

ABOUT THE BLASCHKE-SANTALÓ DIAGRAM OF AREA, PERIMETER AND MOMENT OF INERTIA

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ABSTRACT. We study the Blaschke-Santaló diagram associated to the area, the perimeter, and the moment of inertia. We work in dimension 2, under two assumptions on the shapes: convexity and the presence of two orthogonal axis of symmetry. We discuss topological and geometrical properties of the diagram. As a by-product we address a conjecture by Pólya, in the simplified setting of double symmetry.

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1. INTRODUCTION

Blaschke-Santaló diagrams represent a powerful tool in shape optimization, to investigate the relations of triples of shape functionals F_1, F_2, F_3 , defined in a class of admissible shapes \mathcal{A} . These diagrams consist in studying the range of the vector shape functional $(F_1, F_2) : \mathcal{A} \rightarrow \mathbb{R}^2$, under the constraint $F_3(\Omega) = 1$, namely the set

$$\{(x, y) \in \mathbb{R}^2 : \exists \Omega \in \mathcal{A}, x = F_1(\Omega), y = F_2(\Omega), F_3(\Omega) = 1\}.$$

When dealing with homogeneous shape functionals, an equivalent diagram is

$$\{(x, y) \in \mathbb{R}^2 : \exists \Omega \in \mathcal{A}, x = F_3^\alpha(\Omega)F_1(\Omega), y = F_3^\beta(\Omega)F_2(\Omega)\},$$

where the powers α and β are chosen so that $F_3^\alpha F_1$ and $F_3^\beta F_2$ are scale invariant.

First used by Santaló in [18], this approach has now become a standard tool in shape optimization. We cite, e.g., [2, 3, 6, 7, 8, 11, 12, 16], in which shape functionals of spectral and geometric type are studied. As it appears from the literature, the theoretical analysis, even if very fine, is in general not enough for an accurate description of the diagram. Therefore it is often accompanied by numerical simulations. In this respect we cite the recent paper [5], in which the authors give a new and interesting way to generate random shapes whose images distribute uniformly in \mathcal{D} .

In the present paper, we work in dimension 2 and we consider the following triple of shape functionals: the area $A(\Omega)$, the perimeter $P(\Omega)$, and the moment of inertia $W(\Omega)$ with respect to the center of gravity

$$W(\Omega) := \int_{\Omega} [(x - x_G)^2 + (y - y_G)^2] dx dy,$$

where (x_G, y_G) are the coordinates of the center of gravity of Ω .

The three functionals under study are invariant under rigid motion and are positively homogeneous. Therefore, as mentioned above, the Blaschke-Santaló diagram associated to (A, P, W)

$$\{(x, y) \in \mathbb{R}^2 : \exists \Omega \in \mathcal{A}, A(\Omega) = 1, x = P(\Omega), y = W(\Omega)\}$$

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can be deduced from the diagram

$$\mathcal{D} := \left\{ (x, y) \in \mathbb{R}^2 : \exists \Omega \in \mathcal{A}, x = \frac{1}{2\pi} \frac{A^2(\Omega)}{W(\Omega)}, y = 4\pi \frac{A(\Omega)}{P^2(\Omega)} \right\},$$

in which the area constraint is enclosed into the coordinate shape functionals, which are now scale invariant. The pre-factors $1/2\pi$ and 4π are normalization constants and serve the purpose of making the diagram fit the unit square (cf. Proposition 2.2).

In the present paper we study \mathcal{D} for the following class of planar shapes:

$$\mathcal{A} := \{ \Omega \subset \mathbb{R}^2 : \Omega \text{ open, convex, with 2 orthogonal axis of symmetry} \}.$$

The double symmetry assumption allows us to rewrite the moment of inertia as

$$W(\Omega) = \int_{\Omega} (x^2 + y^2) dx dy,$$

since we may assume that the two axis of symmetry coincide with the coordinate axis so that the center of gravity is located at the origin.

We prove that the diagram is simply connected and coincides with the planar region enclosed between two increasing curves which connect the point $(0, 0)$ to the point $(1, 1)$. The point $(0, 0)$ is attained asymptotically by the image of thin shapes, whereas the point $(1, 1)$ corresponds (only) to the disks. The behavior of \mathcal{D} near these two points is investigated using the technique of shape derivatives, either of thin domains or of nearly spherical sets, and it is described by the slopes of the boundary curves in the two extremal points of the diagram.

The two boundary curves are the graphs of

$$L^{\pm}(x) := \max / \min \left\{ 4\pi \frac{A(\Omega)}{P^2(\Omega)} : \Omega \in \mathcal{A}, \frac{1}{2\pi} \frac{A^2(\Omega)}{W(\Omega)} = x \right\}.$$

Optimal shapes for L^+ turn out to be $C^{1,1}$ shapes, whereas that for L^- are polygons. For values of x less than $3/\pi$ we are able to characterize optimal shapes for $L^-(x)$: they are rhombi, going from the segment, in the limit as $x \rightarrow 0^+$, to the square, for $x = 3/\pi$. The key argument here is the minimization of the ratio y/x for $(x, y) \in \mathcal{D}$, namely the maximization of $F(\Omega) := P^2(\Omega)A(\Omega)/W(\Omega)$ for $\Omega \in \mathcal{A}$. We mention that the same issue, in the wider class of convex sets, was addressed by G. Pólya in [17]: according to the conjecture, still unsolved, the maximiser of F should be the equilateral triangle.

For all the remaining cases, namely for $L^+(x)$, $x \in (0, 1]$, and for $L^-(x)$, $x \in (3/\pi, 1]$, optimizers are searched numerically. We look for the optimal shapes corresponding to points on $L^+(x)$ through their support functions. We choose to decompose the support function in Fourier series, the unknowns being the Fourier coefficients. For the optimal shapes corresponding to points on $L^-(x)$, since we know that they are polygonal, we choose a different strategy: the unknowns being the coordinates of the vertices.

The paper is organized as follows. In Section 2 we present the notation and we gather the first results: in Proposition 2.8 we show that the diagram is simply connected and its boundary is the union of two graphs of functions $L^{\pm} : (0, 1] \rightarrow \mathbb{R}^2$, which have the same limit as $x \rightarrow 0^+$ and coincide (only) for $x = 1$. The properties of the shapes associated to boundary points are summarized in Theorem 2.9. The continuity and monotonicity of L^{\pm} is investigated in the two subsequent sections, in Proposition 3.5 and 4.4. Section 5 is dedicated to the study of the ratio P^2A/W , which allows us to give the explicit expression of $L^-(x)$ for $x \in [0, 3/\pi]$. In Section 6 we analyse the diagram near its ‘‘corners’’, i.e. near $x = 0$ and $x = 1$. The final section is devoted to numerical shape optimization.

2. DEFINITIONS AND FIRST PROPERTIES

Notation. Throughout the paper, we denote by $x(\cdot)$ and $y(\cdot)$ the coordinate shape functionals defining the diagram, that is

$$x(\Omega) := \frac{1}{2\pi} \frac{A^2(\Omega)}{W(\Omega)}, \quad y(\Omega) := 4\pi \frac{A(\Omega)}{P^2(\Omega)}.$$

We use the letter \mathbb{D} to refer to the disk of radius 1, centered at the origin.

We will also assume that the shapes of \mathcal{A} are centered at the origin and that the coordinate system is oriented so that the axis of symmetry are the horizontal and vertical axis.

We endow the class of admissible shapes of the complementary Hausdorff distance, denoted by $d_{\mathcal{H}}$. Neighborhoods of shapes are intended with respect to the complementary Hausdorff distance.

With the symbol $f_{\Omega} f$ we denote the average of the function f over the set Ω , that is, $f_{\Omega} f = (\int_{\Omega} f) / (|\Omega|)$.

Definition 2.1. Let Ω be an admissible set. We define *small deformation* of Ω a family $\{\Omega_{\epsilon}\}_{\epsilon \in [0, \epsilon_0]}$, $\epsilon_0 > 0$, with the following properties:

- for $\epsilon = 0$ there holds $\Omega_0 = \Omega$
- the map $\epsilon \mapsto \Omega_{\epsilon}$ is continuous from $[0, \epsilon_0)$ to the class of admissible shapes \mathcal{A} , with respect to the Hausdorff distance.

Results on $x(\cdot)$ and $y(\cdot)$. Here we gather some properties concerning the two functionals $x(\cdot)$ and $y(\cdot)$, separately. More precisely, we investigate their bounds and continuity.

Proposition 2.2. *The shape functionals $x(\cdot)$ and $y(\cdot)$ take value in the interval $(0, 1]$ and the value 1 is attained only at the disk. In particular, the diagram \mathcal{D} is contained into the square $(0, 1] \times (0, 1]$.*

Proof. For every admissible shape Ω both $x(\Omega)$ and $y(\Omega)$ are strictly positive, implying that $x(\cdot) > 0$ and $y(\cdot) > 0$. The (sharp) upper bound on $y(\cdot)$ is nothing but the classical isoperimetric inequality. As for $x(\cdot)$, this isoperimetric inequality asserting that the ball minimizes the moment of inertia is also well-known. For sake of completeness, let us give an elementary proof: we use polar coordinates with respect to the center of gravity. A convex shape Ω containing the origin can be described as $\Omega = \{(\rho, \theta) : \theta \in [0, 2\pi], \rho \in [0, \rho_{max}(\theta)]\}$ for some function ρ_{max} . Computing the area and moment of inertia in polar coordinates yields:

$$x(\Omega) = \frac{1}{2\pi} \frac{\left[\int_0^{2\pi} \int_0^{\rho_{max}(\theta)} \rho d\rho d\theta \right]^2}{\int_0^{2\pi} \int_0^{\rho_{max}(\theta)} \rho^3 d\rho d\theta} = \frac{1}{2\pi} \frac{\left[\int_0^{2\pi} \rho_{max}^2(\theta) d\theta \right]^2}{\int_0^{2\pi} \rho_{max}^4(\theta) d\theta}.$$

Finally, using Cauchy-Schwarz inequality, we infer that the last term is bounded above by 1. The threshold 1 is sharp and it is attained if and only if ρ_{max} is constant, namely if Ω is a disk. This concludes the proof. \square

As already pointed out in the Introduction, the shape functionals $x(\cdot)$ and $y(\cdot)$ are scale invariant, since for every $t > 0$

$$P(t\Omega) = tP(\Omega), \quad A(t\Omega) = t^2A(\Omega), \quad W(t\Omega) = t^4W(\Omega).$$

This allows us to replace, if needed, the class of admissible sets with the subclass of shapes contained into the same compact set \mathbb{R}^2 , obtaining the same diagram (for any choice of such a compact set), as we state in the next lemma.

Lemma 2.3. *Let K be a compact set. Then*

$$\mathcal{D} = \{(x(\Omega), y(\Omega)) : \Omega \in \mathcal{A}, \Omega \subset K\}.$$

The advantage of working with shapes in a given box is that the classes of admissible shapes have now some compactness, in the following sense. Exploiting the compactness of the space $\{K \setminus \Omega : \Omega \in \mathcal{A}\}$ endowed with the Hausdorff distance (Blaschke selection theorem), we deduce that for every sequence $\Omega_n \subset K$, $\Omega_n \in \mathcal{A}$, there exists a subsequence (not relabeled) such that

- either $\Omega_n \rightarrow \Omega \in \mathcal{A}$, with respect to the complementary Hausdorff distance,
- or $\overline{\Omega}_n$ collapses to a segment or shrinks to a point.

Proposition 2.4. *The shape functionals $x(\cdot)$ and $y(\cdot)$ are continuous in \mathcal{A} with respect to the complementary Hausdorff convergence.*

Proof. Take $\Omega_n \rightarrow \Omega$ in \mathcal{A} , with respect to the Hausdorff convergence, as $n \rightarrow \infty$. By continuity of A , P , W among convex sets, we have $A(\Omega_n) \rightarrow A(\Omega)$, $P(\Omega_n) \rightarrow P(\Omega)$, $W(\Omega_n) \rightarrow W(\Omega)$. If Ω has non empty interior, we infer that all the quantities involved (in particular $P(\Omega)$ and $W(\Omega)$) do not vanish for n large enough, allowing to obtain the continuity of $x(\cdot)$ and $y(\cdot)$ at Ω . This concludes the proof. \square

As we have already shown in Proposition 2.2, the origin does not belong to the diagram. However, it has an important role in the description of its closure.

Proposition 2.5. *The origin O belongs to the closure of the diagram. More precisely, $\overline{\mathcal{D}} = \mathcal{D} \cup \{O\}$.*

Proof. We start the proof with a direct computation of $x(\cdot)$ and $y(\cdot)$ on two particular families of shapes: rectangles R_ℓ and rhombi S_ℓ , with semiaxis 1 and ℓ , with $\ell > 0$. It is straightforward that:

$$\begin{aligned} A(R_\ell) &= 4\ell, & P(R_\ell) &= 4(1 + \ell), & W(R_\ell) &= \frac{4}{3}(\ell + \ell^3), \\ A(S_\ell) &= 2\ell, & P(S_\ell) &= 4\sqrt{1 + \ell^2}, & W(S_\ell) &= \frac{1}{3}(\ell + \ell^3), \end{aligned}$$

so that

$$x(R_\ell) = \frac{6}{\pi} \frac{\ell}{1 + \ell^2}, \quad y(R_\ell) = \pi \frac{\ell}{(1 + \ell)^2}, \quad x(S_\ell) = \frac{6}{\pi} \frac{\ell}{1 + \ell^2}, \quad y(S_\ell) = \frac{\pi}{2} \frac{\ell}{1 + \ell^2}.$$

The two constructed families of points of the diagram converge to the origin as $\ell \rightarrow 0$ or $+\infty$. The former limit corresponds to shapes (rectangles/rhombi) collapsing to horizontal segments, the latter to shapes (rectangles/rhombi) collapsing to vertical segments. This implies that the origin is in the closure of the diagram.

Let $\{Q_n\} \subset \mathcal{D}$ be a sequence converging to a point $Q \in \mathbb{R}^2$ (in norm). Let $\Omega_n \in \mathcal{A}$ be a sequence of associated admissible shapes. Without loss of generality (see Lemma 2.3), we may assume that the sequence is uniformly bounded. Then two situations may occur: either a subsequence converges with respect to the complementary Hausdorff distance to some admissible shape Ω , or $\overline{\Omega}_n$ collapses/shrinks to a segment/point. In the former, we infer that, by continuity of $x(\cdot)$ and $y(\cdot)$ in \mathcal{A} , $Q = (x(\Omega), y(\Omega)) \in \mathcal{D}$. Let us examine the second situation. The case in which the chosen Ω_n shrinks to a point can be easily treated as before, since we may replace every Ω_n with a homothetic copy $\hat{\Omega}_n$, obtaining a sequence which has inradii bounded from below by a positive constant. The previous result then applies to $\hat{\Omega}_n$: we find a subsequence (not relabeled) such that $\hat{\Omega}_n \rightarrow \Omega \in \mathcal{A}$ and $Q_n = (x(\hat{\Omega}_n), y(\hat{\Omega}_n)) \rightarrow Q = (x(\Omega), y(\Omega)) \in \mathcal{D}$. The last case that we have to consider is that of $\{\Omega_n\}$ thin domains, collapsing to a segment. Without loss of generality (by the scale invariance and the invariance under rotations of the functionals) may assume that Ω_n all cross the horizontal axis exactly in the segment of length 1. Moreover, they all cross the vertical axis on a centered segment of some length $\ell(n)$ which goes to 0 as $n \rightarrow \infty$. We easily obtain (by convexity) that Ω_n is contained into the centered rectangle $R_{\ell(n)}$ with sides 2 and $2\ell(n)$ and it contains the centered rhombus $S_{\ell(n)}$ with semiaxis of length 1 and $\ell(n)$. In particular,

$$\begin{aligned} 0 < x(\Omega_n) &\leq \frac{A^2(R_{\ell(n)})}{2\pi W(S_{\ell(n)})} = \frac{24}{\pi} \frac{\ell(n)}{1 + \ell^2(n)} \rightarrow 0 \\ 0 < y(\Omega_n) &\leq \frac{4\pi A(R_{\ell(n)})}{P^2(S_{\ell(n)})} = \pi \frac{\ell(n)}{(1 + \ell^2(n))} \rightarrow 0. \end{aligned}$$

This means that the limit point is $Q = (0, 0)$. \square

Proposition 2.6. *The diagrams \mathcal{D} is connected by arcs.*

Proof. Let Q_0 and Q_1 be two points of \mathcal{D} . Take Ω_0 and Ω_1 two associated shapes. Consider the 1-parameter family of shapes obtained with the following Minkowski sum:

$$\Omega_t := t\Omega_1 \oplus (1 - t)\Omega_0 := \{ta + (1 - t)b : a \in \Omega_1, b \in \Omega_0\}, \quad t \in [0, 1].$$

The map $t \mapsto \Omega_t$ preserves convexity and it is continuous with respect to the (complementary) Hausdorff distance. We know that Hausdorff convergence is equivalent to uniform

convergence of support functions (see [19]) and that the support function of a Minkowski sum is the combination of support functions. If we work in \mathcal{A} , in order to preserve also the double symmetry, we have to choose the orientation of Ω_0 and Ω_1 so that their (two orthogonal) axis of symmetry coincide with the axis of the coordinate system. This implies that Ω_t is an admissible shape for every $t \in [0, 1]$. In view of Proposition 2.4, we infer that the map $t \mapsto (x(\Omega_t), y(\Omega_t))$ is continuous in \mathcal{D} and connects Q_0 to Q_1 with an arc. This concludes the proof. \square

The upper and lower boundaries. In order to describe the boundary of the diagram, we introduce the two functions: given $x \in (0, 1]$ we set

$$L^+(x) := \sup\{y(\Omega) : \Omega \in \mathcal{A}, x(\Omega) = x\}; \quad (2.1)$$

$$L^-(x) := \inf\{y(\Omega) : \Omega \in \mathcal{A}, x(\Omega) = x\}. \quad (2.2)$$

These functions satisfy the following properties.

Proposition 2.7. *The supremum defining L^+ and the infimum defining L^- are attained, namely they are a maximum and a minimum, respectively. The functions L^\pm coincide at $x = 1$ and in the limit as $x \rightarrow 0^+$.*

Proof. Let $x \in (0, 1]$ be fixed. Let $y(\Omega_n)$ be a maximizing sequence for $L^+(x)$, with $\Omega_n \in \mathcal{A}$ and such that $x(\Omega_n) = x$ for every n . By Blaschke selection theorem (see also Lemma 2.3), we may extract a subsequence (not relabeled), converging to some $\Omega \in \mathcal{A}$ (the fact that $x \neq 0$ ensures that the subsequence does not collapse to a segment). By continuity of $x(\cdot)$ and $y(\cdot)$, we infer that the limit shape satisfies $x(\Omega) = x$ and $y(\Omega) = \lim_{n \rightarrow \infty} y(\Omega_n)$. Since by assumption $\lim_{n \rightarrow \infty} y(\Omega_n) = L^+(x)$, we conclude that Ω is a maximizer. The same strategy applies for the existence of a minimizer defining $L^-(x)$.

The fact that L^\pm coincide at 1 and 0 (the latter as a limit), comes from Propositions 2.2, 2.4, and 2.5. \square

We are now in a position to state the next result.

Proposition 2.8. *The diagram \mathcal{D} is simply connected.*

Proof. The proof is borrowed from [8, Theorem 3.14], in which the authors deal with the Blaschke-Santaló diagram of volume, perimeter, and first Dirichlet eigenvalue of the Laplacian λ_1 . Here λ_1 is replaced by the moment of inertia W , which is a more tractable functional. For the benefit of the reader, let us summarize the main steps of the proof.

Step 1. Loops in the plane. Let $Q_0, Q_1 \in \mathcal{D}$ be two points of the diagram with the same y -coordinate, say $\bar{y} \in]0, 1[$. We construct a closed (continuous) curve $\Gamma : [0, 2] \rightarrow \mathbb{R}^2$ going first from Q_0 to Q_1 and then back from Q_1 to Q_0 . In order to present the construction, let us consider $\Omega_0, \Omega_1 \in \mathcal{A}$ associated to Q_0 and Q_1 , respectively. Let $\Omega_t, t \in [0, 1]$, denote the normalized Minkowski sum

$$\Omega_t := \frac{t\Omega_1 \oplus (1-t)\Omega_0}{A^{1/2}(t\Omega_1 \oplus (1-t)\Omega_0)}.$$

Without loss of generality, up to a rotation or $\pi/2$, we may assume that the two semi-axes are chosen so that the horizontal one is greater than the vertical one. As we have already shown in Proposition 2.6, $\Omega_t \in \mathcal{A}$. We set

$$\Gamma_{\Omega_0, \Omega_1}(t) := \begin{cases} (x(\Omega_t), y(\Omega_t)) & \text{if } t \in [0, 1], \\ ((2-t)x(\Omega_1) + (t-1)x(\Omega_0), \bar{y}) & \text{if } t \in [1, 2]. \end{cases}$$

The curve is clearly closed and continuous (see the proof of Proposition 2.6). For $t \in [0, 1]$ the support is contained in the diagram (see Proposition 2.6), whereas for $t \in [1, 2]$ it is a horizontal segment, not necessarily contained into the diagram.

We claim that the curve is contained in the stripe $\mathbb{R} \times [\bar{y}, \min(1, 4\bar{y})]$. This is clearly true for the horizontal part. We only need to verify that

$$\bar{y} \leq y(\Omega_t) \leq \min(1, 4\bar{y}) \quad \forall t \in [0, 1]. \quad (2.3)$$

Let us prove the first inequality. Without loss of generality, we may assume that $A(\Omega_0) = A(\Omega_1) = 1$. In particular $P(\Omega_0) = P(\Omega_1) = \sqrt{4\pi/\bar{y}} =: \bar{p}$. Thanks to the linearity

of the perimeter for the Minkowski sum, as well as the Brunn-Minkowski inequality for the area, see [19], we have

$$\begin{aligned} P(t\Omega_1 \oplus (1-t)\Omega_0) &= tP(\Omega_0) + (1-t)P(\Omega_1) = \bar{p}, \\ A^{1/2}(t\Omega_1 \oplus (1-t)\Omega_0) &\geq tA^{1/2}(\Omega_1) + (1-t)A^{1/2}(\Omega_0) = 1, \end{aligned}$$

which immediately gives the desired estimate. Let us now prove the second inequality in (2.3) (the upper bound 1 is trivial). Let α_i and β_i denote the lengths of the semi-axis of Ω_i , $i = 0, 1$. By convexity, since Ω_i contains a rhombus and is contained into a rectangle, we infer that $A(\Omega_i) \geq 2\alpha_i\beta_i$ and $4\sqrt{\alpha_i^2 + \beta_i^2} \leq P(\Omega_i) \leq 4(\alpha_i + \beta_i)$. In particular, recalling that by construction $\alpha_i \geq \beta_i$, $A(\Omega_i) = 1$ and $P(\Omega_i) = \bar{p}$, we deduce that for $i = 0, 1$,

$$\beta_i \leq \frac{1}{2\alpha_i} \leq \frac{4}{\bar{p}}, \quad \alpha_i \leq \frac{\bar{p}}{4}.$$

The previous inequalities allow us to deduce the following upper bound on the area of the (non-normalized) Minkowski sum:

$$A(t\Omega_1 \oplus (1-t)\Omega_0) \leq 4(t\alpha_1 + (1-t)\alpha_0)(t\beta_1 + (1-t)\beta_0) \leq 4,$$

so that

$$y(\Omega_t) = 4\pi \frac{1}{P^2(\Omega_t)} = 4\pi \frac{A(t\Omega_1 \oplus (1-t)\Omega_0)}{\bar{p}^2} \leq 4\bar{y}.$$

Step 2. Sequences of loops. In this step we show that Hausdorff convergence of shapes entails uniform convergence of associated loops. Let us write the statement. Let Ω_0 and Ω_1 be two admissible shapes, different from the disk, satisfying

$$A(\Omega_0) = A(\Omega_1) = 1, \quad P(\Omega_0) = P(\Omega_1) = \bar{p}.$$

Assume to have two sequences of admissible shapes $\{\Omega_{0,n}\}_{n \in \mathbb{N}}$ and $\{\Omega_{1,n}\}_{n \in \mathbb{N}}$ satisfying the following properties: for every $n \in \mathbb{N}$

$$A(\Omega_{0,n}) = A(\Omega_{1,n}) = 1, \quad P(\Omega_{0,n}) = P(\Omega_{1,n}) =: p_n$$

and, in the limit as $n \rightarrow \infty$,

$$\Omega_{0,n} \rightarrow \Omega_0, \quad \Omega_{1,n} \rightarrow \Omega_1$$

with respect to the complementary Hausdorff distance. Step 1 allows us to construct a closed path associated to Ω_0 and Ω_1 , and a family of closed paths associated to the pairs $\Omega_{0,n}, \Omega_{1,n}$. For brevity, we set $\Gamma(t) := \Gamma_{\Omega_0, \Omega_1}(t)$ and $\Gamma_n(t) := \Gamma_{\Omega_{0,n}, \Omega_{1,n}}(t)$. We claim that for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$\forall n \geq n_\epsilon, \quad \forall t \in [0, 2], \quad \|\Gamma(t) - \Gamma_n(t)\| < \epsilon.$$

For $t \in [1, 2]$ we have

$$\|\Gamma(t) - \Gamma_n(t)\| \leq 4\pi \left| \frac{1}{\bar{p}^2} - \frac{1}{p_n^2} \right| + \frac{1}{2\pi} \left| \frac{1}{W(\Omega_1)} - \frac{1}{W(\Omega_{1,n})} \right| + \frac{1}{2\pi} \left| \frac{1}{W(\Omega_0)} - \frac{1}{W(\Omega_{0,n})} \right|.$$

The right-hand side does not depend on t and, thanks to the Hausdorff convergence of the shapes and to the continuity of the shape functionals under study, it is arbitrarily small for n large enough. For $t \in [0, 1]$ we have

$$\|\Gamma(t) - \Gamma_n(t)\| \leq |x(\Omega_t) - x(\Omega_{n,t})| + |y(\Omega_t) - y(\Omega_{n,t})|. \quad (2.4)$$

Set

$$w_n(t) := W(t\Omega_{1,n} \oplus (1-t)\Omega_{0,n}), \quad w(t) := W(t\Omega_1 \oplus (1-t)\Omega_0),$$

and

$$a_n(t) := A(t\Omega_{1,n} \oplus (1-t)\Omega_{0,n}), \quad a(t) := A(t\Omega_1 \oplus (1-t)\Omega_0).$$

Let us consider the second term in the right-hand side of (2.4). Using the notation above we have

$$|y(\Omega_t) - y(\Omega_{n,t})| \leq \frac{4\pi a(t)}{p_n^2 \bar{p}^2} |p_n^2 - \bar{p}^2| + \frac{4\pi}{p_n^2} |a(t) - a_n(t)|.$$

The factors $a(t)$ and p_n are uniformly bounded (and away from 0) in t and n . The term $|p_n^2 - \bar{p}^2|$ is infinitesimal by assumption. The difference $a(t) - a_n(t)$ is a polynomial of degree 2 in t (see the properties of Minkowski mixed volumes, e.g., in [19]); its coefficients depend on n and they go to 0 as $n \rightarrow \infty$, by the Hausdorff convergence of shapes. These

estimates imply that $|y(\Omega_t) - y(\Omega_{n,t})|$ is arbitrarily small for n large enough, uniformly in t .

The same strategy applies to the first term in the right-hand side of (2.4):

$$\begin{aligned} |x(\Omega_t) - x(\Omega_{n,t})| &= \frac{1}{2\pi} \left| \frac{a_n^2(t)}{w_n(t)} - \frac{a^2(t)}{w(t)} \right| = \frac{|a_n^2(t)w(t) - a^2(t)w_n(t)|}{2\pi w_n(t)w(t)} \\ &\leq \frac{a_n(t) + a(t)}{2\pi w_n(t)} |a_n(t) - a(t)| + \frac{a^2(t)}{2\pi w_n(t)w(t)} |w(t) - w_n(t)|. \end{aligned}$$

As before, the factors $a_n(t), a(t), w_n(t), w(t)$ are uniformly bounded (and away from 0) in t and n , and the difference $a_n(t) - a(t)$ is arbitrarily small, uniformly in t . Using polar coordinates, it is immediate to check that $|w_n(t) - w(t)|$ is bounded above by the complementary Hausdorff distance

$$d_n(t) := d_H(t\Omega_{1,n} \oplus (1-t)\Omega_{0,n} ; t\Omega_1 \oplus (1-t)\Omega_0).$$

Since the distance d_H of two shapes coincides with the L^∞ norm of the difference of the associated support functions, and since the support function is linear for a Minkowski sum, we deduce that

$$d_n(t) = \|th_{1,n} + (1-t)h_{0,n} - th_1 - (1-t)h_0\|_\infty \leq t\|h_1 - h_{1,n}\| + (1-t)\|h_0 - h_{0,n}\|_\infty,$$

where $h_0, h_{0,n}, h_1$, and $h_{1,n}$ denote the support functions of $\Omega_0, \Omega_{0,n}, \Omega_1$, and $\Omega_{1,n}$, respectively. Since Hausdorff convergence is equivalent to uniform convergence of support functions, we conclude that $d_n(t)$, and then $|w(t) - w_n(t)|$, are arbitrarily small for n large enough, uniformly in t .

This concludes the proof of the uniform continuity.

Step 3. Loops around holes. We claim that \mathcal{D} coincides with the set

$$\{(x, y) : x \in (0, 1], y \in [L^-(x), L^+(x)]\}.$$

This fact, together with the connectedness obtained in Proposition 2.6, implies the simple connectedness of the diagram. Assume by contradiction that this is not true. Then there exists a point $Q = (x_Q, y_Q) \in \mathbb{R}^2 \setminus \mathcal{D}$ such that $x_Q \in (0, 1)$ and $y_Q \in (L^-(x_Q), L^+(x_Q))$. Since $Q \neq O$, we infer that $Q \in \mathbb{R}^2 \setminus (\mathcal{D} \cup \{O\}) = \mathbb{R}^2 \setminus \overline{\mathcal{D}}$, which is an open set. Thus there exists a radius $r > 0$ such that $B_r(Q) \cap \mathcal{D} = \emptyset$. Given a continuous closed curve in \mathbb{R}^2 and a point not belonging to its support, it is well defined the winding number, invariant under homotopy. Let us denote by $w_\gamma(Q)$ the winding number of a closed curve $\gamma : I \rightarrow \mathbb{R}^2$, being $I \subset \mathbb{R}$ an interval. Given $x \in (0, 1)$, we introduce the following notation

$$\mathcal{A}_1(y) := \{\Omega \in \mathcal{A} : A(\Omega) = 1, y(\Omega) = y\}.$$

Given two shapes $\Omega_0, \Omega_1 \in \mathcal{A}_1(y)$ we can construct, by Step 2, a continuous curve in \mathbb{R}^2 , with support $\Gamma_{\Omega_0, \Omega_1}$. We notice that if $y \leq y_Q - \frac{r}{2}$, this path does not cross the point Q : indeed, for $t \in [0, 1]$ the path is contained into \mathcal{D} , whereas for $t \in [1, 2]$ it is horizontal with ordinate $y < y_Q$. Therefore, the winding number of the path around Q is well defined.

Let us consider the set

$$J = \left\{ 0 < y \leq y_Q - \frac{r}{2} : \exists \Omega_0, \Omega_1 \in \mathcal{A}_1(y), w_{\Gamma_{\Omega_0, \Omega_1}}(Q) \neq 0 \right\}.$$

It is easy to see that $J \neq \emptyset$ (the point $y_Q - r/2$ belongs to it) and that J is bounded from below by a positive constant (by (2.3)). Let $\bar{y} := \inf J$. Two possibilities may occur: either $\bar{y} \in J$ or $\bar{y} \notin J$. Consider, e.g., the sequence $y_n = \bar{y} - 1/n$. We clearly have $y_n \notin J$. Using [8, Corollary 3.6] (that can be easily adapted to the case of double symmetric sets), we can construct a family of pairs $(\Omega_{0,n}, \Omega_{1,n}) \in \mathcal{A}_1(y_n)$ such that $\Omega_{0,n} \rightarrow \Omega_0$ and $\Omega_{1,n} \rightarrow \Omega_1$ with respect to the complementary Hausdorff distance. Since $y_n \notin J$, we have $w_{\Gamma_{\Omega_{0,n}, \Omega_{1,n}}}(Q) = 0$. The winding number being an invariant under homotopy, we get a contradiction. The second case is similar, taking a sequence $y_n \rightarrow \bar{y}^+$. This concludes the proof. \square

Now, using recent results of Lamboley, Novruzzi, Pierre, see [14] we prove the following

Theorem 2.9. *Any convex domain, different from the disk, that minimizes $y(\Omega)$ with $x(\Omega) = x_0$ fixed is a polygon.*

Any convex domain, different from the disk, that maximizes $y(\Omega)$ with $x(\Omega) = x_1$ fixed is $C^{1,1}$.

Proof. First assertion: Minimizing $y(\Omega)$ with $x(\Omega) = x_0$ is equivalent, according to the scale invariance of the functionals x and y to

$$\min \left\{ -P(\Omega) \mid \Omega \text{ convex}, A(\Omega) = 1, W(\Omega) = \frac{1}{2\pi x_0} =: w_0 \right\}.$$

Now we want to use Theorem 4 in [14] with the constraint $m(\Omega) = (0, 0)$ where, using the parametrization with the gauge function u :

$$m(\Omega) = m(u) = \left(\frac{1}{2} \int_0^{2\pi} \frac{1}{u^2(\theta)} d\theta - 1, \frac{1}{4} \int_0^{2\pi} \frac{1}{u^4(\theta)} d\theta - w_0 \right)$$

and the functional to minimize is

$$-P(\Omega) = -P(u) = - \int_0^{2\pi} \frac{\sqrt{u^2 + u'^2}}{u^2(\theta)} d\theta.$$

In order to apply this Theorem 4, we just need to prove (u_0 denotes the minimizer)

- that $m'(u_0)$ is onto on \mathbb{R}^2
- that $\langle m''(u_0)v, v \rangle$ is dominated by a norm of v weaker than the H^1 -norm.

The first point comes from the fact that

$$\langle m'(u_0), v \rangle = \left(- \int_0^{2\pi} \frac{v}{u_0^3(\theta)} d\theta, - \int_0^{2\pi} \frac{v}{u_0^5(\theta)} d\theta \right)$$

and the two linear forms are independent since u_0 is not constant.

The second point comes from the fact that

$$\langle m''(u_0)v, v \rangle = \left(3 \int_0^{2\pi} \frac{v^2}{u_0^4(\theta)} d\theta, 5 \int_0^{2\pi} \frac{v^2}{u_0^6(\theta)} d\theta \right)$$

and we can estimate both terms by the L^2 -norm of v .

Second assertion: Now we use, exactly in the same spirit, Theorem 2 in [14], see also Theorem 1.1 in [15]. Now, we minimize $P(u)$ with the same constraint, and the "convexity" of the functional $u \mapsto P(u)$ provides the good assumptions. \square

From the previous result we immediately have the following

Corollary 2.10. *The two functions L^+ and L^- only coincide for $x = 1$.*

3. THE LOWER BOUNDARY OF THE DIAGRAM

In this section we investigate the properties of the lower boundary of the diagram, described by the graph of the function L^- (see (2.2)). We recall that $L^-(x)$ is defined as the minimum of the shape functional $y(\cdot)$ when the shape functional $x(\cdot)$ is fixed (equal to x).

As we have already proved in Theorem 2.9, the optimal shapes associated to $L^-(x)$ are polygons.

To prove the regularity and monotonicity of L^- , we need some preparatory results.

Lemma 3.1. *Let Ω be a polygon in \mathcal{A} . Then there exists $\epsilon_0 > 0$ and a map*

$$[0, \epsilon_0] \ni \epsilon \mapsto \Omega_\epsilon \in \mathcal{A}$$

continuous with respect to the complementary Hausdorff distance, such that $\Omega_0 = \Omega$ and such that $\epsilon \mapsto x(\Omega_\epsilon)$ is strictly increasing.

Proof. Let V_1 be one of the vertexes of Ω in the first quadrant with boundary angle $\alpha \in]0, \pi[$, and let V_2, V_3, V_4 be the corresponding (symmetric) points in the other three quadrants. For $\epsilon > 0$ small, let T_ϵ denote the union of the four isosceles triangles, each of them with vertex at V_i , two legs lying on $\partial\Omega$, and base of length ϵ .

We define the set Ω_ϵ as the set obtained by removing from Ω the four triangles, namely $\Omega_\epsilon := \Omega \setminus T_\epsilon$. This operation clearly makes A and W decrease:

$$A(\Omega_\epsilon) = A(\Omega) \left(1 - \frac{\int_{T_\epsilon} 1}{\int_\Omega 1} \right), \quad W(\Omega_\epsilon) = W(\Omega) \left(1 - \frac{\int_{T_\epsilon} (x^2 + y^2)}{\int_\Omega (x^2 + y^2)} \right).$$

Therefore

$$\begin{aligned} x(\Omega_\epsilon) &= x(\Omega) \left(1 - 2 \frac{\int_{T_\epsilon} 1}{\int_\Omega 1} + o(\epsilon^2) \right) \left(1 + \frac{\int_{T_\epsilon} (x^2 + y^2)}{\int_\Omega (x^2 + y^2)} + o(\epsilon^2) \right) \\ &= x(\Omega) \left(1 - 2 \frac{\int_{T_\epsilon} 1}{\int_\Omega 1} + \frac{\int_{T_\epsilon} (x^2 + y^2)}{\int_\Omega (x^2 + y^2)} + o(\epsilon^2) \right). \end{aligned}$$

The integrals over T_ϵ are of order ϵ and the sign of their sum, in the limit as $\epsilon \rightarrow 0$, gives the sign of the derivative of $\epsilon \mapsto x(\Omega_\epsilon)$ at $\epsilon = 0$. Therefore, if we are able to show that for ϵ small enough there holds

$$\int_\Omega (x^2 + y^2) < \frac{1}{2} \int_{T_\epsilon} (x^2 + y^2),$$

we are done. Here the symbol f stands for the average. Let us now make a choice on the V_i s: we cut four triangles near the boundary points V_i satisfying the maximal distance from the origin, namely

$$\|V_1\|^2 = \|V_2\|^2 = \|V_3\|^2 = \|V_4\|^2 = \max_{S \in \Omega} \|S\|^2 = \max_{(x,y) \in \Omega} (x^2 + y^2).$$

Passing to the limit as $\epsilon \rightarrow 0$ we infer that each triangle in T_ϵ shrinks to the its vertex V_i and, by the Lebesgue theorem,

$$\int_{T_\epsilon} (x^2 + y^2) \rightarrow \|V_1\|^2.$$

Therefore it is enough to prove that

$$\int_\Omega (x^2 + y^2) < \frac{\|V_1\|^2}{2}.$$

Let us use a parametrization of Ω in polar coordinates: $\theta \in [0, 2\pi]$ and $\rho \in [0, \rho(\theta)]$, so that

$$\begin{aligned} \int_\Omega (x^2 + y^2) &= \frac{\int_0^{2\pi} \int_0^{\rho(\theta)} \rho^3(\theta) \, d\rho \, d\theta}{\int_0^{2\pi} \int_0^{\rho(\theta)} \rho(\theta) \, d\rho \, d\theta} \\ &= \frac{1}{2} \frac{\int_0^{2\pi} \rho^4(\theta) \, d\theta}{\int_0^{2\pi} \rho^2(\theta) \, d\theta} \leq \frac{1}{2} \max_\theta \|\rho(\theta)\|^2 = \frac{\|V_1\|^2}{2}. \end{aligned}$$

The equality holds true only if $\rho(\theta) = \|V_1\|$ for every θ , namely when Ω is the disk. In particular the inequality is strict in the case under study. This concludes the proof. \square

Lemma 3.2. *Let Ω be a polygon in \mathcal{A} . Then there exists $\epsilon_0 > 0$ and a map*

$$[0, \epsilon_0] \ni \epsilon \mapsto \Omega_\epsilon \in \mathcal{A}$$

continuous with respect to the complementary Hausdorff distance, such that $\Omega_0 = \Omega$ and such that $\epsilon \mapsto x(\Omega_\epsilon)$ is strictly decreasing.

Proof. The shape Ω_ϵ will be constructed as a small (continuous) deformation of Ω . The deformation will be explicitly described in the first quadrant and repeated in the other three quadrants, in order to preserve the double symmetry.

Let Ω be a given polygon in \mathcal{A} . Let k be the number of boundary vertexes falling in the first quadrant $\{x \geq 0, y \geq 0\}$. Let $\{Q_1, \dots, Q_k\}$ be such points, ordered in counter-clockwise sense.

Case 1: $k = 1$ or $k = 2$ and Q_1, Q_2 on the axis. This two cases correspond to rectangles and rhombi. They can be described, respectively, as the 1-parameter families $\{R_\ell\}_{\ell \in [0,1]}$

and $\{S_\ell\}_{\ell \in]0,1]}$ defined as follows: R_ℓ is the rectangle associated to $k = 1$ with $Q_1 = (1, \ell)$; S_ℓ is the rhombus associated to $k = 2$ with $Q_1 = (1, 0), Q_2 = (0, \ell)$. The functions

$$]0, 1] \ni \ell \mapsto R_\ell \in \mathcal{A}, \quad]0, 1] \ni \ell \mapsto S_\ell \in \mathcal{A}$$

are continuous with respect to the complementary Hausdorff distance, moreover (see also the proof of Proposition 2.5) the maps

$$]0, 1] \ni \ell \mapsto x(R_\ell) = x(S_\ell) = \frac{6\ell}{\pi(1 + \ell^2)}$$

are strictly increasing. This means that we can always increase/decrease ℓ to make $x(\cdot)$ decrease in a continuous way.

Case 2: $k = 2$ and either Q_1 or Q_2 on the coordinate axis. Without loss of generality, we can assume that Q_1 is on the horizontal axis $y = 0$, say $Q_1 = (1, 0)$, whereas $Q_2 = (s, t)$ for some $s \in]0, 1]$ and $t > 0$. In this case Ω is a hexagon with boundary points $Q_1, -Q_1, Q_2$ and the three reflections of Q_2 in the other three quadrants. Let us denote by $H_{s,t}$ such hexagon. As before, the map

$$]0, 1[\times]0, +\infty[\ni (s, t) \mapsto H_{s,t} \in \mathcal{A}$$

is continuous with respect to the complementary Hausdorff distance. A direct computation leads to

$$x(H_{s,t}) = \frac{24}{\pi} \frac{t(s+1)^2}{t^2(3s+1) + s^3 + s^2 + s + 1}.$$

By computing the gradient of this function in the variables s and t , it is immediate to check that there is no critical point in the stripe $(s, t) \in]0, 1[\times]0, +\infty[$. As in the previous case, this implies that we can always find a direction which makes the directional derivative of $x(\cdot)$ negative.

Case 3: $k = 2$ and neither Q_1 nor Q_2 are on the coordinate axis. In this case Ω is an octagon. This class in \mathcal{A} can be described as the three-parameters family $O_{s,t,u}$ associated to the two points in the first quadrant $Q_1 = (1, s), Q_2 = (t, u)$, where $t < 1$ and $u > s$. We have:

$$x(O_{s,t,u}) = \frac{24}{\pi} \frac{N(s, t, u)}{D(s, t, u)},$$

with

$$\begin{aligned} N(s, t, u) &= [tu + (1-t)(u+s)/2]^2, \\ D(s, t, u) &= tu(t^2 + 4u^2 + t + 2) \\ &\quad + (1-t)(s^3 + s^2u + st^2 + 2st + su^2 + 3s + u^3 + u). \end{aligned}$$

Once again this function has no critical point in the region of \mathbb{R}^3 where s, t, u live.

Case 4: $k \geq 3$. Here we proceed in a different way with respect to the previous cases: we will perform a parallel chord movement, consisting in making Q_2 slide on a line parallel to Q_1Q_3 passing through Q_2 . To preserve symmetry, we do the same thing in the other three quadrants. Thanks to the standing assumptions, since Q_2 is not aligned to Q_1 vertically nor to Q_3 horizontally, we can perform such sliding in two directions. The resulting shape is still convex, double symmetric, and has the same area (since we keep the point Q_2 at the same distance from the line passing through Q_1Q_3). Since this deformation does not affect the area, it is enough to study the behavior of W : to get the thesis we need to make W increase. More precisely, since we only perturb the shape in the triangle $Q_1Q_2Q_3$, it is enough to study the behavior of $W(Q_1Q_2Q_3)$. Once clarified that we do four identical deformations in the four quadrants, we focus ourselves to what happens in one of the triangles. Since W is invariant under rotation, we may assume that the segment Q_1Q_3 is vertical. Without loss of generality we may assume that Q_2 is at distance 1 from the vertical line through Q_1Q_3 . Thus Q_2 is of the form $Q_2 = Q_1 + (1, t)$ for some positive t . Let $c > 0$ be the distance of Q_3 from Q_1 , namely $Q_3 = Q_1 + (0, c)$. The admissible values of t depend on the sides of Ω adjacent to Q_1Q_2 and Q_2Q_3 , however, given an admissible t , there exists a neighborhood of t for which the associated shape is

still admissible. Let $Q_1 = (a, b)$. A direct computation shows that

$$\begin{aligned} W(Q_1 Q_2 Q_3) &= \int_0^1 \int_{tx}^{(t-c)x+c} [(x+a)^2 + (y+b)^2] dy dx \\ &= \varphi(a, b, c) + \frac{c(c+4b)}{12}t + \frac{c}{12}t^2, \end{aligned}$$

for some (explicit) function φ not depending on t . As a function of t , there is only one critical point, which is a local minimizer. This implies that for every t , we can always increase or decrease t in such a way to make W increase. This concludes the proof. \square

Corollary 3.3. *Let $x_0 \in]0, 1[$ and $\Omega_0 \in \mathcal{A}$ with $x(\Omega_0) = x_0$ and $y(\Omega_0) = L^-(x_0)$. Let $\{x_n\}_{n \in \mathbb{N}} \subset]0, 1[$ be a sequence such that $x_n \rightarrow x_0$ as $n \rightarrow \infty$. Then there exists a sequence of shapes $\{\Omega_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ such that for n sufficiently large $x(\Omega_n) = x_n$ and in the limit as $n \rightarrow \infty$ there holds $\Omega_n \rightarrow \Omega_0$ with respect to the complementary Hausdorff distance.*

Proof. In view of the assumption $L^-(x_0) = y(\Omega_0)$ and Theorem 2.9 we infer that Ω_0 is a polygon. We will distinguish the two cases of $x_n \rightarrow x_0^+$ and $x_n \rightarrow x_0^-$. Let us detail the case $x_n \rightarrow x_0^+$. In view of Lemma 3.1, there exists some $\epsilon_0 > 0$ and a continuous map $[0, \epsilon_0] \ni \epsilon \mapsto \Omega_\epsilon \in \mathcal{A}$ such that $\epsilon \mapsto x(\Omega_\epsilon)$ is continuous and increasing. The image of $\epsilon \mapsto x(\Omega_\epsilon)$ is the whole interval $[x_0, x(\Omega_{\epsilon_0})]$. For every n large enough, $x_n \in [x_0, x(\Omega_{\epsilon_0})]$ and there exists $\epsilon_n \in [0, \epsilon_0]$ such that $x(\epsilon_n) = x_n$. As $x_n \rightarrow x_0$ we have $\epsilon_n \rightarrow 0$. The thesis then follows by taking $\Omega_n := \Omega_{\epsilon_n}$. The same strategy applies to the case $x_n \rightarrow x_0^-$ using Lemma 3.2. \square

Lemma 3.4. *Let $\Omega \in \mathcal{A}$ be a polygon. Then it is not a local minimizer of $y(\cdot)$.*

Proof. In order to prove the statement, it is enough to show that given a polygon $\Omega \in \mathcal{A}$, we can construct a small deformation (see Definition 2.1) which makes $y(\cdot)$ strictly decrease. The deformation will be explicitly described in the first quadrant and repeated in the other three quadrants, in order to preserve the double symmetry.

Let Ω be a given polygon in \mathcal{A} . Let k be the number of boundary vertexes falling in the first quadrant $\{x \geq 0, y \geq 0\}$. Let $\{Q_1, \dots, Q_k\}$ be such points, ordered in counter-clockwise sense.

Case 1: $k = 1$ or $k = 2$ and Q_1, Q_2 on the axis. This two cases correspond to rectangles and rhombi. They can be described, respectively, as the 1-parameter families $\{R_\ell\}_{\ell \in]0, 1]}$ and $\{S_\ell\}_{\ell \in]0, 1]}$ defined as follows: R_ℓ is the rectangle associated to $k = 1$ with $Q_1 = (1, \ell)$; S_ℓ is the rhombus associated to $k = 2$ with $Q_1 = (1, 0), Q_2 = (0, \ell)$. The functions

$$]0, 1] \ni \ell \mapsto R_\ell \in \mathcal{A}, \quad]0, 1] \ni \ell \mapsto S_\ell \in \mathcal{A}$$

are continuous with respect to the complementary Hausdorff distance, moreover (see also the proof of Proposition 2.5) the maps

$$]0, 1] \ni \ell \mapsto y(R_\ell) = \frac{\pi \ell}{(1 + \ell)^2}, \quad]0, 1] \ni \ell \mapsto y(S_\ell) = \frac{\pi \ell}{2(1 + \ell^2)}$$

are strictly increasing. This means that we can always increase/decrease ℓ to make $y(\cdot)$ decrease.

Case 2: $k = 2$ and either Q_1 or Q_2 on the coordinate axis. Without loss of generality, we can assume that Q_1 is on the horizontal axis $y = 0$, say $Q_1 = (1, 0)$, whereas $Q_2 = (s, t)$ for some $s \in]0, 1]$ and $t > 0$. In this case Ω is a hexagon with boundary points $Q_1, -Q_1, Q_2$ and the three reflections of Q_2 in the other three quadrants. Let us denote by $H_{s,t}$ such hexagon. As before, the map

$$]0, 1[\times]0, +\infty[\ni (s, t) \mapsto H_{s,t} \in \mathcal{A}$$

is continuous with respect to the complementary Hausdorff distance. A direct computation leads to

$$y(H_{s,t}) = \frac{\pi}{2} \frac{s(1+t)}{[s + \sqrt{(1-s)^2 + t^2}]^2}.$$

By computing the gradient of this function in the variables s and t , it is immediate to check that there is no critical point in the stripe $(s, t) \in]0, 1[\times]0, +\infty[$. As in the previous case,

this implies that we can always find a direction which makes the directional derivative of $y(\cdot)$ negative.

Case 3: $k = 2$ and neither Q_1 nor Q_2 are on the coordinate axis. In this case Ω is an octagon. This class in \mathcal{A} can be described as the three-parameters family $O_{s,t,u}$ associated to the two points in the first quadrant $Q_1 = (1, s)$, $Q_2 = (t, u)$, where $t < 1$ and $u > s$. We have:

$$y(O_{s,t,u}) = \frac{\pi}{2} \frac{tu + (1-t)(u+s)/2}{[t+s + \sqrt{(u-s)^2 + (1-t)^2}]^2}$$

Once again this function has no critical point in the region of \mathbb{R}^3 where s, t, u live.

Case 4: $k \geq 3$. Here we proceed in a different way with respect to the previous cases: we will perform a parallel chord movement, consisting in making Q_2 slide on a line parallel to Q_1Q_3 passing through Q_2 . To preserve symmetry, we do the same thing in the other three quadrants. Thanks to the standing assumptions, since the Q_2 is not aligned to Q_1 vertically nor to Q_3 horizontally, we can perform such sliding in two directions. The resulting shape is still convex, double symmetric, and has the same area (since we keep the point Q_2 at the same distance from the line passing through Q_1Q_3). The deformation can be done in two senses and at least one of the two makes the perimeter of the triangle $Q_1Q_2Q_3$ (and then of the entire shape) increase, namely the functional $y(\cdot)$ decreases. When the triangle $Q_1Q_2Q_3$ is isosceles, both deformations make the perimeter increase. This concludes the proof. \square

We are now in a position to state the following.

Proposition 3.5. *The function $L^- :]0, 1] \rightarrow \mathcal{D}$ is continuous and strictly increasing.*

Proof. The proof is divided into three steps.

Step 1: L^- is l.s.c. This property follows by construction. Let $x \in]0, 1]$ and $\{x_n\} \subset (0, 1]$ an arbitrary sequence converging to x . We need to prove that

$$L^-(x) \leq \liminf_n L^-(x_n).$$

Assume (without loss of generality) that the \liminf is a limit. Let Ω_n be the sequence of optimal shapes for $L^-(x_n)$. Since $x > 0$, we infer that also x_n are (uniformly) far from 0. This ensures the existence of a subsequence (not relabeled) Ω_n which converges to some admissible $\Omega \in \mathcal{A}$. By continuity $x(\Omega) = \lim_n x(\Omega_n) = \lim_n x_n = x$. Thus by definition of $L^-(x)$ we infer that $y(\Omega) \geq L^-(x)$. Since $y(\Omega) = \lim_n y(\Omega_n)$ we conclude that

$$L^-(x) \leq y(\Omega) = \lim_n y(\Omega_n).$$

Step 2: L^- is u.s.c. Take $x_0 \in]0, 1]$ and take $x_n \rightarrow x_0$ satisfying

$$\limsup_{x \rightarrow x_0} L^-(x) = \lim_{n \rightarrow \infty} L^-(x_n).$$

Take Ω_0 an optimal shape for $L^-(x_0)$, namely such that $L^-(x_0) = y(\Omega_0)$. Its existence is ensured by Proposition 2.7. In view of Corollary 3.3, there exists a sequence of shapes Ω_n with the following properties: for every $n \in \mathbb{N}$

$$\Omega_n \in \mathcal{A}, \quad x(\Omega_n) = x_n;$$

moreover, in the limit as $n \rightarrow \infty$, $\Omega_n \rightarrow \Omega_0$ with respect to the complementary Hausdorff distance. In particular, $\lim_{n \rightarrow \infty} y(\Omega_n) = y(\Omega_0)$. Exploiting now the definition of L^- , we obtain

$$\limsup_{x \rightarrow x_0} L^-(x) = \lim_{n \rightarrow \infty} L^-(x_n) \leq \lim_{n \rightarrow \infty} y(\Omega_n) = y(\Omega_0) = L^-(x_0).$$

Step 3: L^- is strictly increasing. By the previous steps, L^- is continuous. Moreover, its infimum is 0 and it attains its maximum 1 for $x = 1$. Assume by contradiction that L^- is not strictly increasing. Then there exists $x_0 \in]0, 1[$ local minimizer of L^- . Therefore there exists a neighborhood $U(x_0)$ of x_0 such that

$$\Omega \in \mathcal{A}, \quad x(\Omega) \in U(x_0) \quad \Rightarrow \quad L^-(x_0) \leq y(\Omega).$$

Therefore any optimal shape Ω_0 for $L^-(\Omega_0)$, namely such that $L^-(x_0) = y(\Omega_0)$, is a local minimizer for $y(\cdot)$. In view of Theorem 2.9, Ω_0 is a polygon. The contradiction comes from Lemma 3.4, which states that a polygon can not be a local minimizer for $y(\cdot)$. \square

4. THE UPPER BOUNDARY OF THE DIAGRAM

In this section we investigate the properties of the upper boundary of the diagram, described by the graph of the function L^+ (see (2.1)). We recall that $L^+(x)$ is defined as the maximum of the shape functional $y(\cdot)$ when the shape functional $x(\cdot)$ is fixed (equal to x). As we have proved in Theorem 2.9, optimal shapes for L^+ are $C^{1,1}$.

In order to prove the regularity and monotonicity of L^+ , we need two preparatory results.

Lemma 4.1. *Let $x_0 \in]0, 1[$ and $\Omega_0 \in \mathcal{A}$ with $x(\Omega_0) = x_0$ and $y(\Omega_0) = L^+(x_0)$. Let $\{x_n\}_{n \in \mathbb{N}} \subset]0, 1[$ be a sequence such that $x_n \rightarrow x_0$ as $n \rightarrow \infty$. Then there exists a sequence of shapes $\{\Omega_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ such that for n sufficiently large $x(\Omega_n) = x_n$ and in the limit as $n \rightarrow \infty$ there holds $\Omega_n \rightarrow \Omega_0$ with respect to the complementary Hausdorff distance.*

Proof. In view of the assumption $y(\Omega_0) = L^+(x_0)$ and Theorem 2.9, we infer that Ω_0 is of class $C^{1,1}$. As already done in the previous Section (cf. Lemmas 3.1 and 3.2, Corollary 3.3), it is enough to provide a deformation field V localized on the strictly convex part of the boundary of Ω_0 for which $y'(\Omega_0, V) \neq 0$. Then, using either V or $-V$, we can make the first order shape derivative positive (or negative): the associated continuous deformation $t \mapsto \Omega_t$, for t small enough, is such that $t \mapsto x(\Omega_t)$ is increasing (resp. decreasing). To conclude it is enough to take $\Omega_n := \Omega_{t_n}$ being t_n such that $x(\Omega_{t_n}) = x_n$.

Let us now prove that such V exists. Assume by contradiction that for every $V : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ smooth, localized on the strictly convex part Γ_+ of the boundary $\partial\Omega_0$. Then, using the Hadamard's formula

$$\begin{aligned} x'(\Omega_0, V) &= \frac{A(\Omega_0)}{2\pi W(\Omega_0)} [2W(\Omega_0)A'(\Omega_0, V) - A(\Omega_0, V)W'(\Omega_0)] \\ &= \frac{A(\Omega_0)}{2\pi W(\Omega_0)} \int_{\partial\Omega_0} [2W(\Omega_0) - A(\Omega_0)\|x\|^2] d\mathcal{H}^1(x), \end{aligned}$$

we deduce that the shape under study satisfies

$$\|x\|^2 = \frac{2W(\Omega_0)}{A(\Omega_0)} \quad \forall x \in \Gamma_+.$$

In other words, Γ_+ is made of arcs of circle of radius $\sqrt{2W(\Omega_0)/A(\Omega_0)}$ centered at the origin. This implies that Ω_0 is the disk. This is excluded by the assumptions. This concludes the proof. \square

Lemma 4.2. *Let $\Omega \in \mathcal{A}$ be of class $C^{1,1}$ different from the disk. Then it is not a local maximiser of $y(\cdot)$.*

Proof. Let $\Omega \in \mathcal{A}$ be of class $C^{1,1}$. The boundary $\partial\Omega$ has the following structure:

$$\partial\Omega = \Gamma_0 \cup \Gamma_+,$$

where Γ_0 is the union of the flat parts (with 0 curvature) and Γ_+ is the union of the strictly convex parts (with positive curvature). In the following we denote by $H(x)$ the curvature at a point $x \in \partial\Omega$. Note that Γ_+ can not be the empty set, by the regularity assumption on Ω .

In the following steps we find a small deformation, acting on Γ_+ which makes $y(\cdot)$ (strictly) increase. This will be done using shape derivatives, namely showing that $y'(\Omega, V) > 0$ for some vector field $V : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Step 1. Generic shape: deformation localized on Γ_+ . Consider a smooth vector fields V with support localized in a part of Γ_+ . Two situations may occur: either there exists V such that $y'(\Omega, V) \neq 0$ or $y'(\Omega, V) = 0$ for every V . In the former, up to changing V

into $-V$, we obtain a small deformation of Ω making $y(\cdot)$ increase, concluding the proof. In the second situation, using the Hadamard's formula

$$\begin{aligned} y'(\Omega, V) &= \frac{4\pi}{P^3(\Omega)} [A'(\Omega, V)P(\Omega) - 2A(\Omega)P'(\Omega, V)] \\ &= \frac{4\pi}{P^3(\Omega)} \int_{\partial\Omega} [P(\Omega) - 2A(\Omega)H(x)] V(x) \cdot n(x) d\mathcal{H}^1(x), \end{aligned}$$

we deduce that the shape Ω under study satisfies the following (optimality) condition:

$$H(x) = \frac{P(\Omega)}{2A(\Omega)} \quad \forall x \in \Gamma_+.$$

In other words, Γ_+ is made of arcs of circle of radius

$$R := \frac{2A(\Omega)}{P(\Omega)}.$$

Since Ω is not the disk, Γ_+ does not cover the whole $\partial\Omega$, so that $\Gamma_0 \neq \emptyset$. We call such domains *generalized stadiums*.

Step 2. The only local maximizer of $y(\cdot)$ among generalized stadiums is the disk. Let Ω be a generalized stadium satisfying the optimality condition described in Step 1. Let $\{\gamma_i\}_{i=1}^k$ denote the boundary arcs, labeled in counter-clockwise sense, for some $n \in \mathbb{N}$. Let C_i denote the center of the circle of radius R to which γ_i belongs to. Furthermore, denote by S_i , $i = 1, \dots, n$, the segment joining the arc γ_i to the arc γ_{i+1} , with the identification $\gamma_{n+1} := \gamma_1$. Using the $C^{1,1}$ regularity of the boundary, we infer that every S_i is tangent to the arcs γ_i and γ_{i+1} , in particular the segments joining C_i and C_{i+1} to the junction points arc-segment are orthogonal to the segment. This means that the quadrilateral obtained by considering the segment S_i and the segment $C_i C_{i+1}$ is a rectangle. Repeating the same procedure to every pair arc-segment, we infer that Ω is the union of the polygon $K := C_1 C_2 \dots C_n$, n circular sectors centered at C_i , and n rectangles constructed on the sides of K . In other words, Ω is the Minkowski sum $\Omega = K \oplus B(0, R)$. By the classical properties of area and perimeter of Minkowski sums, we have

$$A(\Omega) = A(K) + RP(K) + \pi R^2, \quad P(\Omega) = P(K) + 2\pi R.$$

Inserting these two equalities into the optimality condition relating R , $A(\Omega)$, and $P(\Omega)$, we get

$$2A(K) + 2P(K)R + 2\pi R^2 = P(K)R + 2\pi R^2 \quad \Rightarrow \quad 2A(K) + P(K)R = 0,$$

implying that $K = \emptyset$. Thus the only possible shape is the disk, excluded by assumption. This concludes the proof. \square

Remark 4.3. We point out that the same result can be also deduced from [8, Lemma 3.5]: the authors prove that the ball is the only local minimizer of P among planar convex sets with area 1. Their proof consists in comparing $P(\Omega)$ with $P(\Omega_\epsilon)$, being Ω_ϵ the Minkowski sum $\Omega \oplus \epsilon B(0, 1)$ normalized with area 1.

We are now in a position to state the following.

Proposition 4.4. *The function $L^+ :]0, 1] \rightarrow \mathcal{D}$ is continuous and strictly increasing.*

Proof. The proof is divided into three steps.

Step 1: L^+ is u.s.c. This property follows by construction. Let $x \in]0, 1]$ and $\{x_n\} \subset (0, 1]$ an arbitrary sequence converging to x . We need to prove that

$$L^+(x) \geq \limsup_n L^+(x_n).$$

Assume (without loss of generality) that the lim sup is a limit. Let Ω_n be the sequence of optimal shapes for $L^+(x_n)$. Since $x > 0$, we infer that also x_n are (uniformly) far from 0. This ensures the existence of a subsequence (not relabeled) Ω_n which converges to some admissible $\Omega \in \mathcal{A}$. By continuity $x(\Omega) = \lim_n x(\Omega_n) = \lim_n x_n = x$. Thus by definition of $L^+(x)$ we infer that $y(\Omega) \leq L^+(x)$. Since $y(\Omega) = \lim_n y(\Omega_n)$ we conclude that

$$L^+(x) \geq y(\Omega) = \lim_n y(\Omega_n).$$

Step 2: L^+ is l.s.c. This property can be proved following the vary same strategy adopted for the u.s.c. of L^- in the previous section. For the benefit of the reader, we rewrite it in this case. Take $x_0 \in]0, 1]$ and take $x_n \rightarrow x_0$ satisfying

$$\liminf_{x \rightarrow x_0} L^+(x) = \lim_{n \rightarrow \infty} L^+(x_n).$$

Take Ω_0 an optimal shape for $L^+(x_0)$, namely such that $L^+(x_0) = y(\Omega_0)$. Its existence is ensured by Proposition 2.7. Moreover, by scale invariance of the shape functionals involved, we may assume that $A(\Omega_0) = 1$. In view of Lemma 4.1, there exists a sequence of shapes Ω_n with the following properties: for every $n \in \mathbb{N}$

$$\Omega_n \in \mathcal{A}, \quad A(\Omega_n) = 1, \quad x(\Omega_n) = x_n;$$

moreover, in the limit as $n \rightarrow \infty$, $\Omega_n \rightarrow \Omega_0$ with respect to the complementary Hausdorff distance. In particular, $\lim_{n \rightarrow \infty} y(\Omega_n) = y(\Omega_0)$. Exploiting now the definition of L^+ , we obtain

$$\liminf_{x \rightarrow x_0} L^+(x) = \lim_{n \rightarrow \infty} L^+(x_n) \geq \lim_{n \rightarrow \infty} y(\Omega_n) = y(\Omega_0) = L^+(x_0).$$

Step 3: L^+ is strictly increasing. By the previous steps, L^+ is continuous. Moreover, its infimum is 0 and it attains its maximum 1 for $x = 1$. Assume by contradiction that L^+ is not strictly increasing. Then there exists $x_0 \in]0, 1[$ local maximizer of L^+ . Therefore there exists a neighborhood $U(x_0)$ of x_0 such that

$$\Omega \in \mathcal{A}, \quad x(\Omega) \in U(x_0) \quad \Rightarrow \quad L^+(x_0) \geq y(\Omega).$$

This is in contradiction with Lemma 4.2. □

5. THE RATIO P^2A/W

In this section, we are interested in the maximization of the ratio P^2A/W among sets in the class \mathcal{A} . We have two different motivations for this study:

- First of all, it will help drawing the Blaschke-Santaló diagram \mathcal{D} since, maximizing this ratio is equivalent to minimizing $y(\Omega)/x(\Omega)$ and therefore will give a limit line below our diagram. Moreover, we will see later that the lower part of our diagram precisely coincides with the line $y = \pi^2 x/12$ for $x \in (0, 3/\pi]$ that corresponds to all rhombi between the segment and the square (see the proof of Theorem 5.1).
- In his paper [17], G. Pólya studies this functional $F(\Omega) := P^2(\Omega)A(\Omega)/W(\Omega)$ as a characteristic example of a shape functional whose maximizer is not the disk. This is an argument against the heuristic claim: "when the problem has many symmetries, the disk should be the optimal domain". Indeed the value of F for the disk is $8\pi^2 < 96$ where 96 is the value for all rhombi. Actually, G. Pólya gives the conjecture that the actual maximizer is the equilateral triangle, for which $F = 108$. Our theorem below gives the optimal domains among convex sets with two orthogonal axis of symmetry.

Let us give the main theorem of this section. We denote by $F(\Omega)$ the ratio $F(\Omega) := P^2(\Omega)A(\Omega)/W(\Omega)$.

Theorem 5.1. *The maximizers of $F(\Omega)$ among sets in the class \mathcal{A} (i.e. convex sets with two orthogonal axis of symmetry) are all rhombi (for example with vertices $(1, 0); (0, H); (-1, 0); (0, -H)$ for any $H > 0$).*

Corollary 5.2. *The Blaschke-Santaló diagram \mathcal{D} coincides on its lower boundary $L^-(x)$ with the line $y = \pi^2 x/12$ for $x \in (0, 3/\pi]$.*

Existence of a maximizer for F is straightforward, using the Blaschke selection theorem and the continuity of the geometric quantities involved for the Hausdorff convergence. The strategy we use to prove Theorem 5.1 is the following:

- (1) First we prove, using the same kind of arguments as in Theorem 2.9 that the maximizer is a polygon.

- (2) Then, we want to exclude vertices of the polygon that are in the interior of the triangle T defined by the three points $(1, 0); (1, H); (0, H)$. For that purpose, we will not use a first order argument (using the first order optimality conditions), but a second order argument (using the second order optimality conditions). Assuming that there is a vertex (x_i, y_i) in the interior of the triangle T , we can move in both directions the coordinates x_i, y_i . We write the Hessian matrix \mathcal{H}_F of F with respect to x_i, y_i and we want to prove that this Hessian matrix is not negative.
- (3) Using an affine change of variables, we are able to consider this Hessian matrix for a triplet of points like $(0, H); (t, uH); (1, 0)$ and we can even fix the value of H .
- (4) At some point, we need to estimate global quantities like $A/P, A/P^2, A/W$ for the optimal domain. This leads to (simpler) extremum problems that we solve analogously.
- (5) We conclude that there are no vertices inside the triangle by proving that the Hessian matrix is not negative. Then it remains to consider the cases of a vertex on the boundary of the triangle and eventually we are led just to compare the rhombus and the rectangle.

Without loss of generality, we consider convex sets with projections 1 and H on the two axis of symmetry. We can represent the profile of the convex set in the first quadrant by a concave function h defined on $[0, 1]$ with $h(0) = H$ and $h(1) = 0$ and $h'(0) \leq 0$. As already mentioned, the existence of a maximizer follows by the direct methods of the calculus of variations, either working with the family of concave functions h or with the convex shapes .

1st step: To prove that the optimizers are polygons, we will follow the idea developed in [13], [14]. For that purpose, let us write the optimality conditions in terms of h and the variation v . First we recall the expressions of A, P, W at h :

$$A(h) = \int_0^1 h, \quad P(h) = \int_0^1 \sqrt{1 + (h')^2}, \quad W(h) = \int_0^1 \left(hx^2 + \frac{h^3}{3} \right).$$

Note that we get the total area, perimeter or moment of inertia by multiplying by 4. A direct computation gives

$$\langle A'(h), v \rangle = \int_0^1 v, \quad \langle P'(h), v \rangle = \int_0^1 \frac{h'v'}{\sqrt{1 + (h')^2}}, \quad \langle W'(h), v \rangle = \int_0^1 (h^2 + x^2)v,$$

and

$$\langle A''(h)v, v \rangle = 0, \quad \langle P''(h)v, v \rangle = \frac{1}{2} \int_0^1 \frac{(v')^2}{(1 + (h')^2)^{3/2}}, \quad \langle W''(h)v, v \rangle = 2 \int_0^1 hv^2.$$

Now, the first and second order optimality conditions (namely $F' = 0$ and $F'' \leq 0$) read

$$\begin{aligned} P' &= \frac{P}{2} \left(\frac{W'}{W} - \frac{A'}{A} \right), \\ P'' &\leq -\frac{P}{2A} A'' + \frac{P}{2W} W'' + \frac{P}{A^2} (A')^2. \end{aligned} \tag{5.1}$$

Using in (5.1) the expressions of $A, P, W, A', P', W', A'', P'', W''$, we obtain

$$\frac{1}{2} \int_0^1 \frac{(v')^2}{(1 + (h')^2)^{3/2}} \leq \frac{P}{W} \int_0^1 hv^2 + \frac{P}{A^2} \left(\int_0^1 v \right)^2.$$

Let I_v denote an interval containing the support of v : using the Cauchy-Schwartz inequality, we get

$$\frac{1}{2} \int_0^1 \frac{(v')^2}{(1 + (h')^2)^{3/2}} \leq \left[\frac{P}{W} \|h\|_{L^\infty(I_v)} + \frac{P}{A^2} |I_v|^2 \right] \|v\|_{L^2(I_v)}^2.$$

This estimate is true for every admissible v with the properties above. This will be crucial for the proof that h is polygonal.

In order to prove the polygonal structure, we follow [13]. We want to prove that the support of the measure h'' is discrete. Assuming, this is not the case: it contains an

accumulation point x_0 and, for any $\varepsilon_n > 0$ (or $\varepsilon_n < 0$) we can find at least four point $x_1^n < x_2^n < x_3^n < x_4^n$ in the interval $[x_0, x_0 + \varepsilon_n]$ such that the support of h'' satisfies

$$\text{supp}^t(h'') \cap (x_j^n, x_{j+1}^n) \neq \emptyset, \quad \text{for } j = 1, 2, 3.$$

Then we construct three functions $v_{n,i}$ with support in $[x_0, x_0 + \varepsilon_n]$ in the following way: $v_{n,i}$ solves the EDO: $v_{n,i}'' = \chi_{(x_i^n, x_{i+1}^n)} \cdot h''$ and $v_{n,i} = 0$ in $(0, \varepsilon_n)^c$, $i = 1, 3$.

Now, we choose three constants $\alpha_1, \alpha_2, \alpha_3$ such that the function v_n defined as

$$v_n = \sum_{i=1}^3 \alpha_i v_{n,i}$$

satisfies $v_n'(x_0) = v_n'(x_0 + \varepsilon_n) = 0$ and then v_n (extended by zero) has its support in $[x_0, x_0 + \varepsilon_n]$ and satisfying $v_n'' = \sum_{i=1}^3 \alpha_i \chi_{(x_i^n, x_{i+1}^n)} \cdot h''$ is admissible as a perturbation of the optimum h .

Then we use the previous inequality, with $I_v = [x_0, x_0 + \varepsilon_n]$:

$$\frac{1}{2} \int_0^1 \frac{(v_n')^2}{(1 + (h')^2)^{3/2}} \leq \left[\frac{P}{W} \|h\|_{L^\infty(I_v)} + \frac{P}{A^2} |I_v|^2 \right] \|v_n\|_{L^2(I_v)}^2,$$

and we can assume that there exist $0 \leq C < +\infty$ such as $|h'| \leq C$ on I_v .

So we have

$$\frac{1}{2(1 + C^2)^{\frac{3}{2}}} \|v_n'\|_{L^2(I_v)}^2 \leq \left[\frac{P}{W} \|h\|_{L^\infty(I_v)} + \frac{P}{A^2} |I_v|^2 \right] \|v_n\|_{L^2(I_v)}^2.$$

By using the Poincaré's inequality on I_v , we have:

$$\frac{\pi^2}{\varepsilon_n^2} \leq 2(1 + C^2)^{\frac{3}{2}} \left[\frac{P}{W} \|h\|_{L^\infty(I_v)} + \frac{P}{A^2} |I_v|^2 \right]$$

This gives us the contradiction since the left part tends to $+\infty$ when n increases and the right part is bounded.

Remark 5.3. Note that the sole admissible v for h linear is $v \equiv 0$. Therefore the optimality conditions are (trivially) satisfied.

In view of the previous proposition, we infer that a maximizer h satisfies either $h'' \equiv 0$ or $h'' = \sum_{i=1}^k \alpha_i \delta_{x_i}$ in the open set $(0, 1)$ for a finite family of $\alpha_i \in \mathbb{R}$ and $x_i \in (0, 1)$. Our purpose is now to exclude this second case.

2nd step Let us assume that the optimal polygon has a "free" vertex $M_1 = (x_1, y_1)$. By free, we mean that we can move infinitesimally x_1 and y_1 in any direction keeping an admissible (convex) polygon. Taking the first interior vertex, we can assume that his two neighbouring vertices are $M_0 = (0, H)$ and $M_2 = (x_2, y_2)$. To compute the successive derivatives $\partial F / \partial x_1; \partial F / \partial y_1; \partial^2 F / \partial x_1^2; \partial^2 F / \partial x_1 \partial x_2; \partial^2 F / \partial x_2^2$, we just need to look at the contribution of x_1, y_1 in the global expression of P, A, W . Let us denote by $\widehat{P}, \widehat{A}, \widehat{W}$ the remaining parts not depending on x_1, y_1 . We have

$$\begin{aligned} P &= \widehat{P} + 4(x_1^2 + (y_1 - H)^2)^{1/2} + 4((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2} \\ A &= \widehat{A} + 2(x_1(H - y_2) + x_2(y_1 + y_2)) \\ W &= \widehat{W} + 4(I_1 + J_1 + I_2 + J_2) \end{aligned}$$

where, denoting by $h_1(x), h_2(x)$ the expression of $h(x)$ on the first and the second interval, namely

$$\begin{aligned} h_1(x) &= \frac{y_1 - H}{x_1} x + H, \quad x \in [0, x_1] \\ h_2(x) &= \frac{y_2 - y_1}{x_2 - x_1} (x - x_2) + y_2, \quad x \in [x_1, x_2] \end{aligned}$$

we have

$$\begin{aligned} I_1 &= \int_0^{x_1} x^2 h_1(x) = \frac{1}{4} y_1 x_1^3 + \frac{1}{12} x_1^3 H \\ I_2 &= \int_{x_1}^{x_2} x^2 h_2(x) = \frac{1}{4} (y_2 x_2^3 - y_1 x_1^3) - \frac{1}{12} (x_1^2 + x_1 x_2 + x_2^2) (y_2 x_1 - y_1 x_2) \\ J_1 &= \int_0^{x_1} \frac{1}{3} h_1^3(x) = \frac{1}{12} x_1 (y_1^3 + y_1^2 H + y_1 H^2 + H^3) \\ J_2 &= \int_{x_1}^{x_2} \frac{1}{3} h_2^3(x) = \frac{1}{12} (x_2 - x_1) (y_1^3 + y_1^2 y_2 + y_1 y_2^2 + y_2^3) \end{aligned}$$

Using the previous formulas, we can compute the first and second derivatives of P, A, W with respect to x_1, y_1 and therefore, the derivatives of $F = P^2A/W$. Now, the first order optimality condition for F writes

$$\frac{1}{W} \frac{\partial W}{\partial x_1} = \frac{2}{P} \frac{\partial P}{\partial x_1} + \frac{1}{A} \frac{\partial A}{\partial x_1}$$

and the same for the derivative in y_1 . We use these relations to simplify the computations of the second derivative. For example, $\partial^2 F / \partial x_1^2$ can be written

$$\frac{\partial^2 F}{\partial x_1^2} = F \left(\frac{2}{P} \frac{\partial^2 P}{\partial x_1^2} + \frac{2}{P^2} \left(\frac{\partial P}{\partial x_1} \right)^2 + \frac{4}{AP} \frac{\partial P}{\partial x_1} \frac{\partial A}{\partial x_1} - \frac{1}{W} \frac{\partial^2 W}{\partial x_1^2} \right)$$

(we use here the fact that the second derivative $\partial^2 A / \partial x_1^2$ vanishes). Similarly for the other derivatives.

Let us give the final expression of the Hessian matrix after a straightforward computation. For sake of simplicity, we introduce the two angles θ_1, θ_2 that the segments M_0M_1 and M_1M_2 make with the horizontal, namely

$$\theta_1 = \arctan \left(\frac{H - y_1}{x_1} \right), \quad \theta_2 = \arctan \left(\frac{y_1 - y_2}{x_2 - x_1} \right),$$

and the relative coordinates

$$t = \frac{x_1}{x_2}, \quad u = \frac{y_1 - y_2}{H - y_2}.$$

We get

$$\begin{aligned} A \frac{\partial^2 F}{\partial x_1^2} &= \frac{8A}{P} \left(\frac{\sin^2 \theta_1}{M_0M_1} + \frac{\sin^2 \theta_2}{M_1M_2} \right) + \frac{32A}{P^2} (\cos \theta_1 - \cos \theta_2)^2 \\ &\quad + \frac{32(H - y_2)}{P} (\cos \theta_1 - \cos \theta_2) - \frac{2A(H - y_2)}{3W} (3x_1 + x_2u) \\ A \frac{\partial^2 F}{\partial x_1 \partial y_1} &= \frac{8Ax_2(H - y_2)}{P} \left(\frac{t(1 - u)}{M_0M_1^3} + \frac{u(1 - t)}{M_1M_2^3} \right) \\ &\quad + \frac{32A}{P^2} (\cos \theta_1 - \cos \theta_2) (\sin \theta_2 - \sin \theta_1) \\ &\quad + \frac{16}{P} (x_2(\cos \theta_1 - \cos \theta_2) + (H - y_2)(\sin \theta_2 - \sin \theta_1)) \\ &\quad - \frac{A}{3W} [x_2(x_2 + 2x_1) + (H - y_2)(H + y_2 + 2y_1)] \\ A \frac{\partial^2 F}{\partial y_1^2} &= \frac{8A}{P} \left(\frac{\cos^2 \theta_1}{M_0M_1} + \frac{\cos^2 \theta_2}{M_1M_2} \right) + \frac{32A}{P^2} (\sin \theta_2 - \sin \theta_1)^2 \\ &\quad + \frac{32x_2}{P} (\sin \theta_2 - \sin \theta_1) - \frac{2Ax_2}{3W} [4H - (H - y_2)((1 - t) + 3(1 - u))]. \end{aligned}$$

Now, let us consider the following affine transformation given by the change of variable

$$x' = \frac{x}{x_2}, \quad y' = \frac{y - y_2}{4(H - y_2)}.$$

It has the effect of transforming the point M_0 in $N_0 = (0, \frac{1}{4})$, the point M_2 in $N_2 = (1, 0)$ and the point M_1 in $N_1 = (x'_1, y'_1)$ in such a way that these three points remain in a "convex" position (we can assume $H > y_2$, otherwise M_1 would not be a vertex). Moreover, when computing the new Hessian matrix inherited with this change of variable, we see that it has the same properties of the original Hessian matrix, for example the determinants of the two Hessian matrices are equal up to the positive factor $1/16x_2^2(H - y_2)^2$. Therefore, we can restrict to this particular situation with the three points $((0, \frac{1}{4}); (t, u/4); (1, 0))$ for which the above formula simplifies. In particular, we will be interested in the following quantity $(1, -1)\mathcal{H}_F(1, -1)^T$ or

$$\mathcal{E} := \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial y_1^2} - 2 \frac{\partial^2 F}{\partial x_1 \partial y_1} \tag{5.2}$$

together with the trace of the Hessian matrix

$$\mathcal{T} := \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial y_1^2}.$$

3rd step In the second derivatives of the functional F , appear some global quantities involving A, P, W . In order to be able to prove that the quantity \mathcal{E} is positive, we need to estimate A, P, W and some ratios. This is the aim of the following proposition. Without loss of generality, we will use the following normalization: we work in the subclass:

$$\mathcal{A}_0 = \{\Omega \in \mathcal{A}, h(0) = H, h(1) = 0\}$$

where we assume $H \leq 1$.

Proposition 5.4. *Let Ω in \mathcal{A}_0 , then we have the following inequalities*

$$\begin{aligned} A(\Omega) &\geq 2H, & P(\Omega) &\leq 4(1+H), & \frac{A(\Omega)}{P(\Omega)} &\geq \frac{H}{2\sqrt{1+H^2}} \\ \frac{A(\Omega)}{P^2(\Omega)} &\geq \frac{H}{8(1+H^2)}, & \frac{A(\Omega)}{W(\Omega)} &\leq \frac{6}{1+H^2}. \end{aligned}$$

Proof. The two first inequalities come immediately from the fact that Ω contains the rhombus of vertices $(\pm 1, 0); (0, \pm H)$ and is contained in the rectangle $(-1, 1) \times (-H, H)$. To prove the three other inequalities, we solve a shape optimization problem in the class \mathcal{A}_0 . Existence of an optimal domain is immediate each time.

Minimizing A/P . Let Ω^* be a minimizer. Working exactly as the beginning of the proof of Theorem 5.1, we infer that Ω^* is a polygon. Let us assume that Ω^* contains a free vertex (namely a vertex (x_1, y_1) with $0 < x_1 < 1, 1 - x_1 < y_1 < H$). Let us write the optimality conditions we obtain by moving this vertex. On the one-hand, the first optimality condition in x_1 can be written

$$\frac{1}{A} \frac{\partial A}{\partial x_1} - \frac{1}{P} \frac{\partial P}{\partial x_1} = 0.$$

Let us now compute the second derivative in x_1 . Taking into account that $\frac{\partial^2 A}{\partial x_1^2} = 0$, we get

$$\frac{\partial^2}{\partial x_1^2} \left(\frac{A}{P} \right) = -\frac{1}{A^2} \left(\frac{\partial A}{\partial x_1} \right)^2 + \frac{1}{P^2} \left(\frac{\partial P}{\partial x_1} \right)^2 - \frac{1}{P} \frac{\partial^2 P}{\partial x_1^2}.$$

Now, using the first order optimality condition, the two first terms cancel out and we get

$$\frac{\partial^2}{\partial x_1^2} \left(\frac{A}{P} \right) = -\frac{1}{P} \frac{\partial^2 P}{\partial x_1^2} = -\frac{(y_1 - y_0)^2}{A_0 A_1^3} - \frac{(y_1 - y_2)^2}{A_2 A_1^3}$$

the second derivative in x_1 being strictly negative, we get a contradiction with the minimality. Therefore, Ω^* cannot have a free vertex and the only possibilities that remain to be considered are

- the rhombus
- the rectangle
- another vertex on the horizontal line $y = H$
- or/and another vertex on the vertical line $x = 1$.

Actually in the two last cases, we can move freely the vertex along the horizontal (or the vertical) line, therefore the previous computation with the variable x_1 (or the variable y_1) still holds and leads to a similar contradiction. Thus it remains only to compare the rectangle and the rhombus for which the ratio A/P equals respectively $H/(1+H)$ and $H/2\sqrt{1+H^2}$ and a direct comparison shows that the rhombus gives the lower value.

Minimizing A/P^2 . We proceed exactly in the same way. We get for the second derivative in x_1 :

$$\frac{\partial^2}{\partial x_1^2} \left(\frac{A}{P^2} \right) = -\frac{1}{2A^2} \left(\frac{\partial A}{\partial x_1} \right)^2 - \frac{2}{P} \frac{\partial^2 P}{\partial x_1^2}$$

and the right-hand side being negative, we conclude in the same way that we just have to compare the rhombus and the rectangle. For the rhombus, the ratio A/P^2 is $H/8(1+H^2)$ while for the rectangle, it is equal to $H/4(1+H)^2$ and the result follows from the comparison of these two numbers.

Maximizing A/W or minimizing W/A . This case is more complicated. The fact that the perimeter is not in the functional makes not clear whether the maximizer is a polygon. Let us write the optimality condition, using the formalism developed in [13] to take into

account the concavity constraint on the function h describing the boundary of the optimal set. Since the derivative of the area and the moment of inertia are

$$\langle A'(h), v \rangle = \int_0^1 v dx \quad \langle W'(h), v \rangle = \int_0^1 (x^2 + h^2)v dx$$

the first order optimality condition writes: there exists a function ξ in $H^1(0, 1)$, $\xi \geq 0$, $\xi = 0$ on S the support of the measure h'' such that

$$-\frac{W}{A} \xi'' = h^2 + x^2 - \frac{W}{A}. \quad (5.3)$$

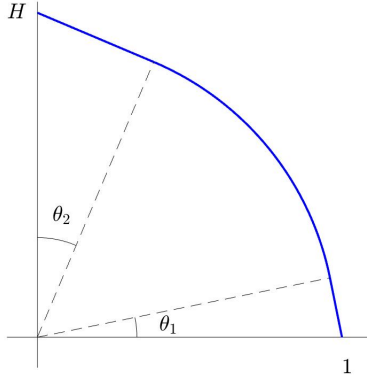
The right-hand side of (5.3) being continuous, we see that the function ξ is indeed in $C^2(0, 1)$. Now, the support S is closed: let us assume that there exists a (maximal) open interval (α, β) in its complement with $0 < \alpha < \beta < 1$. On this interval, we have

$$-\frac{W}{A} \xi'' = h^2 + x^2 - \frac{W}{A}, \quad x \in (\alpha, \beta)$$

with the boundary conditions $\xi(\alpha) = \xi(\beta) = 0$. Moreover, since ξ is C^2 and $\xi \geq 0$, we must have also $\xi'(\alpha) = \xi'(\beta) = 0$. Now, since $h'' = 0$ on (α, β) we see that h is affine on this interval and therefore the right-hand side of the equation (5.3) is a polynomial of degree 2. Taking into account the boundary conditions, this implies that $\xi(x) = T(x - \alpha)^2(x - \beta)^2$ on (α, β) . Now, coming back to the equation, we see that the term in x^2 of ξ'' should be, on the one hand, equal to $12T$ and, on the other hand, negative (because it is positive for $h^2 + x^2$). The negativity of T is in contradiction with $\xi \geq 0$. In conclusion, the complement of S cannot have an internal open interval. In other words, we are led to the following possibilities

$$S^c = (0, \alpha) \cup (\beta, 1) \quad \text{with } 0 \leq \alpha \leq \beta \leq 1.$$

Let us start with the case $0 < \alpha < \beta < 1$ that means that h is affine on the intervals $[0, \alpha] \cup [\beta, 1]$ and is strictly convex for $x \in (\alpha, \beta)$. On this strictly convex part, since here $\xi = 0$, we have $h^2 + x^2 = W/A$: so the boundary is an arc of circle of radius $R = \sqrt{W/A}$. We parametrize the boundary with two angles θ_1, θ_2 such that (see Figure)



- the first segment on $[0, \alpha]$ is given in polar coordinates as $\rho = R/\sin(\theta + \theta_2)$, $\theta \in [\pi/2 - \theta_2, \pi/2]$,
- the arc of circle is given by $\rho = R, \theta \in [\theta_1, \pi/2 - \theta_2]$,
- the second segment on $[\beta, 1]$ is given in polar coordinates as $\rho = R/\cos(\theta - \theta_1)$, $\theta \in [0, \theta_1]$.

We also have the relations $R = \sqrt{\frac{W}{A}} = \cos \theta_1$ and $H = R/\cos \theta_2$. Using the expressions in polar coordinates, we immediately get

$$A = 2R^2 \left(\tan \theta_1 + \frac{\pi}{2} - \theta_2 - \theta_1 + \tan \theta_2 \right),$$

$$W = R^4 \left(\int_0^{\theta_1} \frac{dt}{\cos^4(t)} + \frac{\pi}{2} - \theta_2 - \theta_1 + \int_0^{\theta_2} \frac{dt}{\cos^4(t)} \right).$$

If $0 < \theta_1$ and $\theta_1 + \theta_2 < \pi/2$, we can differentiate with respect to θ_1 and we should get 0 for this derivative. Now

$$\frac{\partial}{\partial \theta_1} \left(\frac{W}{A} \right) = \frac{R^4}{A} \left(\frac{1}{\cos^4 \theta_1} - 1 \right) - \frac{2WR^2}{A^2} \left(\frac{1}{\cos^2 \theta_1} - 1 \right)$$

Using $\cos \theta_1 = R$ and $R^2 = W/A$, this derivative can be written

$$\frac{\partial}{\partial \theta_1} \left(\frac{W}{A} \right) = \frac{(1 - R^2)^2}{A}.$$

But if R would be equal to 1, we would have $\theta_1 = 0$ that is not the assumption. This computation shows that the case $0 < \theta_1$ and $\theta_1 + \theta_2 < \pi/2$ cannot occur. We can prove exactly in the same way that $0 < \theta_2$ and $\theta_1 + \theta_2 < \pi/2$ cannot occur. Therefore, it remains only two cases:

- either $\theta_1 = 0$ and $\theta_2 = 0$ that corresponds to the unit disk for which $W/A = R^2/2$,
- or $\theta_1 + \theta_2 = \pi/2$ that corresponds to a polygon with one interior vertex, say (x_1, y_1) .

In that case, it is more convenient to express all the quantities in the cartesian coordinates. Let us introduce $t = x_1$ and $u = y_1/H$ both in the interval $[0, 1]$ The concavity assumption is $t + u \geq 1$. With these variable, we have

$$A = 2H(t + u)$$

$$W = \frac{H}{3} [t^3 + (1 + t + t^2)u + H^2(t(1 + u + u^2) + u^3)].$$

As previously, we want to prove that the vertex (x_1, y_1) cannot be "free". This leads us to study the function with two variables t, u

$$\psi(t, u) = \frac{t^3 + (1 + t + t^2)u + H^2(t(1 + u + u^2) + u^3)}{t + u},$$

and prove that it cannot have a minimizer in the open set $\{0 < t < 1, 1 - t < u < 1\}$. Let us compute the derivative of ψ with respect to u :

$$(t + u)^2 \frac{\partial \psi}{\partial u} = t + t^2 + H^2(t(4u^2 - 1) + t^2(1 + 2u) + 2u^3).$$

Since $t + H^2t(4u^2 - 1) > t(1 - H^2) \geq 0$, we see that the right-hand side is positive and cannot vanish. This means that u has to be equal either to $1 - t$, that corresponds to a rhombus, or $u = 1$. In this last case, we compute the derivative $\partial \psi / \partial t$ and we get for $u = 1$

$$(t + 1)^2 \frac{\partial \psi}{\partial t}(t, 1) = 2t^3 + 4t^2 + 2t + 2H^2$$

that is positive, too. Therefore, when $u = 1$, the only new case is $t = 1$ that corresponds to a rectangle. We conclude by comparing W/A for the rhombus (whose value is $(1 + H^2)/6$) and for the rectangle (whose value is $(1 + H^2)/3$). This gives the desired result. \square

4th step: Conclusion We come back to the quantity \mathcal{E} defined in (5.2) that we write with the three points (we keep the notations t, u introduced previously to denote the interior point) $N_0 = (0, H)$; $N_1 = (t, uH)$; $N_2 = (1, 0)$ and we use the angles θ_1, θ_2 , in particular to write

$$t = N_0 N_1 \cos \theta_1, (1 - u)H = N_0 N_1 \sin \theta_1, 1 - t = N_1 N_2 \cos \theta_2, uH = N_1 N_2 \sin \theta_2.$$

We recall that $\theta_1 \leq \theta \leq \theta_2$ by convexity. The quantity $A\mathcal{E}$ is the sum of four terms that we denote $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4$:

$$\mathcal{E}_1 = \frac{8A}{P} \left(\frac{1 - \sin(2\theta_1)}{N_0 N_1} + \frac{1 - \sin(2\theta_2)}{N_1 N_2} \right).$$

For the second and third terms, it is convenient to introduce $\delta := (\theta_2 - \theta_1)/2$ and $\mu := (\theta_2 + \theta_1)/2$. With these notations, we can write

$$\begin{aligned}\mathcal{E}_2 &= \frac{128A}{P^2} \sin^2 \delta (\cos \mu - \sin \mu)^2 \\ \mathcal{E}_3 &= \frac{64}{P} (1 - H) \sin \delta (\cos \mu - \sin \mu)\end{aligned}$$

and the last term is

$$\mathcal{E}_4 = \frac{2A}{3W} (1 + 2t + H^2(1 + 2u) - 4H(t + u)).$$

It is clear that $\mathcal{E}_1, \mathcal{E}_2$ are always positive. This is also the case for \mathcal{E}_3 as soon as $\mu \leq \pi/4$. Now, since H has been supposed to be less than $1/2$ (we have chosen $H = 1/4$ previously, the minimum of \mathcal{E}_4 is certainly obtained by taking $t = 0$ and $u = 1$ (since $2 - 4H \geq 0$ and $2H^2 - 4H \leq 0$). This means

$$\mathcal{E}_4 \geq \frac{2A}{3W} (1 + 2H^2 - 4H) > 0 \quad \text{when } H < 1 - \frac{1}{\sqrt{2}}.$$

In conclusion $\mathcal{E} > 0$ for $H = 1/4$ and $\mu \leq \pi/4$.

It remains to consider the case $\mu \geq 1/4$. In that case, it is easier to work with the trace of the Hessian matrix, namely

$$\mathcal{T} = \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial y_1^2}.$$

It can be written

$$\mathcal{T} = \frac{8A}{P} \left(\frac{1}{N_0 N_1} + \frac{1}{N_1 N_2} \right) + \frac{128A}{P^2} \sin^2 \delta + \frac{64}{P} \sin \delta (H \sin \mu + \cos \mu) - \frac{8AH}{3W} (t + u).$$

Here we have also four terms $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ that we estimate separately. Taking into account $\mu \geq \pi/4$, we infer

$$\frac{\pi}{4} \leq \frac{\theta_1 + \theta_2}{2} \leq \frac{\theta + \theta_2}{2}$$

therefore $\theta_2 \geq \frac{\pi}{2} - \theta$. This implies that the point (t, uH) lies into the triangle defined by the three points $(1, 0); (1, H); (1 - H \tan \theta, H)$. In particular, $N_0 N_1 \leq \sqrt{1 + H^2}$, while $N_1 N_2 \leq H / \cos \theta$. Using now $8A/P \geq 4H/\sqrt{1 + H^2}$ coming from Proposition 5.4, we see that

$$\mathcal{T}_1 = \frac{8A}{P} \left(\frac{1}{N_0 N_1} + \frac{1}{N_1 N_2} \right) \geq \frac{4(1 + H)}{1 + H^2}$$

where we used $\cos \theta = 1/\sqrt{1 + H^2}$.

We consider the second term. From $\theta_2 \geq \pi/2 - \theta$, we deduce $\delta \geq \pi/4 - \theta$, thus $\sin \delta \geq (\cos \theta - \sin \theta)/\sqrt{2}$. We use $A/P^2 \geq H/(8(1 + H^2))$ coming from Proposition 5.4 to infer

$$\mathcal{T}_2 = \frac{128A}{P^2} \sin^2 \delta \geq \frac{8H(1 - H)^2}{(1 + H^2)^2}.$$

For the third term, we use the identity $H \sin \mu + \cos \mu = \cos(\mu - \theta)/\cos \theta$ with the inequality $\mu - \theta \leq \pi/4 - \theta/2$ to have $\cos(\mu - \theta) \geq (\cos \frac{\theta}{2} + \sin \frac{\theta}{2})/\sqrt{2} \geq 1/\sqrt{2}$, and we also use $1/P \geq 1/4(1 + H)$ and $\sin \delta \geq (\cos \theta - \sin \theta)/\sqrt{2}$ to infer

$$\mathcal{T}_3 = \frac{64}{P} \sin \delta (H \sin \mu + \cos \mu) \geq \frac{8(1 - H)}{1 + H}.$$

At last, we use $8A/3W \leq 16/(1 + H^2)$ coming from Proposition 5.4 and $t + u \leq 2$ to deduce that the fourth term is estimated by

$$\mathcal{T}_4 = -\frac{8AH}{3W} (t + u) \geq -\frac{32H}{1 + H^2}.$$

Summing these four terms, we finally get that the trace of the Hessian matrix is estimated from below by

$$\mathcal{T} \geq \frac{4}{(1 + H)(1 + H^2)^2} (3 - 6H - 4H^2 - 12H^3 - 3H^4 - 2H^5)$$

therefore $\mathcal{T} > 0$ for $H = 1/4$ yielding the desired contradiction.

We are led to the only following possibilities for the optimal domain

- (1) it is the rhombus
- (2) it is the rectangle
- (3) it has another vertex on the horizontal line $y = H$
- (4) or/and it has another vertex on the vertical line $x = 1$.

We are going to prove that we are necessarily in the two first cases. Let us assume that the optimal domain has two vertices $M_1 = (t, H)$ and $M_2 = (1, uH)$ with $(t, u) \in [0, 1]^2$. In that case, we have

$$A = 2H(1 + t + u - tu), \quad P = 4 \left(t + uH + \sqrt{(1-t)^2 + H^2(1-u)^2} \right)$$

$$W = \frac{H}{3} (4u + (1-u)(1+t+t^2+t^3)) + \frac{H^3}{3} (4t + (1-t)(1+u+u^2+u^3))$$

Let us introduce the quantity $G = P^2A - 96W$ and we want to prove that $G \leq 0$ for any $(t, u) \in [0, 1]^2$ and any $H \in (0, 1]$. For that purpose, we compute the first and second derivative of G with respect to t, u . We denote by d the distance between M_1 and M_2 , namely $d = \sqrt{(1-t)^2 + H^2(1-u)^2}$

$$\frac{\partial G}{\partial t} = 8AP \left(1 - \frac{1-t}{d} \right) + 2H(1-u)P^2 - 32H(1-u)(1+2t+3t^2+H^2(3+2u+u^2)).$$

$$\frac{\partial G}{\partial u} = 8APH \left(1 - \frac{H(1-u)}{d} \right) + 2H(1-t)P^2 - 32H(1-t)(3+2t+t^2+H^2(1+2u+3u^2)).$$

The second derivative of G in u can be written as the sum of four terms as

$$\frac{\partial^2 G}{\partial u^2} = 32H(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4)$$

with

$$\begin{aligned} \mathcal{E}_1 &= (1+t+u-tu)H^2 \left(1 - \frac{H(1-u)}{d} \right)^2 \\ \mathcal{E}_2 &= (1+t+u-tu)H^2(t+uH+d)(1-t)^2/d^3 \\ \mathcal{E}_3 &= 2(1-t)H(t+uH+d) \left(1 - \frac{H(1-u)}{d} \right) \\ \mathcal{E}_4 &= -H^2(1+3u)(1-t). \end{aligned}$$

We follow now the same strategy than previously: thanks to the affine change of variable $x' = x, y' = y/(4H)$ we are led to examine the situation of the second derivatives where $H = 1/4$. Let us look at $\mathcal{E}_3 + \mathcal{E}_4$. Using $t+uH+d \geq \sqrt{1+H^2}$ (estimate of the perimeter by the perimeter of the rhombus) and $d \leq \sqrt{1+H^2}$, we get

$$\mathcal{E}_3 + \mathcal{E}_4 \geq 2H(1-t) \left(\sqrt{1+H^2} - H(1-u) - H^2(1+3u) \right)$$

now, it is clear that the right-hand side is positive for $H = 1/4$ for all t, u . Since the two first terms $\mathcal{E}_1, \mathcal{E}_2$ are positive, we infer that the second derivative $\frac{\partial^2 G}{\partial u^2}$ is positive: that implies that G cannot have a maximizer for $0 < u < 1$. Thus, for a maximizer, we necessarily have $u = 0$ or $u = 1$. The case $u = 1$ corresponds to a rectangle. It remains to look at the case $u = 0$. In that case, we are going to prove directly that $G(t, 0) < 0$ for $0 < t < 1$ it will imply that in that case, the maximizing domain is a rhombus and a simple comparison of G in these two cases provides $G = 0$ for the rhombus and $G = -64H(H-1)^2$ that shows that the maximizing domain is any rhombus (including the square that we recover in the case of a rectangle with $H = 1$).

Let us prove that $G(t, 0) < 0$ when $0 < t < 1$. For that purpose, we use first the convexity of the function $t \mapsto \sqrt{(1-t)^2 + H^2}$ to claim that

$$\sqrt{(1-t)^2 + H^2} \leq \sqrt{1+H^2} + t(H - \sqrt{1+H^2}).$$

Then, we can estimate the perimeter squared by

$$P^2 \leq 16 \left(1 + H^2 - 2t + 2t^2 + 2t(\sqrt{1+H^2} + t(H - \sqrt{1+H^2})) \right).$$

This allows to estimate $G = P^2A - 96W$ by

$$G(t, 0) \leq t(a_1t^2 + a_2t + a_3)$$

with

$$a_1 = 1 + 2H - 2\sqrt{1 + H^2}, \quad a_2 = 2H - 1, \quad a_3 = 2\sqrt{1 + H^2} - 2 - 2H^2.$$

We have $a_1 \geq 0 \Leftrightarrow H \geq 3/4$, $a_2 \geq 0 \Leftrightarrow H \geq 1/2$ and $a_3 < 0$. Then, $P_H(t) = a_1 t^2 + a_2 t + a_3$ is clearly negative when $H \leq 1/2$. When $H > 3/4$, $P_H(t)$ is increasing on $[0, 1]$ since its minimum is attained at a negative $-a_2/2a_1$. But $P_H(1) = -2(H - 1)^2 \leq 0$; In the last case, $1/2 < H \leq 3/4$, P_H has a maximum at $t_1 = -a_2/2a_1$ that is positive. But when $H \geq (-3 + 2\sqrt{21})/10$ this maximum point being outside $[0, 1]$, P_H is still increasing, then negative on $[0, 1]$. At last, when $H \in [1/2, (-3 + 2\sqrt{21})/10]$, $t_1 \in [0, 1]$ and the maximum of P_H is

$$P_H(t_1) = -\frac{a_2^2}{4a_1} + a_3$$

and it is straightforward to check that this number is negative. Therefore, we have proved that $P_H(t) = a_1 t^2 + a_2 t + a_3$ is always negative when $0 < t < 1$. \square

6. BEHAVIOR NEAR $(0, 0)$ AND $(1, 1)$

In this section we investigate the behavior of the diagram near the two ‘‘extremal’’ points, on the left and on the right of the diagram. These points are: the origin, associated to thin domains collapsing to a segment, and the point $(1, 1)$ associated to the ball.

Behavior near the origin $(0, 0)$. We will compute the minimal slope γ_O^- and the maximal slope γ_O^+ at the origin, defined as

$$\begin{aligned} \gamma_O^- &:= \liminf \left\{ \frac{y_\epsilon}{x_\epsilon} : (x_\epsilon, y_\epsilon) \in \mathcal{D}, (x_\epsilon, y_\epsilon) \rightarrow (0, 0) \right\}, \\ \gamma_O^+ &:= \limsup \left\{ \frac{y_\epsilon}{x_\epsilon} : (x_\epsilon, y_\epsilon) \in \mathcal{D}, (x_\epsilon, y_\epsilon) \rightarrow (0, 0) \right\}. \end{aligned}$$

To this aim, let us first rewrite in a more tractable way the ratios y/x for $(x, y) \in \mathcal{D}$. By definition, the pair $(x, y) \in \mathcal{D}$ is associated to an admissible shape $\Omega \in \mathcal{A}$, which in turn is characterized by its intersection with the first quadrant, say $\Omega^+ := \Omega \cap \{x \geq 0, y \geq 0\}$. This set can be described as

$$\Omega^+ = \{0 \leq x \leq x_0, \quad 0 \leq y \leq h(x)\}, \quad (6.1)$$

for some $x_0 > 0$ and h concave decreasing. Therefore we can rewrite

$$\frac{y}{x} = \frac{y(\Omega)}{x(\Omega)} = \frac{\pi^2}{2} \frac{\int_0^{x_0} (x^2 h(x) + h^3(x)/3) dx}{\left[\int_0^{x_0} \sqrt{1 + (h'(x))^2} dx + h(x_0) \right]^2 \int_0^{x_0} h(x) dx}.$$

Proposition 6.1. *The minimal and maximal slopes at the origin are*

$$\gamma_O^- = \frac{\pi^2}{12}, \quad \gamma_O^+ = \frac{\pi^2}{6}.$$

Moreover, they are attained, e.g., by sequences of thin rhombi and thin rectangles, respectively.

Proof. In view of Corollary 5.2, we immediately get $\gamma_O^- \geq \pi^2/12$. The equality sign is obtained e.g. by taking a sequence of thin rhombi shrinking to a segment.

Let us study the upper bound. Let Ω_ϵ be a sequence satisfying the lim sup characterized by some h_ϵ according to (6.1). Without loss of generality, up to a rotation of $\pi/2$, we may assume that h_ϵ is defined in the interval $[0, 1]$ and that $h_\epsilon(0) \rightarrow 0$. Thus h_ϵ is of the form $h_\epsilon(x) = c_\epsilon f_\epsilon(x)$, with $c_\epsilon \rightarrow 0$ and f_ϵ in the following class:

$$\mathcal{F} := \{f : [0, 1] \rightarrow \mathbb{R} : f(0) = 1, f \text{ concave, decreasing, non negative}\}.$$

Then

$$\begin{aligned} \frac{y(\Omega_\epsilon)}{x(\Omega_\epsilon)} &= \frac{\pi^2}{2} \frac{\int_0^1 (x^2 c_\epsilon f_\epsilon(x) + c_\epsilon^3 f_\epsilon^3(x)/3) dx}{\left[\int_0^1 \sqrt{1 + (c_\epsilon f_\epsilon'(x))^2} dx + c_\epsilon f_\epsilon(1) \right]^2 \int_0^1 c_\epsilon f_\epsilon(x) dx} \\ &= \frac{\pi^2}{2} \frac{\int_0^1 x^2 f_\epsilon(x) dx + o(1)}{\int_0^1 f_\epsilon(x) dx + o(1)}. \end{aligned}$$

This ratio is clearly bounded above by $\pi^2/2$. To get a finer estimate, we show that for every $f \in \mathcal{F}$

$$\int_0^1 (x^2 - 1/3)f(x) dx \leq 0. \quad (6.2)$$

This fact easily comes by splitting the interval of integration as $[0, 1] = [0, 1/\sqrt{3}] \cup [1/\sqrt{3}, 1]$. Using the monotonicity and positivity of f , we infer that the integrand is negative in $[0, 1/\sqrt{3}]$ and positive in $[1/\sqrt{3}, 1]$. In both cases the integrand is bounded above by the function $f(1/\sqrt{3})(x^2 - 1/3)$, which has zero average in $[0, 1]$. This gives (6.2), in particular the estimate

$$\gamma_{\bar{O}} \leq \frac{\pi^2}{6}.$$

In order to prove that $\gamma_{\bar{O}} = \pi^2/6$ it is enough to exhibit a sequence of admissible shapes for which the ratio $y_\epsilon/x_\epsilon \rightarrow \pi^2/6$. This is the case of thin rectangles, in which $f_\epsilon \equiv 1$ for every ϵ . \square

Remark 6.2. Notice that the highest slope associated to thin domains coincides with the slope of the curve associated to the stadiums. Let Ω_L be the stadium associated to two half disks of centers $\pm L$ and radius 1, then as $L \rightarrow +\infty$, $(x(\Omega_L), y(\Omega_L)) \rightarrow (0, 0)$, since

$$A(\Omega_L) = \pi + 4L, \quad P(\Omega_L) = 2\pi + 4L, \quad W(\Omega_L) = \frac{\pi}{2} + 4L + \pi L^2 + \frac{4}{3}L^3.$$

The slope at the origin is

$$\lim_{L \rightarrow +\infty} \frac{y(\Omega_L)}{x(\Omega_L)} = \frac{\pi^2}{6}.$$

Behavior near (1, 1). As in the previous section, we aim to compute the minimal slope $\gamma_{\mathbb{D}}^-$ and the maximal slope $\gamma_{\mathbb{D}}^+$ at the point (1, 1), defined as the left derivatives of L^+ and L^- in $x = 1$, respectively. We will prove that the two following limits exist

$$\gamma_{\mathbb{D}}^- := \lim_{x \rightarrow 1^-} \frac{L^+(x) - 1}{x - 1}, \quad \gamma_{\mathbb{D}}^+ := \lim_{x \rightarrow 1^-} \frac{L^-(x) - 1}{x - 1},$$

and we will compute their value.

Proposition 6.3. *The minimal and maximal slopes at (1, 1) are*

$$\gamma_{\mathbb{D}}^- = \frac{3}{4}, \quad \gamma_{\mathbb{D}}^+ = +\infty.$$

Proof. The proof is divided into several steps.

Step 1: the sequence of regular polygons. Let $\widehat{\Omega}_n$ be the regular polygons with $2n$ sides, $n \geq 2$, with outer radius 1. Thus we have

$$A(\widehat{\Omega}_n) = 2n \sin(\pi/2n) \cos(\pi/2n), \quad P(\widehat{\Omega}_n) = 4n \sin(\pi/2n),$$

$$W(\widehat{\Omega}_n) = n \cos^4(\pi/2n) \left(\tan(\pi/2n) + \frac{\tan^3(\pi/2n)}{3} \right).$$

So that

$$\widehat{x}_n := x(\widehat{\Omega}_n) = \frac{\sin(\pi/2n)}{(\pi/2n) \cos(\pi/2n) (1 + \tan^2(\pi/2n)/3)}$$

and

$$\widehat{y}_n := y(\widehat{\Omega}_n) = \frac{(\pi/2n) \cos(\pi/2n)}{\sin(\pi/2n)}.$$

Along this sequence, it is immediate to check that

$$\lim_{n \rightarrow \infty} \frac{\widehat{y}_n - 1}{\widehat{x}_n - 1} = +\infty.$$

Step 2: the maximal slope. Let $x_n \rightarrow 1^-$ be a sequence realizing

$$\lim_{n \rightarrow \infty} \frac{1 - L^-(x_n)}{1 - x_n} = \liminf_{x \rightarrow 1} \frac{1 - L^-(x)}{1 - x}.$$

We can extract an increasing subsequence x_{n_k} such that $\widehat{x}_{n_k} \leq x_{n_k} \leq \widehat{x}_{n_k+1}$. On the one hand, we clearly have $1 - x_n \leq 1 - \widehat{x}_{n_k}$. On the other hand, since L^- is increasing and

defined as a minimum, we have $1 - L^-(x_n) \geq 1 - \widehat{y}_{n_k+1}$. Using these estimates together with Step 1, we conclude that

$$\lim_{n \rightarrow \infty} \frac{1 - L^-(x_n)}{1 - x_n} \geq \lim_{k \rightarrow \infty} \frac{1 - \widehat{y}_{n_k+1}}{1 - \widehat{x}_{n_k}} = +\infty.$$

Step 3: the minimal slope. In this part we use the representation of shapes via polar coordinates: $\Omega = \{(\rho \cos \theta, \rho \sin \theta) : \theta \in [0, 2\pi], \rho \in [0, \rho_{max}(\theta)]\}$, for a suitable function ρ_{max} .

Let $x_\epsilon \rightarrow 1^-$ as $\epsilon \rightarrow 0$ be a sequence realizing

$$\lim_{\epsilon \rightarrow 0} \frac{1 - L^+(x_\epsilon)}{1 - x_\epsilon} = \liminf_{x \rightarrow 1} \frac{1 - L^+(x)}{1 - x}.$$

Let Ω_ϵ be the associated shapes in \mathcal{A} . So that

$$\frac{1 - L^+(x_\epsilon)}{1 - x_\epsilon} = \frac{1 - y(\Omega_\epsilon)}{1 - x(\Omega_\epsilon)}. \quad (6.3)$$

Using the representation in polar coordinates, the shapes are characterized by a function $r_\epsilon(\theta)$ as follows:

$$\Omega_\epsilon = \{(\rho \cos \theta, \rho \sin \theta) : \theta \in [0, 2\pi], \rho \in [0, r_\epsilon(\theta)]\}.$$

Since $x_\epsilon \rightarrow 1^-$, without loss of generality, we may assume that Ω_ϵ converges to the disk of unit radius \mathbb{D} , with respect to the complementary Hausdorff distance. Therefore we infer that $r_\epsilon \rightarrow 1$ uniformly and it can be written as $r_\epsilon(x) = 1 + c_\epsilon f_\epsilon(x)$, for some $c_\epsilon \rightarrow 0$ and $f_\epsilon \in C^0([0, 2\pi])$ with $\|f_\epsilon\|_\infty = 1$. Let us write the Taylor developments in $\epsilon = 0$ of A, P, W computed at Ω_ϵ :

$$\begin{aligned} A(\Omega_\epsilon) &= \int_0^{2\pi} \frac{(1 + c_\epsilon f_\epsilon)^2}{2} = \pi + c_\epsilon \int_0^{2\pi} f_\epsilon + \frac{c_\epsilon^2}{2} \int_0^{2\pi} f_\epsilon^2 \\ &= A(\mathbb{D}) + 4c_\epsilon \int_0^{\pi/2} f_\epsilon + 2c_\epsilon^2 \int_0^{\pi/2} f_\epsilon^2, \\ P(\Omega_\epsilon) &= \int_0^{2\pi} \sqrt{r_\epsilon^2 + \dot{r}_\epsilon^2} = \int_0^{2\pi} \sqrt{1 + 2c_\epsilon f_\epsilon + c_\epsilon^2(f_\epsilon^2 + \dot{f}_\epsilon^2)} \\ &= \int_0^{2\pi} \left[1 + c_\epsilon f_\epsilon + \frac{c_\epsilon^2}{2}(f_\epsilon^2 + \dot{f}_\epsilon^2) - \frac{1}{8}(2c_\epsilon f_\epsilon)^2 \right] + o(c_\epsilon^2) \\ &= P(\mathbb{D}) + 4c_\epsilon \int_0^{\pi/2} f_\epsilon + 2c_\epsilon^2 \int_0^{\pi/2} \dot{f}_\epsilon^2 + o(c_\epsilon^2), \\ W(\Omega_\epsilon) &= \int_0^{2\pi} \frac{(1 + c_\epsilon f_\epsilon)^4}{4} = \frac{1}{4} \int_0^{2\pi} [1 + 4c_\epsilon f_\epsilon + 6c_\epsilon^2 f_\epsilon^2] + o(c_\epsilon^2) \\ &= W(\mathbb{D}) + 4c_\epsilon \int_0^{\pi/2} f_\epsilon + 6c_\epsilon^2 \int_0^{\pi/2} f_\epsilon^2 + o(c_\epsilon^2). \end{aligned}$$

Note that here we have used the double symmetry to replace the integrals over $[0, 2\pi]$ as integrals over $[0, \pi/2]$. Inserting these expressions in (6.3), and denoting by $g_\epsilon(x) := f_\epsilon(x) - \dot{f}_\epsilon$, we obtain

$$\frac{1 - y(\Omega_\epsilon)}{1 - x(\Omega_\epsilon)} = \frac{1}{4} \frac{\left(\int_0^{\pi/2} f_\epsilon \right)^2 + 2\pi \int_0^{\pi/2} (\dot{f}_\epsilon^2 - f_\epsilon^2)}{2\pi \int_0^{\pi/2} f_\epsilon^2 - \left(\int_0^{\pi/2} f_\epsilon \right)^2} + o(1) = \frac{1}{4} \left(\frac{\int_0^{\pi/2} \dot{g}_\epsilon^2}{\int_0^{\pi/2} g_\epsilon^2} - 1 \right) + o(1).$$

The last expression can be bounded from below with the first non trivial Neumann eigenvalue $\mu_1((0, \pi/2))$ of the interval $(0, \pi/2)$:

$$\frac{\int_0^{\pi/2} \dot{g}_\epsilon^2}{\int_0^{\pi/2} g_\epsilon^2} \geq \inf_{\int_0^{\pi/2} g = 0, g \neq 0} \frac{\int_0^{\pi/2} \dot{g}^2}{\int_0^{\pi/2} g^2} =: \mu_1((0, \pi/2)) = 4.$$

Thus, passing to the limit $\epsilon \rightarrow 0$, we obtain

$$\gamma_{\mathbb{D}}^- \geq \frac{3}{4}.$$

If we are able to find a sequence of shapes converging to the disk with slope $3/4$ we are done. A sequence with this property is the sequence of ellipses E_ϵ with semi-axes 1 and $1 + \epsilon$. For these shapes we have

$$A(E_\epsilon) = \pi(1 + \epsilon), \quad W(\Omega_\epsilon) = \frac{\pi}{4}[(1 + \epsilon)^3 + (1 + \epsilon)],$$

$$P(E_\epsilon) = \int_0^{2\pi} \sqrt{1 - 2\epsilon \sin^2(\theta) + \epsilon^2 \sin^2(\theta)} \, d\theta.$$

A direct computation gives

$$1 - x(E_\epsilon) = \frac{\epsilon^2}{2} + o(\epsilon^2), \quad 1 - y(E_\epsilon) = \frac{3\epsilon^2}{8} + o(\epsilon^2),$$

therefore, in the limit as $\epsilon \rightarrow 0$, we have the desired result:

$$\lim_{\epsilon \rightarrow 0^+} \frac{1 - y(E_\epsilon)}{1 - x(E_\epsilon)} = \frac{3}{4}.$$

□

7. NUMERICS

In this section, we perform simple numerical methods to try to identify the optimal domains on the upper and lower boundaries of the Blaschke-Santaló diagram. We choose a different methods for each boundary, since we already know that the optimal domains are "smooth" (at least $C^{1,1}$) on the upper boundary while they are polygonal on the lower boundary. We also compare the best domains we get numerically with the candidates as they appear for example in Reference [5].

7.1. The upper boundary L^+ . For a given abscissa $x_0 \in (0, 1)$, the problem consists in *minimizing* the perimeter among convex sets Ω that satisfy for example $A(\Omega) = \pi$ and $W(\Omega) = \pi/2x_0$. Since we know that the optimal domain is smooth, we choose to represent the convex domain by its support function $p(\theta)$ as suggested for example in [1] and [4]. Then we have the choice

- (1) either to decompose the support function in Fourier series, and the unknown are the Fourier coefficients.
- (2) or to discretize the support function by looking for its value p_i on a discrete grid θ_i .

These two methods are discussed and implemented (in particular for spectral problems) in [1] and [4]. Here we choose the Fourier series decomposition. Due to the symmetries, the Fourier series of p writes

$$p(\theta) = \sum_{k=0}^N a_{2k} \cos 2k\theta.$$

The reason of our choice is the following. First, the geometric quantities are either exactly calculable (perimeter, area) in terms of the Fourier coefficients:

$$P(\Omega) = 2\pi a_0, \quad A(\Omega) = \frac{\pi}{2} \left(2a_0^2 + \sum_{k=1}^N (1 - 4k^2) a_{2k}^2 \right)$$

or can be computed with a very good accuracy (Simpson rule for example) for the moment of inertia:

$$W(\Omega) = \frac{1}{12} \int_0^{2\pi} [3p^4 - 6p^2 p'^2 - p'^4] \, d\theta.$$

Then, the most important point is that this choice allows us to consider a small-medium scale optimization problem. Indeed, we have chosen to work with 16 Fourier coefficients (where a discretization method would require several hundred unknowns for the same accuracy). Of course, this choice has drawbacks, since it is impossible to capture shapes with segments by considering a truncated Fourier series: our support function being C^∞ , the shape is strictly convex (see [19]). As we will see, the stadium (or a stadium-like shape) seems to be the optimal domain for a range of values $x_0 \in [0, 0.72]$ and it is slightly difficult to capture: we probably need to work with more Fourier coefficients here.

In [5], for the same problem, some numerical results are presented and the authors seem to identify two families of optimal domains for this problem: stadium-like and ellipse-like. We recover a similar result here, but let us mention two points:

- it is not because the shape looks like a stadium and the numerical values are very similar that it is a stadium! We have a famous counter-example in [9]
- The ellipse **cannot** be an optimal domain, as it is proved in the following proposition.

Proposition 7.1. *An ellipse (different from a disk) cannot be a solution of the problem*

$$\min\{P(\Omega), \text{ with } A(\Omega) = A_0, W(\Omega) = W_0\}.$$

Proof. Let us consider an ellipse of semi-axes $a > b$ whose parametrization is $x(t) = a \cos t, y(t) = b \sin t, t \in [0, 2\pi]$. Since it is a convex domain C_+^2 (in the sense that the curvature is bounded from below by a positive constant), we can perform any variation of its boundary. The shape derivative techniques, see [10, Chapter 5] shows that there exist two Lagrange multipliers λ_1, λ_2 such that

$$\forall V \in C^0(\mathbb{R}^2, \mathbb{R}^2), \int_{\partial\Omega} \mathcal{C}V \cdot \nu = \lambda_1 \int_{\partial\Omega} V \cdot \nu + \int_{\partial\Omega} (x^2 + y^2)V \cdot \nu$$

where \mathcal{C} is the curvature and ν the exterior normal vector. Since this is true for any V , this implies

$$\mathcal{C} = \lambda_1 + \lambda_2(x^2 + y^2). \quad (7.1)$$

Using the parametrization of the boundary, (7.1) can be written

$$\frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} = \lambda_1 + \lambda_2(a^2 \cos^2 t + b^2 \sin^2 t). \quad (7.2)$$

Differentiating (7.2) yields

$$\frac{3ab(b^2 - a^2) \sin 2t}{2(a^2 \sin^2 t + b^2 \cos^2 t)^{5/2}} = \lambda_2(b^2 - a^2) \sin 2t$$

and we see that the previous equality can hold for any t if and only if $a = b$, that proves the claim. \square

For any given abscissa x_0 there is only one (up to scaling) stadium and ellipse satisfying $A^2/(2\pi W) = x_0$. In the following table, we represent for different values of x_0 , the corresponding value of $y = 4\pi A/P^2$ for the ellipse (2nd column), the stadium (3rd column) and the best domain we got with our numerical procedure where we use the routine *fmincon* of Matlab, providing the gradients of the objective function and the constraints (with 16 Fourier coefficients). The convexity constraint is linear in the unknowns a_{2k} and is simply written as $(p + p'')(\theta_j) \geq 0$ for the same discretization of $(0, \pi/2)$ used for computing the moment of inertia. We recall that minimizing the perimeter is equivalent here to maximize y , so we are looking for the largest value of y .

x_0	ellipse	stadium	best numerical
0.1	0.1225	0.2522	?
0.2	0.2413	0.2839	0.2420
0.3	0.3551	0.3998	0.3870
0.4	0.4634	0.5036	0.4861
0.5	0.5660	0.5977	0.5841
0.6	0.6630	0.6844	0.6777
0.7	0.7546	0.7654	0.7641
0.75	0.7985	0.8043	0.8050
0.8	0.8411	0.8425	0.8456
0.85	0.8825	0.8804	0.8851
0.9	0.9228	0.9184	0.9241
0.95	0.9619	0.9572	0.9623

Our observations are the following:

- For $x_0 \leq 0.72$, we are not able to get numerically a better domain (with 16 Fourier coefficients) than the stadium. After 0.72 we are always able to get a domain which is better than the stadium, but also better than the ellipse.
- When x_0 approaches 1, the domain we get numerically gives a value of y closer and closer to the one given by the ellipse, confirming Proposition 6.3 and the fact that ellipses provide the slope of the curve L^+ when we approach the point $(1, 1)$.

Here we plot two possible optimal domains for the values $x_0 = 0.8$ and 0.9 .

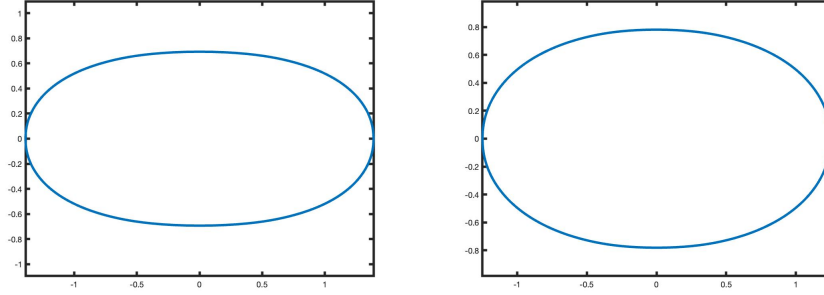


FIGURE 1. A possible optimal domain for $x_0 = 0.8$ (left), for $x_0 = 0.9$ (right).

7.2. The lower boundary L^- . We already know that the optimal domains are polygons. We also know that for $x_0 \leq 3/\pi \simeq 0.9549$ the optimal domain is the unique rhombus (up to scaling) satisfying $A^2/(2\pi W) = x_0$. It remains to find the optimal (polygonal) domains for $\frac{3}{\pi} < x < 1$. For that purpose, we consider $k+1$ unknown vertices $M_i = (x_i, y_i)$, $i = 0, \dots, k$ where we assume $x_0 = 0$ and the last vertex is known and taken as $M_{k+1} = (1, 0)$. We have $2k+1$ unknown, the quantity under consideration are

$$P(\Omega) = 4 \sum_{i=0}^k \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}, \quad A(\Omega) = 2 \left(y_k + \sum_{i=1}^k x_i (y_{i-1} - y_{i+1}) \right)$$

$$W(\Omega) = \frac{1}{3} \sum_{i=0}^k (x_{i+1} - x_i) (y_i^3 + y_i^2 y_{i+1} + y_i y_{i+1}^2 + y_{i+1}^3) \\ - \frac{1}{3} \sum_{i=0}^k (x_i^2 + x_i x_{i+1} + x_{i+1}^2) (x_i y_{i+1} - x_{i+1} y_i).$$

In the paper [5], the optimal domains looks like some "regular" octagon, therefore we have compared in the array below:

- this family of regular octagons (whose vertices in the first quadrant are $(0, 1)$; (x_1, x_1) ; $(1, 0)$ where x_1 is chosen in such a way that $A^2/(2\pi W) = x_0$)
- the best hexagon we get in the family of hexagons whose vertices in the first quadrant are $(0, H)$; (x_1, H) ; $(1, 0)$,
- the best polygon we get with our numerical procedure (we still use the routine *fmincon* of Matlab with a maximal value of k equal to 5).

The convexity constraint is non-linear in the unknowns and writes, for any internal vertex $1 \leq i \leq k$:

$$(x_{i+1} - x_i)y_{i-1} + (x_i - x_{i-1})y_{i+1} - (x_{i+1} - x_{i-1})y_i \leq 0$$

Since here we want to maximize the perimeter, this corresponds to the lower value of $y = 4\pi A/P^2$.

x_0	"regular" octagon	best hexagon	best numerical
0.956	0.78893	0.78892	0.78892
0.96	0.8023	0.8021	0.8021
0.97	0.8376	0.8353	0.8353
0.98	0.8794	0.8682	0.8682
0.985	0.8946	0.8843	0.8843
0.99	0.9150	0.8998	0.8998
0.995	0.9362	<i>none</i>	0.9290

Comments: hexagons exist up to $x_0 = 9\sqrt{3}/(5\pi) \simeq 0.9924$ that is the value for the regular hexagon. By the way, the authors do not know whether the maximization of A^2/W among polygons with a given number of sides is always given by the regular polygon. When hexagons exist, i.e. when $x_0 < 0.9924$, the best hexagons seem to be the optimal sets and, in particular are better than the regular octagons. After 0.9924 and before $12(\sqrt{2} + 1)/(\pi(5 + 3\sqrt{2})) \simeq 0.9977$ that is the value of x for the regular octagon, see the beginning of the proof of Proposition 6.3, we get optimal domains that are non regular octagons. Here we plot two possible optimal domains for the values $x_0 = 0.985$ and 0.995.

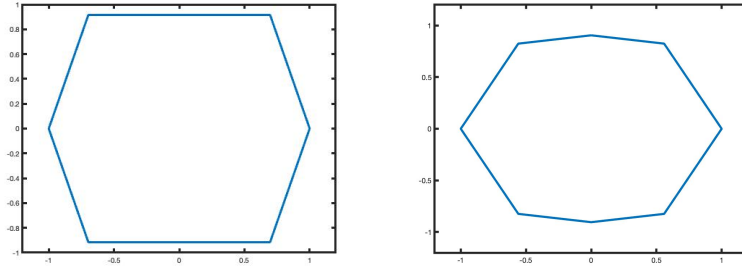
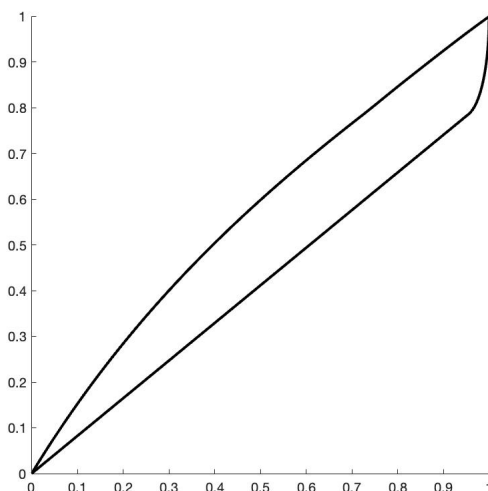


FIGURE 2. A possible optimal domain for $x_0 = 0.985$ (left), for $x_0 = 0.995$ (right).

7.3. The Blaschke-Santaló diagram. We plot now the diagram, taking into account all the previous information.

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FIGURE 3. The Blaschke-Santaló diagram (A, P, W) .

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