I}\int_0^{2\pi} \frac{|\boldsymbol{m}_\perp^\varepsilon(\phi,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t \boldsymbol{m}^\varepsilon(\phi,t)|^2}{\mathfrak{h}_2^2(t)} \sqrt{\mathfrak{g}(t)}\mathrm{d}\phi\mathrm{d}t$$

$$+ \int_0^{2\pi}\int_{t\in I} g(\boldsymbol{m}^\varepsilon(\phi,t)\cdot \boldsymbol{a}(\phi,t))\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t \tag{3.52}$$

$$\leqslant \int_{t\in I}\int_0^{2\pi} \frac{|\partial_\phi\boldsymbol{m}_\perp^\varepsilon(\phi,t)|^2}{\mathfrak{h}_1^2(t)} + \frac{|\partial_t \boldsymbol{m}^\varepsilon(\phi,t)|^2}{\mathfrak{h}_2^2(t)} \mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t \tag{3.53}$$

$$+ \int_{t\in I} W^2(t)|\langle \boldsymbol{m}_\perp^\varepsilon\rangle(t)|^2\sqrt{\mathfrak{g}(t)}\mathrm{d}t$$

$$+ \int_0^{2\pi}\int_{t\in I} g(\boldsymbol{m}^\varepsilon(\phi,t)\cdot \boldsymbol{a}(\phi,t))\mathrm{d}\phi\sqrt{\mathfrak{g}(t)}\mathrm{d}t \tag{3.54}$$

$$\leqslant \mathcal{E}_\omega(\boldsymbol{m}^\varepsilon). \tag{3.55}$$

Now, we observe that since $\mathcal{E}_\omega(\boldsymbol{m}^\varepsilon)$ is uniformly bounded, so is $\mathcal{E}_\omega(\boldsymbol{u}_\varepsilon)$. Therefore, there exists $\boldsymbol{u} \in H^1(S,T)$ such that $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}$ weakly in $H^1(S,T)$ and, by compactness, up to a subsequence, one also has that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ strongly in $L^2(S,T)$ and $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ a.e. in S. But also, the family $(A(\phi)\boldsymbol{u}_\varepsilon(\phi,t) = A(\phi_\varepsilon)\boldsymbol{m}^\varepsilon(\phi_\varepsilon,t))_\varepsilon$ is bounded in $H^1(I,T)$ and, therefore, there exists $\boldsymbol{u}_* \in H^1(I,T)$ such that $A(\phi_\varepsilon)\boldsymbol{m}^\varepsilon(\phi_\varepsilon,t) \rightharpoonup \boldsymbol{u}_*$ weakly in $H^1(I,T)$. By compactness, up to a subsequence, one also has that $A(\phi_\varepsilon)\boldsymbol{m}^\varepsilon(\phi_\varepsilon,t) \to \boldsymbol{u}_*$ strongly in $L^2(I,T)$ and $A(\phi_\varepsilon)\boldsymbol{m}^\varepsilon(\phi_\varepsilon,t) \to \boldsymbol{u}_*$ a.e. in I and, due to the independence of the family $(A(\phi_\varepsilon)\boldsymbol{m}^\varepsilon(\phi_\varepsilon,t))_\varepsilon$ on the ϕ-variable, the same convergence relations also hold, respectively, weakly in $H^1(S,T)$, strongly in $L^2(S,T)$ and a.e. in S.

A first consequence of the previous convergence relations is that the limit vector field $\boldsymbol{u}$ is axially symmetric. Indeed, we know that

$$A(\phi)\boldsymbol{u}_\varepsilon(\phi,t) \to A(\phi)\boldsymbol{u}(\phi,t) \text{ a.e. in } S, \tag{3.56}$$

$$A(\phi)\boldsymbol{u}_\varepsilon(\phi,t) \to \boldsymbol{u}_*(t) \text{ a.e. in } S, \tag{3.57}$$

and, therefore, also that $\boldsymbol{u}(\phi,t) = A^\top(\phi)\boldsymbol{u}_*(t)$. This shows that $\boldsymbol{u}$ is axially symmetric.

Also, by the weak lower semicontinuity of the norm, the strong convergence in $L^2(S,T)$, from (3.48) and (3.55) we conclude that

$$\mathcal{E}_\omega(\boldsymbol{u}) \leqslant \liminf_{\varepsilon\to 0}\mathcal{E}_\omega(\boldsymbol{u}_\varepsilon) \leqslant \lim_{\varepsilon\to 0}\,(3.54)\, \leqslant \lim_{\varepsilon\to 0}\mathcal{E}_\omega(\boldsymbol{m}^\varepsilon) = \mathcal{E}_\omega(\boldsymbol{m}). \tag{3.58}$$

By the minimality of $\boldsymbol{m}$, all previous inequalities are actually equalities, and we can pass to the limit under the integral sign in (3.54). Hence, we are back to the hypotheses necessary to resume the argument from STEP 4 and conclude the proof. Indeed, STEP 4 starts from the assumptions that if $\boldsymbol{m}$ is a minimizer of $\mathcal{E}_\omega$ then there exists an axially symmetric vector field $\boldsymbol{u} \in H^1(S,T)$, built from $\boldsymbol{m}$, such that $\mathcal{E}_\omega(\boldsymbol{u}) = \mathcal{E}_\omega(\boldsymbol{m})$, and this step does not rely on any smoothness assumption on $\boldsymbol{m}$.

4. Further results and applications

In this last section, we want to emphasize a particular case of Theorem 2. Formally, when $\omega(\xi) := \lambda \in \mathbb{R}_+$ is constant and $\lambda \to +\infty$, the energy functional $\mathcal{E}_\lambda$ converges, in the sense of Γ-convergence, to the energy functional

$$\mathcal{E}(\boldsymbol{m}) := \mathcal{D}_S(\boldsymbol{m}) + \mathcal{A}_S(\boldsymbol{m}) \tag{4.1}$$

subject to the constraint that $\langle \boldsymbol{m} \rangle_{S_\xi} \times \boldsymbol{e}_3 = 0$ for every $\xi \in S$. It is convenient for us to introduce the following notion.

Definition 3. *We say that a vector field $\boldsymbol{m} \in H^1(S,T)$ is* axially null-average*, along the $\boldsymbol{e}_3$-axis, if $\langle \boldsymbol{m} \rangle_{S_\xi} \times \boldsymbol{e}_3 = 0$ for every $\xi \in S$. Equivalently, given that $|\langle \boldsymbol{m} \rangle_{S_\xi} \times \boldsymbol{e}_3|^2 = |\langle \boldsymbol{m}_\perp \rangle_{S_\xi}|^2$, the vector field $\boldsymbol{m}$ is axially null-average when $\langle \boldsymbol{m}_\perp \rangle_{S_\xi} = 0$ for every $\xi \in S$. In local coordinates, $\boldsymbol{m}$ is axially null-average if, and only if, for every $t \in I$ there holds that*

$$\langle \boldsymbol{m}(\cdot,t) \rangle := \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{m}_\perp(\theta,t) \mathrm{d}\theta = 0\,, \tag{4.2}$$

where $\boldsymbol{m}_\perp(\theta,t) := (\boldsymbol{m}_\perp \circ \boldsymbol{\xi})(\theta,t)$.

Remark 7. Vector fields with line symmetry are axially null-average (i.e., satisfies (4.2)). In particular, so are axially symmetric and antisymmetric configurations. Indeed, if $\boldsymbol{m}$ is axially symmetric with respect to the $\boldsymbol{e}_3$-axis then, in local coordinates, we have that

$$\boldsymbol{m}(\phi,t) = A^\top(\phi)\boldsymbol{\alpha}(t) \quad \forall (\phi,t) \in \mathbb{R} \times I \tag{4.3}$$

for some profile $\boldsymbol{\alpha} \in H^1(I,T)$. Hence, $\langle \boldsymbol{m}(\cdot,t) \rangle = (\boldsymbol{\alpha}(t) \cdot \boldsymbol{e}_3)\boldsymbol{e}_3$ for every $t \in I$, and this implies that $\langle \boldsymbol{m}_\perp \rangle_{S_\xi} = 0$ for every $\xi \in S$. A similar argument shows that, more generally, vector fields with line symmetry are axially null-average.

Note that the class of axially null-average vector fields is not directly related to the class of null-average configurations in $H^1(S,T)$. Even if $\boldsymbol{m}$ is t-invariant, (4.2) does not imply that $\boldsymbol{m}$ is null-average, but only that its projection $\boldsymbol{m}_\perp$ is null-average.

The main idea behind the proofs of Theorems 1 and 2 is geometric and relatively intuitive. Yet, it brings up many interesting consequences. Specifically, as a byproduct of our analysis, we get that if $\boldsymbol{m}$ is a minimizer of $\mathcal{E}$ in $H^1(S,T)$, which, a posteriori, turns out to be axially null-average, then any minimizer of $\mathcal{E}$ is of the form (2.13) and, if in addition, T is never-flat, then any minimizer of $\mathcal{E}$ has line symmetry. Indeed, one can repeat verbatim the argument used to prove Theorems 1 and 2 to obtain the following result.

Theorem 3. *For given surfaces of revolution $S, T \subseteq \mathbb{R}^3$, we consider the energy functional defined for every $\boldsymbol{m} \in H^1(S,T)$ by (cf. (4.1))*

$$\mathcal{E}(\boldsymbol{m}) := \int_S |\nabla \boldsymbol{m}(\xi)|^2 \mathrm{d}\xi + \int_S g(\boldsymbol{m}(\xi) \cdot \boldsymbol{a}(\xi)) \mathrm{d}\xi$$

with $g : \mathbb{R} \to \mathbb{R}_+$ a Lipschitz function and $\boldsymbol{a} : S \to \mathbb{R}^3$ a Lipschitz vector field. The following assertions hold:

i. If the vector field $\boldsymbol{a}$ is axially symmetric or antisymmetric, and $\boldsymbol{m} \in H^1(S,T)$ is an axially null-average minimizer of $\mathcal{E}$, then $\boldsymbol{m}$ is of the form (cf. (2.13))

$$\boldsymbol{m}(\phi,t) = \boldsymbol{\alpha}_\perp(t)\cos\phi + \boldsymbol{\beta}_\perp(t)\sin\phi + \eta(t)\boldsymbol{e}_3$$

for some $\boldsymbol{\alpha}_\perp, \boldsymbol{\beta}_\perp \in H^1(I,\mathbb{R}^2)$, $\eta \in H^1(I,\mathbb{R})$, subject to the constraint of the resulting $\boldsymbol{m}$ being T-valued.

ii. If T is never flat, $\boldsymbol{a}$ is axially symmetric, and $\boldsymbol{m} \in H^1(S,T)$ is an axially null-average minimizer of $\mathcal{E}$, then either every minimizer is axially symmetric, or there also coexist axially antisymmetric minimizers.

If T is never flat, $\boldsymbol{a}$ is axially antisymmetric, and $\boldsymbol{m} \in H^1(S,T)$ is an axially null-average minimizer of $\mathcal{E}$, then either every minimizer is axially antisymmetric, or there also coexist axially symmetric minimizers.

In any case, if the vector field $\boldsymbol{a}$ is axially symmetric or antisymmetric, then any axially null-average minimizer $\boldsymbol{m} \in H^1(S,T)$ of $\mathcal{E}$ has line symmetry and is of the form

$$\boldsymbol{m}(\phi,t) = \boldsymbol{\alpha}_\perp(t)\cos\phi + \boldsymbol{\beta}_\perp(t)\sin\phi + \eta(t)\boldsymbol{e}_3.$$

for some $\boldsymbol{\alpha}_\perp, \boldsymbol{\beta}_\perp \in H^1(I,\mathbb{R}^2)$, $\eta \in H^1(I,\mathbb{R})$, subject to the orthogonality conditions $\boldsymbol{\beta}_\perp(t)\cdot\boldsymbol{\alpha}_\perp(t) \equiv 0$ and $|\boldsymbol{\beta}_\perp(t)| \equiv |\boldsymbol{\alpha}_\perp(t)|$, and to the conditions of the resulting $\boldsymbol{m}$ being T-valued.

Remark 8. We want to highlight the main message contained in Theorem 3: while it is always the case that axially symmetric configurations are axially null average, minimality allows for the converse implication.

Remark 9. Note that in the statement of Theorem 3 there is no more reference to the weight ω and to the circular integral weight W.

Remark 10. We are presenting Theorem 3 in a concise form that particularizes only the main assertions contained in Theorems 1 and 2 to the current context of axially null-average configurations. However, *every* claim in Theorems 1 and 2 transposes to the current setting. In particular, the results about the existence of axially symmetric (and axially antisymmetric) minimizers specified in Theorem 1.*i* and 1.*ii* still apply to axially null-average minimizers.

To better explain the interest in Theorem 3, one can imagine the following scenario, which will be explained through a concrete example soon, in which by writing down the Euler–Lagrange equations associated with $\mathcal{E}$ in (4.1), one infers that every stationary point of $\mathcal{E}$ has to be *axially null-average*. In other words, if by any means one can prove that for some specific choices of S and T minimizers of $\mathcal{E}$ are necessarily axially null-average, then the statements of Theorems 1 and 2 still hold *regardless* of whether or not the condition $\mathfrak{h}_1(t)W(t) > \sqrt{2\pi}$ on the circular integral weight W is satisfied.

Example 1. Let us consider the simplest case in which $S := \mathbb{A}$ is the annulus of $\mathbb{R}^2$ of inner radius t_1 and outer radius t_2, T is the whole space $\mathbb{R}^2$, and $g(\boldsymbol{m}(\xi)\cdot\boldsymbol{a}(\xi)) = \kappa(\boldsymbol{m}(\xi)\cdot\boldsymbol{e}_3)^2$ for some $\kappa \in \mathbb{R}$. The functional $\mathcal{E}$ has to be minimized among all possible vector fields in $H^1(\mathbb{A},\mathbb{R}^2)$ that satisfy the boundary condition $\boldsymbol{m} = \boldsymbol{b}_1$ on $t_1\mathbb{S}^1$ and $\boldsymbol{m} = \boldsymbol{b}_2$ on $t_2\mathbb{S}^1$ for axially symmetric vector fields $\boldsymbol{b}_1, \boldsymbol{b}_2$. We want to understand the symmetry properties of the minimizers. For that, we observe that any minimizer of $\mathcal{E}$ has to satisfy the Euler–Lagrange equations

$$-\Delta\boldsymbol{m} + \kappa(\boldsymbol{m}\cdot\boldsymbol{e}_3)\boldsymbol{e}_3 = 0 \quad \text{in } \mathbb{A}, \tag{4.4}$$

under the prescribed boundary conditions $\boldsymbol{m} = \boldsymbol{b}_1$ on $t_1\mathbb{S}^1$ and $\boldsymbol{m} = \boldsymbol{b}_2$ on $t_2\mathbb{S}^1$. By the previous relation, it follows that any minimizer of $\mathcal{E}$ is such that $-\Delta\boldsymbol{m}_\perp = 0$ in $\mathbb{A}$. Expressing $\mathbb{A}$ in local coordinates through the classical polar map $\boldsymbol{\xi}(\phi,t) = A^\top(\phi)\gamma(t)$, $\gamma(t) := t\boldsymbol{e}_1$, case in which the metric coefficients are $\mathfrak{h}_1^2(t) = t^2$, $\mathfrak{h}_2^2(t) = 1$, and then integrating in the ϕ-variable, we conclude that any minimizer of $\mathcal{E}$ is such that

$$-\partial_t(t\partial_t\langle\boldsymbol{m}_\perp\rangle(t)) = 0 \quad \text{in } [t_1,t_2]\,, \tag{4.5}$$

and subject to the boundary conditions $\langle \boldsymbol{m}_\perp \rangle(t_1) = \langle \boldsymbol{m}_\perp \rangle(t_2) = 0$. But the only solution of this boundary value problem is the zero solution, i.e., the solution $\langle \boldsymbol{m}_\perp \rangle(t) = 0$ for every $t \in [t_1, t_2]$. It follows that any solution of the boundary value problem (4.4) is *axially null-average* and, therefore, the conclusions of Theorem 3 apply. But as pointed out in Remark 10, also the conclusions of Theorem 1 apply from which we infer that there always exist axially symmetric solutions of the boundary value problem (4.4).

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