Non-local approximation of free-discontinuity problems with linear growth

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Abstract. We approximate, in the sense of Γ-convergence, free-discontinuity functionals with linear growth in the gradient by a sequence of non-local integral functionals depending on the average of the gradients on small balls. The result extends to higher dimension what we already proved in the one-dimensional case.

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1 Introduction

A number of variational problems recently under consideration involves integral functionals with "free discontinuities" (according to a terminology introduced in [20]): the variable function u is required to be smooth only outside a surface K, depending on u, and both u and K enter the structure of the functional. Hence, a typical form is:

$$F(u,K) = \int_{\Omega\setminus K} \phi(|\nabla u|)\,dx + \int_{K\cap\Omega} \vartheta(|u^+ - u^-|)\,d\mathcal{H}^{n-1}\,,$$

where Ω is an open subset of $\mathbb{R}^n$, K is a $(n-1)$-dimensional compact set, $|u^+ - u^-|$ is the jump of u across K, while ϕ and ϑ are given positive functions (and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure).

The natural weak formulation is obtained looking at K as the set of discontinuities of u, thus working in spaces of functions allowing hypersurfaces of discontinuities, such as the space $BV(\Omega)$ of functions of bounded variation.

The main difficulty in the actual minimization of F is the presence of the $(n-1)$-dimensional integral: the need of suitable approximations (leading to the convergence of minimum points) by means of more manageable functionals naturally arises. The method introduced in [12], when $\phi(t) = t^2$ and ϑ is constant, makes use of integral functionals whose density depends on the average of the gradient on small balls. Here we apply this model to the case of ϕ having linear growth at infinity.

The aforementioned weak form of F in $BV(\Omega)$ takes the form:

$$F(u) = \int_\Omega \phi(|\nabla u|)\,dx + \int_{S_u} \vartheta(|u^+ - u^-|)\,d\mathcal{H}^{n-1} + c_0|D^c u|(\Omega)\,, \tag{1.1}$$

where $Du = \nabla u\,dx + (u^+ - u^-)\,d\mathcal{H}^{n-1} + D^c u$ is the decomposition of the measure derivative of u in its absolutely continuous, jump and Cantor part, respectively, and S_u denotes the set of discontinuity points of u. Assuming that ϕ is convex and ϑ is concave, with

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = c_0 = \lim_{t\to 0^+} \frac{\vartheta(t)}{t}\,, \tag{1.2}$$

it turns out that F is lower semicontinuous with respect to the L^1-topology. Note that if ϕ has superlinear growth at infinity then $c_0 = +\infty$ and $F(u)$ is finite only if Du has no Cantor part (i.e. u belongs to the so-called space of *special functions with bounded variation*). The well-known Mumford-Shah functional falls within this case: $\int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S_u)$.

As pointed out in [12], it is not possible to obtain a variational approximation of F by usual integral functionals of the form

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(\nabla u)\,dx$$

on Sobolev spaces; indeed, passing to the lower semicontinuous envelopes, this would lead to a convex limit, which contrasts with the non-convexity of F.

Heuristic arguments suggest that to get around the difficulty we have to prevent the consideration or the optimality of approximation gradients which are "too large" (with respect to $1/\varepsilon$), or to prevent that the effect of "large" gradients is concentrated on "small" regions. Several approximation methods fit this requirements: see, e.g., the case where the functionals F_ε are restricted to finite elements spaces on regular triangulations of size ε ([9], [13], [23]); or the implicit constraint on the gradient through the addition of a higher order penalization ([1], [3], [22]); or the study of non-local models, where the effect of a "large" gradient is "spread" onto a set of size ε: this is the method which was first applied to the Mumford-Shah functional by Braides & Dal Maso in [12] (see also [11], [14], [15], [16], [18]), and that we follow in this paper for the case of linear growth. We also have to mention the Ambrosio & Tortorelli approximation (see [6] and [7]) of the Mumford-Shah functional via elliptic functionals, where an additional variable, say v, which approaches the characteristic function of the complement of the discontinuity set, is introduced.

A variant of this last method was studied in [25], [26] and [2] for functionals with linear growth in the gradient: the attempt is to unify the *curve evolution* method used in Computer Vision to detect boundaries, and the pre-processing of the image to provide an "edge-strength" function v, which indicates the likelihood of an object boundary being present at any point of the domain (compare with the additional variable v in the Ambrosio-Tortorelli functional). Indeed, in the method of shape recovery by curve evolution, we try to detect the boundaries as curves Γ where the image intensity gradient (hence v) is high; therefore, we apply gradient descent to the functional $\int_\Gamma (1-v)^2\, d\mathcal{H}^1$ (see [25] and [26]). Following OSHER & SETIAN [24] it is convenient to embed the initial curve in a surface (the graph of a function u) as a level curve, and to apply the evolution to the surface, so that all of its level curves evolve simultaneously. Hence the functional $\int_{-\infty}^{+\infty}\int_{\{u=c\}}(1-v)^2\, d\mathcal{H}^1\, dc$ is taken into account. By the coarea formula this is nothing but

$$\int_\Omega (1-v)^2|\nabla u|\, dx.$$

In [25] and [26] a new segmentation functional was proposed by inserting this term in place of the square-gradient term of the Ambrosio-Tortorelli functional. In [2] the limit functional is proved to be of the form (1.1), with ϕ satisfying (1.2).

Here we tackle the problem of the approximation of a functional F as in (1.1) with ϕ of linear growth, by the method of non-local functionals. More precisely, we consider the problem of the convergence of

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)| dy\right) dx$$

as $\varepsilon \to 0$ (here $B_\varepsilon(x)$ denotes the ball of centre x and radius ε). Unlike the one-dimensional case we dealt with in [21], here we restrict the study to a fixed integrand function f, independent of ε. In the limit (see Theorem 3.1) the bulk and Cantor parts are completely determined by the behaviour of f at 0 (namely, we get $f'(0)(|D^a u|(\Omega) + |D^c u|(\Omega))$ while the surface energy density can be explicitely computed as $\vartheta(s) = 2\int_0^1 f\left(\frac{\omega_{n-1}}{\omega_n} s(\sqrt{1-t^2})^{n-1}\right) dt$ (here ω_k denotes the volume of the k-dimensional ball in $\mathbb{R}^k$).

2 Notation and preliminaries

Let $n \geq 1$ be a fixed integer. The scalar product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y\rangle$ and the euclidean norm by $|x|$. The open ball with centre x and radius r is indicated by $B_r(x)$; the boundary of the unit ball $B_1(0)$ is denoted by S^{n-1}. The Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure of a Borel set $B \subseteq \mathbb{R}^n$ are denoted by $|B|$ (or $\mathcal{L}^n(B)$) and $\mathcal{H}^{n-1}(B)$, respectively. We use standard notation for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{1,p}(\Omega)$.

Functions of bounded variation. For a thorough treatment of BV functions we refer to [5]. Let Ω be an open subset of $\mathbb{R}^n$. We recall that the space $BV(\Omega)$ of real *functions of bounded variation* is the space of the functions $u \in L^1(\Omega)$ whose distributional derivative is representable by a measure in Ω, i.e.,

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \varphi\, dD_i u \qquad \text{for every } \varphi \in C_c^\infty(\Omega) \text{ and } i = 1, \dots, n$$

for some $\mathbb{R}^n$-valued measure $Du = (D_1 u, \dots, D_n u)$ on Ω. Clearly, the Sobolev space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$.

Let $u \in BV(\Omega)$. We say that u has *approximate limit* at $x \in \Omega$ if there exists $z \in \mathbb{R}$ such that

$$\lim_{\varrho \to 0^+} \fint_{B_\varrho(x)} |u(y) - z|\, dy = 0.$$

The set S_u where this property fails is called *approximate discontinuity set* of u. The vector z is uniquely determined for any point $x \in \Omega \setminus S_u$ and is called the *approximate limit* of u at x and denoted by $\tilde{u}(x)$.

We say that x is an *approximate jump point* of the function $u \in BV(\Omega)$ if there exist $a, b \in \mathbb{R}$ and $\nu \in \mathbb{R}^n$ with $|\nu| = 1$, such that $a \neq b$ and

$$\lim_{\varrho \to 0^+} \fint_{B^+_\varrho(x,\nu)} |u(y) - a|\, dy = 0, \qquad \lim_{\varrho \to 0^+} \fint_{B^-_\varrho(x,\nu)} |u(y) - b|\, dy = 0\,, \quad (2.1)$$

where $B^+_\varrho(x,\nu) = \{y \in B_\varrho(x) :\ \langle y - x, \nu\rangle > 0\}$ and $B^-_\varrho(x,\nu) = \{y \in B_\varrho(x) : \langle y - x, \nu\rangle < 0\}$. The set of approximate jump points of u is denoted by J_u. The triplet (a, b, ν), which turns out to be uniquely determined up to a permutation of a and b and a change of sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$. On $\Omega \setminus S_u$ we set $u^+ = u^- = \tilde{u}$.

Theorem 2.1 (Federer-Vol'pert) *For any $u \in BV(\Omega)$ the set S_u is countably $(n-1)$-rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$. Moreover, $Du \llcorner J_u = (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u$, and $\nu_u(x)$ gives the approximate normal direction to J_u for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$.*

For a function $u \in BV(\Omega)$ let $Du = D^a u + D^s u$ be the (Lebesgue) decomposition of Du into absolutely continuous and singular part. We denote by ∇u the density of $D^a u$; the measures

$$D^j u := D^s u \llcorner J_u\,, \qquad D^c u := D^s u \llcorner (\Omega \setminus S_u)$$

are called the *jump part* and the *Cantor part* of the derivative, respectively. Since Du vanishes on $\mathcal{H}^{n-1}$-negligible Borel sets (see [5], Lemma 3.76), from the Federer-Vol'pert Theorem we can write:

$$Du = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u + D^c u.$$

Finally, let us recall the following important compactness Theorem in BV (see Theorem 3.23 and Proposition 3.21 in [5]):

Theorem 2.2 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Every sequence (u_h) in $BV(\Omega)$ which is bounded in $BV(\Omega)$ admits a subsequence converging in $L^1(\Omega)$ to a function $u \in BV(\Omega)$.*

Spaces SBV and $GSBV$. Let Ω be an open subset of $\mathbb{R}^n$. We say that a function $u \in BV(\Omega)$ is a *special function of bounded variation* ($u \in SBV(\Omega)$) if $|D^c u|(\Omega) = 0$.

We say that a function $u \in L^1(\Omega)$ is a *generalized function of bounded variation* ($u \in GBV(\Omega)$) if $u^T := (-T) \vee u \wedge T$ belongs to $BV(\Omega)$ for every $T \geq 0$.

If $u \in GBV(\Omega)$, the function ∇u given by

$$\nabla u = \nabla u^T \qquad \mathcal{L}^n\text{-a.e. on } \{|u| \leq T\}, \quad (2.2)$$

turns out to be well-defined ([5], Theorem 4.34). Moreover, the set function $T \mapsto S_{u^T}$ is monotone increasing; therefore, if we set [1] $S_u = \bigcup_{T>0} S_{u^T}$, by the Federer-Vol'pert Theorem, for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ we can consider the functions of T:

$$(u^T)^-(x)\,, \qquad (u^T)^+(x)\,,$$

which turn out to be monotone; then the traces:

$$u^-(x) = \lim_{T\to+\infty}(u^T)^-(x)\,, \qquad u^+(x) = \lim_{T\to+\infty}(u^T)^+(x) \tag{2.3}$$

are well-defined for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$.

Finally, for a function $u \in GBV(\Omega)$, we define (see [5], Definition 4.33) $|D^c u|$ as the supremum (in the sense of measures) of $|D^c u^T|$ for $T > 0$. It can be proved that for any Borel subset B of Ω

$$|D^c u|(B) = \lim_{T\to+\infty} |D^c u^T|(B)\,. \tag{2.4}$$

Slicing. Let us now recall some basic properties of one-dimensional sections of BV functions; they will enter the so-called *slicing* methods to reduce to lower-dimensional statements (see, e.g., [4]). We first introduce some notation. Let $\xi \in S^{n-1}$, and let $\xi^\perp = \{y \in \mathbb{R}^n : \langle y, \xi\rangle = 0\}$ be the linear hyperplane orthogonal to ξ. If $y \in \xi^\perp$ and $E \subseteq \mathbb{R}^n$ we set $E_{\xi,y} = \{t \in \mathbb{R} : y + t\xi \in E\}$. Moreover, for any given function $u\colon \Omega \to \mathbb{R}$ we define $u_{\xi,y}\colon \Omega_{\xi,y} \to \mathbb{R}$ by $u_{\xi,y}(t) = u(y + t\xi)$. For the results collected in the following theorem see [5], § 3.11.

Theorem 2.3 *Let $u \in BV(\Omega)$. Then $u_{\xi,y} \in BV(\Omega_{\xi,y})$ for every $\xi \in S^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \xi^\perp$. For such values of y we have (denoting by $u'_{\xi,y}$ the absolutely continuous part of the measure derivative of $u_{\xi,y}$):*

$$u'_{\xi,y}(t) = \langle \nabla u(y + t\xi), \xi\rangle \quad \textit{for a.e. } t \in \Omega_{\xi,y}\,; \qquad S_{u_{\xi,y}} = (S_u)_{\xi,y}\,.$$

Moreover, for every open subset A of Ω we have

$$\int_{\xi^\perp} |D^c u_{\xi,y}|(A_{\xi,y})\,d\mathcal{H}^{n-1}(y) = |\langle D^c u, \xi\rangle|(A)\,,$$

and for every positive Borel function g

$$\int_{\xi^\perp} \sum_{t\in S_{u_{\xi,y}}} g(y + t\xi)\,d\mathcal{H}^{n-1}(y) = \int_{S_u} g(x)|\langle \nu_u, \xi\rangle|\,d\mathcal{H}^{n-1}.$$

Conversely, if $u \in L^1(\Omega)$ and for all $\xi \in \{e_1, \dots, e_n\}$ and for a.e. $y \in \xi^\perp$, $u_{\xi,y} \in BV(\Omega_{\xi,y})$ and

$$\int_{\xi^\perp} |Du_{\xi,y}|(\Omega_{\xi,y})\,d\mathcal{H}^{n-1}(y) < +\infty,$$

then $u \in BV(\Omega)$.

[1] It turns out that if $x \in \Omega \setminus S_u$ then u has an *approximate limit* (in the sense of Federer) at x, i.e. there exists $z \in \mathbb{R}$ such that for any $\varepsilon > 0$ the set $\{y \in \Omega : |u(y) - z| > \varepsilon\}$ has density 0 at x.

Relaxation. We recall that the *relaxed* functional $\overline{F}$ of a given functional F is the largest lower semicontinuous functional smaller than F. We will need the following relaxation theorem, which can be obtained from the results contained in [8] (see, in particular, the proof of Theorem 3.1 and use the case $f(t) = t + t^2$, too); see also [2], Theorem 3.2. Here $SBV^2(\Omega) = \{u \in SBV(\Omega) : \ |\nabla u| \in L^2(\Omega),\ \mathcal{H}^{n-1}(S_u) < +\infty\}$.

Theorem 2.4 *Let $\vartheta\colon \mathbb{R} \to [0, +\infty)$ be a continuous, concave function with $\vartheta(0) = 0$ and $\lim_{t\to 0} \vartheta(t)/t = 1$. Let Ω be a bounded open subset of $\mathbb{R}^n$, and consider the functional $F : BV(\Omega) \to [0, +\infty]$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u(x)| dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1} & \\ & \text{if } u \in SBV^2(\Omega) \cap L^\infty(\Omega)\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the relaxed functional $\overline{F}$ of F on $BV(\Omega)$ with respect to the L^1-topology is given by

$$\overline{F}(u) = \int_\Omega |\nabla u(x)|\, dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1} + |D^c u|(\Omega).$$

Γ-*convergence.* For the general theory see [19]. Let (X, d) be a metric space. Let $(F_j)_{j\in\mathbb{N}}$ be a sequence of functions $X \to \overline{\mathbb{R}}$. We say that (F_j) Γ-*converges*, as $j \to +\infty$, to $F\colon X \to \overline{\mathbb{R}}$, if for all $u \in X$ we have:

i) (*lower semicontinuity inequality*) for every sequence (u_j) converging to u

$$F(u) \le \liminf_{j\to+\infty} F_j(u_j);$$

ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u such that:

$$F(u) \ge \limsup_{j\to+\infty} F_j(u_j);$$

If F_j equals the same functional G for every j, then the Γ-limit is nothing but the relaxed functional of G.

The *lower* and *upper* Γ-*limits* of (F_j) in $u \in X$ are defined as

$$F'(u) = \inf\bigl\{\liminf_{j\to+\infty} F_j(u_j) : \ u_j \to u\bigr\}, \tag{2.5}$$

$$F''(u) = \inf\bigl\{\limsup_{j\to+\infty} F_j(u_j) : \ u_j \to u\bigr\}, \tag{2.6}$$

respectively.

We extend this definition of convergence to families depending on a real parameter. Given a family $(F_\varepsilon)_{\varepsilon>0}$ of functions $X \to \overline{\mathbb{R}}$, we say that it Γ-*converges*, as $\varepsilon \to 0$, to $F\colon X \to \overline{\mathbb{R}}$ if for every positive infinitesimal sequence (ε_j) the sequence (F_{ε_j}) Γ-converges to F.

If we define the *lower* and *upper* Γ-*limits* of (F_ε) as

$$F'(u) = \inf\bigl\{\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\bigr\}, \tag{2.7}$$

$$F''(u) = \inf\bigl\{\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\bigr\}, \tag{2.8}$$

respectively, then (F_ε) Γ-converges to F in u if and only if $F'(u) = F''(u) = F(u)$. Both F' and F'' are lower semicontinuous on X. In the estimate of F' we shall use the following immediate consequence of the definition:

$$F'(u) \quad = \quad \inf\big\{\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j) :\ \varepsilon_j \to 0^+,\ u_j \to u\big\}. \tag{2.9}$$

It turns out that the infimum is attained.

We will also use the fact (see [19], Proposition 6.11) that the lower and upper Γ-limits of a sequence of functionals coincide with the lower and upper Γ-limits, respectively, of the sequence of the corresponding relaxed functionals.

An important consequence of the definition of Γ-convergence is the following result about the convergence of minimizers (see, e.g., [19], Corollary 7.20):

Theorem 2.5 *Let $F_j\colon X \to \overline{\mathbb{R}}$ $(j \in \mathbb{N})$ be a sequence of functions which Γ-converge to some $F : X \to \overline{\mathbb{R}}$; assume that $\inf_{v\in X} F_j(v) > -\infty$ for every j. Let (ε_j) be a positive infinitesimal sequence, and for every j let $u_j \in X$ be an ε_j-minimizer of F_j, i.e.*

$$F_j(u_j) \leq \inf_{v\in X} F_j(v) + \varepsilon_j\,.$$

Assume that $u_j \to u$ for some $u \in X$. Then u is a minimum point of F, and $F(u) = \lim_{j\to+\infty} F_j(u_j)$.

Remark 2.6 We also point out the following property, which is a direct consequence of the definition of Γ-convergence: if $F_\varepsilon \xrightarrow{\Gamma} F$ then $F_\varepsilon + G \xrightarrow{\Gamma} F + G$ whenever G is continuous.

In conclusion we recall the following useful tool, which can be found in [10].

Lemma 2.7 (supremum of measures) *Let Ω be an open subset of $\mathbb{R}^n$ and denote by $\mathcal{A}(\Omega)$ the family of its open subsets. Let λ be a positive Borel measure on Ω, and $\mu\colon \mathcal{A}(\Omega) \to [0,+\infty)$ a set function which is superadditive on open sets with disjoint compact closures (i.e. if $A, B \subset\subset \Omega$ and $\overline{A}\cap\overline{B} = \emptyset$, then $\mu(A\cup B) \geq \mu(A) + \mu(B)$). Let $(\psi_i)_{i\in I}$ be a family of positive Borel functions. Suppose that*

$$\mu(A) \geq \int_A \psi_i d\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega) \textit{ and } i \in I;$$

then

$$\mu(A) \geq \int_A \sup_i \psi_i d\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega).$$

Corollary 2.8 *Let μ be as in Lemma 2.7. Let λ_1, λ_2 be mutually singular Borel measures, and ψ_1, ψ_2 positive Borel functions. Assume that*

$$\mu(A) \geq \int_A \psi_i d\lambda_i \quad \textit{for every } A \in \mathcal{A}(\Omega) \textit{ and } i = 1, 2\,.$$

Then

$$\mu(A) \geq \int_A \psi_1 d\lambda_1 + \int_A \psi_2 d\lambda_2 \quad \textit{for every } A \in \mathcal{A}(\Omega)\,.$$

Proof. Let $E \subseteq \Omega$ be such that $\lambda_1(\Omega\setminus E) = 0$ and $\lambda_2(E) = 0$. Then we can suppose that $\psi_1 = 0$ on $\Omega\setminus E$ and $\psi_2 = 0$ on E. Then $\psi_1\vee\psi_2 = \psi_1 + \psi_2$. We now conclude by applying the preceeding lemma with $\lambda = \lambda_1 + \lambda_2$. ∎

3 Statement of the results

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and $f : [0, +\infty) \to [0, +\infty)$ be a non-decreasing, strictly concave and C^2 function such that

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 1 \,.$$

For every $\varepsilon > 0$ let $F_\varepsilon \colon L^1(\Omega) \to \mathbb{R}$, be defined by

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)|dy\right) dx & \text{if } u \in W^{1,1}(\Omega) \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

The main result is the following theorem:

Theorem 3.1 *The family $(F_\varepsilon)_{\varepsilon>0}$, defined in (3.1), Γ-converges, in the L^1-topology as $\varepsilon \to 0$, to the functional $\mathcal{F} \colon L^1(\Omega) \to [0, +\infty]$ given by*

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|)d\mathcal{H}^{n-1} + |D^c u|(\Omega) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in GBV(\Omega) \\ \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \text{otherwise,} \end{cases}$$

with

$$\vartheta(s) = 2\int_0^1 f\left(\frac{\omega_{n-1}}{\omega_n} s(\sqrt{1-t^2})^{n-1}\right) dt \qquad (s > 0), \tag{3.2}$$

where ω_k denotes the volume of the k-dimensional ball in $\mathbb{R}^k$ (with $\omega_0 = 1$).

If $n = 1$ and $\Omega = (a, b)$, this result follows as a particular case of Theorem 3.3 in [21]; the corresponding form of the jump energy density is

$$\vartheta(s) = 2f\left(\frac{s}{2}\right).$$

Theorem 3.2 (Compactness) *Let (ε_j) be a positive infinitesimal sequence, and let (u_j) be a sequence in $L^1(\Omega)$ such that $\|u_j\|_\infty \le M$, and $F_{\varepsilon_j}(u_j) \le M$ for a suitable constant M independent of j. Then there exists a subsequence (u_{j_k}) converging in $L^1(\Omega)$ to a function $u \in BV(\Omega)$.*

As an example of application of these results we consider the following corollary.

Corollary 3.3 *Let (ε_j) be a positive infinitesimal sequence and $g \in L^\infty(\Omega)$. For every $u \in L^1(\Omega)$ and $j \in \mathbb{N}$, define:*

$$G_j(u) = F_{\varepsilon_j}(u) + \int_\Omega |u(x) - g(x)|\, dx \,,$$

and

$$\mathcal{G}(u) = \mathcal{F}(u) + \int_\Omega |u(x) - g(x)|\, dx \,.$$

For every j let u_j be a σ_j-minimizer of G_j in $L^1(\Omega)$ (with $\sigma_j \searrow 0$), i.e. $G_j(u_j) \le \inf_{L^1(\Omega)} G_j + \sigma_j$. Then (u_j) converges, up to a subsequence, to a minimizer of $\mathcal{G}$ in $L^1(\Omega)$.

Proof. Since $g \in L^\infty(\Omega)$ and F_{ε_j} decreases by truncation, we can assume that (u_j) is equibounded. By Theorem 3.2 there exists $u \in BV(\Omega)$ such that (up to a subsequence) $u_j \to u$ in $L^1(\Omega)$. By Theorem 2.5, since (G_j) Γ-converge to $\mathcal{G}$ (recall Remark 2.6), u is a minimum point of $\mathcal{G}$ on $L^1(\Omega)$. ■

REMARK 3.4 We will need the following "localization" of the functional F_ε: for every open subset A of Ω, we set

$$F_\varepsilon(u,A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)|dy\right) dx & \text{if } u \in W^{1,1}(\Omega) \\ \\ +\infty & \text{if } u \in L^1(\Omega)\setminus W^{1,1}(\Omega). \end{cases}$$

Clearly, $F_\varepsilon\big(\cdot,\Omega\big)$ coincides with the functional F_ε defined in (3.1). The lower and upper Γ-limits of $\big(F_\varepsilon(\cdot,A)\big)$ (see § 2) will be denoted by $F'(\cdot,A)$ and $F''(\cdot,A)$, respectively.

Note that, if $A \subset\subset \Omega$, the lower and upper Γ-limits of $(F_\varepsilon(\cdot,A))$ do not change by replacing Ω with any $\Omega' \supset\supset A$.

REMARK 3.5 The one-dimensional version of Theorem 3.1 will be used in the volume estimate of the lower Γ-limit (see Proposition 4.7) when applying the so-called *slicing* method. Actually, we will need the stronger form of the lower Γ-limit estimate contained in Remark 3.4 of [21]: if $n = 1$ and $\Omega = (a,b)$, then

$$F'(u,A) \geq \int_A |u'(x)|\,dx + 2\sum_{x\in S_u\cap A} f\left(\frac{1}{2}|u^+(x)-u^-(x)|\right) + |D^c u|(A)$$

for every $u \in GBV(\Omega)$ and A open subset of Ω. The extension to an arbitrary bounded open subset Ω of $\mathbb{R}$ and $A \subset\subset \Omega$ is immediate.

In the computation of the upper Γ-limit we will use the following result. Here the Lipschitz regularity of $\partial\Omega$ guarantees that a function $u \in BV(\Omega)$ admits a BV-extension on a neighborhood of Ω (see, e.g., [5] Prop. 3.21); and that there exists a constant $\gamma > 0$ such that:

$$|B_\varepsilon(x)\cap\Omega| \geq \gamma\varepsilon^n \tag{3.3}$$

for any $x \in \Omega$ and $\varepsilon < \mathrm{diam}(\Omega)$.

Proposition 3.6 *For every $\varepsilon > 0$, the relaxed functional of F_ε in the L^1-topology is given by*

$$\overline{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\left(\frac{\varepsilon}{|B_\varepsilon(x)\cap\Omega|}|Du|(B_\varepsilon(x)\cap\Omega)\right) dx \tag{3.4}$$

for every $u \in BV(\Omega)$.

Proof. Denote by H_ε the functional on the right-hand side of (3.4), defined with value $+\infty$ on $L^1\setminus BV$. Let $c_\varepsilon(x) = \varepsilon/|B_\varepsilon(x)\cap\Omega|$. It is easy to prove the L^1-lower semicontinuity of H_ε at every $u \in BV$. Indeed: let (u_h) be a sequence in $L^1(\Omega)$

converging to a function $u \in BV(\Omega)$; we can suppose that $\liminf_{h\to+\infty} H_\varepsilon(u_h)$ is finite and is a limit, and that each u_h is in BV; then by Fatou's Lemma, the monotonicity of f and the lower semicontinuity of the total variation, we have:

$$\begin{aligned}\liminf_{h\to+\infty} H_\varepsilon(u_h) &\geq \frac{1}{\varepsilon}\int_\Omega \liminf_{h\to+\infty} f\left(c_\varepsilon(x)|Du_h|\big(B_\varepsilon(x)\cap\Omega\big)\right)dx \\ &\geq \frac{1}{\varepsilon}\int_\Omega f\left(c_\varepsilon(x)\liminf_{h\to+\infty}|Du_h|\big(B_\varepsilon(x)\cap\Omega\big)\right)dx \\ &\geq \frac{1}{\varepsilon}\int_\Omega f\left(c_\varepsilon(x)|Du|\big(B_\varepsilon(x)\cap\Omega\big)\right)dx = H_\varepsilon(u).\end{aligned}$$

Since $H_\varepsilon(u) \leq F_\varepsilon(u)$ for all $u \in L^1(\Omega)$, the relaxed functional $\overline{F}_\varepsilon$ is estimated from below by H_ε on $BV(\Omega)$. Consider now the opposite inequality. Given $u \in BV(\Omega)$, if (v_h) denotes the sequence obtained from u (extended to a neighborhood of Ω) by standard mollification, then $v_h \to u$ in $L^1(\Omega)$ and

$$|Dv_h|\big(B_\varepsilon(x)\cap\Omega\big) \to |Du|\big(B_\varepsilon(x)\cap\Omega\big)$$

for a.e. $x \in \Omega$ (see, e.g., [5] Proposition 3.7 and Remark 3.8: note that $D^c u$ vanishes on the sets with finite $\mathcal{H}^{n-1}$ measure; moreover, if S is σ-finite with respect to $\mathcal{H}^{n-1}$, then $\{x \in \Omega :\ \mathcal{H}^{n-1}(S \cap \partial B_\varepsilon(x)) > 0\}$ is at most countable). Then by the Dominated Convergence Theorem (recall (3.3) and that $f(t) \leq t$):

$$\lim_{h\to+\infty} F_\varepsilon(v_h) = \frac{1}{\varepsilon}\int_\Omega f\left(c_\varepsilon(x)|Du|\big(B_\varepsilon(x)\cap\Omega\big)\right)dx = H_\varepsilon(u).$$

This shows that H_ε coincides with the relaxed functional $\overline{F}_\varepsilon$ on $BV(\Omega)$. ■

Remark 3.7 From the preceeding proposition we easily deduce that $F''(u) < +\infty$ if $u \in BV(\Omega)$. Indeed, the upper Γ-limit of (F_ε) coincides with the upper Γ-limit of $(\overline{F}_\varepsilon)$; therefore, since $f(t) \leq t$, if we set $\mu = |Du|$, we have:

$$F''(u) \leq \limsup_{\varepsilon\to0} \overline{F}_\varepsilon(u) \leq \limsup_{\varepsilon\to0} \int_\Omega \frac{1}{|B_\varepsilon(x)\cap\Omega|}\int_\Omega 1_{B_\varepsilon(x)}\,d\mu\,dx\,,$$

where $1_{B_\varepsilon(x)}$ is the characteristic function of $B_\varepsilon(x)$; therefore, by (3.3):

$$F''(u) \leq \limsup_{\varepsilon\to0} \frac{1}{\gamma\varepsilon^n}\int_\Omega d\mu(y)\int_\Omega 1_{B_\varepsilon(x)}(y)\,dx \leq \frac{\omega_n}{\gamma}\mu(\Omega) < +\infty\,.$$

4 Compactness. Lower bound in terms of the volume and Cantor parts

In [12] a crucial point in the proof of a lower bound for the Γ-limit of (F_ε) in terms of the volume part of the Mumford-Shah functional is the estimate from below of $F_\varepsilon(u)$ through $(1-\delta)\int_\Omega |\nabla v|^2\,dx$, where v is a function close to u in L^1 and δ is arbitrary in $(0,1)$ (see [12], Proposition 4.1) The presence of the L^2-norm of the gradient yields, via the SBV compactness theorem, separate semicontinuity inequalities for the absolutely continuous and the singular parts of the derivatives, thus giving the required bound in terms of the volume part. The proof of the cited proposition relies on a delicate partitioning of $\mathbb{R}^n$ by means of coordinate squares which are well-behaved with respect to the balls over which the gradients are averaged. In a L^1-context a perfectly analogous result, contained in Lemma 4.1 below, can thus be obtained with essentially the same proof, but now this does not imply the needed lower semicontinuity inequality for the volume parts. However, from Lemma 4.1 the compactness property of Theorem 3.2 for (F_ε) easily follows.

The proof of the correct lower bound for the Γ-limit (see Proposition 4.7) will be obtained by a slicing tecnique. The main difficulty is that $F_\varepsilon(u)$, due to its non-local character, can not be simply expressed through the one-dimensional sections of u. Thus, for any direction $\xi \in S^{n-1}$ we estimate $F_\varepsilon(u)$ through a suitable functional where the average on balls is replaced by the average on squares with a face orthogonal to ξ (Lemma 4.3); this allows to split the average itself into a part along ξ and another in the orthogonal space $\xi^\perp$. A slicing method can now be applied: the one-dimensional sections of u are replaced by its averages along $\xi^\perp$ (see Lemma 4.4).

For every open subset A of Ω and $\varrho > 0$ we set:

$$A_\varrho = \{x \in A : \ d(x, \partial A) > \varrho\}\,.$$

Lemma 4.1 *Let A be an open subset of Ω, and let $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. For every $\varepsilon > 0$ and $\delta > 0$, there exists a function $v \in SBV(A) \cap L^\infty(A)$ such that:*

$$(1-\delta)\int_A |\nabla v(x)|dx \leq F_\varepsilon(u, A)\,, \quad \mathcal{H}^{n-1}(S_v \cap A_{6\varepsilon}) \leq cF_\varepsilon(u, A),$$

$$\|v\|_{L^\infty(A)} \leq \|u\|_{L^\infty(A)}\,, \quad \|v - u\|_{L^1(A_{6\varepsilon})} \leq c\varepsilon F_\varepsilon(u, A)\|u\|_{L^\infty(A)},$$

where c is a constant depending only on n, δ and f.

Proof of Theorem 3.2. Let $A \subset\subset \Omega$, with ∂A smooth. By Lemma 4.1 there exists a sequence (v_j) in $SBV(A)$ and a constant C independent of A such that $\|v_j\|_{BV(A)} \leq C$ and $\|v_j\|_{L^\infty(A)} \leq M$; moreover, $\|v_j - u_j\|_{L^1(A)} \to 0$. Therefore, there exists a subsequence (v_{j_k}) which converges to a function $u \in BV(A)$, with $\|u\|_{BV(A)} \leq C$. Hence $u \in BV(\Omega)$, too. Clearly, also (u_{j_k}) converges to u in $L^1(A)$. The arbitrariness of A and a diagonal argument allow to find a subsequence (u_{j_k}) which converges in $L^1_{\rm loc}(\Omega)$ to a function $u \in BV_{\rm loc}(\Omega)$; the uniform bound of $\|u_j\|_{L^\infty(\Omega)}$ implies the $L^1(\Omega)$-convergence of (u_{j_k}) to u. ■

Corollary 4.2 *Let (ε_j) be a positive infinitesimal sequence and let (u_j) be sequence in $L^1(\Omega)$ which converges to a function u. If $\big(F_{\varepsilon_j}(u_j)\big)$ is bounded, then $u \in GBV(\Omega)$. In particular, if $F'(u) < +\infty$ then $u \in GBV(\Omega)$.*

Proof. For each $T > 0$ apply Theorem 3.2 to $u_j^T = (-T) \vee u_j \wedge T$: we get $(-T) \vee u \wedge T \in BV(\Omega)$; hence $u \in GBV(\Omega)$. ■

Let $\xi \in S^{n-1}$ and let $\nu_1, \dots, \nu_n$ be an orthonormal basis of $\mathbb{R}^n$, with $\nu_1 = \xi$. For every $r > 0$ and $x \in \mathbb{R}^n$, define:

$$Q_r^\xi(x) = \{y \in \mathbb{R}^n : \ |\langle y - x, \nu_i\rangle| < r, \quad i = 1, \dots, n\}.$$

Lemma 4.3 *There exists a sequence $(c_h)_{h\in\mathbb{N}}$ of positive real numbers, with $c_h \stackrel{h}{\to} 1$, such that, for every $u \in W^{1,1}(\Omega)$, $\xi \in \mathbb{R}^n$ and A, A' open subsets of Ω, with $A' \subset\subset A$, and for every $\varepsilon < d(A', \partial A)/2$, the following inequality holds:*

$$F_\varepsilon(u, A) \geq \frac{1}{h\sigma_\varepsilon^h} \int_{A'} f^h \left(\sigma_\varepsilon^h \fint_{Q^\xi_{\sigma_\varepsilon^h}(z)} |\nabla u(y)| dy \right) dz,$$

where $f^h(t) = f(c_h ht)$ and $\sigma_\varepsilon^h = \varepsilon/h$.

Proof. For ease of notation we drop the superscript ξ in $Q_r^\xi(z)$ ($\xi \in S^{n-1}$ is fixed). For every $h \in \mathbb{N}$ let

$$Z_h = \{\alpha \in \mathbb{Z}^n : \ Q_{1/h}(2\alpha/h) \subseteq B_1(0)\}, \quad N_h = \# Z_h.$$

Clearly

$$c_h := \frac{2^n N_h}{\omega_n h^n} \stackrel{h}{\to} 1\,,$$

and, for every $\varepsilon > 0$ and $x \in \mathbb{R}^n$

$$Z_h = \{\alpha \in \mathbb{Z}^n : \ Q_\varepsilon^h(x, \alpha) := x + Q_{\varepsilon/h}(2\alpha\varepsilon/h) \subseteq B_\varepsilon(x)\}.$$

Fix now $u \in W^{1,1}(\Omega)$; then for every $\varepsilon > 0$ and $x \in \Omega$, with $d(x, \partial\Omega) > \varepsilon$:

$$\begin{aligned} \fint_{B_\varepsilon(x)} |\nabla u(y)| dy &\geq \sum_{\alpha \in Z_h} \frac{\left(\frac{2\varepsilon}{h}\right)^n}{\omega_n \varepsilon^n} \fint_{Q_\varepsilon^h(x,\alpha)} |\nabla u(y)| dy \\ &= \sum_{\alpha \in Z_h} \frac{c_h}{N_h} \fint_{Q_\varepsilon^h(x,\alpha)} |\nabla u(y)| dy. \end{aligned}$$

By the concavity of f:

$$f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u(y)| dy \right) \geq \sum_{\alpha \in Z_h} \frac{1}{N_h} f\left(\varepsilon c_h \fint_{Q_\varepsilon^h(x,\alpha)} |\nabla u(y)| dy \right).$$

Let $A' \subset\subset A$, and $2\varepsilon < d(A', \partial A)$; we can find an open subset A'' of A such that

$$A' \subset\subset A'' \subset\subset A, \quad d(A', \partial A'') > \varepsilon, \quad d(A'', \partial A) > \varepsilon.$$

Thus, if we set $\sigma_\varepsilon^h = \varepsilon/h$, we get:

$$F_\varepsilon(u, A'') \geq \sum_{\alpha \in Z_h} \frac{1}{N_h} \left(\frac{1}{h\sigma_\varepsilon^h} \int_{A''} f^h \left(\sigma_\varepsilon^h \fint_{Q_\varepsilon^h(x,\alpha)} |\nabla u(y)| dy \right) dx \right),$$

with f^h as in the statement of the lemma. The change of variables $z = x + 2\sigma_\varepsilon^h \alpha$ now yields:

$$F_\varepsilon(u, A) \geq \sum_{\alpha \in Z_h} \frac{1}{N_h} \left(\frac{1}{h\sigma_\varepsilon^h} \int_{A'} f^h \left(\sigma_\varepsilon^h \fint_{Q_{\sigma_\varepsilon^h}(z)} |\nabla u(y)| dy \right) dz \right).$$

Since the N_h terms of the sum do not depend on α, we conclude. ■

Given $y \in \mathbb{R}^{n-1}$ and $r > 0$, define

$$\tilde{Q}_r(y) = \{z \in \mathbb{R}^{n-1} : \ |z_i - y_i| < r, \ i = 1, \ldots, n-1\}. \tag{4.1}$$

Lemma 4.4 *Let (ε_j) be a positive infinitesimal sequence. Let A be an open subset of $\mathbb{R}^{n-1}$, and $a, b \in \mathbb{R}$, with $a < b$. Let $u_j, u \in L^1\big((a,b) \times A\big)$ $(j \in \mathbb{N})$ and $u_j \to u$ in $L^1\big((a,b) \times A\big)$. For a.e. $t \in (a,b)$, for every $y \in A$ and $j \in \mathbb{N}$ with $\varepsilon_j < \frac{1}{\sqrt{n-1}} d(y, \partial A)$ we can define*

$$v_j(t,y) = \fint_{\tilde{Q}_{\varepsilon_j}(y)} u_j(t,z)\, dz.$$

Then there exists a subsequence $(v_{j_k})_k$ such that $v_{j_k}(\cdot, y) \to u(\cdot, y)$ in $L^1(a,b)$ for a.e. $y \in A$.

Proof. There exists $N \subseteq (a,b)$, with $|N| = 0$ such that $u_j(t,\cdot), u(t,\cdot) \in L^1(A)$ for every $t \in (a,b) \setminus N$. In particular, $v_j(t,y)$ is well-defined for every $t \in (a,b) \setminus N$, $y \in A$, and $\varepsilon_j < d(y, \partial A)/\sqrt{n-1}$. Let

$$\phi_j(z) = \frac{1}{|\tilde{Q}_{\varepsilon_j}(0)|} \chi_{\tilde{Q}_{\varepsilon_j}(0)}(z),$$

where $\chi_{\tilde{Q}_{\varepsilon_j}(0)}$ denotes the characteristic function of $\tilde{Q}_{\varepsilon_j}(0)$. We have

$$\int_A \left(\int_a^b |v_j(t,y) - u(t,y)|\, dt \right) dy =$$

$$= \int_A dy \int_a^b \left| \int_{\mathbb{R}^{n-1}} u_j(t,z)\phi_j(z-y)\, dz - u(t,y) \right| dt \leq$$

$$\leq \int_A dy \int_a^b dt \int_{\mathbb{R}^{n-1}} |u_j(t,z) - u(t,z)| \phi_j(z-y)\, dz +$$

$$+ \int_A dy \int_a^b \left| \int_{\mathbb{R}^{n-1}} u(t,z)\phi_j(z-y)\, dz - u(t,y) \right| dt.$$

Denote these two integral terms by I'_j and I''_j respectively. Then

$$I''_j \leq \int_a^b dt \int_A dy \int_{\mathbb{R}^{n-1}} |u(t,z) - u(t,y)| \phi_j(z-y)\, dz =$$

$$= \int_a^b dt \int_A dy \int_{\mathbb{R}^{n-1}} |u(t,y+w) - u(t,y)| \phi_j(w)\, dw =$$

$$= \int_{\mathbb{R}^{n-1}} \phi_j(w) \|u(\cdot, \cdot + w) - u\|_{L^1((a,b)\times A)} dw \leq \sup_{|w| \leq \varepsilon_j \sqrt{n-1}} \|u(\cdot, \cdot + w) - u\|_{L^1((a,b)\times A)}.$$

Since translation is continuous in the L^1 norm, I''_j tends to 0 as $j \to +\infty$. Let us now consider I'_j;

$$I'_j = \int_A \left(\int_a^b (|u_j - u|(t,\cdot) * \phi_j)(y)\, dt \right) dy \leq$$

$$\leq \int_a^b \|(u_j - u)(t,\cdot)\|_{L^1(A)} \|\phi_j\|_{L^1(\mathbb{R}^{n-1})} dt = \|u_j - u\|_{L^1((a,b)\times A)},$$

which tends to 0 as $j \to +\infty$. Thus we conclude that for every A open subset of $\mathbb{R}^{n-1}$

$$\lim_{j\to+\infty} \int_A \left(\int_a^b |v_j(t,y) - u(t,y)|\, dt \right) dy = 0.$$

In particular, there exists a subsequence $(v_{j_k})_k$ such that $v_{j_k}(\cdot, y) \to u(\cdot, y)$ in $L^1(a,b)$ for a.e. $y \in A$. ■

Remark 4.5 The result of the previous lemma can be immediately generalized to the case where (a,b) is replaced by any bounded open subset of $\mathbb{R}$.

Remark 4.6 Given $u \in L^1(a,b)$, the set functions $A \mapsto F_\varepsilon(u,A)$ are increasing and superadditive. Consequently, also the set function $A \mapsto F'(u,A)$ is increasing and superadditive, i.e.

(i) $F'(u,A_1) \le F'(u,A_2)$, whenever $A_1 \subseteq A_2 \subseteq \Omega$;

(ii) $F'(u, A_1 \cup A_2) \ge F'(u,A_1) + F'(u,A_2)$, whenever $A_1 \cap A_2 = \emptyset$.

Proposition 4.7 *For every $u \in BV(\Omega)$ and $A \in \mathcal{A}(\Omega)$*

$$F'(u,A) \ge \int_A |\nabla u(x)| dx\,, \quad F'(u,A) \ge |D^c u|(A).$$

Proof. Let (ε_j) be a positive infinitesimal sequence and let (u_j) be a sequence in $W^{1,1}(\Omega)$ converging to $u \in L^1(\Omega)$ and such that $F_{\varepsilon_j}(u_j,A) \to F'(u,A)$ as $j \to +\infty$. Let $\xi \in S^{n-1}$. By Lemma 4.3, applied with $\varepsilon = \varepsilon_j$ and $u = u_j$, and by Fubini's Theorem, if A and A' are open subsets of Ω, with $A' \subset\subset A$, and $2\varepsilon_j < d(A', \partial A)$, we have (here $\sigma_j^h = \varepsilon_j/h$):

$$F_{\varepsilon_j}(u_j,A) \ge \frac{1}{h\sigma_j^h} \int_{A'} f^h \left(\sigma_j^h \fint_{Q^\xi_{\sigma_j^h}(z)} |\nabla u_j(x)| dx \right) dz =$$

$$= \int_{\xi^\perp} \left(\frac{1}{h\sigma_j^h} \int_{A'_{\xi,y}} f^h \left(\sigma_j^h \fint_{Q^\xi_{\sigma_j^h}(y+t\xi)} |\nabla u_j(x)| dx \right) dt \right) d\mathcal{H}^{n-1}(y)$$

where $\xi^\perp$ and $A'_{\xi,y}$ stand for the subspace orthogonal to ξ and for the one-dimensional section of A' in the direction ξ, as in Theorem 2.3. It is not restrictive to assume $\xi = e_1$ (and we will drop the superscript ξ). Then we can rewrite:

$$F_{\varepsilon_j}(u_j,A) \ge \int_{\tilde{A}'} \left(\frac{1}{h\sigma_j^h} \int_{A'_y} f^h \left(\sigma_j^h \fint_{Q_{\sigma_j^h}(t,y)} |\nabla u_j(x)| dx \right) dt \right) d\mathcal{H}^{n-1}(y)\,,$$

where $A'_y = \{t \in \mathbb{R} :\ (t,y) \in A'\}$ and $\tilde{A}' = \{y \in \mathbb{R}^{n-1} :\ A'_y \neq \emptyset\}$. In view of the definition (4.1) of $\tilde{Q}$, for every $(t,y) \in A'$:

$$\begin{aligned} \fint_{Q_{\sigma_j^h}(t,y)} |\nabla u_j(x)| dx \ &\ge \fint_{Q_{\sigma_j^h}(t,y)} \left| \frac{\partial u_j}{\partial x_1}(x) \right| dx \\ &\ge \fint_{t-\sigma_j^h}^{t+\sigma_j^h} \left| \fint_{\tilde{Q}_{\sigma_j^h}(y)} \frac{\partial u_j}{\partial s}(s,z)\, dz \right| ds\,; \end{aligned} \tag{4.2}$$

Consider u_j extended with value 0 outside A. Then, as in Lemma 4.4, for a.e. $s \in \mathbb{R}$ and for every $y \in \mathbb{R}^{n-1}$ we can define

$$v_j^y(s) = v_j(s,y) = \fint_{\tilde{Q}_{\sigma_j^h}(y)} u_j(s,z)\, dz\,.$$

It turns out that if $y \in \tilde{A}'$, then $v_j^y \in W^{1,1}(V_y)$ for a suitable open neighborhood V_y of $\overline{A'_y}$ (recall that $A' \subset\subset A$); furthermore, (4.2) gives:

$$\fint_{Q_{\sigma_j^h}(t,y)} |\nabla u_j(x)| dx \geq \fint_{t-\sigma_j^h}^{t+\sigma_j^h} \left| \frac{dv_j^y}{ds}(s) \right| ds.$$

By Lemma 4.4 and Remark 4.5, the sequence (v_j^y) converges (up to a subsequence, which does not affect the rest of the proof) to $u^y := u(\cdot, y)$ in $L^1(V_y)$ for a.e. $y \in \tilde{A}'$. By Fatou's Lemma:

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, A) \geq \int_{\tilde{A}'} \frac{1}{h} \left(\liminf_{j\to+\infty} \frac{1}{\sigma_j^h} \int_{A'_y} f^h \left(\sigma_j^h \fint_{t-\sigma_j^h}^{t+\sigma_j^h} \left| \frac{dv_j^y}{ds}(s) \right| ds \right) dt \right) dy;$$

thus, taking the one-dimensional Γ-convergence result (Remark 3.5) into account:

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, A) \geq \int_{\tilde{A}'} \frac{1}{h} \left(\int_{A'_y} c_h h |(u^y)'(s)| ds + c_h h |D^c u^y|(A'_y) \right) dy.$$

By Theorem 2.3 we deduce that

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, A) \geq c_h \left(\int_{A'} |\langle \nabla u(x), e_1 \rangle| dx + |\langle D^c u, e_1 \rangle|(A') \right).$$

As mentioned above, this result holds with any ξ in place of e_1; therefore, since $c_h \to 1$ as $h \to +\infty$ and $A' \subset\subset A$ is arbitrary, we get

$$F'(u, A) \geq \int_A |\langle \nabla u(x), \xi \rangle| dx\,, \quad F'(u,A) \geq |\langle D^c u, \xi \rangle|(A) \tag{4.3}$$

for every $\xi \in S^{n-1}$. From the first inequality, the superadditivity of F' and Lemma 2.7 we get $F'(u, A) \geq \int_A |\nabla u(x)| dx$ for every A.

Let $\lambda = D^c u$ and $\lambda_\xi = \langle \lambda, \xi \rangle$; note that:

$$|\lambda_\xi| = \big| \frac{d\lambda_\xi}{d|\lambda|} \big|\, d|\lambda| = \big| \langle \frac{d\lambda}{d|\lambda|}, \xi \rangle \big|\, d|\lambda|\,.$$

The second inequality in (4.3) can now be rewritten as $F'(u, A) \geq \int_A \psi_\xi\, d|\lambda|$, where $\psi_\xi = |\langle d\lambda/d|\lambda|, \xi \rangle|$; an application of Lemma 2.7 yields:

$$F'(u, A) \geq \int_A \sup_\xi \big| \langle \frac{d\lambda}{d|\lambda|}, \xi \rangle \big|\, d|\lambda| \geq \int_A \big| \sup_\xi \langle \frac{d\lambda}{d|\lambda|}, \xi \rangle \big|\, d|\lambda| = |\lambda|(A) = |D^c u|(A)\,.$$

This concludes the proof. ■

5 Lower bound in terms of the surface energy

In this section for any sequence (F_{ε_j}) and any given function $u \in BV$ we apply Besicovitch's Differentiation Theorem, with respect to $\mathcal{H}^{n-1} \llcorner S_u$, to the (inner regular envelope of the) lower Γ-limit considered as a set function (Proposition 5.1). Through a rescaling argument the density of this bound will be estimated in terms of F' on the functions u_0 obtained by "blowing-up" u around the jump points (Proposition 5.2 and subsequent Corollary). We next need to express this bound in terms of $F'(u_0, C_1^\nu(x_0))$ where $C_1^\nu(x_0)$ is a cilinder of unit size with axis normal to the jump set in x_0 (Proposition 5.5): this allows an explicit computation of the lower bound (see Proposition 5.6 together with 5.7).

Denote by $\mathcal{A}(\Omega)$ the family of the open subsets of Ω. Let (ε_j) be a positive infinitesimal sequence and, for every $A \in \mathcal{A}(\Omega)$, let $\Phi'(\cdot, A)$ be the lower Γ-limit of $\big(F_{\varepsilon_j}(\cdot, A)\big)$. We define the *inner regular envelope* of Φ' as:

$$\Phi'_-(\cdot, A) = \sup\{\Phi'(\cdot, B) : \ B \in \mathcal{A}(\Omega), B \subset\subset A\}\,.$$

It is easy to see that $\Phi'_-(\cdot, A)$ is lower semicontinuous in $L^1(\Omega)$ for every A, and that $\Phi'_-(u, \cdot)$ is superadditive for every u.

Proposition 5.1 *For every $u \in L^1(\Omega)$ the function $\Phi'_-(u, \cdot)\colon \mathcal{A}(\Omega) \to \mathbb{R}$ is the trace on $\mathcal{A}(\Omega)$ of a Borel measure on Ω; moreover, for every $A \in \mathcal{A}(\Omega)$ and $u \in BV(\Omega)$:*

$$\Phi'_-(u, A) \geq \int_{S_u \cap A} h(x) d\mathcal{H}^{n-1}\,,$$

where, for $\mathcal{H}^{n-1}$ a.e. $x \in S_u$:

$$h(x) = \lim_{\varrho \to 0} \frac{\Phi'_-(u, B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}}.$$

Proof. An increasing set function $\alpha\colon \mathcal{A}(\Omega) \to [0, +\infty]$, which satisfies $\alpha(0) = 0$ and which is subadditive, superadditive and inner regular, can be extended to a Borel measure on Ω: see, e.g. [19], Th. 14.23. The result can be applied to Φ'_- if we check the subadditivity: this can be shown following the proof of Proposition 4.3 and Theorem 4.6 of [14] (these results are established in the case $p > 1$, but the same arguments work if $p = 1$).

Given $u \in BV(\Omega)$, for every $k \in \mathbb{N}$ let $S_k = \{x \in S_u : \ |u^+(x) - u^-(x)| > 1/k\}$. Clearly, $\mathcal{H}^{n-1}(S_k) < +\infty$; let $\nu_k = \mathcal{H}^{n-1} \llcorner S_k$, and denote by μ the Borel measure which extends $\Phi'_-(u, \cdot)$: in view of Remark 3.7, μ is a finite measure. By Besicovitch's Differentiation Theorem (see, e.g. [27], Th. 4.7), the limit

$$g(x) = \lim_{\varrho \to 0} \frac{\mu(B_\varrho(x))}{\nu_k(B_\varrho(x))}$$

exists and is finite for ν_k-a.e. $x \in \Omega$, and is ν_k-measurable; moreover, the Radon-Nikodym decomposition of μ is given by $\mu = g\nu_k + \mu^s$, with $\mu^s \perp \nu_k$. Since S_k is $\mathcal{H}^{n-1}$-rectifiable, for $\mathcal{H}^{n-1}$-a.e. $x \in S_k$ we have

$$\frac{\nu_k(B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}} = \frac{\mathcal{H}^{n-1}(B_\varrho(x) \cap S_k)}{\omega_{n-1}\varrho^{n-1}} \to 1, \qquad \varrho \to 0$$

(see, e.g., [5] Theorem 2.63). Thus, for $\mathcal{H}^{n-1}$-a.e. $x \in S_k$

$$g(x) = \lim_{\varrho \to 0} \frac{\mu(B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}} = \lim_{\varrho \to 0} \frac{\Phi'_-(u, B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}} = h(x).$$

Taking into account that μ^s is non-negative, we deduce that for every $A \in \mathcal{A}(\Omega)$

$$\Phi'_-(u, A) \geq \int_A h(x) d\nu_k = \int_{S_k \cap A} h(x) d\mathcal{H}^{n-1}.$$

The conclusion follws by considering the supremum for $k \in \mathbb{N}$. ■

When considering F' for the blow-up u_0 on a unit ball B_1 as below (or on a cilinder C_1 of unit size, as in Proposition 5.5), we assume as Ω any set strictly containing B_1 (or C_1): see Remark 3.4.

Proposition 5.2 *For every $u \in BV(\Omega)$*

$$\liminf_{\varrho \to 0} \frac{\Phi'_-(u, B_\varrho(x_0))}{\varrho^{n-1}} \geq F'(u_0, B_1(x_0)), \qquad \textit{for } \mathcal{H}^{n-1}\textit{-a.e.} \quad x_0 \in S_u,$$

where u_0 is the function given by

$$u_0(x) = \begin{cases} u^+(x_0) & \textit{if } \langle x - x_0, \nu \rangle \geq 0\,, \\ u^-(x_0) & \textit{if } \langle x - x_0, \nu \rangle < 0\,, \end{cases}$$

with $\nu = \nu_u(x_0)$.

Proof. Let $\delta \in (0,1)$ and note that $\Phi'_-\big(u, B_\varrho(x_0)\big) \geq \Phi'\big(u, B_{\delta\varrho}(x_0)\big)$ for every ϱ; then

$$\liminf_{\varrho \to 0} \frac{\Phi'_-(u, B_\varrho(x_0))}{\varrho^{n-1}} \geq \delta^{n-1} \liminf_{r \to 0} \frac{\Phi'(u, B_r(x_0))}{r^{n-1}}\,. \tag{5.1}$$

Let us now estimate the lower limit in the right-hand side.

We can assume $x_0 = 0$. Let (r_k) be a decreasing infinitesimal sequence; for every $k \in \mathbb{N}$ there exists $w_j \in W^{1,1}(\Omega)$ such that $w_j \to u$ in $L^1(\Omega)$ and

$$\liminf_{j \to +\infty} F_{\varepsilon_j}(w_j, B_{r_k}(0)) \leq \Phi'(u, B_{r_k}(0)) + \frac{r_k^{n-1}}{2k}.$$

Let $\bar{j} = j(k)$ be such that $\varepsilon_{\bar{j}}/r_k \leq 1/k$ and

$$F_{\varepsilon_{\bar{j}}}(w_{\bar{j}}, B_{r_k}(0)) \leq \Phi'(u, B_{r_k}(0)) + \frac{r_k^{n-1}}{k},$$

$$\|w_{\bar{j}} - u\|_{L^1(\Omega)} \leq \frac{1}{k}\,, \qquad \int_{B_2(0)} |w_{\bar{j}}(r_k x) - u(r_k x)| dx \leq \frac{1}{k}.$$

Let $u_k = w_{j(k)}$. We can suppose that the sequence $\big(j(k)\big)$ is increasing, and we set $\sigma_k = \varepsilon_{j(k)}$; then $u_k \to u$ in $L^1(\Omega)$ and

$$F_{\sigma_k}(u_k, B_{r_k}(0)) \leq \Phi'(u, B_{r_k}(0)) + \frac{r_k^{n-1}}{k}\,, \qquad \int_{B_2(0)} |u_k(r_k x) - u(r_k x)| dx \leq \frac{1}{k}\,.$$

The first inequality gives:

$$\liminf_{k\to+\infty}\frac{\Phi'(u,B_{r_k}(0))}{r_k^{n-1}}\geq\liminf_{k\to+\infty}\frac{F_{\sigma_k}(u_k,B_{r_k}(0))}{r_k^{n-1}}\,;$$

while, from the second inequality and the definition (2.1) of $u^{\pm}$ we obtain:

$$\int_{B_2(0)}|u_k(r_kx)-u_0(r_kx)|dx\leq\frac{1}{k}+\int_{B_2(0)}|u(r_kx)-u_0(r_kx)|dx\xrightarrow{k}0\,.$$

If we define $v_k(t)=u_k(r_kt)$, then $v_k\to u_0$ in $L^1(B_2(0))$; moreover, for every $x\in B_{r_k}(0)$

$$⨍_{B_{\sigma_k}(x)}|\nabla u_k(y)|dy=\frac{r_k^{n-1}}{\omega_n\sigma_k^n}\int_{B_{\sigma_k/r_k}(x/r_k)}|\nabla v_k(t)|\,dt.$$

Therefore, setting $x/r_k=z$, we have:

$$\frac{F_{\sigma_k}(u_k,B_{r_k}(0))}{r_k^{n-1}}=\frac{1}{\sigma_k/r_k}\int_{B_1(0)}f\left(\frac{\sigma_k}{r_k}⨍_{B_{\sigma_k/r_k}(z)}|\nabla v_k(t)|dt\right)\,dz\,.$$

Since $\gamma_k:=\sigma_k/r_k\to 0$, and $v_k\to u_0$ in $L^1(B_2(0))$, by the arbitrariness of (r_k) and the definition of F', we conclude that

$$\liminf_{r\to 0}\frac{\Phi'(u,B_r(0))}{r^{n-1}}\geq F'(u_0,B_1(0))\,.$$

This, together with (5.1) and the arbitrariness of δ, yields the conclusion. ■

From the two propositions we have just proved the next result immediately follows.

Corollary 5.3 *Let $u\in BV(\Omega)$ and let A be an open subset of Ω. Then*

$$F'(u,A)\geq\int_{S_u\cap A}\sigma(x)\,d\mathcal{H}^{n-1},$$

where

$$\sigma(x_0)=\omega_{n-1}^{-1}F'\big(u_0,B_1(x_0)\big),\qquad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x_0\in S_u,$$

with ($\nu=\nu_u(x_0)$)

$$u_0(x)=\begin{cases}u^+(x_0) & \textit{if }\ \langle x-x_0,\nu\rangle\geq 0\,,\\ u^-(x_0) & \textit{if }\ \langle x-x_0,\nu\rangle<0\,.\end{cases}$$

Let $\nu\in S^{n-1}$. For any $y\in\mathbb{R}^n$ denote by y_ν and $y_{\nu^\perp}$ the projections onto the subspaces $V=\{t\nu: t\in\mathbb{R}\}$ and $V^\perp$, respectively. For $\varrho>0$ and $x\in\mathbb{R}^n$ define

$$C_\varrho^\nu(0)=\{y\in\mathbb{R}^n:\ |y_\nu|<\varrho,\ |y_{\nu^\perp}|<\varrho\},\qquad C_\varrho^\nu(x)=x+C_\varrho^\nu(0).$$

The next lemma proves that the "'transition set" between two constant values shrinks onto the interface.

Lemma 5.4 *Let $\nu \in S^{n-1}$ and*

$$u_0(x) = \begin{cases} a & if\ \langle x, \nu\rangle \geq 0, \\ b & if\ \langle x, \nu\rangle < 0\ . \end{cases} \tag{5.2}$$

Let $\Omega' \supset\supset C_1^\nu(0)$. For any A open subset of $C_1^\nu(0)$, there exist a positive infinitesimal sequence (ε_j) and a sequence (u_j) in $W^{1,1}(\Omega')$ converging to u_0 in $L^1(\Omega')$ and such that

$$\lim_{j\to+\infty} F_{\varepsilon_j}(u_j, A) = F'(u_0, A)\,,$$

$$u_j(x) = a \quad if\ \langle x, \nu\rangle \geq a_j; \qquad u_j(x) = b \quad if\ \langle x, \nu\rangle \leq -b_j,$$

where (a_j) and (b_j) are suitable positive infinitesimal sequences.

Proof. It is not restrictive to assume $\nu = e_1$. Fix A open subset of $C_1^\nu(0)$.
Step 1. Let $0 < \varepsilon < d(C_1^\nu(0), \partial\Omega')$, and $\sigma > 0$. Let φ be the continuous function given by

$$\varphi(x) = \begin{cases} 0 & \text{if } x_1 \leq -2\varepsilon - \sigma \\ \text{affine} & \text{if } -2\varepsilon - \sigma < x_1 < -2\varepsilon \\ 1 & \text{if } x_1 \geq -2\varepsilon \end{cases}$$

Clearly, $|\nabla\varphi| \leq 1/\sigma$. Define:

$$A_\varepsilon = \{x \in \mathbb{R}^n :\ x_1 < -3\varepsilon - \sigma\}\,, \qquad B_\varepsilon = \{x \in \mathbb{R}^n :\ x_1 > -\varepsilon\}\,,$$

$$S_\varepsilon = \{x \in \mathbb{R}^n :\ -3\varepsilon - \sigma < x_1 < -\varepsilon\}\,.$$

Let $u_1, u_2 \in W^{1,1}(\Omega')$ and $v = \varphi u_1 + (1-\varphi)u_2$; then

$$\varepsilon F_\varepsilon(v, A) = \int_{A\cap A_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u_2(y)| dy\right) dx +$$

$$+ \int_{A\cap B_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u_1(y)| dy\right) dx + \int_{A\cap S_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla v(y)| dy\right) dx,$$

By the subadditivity of f,

$$\int_{A\cap S_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla v(y)|\, dy\right) \leq$$

$$\leq \int_{A\cap S_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} \Big[\varphi(y)|\nabla u_1(y)| + (1-\varphi(y))|\nabla u_2(y)| + \right.$$

$$\left. + |u_1(y) - u_2(y)||\nabla\varphi(y)|\Big]\, dy\right) dx \leq \int_{A\cap S_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} \varphi(y)|\nabla u_1(y)| dy\right) dx +$$

$$+ \int_{A\cap S_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} (1-\varphi(y))|\nabla u_2(y)|\, dy\right) dx +$$

$$+ \int_{A\cap S_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |u_1(y) - u_2(y)||\nabla\varphi(y)| dy\right) dx.$$

Then, since $f(t) \le t$,

$$F_\varepsilon(v, A) \le F_\varepsilon(u_1, A \cap (B_\varepsilon \cup S_\varepsilon)) + F_\varepsilon(u_2, A \cap (A_\varepsilon \cup S_\varepsilon)) + I_\varepsilon \qquad (5.3)$$

with

$$I_\varepsilon \le \frac{1}{\sigma} \int_{C_1^\nu(0) \cap S_\varepsilon} \fint_{B_\varepsilon(x)} |u_1(y) - u_2(y)| \, dy \, dx.$$

A simple application of Fubini's theorem now gives

$$I_\varepsilon \le \frac{1}{\sigma} \int_{\Omega'_-} |u_1(y) - u_2(y)| \, dy,$$

where $\Omega'_- = \{x \in \Omega' : \ x_1 < 0\}$.

Step 2. Let (ε_j) be a positive infinitesimal sequence, and (u_j) in $W^{1,1}(\Omega')$ converging to u_0 in $L^1(\Omega')$ such that $F_{\varepsilon_j}(u_j, A) \to F'(u_0, A)$. For every $j \in \mathbb{N}$ apply estimate (5.3) with $\varepsilon = \varepsilon_j$, $u_1 = u_j$ and $u_2 = b$. If $v_j = \varphi u_j + (1 - \varphi)b$, since $F_{\varepsilon_j}(b, A) = 0$, we have:

$$F_{\varepsilon_j}(v_j, A) \le F_{\varepsilon_j}(u_j, A) + \frac{1}{\sigma} \int_{\Omega'_-} |u_j(y) - b| \, dy.$$

Note that

$$\int_{\Omega'_-} |u_j(y) - b| \, dy = \int_{\Omega'_-} |u_j(y) - u_0(y)| \, dy \xrightarrow{j} 0\,.$$

Therefore, if (σ_h) is a positive infinitesimal sequence, we can find a strictly increasing sequence (j_h) in $\mathbb{N}$ such that

$$\frac{1}{\sigma_h} \int_{\Omega'_-} |u_{j_h} - b| \, dy \xrightarrow{h} 0\,.$$

Then

$$\limsup_{h \to +\infty} F_{\varepsilon_{j_h}}(v_{j_h}, A) \le \limsup_{h \to +\infty} F_{\varepsilon_{j_h}}(u_{j_h}, A) = F'(u_0, A).$$

Moreover,

$$v_{j_h}(x) = \begin{cases} b & x_1 \le -2\varepsilon_{j_h} - \sigma_h \\ u_{j_h} & x_1 \ge 0\,. \end{cases}$$

Clearly $v_{j_h} \to u_0$ in $L^1(\Omega')$, and $\liminf F_{\varepsilon_{j_h}}(v_{j_h}, A) \ge F'(u_0, A)$. An analogous procedure allows to modify v_{j_h} so that, while mantaining the convergence of the functionals to $F'(u_0, A)$, it takes the value a at the points x with x_1 positive and outside a neighborhood of 0, shrinking to 0. ∎

Proposition 5.5 *For $\mathcal{H}^{n-1}$ a.e. $x_0 \in S_u$*

$$F'(u_0, B_1(x_0)) = F'(u_0, C_1^\nu(x_0))$$

where $\nu = \nu_u(x_0)$ and u_0 is as (5.2).

Proof. Clearly, the non-trivial inequality is $F'(u_0, B_1(x_0)) \geq F'(u_0, C_1^\nu(x_0))$. It is not restrictive to assume $x_0 = 0$ and $\nu = e_1$. Fix $0 < \delta < 1$. Let (ε_j) and (u_j) be given by the previous lemma, applied with $A = B_1(0)$. Then $u_j(x) = a$ if $x_1 \geq a_j$, and $u_j(x) = b$ if $x_1 \leq -b_j$, where (a_j) and (b_j) are suitable positive infinitesimal sequences. Let $S_j = (-b_j, a_j) \times \mathbb{R}^{n-1}$; then, for j sufficiently large, we have $C_\delta^\nu(0) \cap S_j \subset\subset B_1(0)$, and therefore $F_{\varepsilon_j}\big(u_j, C_\delta^\nu(0) \cap B_1(0)\big) = F_{\varepsilon_j}\big(u_j, C_\delta^\nu(0)\big)$. Hence:

$$F_{\varepsilon_j}(u_j, B_1(0)) \geq F_{\varepsilon_j}(u_j, B_1(0) \cap C_\delta^\nu(0)) = F_{\varepsilon_j}(u_j, C_\delta^\nu(0))\,. \tag{5.4}$$

Moreover, if we set $v_j(x) = u_j(\delta x)$, then $F_{\varepsilon_j}(u_j, C_\delta^\nu(0)) = \delta^{n-1} F_{\varepsilon_j/\delta}(v_j, C_1^\nu(0))$, as one can easily check by a change of variable. Thus, passing to the limit in (5.4), we get $F'(u_0, B_1(0)) \geq \delta^{n-1} \liminf_j F_{\varepsilon_j/\delta}(v_j, C_1^\nu(0)) \geq \delta^{n-1} F'(u_0, C_1^\nu(0))$. The conclusion now follows by taking the supremum on $\delta \in (0,1)$. ∎

Proposition 5.6 *Let u_0 be as in (5.2), with $a \neq b$ and $\nu = e_1$. Then*

$$F'(u_0, C_1^{e_1}(0)) \geq \omega_{n-1} \inf_X G\,,$$

where

$$G(v) = \int_{\mathbb{R}} f\Big(a_n \int_0^1 \big[v(\xi + \sqrt{1-\eta^2}, \eta) - v(\xi - \sqrt{1-\eta^2}, \eta)\big]\eta^{n-2} d\eta\Big) d\xi\,,$$

$a_n = (n-1)\omega_{n-1}/\omega_n$, and X is the space of all functions $v \in W^{1,1}_{loc}(\mathbb{R} \times (0,1))$ which are non-decreasing in the first variable, such that there exists $\xi_0 < \xi_1$ (depending on v) with $v(\xi, \eta) = b$ if $\xi \leq \xi_0$, and $v(\xi, \eta) = a$ if $\xi \geq \xi_1$.

Proof. Denote $C_1^{e_1}(0)$ by C. We can suppose $a > 0$ and $b = 0$.

Let (ε_j) and (u_j) be the sequences given by Lemma 5.4, applied with $A = C$ and any $\Omega' \supset\supset C$. In particular, $u_j(x) = 0$ if $x_1 \leq -b_j$, and $u_j(x) = a$ if $x_1 \geq a_j$, for some positive $a_j, b_j \to 0$.

Let

$$\alpha_j(x_1, x_2, \ldots, x_n) = \int_{-1}^{x_1} \left(\frac{\partial}{\partial t} u_j(t, x_2, \ldots, x_n)\right)^+ dt\,,$$

and $v_j = \alpha_j \wedge a$. Then $v_j \in W^{1,1}(\Omega')$, $v_j(x) = u_0(x)$ if $x_1 \notin (-b_j, a_j)$, and v_j is non-decreasing in the first variable. Moreover

$$|\nabla u_j| \geq \left|\frac{\partial u_j}{\partial x_1}\right| \geq \frac{\partial v_j}{\partial x_1} \geq 0.$$

Then, for j sufficiently large,

$$F_{\varepsilon_j}(u_j, C) \geq \frac{1}{\varepsilon_j} \int_C f\left(\varepsilon_j \ \fint_{B_{\varepsilon_j}(x)} \frac{\partial v_j}{\partial s_1}(s) ds\right) dx.$$

Let B_ϱ^{n-1} be the $(n-1)$-dimensional ball of centre 0 and radius ϱ. Since $C = (-1,1) \times B_1^{n-1}$, we have

$$F_{\varepsilon_j}(u_j, C) \geq \frac{1}{\varepsilon_j} \int_{B_1^{n-1}} dx_2 \cdots dx_n \int_{-1}^1 f\left(\varepsilon_j \ \fint_{B_{\varepsilon_j}(x)} \frac{\partial v_j}{\partial s_1}(s) ds\right) dx_1.$$

Let $(\bar{x}_2, \cdots, \bar{x}_n) \in$ in the closure $\overline{B}_1^{n-1}$ of B_1^{n-1} be such that

$$\int_{-1}^{1} f\left(\varepsilon_j \fint_{B_{\varepsilon_j}(x_1, \bar{x}_2, \cdots, \bar{x}_n)} \frac{\partial v_j}{\partial s_1}(s) ds\right) dx_1 =$$

$$= \min_{(x_2, \cdots, x_n) \in \overline{B}_1^{n-1}} \int_{-1}^{1} f\left(\varepsilon_j \fint_{B_{\varepsilon_j}(x_1, \cdots, x_n)} \frac{\partial v_j}{\partial s_1}(s) ds\right) dx_1 \, .$$

In the argument below it will not be restrictive to assume $\bar{x}_i = 0$ for every i. Then

$$F_{\varepsilon_j}(u_j, C) \geq \frac{\omega_{n-1}}{\varepsilon_j} \int_{-1}^{1} f\left(\frac{1}{\omega_n \varepsilon_j^{n-1}} \int_{B_{\varepsilon_j}(x_1, 0)} \frac{\partial v_j}{\partial s_1}(s) ds\right) dx_1 \, .$$

By Fubini's Theorem,

$$\int_{B_{\varepsilon_j}(x_1, 0)} \frac{\partial v_j}{\partial s_1}(s) ds = \int_{B_{\varepsilon_j}^{n-1}} \left[v_j\big(x_1 + \sqrt{\varepsilon_j^2 - |y|^2}, y\big) - v_j\big(x_1 - \sqrt{\varepsilon_j^2 - |y|^2}, y\big)\right] dy \, .$$

Define

$$\hat{v}_j(t, \varrho) = \fint_{\partial B_\varrho^{n-1}} v_j(t, y) d\mathcal{H}^{n-2}(y), \quad \varrho \in (0, \varepsilon_j) \, .$$

Then

$$\int_{B_{\varepsilon_j}(x_1, 0)} \frac{\partial v_j}{\partial s_1}(s) ds =$$

$$= \int_0^{\varepsilon_j} d\varrho \int_{\partial B_\varrho^{n-1}} \left[v_j\big(x_1 + \sqrt{\varepsilon_j^2 - |y|^2}, y\big) - v_j\big(x_1 - \sqrt{\varepsilon_j^2 - |y|^2}, y\big)\right] d\mathcal{H}^{n-2}(y)$$

$$= \int_0^{\varepsilon_j} \mathcal{H}^{n-2}(\partial B_\varrho^{n-1}) [\hat{v}_j(x_1 + \sqrt{\varepsilon_j^2 - \varrho^2}, \varrho) - \hat{v}_j(x_1 - \sqrt{\varepsilon_j^2 - \varrho^2}, \varrho)] d\varrho \, .$$

By the change of variables $x_1/\varepsilon_j = \xi$ and $\varrho = \varepsilon_j \eta$, we obtain, for j sufficiently large:

$$F_{\varepsilon_j}(u_j, C) \geq \omega_{n-1} \int_{-1/\varepsilon_j}^{1/\varepsilon_j} f\Bigg(\frac{(n-1)\omega_{n-1}}{\omega_n} \int_0^1 [\hat{v}_j(\varepsilon_j \xi + \varepsilon_j \sqrt{1 - \eta^2}, \varepsilon_j \eta) +$$

$$- \hat{v}_j(\varepsilon_j \xi - \varepsilon_j \sqrt{1 - \eta^2}, \varepsilon_j \eta)] \eta^{n-2} d\eta\Bigg) d\xi \, .$$

Let $w_j(x) = \hat{v}_j(\varepsilon_j x)$. Clearly w_j is non-decreasing in the first variable, and there exist $\xi_0 < \xi_1$ such that $w_j(x) = a$ if $x \geq \xi_1$ and $w_j(x) = 0$ if $x \leq \xi_0$. Then w_j can be extended to all $\mathbb{R} \times (0, 1)$ (with values 0 and a) and thus $w_j \in X$. Hence $F'(u_0, C) \geq \omega_{n-1} G(w_j)$. ■

We now need a precise estimate of the infimum of the functional G introduced in Proposition 5.6. To bypass the problem of the existence of the minimum of G we consider a discrete version G_k ($k \in \mathbb{N}$) of G. This leads to a minimum problem in a finite dimensional space, and a careful analysis of the properties of the minimizers allows to compute explicitely the infimum.

Proposition 5.7 *With the notation of Proposition 5.6 we have*

$$\inf_X G \geq \vartheta(|a-b|)\,,$$

where ϑ is defined in (3.2).

Proof. We can suppose $a > 0$ and $b = 0$. For each $k \in \mathbb{N}$ we now consider a discrete version G_k of G defined on the space of the functions on $S = \mathbb{R} \times [0,1]$ which are constant on each of the squares determined by a coordinate partition of S and with sides of length $1/k$. We also require the monotonicity in the first variable and the constant value 0 and a on the left and right of $[\xi_0, \xi_1] \times [0,1]$, respectively, for some $\xi_0 < \xi_1$.

Clearly, we can deal only with the values on the nodes; thus, for any $N \in \mathbb{N}$, we define Y_k^N as the set of the functions

$$v = (v^{i,j})_{i,j} \colon \mathbb{Z} \times \{1, \ldots, k-1\} \to [0,a],$$

such that:

a) for every j the function $i \mapsto v^{i,j}$ is increasing;

b) $v^{i,j} = 0$ if $i < -Nk$ and $v^{i,j} = a$ if $i \geq Nk$.

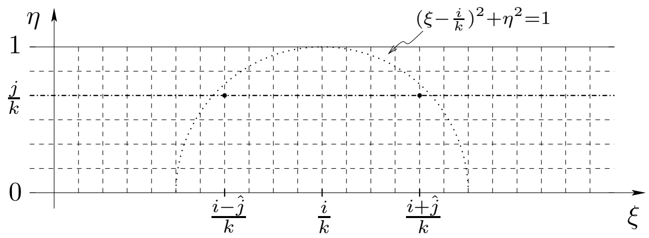
We think $v^{i,j}$ as the value in $(i/k, j/k)$ of the input function of G. Let $Y_k = \bigcup_{N \in \mathbb{N}} Y_k^N$, and let $G_k \colon Y_k \to \mathbb{R}$ be defined by:

$$G_k(v) = \sum_{i \in \mathbb{Z}} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k} [v]_{i,j}\right),$$

where

$$[v]_{i,j} = \left(v^{i+\hat{j},j} - v^{i-\hat{j},j}\right)(j/k)^{n-2},$$

with $\hat{j}$ denoting the integer part of $\sqrt{1-(j/k)^2}/(1/k) = \sqrt{k^2 - j^2}$ (see the figure below).



Note that:

$$G_k(v) = \sum_{i=-(N+1)k}^{(N+1)k} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k} [v]_{i,j}\right), \qquad \text{on } Y_k^N\,.$$

Step 1. Each minimizer of G_k on Y_k^N takes only the values 0 and a.

Let v be a minimizer of G_k on Y_k^N. Suppose, by contradiction, that there exists i_0, j_0 with $v^{i_0,j_0} = c \in (0,a)$. We can assume that for a suitable $s \in \mathbb{N}$:

$$v^{i_0-1,j_0} < c\,, \qquad c = v^{i_0,j_0} = v^{i_0+1,j_0} = \dots = v^{i_0+s,j_0}\,, \qquad v^{i_0+s+1,j_0} > c\,.$$

Then, given $t \in \mathbb{R}$, we define $v_t = (v_t^{i,j})_{i,j}$ as $v_t^{i_0+l,j_0} = c+t$, if $0 \le l \le s$, and $v_t = v$ otherwise. For $|t|$ sufficiently small, $v_t \in Y_k^N$. Let

$$I_1 = \{i \in \mathbb{Z}:\ i + \hat{j}_0 \in [i_0, i_0+s]\}\,, \quad I_2 = \{i \in \mathbb{Z}:\ i - \hat{j}_0 \in [i_0, i_0+s]\}\,.$$

Note that if $i \notin I_1 \Delta I_2$ then $[v_t]_{i,j} = [v]_{i,j}$ for every j. Therefore:

$$G_k(v_t) = \sum_{i \in I_1 \setminus I_2} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v_t]_{i,j}\right) + \sum_{i \in I_2 \setminus I_1} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v_t]_{i,j}\right)$$

$$+ \sum_{i \notin I_1 \Delta I_2} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i,j}\right) = \sum_{i \in I_1 \setminus I_2} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i,j} + \frac{a_n}{k}\Big(\frac{j_0}{k}\Big)^{n-2} t\right)$$

$$+ \sum_{i \in I_2 \setminus I_1} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i,j} - \frac{a_n}{k}\Big(\frac{j_0}{k}\Big)^{n-2} t\right) + \sum_{i \notin I_1 \Delta I_2} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i,j}\right).$$

The function $t \mapsto G_k(v_t)$ is twice continuously differentiable in $t = 0$ (due to the smoothness of f), and:

$$\frac{d^2}{dt^2} G_k(v_t)\big|_{t=0} =$$

$$= \frac{a_n^2}{k^3}\Big(\frac{j_0}{k}\Big)^{2(n-2)} \left[\sum_{i \in I_1 \setminus I_2} f''\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i,j}\right) + \sum_{i \in I_2 \setminus I_1} f''\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i,j}\right)\right] < 0\,.$$

by the strict concavity of f; this is a contradiction, since v is a minimizer for G_k.

Step 2. We claim that if $v \in Y_k^N$ takes only the values 0 and a, then $G_k(v) \ge G_k(\overline{v})$, where

$$\overline{v}^{i,j} = \begin{cases} 0 & i < Nk\,, \\ a & i \ge Nk\,. \end{cases}$$

To this aim we can assume that

$$E_v := \{i \in \mathbb{Z}:\ \exists j\ \ v^{i+\hat{j},j} = a,\ i + \hat{j} < Nk\} \neq \emptyset$$

(otherwise $v = \overline{v}$). Let $i_0 = \min E_v$ and

$$j_M = \max\{j:\ \ v^{i_0+\hat{j},j} = a,\ i_0 + \hat{j} < Nk\},$$
$$J_M = \{j:\ \hat{j} = \hat{j}_M,\ v^{i_0+\hat{j},j} = a\}, \qquad j_0 = \min J_M.$$

Then $v^{i_0+\hat{j}_0,j_0} = a$ and

$$j < j_0 \quad \Rightarrow \quad v^{i_0+\hat{j}_0,j} = 0. \tag{5.5}$$

Indeed, if we had $j < j_0$ with the property $v^{i_0+\hat{j}_0,j} = a$, then we would have $\hat{j} > \hat{j}_0 = \hat{j}_M$, and $Nk > i_0 + \hat{j}_0 = i_0 - l + \hat{j}$, with $l = \hat{j} - \hat{j}_0 > 0$; therefore $v^{i_0-l+\hat{j},j} = a$, and $i_0 - l \in E$, which contrasts with the definition of i_0.

Denote by w the function obtained by modifying v in $(i_0 + \hat{j}, j)$ for $j \in J_M$:

$$w^{i_0+\hat{j},j} = 0 \quad \text{for } j \in J_M, \qquad w^{i,j} = v^{i,j} \quad \text{otherwise.}$$

We want to show that

$$G_k(v) \geq G_k(w). \tag{5.6}$$

Note that in the sum over i defining $G_k(v)$ the terms $v^{i_0+\hat{j},j}$ $(j \in J_M)$ appear only if $i = i_0$ or $i = i_0 + 2\hat{j}_0$. Accordingly, let us write $kG_k(v)$ as:

$$kG_k(v) = f(q+\delta) + f(p) + \sum_{i \notin \{i_0, i_0+2\hat{j}_0\}} f\left(\sum_j \frac{a_n}{k}[v]_{i,j}\right), \tag{5.7}$$

where:

$$q = \sum_{j \notin J_M} \frac{a_n}{k}[v]_{i_0,j}, \qquad \delta = \sum_{j \in J_M} \frac{a_n a}{k}\left(\frac{j}{k}\right)^{n-2}, \qquad p = \sum_{j=0}^{k-1} \frac{a_n}{k}[v]_{i_0+2\hat{j}_0,j}.$$

An analogous splitting can be written for $kG_k(w)$: clearly, the last term is the same as in (5.7). Thus:

$$kG_k(v) - kG_k(w) = f(q+\delta) - f(q) - [f(p+\delta) - f(p)]. \tag{5.8}$$

By the definition of j_0 it turns out that:

$$q = \sum_{j<j_0} \frac{a_n}{k}[v]_{i_0,j}.$$

Moreover, if $j < j_0$ then $\hat{j} \geq \hat{j}_0$; the monotonicity in the first variable and (5.5) yield:

$$v^{i_0+2\hat{j}_0-\hat{j},j} \leq v^{i_0+\hat{j}_0,j} = 0.$$

Hence, $j < j_0$ implies that $v^{i_0+2\hat{j}_0-\hat{j},j} = 0$, so that

$$j < j_0 \quad \Rightarrow \quad [v]_{i_0+2\hat{j}_0,j} = v^{i_0+2\hat{j}_0+\hat{j},j} - v^{i_0+2\hat{j}_0-\hat{j},j} \geq v^{i_0+\hat{j},j} = [v]_{i_0,j}.$$

Therefore

$$p \geq \sum_{j<j_0} \frac{a_n}{k}[v]_{i_0+2\hat{j}_0,j} \geq q.$$

Note now that the concavity of f implies:

$$q \leq p \quad \Rightarrow \quad f(q+\delta) - f(q) \geq f(p+\delta) - f(p).$$

This, together with (5.8), proves (5.6). If E_w is not empty, it has a minimum strictly greater than $i_0 = \min E_v$. A finite iteration of the above argument proves the claim.

Step 3. We have shown that for every $N>0$, $\inf_{Y_k^N} G_k = G_k(\overline{v})$, where $\overline{v}$ does not depend on N. Then $\inf_{Y_k} G_k = \inf_{N>0}\inf_{Y_k^N} G_k = G_k(\overline{v})$. Let us estimate $G_k(\overline{v})$, where, by translation, we can suppose $\overline{v}$ given by

$$\overline{v}^{i,j} = \begin{cases} 0 & i<0 \\ a & i\geq 0. \end{cases}$$

We have

$$G_k(\overline{v}) \geq 2\sum_{i=-k}^{-1}\frac{1}{k} f\left(a_n \sum_{\substack{j=1\\ i+j\geq 0}}^{k-1} \frac{a}{k}\left(\frac{j}{k}\right)^{n-2}\right) = 2\sum_{i=-k}^{-1}\frac{1}{k} f\left(\frac{a_n a}{k}\sum_{j=1}^{\hat{i}}\left(\frac{j}{k}\right)^{n-2}\right),$$

where $\hat{i}$ denotes the integer part of $\sqrt{k^2-i^2}$. Fix $\sigma>0$; then for $k\geq 1/\sigma$:

$$\begin{aligned} \frac{1}{k}\sum_{j=1}^{\hat{i}}\left(\frac{j}{k}\right)^{n-2} &\geq \frac{1}{k}\sum_{j=1}^{\hat{i}+1}\left(\frac{j}{k}\right)^{n-2} - \sigma \geq \int_0^{(\hat{i}+1)/k} t^{n-2}dt - \sigma \\ &\geq \int_0^{\sqrt{1-(i/k)^2}} t^{n-2}dt - \sigma = \frac{1}{n-1}\big(\sqrt{1-(i/k)^2}\,\big)^{n-1} - \sigma\,. \end{aligned}$$

Then

$$G_k(\overline{v}) \geq 2\sum_{i=1}^{k}\frac{1}{k} f\left(a_n a\big[\frac{1}{n-1}\big(\sqrt{1-(i/k)^2}\,\big)^{n-1} - \sigma\big]\right).$$

Thus, by the definition of the Riemann integral as the limit of the Riemann sums, we have:

$$\liminf_{k\to+\infty}\inf_{Y_k} G_k \geq 2\int_0^1 f\left(\frac{\omega_{n-1}}{\omega_n} a(\sqrt{1-t^2}\,)^{n-1} - a_n a\sigma\right)dt\,.$$

By the arbitrariness of σ we conclude that:

$$\liminf_{k\to+\infty}\inf_{Y_k} G_k \geq 2\int_0^1 f\left(\frac{\omega_{n-1}}{\omega_n} a(\sqrt{1-t^2}\,)^{n-1}\right)dt\,. \tag{5.9}$$

Step 4. We now relate G with the discrete functionals G_k.

Let $v\in X$: then there exist $\xi_0<\xi_1$ such that $v(\xi,\eta)=0$ if $\xi\leq\xi_0$ and $v(\xi,\eta)=a$ if $\xi\geq\xi_1$. Extend v on $\mathbb{R}^2$ with $v(\xi,\eta)=a$ if $\xi\geq\xi_1$ and $v(\xi,\eta)=0$ otherwise outside $\mathbb{R}\times[0,1]$. If $(\varrho_\tau)_{\tau>0}$ is a family of mollifiers, it is easy to see that $v*\varrho_\tau\in X\cap C^\infty$, and $G(v*\varrho_\tau)\to G(v)$ as $\tau\to 0$.

Then, given $\sigma>0$ there exists $v_\sigma\in X\cap C^\infty$ such that

$$\inf_X G \geq G(v_\sigma)-\sigma. \tag{5.10}$$

Assume that $v_\sigma(\xi,\eta)=0$ if $\xi\leq -N$ and $v_\sigma(\xi,\eta)=a$ if $\xi\geq N$, for a suitable $N\in\mathbb{N}$. Since v_σ is uniformly continuous, there exists $\delta>0$ such that

$$|(\xi,\eta)-(\xi',\eta')|<\delta \Rightarrow |v_\sigma(\xi,\eta)-v_\sigma(\xi',\eta')|<\sigma/N\,.$$

Fix $k \in \mathbb{N}$, and define $Q_k^{i,j} = \big[i/k,(i+1)/k\big) \times \big[j/k,(j+1)/k\big)$, for $i \in \mathbb{Z}$ and $j \in \{1,\dots,k-1\}$. If $(\xi,\eta) \in Q_k^{i,j}$ then

$$\frac{i+\widehat{(j+1)}}{k} \le \xi + \sqrt{1-\eta^2} \le \frac{i+\hat{j}+2}{k}\,.$$

Note that

$$0 \le \frac{\hat{j}}{k} - \frac{\widehat{(j+1)}}{k} \le \frac{1+\sqrt{2k-1}}{k} \xrightarrow{k} 0\,.$$

Then there exists k_δ such that for every $k \ge k_\delta$, $i \in \mathbb{Z}$, $j \in \{1,\dots,k-1\}$:

$$(\xi,\eta) \in Q_k^{i,j}\,,\ (\xi',\eta') \in Q_k^{i+\hat{j},j} \ \Rightarrow\ |(\xi+\sqrt{1-\eta^2},\eta) - (\xi',\eta')| < \delta\,.$$

It follows that

$$(\xi,\eta) \in Q_k^{i,j} \ \Rightarrow\ |v_\sigma(\xi+\sqrt{1-\eta^2},\eta) - \fint_{Q_k^{i+\hat{j},j}} v_\sigma\,dx| < \sigma/N\,.$$

Analogously, we have

$$(\xi,\eta) \in Q_k^{i,j} \ \Rightarrow\ |v_\sigma(\xi-\sqrt{1-\eta^2},\eta) - \fint_{Q_k^{i-\hat{j},j}} v_\sigma\,dx| < \sigma/N\,.$$

For every $k \in \mathbb{N}$ and $(i,j) \in \mathbb{Z}\times\{1,\dots,k-1\}$ let

$$w_\sigma^{i,j} = \fint_{Q_k^{i,j}} v_\sigma\,dx\,.$$

Clearly, $w_\sigma = (w_\sigma^{i,j}) \in Y_k$. For every i,j and $(\xi,\eta) \in Q_k^{i,j}$

$$\big[v_\sigma(\xi+\sqrt{1-\eta^2},\eta) - v_\sigma(\xi-\sqrt{1-\eta^2},\eta)\big]\eta^{n-2} \ge [w_\sigma]_{i,j} - 2\sigma/N\,.$$

This implies that for every $i \in \mathbb{Z}$ and $\xi \in \big[i/k,(i+1)/k\big)$

$$\int_0^1 \big[v_\sigma(\xi+\sqrt{1-\eta^2},\eta) - v_\sigma(\xi-\sqrt{1-\eta^2},\eta)\big]\eta^{n-2}d\eta \ge \frac{1}{k}\sum_{j=1}^{k-1}[w_\sigma]_{i,j} - 2\sigma/N\,.$$

Finally, since f (extended with value 0 on $\mathbb{R}^-$) is Lipschitz continuous with Lipschitz constant 1, we have:

$$G(v_\sigma) \ge \sum_{i=-(N+1)k}^{(N+1)k} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1}\frac{1}{k}[w_\sigma]_{i,j} - \frac{2\sigma a_n}{N}\right) \ge G_k(w_\sigma) - 2\sigma a_n(2N+3)/N\,.$$

By (5.10), and the arbitrariness of σ, we obtain

$$\inf_X G \ge \liminf_{k\to+\infty} \inf_{Y_k} G_k$$

and this, by (5.9) of *Step 3*, concludes the proof. ■

6 Estimate from below of the lower Γ-limit

We collect here the results of §§ 4 and 5, proving the lower bound $F' \geq \mathcal{F}$. Recall that, by Corollary 4.2, if $F'(u) < +\infty$ then $u \in GBV(\Omega)$.

Theorem 6.1 *For every $u \in GBV(\Omega)$*

$$F'(u) \geq \int_\Omega |\nabla u(x)| dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1} + |D^c u|(\Omega).$$

Proof. Let $u \in BV(\Omega)$ and let A be an open subset of Ω. From Proposition 4.7 we have

$$F'(u, A) \geq \int_A |\nabla u(x)| dx \,, \quad F'(u, A) \geq |D^c u|(A);$$

moreover, Corollary 5.3 and Propositions 5.5, 5.6 and 5.7 give

$$F'(u, A) \geq \int_{S_u \cap A} \vartheta\big(|u^+(x) - u^-(x)|\big) d\mathcal{H}^{n-1} \,.$$

Apply now Corollary 2.8 (which clearly extends to a finite number of measures λ_i), with $\lambda_1 = \mathcal{L}^n$, $\lambda_2 = \mathcal{H}^{n-1} \llcorner S_u$ and $\lambda_3 = |D^c u|$. Therefore $F'(u) \geq \mathcal{F}(u)$.

Let us now consider the case $u \in GBV(\Omega)$. We can find a positive infinitesimal sequence (ε_j) and a sequence (u_j) in $W^{1,1}(\Omega)$ converging to u in $L^1(\Omega)$ and such that $F_{\varepsilon_j}(u_j) \xrightarrow{j} F'(u)$. Define $u_j^T = (-T) \vee u_j \wedge T$, and $u^T = (-T) \vee u \wedge T$. Since $u_j^T \to u^T$ in $L^1(\Omega)$, and $u^T \in BV(\Omega)$, we have

$$F'(u) = \liminf_{j \to +\infty} F_{\varepsilon_j}(u_j) \geq \liminf_{j \to +\infty} F_{\varepsilon_j}(u_j^T) \geq$$

$$\geq \int_\Omega |\nabla u^T(x)| dx + \int_{S_u \cap \Omega} \vartheta\big(|(u^T)^+(x) - (u^T)^-(x)|\big)\, d\mathcal{H}^{n-1} + |D^c u^T|(\Omega).$$

We conclude by taking the limit as $j \to +\infty$ and recalling the definitions of ∇u, $u^\pm$ and $|D^c u|$ (see (2.2), (2.3) and (2.4)). ■

7 Estimate from above of the upper Γ-limit

In this last section we conclude the proof of Theorem 3.1 by proving the upper estimate for the Γ-limit (Proposition 7.3). The right bound will be first obtained for a suitable dense subset of $SBV(\Omega)$: let $\mathcal{W}(\Omega)$ be the space of all functions $w \in SBV(\Omega)$ satisfying the following properties:

i) $\mathcal{H}^{n-1}(\overline{S}_w \setminus S_w) = 0$;

ii) $\overline{S}_w$ is the intersection of Ω with the union of a finite member of $(n-1)$-dimensional simplexes;

iii) $w \in W^{k,\infty}(\Omega \setminus \overline{S}_w)$ for every $k \in \mathbb{N}$.

From [17] we get the density property of $\mathcal{W}(\Omega)$ we need; we recall that $SBV^2(\Omega) = \{u \in SBV(\Omega) : \ |\nabla u| \in L^2(\Omega), \ \mathcal{H}^{n-1}(S_u) < +\infty\}$.

Theorem 7.1 ([17] Theorem 3.1) *Assume that $\partial\Omega$ is Lipschitz. Let $u \in SBV^2(\Omega)\cap L^\infty(\Omega)$. Then there exists a sequence (w_j) in $\mathcal{W}(\Omega)$ such that $w_j \to u$ strongly in $L^1(\Omega)$, $\nabla w_j \to \nabla u$ strongly in $L^2(\Omega,\mathbb{R}^n)$, $\limsup_h \|w_j\|_\infty \le \|u\|_\infty$ and*

$$\limsup_{j\to+\infty} \int_{S_{w_j}} \phi(w_j^+, w_j^-, \nu_{w_j}) d\mathcal{H}^{n-1} \le \int_{S_u} \phi(u^+, u^-, \nu_u) d\mathcal{H}^{n-1}$$

for every upper semicontinuous function ϕ such that $\phi(a,b,\nu) = \phi(b,a,-\nu)$ whenever $a, b \in \mathbb{R}$ and $\nu \in S^{n-1}$.

Lemma 7.2 *Let $g \in L^1(\Omega)$ and let A be an open subset of Ω. Then*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_A f\left(\varepsilon ⨍_{B_\varepsilon(x)\cap\Omega} g(y)\,dy\right) dx = \int_A g(x)\,dx.$$

Proof. Let $g_\varepsilon(x) = ⨍_{B_\varepsilon(x)\cap\Omega} g(y)\,dy$. Since $g \in L^1(\Omega)$, from Lebesgue's Differentation Theorem g_ε converges to g in $L^1(\Omega)$ and a.e. in Ω. We can now conclude by the Dominated Convergence Theorem, recalling that $f(t) \le t$ and $f(t)/t \to 1$ as $t \to 0^+$. ■

Proposition 7.3 *Let $u \in GBV(\Omega)$; then*

$$F''(u) \le \int_\Omega |\nabla u(x)| dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1} + |D^c u|(\Omega)\,,$$

where ϑ is as in Theorem 3.1.

Proof. Step 1. Assume first that u is in the space $\mathcal{W}(\Omega)$ introduced above. In particular S_u is contained in the union of a finite collection $K_1, \ldots, K_m$ of $(n-1)$-dimensional simplexes. Since the upper Γ-limit of (F_ε) coincides with the upper Γ-limit of $(\overline{F}_\varepsilon)$, we have $F''(u) \le \limsup_\varepsilon \overline{F}_\varepsilon(u)$: we intend to estimate this limit.

Fix $\sigma > 0$. Let S be the union of the relative boundaries of K_i (note that $\mathcal{H}^{n-1}(S) = 0$.) Define

$$\Omega_\sigma = \{x \in \Omega :\ d(x, \partial\Omega) > \sigma,\ d(x, S) > \sigma\}\,.$$

By the integral representation of $\overline{F}_\varepsilon$ given in Proposition 3.6, and by the same argument used in Remark 3.7, we get:

$$\overline{F}_\varepsilon(u) \le \overline{F}_\varepsilon(u, \Omega_\sigma) + \frac{\omega_n}{\gamma} |Du|(A_{\varepsilon,\sigma})\,,$$

where $A_{\varepsilon,\sigma} = \{x \in \Omega :\ d(x, \Omega\setminus\Omega_\sigma) < \varepsilon\}$. Clearly, $\lim_\sigma \limsup_\varepsilon |Du|(A_{\varepsilon,\sigma}) = 0$. To deal with $\overline{F}_\varepsilon(u, \Omega_\sigma)$ it will not be restrictive to assume $m = 1$ and $K := K_1 \subseteq \{x \in \mathbb{R}^n :\ x_1 = 0\}$.

By the subadditivity of f, if $\varepsilon < \sigma$:

$$\begin{aligned}
\overline{F}_\varepsilon(u,\ \Omega_\sigma) &\le \frac{1}{\varepsilon} \int_{\Omega_\sigma} f\left(\varepsilon ⨍_{B_\varepsilon(x)} |\nabla u(y)|\,dy + \frac{1}{\omega_n \varepsilon^{n-1}} |D^s u|\big(B_\varepsilon(x)\big)\right) dx \\
&\le \frac{1}{\varepsilon} \int_{\Omega_\sigma} f\left(\varepsilon ⨍_{B_\varepsilon(x)} |\nabla u(y)|\,dy\right) dx + \frac{1}{\varepsilon} \int_{\Omega_\sigma} f\left(\frac{|D^s u|\big(B_\varepsilon(x)\big)}{\omega_n \varepsilon^{n-1}}\right) dx\,.
\end{aligned}$$

In view of Lemma 7.2, the first of these last two terms tends to $\int_{\Omega_\sigma} |\nabla u|\, dx$. Denote by R_ε the other integral term, and let $K_\varepsilon = \{x \in \mathbb{R}^n : \ d(x, K) < \varepsilon\}$. Since u is smooth outside $\{x_1 = 0\}$, we have:

$$R_\varepsilon = \frac{1}{\varepsilon} \int_{\Omega_\sigma \cap K_\varepsilon} f\left(\frac{1}{\omega_n \varepsilon^{n-1}} |D^s u|\big(B_\varepsilon(x)\big)\right) dx\,.$$

If $K_{\varepsilon,\sigma} \subseteq \mathbb{R}^{n-1}$ denotes the projection of $\Omega_\sigma \cap K_\varepsilon$ onto $\{x_1 = 0\}$, it turns out that $\Omega_\sigma \cap K_\varepsilon$ is contained in $(-\varepsilon, \varepsilon) \times K_{\varepsilon,\sigma}$, which is a subset of $\{x \in \Omega : \ d(x, \partial\Omega) > \varepsilon\}$ for ε sufficiently small. Then:

$$R_\varepsilon \le \frac{1}{\varepsilon} \int_{K_{\varepsilon,\sigma}} dy \int_{-\varepsilon}^{\varepsilon} f\left(\frac{1}{\omega_n \varepsilon^{n-1}} |D^s u|\big(B_\varepsilon(s, y)\big)\right) ds\,.$$

Set

$$h(x) = u^+(x) - u^-(x), \quad x \in \Omega \cap \{x_1 = 0\}.$$

Since $u \in \mathcal{W}(\Omega)$, the function h is continuous, hence uniformly continuous on any compact subset of $\Omega \cap \{x_1 = 0\}$. Then, for ε sufficiently small:

$$|D^s u|\big(B_\varepsilon(s, y)\big) = \int_{B_\varepsilon(s,y) \cap \{x_1 = 0\}} |h|\, d\mathcal{H}^{n-1} \le \omega_{n-1}(\sqrt{\varepsilon^2 - s^2})^{n-1}(|h(0, y)| + \sigma).$$

Therefore

$$R_\varepsilon \le \frac{1}{\varepsilon} \int_{K_{\varepsilon,\sigma}} \left(\int_{-\varepsilon}^{\varepsilon} f\Big(\frac{\omega_{n-1}}{\omega_n} \big(\sqrt{1 - (s/\varepsilon)^2}\big)^{n-1} (|h(0, y)| + \sigma)\Big) ds\right) dy.$$

By the change of variable $t = s/\varepsilon$ we deduce that

$$\limsup_{\varepsilon \to 0} R_\varepsilon \le \int_{\Omega \cap \{x_1 = 0\}} \left(\int_{-1}^{1} f\Big(\frac{\omega_{n-1}}{\omega_n} (\sqrt{1 - t^2})^{n-1} (|h(z)| + \sigma)\Big) dt\right) d\mathcal{H}^{n-1}(z).$$

Now we conclude by taking the limit as $\sigma \to 0$, and noting that $h(z)$ vanishes $\mathcal{H}^{n-1}$-a.e. outside S_u.

Step 2. In the case $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, we can apply Theorem 7.1, with $\phi(a, b, \nu) = \vartheta(|a - b|)$. Then there exists a sequence $w_j \to u$ in $L^1(\Omega)$, with $w_j \in \mathcal{W}(\Omega)$, such that $\nabla w_j \to \nabla u$ in $L^2(\Omega, \mathbb{R}^n)$ and

$$\limsup_{j \to +\infty} \int_{S_{w_j}} \vartheta(|w_j^+(x) - w_j^-(x)|) d\mathcal{H}^{n-1} \le \int_{S_u} \vartheta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1};$$

thus, by the lower semicontinuity of F'' and by *Step 1*,

$$F''(u) \le \liminf_{j \to +\infty} F''(w_j) \le \int_\Omega |\nabla u(x)| dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1}.$$

Using now the relaxation Theorem 2.4, we have

$$F''(u) \le \int_\Omega |\nabla u(x)| dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1} + |D^c u|(\Omega)$$

for every $u \in BV(\Omega)$. Finally, by a truncation argument and again the lower semicontinuity of F'' we obtain the desired inequality in $GBV(\Omega)$. ■

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