Monotonicity formula and stratification of the singular set of perimeter minimizers in RCD spaces

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Abstract

The goal of this paper is to establish a monotonicity formula for perimeter minimizing sets in $\mathrm{RCD}(0,N)$ metric measure cones, together with the associated rigidity statement. The applications include sharp Hausdorff dimension estimates for the singular strata of perimeter minimizing sets in non collapsed RCD spaces and the existence of blow-down cones for global perimeter minimizers in Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growth.

Contents

1 Introduction

The main goal of this paper is to prove a monotonicity formula for perimeter minimizing sets in $\mathrm{RCD}(0,N)$ metric measure cones, together with the associated rigidity statement.
Among the applications, we establish sharp Hausdorff dimension estimates for the singular strata of perimeter minimizing sets in non collapsed RCD spaces, and the existence of blow-down cones for global perimeter minimizers in Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growth.

Below we briefly introduce the setting and then discuss more in detail the main results and their relevance.

The class of $\mathrm{RCD}(K,N)$ spaces consists of infinitesimally Hilbertian metric measure spaces spaces with synthetic Ricci curvature lower bounds and dimension upper bounds. More specifically, $N \in [1,\infty)$ represents a synthetic upper bound on the dimension and $K \in \mathbb{R}$ represents a synthetic lower bound on the Ricci curvature. This class includes finite dimensional Alexandrov spaces with curvature bounded from below and (possibly pointed) measured Gromov Hausdorff limits of smooth Riemannian manifolds with Ricci curvature lower bounds and dimension upper bounds, the so-called Ricci limit spaces. Many of the results in this paper are new also in these settings, to the best of our knowledge. We address the reader to Section 2 and to the references therein indicated for the relevant background on RCD spaces.

Sets of finite perimeter have been a very important tool in the developments of Geometric Measure Theory in Euclidean and Riemannian contexts in the last seventy years. In [3, 18, 17], and the more recent [15, 12], most of the classical Euclidean theory of sets of finite perimeter has been generalized to $\mathrm{RCD}(K, N)$ metric measure spaces. Moreover in [48] the second and the third author started a study of locally perimeter minimizing sets in the same setting (see also [37]). Due to the compactness of the class of $\mathrm{RCD}(K, N)$ spaces with respect to the (pointed) measured Gromov-Hausdorff topology, these developments have been important to address some questions of Geometric Measure Theory on smooth Riemannian manifolds, e.g. see [13, 14].

1.1 Monotonicity Formula

The first main result of this work is a monotonicity formula for perimeter minimizers in cones over $\mathrm{RCD}(N-2, N-1)$ spaces, with the associated conical rigidity statement. We recall that, by [40], the metric measure cone over a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(0, N)$ metric measure space if and only if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(N-2, N-1)$ metric measure space.

For the sake of clarity, we introduce below the relevant notion of perimeter minimizing set in an $\mathrm{RCD}(K, N)$ space.

Definition 1.1 (Local and Global Perimeter Minimizer). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. A set of locally finite perimeter $E \subset X$ is a*

- Global perimeter minimizer *if it minimizes the perimeter for every compactly supported perturbation, i.e.*
$$\mathrm{Per}(E; B_R(x)) \leq \mathrm{Per}(F; B_R(x))$$
for all $x \in X$, $R > 0$ and $F \subset X$ with $F = E$ outside $B_R(x)$;
- Local perimeter minimizer *if for every $x \in X$ there exists $r_x > 0$ such that E minimizes the perimeter in $B_{r_x}(x)$, i.e. for all $F \subset X$ with $F = E$ outside $B_{r_x}(x)$ it holds*
$$\mathrm{Per}(E; B_{r_x}(x)) \leq \mathrm{Per}(F; B_{r_x}(x))\,.$$

Our main result is the following:

Theorem 1.2 (Monotonicity Formula). *Let $N \geq 2$ and let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(N-2, N-1)$ space (with $\mathrm{diam}(X) \leq \pi$, if $N = 2$). Let $C(X)$ be the metric measure cone over $(X, \mathsf{d}, \mathfrak{m})$ and let O denote its tip. Let $E \subset C(X)$ be a global perimeter minimizer. Then the function $\Phi : (0, \infty) \to \mathbb{R}$ defined by*
$$\Phi(r) := \frac{\mathrm{Per}(E; B_r(O))}{r^{N-1}}\,, \tag{1.1}$$
is non-decreasing. Moreover, if there exist $0 < r_1 < r_2 < \infty$ such that $\Phi(r_1) = \Phi(r_2)$, then $E \cap \left(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\right)$ is a conical annulus, in the sense that there exists $A \subset X$ such that
$$E \cap \left(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\right) = C(A) \cap \left(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\right),$$
where $C(A) = \{(t, x) \in C(X) \colon x \in A\}$ is the cone over $A \subset X$. In particular, if Φ is constant on $(0, \infty)$, then E is a cone (in the sense that there exists $A \subset X$ such that $E = C(A)$).

The above monotonicity formula with rigidity generalizes the analogous, celebrated result in the Euclidean setting, see for instance [30, 50, 45]. On smooth Riemannian manifolds, it is well known that an almost monotonicity formula holds, with error terms depending on two sided bounds on the Riemann curvature tensor and on lower bounds on the injectivity radius. For cones over smooth cross sections, the monotonicity formula is a folklore result, see for instance [11, 28]. Some special cases of Theorem 1.2 have been discussed recently in [27, 48]. We also mention that in [22] an analogous monotonicity formula in RCD metric measure cones has been obtained for solutions of free boundary problems, generalizing a well known Euclidean result.
In the proof, we adapt one of the classical strategies in the Euclidean setting. The implementation is of course technically more demanding, in particular for the rigidity part, due to the low regularity of the present context.

The relevance of Theorem 1.2 for the applications, that we are going to discuss below, comes from the fact that tangent cones of non collapsed $\mathrm{RCD}(K,N)$ metric measure spaces $(X,\mathsf{d},\mathcal{H}^N)$ and blow-downs of $\mathrm{RCD}(0,N)$ spaces $(X,\mathsf{d},\mathcal{H}^N)$ with Euclidean volume growth are metric measure cones, see [25, 26, 40] for the present setting and the earlier [23, 24, 20] for previous results in the case of Ricci limit spaces and Alexandrov spaces.

It is an open question whether an almost monotonicity formula holds for perimeter minimizers in general $\mathrm{RCD}(K,N)$ spaces, possibly under the non collapsing assumption. In particular, we record the following:

Open question: let $(X,\mathsf{d},\mathcal{H}^N)$ be an $\mathrm{RCD}(K,N)$ space and let $E\subset X$ be a local perimeter minimizing set. Is it true that the limit

$$\lim_{r\to 0}\frac{\mathrm{Per}(E;B_r(x))}{r^{N-1}} \tag{1.2}$$

exists for all $x\in\partial E$?

1.2 Stratification of the singular set and other applications

It is well known that monotonicity formulas are an extremely powerful tool in the analysis of singularities of several problems in Geometric Analysis. We just mention here, for the sake of illustration and because of the connection with the developments of the present work:

- the Hausdorff dimension estimates for the singular strata of area minimizing currents in codimension one, originally obtained in [31];
- the Hausdorff dimension estimates for the singular strata of $\mathrm{RCD}(K,N)$ spaces $(X,\mathsf{d},\mathcal{H}^N)$, obtained in [26] and earlier in [24] in the case of non collapsed Ricci limit spaces.

The proofs of the aforementioned results are based on the so-called dimension reduction technique, which relies in turn on the validity of a monotonicity formula with associated conical rigidity statement.
In the present work, building on the top of Theorem 1.2 we establish analogous Hausdorff dimension estimates for the singular strata of perimeter minimizing sets in $\mathrm{RCD}(K,N)$ spaces $(X,\mathsf{d},\mathcal{H}^N)$. Below we introduce the relevant terminology and state our main results.

Definition 1.3 (Singular Strata)**.** *Let $(X,\mathsf{d},\mathcal{H}^N)$ be an $\mathrm{RCD}(K,N)$ space, $E\subset X$ a locally perimeter minimizing set and $0\le k\le N-3$ an integer. The k-singular stratum of E, $\mathcal{S}^E_k$, is* *defined as*

$$\begin{aligned}\mathcal{S}^E_k := \{x\in\partial E\colon & \textit{ no tangent space to } (X,\mathsf{d},\mathcal{H}^N,E,x) \textit{ at } x \textit{ is of the form } (Y,\rho,\mathcal{H}^N,F,y),\\ & \textit{with } (Y,\rho,y) \textit{ isometric to } (Z\times\mathbb{R}^{k+1},\mathsf{d}_Z\times\mathsf{d}_{\mathrm{eucl}},(z,0)) \textit{ for some pointed } (Z,\mathsf{d}_Z,z)\\ & \textit{and } F=G\times\mathbb{R}^{k+1} \textit{ with } G\subset Z \textit{ global perimeter minimizer}\}.\end{aligned}$$

The above definition would make sense also in the cases when $k\ge N-2$. However, it seems more appropriate not to adopt the terminology *singular strata* in those instances.

Definition 1.4 (Interior and Boundary Regularity Points)**.** *Let $(X,\mathsf{d},\mathcal{H}^N)$ be an* RCD(K,N) *space and let $E\subset X$ be a locally perimeter minimizing set. Given $x\in\partial E$, we say that x is an* interior regularity *point if*

$$\mathrm{Tan}_x(X,\mathsf{d},\mathcal{H}^N,E,x)=\{(\mathbb{R}^N,\mathsf{d}_{\mathrm{eucl}},\mathcal{H}^N,\mathbb{R}^N_+,0)\}\,. \tag{1.3}$$

The set of interior regularity points of E will be denoted by $\mathcal{R}^E$.
Given $x\in\partial E$, we say that x is a boundary regularity *point if*

$$\mathrm{Tan}_x(X,\mathsf{d},\mathcal{H}^N,E,x)=\{(\mathbb{R}^N_+,\mathsf{d}_{\mathrm{eucl}},\mathcal{H}^N,\{x_1\ge 0\},0)\}\,, \tag{1.4}$$

where x_1 is one of the coordinates of the $\mathbb{R}^{N-1}$ factor in $\mathbb{R}^N_+=\mathbb{R}^{N-1}\times\{x_N\ge 0\}$. The set of boundary regularity points of E will be denoted by $\mathcal{R}^E_{\partial X}$.

It was proved in [48] that the interior regular set $\mathcal{R}^E$ is topologically regular, in the sense that it is contained in a Hölder open manifold of dimension $N-1$. By a blow-up argument it is not hard to show that $\dim_{\mathcal{H}}\mathcal{R}^E_{\partial X}\le N-2$ (see Proposition 4.3). Our main results about the stratification of the singular set for perimeter minimizers are that the complement of $\mathcal{S}^E_{N-3}$ in ∂E consists of either interior or boundary regularity points, and that the classical Hausdorff dimension estimate $\dim(\mathcal{S}^E_k)\le k$ holds for any $0\le k\le N-3$.

Theorem 1.5. *Let $(X,\mathsf{d},\mathcal{H}^N)$ be an* RCD(K,N) *space and let $E\subset X$ be a locally perimeter minimizing set. Then*

$$\partial E\setminus\mathcal{S}^E_{N-3}=\mathcal{R}^E\cup\mathcal{R}^E_{\partial X}\,. \tag{1.5}$$

Theorem 1.6 (Stratification of the singular set)**.** *Let $(X,\mathsf{d},\mathcal{H}^N)$ be an* RCD(K,N) *space and $E\subset X$ a locally perimeter minimizing set. Then, for any $0\le k\le N-3$ it holds*

$$\dim_{\mathcal{H}}\mathcal{S}^E_k\le k\,. \tag{1.6}$$

We point out that the Hausdorff dimension estimate for the top dimensional singular stratum had already been established in [27] (for limits of sequences of codimension one area minimizing currents in smooth Riemannian manifolds with Ricci curvature and volume lower bounds) and independently by the second and the third author in [48] (in the same setting of the present paper, under the additional assumption that $\partial X=\emptyset$). Elementary examples illustrate that the Hausdorff dimension estimates above are sharp in the present setting.

With respect to the classical [31] or [24, 26], in the proof of Theorem 1.6 we need to handle the additional difficulty that a monotonicity formula does not hold directly on the ambient space.

Another application of the monotonicity formula with the associated rigidity is that if an RCD$(0,N)$ space $(X,\mathsf{d},\mathcal{H}^N)$ with Euclidean volume growth contains a global perimeter minimizer, then any asymptotic cone contains a perimeter minimizing cone.

Theorem 1.7. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an* $\mathrm{RCD}(0, N)$ *metric measure space with Euclidean volume growth, i.e. satisfying for some (and thus for every) $x \in X$:*

$$\lim_{r \to \infty} \frac{\mathcal{H}^N(B_r(x))}{r^N} > 0. \tag{1.7}$$

Let $E \subset X$ be a global perimeter minimizer. Then for any blow-down $(C(Z), \mathsf{d}_{C(Z)}, \mathcal{H}^N)$ of $(X, \mathsf{d}, \mathcal{H}^N)$ there exists a cone $C(W) \subset C(Z)$ global perimeter minimizer.

The conclusion of Theorem 1.7 above seems to be new also in the more classical case of smooth Riemannian manifolds with nonnegative sectional curvature, or nonnegative Ricci curvature. We refer to [11] for earlier progress in the case of smooth manifolds with nonnegative sectional curvature satisfying additional conditions on the rate of convergence to the tangent cone at infinity and on the regularity of the cross section, and to the more recent [28] for the case of smooth Riemannian manifolds with nonnegative Ricci curvature and quadratic curvature decay.

Related to the open question that we raised above, to the best of our knowledge it is not currently known whether in the setting of Theorem 1.7 any blow down of the perimeter minimizing set must actually be a cone.

Acknowledgements.
The first author is supported by the EPSRC-UKRI grant "Maths DTP 2021-22", at the University of Oxford, with reference EP/W523781/1.

The second author is supported by the European Research Council (ERC), under the European's Union Horizon 2020 research and innovation programme, via the ERC Starting Grant "CURVATURE", grant agreement No. 802689.

The last author was supported by the European Research Council (ERC), under the European's Union Horizon 2020 research and innovation programme, via the ERC Starting Grant "CURVATURE", grant agreement No. 802689, while he was employed at the University of Oxford until August 2022. He was supported by the Fields Institute for Research in Mathematical Sciences with a Marsden Fellowship from September 2022 to December 2022. He is currently supported by the FIM-ETH Zürich with a Hermann Weyl Instructorship. He is grateful to these institutions for the excellent working conditions during the completion of this work.

2 Preliminaries

In this work, a *metric measure space* (m.m.s. for short) is a triple $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a non negative Borel measure on X of full support (i.e. $\operatorname{supp} \mathfrak{m} = X$), called the ambient or reference measure, which is finite on metric balls. We write $B_r(x)$ for the open ball centered at $x \in X$ of radius $r > 0$. Under our working conditions, the closed metric balls are compact, so we assume from the beginning the metric space (X, d) to be proper. Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. and fix $x \in X$. The quadruple $(X, \mathsf{d}, \mathfrak{m}, x)$ is called *pointed* metric measure space.

We will denote with $L^p(X; \mathfrak{m}) := \left\{ u : X \to \mathbb{R} : \int |u|^p \mathrm{d}\mathfrak{m} < \infty \right\}$ the space of p-integrable functions; sometimes, if it is clear from the context which space and measure we are considering, we will simply write L^p or $L^p(X)$.

Given a function $u : X \to \mathbb{R}$, we define its local Lipschitz constant at $x \in X$ by

$$\operatorname{lip}(u)(x) := \limsup_{y \to x} \frac{|u(x) - u(y)|}{\mathsf{d}(x, y)} \qquad \text{if } x \in X \text{ is an accumulation point,}$$

and $\mathrm{lip}(u)(x) = 0$ otherwise. We indicate by $\mathrm{LIP}(X)$ and $\mathrm{LIP}_{\mathrm{loc}}(X)$ the space of Lipschitz functions, and locally Lipschitz functions respectively. We also denote by $\mathrm{C}_b(X)$ and $\mathrm{C}_{\mathrm{bs}}(X)$ the space of bounded continuous functions, and the space of continuous functions with bounded support, respectively.

We assume that the reader is familiar with the notion and basic properties of RCD spaces. Let us just briefly recall [52, 53, 44] that a $\mathrm{CD}(K,N)$ metric measure space $(X,\mathsf{d},\mathfrak{m})$ has Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in [1,+\infty]$ in a synthetic sense, via optimal transport. The $\mathrm{RCD}(K,N)$ condition is a refinement of the $\mathrm{CD}(K,N)$ one, obtained by adding the assumption that the heat flow is linear or, equivalently, that the Sobolev space $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ is a Hilbert space or, equivalently, that the Laplacian is a linear operator. The RCD condition was first introduced in the $N=\infty$ case in [6] and then proposed in the $N<\infty$ case in [32]. We refer the reader to the original papers [6, 32, 10, 29, 9, 21], or to the survey [1] for more details.

2.1 Metric-measure cones

An N-metric measure cone over a measure metric space $(X,\mathsf{d}_X,\mathfrak{m}_X)$ is defined as the warped product

$$C(X) := ([0,\infty) \times_C X, \mathrm{d}_C, \mathfrak{m}_C)$$

obtained with $C_{\mathsf{d}}(r) = r^2$ and $C_{\mathfrak{m}} = r^{N-1}$. See [34] for some background.
We will denote by $O \in C(X)$ the tip of the cone, given by $\{O\} := \pi(\{0\} \times X) \subset C(X)$. In what follows we will use a slight abuse of notation and denote $\pi(t,x)$ by (t,x). In particular, $O = (0,x)$ for any $x \in X$. Moreover, we shall adopt the more intuitive notation d_C, $\mathfrak{m}_C$ to denote the distance and the reference measure on $C(X)$, when there is no risk of confusion.

There is an explicit expression for the distance between two points on a cone:

$$\mathsf{d}_C^2((r_1,x_1),(r_2,x_2)) = r_1^2 + r_2^2 - 2r_1r_2\cos(\mathsf{d}_X(x_1,x_2)\wedge\pi). \tag{2.1}$$

In particular, we have

$$\mathsf{d}_C(O,(r,x)) = r\,. \tag{2.2}$$

The following result was obtained in [25].

Proposition 2.1. *Let $(X,\mathsf{d}_X,\mathfrak{m}_X)$ be a metric measure space and let $N \geq 2$. Then $(C(X),\mathsf{d}_C,\mathfrak{m}_C)$ is an $\mathrm{RCD}(0,N)$ m.m.s. if and only if $(X,\mathsf{d}_X,\mathfrak{m}_X)$ is $\mathrm{RCD}(N-2,N-1)$ and, in the case $N=2$, $\mathrm{diam}(X) \leq \pi$.*

Remark 2.2. *If $N > 2$, then the diameter bound $\mathrm{diam}(X) \leq \pi$ follows already from the $\mathrm{RCD}(N-2,N-1)$ condition, by the Bonnet-Meyers theorem for CD spaces.*

Next, we show a result relating radial derivatives of functions defined on cones and the distance from the tip of the cone. The gradient of such distance function, in some sense, corresponds to the position vector field in the Euclidean setting. Some of the identities of the proof, obtained using basic calculus in the metric setting, will be used later in this work. The following result uses the BL characterization found in [34], section 3.2.

Proposition 2.3. *Let $(C(X),\mathsf{d}_C,\mathfrak{m}_C)$ be an $\mathrm{RCD}(0,N)$ cone over some $\mathrm{RCD}(N-2,N-1)$ space $(X,\mathrm{d}_X,\mathfrak{m}_X)$ and $f \in W^{1,2}(X)$. Let $f^{(t)}(x) := f(t,x)$ and $f^{(x)}(t) := f(t,x)$, for $(t,x) \in C(X)$. Then*

$$|\nabla f^{(x)}|(t) = \frac{1}{2t}|\nabla f(t,x)\cdot\nabla\mathsf{d}_C^2(O,\cdot)(t,x)| \quad \textit{for } \mathfrak{m}\textit{-a.e. } x \in X \textit{ and } \mathcal{L}^1\textit{-a.e. } r \in (0,\infty)\,. \tag{2.3}$$

Proof. By [34], section 3.2, $f^{(t)} \in W^{1,2}(X)$ and $f^{(x)} \in W^{1,2}([0,\infty))$ for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$ and $\mathfrak{m}$-a.e. $x \in X$, respectively. By using the BL characterization of $W^{1,2}(C(X))$ functions (cf. [34], section 3.2) and the polarization identity, we have

$$\begin{aligned}
\nabla f(t,x) \cdot \nabla \mathsf{d}_C^2(O,\cdot)(t,x) = & \\
=&\frac{1}{2}|\nabla(f+\mathsf{d}_C^2(O,\cdot))|^2(t,x) - \frac{1}{2}|\nabla(f-\mathsf{d}_C^2(O,\cdot))|^2(t,x) \\
=&\frac{1}{2}|\partial_r(f^{(x)}(t)+\mathsf{d}_{C,(x)}^2(t))|^2 - \frac{1}{2}|\partial_r(f^{(x)}(t)-\mathsf{d}_{C,(x)}^2(t))|^2 \qquad (2.4)\\
=&\frac{1}{2}|\partial_r f^{(x)}+2t|^2 - \frac{1}{2}|\partial_r f^{(x)}(t)-2t|^2 \qquad (2.5)\\
=&2t\,\partial_r f^{(x)}(t) = 2t\,\mathrm{sign}(\partial_r f^{(x)}(t))\,|\nabla f^{(x)}|(t)\,;
\end{aligned}$$

where we have denoted $\mathsf{d}_C(O,(t,x))^{(x)}$ by $\mathsf{d}_{C,(x)}(t)$. Moreover, we have used the fact that $|\nabla(f^{(t)}+(\mathsf{d}_C^2(O,\cdot))^{(t)})|^2(x) = |\nabla f^{(t)}|(x)$ since $(\mathsf{d}_C(O,\cdot))^{(t)}$ is constant. We have also exploited the identification between different notions of derivatives (by, say, [38, Th. 2.1.37]) $|\nabla g| = |\partial_r g|$ for smooth functions $g : [0,\infty) \to \mathbb{R}$ and the explicit formula for the radial sections of the distance function from the origin given by (2.2). Let us also point out that (2.4) shows that

$$\nabla f(t,x) \cdot \nabla(\mathsf{d}_C^2(O,(t,x)) = \nabla f^{(x)}(t) \cdot \nabla(\mathsf{d}_{C,(x)}^2)(t)\,. \qquad (2.6)$$

Lastly, we point out that the following equality

$$|\nabla \mathsf{d}_C(O,\cdot)|(t,x) = |\partial_r \mathsf{d}_{C,(x)}(t)| = \frac{1}{2t}|\nabla \mathsf{d}_C^2(O,\cdot)|(t,x)\,, \qquad (2.7)$$

implies

$$|\nabla f^{(x)}|(t) = |\nabla f(t,x) \cdot \nabla(\mathsf{d}_C(O,(t,x)))|\,. \qquad (2.8)$$

See [25, Cor. 3.6]. □

2.2 Finite perimeter sets in RCD spaces

Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space, $u \in L^1_{\mathrm{loc}}(X)$ and $\Omega \subset X$ an open set. The total variation norm of u evaluated on Ω is defined by

$$|Du|(\Omega) := \inf\left\{\liminf_{j\to\infty} \int_\Omega \mathrm{lip}(u)(y)\,\mathrm{d}\mathfrak{m}\right\}\,, \qquad (2.9)$$

where the infimum is taken over all sequences $(u_j) \subset \mathrm{Lip}_{\mathrm{loc}}(X)$ such that $u_j \to u$ in $L^1_{\mathrm{loc}}(X)$.

A function $u \in L^1(X)$ is said to have bounded variation if its total variation $|Du|(X)$ is finite. In this case one can prove that $|Du|$ can be extended to a Borel measure on X. The space of functions of bounded variation is denoted by $\mathrm{BV}(X)$.

A set $E \subset X$ is of locally finite perimeter if, for all $x \in X$ and $R > 0$ there holds

$$\mathrm{Per}(E;B_R(x)) := \inf\left\{\liminf_{i\to\infty} \int_{B_r(x)} \mathrm{lip}(f_i)\,\mathrm{d}\mathfrak{m}\ :\ \{f_i\} \subset \mathrm{Lip}_{\mathrm{loc}}(X),\ f_i \xrightarrow{L^1_{loc}} \chi_E\right\} < \infty\,.$$

We denote the perimeter measure of a locally finite perimeter set E by $\mathrm{Per}(E)$. Let us point out that this coincides with the variational measure $|D\chi_E|$ defined in (2.9). We adopt the following notation: given a Borel set $A \subset X$, $\mathrm{Per}(E;A) := |D\chi_E(A)|$.

An important tool for what follows is the coarea formula (cf. [46], theorem 2.16):

Theorem 2.4 (Coarea Formula)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space and $v \in \mathrm{BV}(X)$. Then $\{v > r\}$ has finite perimeter for $\mathcal{L}^1$-a.e. r and for any Borel function $f : X \to \mathbb{R}$ it holds*

$$\int_X f \, \mathrm{d}|Dv| = \int_{\mathbb{R}} \left(\int_X f \, \mathrm{dPer}(\{v > r\}) \right) \mathrm{d}r \,. \tag{2.10}$$

Given a function $u \in \mathrm{BV}(X)$, we define the set $E_t := \{x \in X : u(x) \geq t, \, t \in \mathbb{R}\}$. As a consequence of the coarea formula, it holds

$$|Du|(C) = \int_{\mathbb{R}} \mathrm{Per}(E_t; C) \, \mathrm{d}t \,.$$

We now look at a particular type of locally finite perimeter sets: cones inside cones. Let $(C(X), \mathsf{d}_C, \mathfrak{m}_C)$ be as in Proposition 2.1, $r > 0$, $x \in X$. An important family of sets we will use later, in particular in the characterization of cones (see Lemma 3.3), is the following

$$C(B_r^X(x)) := \{(t, y) \in C(X) : y \in B_r^X(x)\} \,. \tag{2.11}$$

Lemma 2.5. *The sets $C(B_r^X(x)) \subset C(X)$, $r > 0$, are of locally finite perimeter.*

Proof. We will prove the claim by exhibiting an explicit sequence of Lipschitz functions converging to $\chi_{C(B_r^X(x))}$. Let

$$f_n(t, y) = \begin{cases} 1 & \text{if } y \in B_r^X(x) \,; \\ nr + 1 - n\mathsf{d}(y, x) & \text{if } y \in B_{r+\frac{1}{n}}^X(x) \setminus B_r^X(x) \,; \\ 0 & \text{if } y \in X \setminus B_{r+\frac{1}{n}}^X(x)) \,. \end{cases} \tag{2.12}$$

Clearly, f_n is Lipschitz with bounded support, $|f_n| \leq 1$, hence $f_n \in W^{1,2}_{\mathrm{loc}}(C(X))$. Moreover, $f_n \to \chi_{C(B_r^X(x))}$ in $L^1_{\mathrm{loc}}(C(X))$. Indeed, for $p = (s, z) \in C(X)$, $R > 0$

$$\begin{aligned} \int_{B_R(p)} |f_n - \chi_{C(B_r^X(x))}| \, \mathrm{d}\mathfrak{m}_C &= \int_{B_R(p)} |f_n| \chi_{C(B_{r+\frac{1}{n}}^X(x) \setminus B_r^X(x))} \mathrm{d}\mathfrak{m}_C \\ &\leq \mathfrak{m}_C \left(B_R(p) \cap C \left(B_{r+\frac{1}{n}}^X(x) \setminus B_r^X(x) \right) \right) . \end{aligned}$$

Using the Bishop Gromov inequality [53], we have

$$\mathfrak{m}(B_{r+\frac{1}{n}}^X(x) \setminus B_r^X(x)) = \mathfrak{m}(B_r^X(x)) \left(\frac{\mathfrak{m}(B_{r+\frac{1}{n}}^X(x)}{\mathfrak{m}(B_r^X(x))} - 1 \right) \leq \mathfrak{m}(B_r^X(x)) \left(\frac{N-1}{rn} + O\left(\frac{1}{n^2} \right) \right) .$$

Therefore,

$$\mathfrak{m}_C \left(B_R(p) \cap C \left(B_{r+\frac{1}{n}}^X(x) \setminus B_r^X(x) \right) \right) = O\left(\frac{1}{n} \right) , \tag{2.13}$$

which implies $f_n \to \chi_{C(B_r^X(x))}$ in $L^1_{\mathrm{loc}}(C(X))$.

Let us now show that

$$\limsup_{n \to \infty} \int_{B_R(p)} \mathrm{lip} f_n \mathrm{d}\mathfrak{m}_C < \infty \,, \tag{2.14}$$

for any $p = (s, z) \in C(X)$, $R > 0$, which directly implies that $C(B_r^X(x))$ is a set of locally finite perimeter.

It is elementary to check that

$$\mathrm{lip} f_n(t,x) = \begin{cases} n & \text{if } y \in B^X_{r+\frac{1}{n}}(x) \setminus B^X_r(x)\,; \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.15}$$

Therefore, using (2.13), we obtain

$$\int_{B_R(p)} \mathrm{lip} f_n \mathrm{d}\mathfrak{m}_C = n\, \mathfrak{m}_C \left(B_R(p) \cap C\left(B^X_{r+\frac{1}{n}}(x) \setminus B^X_r(x)\right)\right) = O(1)\,, \quad \text{as } n \to \infty\,.$$

□

Let us recall some useful notions of convergence, in order to study blow-ups of sets of (locally) finite perimeter. We refer to [36, 8, 3] for more details.

Definition 2.6 (L^1 strong convergence of sets)**.** *Let $\{(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)\}_i$ be a sequence of pointed metric measure spaces converging in the pointed measured Gromov Hausdorff sense to (Y, ρ, μ, y). Let (Z, d_Z) be the ambient space realizing the convergence. Moreover, let $E_i \subset X_i$ be a sequence of Borel sets with $\mathfrak{m}_i(E_i) < \infty$ for every $i \in \mathbb{N}$. We say that $\{E_i\}_i$ converges to a Borel set $F \subset Y$ in the strong L^1 sense if the measures $\chi_{E_i}\mathfrak{m}_i \rightharpoonup \chi_F \mu$ in duality with $\mathrm{C_{bs}}(Z)$ and $\mathfrak{m}_i(E_i) \to \mu(F)$.*

We also say that the convergence of E_i is strong in L^1_{loc} if $E_i \cap B_r(x_i)$ converges in the strong L^1 sense to $F \cap B_r(y)$ for all $r > 0$.

Such a convergence can be metrized, by a distance $\mathcal{D}$ defined on (isomorphsim classes of) quintuples $(X, \mathsf{d}, \mathfrak{m}, x, E)$, see [3, Lemma A.4].

We use that notion of convergence to define the tangent to a set of locally finite perimeter contained in an $\mathrm{RCD}(K,N)$ metric measure space. Before, let us recall that given an $\mathrm{RCD}(K,N)$ and a point $x \in X$, by Gromov pre-compactness theorem there is a not-empty set $\mathrm{Tan}_x(X)$ of tangent spaces at x obtained by considering the pmGH limits of blow-up rescalings of X centered at x; moreover, for $\mathfrak{m}$-a.e. $x \in X$, the tangent space is unique and Euclidean [35, 47, 19].

Definition 2.7 (Tangent to a set of locally finite perimeter)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and $E \subset X$ be a set of locally finite perimeter.*
We say that $(Y, \rho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ if $(Y, \rho, \mu, y) \in \mathrm{Tan}_x(X)$ and $F \subset Y$ is a set of locally finite perimeter of positive measure such that χ_E converges in the L^1_{loc} sense of Definition 2.6 to F along the blow up sequence associated to the tangent Y.

An important tool for our analysis is the Gauss-Green formula for sets of finite perimeter in the RCD setting. We refer to [18] for the proof and for background material; here let us briefly mention that, given a set of finite perimeter $E \subset X$, it is possible to define the space of L^2-vector fields with respect to the perimeter measure, denoted by $L^2_E(TX)$.

Theorem 2.8 (Gauss-Green formula)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and $E \subset X$ a set of finite perimeter with $\mathfrak{m}(E) < \infty$. Then there exists a unique vector field $\nu_E \in L^2_E(TX)$ such that $|\nu_E| = 1$ $|D\chi_E|$-almost everywhere and*

$$\int_E \mathrm{div}(v) \mathrm{d}\mathfrak{m} = -\int \mathrm{tr}_E(v) \cdot \nu_E \,\mathrm{dPer}(E),$$

for all $v \in W^{1,2}_C(TX) \cap D(\mathrm{div})$ with $|v| \in L^\infty(|D\chi_E|)$.

Let us also recall the following useful cut and paste result proved in [17, Theorem 4.11]. We use the following notation: $D\chi_E = \nu_E |D\chi_E|$. Moreover, we denote by $\mathcal{H}^h$ the codimension one Hausdorff type measure induced by $\mathfrak{m}$ with gauge function $h(B_r(x)) := \mathfrak{m}(B_r(x))/r$, see [18] for further details.

Theorem 2.9 (Cut and paste)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *m.m.s. and E, $F \subset X$ be sets of finite perimeter. Then $E \cap F$, $E \cup F$ and $E \setminus F$ are sets of finite perimeter. Moreover, there holds*

$$\begin{aligned} D\chi_{E\cap F} &= D\chi_E|_{F^{(1)}} + D\chi_F|_{E^{(1)}} + \nu_E \mathcal{H}^h|_{\{\nu_E=\nu_F\}}\,; \\ D\chi_{E\cup F} &= D\chi_E|_{F^{(0)}} + D\chi_F|_{E^{(0)}} + \nu_E \mathcal{H}^h|_{\{\nu_E=\nu_F\}}\,; \\ D\chi_{E\setminus F} &= D\chi_E|_{F^{(0)}} - D\chi_F|_{E^{(1)}} + \nu_E \mathcal{H}^h|_{\{\nu_E=-\nu_F\}}\,. \end{aligned}$$

We now recall the notion of perimeter minimizing sets. To avoid discussing trivial cases, we will always assume $\mathfrak{m}(E) > 0$ and $\mathfrak{m}(X \setminus E) \neq 0$.

Definition 2.10 (Local and Global Perimeter Minimizer)**.** *A set of locally finite perimeter $E \subset X$, with $\mathfrak{m}(E) > 0$ and $\mathfrak{m}(X \setminus E) > 0$, is a*

- Global perimeter minimizer *if it minimizes the perimeter for every compactly supported perturbation, i.e.*
$$\mathrm{Per}(E; B_R(x)) \leq \mathrm{Per}(F; B_R(x))$$
for all $x \in X$, $R > 0$ and $F \subset X$ with $F = E$ outside $B_R(x)$;
- Local perimeter minimizer *if for every $x \in X$ there exists $r_x > 0$ such that E minimizes the perimeter in $B_{r_x}(x)$, i.e. for all $F \subset X$ with $F = E$ outside $B_{r_x}(x)$ it holds*
$$\mathrm{Per}(E; B_{r_x}(x)) \leq \mathrm{Per}(F; B_{r_x}(x))\,.$$

For a proof of the following density result see [41].

Lemma 2.11 (Density)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *m.m.s. and let $E \subset X$ be a local perimeter minimizer set and let $x \in \partial E$. Then there exists constants r_0, $C > 0$ such that*

$$C^{-1}\frac{\mathfrak{m}(E \cap B_r(x))}{r} \leq \mathrm{Per}(E; B_r(x)) \leq C\frac{\mathfrak{m}(E \cap B_r(x))}{r}\,,$$

for all $0 < r < r_0$.

We report here a few results on BV functions and their associated vectorial variational measures, for their proof and background on notation see [15].

Proposition 2.12. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, \infty)$ *space and $f \in \mathrm{BV}(X)$. Then there exists a unique vector field $\nu_F \in L_{|DF|}(TX)$ such that*

$$\int_X f\,\mathrm{div}v\,\mathrm{d}\mathfrak{m} = -\int_X v \cdot \nu_f\,\mathrm{d}|Df|\,, \tag{2.16}$$

for all $v \in QC^\infty(TX) \cap D(\mathrm{div})$.

In what follows, for $f \in \mathrm{BV}(X)$ we will denote $Df := \nu_f|Df|$. If E is a set of locally finite perimeter, we denote $\nu_E := \nu_{\chi_E}$. This definition of unit normal is consistent with the one introduced above via the Gauss-Green formula, see [18]. For a function $f : X \to \mathbb{R}$, define

$$\begin{aligned} f^\wedge &= \mathrm{ap}\ \liminf_{y\to x} f(y) = \sup\left\{t \in \overline{\mathbb{R}} : \lim_{r\searrow 0}\frac{\mathfrak{m}(B_r(x) \cap \{f < t\})}{\mathfrak{m}(B_r(x))} = 0\right\} \\ f^\vee &= \mathrm{ap}\ \limsup_{y\to x} f(y) = \inf\left\{t \in \overline{\mathbb{R}} : \lim_{r\searrow 0}\frac{\mathfrak{m}(B_r(x) \cap \{f > t\})}{\mathfrak{m}(B_r(x))} = 0\right\}\,, \end{aligned}$$

and, lastly,

$$\bar{f} = \frac{f^\wedge + f^\vee}{2}\,, \tag{2.17}$$

with the convention $\infty - \infty = 0$.

Lemma 2.13 (Leibniz rule for BV)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space and $f, g \in \mathrm{BV}(X) \cap L^\infty(X)$. Then $fg \in \mathrm{BV}(X)$ and*

$$D(fg) = \bar{f} Dg + \bar{g} Df\,. \tag{2.18}$$

In particular, $|D(fg)| \le |\bar{f}||Dg| + |\bar{g}||Df|$.

Proposition 2.14 (BV extension)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space, E a set of locally finite perimeter and $f \in \mathrm{BV}(X) \cap L^\infty(E)$. Then*

$$\tilde{f}(x) := \begin{cases} \bar{f}(x) & \text{if } x \in E\,, \\ 0 & \text{elsewhere} \end{cases}$$

belongs to $\mathrm{BV}(X)$ and $D\tilde{f} = \bar{f} D\chi_E + Df|_E$.

Proof. The result immediately follows by applying (2.18) with $g = \chi_E$. □

Lemma 2.15 (Cut and paste of BV Functions)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ m.m.s. and E a set of locally finite perimeter. Let $f \in \mathrm{BV}(E)$ and $g \in \mathrm{BV}(X \setminus E)$. Let $h : X \to \mathbb{R}$ be defined as*

$$h(x) := \begin{cases} f(x) & \text{if } x \in E\,; \\ g(x) & \text{if } x \in X \setminus E\,. \end{cases}$$

Then, $h \in \mathrm{BV}(X)$. Moreover, called $\bar{f}, \bar{g}$ the representatives given by (2.17), it holds

$$Dh = Df|_E + Dg|_{X \setminus E} + (\bar{f} - \bar{g}) D\chi_E\,.$$

Proof. Let $\tilde{f}$ and $\tilde{g}$ be the extensions by zero given by Proposition 2.14. Then $h = \tilde{f} + \tilde{g}$. □

3 Monotonicity Formula

A classical and extremely powerful tool for studying sets which locally minimize the perimeter in Euclidean spaces is the monotonicity formula for the perimeter. The goal of this section is to generalize such monotonicity formula (with the associated rigidity statement) to perimeter minimizers in cones over RCD spaces. In the next section, we will draw some applications on the structure of the singular set of local perimeter minimizers.

Recall that given an $\mathrm{RCD}(N-2, N-1)$ space $(X, \mathsf{d}_X, \mathfrak{m}_X)$ then the metric-measure cone over X, denoted by $(C(X), \mathsf{d}_C, \mathfrak{m}_C)$, is an $\mathrm{RCD}(0, N)$ space (if $N = 2$, we also assume that $\mathrm{diam}(X) \le \pi$). We denote by $O = (0, x) \in C(X)$ the tip of the cone (see Section 2.1 for more details) and $B_r(O)$ the open metric ball centered at O of radius $r > 0$.

When we consider a local perimeter minimizer E, we shall always assume that $E = E^{(1)}$ is the open representative, given by the measure theoretic interior. See [42] for the relevant background.

Theorem 3.1 (Monotonicity Formula)**.** *Let $N \geq 2$ and let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(N-2, N-1)$ *space (with* $\mathrm{diam}(X) \leq \pi$*, if $N = 2$). Let $C(X)$ be the metric measure cone over $(X, d, \mathfrak{m})$. Let $E \subset C(X)$ be a global perimeter minimizer in the sense of Definition 2.10. Then the function $\Phi : (0, \infty) \to \mathbb{R}$ defined by*

$$\Phi(r) := \frac{\mathrm{Per}(E; B_r(O))}{r^{N-1}}\,, \tag{3.1}$$

is non-decreasing. Moreover, if there exist $0 < r_1 < r_2 < \infty$ such that $\Phi(r_1) = \Phi(r_2)$, then $E \cap \big(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\big)$ is a conical annulus, in the sense that there exists $A \subset X$ such that

$$E \cap \big(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\big) = C(A) \cap \big(B_{r_2}(O) \setminus \overline{B_{r_1}(O)}\big)\,,$$

where $C(A) = \{(t, x) \in C(X) \colon x \in A\}$ is the cone over $A \subset X$. In particular, if Φ is constant on $(0, \infty)$, then E is a cone (in the sense that there exists $A \subset X$ such that $E = C(A)$).

Remark 3.2. *In the case where $E \subset C(X)$ is a locally finite perimeter set, minimizing the perimeter for perturbations supported in $B_{R+1}(O)$, then the monotonicity formula holds on $(0, R)$, i.e. the function Φ defined in (3.1) is non-decreasing on $(0, R)$. Also the rigidity statement holds, for $0 < r_1 < r_2 < R$. The proofs are analogous.*

Proof of Theorem 3.1. Let us first give an outline of the argument. The first two steps are inspired by the approach used in the lecture notes [49], which provide a proof of the monotonicity formula for local perimeter minimizers in Euclidean spaces by-passing the first variation formula. Classical references for this approach are [30, 50].
The main idea is to approximate the characteristic function of E by regular functions f_k and approximate Φ by the corresponding Φ_{f_k}; show an almost-monotonicity formula for Φ_{f_k} and finally pass to the limit and get the monotonicity of Φ. This will be achieved in steps 1-3. In step 4 we relate the derivative of Φ with a quantity characterizing cones as in Lemma 3.3.

Throughout the proof, we will write B_r in place of $B_r(O)$ for the ease of notation.

Step 1: Approximation preliminaries.
In this step we show that, up to error terms, regular functions approximating χ_E preserve the minimality condition. The argument requires an initial approximation. Let $f \in \mathrm{LIP}(C(X)) \cap \mathrm{D_{loc}}(\Delta)(C(X))$ be non-negative. We introduce two functions a, $b : [0, \infty) \to [0, \infty)$ to quantify the errors in the approximation:

$$a(r) := \big||Df|(B_r) - \mathrm{Per}(E; B_r)\big|\,, \qquad b(r) := \int_{\partial B_r} |\mathrm{tr}^{\mathrm{ext}}_{\partial B_r} \chi_E - \mathrm{tr}_{\partial B_r} f|\, \mathrm{dPer}(B_r)\,, \tag{3.2}$$

where $\mathrm{tr}^{\mathrm{ext}}_{\partial B_r} \chi_E$ is the trace of χ_E from the exterior of the ball B_r.
We remark that the interior and exterior normal traces can be defined by considering the precise representative of $\chi_E \cdot \chi_{B_r}$ and $\chi_E \cdot \chi_{X \setminus B_r}$ respectively. See [15, Lemma 3.23] and [4] for the Euclidean theory.

Notice that $\mathrm{tr}^{\mathrm{ext}}_{\partial B_r} f = \mathrm{tr}_{\partial B_r} f = f|_{\partial B_r}$, since f is continuous. Fix $R > 0$. Let $0 < r < R$ and $g \in \mathrm{BV_{loc}}(C(X))$ be any function such that

$$\mathrm{tr}^{\mathrm{int}}_{\partial B_r} g = \mathrm{tr}_{\partial B_r} f \quad \text{and} \quad g = \chi_E \text{ on } C(X) \setminus B_r\,,$$

where $\mathrm{tr}^{\mathrm{int}}_{\partial B_r} g$ is the trace of g from the interior of the ball B_r. The minimality of E implies

$$\mathrm{Per}(E; B_R) \leq \mathrm{Per}(\{q \in C(X) \,:\, g(q) > t\}; B_R),$$

for any $0 < t < 1$. Integrating in t and using the coarea formula (2.10) we obtain

$$\mathrm{Per}(E; B_R) \le \int_0^1 \mathrm{Per}(\{q \in C(X) \,:\, g(q) > t\}; B_R)\,\mathrm{d}t \le |Dg|(B_R)\,.$$

Therefore, using Lemma 2.15 and the definition of g, we obtain

$$\begin{aligned}
\mathrm{Per}(E; B_r) &= \mathrm{Per}(E; B_R) - \mathrm{Per}(E; B_R \setminus B_r) \le |Dg|(B_R) - \mathrm{Per}(E; B_R \setminus B_r) \\
&= |Dg|(B_r) + \int_{\partial B_r} |\mathrm{tr}^{\mathrm{ext}}_{\partial B_r}\chi_E - \mathrm{tr}_{\partial B_r} f|\,\mathrm{dPer}(B_r) = |Dg|(B_r) + b(r)\,.
\end{aligned}$$

Finally, for any such g there holds

$$|Df|(B_r) \le \mathrm{Per}(E; B_r) + a(r) \le |Dg|(B_r) + a(r) + b(r)\,. \tag{3.3}$$

Step 2. Main computation.

In this step we show the monotonicity, up to error terms, of an approximation of Φ, denoted below by Φ_f, obtained by replacing χ_E with the regular approximation f of step 1.

Fix f as in step 1 and $r > 0$. By [34],

$$|\nabla f|^2(t,x) = |\nabla f^{(x)}|^2(t) + t^{-2}|\nabla f^{(t)}|^2(x)\,, \text{for } \mathfrak{m}\text{-a.e. } x \in X \text{ and } \mathcal{L}^1\text{-a.e. } t > 0\,.$$

Let $h : C(X) \to \mathbb{R}$ be defined by $h(t,x) := f^{(r)}(x)$ for all $t > 0$. Notice that h is locally Lipschitz away from the origin and it is elementary to check that it has locally bounded variation. By [34, 33], it holds

$$|Dh|(t,x) = \frac{r}{t}|\nabla f^{(r)}|(x)\,, \quad \text{for } \mathfrak{m}\text{-a.e. } x \text{ and } \mathcal{L}^1\text{-a.e. } t\,.$$

By integrating over B_r and using the coarea formula, we obtain

$$\begin{aligned}
\int_{B_r} |Dh|(t,x)\,\mathrm{d}\mathfrak{m}_C &= \int_0^r \int_{\partial B_t} |Dh|(t,x)\,\mathrm{dPer}(B_t)\,dt = \int_0^r t^{N-1}\int_X \frac{r}{t}|\nabla f^{(r)}|(x)\mathrm{d}\mathfrak{m}\,\mathrm{d}t \\
&= \int_0^r \frac{t^{N-2}}{r^{N-2}} \int_{\partial B_r} |\nabla f^{(r)}|(x)\,\mathrm{dPer}(B_r)\,\mathrm{d}t \\
&= \frac{r}{N-1}\int_{\partial B_r} |\nabla f^{(r)}|(x)\,\mathrm{dPer}(B_r)\,.
\end{aligned} \tag{3.4}$$

Let us point out that the latter expression can be viewed as the integral on ∂B_r of the analog of the tangential derivative of f in the smooth case, while h is the radial extension of the values of f on ∂B_r to the whole of $C(X)$. Given $r > 0$, let us introduce the quantity

$$J(r) := \int_{B_r} |\nabla f|(t,x)\,\mathrm{d}\mathfrak{m}_C = \int_0^r t^{N-1}\int_X |\nabla f|(t,x)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t\,,$$

which will approximate $r^{N-1}\Phi(r)$. Notice that J is a Lipschitz function, hence it is almost everywhere differentiable. Using the identity (3.4), we obtain that for a.e. r it holds

$$\begin{aligned}
J'(r) &= \int_{\partial B_r} |\nabla f|(r,x)\,\mathrm{dPer}(B_r) \\
&= \frac{N-1}{r}\int_{B_r} |Dh|(t,x)\,\mathrm{d}\mathfrak{m}_C + \int_{\partial B_r} \Big(|\nabla f|(r,x) - |\nabla f^{(r)}|(x)\Big)\,\mathrm{dPer}(B_r)\,.
\end{aligned} \tag{3.5}$$

We notice that $\mathrm{tr}_{\partial B_r} h = \mathrm{tr}_{\partial B_r} f$. By defining $\tilde{h} : C(X) \to \mathbb{R}$ to be equal to h inside B_r and χ_E outside, we observe that (3.4) and (3.5) still hold if we replace h by $\tilde{h}$ (here it is key that B_r is the open ball). Therefore, in step 1 we can choose $g = \tilde{h}$ and (3.3) reads as

$$\int_{B_r} |D\tilde{h}|(t,x)\, \mathrm{d}\mathfrak{m}_C + a(r) + b(r) \geq J(r)\,. \tag{3.6}$$

Substituting (3.6) into (3.5) and rearranging, yields

$$J'(r) - \frac{N-1}{r} J(r) \geq \int_{\partial B_r} \Big(|\nabla f|(r,x) - |\nabla f^{(r)}|(x) \Big)\, \mathrm{dPer}(B_r) - \frac{N-1}{r}(a(r)+b(r))\,. \tag{3.7}$$

With a slight abuse, in order to keep notation simple, below we will write $\nabla \mathsf{d}_C(O,(r,x))$ to denote $\nabla(\mathsf{d}_C(O,\cdot))(r,x)$. Using Proposition 2.3 together with the fact that $1-\sqrt{1-s} \geq \frac{s}{2}$, for $0 \leq s \leq 1$, the BL characterization of the norm [34, Def. 3.8] of $|\nabla f|$ and the indentification between minimal weak upper gradients for different exponents on RCD spaces from [33], we have

$$\begin{aligned} \frac{|\nabla f|(r,x) - |\nabla f^{(r)}|(x)}{|\nabla f|(r,x)} &= 1 - \sqrt{1 - \frac{(\nabla f(r,x)\cdot \nabla \mathsf{d}_C(O,(r,x)))^2}{|\nabla f|(r,x)^2}} \\ &\geq \frac{(\nabla f(r,x)\cdot \nabla \mathsf{d}_C(O,(r,x)))^2}{2|\nabla f|(r,x)^2} \\ &= \frac{\big(\nabla f(r,x)\cdot \nabla(\frac{1}{2}\mathsf{d}_C^2(O,(r,x)))\big)^2}{2r^2\,(|\nabla f|(r,x))^2}\,, \end{aligned} \tag{3.8}$$

for $\mathfrak{m}$-a.e. $x \in X$ and a.e. $r \in (0,+\infty)$. Above, we understand that all the term vanish on the set where $|\nabla f| = 0$.

Let us now define the function

$$\Phi_f(r) := \frac{\int_{B_r} |\nabla f|(t,x)\, \mathrm{d}\mathfrak{m}_C}{r^{N-1}} = \frac{J(r)}{r^{N-1}}$$

which will approximate the function Φ in the statement of the theorem. Notice that Φ_f is Lipschitz and differentiable almost everywhere by the coarea formula (2.10). Taking its derivative and using (3.7) and (3.8) we obtain that for a.e. r it holds

$$\begin{aligned} \Phi_f'(r) &= \frac{J'(r) - \frac{N-1}{r}J(r)}{r^{N-1}} \\ &\geq \int_{\partial B_r} \frac{\left(\nabla f(r,x)\cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}|\nabla f|(r,x)}\, \mathrm{dPer}(B_r) - \frac{N-1}{r^N}(a(r)+b(r))\,. \end{aligned} \tag{3.9}$$

Integrating (3.9) from $0 < r_1 < r_2 < \infty$, and using coarea formula, we get

$$\begin{aligned} \Phi_f(r_2) - \Phi_f(r_1) \geq &\int_{B_{r_2}\setminus \overline{B_{r_1}}} \frac{\left(\nabla f(r,x)\cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}|\nabla f|(r,x)}\, \mathrm{d}\mathfrak{m}_C \\ &- \int_{r_1}^{r_2} \frac{N-1}{r^N}(a(r)+b(r))\, \mathrm{d}r\,. \end{aligned} \tag{3.10}$$

Step 3. Approximation.
In this step we carry out an approximating argument, using step 2 and in particular (3.10). This allows us to conclude the monotonicity part of the theorem.

Let $\{f_k\}_{k\in\mathbb{N}} \subset \mathrm{LIP}(C(X)) \cap \mathrm{D}_{\mathrm{loc}}(\Delta)$ be a sequence of non-negative functions converging in $\mathrm{BV}_{\mathrm{loc}}(X)$ to χ_E. That is,

$$|f_k - \chi_E|_{L^1(B_r)} \overset{k\to\infty}{\longrightarrow} 0, \qquad |Df_k|(B_r) \overset{k\to\infty}{\longrightarrow} |D\chi_E|(B_r), \qquad \text{for all } r>0\,. \tag{3.11}$$

Such sequence can be easily constructed by approximation via the heat flow, see for instance [5] for analogous arguments.

Let us start by showing that the errors defined in (3.2) relative to f_k go to zero as k tends to ∞.

The term $a_k(r) := ||Df_k|(B_r) - \mathrm{Per}(E;B_r)| \overset{k\to\infty}{\longrightarrow} 0$ by $\mathrm{BV}_{\mathrm{loc}}$-convergence of f_k to χ_E, i.e. (3.11).

To deal with the error term $b_k(r) := \int_{\partial B_r} |\mathrm{tr}^{\mathrm{ext}}_{\partial B_r}\chi_E - \mathrm{tr}_{\partial B_r} f_k| \mathrm{dPer}(B_r)$, we can use the coarea formula (2.10) to show that

$$\int_{B_r} |f_k - \chi_E| \,\mathrm{d}\mathfrak{m}_C = \int_0^r b_k(s)\,\mathrm{d}s\,.$$

Together with the L^1-convergence of f_k to χ_E, this shows that $b_k(r)\to 0$ for $\mathcal{L}^1$-a.e. $r>0$.

Lastly, let us show that $\Phi_f(r)\to\Phi(r)$ for $\mathcal{L}^1$-a.e. $r>0$. By $\mathrm{BV}_{\mathrm{loc}}$ convergence of f_k to χ_E (3.11), we have

$$\lim_{k\to\infty} \Phi_{f_k}(r) = \frac{1}{r^{N-1}} \lim_{k\to\infty} \int_{B_r} |Df_k| \mathrm{d}\mathfrak{m}_C = \frac{1}{r^{N-1}} \int_{B_r} \mathrm{dPer}(E) = \Phi(r)\,. \tag{3.12}$$

Consequently, letting $k\to\infty$ in the estimate (3.10) with f replaced by f_k, we obtain

$$\Phi(r_2) - \Phi(r_1) \geq 0\,, \qquad \text{for } \mathcal{L}^1\text{-almost every } r_2 > r_1 > 0\,, \tag{3.13}$$

thanks to the non-negativity of the term

$$\int_{B_{r_2}\setminus\overline{B_{r_1}}} \frac{\left(\nabla f_k(r,x)\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}|\nabla f_k|(r,x)}\,\mathrm{d}\mathfrak{m}_C \geq 0\,. \tag{3.14}$$

To conclude that Φ is monotone, we need to extend (3.13) to every $r_2 > r_1 > 0$.

Let $\{r_k\}_{k\in\mathbb{N}}$ be any sequence such that $r_k \uparrow r$. Since B_r is open, $B_{r_k}\uparrow B_r$. Hence, by the inner regularity of measures

$$\mathrm{Per}(E;B_{r_k}) \to \mathrm{Per}(E;B_r)\,.$$

Let r_2, $r_1>0$. Since the set of radii for which (3.13) holds is dense, we can find $\{r_{1,k}\}_{k\in\mathbb{N}}$ and $\{r_{2,l}\}_{l\in\mathbb{N}}$ for which (3.13) holds and such that $r_{1,k}\uparrow r_1$ and $r_{2,l}\uparrow r_2$. Then

$$0 \leq \lim_{l\to\infty}\Phi(r_{2,l}) - \lim_{k\to\infty}\Phi(r_{1,k}) = \Phi(r_2)-\Phi(r_1)\,.$$

Step 4. Rigidity.

In this step we focus on the rigidity part of the statement. We show that if there exist $r_2>r_1>0$ such that $\Phi(r_1)=\Phi(r_2)$, then $E\cap\left(B_{r_2}\setminus\overline{B_{r_1}}\right)$ is a cone.

The first step is to prove the following claim:

$$\begin{aligned}\liminf_{k\to\infty} &\int_{B_{r_2}\setminus\overline{B_{r_1}}} \frac{\left(\nabla f_k(r,x)\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}|\nabla f_k|(r,x)}\,\mathrm{d}\mathfrak{m}_C \\ &\geq \int_{B_{r_2}\setminus\overline{B_{r_1}}} \frac{\left(\nu_E(r,x)\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}}\,\mathrm{dPer}(E)\,,\end{aligned} \tag{3.15}$$

where ν_E is the unit normal to E, see Theorem 2.8. Subsequently, we will be able to conclude using the characterization of cones provided by Lemma 3.3.

The plan is to apply Lemma 3.5 to prove (3.15). Using the notation in Lemma 3.5, we define the measures

$$\mu_k := |\nabla f_k| \mathfrak{m}_C \llcorner_{(B_{r_2}\setminus \overline{B_{r_1}})}, \qquad \mu := \mathrm{Per}(E;\cdot)\llcorner_{(B_{r_2}\setminus \overline{B_{r_1}})}. \tag{3.16}$$

The $\mathrm{BV}_{\mathrm{loc}}$-convergence of f_k to χ_E (see (3.11)) ensures that $\mu_k \rightharpoonup \mu$ in duality with $\mathrm{C_b}(C(X))$. The functions

$$g_k := \frac{\nabla f_k \cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)}{\sqrt{2}r^{\frac{N+1}{2}}|\nabla f_k|} \cdot \chi_{\{|\nabla f_k|>0\}} \in L^2(C(X);\mu_k)$$

satisfy (3.47). Indeed,

$$\nabla f_k(r,x)\cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right) \le \frac{1}{2}|\nabla f_k|(r,x)|\nabla \mathsf{d}_C^2(O,(r,x))| = r|\nabla f_k|(r,x) \quad \mathfrak{m}_C\text{-a.e.}\,.$$

Therefore, using (2.7),

$$\begin{aligned}
\|g_k\|^2_{(X;\mu_k)} &= \int_{B_{r_2}\setminus \overline{B_{r_1}}} \frac{\left(\nabla f_k\cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\right)^2}{2r^{N+1}|\nabla f_k|}\cdot \chi_{\{|\nabla f_k|>0\}} \mathrm{d}\mathfrak{m}_C \\
&\le \int_{B_{r_2}\setminus \overline{B_{r_1}}} \frac{1}{2r^{N-1}}|\nabla f_k|\,\mathrm{d}\mathfrak{m}_C < C < +\infty\,,
\end{aligned}$$

for some $C>0$ independent of $k\in\mathbb{N}$ thanks to the $\mathrm{BV}_{\mathrm{loc}}$-convergence (3.11).
Consequently, Lemma 3.5 provides the existence of $g \in L^2(C(X);\mu)$ and a subsequence $k(l)$ such that

$$\frac{\nabla f_{k(l)}\cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)}{\sqrt{2}r^{\frac{N+1}{2}}|\nabla f_{k(l)}|}\cdot \chi_{\{|\nabla f_{k(l)}|>0\}}\,\mu_{k(l)} \rightharpoonup g\;\mathrm{Per}(E;\cdot)\llcorner_{(B_{r_2}\setminus \overline{B_{r_1}})}, \tag{3.17}$$

in duality with $\mathrm{C_b}(C(X))$. Up to relabelling the approximating sequence f_k, we can suppose that the whole sequence satisfies (3.17). We next determine the limit function g.

Fix a test function $\varphi \in \mathrm{LIP}\cap \mathrm{D}(\Delta)(B_{r_2}\setminus \overline{B_{r_1}})$ with compact support contained in $B_{r_2}\setminus \overline{B_{r_1}}$.

We apply the Gauss-Green formula (Theorem 2.8) and use that φ has compact support in $B_{r_2}\setminus \overline{B_{r_1}}$ to obtain

$$\begin{aligned}
&\int_{B_{r_2}\setminus \overline{B_{r_1}}} f_k \,\mathrm{div}\left(\frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\right)\,\mathrm{d}\mathfrak{m}_C \\
&\qquad = -\int_{B_{r_2}\setminus \overline{B_{r_1}}} \frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\left(\nabla f_k\cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\right)\,\mathrm{d}\mathfrak{m}_C \\
&\qquad = -\int_{B_{r_2}\setminus \overline{B_{r_1}}} \varphi\frac{\nabla f_k\cdot \nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)}{\sqrt{2}r^{\frac{N+1}{2}}|\nabla f_k|}\mathrm{d}\mu_k\,.
\end{aligned} \tag{3.18}$$

Using the L^1-convergence of f_k to χ_E and that

$$\left\|\frac{\mathrm{div}(\varphi\nabla(\frac{1}{2}\mathsf{d}_C^2(O,\cdot))}{\sqrt{2}r^{\frac{N+1}{2}}}\right\|_{L^\infty(B_{r_2}\setminus \overline{B_{r_1}})} < \infty\,,$$

we infer that

$$\begin{aligned}
&\lim_{k\to\infty}\int_{B_{r_2}\setminus \overline{B_{r_1}}} f_k\,\mathrm{div}\left(\frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\right)\,\mathrm{d}\mathfrak{m}_C = \int_E \mathrm{div}\left(\frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\right)\,\mathrm{d}\mathfrak{m}_C \\
&\quad = -\int_{\partial^* E}\frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\,\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\cdot \nu_E\,\mathrm{d}\mathrm{Per}(E)\,,
\end{aligned} \tag{3.19}$$

where in the last equality we used the Gauss-Green formula (Theorem 2.8). Combining (3.18) and (3.19) we obtain

$$\lim_{k\to\infty}\int_{C(X)}\varphi\frac{\nabla f_k\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)}{\sqrt{2}r^{\frac{N+1}{2}}|\nabla f_k|}\mathrm{d}\mu_k=\int_{\partial^*E}\frac{\varphi}{\sqrt{2}r^{\frac{N+1}{2}}}\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(\cdot))\right)\cdot\nu_E\,\mathrm{dPer}(E).\tag{3.20}$$

That is,

$$\frac{\nabla f_k\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)}{\sqrt{2}r^{\frac{N+1}{2}}|\nabla f_k|}\mu_k\rightharpoonup\frac{\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(\cdot))\right)\cdot\nu_E}{\sqrt{2}r^{\frac{N+1}{2}}}\mathrm{Per}(E)\tag{3.21}$$

in duality with $\mathrm{C}_c(B_{r_2}\setminus\overline{B_{r_1}})$, by approximation. By the uniqueness of the weak limit and from (3.17) we can conclude that

$$g=\frac{\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,\cdot)\right)\cdot\nu_E}{\sqrt{2}r^{\frac{N+1}{2}}}\qquad\mathrm{Per}(E)|_{B_{r_2}\setminus\overline{B_{r_1}}}\text{-almost everywhere}\,.$$

From (3.49) in Lemma 3.5, we have

$$\liminf_{k\to\infty}\|g_k\|^2_{L^2(C(X);\mu_k)}\geq\|g\|^2_{L^2(C(X);\mu)}\,.$$

That is, we have shown the claim (3.15).

We are now in position to improve the estimate (3.13) and use it to show the rigidity part of the theorem. By taking the inferior limit in (3.10), recalling (3.12) and that the error terms go to zero from step 3, we use (3.15) to infer

$$\Phi(r_2)-\Phi(r_1)\geq\int_{B_{r_2}\setminus\overline{B_{r_1}}}\frac{\left(\nu_E(r,x)\cdot\nabla\left(\frac{1}{2}\mathsf{d}_C^2(O,(r,x))\right)\right)^2}{2r^{N+1}}\,\mathrm{dPer}(E)\geq 0,\tag{3.22}$$

for every $r_2>r_1>0$. Since we are assuming $\Phi(r_1)=\Phi(r_2)$, if follows that

$$\nabla(\mathsf{d}_C(O,\cdot))\cdot\nu_E=0\quad\mathrm{Per}(E)\text{-a.e. on }B_{r_2}\setminus\overline{B_{r_1}}\,.$$

By applying Lemma 3.3, we can conclude that $E\cap\left(B_{r_2}\setminus\overline{B_{r_1}}\right)$ is a conical annulus.

□

Let us now prove a useful characterization of conical annuli contained in cones over RCD spaces. The characterization is based on the properties of the normal to the boundary of the subset: roughly the subset is conical if and only if its normal is orthogonal to the gradient of the distance function from the tip of the ambient conical space. In case the ambient space is Euclidean, the result is classical (see for instance [45, Proposition 28.8]).

Lemma 3.3 (Characterization of conical annuli)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(N-2,N-1)$ *space and let* $C(X)$ *be the cone over* X*. Let* $E\subset C(X)$ *be a locally finite perimeter set and let* $0<r_1<r_2<\infty$*. Then the measure theoretic interior* $E^{(1)}\cap\left(B_{r_2}(O)\setminus\overline{B_{r_1}(O)}\right)$ *is a conical annulus if and only if*

$$\nabla(\mathsf{d}_C(O,\cdot))\cdot\nu_E=0\quad\mathrm{Per}(E)\text{-}a.e.\ on\ B_{r_2}(O)\setminus\overline{B_{r_1}(O)}.\tag{3.23}$$

Proof. As in the proof Theorem 3.1, to keep notation short we will write B_r to denote the open ball of radius $r>0$ and centered at the tip of the cone, i.e. $B_r=B_r(O)$. Also, we will write $B_r^X(x)$ for the open ball in X, of center x and radius $r>0$. For simplicity of presentation we will show the equivalence only in the case $r_1 = 0$, $r_2 = \infty$. The general case requires minor modifications. Moreover, in order to simplify the notation, we assume without loss of generality that $E = E^{(1)}$, as the condition (3.23) is clearly independent of the chosen representative.

Step 1.
We start with some preliminary computations aimed to establish the identity (3.29) below, which will be key in showing the characterization of conical annuli in $C(X)$.

Using the Gauss-Green and the coarea formulas, we will express the derivative of the function

$$u(s) := \mathfrak{m}_C(E \cap C(B_r^X(x)) \cap B_s)$$

(suitably rescaled) with the product between the unit normal of E and the gradient of the distance function from the tip of $C(X)$.

Let $x \in X$, $r, s > 0$. By Lemma 2.5 and Theorem 2.9 the set $F := E \cap C(B_r^X(x)) \cap B_s$ is a set of finite perimeter with

$$\begin{aligned} D\chi_F =& D\chi_{E \llcorner C(B_r^X(x)) \cap B_s} + \nabla(\mathsf{d}_C(O,\cdot)) \mathrm{Per}(B_s)_{\llcorner E \cap C(B_r^X(x))} \\ &+ \nu_{C(B_r^X(x))} \mathrm{Per}(C(B_r^X(x)))_{\llcorner E \cap B_s} . \end{aligned}$$

Using the Gauss-Green formula (Theorem 2.8), the equality for the laplacian of the distance function from the tip on cones [25, Prop. 3.7], and cut and paste of sets of locally finite perimeter (Theorem 2.9), we obtain

$$\begin{aligned} N \cdot u(s) =& \int_F \Delta\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \mathrm{d}\mathfrak{m}_C \\ =& \int_{C(B_r^X(x)) \cap B_s \cap \partial^* E} \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_E \, \mathrm{dPer}(E) \\ &+ \int_{E \cap C(B_r^X(x)) \cap \partial B_s} \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_{\partial B_s} \, \mathrm{dPer}(B_s) \\ &+ \int_{E \cap B_s \cap \partial C(B_r^X(x))} \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_{C(B_r^X(x))} \, \mathrm{dPer}(C(B_r^X(x))) . \end{aligned} \tag{3.24}$$

We now study separately the three integrals in the right hand side of (3.24), starting from the last one.
Fix a function $\varphi \in \mathrm{LIP}(C(X)) \cap \mathrm{D}(\Delta)$ with compact support. By applying the Gauss-Green Theorem 2.8 on the set of locally finite perimeter $C(B_r^X(x))$, we obtain

$$\begin{aligned} &\int_{\partial C(B_r^X(x))} \varphi \, \nu_{C(B_r^X(x))} \cdot \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \mathrm{dPer}(C(B_r^X(x))) \\ &= -\int_{C(B_r^X(x))} \nabla\varphi \cdot \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \mathrm{d}\mathfrak{m}_C + \int_{C(B_r^X(x))} \varphi\Delta\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \mathrm{d}\mathfrak{m}_C \\ &= \int_{B_r^X(x)} \int_0^\infty \partial_r \varphi^{(y)}(r) \cdot \partial_r\left(\frac{1}{2}\mathsf{d}^2_{(y)}(r)\right) r^{N-1} \, \mathrm{d}r \, \mathrm{d}\mathfrak{m}(y) + \int_{C(B_r^X(x))} \varphi N \mathrm{d}\mathfrak{m}_C \\ &= -\int_{B_r^X(x)} \int_0^\infty \varphi^{(y)}(r) N r^{N-1} \, \mathrm{d}r \, \mathrm{d}\mathfrak{m}(y) + \int_{C(B_r^X(x))} \varphi N \mathrm{d}\mathfrak{m}_C = 0, \end{aligned}$$

where we have used (2.6), the definition of $\mathfrak{m}_C$ and integration by parts on $\mathbb{R}$. Since φ was arbitrary, we infer that

$$\nu_{C(B_r^X(x))} \cdot \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) = 0 \quad \mathrm{Per}(C(B_r^X(x)))\text{-a.e.}$$

and thus

$$\int_{E\cap B_s\cap\partial C(B_r^X(x))} \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right)\cdot\nu_{C(B_r^X(x))}\,\mathrm{dPer}(C(B_r^X(x)))=0\,. \tag{3.25}$$

Let us now deal with the second integral appearing the right hand side of (3.24). By the chain rule, we have $\nabla\left(\frac{1}{2}\mathsf{d}^2(O,q)\right)=\mathsf{d}(O,q)\nabla\mathsf{d}(O,q)$. Therefore, we obtain

$$\begin{aligned}&\int_{E\cap C(B_r^X(x))\cap\partial B_s} \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right)\cdot\nu_{\partial B_s}\,\mathrm{dPer}(B_s)\\&=\int_{E\cap C(B_r^X(x))\cap\partial B_s}\mathsf{d}(O,\cdot)\,\nabla\mathsf{d}(O,\cdot)\cdot\nu_{\partial B_s}\,\mathrm{dPer}(B_s)\\&=s\mathrm{Per}(B_s;E\cap C(B_r^X(x)))\,,\end{aligned} \tag{3.26}$$

where we used [17, Prop. 6.1]. Inserting (3.25) and (3.26) into (3.24), yields

$$u(s)=\frac{1}{N}\int_{C(B_r^X(x))\cap B_s}\nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right)\cdot\nu_E\,\mathrm{dPer}(E)+\frac{s}{N}\mathrm{Per}(B_s;E\cap C(B_r^X(x)))\,. \tag{3.27}$$

By the coarea formula (2.10), u is Lipschitz and differentiable almost everywhere and it holds

$$u(s)=\int_0^s\mathrm{Per}(B_t;E\cap C(B_r^X(x)))\,\mathrm{d}t\,. \tag{3.28}$$

We now compute the derivative of $\frac{u(s)}{s^N}$. Combining (3.27) and (3.28), we obtain that for a.e. s it holds

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{u(s)}{s^N}=\frac{u'(s)}{s^N}-N\frac{u(s)}{s^{N+1}}=-\frac{\int_{C(B_r^X(x))\cap B_s}\nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right)\cdot\nu_E\,\mathrm{dPer}(E)}{s^{N+1}}. \tag{3.29}$$

Fix $0<r<s$ and consider the sets

$$A((s,x),r):=C(B_r^X(x))\cap B_{s(1+r)}\setminus B_{s(1-r)}\,. \tag{3.30}$$

By Lemma 3.4 below, the family of sets $\{A(q,r)|\,q\in C(X),\,r>0\}$ generates the Borel σ-algebra of $C(X)$, since for any $q\in C(X)$, $r>0$ there exist r_a, $r_b>0$ such that

$$B(q,r_a)\subset A(q,r)\subset B(q,r_b)\,. \tag{3.31}$$

Define

$$v(s,r):=\mathfrak{m}_C(E\cap A((s,x),r))=u(s(1+r))-u(s(1-r))\,.$$

The identity (3.29) yields that for a.e. s,

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{v(s)}{s^N}=-\frac{\int_{A((s,x),r)}\nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right)\cdot\nu_E\,\mathrm{dPer}(E)}{s^{N+1}}\,. \tag{3.32}$$

Step 2.
In this step we show that if E is a cone, then (3.23) holds with $r_1=0,r_2=\infty$. We will first show that $\frac{v(s)}{s^N}$ is constant, and then conclude using (3.32).

Since E is a cone, there exists a set $F \subset X$ such that $E = \{(t,x) \in C(X) : x \in F,\ t \geq 0\}$. Note that $E \cap C(B_r^X(x))$ is a cone for any $x \in X$, $r > 0$. Thus, for any $s > 0$, it holds:

$$\mathfrak{m}_C(E \cap C(B_r^X(x)) \cap B_s) = \mathfrak{m}(F \cap B_r^X(x)) \int_0^s \rho^{N-1}\,\mathrm{d}\rho = \frac{s^N}{N}\mathfrak{m}(F \cap B_r^X(x))\,,$$

yielding that $s \mapsto s^{-N}u(s)$ is constant.
By (3.32), for all $r > 0$, $p \in C(X)$ we have

$$\int_{A(p,r)} \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_E \,\mathrm{dPer}(E) = 0. \tag{3.33}$$

By the Lebesgue differentiation Theorem (see for instance [12, Rem. 2.19]), for $\mathrm{Per}(E)$-a.e. $p \in C(X)$ it holds

$$\lim_{r\to 0} \fint_{B_r(p)} \left|\nabla\left(\frac{1}{2}\mathsf{d}^2(O,q)\right) \cdot \nu_E(q) - \nabla\left(\frac{1}{2}\mathsf{d}^2(O,p)\right) \cdot \nu_E(p)\right| \mathrm{dPer}(E)(q) = 0\,.$$

From (3.31) and the asymptotic doubling property of the perimeter we infer

$$\begin{aligned}
&\lim_{r\to 0} \frac{1}{\mathrm{Per}(E, A(p,r))} \int_{A(p,r)} \left|\nabla\left(\frac{1}{2}\mathsf{d}^2(O,q)\right) \cdot \nu_E(q) - \nabla\left(\frac{1}{2}\mathsf{d}^2(O,p)\right) \cdot \nu_E(p)\right| \mathrm{dPer}(E)(q)\\
&\quad\leq C \lim_{r\to 0} \frac{1}{\mathrm{Per}(E, B_{r_b}(p))} \int_{B_{r_b}(p)} \left|\nabla\left(\frac{1}{2}\mathsf{d}^2(O,q)\right) \cdot \nu_E(q) - \nabla\left(\frac{1}{2}\mathsf{d}^2(O,p)\right) \cdot \nu_E(p)\right| \mathrm{dPer}(E)(q)\\
&\quad= 0\,,
\end{aligned}$$

where r_a and r_b are as in Lemma 3.4. We can now conclude recalling (3.33):

$$\begin{aligned}
0 &= \lim_{r\to 0} \frac{1}{\mathrm{Per}(E, A(p,r))} \int_{A(p,r)} \nabla\left(\frac{1}{2}\mathsf{d}^2(O,\cdot)\right) \cdot \nu_E \,\mathrm{dPer}(E)\\
&= \nabla\left(\frac{1}{2}\mathsf{d}^2(O,p)\right) \cdot \nu_E(p)\,, \quad \text{for} \quad \mathrm{Per}(E)\text{-a.e. } p\,.
\end{aligned}$$

Step 3.
In this last step we show that (3.23) for $r_1 = 0, r_2 = \infty$ implies that E is a cone. We will show that given $(s,x) \in E$ and $\lambda > 0$, then $(\lambda s, x) \in E$.

Using the assumption (3.23) and (3.32), we obtain that $s \mapsto v(s) = \mathfrak{m}_C(E \cap A((s,x),r))/s^N$ is constant. Therefore, for $\lambda > 0$

$$\mathfrak{m}_C(E \cap A((s,x),r)) = \frac{\mathfrak{m}_C(E \cap A((\lambda s,x),r))}{\lambda^N}. \tag{3.34}$$

Moreover,

$$\begin{aligned}
\mathfrak{m}_C(A((\lambda t,x),r)) &= \mathfrak{m}(B_r^X(x)) \int_{\lambda t(1-r)}^{\lambda t(1+r)} s^{N-1}\,\mathrm{d}s = \lambda^N \mathfrak{m}(B_r^X(x)) \int_{t(1-r)}^{t(1+r)} s^{N-1}\,ds\\
&= \lambda^N \mathfrak{m}_C(A((t,x),r))),
\end{aligned} \tag{3.35}$$

The combination of (3.34) and (3.35) gives

$$\frac{\mathfrak{m}_C(E \cap A((s,x),r))}{\mathfrak{m}_C(A((s,x),r))} = \frac{\mathfrak{m}_C(E \cap A((\lambda s,x),r))}{\mathfrak{m}_C(A((\lambda s,x),r))}\,. \tag{3.36}$$

We next show that if $(s,x) \in E$ then $(\lambda s, x) \in E$. Thanks to (3.36), it is enough to show that $q \in E$ if and only if

$$\lim_{r\to 0} \frac{\mathfrak{m}_C(E \cap A(q,r))}{\mathfrak{m}_C(A(q,r))} = 1 \,. \tag{3.37}$$

Assume by contradiction that (3.37) holds but $q \notin E$. Then,

$$\liminf_{r\to 0} \frac{\mathfrak{m}_C((X\setminus E) \cap B_r(q))}{\mathfrak{m}_C(B_r(q))} \geq \varepsilon > 0 \,.$$

Using Lemma 3.4, we infer that

$$\begin{aligned}
&\liminf_{r\to 0} \frac{\mathfrak{m}_C((X\setminus E) \cap A(q,r))}{\mathfrak{m}_C(A(q,r))} \\
&\geq \liminf_{r\to 0} \frac{\mathfrak{m}_C((X\setminus E) \cap B_{r_a(q,r)}(q))}{\mathfrak{m}_C(B_{r_a(q,r)}(q))} \cdot \frac{\mathfrak{m}_C(B_{r_a(q,r)}(q))}{\mathfrak{m}_C(B_{r_b(q,r)}(q))} \,.
\end{aligned} \tag{3.38}$$

Since $C(X)$ is an $\mathrm{RCD}(0,N)$ space, the Bishop-Gromov monotonicity formula [53] gives

$$\mathfrak{m}_C(B_{r_a}(q)) \geq \left(\frac{r_a}{r_b}\right)^N \mathfrak{m}_C(B_{r_b}(q)) \,.$$

Therefore, from (3.38) we may conclude

$$\begin{aligned}
\liminf_{r\to 0} \frac{\mathfrak{m}_C((X\setminus E) \cap A(q,r))}{\mathfrak{m}_C(A(q,r))} &\geq \liminf_{r\to 0} \left(\frac{r_a(q,r)}{r_b(q,r)}\right)^N \cdot \liminf_{r\to 0} \frac{\mathfrak{m}_C((X\setminus E) \cap B_{r_a(q,r)}(q))}{\mathfrak{m}_C(B_{r_a(q,r)}(q)))} \\
&\geq C\varepsilon > 0 \,,
\end{aligned} \tag{3.39}$$

where $C := \liminf_{r\to 0} \left(\frac{r_a(q,r)}{r_b(q,r)}\right)^N > 0$ thanks to (3.41). Clearly, (3.39) contradicts (3.37). The proof that $q \in E$ implies (3.37) is analogous.

$\square$

The following technical lemma was used in the proof of Lemma 3.3 above.

Lemma 3.4. *Let $N \geq 2$, let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(N-2, N-1)$ space and let $(C(X), \mathsf{d}_C, \mathfrak{m}_C)$ be the cone over it. If $N = 2$, assume also that $\mathrm{diam}(X) \leq \pi$. For $x \in X$ and $0 < r < s$, consider the sets*

$$A((s,x),r) := C(B^X_r(x)) \cap B_{s(1+r)}(O) \setminus B_{s(1-r)}(O). \tag{3.40}$$

Then:

- *The family of sets $\{A(q,r) |\, q \in C(X),\, r > 0\}$ generates the Borel σ-algebra of $C(X)$;*
- *For any $q \in C(X)$, $r > 0$ there exist $r_a = r_a(q,r)$ and $r_b = r_b(q,r) > 0$ such that*

$$B_{r_a}(q) \subset A(q,r) \subset B_{r_b}(q)$$

and

$$\lim_{r\to 0} \frac{r_a}{r_b} = \frac{1}{4\sqrt{2}}. \tag{3.41}$$

Proof. The first claim follows from the second one; thus let us determine r_a and $r_b > 0$ that satisfy the second statement. To this aim, we compute the minimal and maximal distance of $q = (t, x)$ from the set $\partial A(q, r)$. Let us start from the minimal distance. We deal with the shell part first: given $(t(1 + r), y) \in \partial B_{t(1+r)} \cap \partial A$ there holds, using (2.2)

$$\begin{aligned} \mathsf{d}_C^2((t, x), (t(1 + r), y)) &= t^2 + t^2(1 + r)^2 - 2t^2(1 + r) \cos(\mathsf{d}(x, y)) \\ &\geq t^2 + t^2(1 + r)^2 - 2t^2(1 + r) = t^2r^2 , \end{aligned} \tag{3.42}$$

where the equality is achieved at $y = x$. Let now $(s, y) \in \partial C(B_r^X(x)) \cap \partial A(q, r)$:

$$\mathsf{d}_C^2((t, x), (s, y)) = t^2 + s^2 - 2st \cos(r) . \tag{3.43}$$

This defines a differentiable function of $s \in [t(1 - r), t(1 + r)]$. Its derivative $\partial_s \mathsf{d}_C^2((t, x), (s, y)) = 2s - 2t \cos(r)$ is increasing and vanishes at $s = t \cos(r)$. Therefore, we have

$$\mathsf{d}_C^2(q, \partial A(q, r)) = t^2 \sin^2(r) . \tag{3.44}$$

Therefore, we may pick

$$r_a = r_a(q, r) := \frac{1}{2}\mathsf{d}_C(q, \partial A(q, r)) = \frac{1}{2}t \sin(r) . \tag{3.45}$$

Next, let us compute the maximal distance of q from $\partial A(q, r)$. Since the maximal distance is attained at the intersection of the shell with the side part of A (by monotonicity of both formulas (3.42) and (3.43) with respect to $\mathsf{d}(x, y)$ and s, respectively), we can simply compute the maximum by looking at the distance from the top shell. We again compute, for $\mathsf{d}(x, y) = r$,

$$\begin{aligned} \mathsf{d}_C^2((t, x), (t(1 + r), y)) &= t^2 + t^2(1 + r)^2 - 2t^2(1 + r) \cos(r) \\ &= t^2(2 + 2r + r^2 - 2(1 + r) \cos(r)) . \end{aligned}$$

Consequently, we may pick

$$r_b = r_b(q, r) := 2t\sqrt{(2 + 2r + r^2 - 2(1 + r) \cos(r))} . \tag{3.46}$$

It is easy to check that $r_a, r_b > 0$ defined in (3.45), (3.46) satisfy (3.41). ☐

A useful technical tool, used to prove the rigidity statement of the monotonicity formula, is the following lemma (see [2, Lemma 10.1] for the proof).

Lemma 3.5 (Joint Lower Semicontinuity)**.** *Let (X, d) be a Polish space. Let $\mu, \mu_k \in \mathcal{M}_+(X)$ with $\mu_k \rightharpoonup \mu$ in duality with $\mathrm{C}_b(X)$. Let $g_k \subset L^2(X; \mu_k)$ be a sequence of functions such that*

$$\sup_{k\in\mathbb{N}} \|g_k\|_{L^2(X;\mu_k)} < \infty. \tag{3.47}$$

Then, there exists a function $g \in L^2(X; \mu)$ and a subsequence $k(l)$ such that

$$g_{k(l)}\, \mu_{k(l)} \rightharpoonup g\mu \tag{3.48}$$

in duality with $\mathrm{C}_b(X)$ and

$$\liminf_{l\to\infty} \|g_{k(l)}\|_{L^2\left(X;\mu_{k(l)}\right)} \geq \|g\|_{L^2(X;\mu)} . \tag{3.49}$$

4 Stratification of the Singular Set and further applications

The first goal of this section is to prove sharp Hausdorff dimension estimates for the singular strata of locally perimeter minimizing sets in $\mathrm{RCD}(K,N)$ spaces $(X,\mathsf{d},\mathcal{H}^N)$. The statement is completely analogous to the classical one for singular strata of minimizing currents in the Euclidean setting, see [31], and for the singular strata of non collapsed Ricci limits [24] and RCD spaces [26]. Also the proof is based on the classical Federer's dimension reduction argument, and builds upon the monotonicity formula and associated rigidity for perimeter minimizing sets in $\mathrm{RCD}(0,N)$ metric measure cones, Theorem 3.1. Though, a difference between the present work and the aforementioned papers is that the monotonicity formula is available only at the level of blow-ups and not in the space X; this creates some challenges that are addressed in the proof.
The second main goal will be to present an application of the monotonicity formula and the associated rigidity for cones, to the existence of perimeter minimizing cones in any blow-down of an $\mathrm{RCD}(0,N)$ space $(X,\mathsf{d},\mathcal{H}^N)$ with Euclidean volume growth.

Below we introduce the relevant definition of singular strata and of interior or boundary regularity points for a locally perimeter minimizing set $E\subset X$, when $(X,\mathsf{d},\mathcal{H}^N)$ is an $\mathrm{RCD}(K,N)$ metric measure space.

Definition 4.1 (Singular Strata). *Let $(X,\mathsf{d},\mathcal{H}^N)$ be an $\mathrm{RCD}(K,N)$ space, $E\subset X$ a locally perimeter minimizing set in the sense of Definition 2.10 and $0\le k\le N-3$ an integer. The k-singular stratum of E, $\mathcal{S}^E_k$, is defined as*

$$\begin{aligned} S^E_k := \{ & x\in\partial E :\ \text{no tangent space to } (X,\mathsf{d},\mathcal{H}^N,E,x) \text{ at } x \text{ is of the form } (Y,\rho,\mathcal{H}^N,F,y),\\ & \text{with } (Y,\rho,y) \text{ isometric to } (Z\times\mathbb{R}^{k+1},\mathsf{d}_Z\times\mathsf{d}_{\mathrm{eucl}},(z,0)) \text{ for some pointed } (Z,\mathsf{d}_Z,z)\\ & \text{and } F=G\times\mathbb{R}^{k+1} \text{ with } G\subset Z \text{ global perimeter minimizer}\}. \end{aligned} \tag{4.1}$$

The above definition would make sense also in the cases when $k\ge N-2$. However, it seems more appropriate not to adopt the terminology *singular strata* in those instances.

Definition 4.2 (Interior and Boundary Regularity Points). *Let $(X,\mathsf{d},\mathcal{H}^N)$ be an $\mathrm{RCD}(K,N)$ space and let $E\subset X$ be a locally perimeter minimizing set in the sense of Definition 2.10. Given $x\in\partial E$, we say that x is an* interior regularity *point if*

$$\mathrm{Tan}_x(X,\mathsf{d},\mathcal{H}^N,E,x)=\{(\mathbb{R}^N,\mathsf{d}_{\mathrm{eucl}},\mathcal{H}^N,\mathbb{R}^N_+,0)\}\,. \tag{4.2}$$

The set of interior regularity points of E will be denoted by $\mathcal{R}^E$.
Given $x\in\partial E$, we say that x is a boundary regularity *point if*

$$\mathrm{Tan}_x(X,\mathsf{d},\mathcal{H}^N,E,x)=\{(\mathbb{R}^N_+,\mathsf{d}_{\mathrm{eucl}},\mathcal{H}^N,\{x_1\ge 0\},0)\}\,, \tag{4.3}$$

where x_1 is one of the coordinates of the $\mathbb{R}^{N-1}$ factor in $\mathbb{R}^N_+=\mathbb{R}^{N-1}\times\{x_N\ge 0\}$. The set of boundary regularity points of E will be denoted by $\mathcal{R}^E_{\partial X}$.

It was proved in [48] that the interior regular set $\mathcal{R}^E$ is topologically regular, in the sense that it is contained in a Hölder open manifold of dimension $N-1$. By a blow-up argument, in the next proposition, we show that $\dim_{\mathcal{H}}\mathcal{R}^E_{\partial X}\le N-2$.

Proposition 4.3. *Let $(X,\mathsf{d},\mathcal{H}^N)$ be an $\mathrm{RCD}(K,N)$ space. Let $E\subset X$ be a locally perimeter minimizing set and let $\mathcal{R}^E_{\partial X}$ be the set of boundary regularity points of E, in the sense of Definition 4.2. Then*

$$\dim_{\mathcal{H}}\mathcal{R}^E_{\partial X}\le N-2\,. \tag{4.4}$$

Proof. We argue by contradiction. Assume there exists $k > N-2$, $k \in \mathbb{R}$ such that

$$\mathcal{H}^k\left(\mathcal{R}^E_{\partial X}\right) > 0\,. \tag{4.5}$$

Let $\varepsilon > 0$. We define the quantitative ε-singular set to be

$$S^\varepsilon(E) := \{x \in X :\ \mathcal{D}((B^X_r(x), \mathsf{d}, \mathcal{H}^N, E, x), (B^{\mathbb{R}^N}_r, \mathsf{d}_{\rm eucl}, \mathbb{R}^N_+, 0)) \geq \varepsilon r,\ \text{ for all } r \in (0, \varepsilon)\}\,. \tag{4.6}$$

Recall that the distance $\mathcal{D}$ was introduced in [3, Definition A.3]. Notice that $S^{\varepsilon_1}(E) \subset S^{\varepsilon_2}(E)$ for $0 < \varepsilon_1 \leq \varepsilon_2$ and that

$$\partial E \setminus \mathcal{R}^E = \bigcup_{n \in \mathbb{N}} S^{\varepsilon_n}(E), \tag{4.7}$$

for any sequence $\varepsilon_n \downarrow 0$. It is also clear that

$$\mathcal{R}^E_{\partial X} \subset \partial E \setminus \mathcal{R}^E. \tag{4.8}$$

The combination of (4.5), (4.7) and (4.8) implies that there exists $\bar\varepsilon > 0$ such that

$$\mathcal{H}^k\left(S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X}\right) > 0\,. \tag{4.9}$$

By [30, Theorem 2.10.17], there exists $x \in S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X}$ such that

$$\limsup_{r \to 0} \frac{\mathcal{H}^k_\infty\left(B_r(x) \cap S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X}\right)}{r^k} \geq 2^k C_k\,, \tag{4.10}$$

where we denoted by $\mathcal{H}^k_\infty$ the k-dimensional ∞-pre-Hausdorff measure.

By the very definition of $\mathcal{R}^E_{\partial X}$, for every sequence $r_i \searrow 0$, $E \subset (X, \mathsf{d}/r_i, \mathcal{H}^N/r_i^N, x)$ converges in the sense of Definition 2.6 to a quadrant $\{x_1 \geq 0\}$, where x_1 is one of the coordinates of the $\mathbb{R}^{N-1}$ factor in $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \{x_N \geq 0\}$.

Embedding the sequence of rescaled spaces X_i and their limit $\mathbb{R}^N_+$ into a proper realization of the pGH-convergence, by Blaschke's theorem (cf. [20, Theorem 7.3.8]) there exist a compact set $A \subset \mathbb{R}^N_+$ and a subsequence, which we do not relabel, such that $S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X} \cap B^i_1(x)$ converges to A in the Hausdorff sense.
Moreover, it is elementary to check that $A \subset S^{\bar\varepsilon}(\{x_1 \geq 0\})$ in $\mathbb{R}^N_+$. Therefore, we obtain

$$\begin{aligned}\mathcal{H}^k_\infty\left(S^{\bar\varepsilon}(\{x_1 \geq 0\})\right) &\geq \mathcal{H}^k_\infty\left(A\right) \geq \limsup_{i \to \infty} \mathcal{H}^k_\infty\left(S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X} \cap B^i_1(x)\right) \\ &= \limsup_{i \to \infty} \frac{\mathcal{H}^k_\infty\left(B_{r_i}(x) \cap S^{\bar\varepsilon}(E) \cap \mathcal{R}^E_{\partial X}\right)}{r_i^k} > 0\,,\end{aligned} \tag{4.11}$$

where we relied on the classical upper semicontinuity of the pre-Hausdorff measure with respect to Hausdorff convergence in the second inequality and on (4.10) in the last one. However, it is easy to check that $S^{\bar\varepsilon}(\{x_1 \geq 0\}) = \{x_1 = x_N = 0\}$ which has Hausdorff co-dimension 2, contradicting (4.11). □

Our main results about the stratification of the singular set for perimeter minimizers are that the complement of $\mathcal{S}^E_{N-3}$ in ∂E consists of either interior or boundary regularity points, and that the classical Hausdorff dimension estimate $\dim(\mathcal{S}^E_k) \leq k$ holds for any $0 \leq k \leq N-3$. Below are the precise statements.

Theorem 4.4. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an* RCD(K,N) *space and let $E \subset X$ be a locally perimeter minimizing set in the sense of Definition 2.10. Then*

$$\partial E \setminus \mathcal{S}^E_{N-3} = \mathcal{R}^E \cup \mathcal{R}^E_{\partial X}\,. \tag{4.12}$$

Theorem 4.5 (Stratification of the singular set)**.** *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an* $\mathrm{RCD}(K, N)$ *space and $E \subset X$ a locally perimeter minimizing set. Then, for any $0 \le k \le N-3$ it holds*

$$\dim_{\mathcal{H}} \mathcal{S}_k^E \le k\,. \tag{4.13}$$

Another application of the monotonicity formula with the associated rigidity is that if an $\mathrm{RCD}(0, N)$ space $(X, \mathsf{d}, \mathcal{H}^N)$ with Euclidean volume growth contains a global perimeter minimizer, then any asymptotic cone contains a perimeter minimizing cone.

Theorem 4.6. *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an* $\mathrm{RCD}(0, N)$ *metric measure space with Euclidean volume growth, i.e. satisfying for some (and thus for every) $x \in X$:*

$$\liminf_{r\to\infty} \frac{\mathcal{H}^N(B_r(x))}{r^N} > 0. \tag{4.14}$$

Let $E \subset X$ be a global perimeter minimizer in the sense of Definition 2.10. Then for any blow-down $(C(Z), \mathsf{d}_{C(Z)}, \mathcal{H}^N)$ of $(X, \mathsf{d}, \mathcal{H}^N)$ there exists a cone $C(W) \subset C(Z)$ global perimeter minimizer.

Remark 4.7. *The conclusion of Theorem 4.6 above seems to be new also in the more classical case of smooth Riemannian manifolds with nonegative sectional curvature, or nonnegative Ricci curvature. We refer to [11] for earlier progress in the case of smooth manifolds with nonnegative sectional curvature satisfying additional conditions on the rate of convergence to the tangent cone at infinity and on the regularity of the cross section and to [28] for the case of smooth Riemannian manifolds with nonnegative Ricci curvature and quadratic curvature decay.*

Proof of Theorem 4.4. Let us consider a point $x \in \partial E \setminus \mathcal{S}^E_{N-3}$. By the very definition of the singular stratum $\mathcal{S}^E_{N-3}$, there exists a tangent space to $(X, \mathsf{d}, \mathcal{H}^N, E, x)$ at x of the form $(\mathbb{R}^{N-2} \times Z, \mathsf{d}_{\mathrm{eucl}} \times \mathsf{d}_Z, \mathcal{H}^N, y, G \times \mathbb{R}^{N-2})$, where $(Z, \mathsf{d}_Z, \mathcal{H}^2)$ is an $\mathrm{RCD}(0, 2)$ metric measure cone (because all tangent cones to any $\mathrm{RCD}(K, N)$ space $(X, \mathsf{d}, \mathcal{H}^N)$ are metric measure cones [26]) and $G \subset Z$ is a globally perimeter minimizing set (in the sense of Definition 2.10) thanks to [48, Theorem 2.42].

By Lemma 4.8 there are only two options. Either x is an interior point and a tangent space is $(\mathbb{R}^N, \mathsf{d}_{\mathrm{eucl}}, \mathcal{H}^N, \mathbb{R}^N_+, 0)$, or x is a boundary point and a tangent space is $(\mathbb{R}^N_+, \mathsf{d}_{\mathrm{eucl}}, \mathcal{H}^N, \{x_1 \ge 0\}, 0)$.
In the first case, it was shown in [48] that the tangent space at x is unique and hence $x \in \mathcal{R}^E$. If the second possibility occurs, then by [16] we infer that the tangent cone to the ambient space $(X, \mathsf{d}, \mathcal{H}^N)$ is unique. The uniqueness of the tangent cone to the set of finite perimeter can be obtained with an argument completely analogous to the one used for interior points in [48], building on top of the classical boundary regularity theory (cf. for instance with [39]) instead of the classical interior regularity theory for perimeter minimizers in the Euclidean setting. Hence $x \in \mathcal{R}^E_{\partial X}$ is a boundary regularity point. □

Proof of Theorem 4.5. We argue by contradiction via Federer's dimension reduction argument. The proof is divided into four steps. In the first step we set up the contradiction argument and reduce to the case of entire perimeter minimizers inside $\mathrm{RCD}(0, N)$ metric measure cones. In the second step we make a further reduction to the case when the perimeter minimizer is a cone itself, building on top of Theorem 3.1. Via additional blow-up arguments we gain a splitting for the ambient space and for the perimeter minimizing set in step three, thus performing a dimension reduction. The argument is completed in step four. A key subtlety with respect to more classical situations is that the monotonicity formula holds only for perimeter minimizers centered at vertexes of metric measure cones, resulting into the necessity of iterating the blow-ups.

Step 1.
We argue by contradiction. Suppose that the statement does not hold for some $0 \le k \le N-3$. Then, there exists $k' > k$, $k' \in \mathbb{R}$ such that

$$\mathcal{H}^{k'}\left(\mathcal{S}^E_k\right) > 0\,. \tag{4.15}$$

Let $\varepsilon > 0$. We define the quantitative (k,ε)-singular stratum to be

$$\begin{aligned} S^E_{k,\varepsilon} := \{x \in X : \mathcal{D}((B^X_r(x), \mathsf{d}, \mathcal{H}^N, E, x), (B^{\mathbb{R}^{k+1}\times Z}_r, \mathsf{d}_{\mathrm{eucl}} \times \mathsf{d}_Z, F, (0,z))) \ge \varepsilon r \\ \text{for all } r \in (0,\varepsilon),\ (Z, \mathsf{d}_Z, z) \text{ pointed spaces and} \\ F = \mathbb{R}^{k+1} \times G \text{ with } G \subset Z \text{ global perimeter minimizer}\}\,. \end{aligned} \tag{4.16}$$

Recall that the distance $\mathcal{D}$ was introduced in [3, Definition A.3]. Moreover, we notice that $S^E_{k,\varepsilon_1} \subset S^E_{k,\varepsilon_2}$ for $0 < \varepsilon_1 \le \varepsilon_2$ and that $S^E_k = \bigcup_{n\in\mathbb{N}} S^E_{k,\varepsilon_n}$, for any sequence $\varepsilon_n \downarrow 0$.

The contradiction assumption (4.15) implies that there exists $\bar\varepsilon > 0$ such that

$$\mathcal{H}^{k'}\left(S^E_{k,\bar\varepsilon}\right) > 0\,. \tag{4.17}$$

By [30, Theorem 2.10.17], there exists $x \in S^E_{k,\bar\varepsilon}$ such that

$$\limsup_{r\to 0} \frac{\mathcal{H}^{k'}_\infty\left(B_r(x) \cap S^E_{k,\bar\varepsilon}\right)}{r^{k'}} \ge 2^{k'} C_{k'}\,, \tag{4.18}$$

where we denoted by $\mathcal{H}^{k'}_\infty$ the k'-dimensional ∞-pre-Hausdorff measure.

Then there exists a sequence $r_i \searrow 0$ such that $E \subset (X, \mathsf{d}/r_i, \mathcal{H}^N/r_i^N, x)$ converges in the sense of Definition 2.6 to a global perimeter minimizer $F \subset (C(Z), \mathsf{d}_C, \mathcal{H}^N)$, in the sense of Definition 2.10. Here we used [3, Corollary 3.4] in combination with Lemma 2.11 for the compactness, [48, Theorem 2.42] for the perimeter minimality of F and [26] to infer that the ambient tangent space is a cone. Here $(Z, \mathsf{d}_Z, \mathcal{H}^{N-1})$ is an $\mathrm{RCD}(N-2, N-1)$ metric measure space.

Embedding the sequence of rescaled spaces X_i and their limit $C(Z)$ into a proper realization of the pGH-convergence, by Blaschke's theorem (cf. [20, Theorem 7.3.8]) there exist a compact set $A \subset C(Z)$ and a subsequence, which we do not relabel, such that $S^E_{k,\bar\varepsilon} \cap B^i_1(x)$ converges to A in the Hausdorff sense.
Moreover, it is elementary to check that $A \subset S^F_{k,\bar\varepsilon}$. Therefore, we obtain

$$\begin{aligned} \mathcal{H}^{k'}_\infty\left(S^F_{k,\bar\varepsilon}\right) \ge \mathcal{H}^{k'}_\infty(A) \ge \limsup_{i\to\infty} \mathcal{H}^{k'}_\infty\left(S^E_{k,\bar\varepsilon} \cap B^i_1(x)\right) \\ = \limsup_{i\to\infty} \frac{\mathcal{H}^{k'}_\infty\left(B_{r_i}(x) \cap S^E_{k,\bar\varepsilon}\right)}{r_i^{k'}} > 0\,, \end{aligned} \tag{4.19}$$

where we relied on the classical upper semicontinuity of the pre-Hausdorff measure with respect to Hausdorff convergence in the second inequality and on (4.18) in the last one.

Lastly, (4.19) implies that

$$\mathcal{H}^{k'}\left(B^{C(Z)}_1 \cap S^F_{k,\bar\varepsilon}\right) > 0\,. \tag{4.20}$$

Step 2.
In this step, by performing a second blow up, we apply Theorem 3.1 to show that we can also suppose that the global perimeter minimizer is a cone (with respect to a vertex of the ambient cone). For the sake of clarity, we recall that the set of vertexes of $C(Z)$ is the collection of all points $y \in C(Z)$ such that $C(Z)$ is a metric cone centered at y. Moreover, we remark that the set of vertexes is isometric to $\mathbb{R}^k$ for some $0 \le k \le N$.

We claim that there is a point $O \in C(Z)$ such that O is a vertex of $C(Z)$ and the following density estimate holds:

$$\limsup_{r\to 0} \frac{\mathcal{H}^{k'}_{\infty}\left(B_r(O) \cap S^F_{k,\bar{\varepsilon}}\right)}{r^{k'}} \ge 2^{k'} C_{k'} \,. \tag{4.21}$$

If the claim does not hold, then by (4.20) there are points of density for $\mathcal{H}^{k'}_{\infty}$ restricted to $S^F_{k,\bar{\varepsilon}}$ and none of them belongs to the set of vertexes of $C(Z)$. Hence we can repeat the argument in step 1, blowing up at a density point for $\mathcal{H}^{k'}_{\infty}$ restricted to $S^F_{k,\bar{\varepsilon}}$ which is not a vertex in the ambient cone. In this way, the dimension of the set of vertexes of the ambient space, which is isometric to a Euclidean space, increases at least by one.
The procedure can be iterated until one of the following two possibilities occurs: the ambient is isometric to $\mathbb{R}^N$, with standard structure, in which case (4.20) contradicts the classical regularity theory, or there is a density point for $\mathcal{H}^{k'}_{\infty}$ restricted to $S^F_{k,\bar{\varepsilon}}$ which is also a vertex of $C(Z)$.

Let now O denote any such vertex of $C(Z)$. By Theorem 3.1 and the density estimates in Lemma 2.11, the map

$$r \mapsto \frac{\mathrm{Per}(F; B_r(O))}{r^{N-1}} \tag{4.22}$$

is monotone non-decreasing, bounded and bounded away from 0. Therefore, there exists the limit

$$0 < a := \lim_{r\to 0} \frac{\mathrm{Per}(F; B_r(O))}{r^{N-1}} < \infty \,. \tag{4.23}$$

We perform a second blow up at the tip $O \in C(Z)$ and obtain a global perimeter minimizer $G \subset C(Z)$. By (4.23) and Theorem 3.1, G is a cone. Moreover, by repeating the arguments in step 1, taking into account that O was chosen to be a density point for $\mathcal{H}^{k'}_{\infty}$ restricted to $S^F_{k,\bar{\varepsilon}}$, it holds

$$\mathcal{H}^{k'}\left(S^G_{k,\bar{\varepsilon}}\right) > 0 \,. \tag{4.24}$$

It follows from (4.24) that there exists a point in $S^G_{k,\bar{\varepsilon}} \setminus \{O\}$.

Step 3.
The goal of this step is to gain a splitting for the ambient and the perimeter minimizer set by considering a blow-up of G at a density point for $\mathcal{H}^{k'}_{\infty}$ restricted to $S^G_{k,\bar{\varepsilon}}$ that is not a vertex. Roughly speaking, we will achieve this by showing that the unit normal of the blow-up is everywhere perpendicular to the gradient of a splitting function obtained with the help of Lemma 4.9 below, cf. [45, Lemma 28.13].

Our setup is that $G \subset C(Z)$ is a globally perimeter minimizing cone with vertex O, a vertex of the ambient cone. Moreover, $\mathcal{H}^{k'}(S^G_{k,\bar{\varepsilon}}) > 0$. In particular, by the very same arguments as in Step 1, there exist a point $O' \in C(Z)$, $O' \neq O$ and a sequence $r_i \downarrow 0$ such that

$$\lim_{i\to\infty} \frac{\mathcal{H}^{k'}_{\infty}\left(B_{r_i}(O') \cap S^G_{k,\bar{\varepsilon}}\right)}{r_i^{k'}} \ge 2^{k'} C_{k'} \,. \tag{4.25}$$

Up to taking a subsequence that we do not relabel, we can assume that the sequence $(C(Z), \mathsf{d}_C/r_i, \mathcal{H}^N, O', G)$ converges to $(C(Z'), \mathsf{d}_{C'}, \mathcal{H}^N, O'', H)$, where $(C(Z'), \mathsf{d}_{C'}, \mathcal{H}^N)$ is an $\mathrm{RCD}(0,N)$ metric measure cone splitting an additional $\mathbb{R}$ factor with respect to $C(Z)$ and $H \subset C(Z')$ is a global perimeter minimizer. Moreover,

$$\mathcal{H}^{k'}\left(B_1(O'') \cap S^H_{k,\bar{\varepsilon}}\right) > 0\,. \tag{4.26}$$

Consider the sequence of functions $f_i : C(Z) \to \mathbb{R}$ defined as

$$f_i(z) := \frac{\mathsf{d}^2_C(O,z) - \mathsf{d}^2_C(O,O')}{r_i}\,, \tag{4.27}$$

that we view as functions on the rescaled metric measure space $(C(Z), \mathsf{d}_C/r_i, \mathcal{H}^N, O')$.
By Lemma 4.9 below, the functions f_i converge to some splitting function $g : C(Z') \to \mathbb{R}$ in $H^{1,2}_{\mathrm{loc}}$, see [8] for the relevant background. Moreover Δf_i converge to 0 uniformly.

We claim that, for any function $\varphi \in \mathrm{LIP}(C(Z')) \cap W^{1,2}(C(Z'))$ it holds

$$\int_H \nabla\varphi \cdot \nabla g \,\mathrm{d}\mathcal{H}^N = 0\,. \tag{4.28}$$

To see this, let $\varphi_i \in \mathrm{LIP}(X_i) \cap W^{1,2}(X_i)$ converging $H^{1,2}$-strongly to φ along the sequence $(C(Z), \mathsf{d}_C/r_i, \mathcal{H}^N, O')$, whose existence was shown in [8]. Then, using the Gauss-Green formula (Theorem 2.8) and the characterization of cones in Lemma 3.3 we obtain

$$\begin{aligned} 0 &= \int_{\partial^* G} \varphi_i \nu^i_G \cdot \nabla_i f_i \,\mathrm{dPer}_i(G) \\ &= -\int_G \nabla_i\varphi_i \cdot \nabla_i f_i \,\mathrm{d}\mathcal{H}^N - \int_G \varphi_i \cdot \Delta_i f_i \,\mathrm{d}\mathcal{H}^N\,, \end{aligned} \tag{4.29}$$

where the Hausdorff measure $\mathcal{H}^N$ is computed with respect to the rescaled distance d_C/r_i.
By (4.46) below and (4.29)

$$\int_G \nabla_i\varphi_i \cdot \nabla_i f_i \,\mathrm{d}\mathcal{H}^N = -\int_G \varphi_i \cdot \Delta_i f_i \,\mathrm{d}\mathcal{H}^N \to 0\,. \tag{4.30}$$

On the other hand, by [8, Theorem 5.7], it follows that

$$\int_G \nabla_i\varphi_i \cdot \nabla_i f_i \,\mathrm{d}\mathcal{H}^N \to \int_H \nabla\varphi \cdot \nabla g \,\mathrm{d}\mathcal{H}^N\,. \tag{4.31}$$

Combining (4.30) and (4.31) we obtain (4.28); cf. [18, 17] for analogous arguments.

Our next goal is to use (4.28) to show that the perimeter minimizer H splits a line in the direction of the ambient splitting induced by the splitting function g.

Let us set $Y := C(Z') = \mathbb{R} \times Y'$, and assume that $\mathbb{R}$ is the splitting induced by g.

Given any $\varphi \in W^{1,2}_{\mathrm{loc}}(Y)$ let us also denote $\varphi^{(t)}(y) := \varphi(t,y)$ and $\varphi^{(y)}(t) := \varphi(t,y)$. If $\varphi \in W^{1,2}_{\mathrm{loc}}(Y)$, then $\varphi^{(t)} \in W^{1,2}_{\mathrm{loc}}(Y')$ and $\varphi^{(y)} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$, for $\mathcal{L}^1$-a.e. t and $\mathcal{H}^{N-1}$-a.e. y respectively (see [34]). Up to the isomorphism given by the splitting induced by g, there holds

$$\nabla\varphi \cdot \nabla g(t,y) = \partial_t \varphi^{(y)}(t)\,, \quad \text{for } \mathcal{H}^N\text{-a.e. } (t,y) \in Y\,.$$

Let P_s denote the heat flow on Y. Then

$$\begin{aligned}\int_Y \mathrm{P}_s\chi_H(t,y)\nabla\varphi\cdot\nabla g(t,y)\,\mathrm{d}\mathcal{H}^N &= \int_Y \mathrm{P}_s\chi_H(t,y)\partial_t\varphi^{(y)}(t)\,\mathrm{d}\mathcal{H}^N \\ &= \int_H \mathrm{P}_s\partial_t\varphi^{(y)}(t)\,\mathrm{d}\mathcal{H}^N = \int_H \partial_t(\mathrm{P}_s\varphi)^{(y)}(t)\,,\end{aligned} \tag{4.32}$$

where in the second equality we have used the self-adjointess of the heat flow and in the last equality we have used Lemma 4.10.

By (4.28) and (4.32) it follows that

$$\int_Y \mathrm{P}_s\chi_H(t,y)\nabla\varphi\cdot\nabla g(t,y)\,\mathrm{d}\mathcal{H}^N = \int_H \nabla(\mathrm{P}_s\varphi)\cdot\nabla g\,\mathrm{d}\mathcal{H}^N = 0\,. \tag{4.33}$$

Since $\varphi\in\mathrm{LIP}(Y)\cap W^{1,2}(Y)$ is arbitrary, an elementary computation using Fubini's theorem and the splitting $Y=\mathbb{R}\times Y'$ shows that

$$\partial_t(\mathrm{P}_s\chi_H)^{(y)}(t)=0\quad\text{for } \mathcal{L}^1\text{-a.e. } t\in\mathbb{R},\text{ for } \mathcal{H}^{N-1}\text{-a.e. } y\in Y'\,.$$

By the $L^1_{\mathrm{loc}}(Y)$ convergence of $\mathrm{P}_s\chi_H$ to χ_H for $s\downarrow 0$ and the closure of H, we conclude that $\chi_H^{(y)}$ is constant in t for every $y\in Y'$. That implies the existence of a set $H'\subset Y'$ such that

$$\chi_H(t,y)=\chi_{H'}(y)\,. \tag{4.34}$$

By Lemma 4.11, $H'\subset Y'$ is a set of locally finite perimeter.
Let us show that H' is a global perimeter minimizer, by following the classical Euclidean argument, cf. [45, Lemma 28.13]. Suppose not. Then there exist $\varepsilon>0$ and a set $H_0'\subset Y'$ such that $H'\Delta H_0'\subset\subset B_r(y)$ for some $r>0$ and $y\in Y'$, such that

$$\mathrm{Per}(H_0';B_r(y))+\varepsilon\le\mathrm{Per}(H';B_r(y))\,. \tag{4.35}$$

Let $t>0$. We define the sets

$$I_t:=\mathbb{R}\setminus(-t,t)\qquad\qquad H_0:=(H_0'\times(-t,t))\cup(H'\times I_t)\,. \tag{4.36}$$

At this stage, we can use the formulas for the cut and paste of sets of finite perimeter (Theorem 2.9), observe that $H\Delta H_0\subset B_r(y)\times(-t,t):=A$ and conclude by Lemma 4.11 that

$$\begin{aligned}\mathrm{Per}(H_0;A)-\mathrm{Per}(H;A) &= 2t(\mathrm{Per}(H_0';B_r(y))-\mathrm{Per}(H;B_r(y)))+2\mathcal{H}^{N-1}(H_0'\Delta H')\\ &\le -2t\varepsilon+2\mathcal{H}^{N-1}(B_r(y))<0\,,\end{aligned}$$

where we have chosen $t>0$ large enough so that $\mathcal{H}^{N-1}(B_r(y))<t\varepsilon$.
Therefore, H' is a global perimeter minimizer, as we claimed.

If $k=0$, the above argument leads to a contradiction. Indeed we found a point in $\mathcal{S}_0^H$ such that some tangent space splits a line.

Step 4.
If $k>0$, then it is straightforward to see that $(t,y)\in S^H_{k,\bar\varepsilon}$ if and only if $y\in S^{H'}_{k-1,\bar\varepsilon}$.
In particular, from the assumption that

$$\mathcal{H}^{k'}\left(B_1(O'')\cap S^H_{k,\bar\varepsilon}\right)>0\,, \tag{4.37}$$

we conclude that

$$\mathcal{H}^{k'-1}\left(B_1(O'')\cap S^{H'}_{k-1,\bar\varepsilon}\right)>0\,. \tag{4.38}$$

Therefore the steps from 1 to 3 prove that if there exist an $\mathrm{RCD}(K,N)$ metric measure space $(X,\mathsf{d},\mathcal{H}^N)$ and locally perimeter minimizing set $E\subset X$ such that for some $0\le k\le N-3$ it holds $\dim_{\mathcal{H}}(\mathcal{S}^E_k)>k$, then there exist an $\mathrm{RCD}(0,N-1)$ space $(X',\mathsf{d}',\mathcal{H}^{N-1})$ and a locally perimeter minimizing set $E'\subset X'$ such that $\dim_{\mathcal{H}}(\mathcal{S}^{E'}_{k-1})>k-1$. The dimension reduction can be iterated a finite number of times until we reduce to the case $k=0$, that we already discussed above. □

Proof of Theorem 4.6. First of all, up to modifying E on a set of measure zero if necessary, we can (and will) assume that E is open.

Step 1. Fix a point $x\in\partial E$. We claim that there exists $C>1$ such that

$$\frac{r^N}{C}\le\mathcal{H}^N(E\cap B_r(x))\le C\,r^N,\quad\text{for all } r>0, \tag{4.39}$$

$$\frac{r^{N-1}}{C}\le\mathrm{Per}(E;B_r(x))\le C\,r^{N-1},\quad\text{for all } r>0. \tag{4.40}$$

Recall that an $\mathrm{RCD}(0,N)$ space is globally doubling (thanks to the Bishop-Gromov inequality [53]) and satisfies a global Poincaré inequality [51]. Since, by assumption, E mimimizes the perimeter on every metric ball then, by [42, Theorem 4.2], there exists a constant $\gamma_0>0$ (depending only on the doubling and Poincaré constants of $(X,\mathsf{d},\mathcal{H}^N)$) such that

$$\frac{\mathcal{H}^N(E\cap B_r(x))}{\mathcal{H}^N(B_r(x))}\ge\gamma_0\quad\text{and}\quad\frac{\mathcal{H}^N(B_r(x)\setminus E)}{\mathcal{H}^N(B_r(x))}\ge\gamma_0\quad\text{for all } r>0 \text{ and } x\in\partial E. \tag{4.41}$$

Recall that the ratio $\mathcal{H}^N(B_r(x))/r^N$ is monotone non-increasing by Bishop-Gromov inequality, it is bounded above by the value in $\mathbb{R}^N$ and it is bounded below by a positive constant thanks to the assumption (4.14). Hence (4.39) follows from (4.41). The perimeter estimate (4.40) follows from (4.39) and [42, Lemma 5.1].

Step 2. The argument is similar to those involved in the proof of Theorem 4.5 above and therefore we only sketch it. Let $r_i\to\infty$ be any sequence such that $(X,\mathsf{d}/r_i,\mathcal{H}^N,x)$ converges to a tangent cone at infinity $(C(Z),\mathsf{d}_{C(Z)},\mathcal{H}^N,O)$ of $(X,\mathsf{d},\mathcal{H}^N)$. By the Ahlfors regularity estimates (4.39)-(4.40) and the compactness and stability [48, Theorem 2.42], the sequence $(X,\mathsf{d}/r_i,\mathcal{H}^N,E,x)$ converges to $(C(Z),\mathsf{d}_{C(Z)},\mathcal{H}^N,F,O)$ for some non-empty perimeter minimizer $F\subset C(Z)$. At this stage, we are in position to apply Theorem 3.1 and obtain a perimeter minimizing cone in $C(Z)$, up to possibly taking an additional blow-down. □

In the remainder of the section, we present some technical results that have been used in the proof of Theorem 4.5.

Lemma 4.8. *Let $(Z,\mathsf{d}_Z,\mathcal{H}^2)$ be an $\mathrm{RCD}(0,2)$ metric measure cone and let $G\subset Z$ be a globally perimeter minimizing set, in the sense of Definition 2.10. Then one of the following two possibilities occur:*

i) $(Z,\mathsf{d}_Z,\mathcal{H}^2)$ is isomorphic to $(\mathbb{R}^2,\mathsf{d}_{\mathrm{eucl}},\mathcal{H}^2)$ and G is a half-plane;

ii) $(Z,\mathsf{d}_Z,\mathcal{H}^2)$ is isomorphic to the half-plane $(\mathbb{R}^2_+,\mathsf{d}_{\mathrm{eucl}},\mathcal{H}^2)$ and G is a quadrant.

Proof. We distinguish two cases: if Z has no boundary, then we prove that it is isometric to $\mathbb{R}^2$ and i) must occur; if Z has non empty boundary, then we prove that it is isometric to $\mathbb{R}^2_+$ and that ii) must occur.

Let us assume that $(Z, \mathsf{d}_Z, \mathcal{H}^2)$ has empty boundary. Then, by [43], Z is isometric to a cone over $S^1(r)$ for some $0 < r \le 1$. Moreover, by Theorem 4.6 there exists a blow-down of G which is a global perimeter minimizing cone $C(A)$, with vertex in the origin and $A \subset S^1(r)$ connected. Indeed, it is elementary to check that if A is not connected, then $C(A)$ is not locally perimeter minimizing.

Let $2\pi r\theta$ be the length of A, where $0 < \theta < 1$. Let $G' \subset Z$ be a set of finite perimeter such that $G' = G$ outside B_1 and $\partial G \cap B_1$ is composed by the geodesic connecting the two points in $\partial G \cap \partial B_1 = \{x_1, x_2\}$. Such geodesic is contained in B_1 as can be verified through the explicit form of the metric. Using (2.1)

$$\begin{aligned} \mathrm{Per}(G'; B_1) &= \sqrt{2(1 - \cos(\mathsf{d}_{Z'}(x_1, x_2) \wedge \pi))} \\ &\le 2 = \mathrm{Per}(G; B_1). \end{aligned} \tag{4.42}$$

Equality in (4.42) is achieved for $2\pi r\theta = \mathsf{d}_{S^1(r)}(x_1, x_2) \ge \pi$, that is for $1 \ge r\theta \ge \frac{1}{2}$. Let us notice, by symmetry of $S^1(r)$, that we may suppose that $\theta \le \frac{1}{2}$. Indeed, for every fixed θ, we may find a comparison set with perimeter equal to the one constructed above corresponding to $1 - \theta$. Hence equality is only achieved at $r = 1$, $\theta = \frac{1}{2}$, corresponding to the case where $Z = \mathbb{R}^2$ and $C(A)$ is a half space. Notice that once we have established that Z is isometric to $\mathbb{R}^2$, it is elementary that G must be a half-space.

In the case where Z has non empty boundary, by [43] again, Z is isometric to a cone over a segment $[0, l]$ for some $0 < l \le \pi$. The upper bound for the diameter is required in order for the cone to verify the $\mathrm{CD}(0, 2)$ condition. We claim that it must hold $l = \pi$.
As above, by Theorem 4.6, there exists a blow-down of G which is a global perimeter minimizing cone $C(A)$, with vertex in the origin and $A \subset [0, l]$ some set of finite perimeter. If A is not connected, then it is elementary to check that $C(A)$ is not globally perimeter minimizing. Notice also that the complement of a global perimeter minimizer is a global perimeter minimizer.
By minimality and symmetry we can suppose $G = C([0, l'])$, for some $0 < l' \le \frac{l}{2}$. By considering a suitably constructed competitor in B_1, let us show that the only possibility is that Z is a half-space and $C(A)$ is a quadrant. Consider the set G' coinciding with G outside of B_1 and whose boundary inside B_1 is the geodesic minimizing the distance between $\partial B_1 \cap \partial G$ and ∂Z. Then $\mathrm{Per}(G'; B_1) \le \mathrm{Per}(G; B_1)$, with equality achieved only if $l = \pi$ and $l' = \frac{\pi}{2}$. As above, once established that Z is isometric to $\mathbb{R}^2_+$, it is elementary to check that G must be a quadrant. $\square$

It is a standard fact that any blow-up of a cone centered at a point different from the vertex splits a line. For our purposes it is important to observe that the blow-up of the squared distance function from the vertex is indeed a splitting function in the blow-up of the cone.

Lemma 4.9. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(N-2, N-1)$ space and let $(C(X), \mathsf{d}_{C(X)}, \mathfrak{m}_{C(X)})$ be the metric measure cone over X, with vertex $O \in C(X)$. Fix $p \in C(X)$ with $p \ne O$. Let $r_i \downarrow 0$ and consider the sequence of rescaled spaces $Y_i := (C(X), \mathsf{d}_{C(X)}/r_i, \mathfrak{m}_{C(X)}/\mathfrak{m}(B_{r_i}(p)), p)$ converging in the pmGH topology to a tangent space Y of $C(X)$ at p. Then the functions*

$$f_i(\cdot) := \frac{\mathsf{d}^2_{C(X)}(O, \cdot) - \mathsf{d}^2_{C(X)}(O, P)}{r_i}, \tag{4.43}$$

viewed as functions $f_i : Y_i \to \mathbb{R}$, have Laplacians uniformly converging to 0 and converge in $H^{1,2}_{\mathrm{loc}}$ to a splitting function $g : Y \to \mathbb{R}$, up to the extraction of a subsequence.

Proof. Let us set

$$f(\cdot) := \mathsf{d}^2_{C(X)}(O,\cdot) - \mathsf{d}^2_{C(X)}(O,P)\,, \tag{4.44}$$

in order to ease the notation. On $C(X)$ it holds (see [25])

$$\Delta f = 2N\,, \quad |\nabla f(x)| = 2\mathsf{d}_{C(X)}(x,O)\,, \quad \text{for a.e. on } x \in C(X)\,. \tag{4.45}$$

By scaling, we obtain that

$$\Delta f_i = 2Nr_i\,, \quad |\nabla f(x)| = 2\mathsf{d}_{C(X)}(x,O)\,, \quad \text{for a.e. } x \in Y_i\,, \tag{4.46}$$

where it is understood that the Laplacian and the minimal relaxed gradient are computed with respect to the metric measure structure $(C(X), \mathsf{d}_{C(X)}/r_i, \mathfrak{m}_{C(X)}/\mathfrak{m}(B_{r_i}(p)), p)$. Notice that $x \mapsto 2\mathsf{d}_{C(X)}(x,O)$ is a $2r_i$-Lipschitz function on Y_i, by scaling.
Hence the functions $f_i : Y_i \to \mathbb{R}$ are locally uniformly Lipschitz, they satisfy $f_i(p) = 0$, and they have Laplacians uniformly converging to 0. Up to the extraction of a subsequence, thanks to a diagonal argument, we can assume that they converge locally uniformly and in $H^{1,2}_{\mathrm{loc}}$ to a function $g : Y \to \mathbb{R}$ in the domain of the local Laplacian, and that Δf_i converge to Δg locally weakly in L^2, thanks to [8, 7]. We claim that g is a splitting function on Y, which amounts to say that $\Delta g = 0$ and $|\nabla g|$ is constant almost everywhere and not 0.

The fact that $\Delta g = 0$ follows from the weak convergence of the Laplacians and the identity $\Delta f_i = 2Nr_i$ that we established above.
Analogously, employing the identity $|\nabla f_i(x)| = 2\mathsf{d}_{C(X)}(x,O)$ a.e. on Y_i, and the local $W^{1,2}$ convergence of f_i to g, it is immediate to check that $|\nabla g| = 2\mathsf{d}(\cdot,O)$ a.e. on Y. □

The next result relates the Heat flow on product spaces with one dimensional derivatives.

Lemma 4.10 (Heat flow and derivative in the splitting direction commute)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space and let $X \times \mathbb{R}$ be endowed with the standard product metric measure space structure. Let $\varphi \in W^{1,2}(X \times \mathbb{R})$. Then for every $s > 0$ it holds*

$$\mathrm{P}_s \partial_t \varphi(x,t) = \partial_t(\mathrm{P}_s\varphi)(x,t)\,, \tag{4.47}$$

for $\mathfrak{m}_X \otimes \mathcal{L}^1$-a.e. $(x,t) \in X \times \mathbb{R}$.

Proof. The statement follows from the tensorization of the Cheeger energy and of the heat flow for products of $\mathrm{RCD}(K,\infty)$ metric measure spaces, see for instance [5, 6], and from the classical commutation between derivative and heat semi-group on $\mathbb{R}$ endowed with the standard metric measure structure. □

It is a well known fact of the Euclidean theory (see for instance [45]) that the perimeter enjoys natural tensorization properties, when taking an isometric product by an $\mathbb{R}$ factor. The next lemma establishes the RCD counterpart of this useful property.

Lemma 4.11 (Perimeter of Cylinders)**.** *Let $(X, \mathsf{d}_X, \mathfrak{m}_X)$ be an $\mathrm{RCD}(K,N)$ space and let $F \subset X$ be a Borel set. Under these assumptions, $E := F \times \mathbb{R} \subset X \times \mathbb{R}$ is a set of locally finite perimeter (where the product $X \times \mathbb{R}$ is endowed with the standard product metric measure structure), if and only if $F \subset X$ is a set of locally finite perimeter. Moreover, for any open set $A \subset X$ and for any $R > 0$ it holds*

$$R\,\mathrm{Per}(F; A) = \mathrm{Per}(E; A \times [0,R])\,. \tag{4.48}$$

Proof. By the very definition of perimeter it holds

$$\mathrm{Per}(E, A \times [0,R]) = \inf_{(\varphi_i)_i} \left\{ \liminf_{i\to\infty} \int_0^R \int_A \mathrm{lip}\, \varphi_i(t,x)\, \mathrm{d}\mathfrak{m}_X\, \mathrm{d}t \right\} ,$$

where the infimum is taken over all sequences $(\varphi_i)_i \subset \mathrm{LIP}_{\mathrm{loc}}(A \times [0,R])$ such that $\varphi_i \to \chi_E$ in $L^1_{\mathrm{loc}}(A \times [0,R])$. We are going to prove (4.48) and the first part of the statement will follow immediately.

Step 1. Let us start by showing the inequality

$$R\, \mathrm{Per}(F; A) \geq \mathrm{Per}(E; A \times [0,R])\,. \tag{4.49}$$

Let $(\psi_i)_i \subset \mathrm{LIP}_{\mathrm{loc}}(A)$ be a competitor for the perimeter of F in A, i.e. $\psi_i \to \chi_F$ in $L^1_{\mathrm{loc}}(A, \mathfrak{m}_X)$ and all the functions ψ_i are locally Lipschitz. Define $\phi(t,x) := \psi(x)$ for $0 \leq t \leq R$ and $x \in A$. Then, by Fubini's Theorem, $\{\phi_i\}_i$ is a competitor for the perimeter of E in $A \times [0,R]$. Therefore,

$$\begin{aligned}
\mathrm{Per}(F; A) &= \inf_{(\psi_i)_i} \left\{ \liminf_{i\to\infty} \int_A \mathrm{lip}\, \psi_i(x)\, \mathrm{d}\mathfrak{m}_X \right\} \\
&= \frac{1}{R} \inf_{(\psi_i)_i} \left\{ \liminf_{i\to\infty} \int_0^R \int_A \mathrm{lip}\, \phi_i^{(t)}(x)\, \mathrm{d}\mathfrak{m}_X\, \mathrm{d}t \right\} \\
&\geq \frac{1}{R} \inf_{(\varphi_i)_i} \left\{ \liminf_{i\to\infty} \int_0^R \int_A \mathrm{lip}\, \varphi_i(t,x)\, \mathrm{d}\mathfrak{m}_X\, \mathrm{d}t \right\} = \frac{1}{R} \mathrm{Per}(E; A \times [0,R])\,,
\end{aligned}$$

where the inequality follows from the fact that, on the right hand side, we are taking the infimum over a larger class.

Step 2. We prove the opposite inequality in (4.48).
Let us fix $\varepsilon > 0$. There exists a sequence $(\varphi_i)_i \subset \mathrm{LIP}_{\mathrm{loc}}(A \times [0,R])$ with $\varphi_i \to \chi_E$ in $L^1_{\mathrm{loc}}(A \times [0,R])$ such that

$$\liminf_{i\to\infty} \int_0^R \int_A \mathrm{lip}\, \varphi_i(t,x)\, \mathrm{d}x\, \mathrm{d}t \leq \mathrm{Per}(E; A \times [0,R]) + \varepsilon\,. \tag{4.50}$$

It is straightforward to check that $\mathrm{lip}\, \varphi_i^{(t)}(x) \leq \mathrm{lip}\, \varphi_i(t,x)$ for every $(t,x) \in \mathbb{R} \times X$. Moreover, the sequence $(\varphi_i^{(t)})_i$ is a competitor for the variational definition of the perimeter of F in A for $\mathcal{L}^1$-almost every t, by the coarea formula. Therefore, by Fatou's lemma,

$$\begin{aligned}
R\, \mathrm{Per}(F; A) &\leq \int_0^R \liminf_{i\to\infty} \int_A \mathrm{lip}\, \varphi_i^{(t)}(x)\, \mathrm{d}\mathfrak{m}_X\, \mathrm{d}t \\
&\leq \liminf_{i\to\infty} \int_0^R \int_A \mathrm{lip}\, \varphi_i^{(t)}(x)\, \mathrm{d}\mathfrak{m}_X\, \mathrm{d}t \\
&\leq \liminf_{i\to\infty} \int_0^R \int_A \mathrm{lip}\, \varphi_i(t,x)\, \mathrm{d}\mathfrak{m}_X\, \mathrm{d}t \leq \mathrm{Per}(E; A \times [0,R]) + \varepsilon\,.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude. □

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