Optimal Solutions for a Class of Set-Valued Evolution Problems

Stefano Bianchini[(1)], Alberto Bressan[(2)], and Maria Teresa Chiri[(3)]

[(1)] S.I.S.S.A., via Bonomea 265, Trieste 34136, Italy.

[(2)] Department of Mathematics, Penn State University, University Park, Pa. 16802, USA.

[(3)] Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L3N6, Canada.

E-mails: bianchin@sissa.it, axb62@psu.edu, maria.chiri@queensu.ca

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Abstract

The paper is concerned with a class of optimization problems for moving sets $t \mapsto \Omega(t) \subset \mathbb{R}^2$, motivated by the control of invasive biological populations. Assuming that the initial contaminated set Ω_0 is convex, we prove that a strategy is optimal if an only if at each given time $t \in [0,T]$ the control is active along the portion of the boundary $\partial\Omega(t)$ where the curvature is maximal. In particular, this implies that $\Omega(t)$ is convex for all $t \geq 0$. The proof relies on the analysis of a one-step constrained optimization problem, obtained by a time discretization.

1 Introduction

Motivated by a model in [6, 7], describing the control of an invasive biological species, we consider here the evolution problem for a set $\Omega(t) \subset \mathbb{R}^2$ of finite perimeter, depending on the normal velocity assigned at every boundary point $x \in \partial\Omega(t)$. We think of $\Omega(t)$ as the contaminated set at time $t \geq 0$. If no control is applied, this set expands with unit speed in all directions. By implementing a control strategy, we assume that one can reduce the area of $\Omega(t)$ at rate M per unit time.

To model this situation, for $t \in [0,T]$ and $x \in \partial\Omega(t)$, we denote by $\beta(t,x)$ the normal speed of the boundary at the point x, in the direction of the interior normal. In other words, if the sets $\Omega(t)$ are described by

$$\Omega(t) \;=\; \Big\{x = (x_1,x_2) \in \mathbb{R}^2\,;\;\; \psi(t,x_1,x_2) > 0\Big\}$$

for some differentiable function ψ, then

$$\beta \;\doteq\; \frac{-\psi_t}{\sqrt{\psi_{x_1}^2 + \psi_{x_2}^2}}\,.$$

We denote by

$$E(\beta) \doteq \max \{1+\beta, 0\} \tag{1.1}$$

the **control effort**, needed to push the boundary of Ω inward with speed β.

Definition 1.1 *Given a constant $M > 0$, we say that a set-valued function $t \mapsto \Omega(t)$ is* **admissible** *if the corresponding characteristic function $t \mapsto \mathbf{1}_{\Omega(t)}$ is Lipschitz continuous from $[0,T]$ into $\mathbf{L}^1(\mathbb{R}^2)$, and moreover*

$$\int_{\partial\Omega(t)} E\big(\beta(t,x)\big)\, \mathcal{H}^1(dx) \ \leq\ M, \qquad \textit{for a.e. } t \in [0,T]. \tag{1.2}$$

Here β denotes the velocity of a boundary point in the inward normal direction, and the integral is taken w.r.t. the 1-dimensional Hausdorff measure along the boundary of $\Omega(t)$.

Given an initial set $\Omega_0 \subset \mathbb{R}^2$ and a constant $M > 0$, three problems will be considered.

(NCP) **Null Controllability Problem.** *Find an admissible set-valued function $t \mapsto \Omega(t)$ and a time $T > 0$ such that*

$$\Omega(0) \ =\ \Omega_0, \qquad\qquad \Omega(T) \ =\ \emptyset. \tag{1.3}$$

(MTP) **Minimum Time Problem.** *Among all admissible strategies that satisfy (1.3), find one that minimizes the time T.*

(OP) **Optimization Problem.** *Given a time interval $[0,T]$ and constants $c_1, c_2 \geq 0$, find an admissible set-valued function $t \mapsto \Omega(t)$ which minimizes the cost*

$$J \ =\ c_1 \int_0^T \mathcal{L}^2\big(\Omega(t)\big)\, dt + c_2\, \mathcal{L}^2\big(\Omega(T)\big), \tag{1.4}$$

subject to $\Omega(0) = \Omega_0$.

Here and in the sequel, we use the notation $\mathcal{L}^2(\Omega)$ to denote the 2-dimensional Lebesgue measure of a set $\Omega \subset \mathbb{R}^2$, while $\mathcal{H}^1(\partial\Omega)$ will be used to denote the 1-dimensional Hausdorff measure of its boundary.

Remark 1.1 When the control effort is zero, the set $\Omega(t)$ expands in all directions with unit speed. On the other hand, the bound (1.2) on the instantaneous control effort allows us reduce the area of $\Omega(t)$ at rate M per unit time. This yields a basic relation between the growth rate of the area of $\Omega(t)$ and its perimeter:

$$\frac{d}{dt}\mathcal{L}^2\big(\Omega(t)\big) \ =\ \mathcal{H}^1\big(\partial\Omega(t)\big) - M. \tag{1.5}$$

The Null Controllability Problem **(NCP)** can thus be solved if and only if we can reduce the perimeter $P(t) = \mathcal{H}^1\big(\partial\Omega(t)\big)$ to a value strictly smaller than M.

A more general class of optimization problems for moving sets was recently considered in [7], proving the existence of optimal strategies and deriving some necessary conditions for optimality.

In this paper we consider the optimization problems **(MTP)** and **(OP)**, assuming that the initial set Ω_0 is convex. Our main result, Theorem 5.1, completely characterizes the optimal strategies. Confirming a conjecture proposed in [7], we prove that a strategy is optimal if an only if, at each given time $t \in [0, T]$, the control is active precisely along the portion of the boundary $\partial\Omega(t)$ where the curvature is maximal. We observe that, with this control, the perimeter $P(t)$ also shrinks at the fastest possible rate. Moreover, all sets $\Omega(t)$ remain convex.

As a preliminary to the proof of the main theorem, Section 2 collects several geometric results concerning r-semiconvex sets, i.e., sets that satisfy the outer sphere condition with radius r. In particular, we prove a sharp bound on their perimeter, and give estimates on how the area of their r-neighborhood changes, when the boundary is perturbed.

In Section 3 we study a one-step minimization problem, derived from the original evolution problem by discretizing time. More precisely, given a compact convex set $U \subset \mathbb{R}^2$ and a constant $0 < a < \mathcal{L}^2(U)$, we seek a subset $\Omega \subset U$ with area $\mathcal{L}^2(\Omega) = a$, such that the area of its r-neighborhood $B_r(\Omega)$ is as small as possible. By a detailed analysis, we prove that this optimal set Ω is always convex, see Theorem 3.1. As a consequence, the set Ω is also optimal for the problem of minimizing the perimeter, subject to the same constraints. In the literature, this second problem has already been studied in [16], and is well understood in dimension $n = 2$. Properties of constrained perimeter-minimizing sets, described in Section 4, play a key role in determining the optimal strategy $t \mapsto \Omega(t)$ for both **(OP)** and **(MTP)**.

For an introduction to geometric measure theory and BV functions we refer to [1, 12, 14]. Various other models of moving sets, subject to external control, have been considered in [4, 5, 8, 9, 10, 11]. More detailed models of the control of an invasive biological species can be found in [2, 3]. See also [15] for related results on controlled reaction-diffusion equations.

2 Preliminary geometric lemmas

Throughout the following, $B_r(x)$ denotes the open ball centered at x with radius r, while

$$\Omega^r \doteq \Omega + B_r(0) = \big\{x\,;\ \mathrm{dist}(x, \Omega) < r\big\}$$

denotes the open neighborhood of radius r around the set $\Omega \subset \mathbb{R}^2$, and we will use the standard notation $\mathcal{L}^2$ for the Lebesgue measure in $\mathbb{R}^2$ and $\mathcal{H}^1$ for the 1-dimensional Hausdorff measure. The closure, the interior, and the convex hull of a set Ω are denoted by $\mathrm{clos}\,\Omega$, $\mathrm{int}\,\Omega$ and $\mathrm{conv}\,\Omega$, respectively. Given a vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, we write $\mathbf{v}^\perp = (-v_2, v_1)$ for the perpendicular vector.

Let $E \subset \mathbb{R}^2$ be a compact convex set with nonempty interior, and let $t \mapsto \gamma(t)$ be an arc-length parametrization of the boundary ∂E, oriented counterclockwise. Call $\mathbb{S}^1 = \{\mathbf{e} \in \mathbb{R}^2\,;\ |\mathbf{e}| = 1\}$ the set of unit vectors in $\mathbb{R}^2$. For every boundary point $x \in \partial E$ consider the set of outer unit normals

$$\mathbf{n}(x) \doteq \big\{\mathbf{e} \in \mathbb{S}^1\,;\quad \mathbf{e}\cdot(x - y) \geq 0 \quad \text{for all } y \in E\big\}.$$

We observe that the (possibly multivalued) map $t \mapsto \mathbf{n}(t) \doteq \mathbf{n}\big(\gamma(t)\big) \subset \mathbb{S}^1$ is a set-valued function with closed graph and connected values. We will sometimes use the notation $\mathbf{n}(t) = e^{i\theta(t)}$. In this case $t \mapsto \theta(t)$ is a monotone multifunction. A few observations are in order.

1. The map $t \mapsto \mathbf{n}(t)$ is a BV function, with total variation $TV(\mathbf{n}(\cdot)) = 2\pi$. The absolutely continuous and the singular part of its measure-valued derivative will be denoted by

$$D\mathbf{n} \;=\; D^{ac}\mathbf{n} + D^{sing}\mathbf{n} \;\in\; \mathcal{M}(\partial\Omega, \mathbb{S}^1).$$

Note that with this notation we require that in the jump points the quantity $\big|\mathbf{n}(x+) - \mathbf{n}(x-)\big|$ is the length of the smaller arc $\big[\mathbf{n}(x-), \mathbf{n}(x+)\big]$, i.e. the jump $\big|\theta(t+) - \theta(t-)\big|$, $x = \gamma(t)$.

2. If $A \subset \partial E$ is a Borel subset, then (see Fig. 1, left)

$$\mathcal{H}^1\bigg(\bigcup_{x\in A} x + h\mathbf{n}(x)\bigg) \;=\; \mathcal{H}^1(A) + h|D\mathbf{n}|(A) \;=\; \mathcal{H}^1(A) + hD\theta(A), \tag{2.1}$$

$$\mathcal{L}^2\bigg(\bigcup_{x\in A,\ \rho\in[0,h]} x + \rho\mathbf{n}(x)\bigg) \;=\; h\mathcal{H}^1(A) + \frac{1}{2}h^2|D\mathbf{n}|(A) \;=\; h\mathcal{H}^1(A) + \frac{1}{2}h^2 D\theta(A). \tag{2.2}$$

These formulas generalize the classical Steiner's formulas for the perimeter and the area of the r-neighborhood around a convex set (see Theorem 10.1 in [13]). They can be proved by approximating E with a polygon and passing to the limit. Notice that the second is the integral of the first one, by the coarea formula. The above identities can be extended to a more general class of sets with finite perimeter.

Definition 2.1 *Given a closed set $E \subset \mathbb{R}^2$, an open set F and a radius $r > 0$, we write*

$$E^r \;\doteq\; \bigcup_{x\in E} B_r(x), \qquad\qquad F^{-r} \;\doteq\; \bigg(F \setminus \bigcup_{y\notin F} B_r(y)\bigg) \;=\; \mathbb{R}^2 \setminus (\mathbb{R}^2 \setminus F)^r.$$

We say that E, F are in r-duality (or simply in duality) if $F = E^r$ and $E = F^{-r}$.

From the above definition, it immediately follows that the two sets E^r and $(E^r)^{-r}$ are in duality. Note that we always have $E \subseteq (E^r)^{-r}$, but equality does not hold, in general.

Definition 2.2 *Let $E \subset \mathbb{R}^2$ be a closed set, and let $r > 0$. We say that E has the* **exterior r-ball property**, *or equivalently that E is* **r-semiconvex**, *if for every boundary point $x \in \partial E$ there is an outer ball $B_r(y) \subset \mathbb{R}^2 \setminus E$ with $|x - y| = r$. When this holds, we say that the segment with endpoints x, y is an* **optimal ray**.
We say that an open set F has the **interior r-ball property**, *or equivalently that F is* **r-semiconcave**, *if every point $x \in F$ is contained in some open ball $B_r(y) \subseteq F$.*

In the following, having fixed the radius r, for shortness we will just say semiconvex/semiconcave. As immediate consequences of the above definitions, one has:

(i) E is r-semiconvex iff $E = (E^r)^{-r}$.

(ii) F is r-semiconcave iff $F = (F^{-r})^r$.

(iii) For every closed set E, every point $x \in \partial E^r$ belongs to the boundary of a ball of radius r contained in E^r, with center in E;

(iv) For every closed set E, every point $x \in \partial (E^r)^{-r}$ belongs to the boundary of a ball of radius r contained in $\mathbb{R}^2 \setminus E$, with center in ∂E^r.

We observe that, if E is compact, r-semiconvex with int E connected but not simply connected, then each of its "holes" must contain an open ball of radius r. Therefore, the complement $\mathbb{R}^2 \setminus E$ can have at most finitely many connected components. To fix ideas, we call V_1 the unbounded component, and $V_2, \ldots, V_N$ the bounded components. Each boundary ∂V_k, $k = 1, \ldots, N$ is a simple closed curve with finite length. Let $t \mapsto \gamma_k(t)$ be an arc-length parameterization of ∂V_k, oriented counterclockwise in the case of V_1 and clockwise for $V_2, \ldots, V_N$. As before, let $\mathbf{n}(x)$ be the set of outer unit normals at the point $x \in \partial E$. Using complex notation, we again write $\mathbf{n}_k(t) = \mathbf{n}(\gamma_k(t)) = e^{i\theta_k(t)}$ for the set of unit outer normal vectors at the point $\gamma_k(t)$. The assumption of r-semiconvexity implies that the (possibly multivalued) map $t \mapsto \theta_k(t)$ has bounded variation. Its distributional derivative satisfies

$$D\theta_k \;\geq\; -\frac{1}{r}\mathcal{L}^1. \tag{2.3}$$

Indeed, the negative part of the measure $D\theta$ is absolutely continuous and has uniformly bounded density w.r.t. 1-dimensional Lebesgue measure $\mathcal{L}^1$ on the unit circumference $\mathbb{S}^1$. For future use, we notice that this yields:

Lemma 2.1

(i) The set $\mathbb{R}^2 \setminus V_1$ is not convex if and only if there exists a point $x^* = \gamma_1(t^*)$ where the function $t \mapsto \theta_1(t)$ is differentiable, with a strictly negative derivative.

(ii) For every $k = 2, \ldots, N$, there exists a point $x^* = \gamma_k(t^*)$ where the function $t \mapsto \theta_k(t)$ is differentiable, with a strictly negative derivative.

The identities (2.1)-(2.2) have counterparts for general r-semiconvex sets. More precisely, consider a Borel subset $A \subseteq \partial V_k \subseteq \partial E$. For $0 < h \leq r$ we then have

$$\mathcal{H}^1\Big(\bigcup_{x\in A} x + h\mathbf{n}(x)\Big) \;\leq\; \mathcal{H}^1(A) + hD\theta_k(A), \tag{2.4}$$

$$\mathcal{L}^2\Big(\bigcup_{x\in A,\ \rho\in[0,h]} x + \rho\mathbf{n}(x)\Big) \;\leq\; h\mathcal{H}^1(A) + \frac{1}{2}h^2 D\theta_k(A). \tag{2.5}$$

These formulas can again be proved by approximating the set E with polygons, then passing to the limit. The second one is obtained from the first by an integration. We observe that, if E is not convex, then there can be distinct points $x, y \in \partial E$ such that the sets $\{x+\rho\mathbf{n}(x)\,;\ \rho \in [0,h]\}$, $\{y+\rho\mathbf{n}(y)\,;\ \rho \in [0,h]\}$ have non-empty intersection (see Fig. 1, right). This motivates the inequality signs in (2.4)-(2.5).

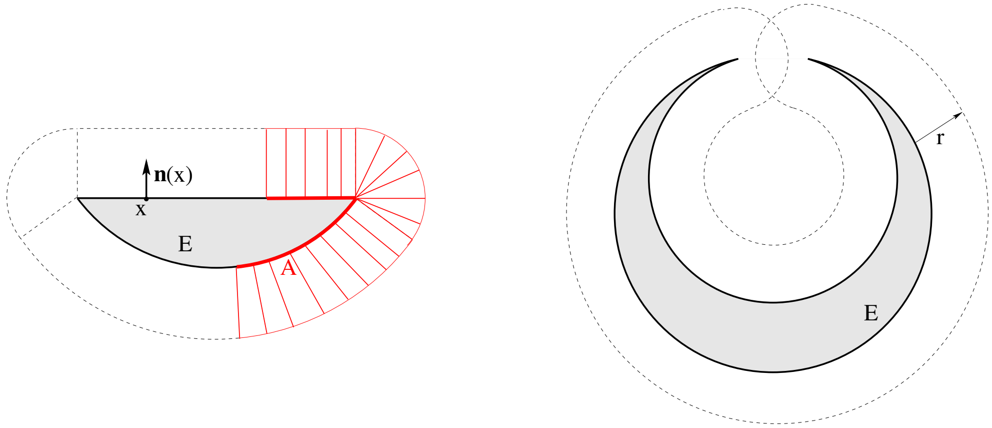

Figure 1: Left: a convex set E and its r-neighborhood. Given a measurable subset of the boundary $A \subset \partial E$, the set of points $y \in E^r$ that project onto A has area computed by (2.2). Right: if the set E is r-convex but not convex, the formula (2.5) may hold only as an inequality, because of the overlap.

2.1 The perimeter of a semiconvex set.

Lemma 2.2 *Any r-semiconvex set E has locally finite perimeter. Indeed, for any ball of radius r, one has*

$$\mathcal{H}^1\big(B_r(\bar{x}) \cap \partial E\big) \;\leq\; 2\pi r. \tag{2.6}$$

Proof. 1. As a first step, we observe that the boundary ∂E is rectifiable with length locally finite. Indeed, for any unit vector $\mathbf{e} \in \mathbb{S}^1$, consider the set of points $x \in \partial E$ which have an optimal ray of direction $\mathbf{n}(x)$ such that $\big|\mathbf{n}(x) - \mathbf{e}\big| < 1/2$. By the outer ball property of r-semiconvex sets, for every such an x there is an open cone C_x with axis in the direction $\pm\mathbf{e}$ and opening $> \pi/6$ such that $\partial E \cap C_x \cap B_r(x) = \emptyset$: indeed, $C_x \cap B_r(x) \subset B_r(x + r\mathbf{e}) \subset \mathbb{R}^2 \setminus E$. A standard rectifiability criterion (see Theorem 2.61 in [1]) shows that ∂E is rectifiable, and moreover the fact that the curves are separated gives that their total length is locally finite.

2. Next, we claim that, for any fixed ball $B_r(\bar{x})$ and any finite family of balls $B_r(x_i)$, $i = 1, \ldots, n$, there holds

$$\mathcal{H}^1\left(B_r(\bar{x}) \cap \partial \bigcup_{i=1}^n B_r(x_i)\right) \;\leq\; \mathcal{H}^1\left(\partial B_r(\bar{x}) \cap \bigcup_{i=1}^n B_r(x_i)\right) \;\leq\; 2\pi r. \tag{2.7}$$

Indeed, the above estimate is trivial when $n = 1$. By induction, assume that it holds for n and consider an additional ball $B_r(x_{n+1})$. By relabeling, we can assume that $\partial B_r(x_{n+1}) \setminus \cup_i^n \mathrm{clos}\, B_r(x_i)$ is an arc intersecting $\partial B_r(\bar{x})$ in at least one point.

- If the intersection
$$\partial B_r(x_{n+1}) \cap \left(\partial B_r(\bar{x}) \setminus \bigcup_{i=1}^n B_r(x_i)\right) \tag{2.8}$$
consists of two distinct points y_1, y_2 (see Fig. 2, center), then either
(i) there is a ball $B_r(x_i)$ such that $B_r(x_i) \cap B_r(\bar{x}) \subset B_r(x_{n+1}) \cap B_r(\bar{x})$. In this case the ball $B_r(x_i)$ can be removed from the family of balls and we can apply the recurrence

hypothesis. Or else
(ii) we observe that

$$
\begin{aligned}
&\mathcal{H}^1\Big(B_r(\bar{x}) \cap \partial \textstyle\bigcup_{i=1}^{n+1} B_r(x_i)\Big) \\
&= \mathcal{H}^1\Big(B_r(\bar{x}) \cap \Big[\Big(\partial B_r(x_{n+1}) \setminus \textstyle\bigcup_{i=1}^{n} B_r(x_i)\Big) \cup \Big(\partial \textstyle\bigcup_{i=1}^{n} B_r(x_i) \setminus B_r(x_{n+1})\Big)\Big]\Big) \\
&\leq \mathcal{H}^1\Big(B_r(\bar{x}) \cap \partial B_r(x_{n+1})\Big) + \mathcal{H}^1\Big(B_r(\bar{x}) \cap \partial \textstyle\bigcup_{i=1}^{n} B_r(x_i)\Big) \\
&\leq \mathcal{H}^1\Big(\partial B_r(\bar{x}) \cap B_r(x_{n+1})\Big) + \mathcal{H}^1\Big(\partial B_r(\bar{x}) \cap \textstyle\bigcup_{i=1}^{n} B_r(x_i)\Big) \\
&= \mathcal{H}^1\Big(\partial B_r(\bar{x}) \cap \textstyle\bigcup_{i=1}^{n+1} B_r(x_i)\Big).
\end{aligned}
\tag{2.9}
$$

Hence the first inequality in (2.7) holds.

- On the other hand, if the intersection (2.8) consists of a single point y_1 (see Fig. 2, right), then it contributes to the measure above by an arc $\widehat{y_1 y_2} \subset \partial B_r(x_{n+1}) \cap B_r(\bar{x})$. In this case, the arc $\widehat{y_1 y_2}$ replaces the set $B_r(x_{n+1}) \cap B_r(\bar{x}) \cap \partial\Big(\bigcup_{i=1}^{n} B_r(x_i)\Big)$, and either
(i) there is ball $B_r(x_i)$ which is not contributing to $B_r(\bar{x}) \cap \partial\big(\bigcup_{i=1}^{n+1} B_r(x_i)\big)$. This happens when two or more balls are involved in the set $B_r(x_{n+1}) \cap B_r(\bar{x}) \cap \partial\Big(\bigcup_{i=1}^{n} B_r(x_i)\Big)$. In this case, the ball $B_r(x_i)$ can be removed, and the result follows by the inductive assumption. Or else
(ii) the arc $\widehat{y_1 y_2}$ replaces an arc $\widehat{y_0 y_2} = B_r(x_{n+1}) \cap \partial B_r(x_j)$. A simple computation shows that
$$\text{length of } \widehat{y_1 y_2} \;\leq\; \text{length of } \widehat{y_0 y_2} + \text{length of } \widehat{y_0 y_1},$$
where $\widehat{y_0 y_1}$ is the the arc of $\partial B_r(\bar{x})$ which belongs to $B_r(x_{n+1}) \cap B_r(\bar{x}) \setminus \cup_i^n B_r(x_i)$. Hence again (2.7) is verified.

By induction, the estimate (2.7) holds for every $n \geq 1$.

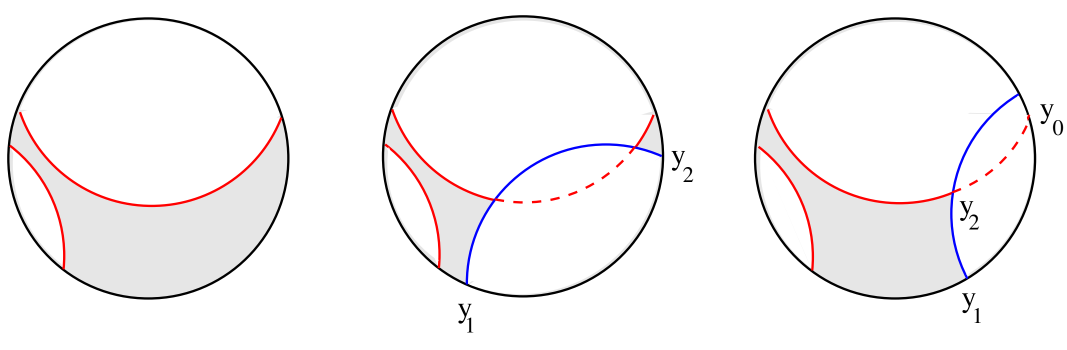

Figure 2: Left: the set $B_r(\bar{x}) \setminus \bigcup_{i=1}^{n} B_r(x_i)$. Center and right: the set $B_r(\bar{x}) \setminus \bigcup_{i=1}^{n+1} B_r(x_i)$, in the two cases considered in step **2** of the proof of Lemma 2.2.

3. To complete the proof, we need to consider the general case where E is any r-semiconvex set. Since $\partial E \cap B_r(\bar{x})$ is rectifiable with finite length, by the Vitali-Besicovitch Covering Theorem [1, Theorem 2.19] for every $\varepsilon > 0$ there is a finite set of points $x_i \in \partial E \cap B_r(\bar{x})$ and radii $r_i < \varepsilon$, $i = 1, \ldots, N$, such that

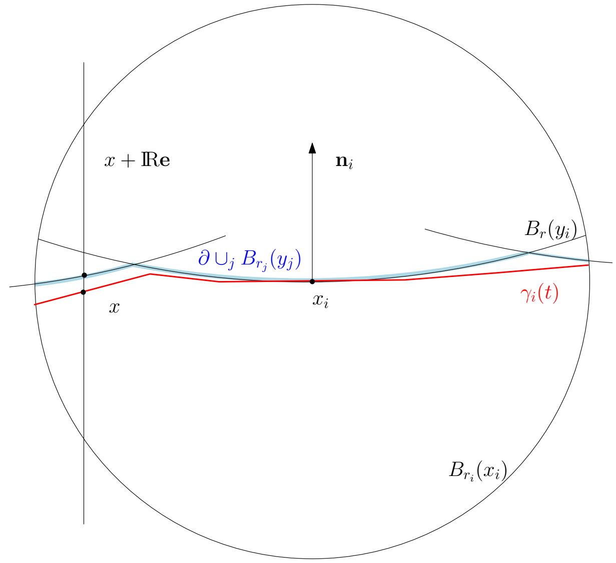

Figure 3: Up to two small end parts, to each point of $x \in \partial E \cap \mathrm{graph}(\gamma_i) \cap B_{r_i}(x_i)$ (red) there corresponds at least one point of $\partial \cup_j B_r(y_j)$ (light blue), giving (2.10).

1. x_i belongs to the reduced boundary $\partial^* E$, i.e. it has a unique normal $\mathbf{n}_i$;
2. $B_{r_i}(x_i) \subset B_r(\bar{x})$;
3. there is a Lipschitz curve γ_i such that
$$|\dot{\gamma}_i - \mathbf{n}_i^\perp| < \varepsilon \quad \text{and} \quad \mathcal{H}^1\big(\mathrm{graph}(\gamma_i) \cap B_{r_i}(x_i)\big) > (1-\varepsilon)2r_i;$$
4. it holds
$$\sum_i \big|\mathcal{H}^1(\partial E \cap B_r(x_i)) - 2r_i\big| < \varepsilon, \qquad \bigg|\mathcal{H}^1(\partial E \cap B_r(\bar{x})) - \sum_i 2r_i\bigg| < \varepsilon.$$

Indeed, the balls satisfying the first 3 statements are a fine covering of $\partial^* E \cap B_r(\bar{x})$.

For each x_i, consider the optimal ray $[x_i, y_i]$ (there is only one because $x_i \in \partial^* E$) and the family of balls $B_r(y_i) \subset \mathbb{R}^2 \setminus E$. The boundary of the set $\cup_i B_r(y_i)$ consists of finitely many curves, and inside every ball $B_{r_i}(x_i)$ its length must be at least

$$\mathcal{H}^1\bigg(B_{r_i}(x_i) \cap \partial \bigcup_j B_r(y_j)\bigg) \;>\; (1-2\varepsilon)(2r_i) \tag{2.10}$$

for $r_i \ll 1$. Indeed, for each point $x \in \partial E \cap \mathrm{graph}(\gamma_i) \cap B_{(1-\varepsilon)r_i}(x_i)$ the line $x + \mathbb{R}\mathbf{n}_i$ must intersect the boundary of $\cup_i B_r(x_i)$ at one point inside $B_{r_i}(x_i)$ (see Fig. 3).

In view of (2.7), we thus conclude that

$$\begin{aligned}\mathcal{H}^1(\partial E \cap B_r(\bar{x})) \leq{}& \varepsilon + \sum_i 2r_i \;\leq\; \varepsilon + \frac{1}{1-2\varepsilon} \sum_i \mathcal{H}^1\bigg(B_{r_i}(x_i) \cap \partial \bigcup_j B_r(y_j)\bigg) \\ \leq{}& \varepsilon + \frac{1}{1-2\varepsilon} \mathcal{H}^1\bigg(\partial B_r(\bar{x}) \cap \bigcup_j B_r(y_j)\bigg) \;<\; \varepsilon + \frac{2\pi r}{1-2\varepsilon}.\end{aligned}$$

Taking the limit $\varepsilon \searrow 0$ we obtain the bound (2.6). $\square$

Remark 2.1 In the case of different radii $0 < \rho < r$, the same arguments used in the proof of Lemma 2.2 show that the estimate (2.7) can be replaced by

$$\mathcal{H}^1\left(B_\rho(\bar{x}) \cap \partial \bigcup_{i=1}^n B_r(x_i)\right) \leq 2\pi\rho. \tag{2.11}$$

2.2 A property of r-semiconvex sets.

Throughout this section we consider a compact set $E \subset \mathbb{R}^2$ whose boundary is a simple closed curve $s \mapsto \gamma(s)$, parameterized by arc-length and oriented counterclockwise. As before, the unit outer normals are denoted by $e^{i\theta(s)} = \mathbf{n}(s) = -\dot{\gamma}(s)^\perp$.

If the set E is r-semiconvex, then the negative part of of the distributional derivative of θ is absolutely continuous w.r.t. one-dimensional Lebesgue measure. Namely,

$$D\theta \geq -\frac{1}{r}\mathcal{L}^1. \tag{2.12}$$

Conversely, assume that $e^{i\theta} = -\dot{\gamma}^\perp$ and that (2.12) holds, so that θ has bounded variation. One can then define the set of unit normal vectors as

$$\mathbf{n}(t) = \big\{e^{i\phi}, \phi \in [\theta(t-), \theta(t+)]\big\}.$$

Equivalently, $\mathbf{n}(t)$ is the set of unit vectors $\mathbf{n} \in \mathbb{S}^1$ such that

$$\limsup_{\delta \searrow 0} \frac{\sup\big\{\mathbf{n}\cdot(y-\gamma(t))\,; \quad y \in E \cap B_\delta(\gamma(t))\big\}}{\delta} \leq 0.$$

Let (2.12) hold and consider any boundary point $x \in \partial E$. As shown in Fig. 4, by moving along the boundary of E, it is possible to get into the interior of the outer tangent ball $B_r\big(x+r\mathbf{n}(x)\big)$, but only after having travelled along an arc of length $> \pi r$.

Lemma 2.3 *In the above setting, if (2.12) holds, then for every $x = \gamma(\bar{t}) \in \partial E$ one has*

$$B_r\big(\gamma(\bar{t}) + r\mathbf{n}(\bar{t})\big) \cap \Big\{\gamma(t)\,; \quad t \in [\bar{t} - \pi r\,,\ \bar{t} + \pi r]\Big\} = \emptyset. \tag{2.13}$$

Proof. If (2.13) fails, then there exists $0 < \rho < r$ such that

$$B_r\big(\gamma(\bar{t}) + \rho\mathbf{n}(\bar{t})\big) \cap \Big\{\gamma(t)\,; \quad t \in [\bar{t} - \pi \rho\,,\ \bar{t} + \pi \rho]\Big\} \neq \emptyset. \tag{2.14}$$

A contradiction can then be obtained in two steps.

1. We claim that, for every $\bar{t}$, there exists $\varepsilon > 0$ such that

$$B_r\big(\gamma(\bar{t}) + r\mathbf{n}(\bar{t})\big) \cap \Big\{\gamma(t)\,; \quad t \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon]\Big\} = \emptyset. \tag{2.15}$$

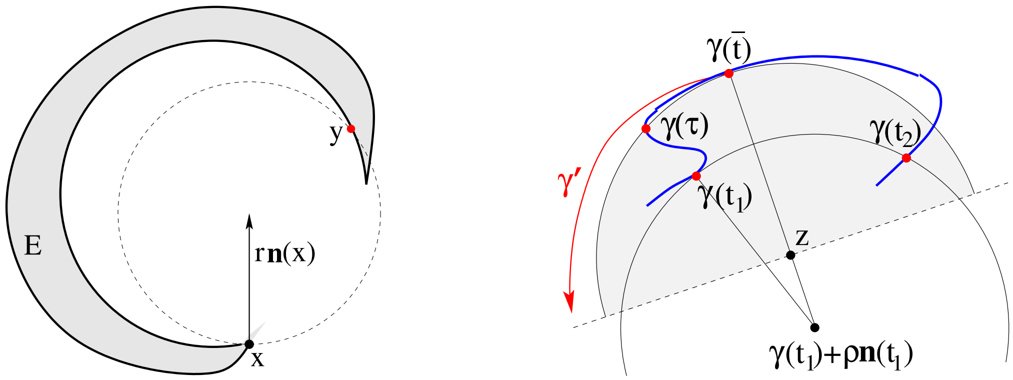

Figure 4: Left: if (2.12) holds, the outer curvature radius is $\geq r$. Hence in a neighborhood of x the set E lies outside the ball $B_r\big(x+r\mathbf{n}(x)\big)$. However, the boundary ∂E can enter this ball at a point y, where the boundary arc $\widehat{xy}$ has length $>\pi r$. Right: the points $\gamma(t_1)$, $\gamma(t_2)$, $\gamma(\bar t)$ and $\gamma(\tau)$, considered in step **2** of the proof of Lemma 2.3. Starting from $\gamma(\bar t)$ and moving toward $\gamma(t_1)$, we enter the ball $B_\rho(z)$ (shaded region) at a point $\gamma(\tau)$, before reaching $\gamma(t_1)$. Indeed, any curve γ' of length $\leq \pi\rho/2$ starting at $\gamma(\bar t)$ and remaining outside the ball $B_\rho(z)$ cannot touch the ball $B_\rho\big(\gamma(t_1)+\rho\mathbf{n}(t_1)\big)$. The inductive process eventually identifies a point $\gamma(t^*)$ where the outer radius of curvature is $\leq \rho < r$, thus obtaining a contradiction.

Indeed, by a translation and rotation of coordinates, we can assume that $\bar t = 0$, $\gamma(0)=0$ and $\dot\gamma = (1,0)$. The equation for γ can be locally written as

$$\dot x(t) = \cos\theta(t), \quad \dot y(t) = \sin\theta(t), \qquad -\frac{\pi}{2} < \theta(0-) \leq \theta(0+) < \frac{\pi}{2},$$

where the map $t\mapsto\theta(t)$ satisfies (2.12).

Since $t\mapsto x(t)$ is Lipschitz, in the interval of invertibility (i.e. as long as $\cos\theta(t)>0$) we obtain

$$\frac{\theta(t_2)-\theta(t_1)}{x(t_2)-x(t_1)} \;\geq\; -\frac{t_2-t_1}{r}\cdot\left(\int_{t_1}^{t_2}\cos\theta(\tau)d\tau\right)^{-1}.$$

This implies that $x\mapsto\theta(t(x))$ satisfies

$$D_x\theta(t(x)) \;\geq\; -\frac{1}{r\cos\theta}, \qquad \sin(\theta(t(x))) \;\geq\; -\frac{x}{r},$$

$$\frac{dy}{dx} \;=\; \tan\theta(t(x)) \;\geq\; -\frac{x}{\sqrt{r^2-x^2}}.$$

Since this argument is valid both for $t>0$ and for $t<0$, one concludes that

$$y(t) \;\geq\; r-\sqrt{r^2-(x(t))^2}$$

in a nonempty interval $t\in[-\varepsilon,\varepsilon]$ where $\cos\theta(t)>0$. This proves (2.15).

2. Next, to prove the global property (2.13), consider any point $\gamma(t_1)\in\partial E$, and choose $t_2>t_1$ so that $\gamma(t_2)$ is the first point where the curve γ touches again the set $\operatorname{clos} B_\rho\big(\gamma(t_1)+\rho\mathbf{n}(t_1)\big)$. In other words (see Fig. 4, right),

$$t_2 \;=\; \min\Big\{t>t_1\,;\;\; \gamma(t)\in\operatorname{clos} B_\rho\big(t_1+\rho\mathbf{n}(t_1)\big)\Big\}.$$

Notice that, by (2.15), one has $t_2 \geq t_1 + \varepsilon$, hence the above minimum is well defined.

If $t_2 - t_1 < \rho\pi$, we will derive a contradiction. As shown in Fig. 4, right, let $\bar{t} \in [t_1, t_2]$ be such that the distance $\big|\gamma(\bar{t}) - (\gamma(t_1) + \rho\mathbf{n}(t_1))\big|$ is maximal. Consider the ball $B_\rho(z)$, where

$$z \;=\; \gamma(\bar{t}) + \rho\mathbf{n}(\bar{t}), \qquad\qquad \mathbf{n}(\bar{t}) \;=\; \frac{\gamma(t_1) + \rho\mathbf{n}(t_1) - \gamma(\bar{t})}{\big|\gamma(t_1) + \rho\mathbf{n}(t_1) - \gamma(\bar{t})\big|}\,.$$

Since $t_2 - t_1 \leq \pi r$, then one of the intervals $[t_1, \bar{t}]$, $[\bar{t}, t_2]$ is shorter than $|t_2 - t_1|/2 \leq \pi r/2$. To fix ideas, assume $\bar{t} - t_1 \leq t_2 - \bar{t}$. Starting from $\gamma(\bar{t})$, we move along this shorter arc $[t_1, \bar{t}]$ toward $\gamma(t_1)$ until we reach a first point $\gamma(\tau) \in \partial B_\rho\big(\gamma(\bar{t}) + \rho\mathbf{n}(\bar{t})\big)$. Notice that this first intersection point is well defined and bounded away from $\gamma(\bar{t})$, because of the argument in step **1**.

Moreover, τ cannot coincide with t_1. Indeed, the point $\gamma(\tau)$ lies on the half circumference

$$\big\{y \in B_\rho(z)\,;\ (z - y)\cdot\mathbf{n}(\bar{t}) \geq 0\big\},$$

while $\gamma(t_1)$ lies on the arc of circumference

$$\Big\{y \in B_\rho\big(\gamma(t_1) + \rho\mathbf{n}(t_1)\big)\,;\ (z - y)\cdot\mathbf{n}(\bar{t}) \geq 0\Big\}.$$

These two arcs do not have any point in common.

We then repeat the above construction, replacing $[t_1, t_2]$ with this new interval $[\bar{t}, \tau]$. By induction, we thus obtain a sequence of nested intervals $[t_{1,n}, t_{2,n}]$, with $t_{2,n} - t_{1,n} < 2^{-n}\pi r$, such that

$$\gamma(t_{2,n}) \in \partial B_\rho(\gamma(t_{1,n}) + \rho\mathbf{n}(t_{1,n})), \quad \gamma\big(\,]t_{1,n}, t_{2,n}[\,\big) \cap \operatorname{clos} B_\rho(\gamma(t_{1,n}) + \rho\mathbf{n}(t_{1,n})) \;=\; \emptyset.$$

Since the length of these intervals is converging to zero, if t^* is the limit of the sequence, then for $n \gg 1$ we obtain a contradiction with step **1**, since $[t_{1,n}, t_{2,n}]$ would be contained in the set where γ does not intersect any of the tangent open balls. This concludes the proof. □

Corollary 2.1 *In the above setting, if (2.12) holds and $\mathcal{H}^1(\partial E) \leq 2\pi r$, then E is r-semiconvex. More precisely, each point $\gamma(t) + r\mathbf{n}(t)$ has distance r from E, and $E = (E^r)^{-r}$.*

Moreover, for every portion of the boundary γ of length $t_2 - t_1 \leq \pi r$, the rays

$$\rho \mapsto \gamma(t) + \rho\mathbf{n}(t)\,, \qquad\qquad \rho \in [0, r]\,, \quad t \in [t_1, t_2]$$

can intersect only at the initial point $\rho = 0$ and final point $\rho = r$, at most. We can thus obtain the same formulas (2.1)-(2.2) over this portion of ∂E. For every Borel subset $A \subset \gamma([t_1, t_2])$, there holds

$$\mathcal{H}^1\bigg(\bigcup_{x\in A} x + h\mathbf{n}(x)\bigg) \;=\; \mathcal{H}^1(A) + h|D\mathbf{n}|(A) \;=\; \mathcal{H}^1(A) + hD\theta(A), \tag{2.16}$$

$$\mathcal{L}^2\bigg(\bigcup_{x\in A} x + h\mathbf{n}(x)\bigg) \;=\; h\mathcal{H}^1(A) + \frac{1}{2}h^2|D\mathbf{n}|(A) \;=\; h\mathcal{H}^1(A) + \frac{1}{2}h^2 D\theta(A). \tag{2.17}$$

2.3 Local perturbations of convex sets.

Let E be a compact convex set, with boundary parameterized by $t \mapsto \gamma(t)$. As before, let $e^{i\theta(t)} = \mathbf{n}(t) = -\dot\gamma(t)^\perp$. Let $\bar t$ be a Lebesgue point for $D\theta(t)$, so that

$$\lim_{\delta\to 0+} \frac{1}{\delta}\big|D\theta - \kappa\mathcal{L}^1\big|\big([\bar t-\delta,\ \bar t+\delta]\big) \ = \ 0.$$

Here $\kappa \geq 0$ is a constant describing the local curvature. This implies that for every $\varepsilon > 0$ one can find $\delta > 0$ such that

$$\big|\mathbf{n}(t) - \mathbf{n}(\bar t) - \kappa\dot\gamma(\bar t)(t-\bar t)\big| \ \leq \ \varepsilon|t-\bar t| \qquad \text{for } t \in [\bar t-\delta,\ \bar t+\delta].$$

Observe that, up to a rigid motion and decreasing δ in case, locally we can write the curve γ as the graph of a positive convex function $f(x)$, $x \in [-\delta,\delta]$, with

$$f(0) = 0, \qquad \big|f'(x) - \kappa x\big| \leq \varepsilon|x|, \qquad \lim_{\delta\to 0}\frac{1}{\delta}\big|D^2 f - \kappa\mathcal{L}^1\big|([-\delta,\delta]) = 0. \tag{2.18}$$

Consider a semiconvex function $\phi \leq 0$ such that

$$\operatorname{supp}\phi \subset [-\delta,\delta], \qquad\qquad D^2\phi \geq -\frac{1}{r}\mathcal{L}^1. \tag{2.19}$$

It is easy to see that these conditions implies that $|\phi'(x)| \leq \delta$.

Define the set E' by

$$E' = E \cup \big\{y \in f(x) + [\phi(x), 0], x \in [-\delta,\delta]\big\}. \tag{2.20}$$

Its boundary is the simple closed curve γ' obtained by replacing the part of its graph equal to $\{(x, f(x)), x \in [-\delta,\delta]\}$ with the graph $\{(x, f(x)+\phi(x)), x \in [-\delta,\delta]\}$. In the interval $x \in [-\delta,\delta]$, the curvature of γ' is clearly a measure and its a.c. part can be easily computed as

$$\frac{f''+\phi''}{\sqrt{1+(f'+\phi')^2}} \geq -\frac{1}{r}.$$

Next, let t be a parametrization of the new curve γ', and for any choice of t and $\mathbf{m} = e^{i\alpha}$ with $\alpha \in [\theta'(t-), \theta'(t+)]$, consider the segment

$$\big\{\gamma(t) + \sigma\mathbf{m}\,; \quad 0 < \sigma < r\big\}.$$

We claim that all these segments are disjoint. Indeed, it is enough to verify this statement for segments when $\gamma'(t)$ belongs to the graph of $f+\phi$ and $\phi \neq 0$. In this case, since the length of the arc is $\mathcal{O}(\delta) < \pi r$, we can apply the analysis in step **1** of the proof of Lemma 2.3.

By (2.17), which can be applied to the whole $\partial E'$ because the optimal rays are disjoint, we obtain

$$\begin{aligned}\mathcal{L}^2((E')^r) - \mathcal{L}^2(E^r) = \ & \mathcal{L}^2(E') - \mathcal{L}^2(E) + r\big(\mathcal{H}^1(\partial E') - \mathcal{H}^1(\partial E)\big) \\ = \ & \int -\phi(x)\,dx + r\int \Big(\sqrt{1+(f'(x)+\phi'(x))^2} - \sqrt{1+(f'(x))^2}\Big)\,dx.\end{aligned}$$

If $\bar{t}$ is a Lebesgue point of $D\theta$ where the derivative is $\dot{\theta}(\bar{t}) = \kappa \ = \frac{1}{\rho} \ \geq 0$, corresponding to the Lebesgue point $x = 0$ for $D^2 f$ where $D^2\phi(0) = \phi''(0) = \kappa = \frac{1}{\rho}$, for $\delta \ll 1$ we have

$$\begin{aligned}
\big(\mathcal{L}^2((E')^r) - \mathcal{L}^2(E^r)\big) - \big(\mathcal{L}^2(E') - \mathcal{L}^2(E)\big) = \ & r\int \Big(\sqrt{1+(f'(x)+\phi'(x))^2} - \sqrt{1+(f'(x))^2}\Big)\,dx \\
= \ & r\int \frac{(f'(x)+\phi'(x))^2 - (f'(x))^2}{\sqrt{1+(f'(x)+\phi'(x))^2}+\sqrt{1+(f'(x))^2}}\,dx \\
= \ & r\int_{-\delta}^{\delta}\bigg(\frac{\big(\phi'(x)\big)^2}{2} + \phi'(x)\frac{x}{\rho}\bigg)\big(1+\mathcal{O}(\delta)\big)\,dx.
\end{aligned} \tag{2.21}$$

To achieve the last estimate in (2.21), notice that $\phi' = \mathcal{O}(\delta)$, $f' = \frac{x}{\rho}(1+\mathcal{O}(\delta))$, because of (2.18,2.19).

A useful choice of the perturbation is

$$\phi(x) \ = \ \begin{cases} -\dfrac{\big(\delta-|x|\big)^2}{2a} & \text{if } \ |x|<\delta, \\[2ex] 0 & \text{if } \ |x|\geq\delta, \end{cases} \tag{2.22}$$

with $a > r$. In this case, if the local curvature is $\kappa = \frac{1}{\rho} > 0$, then the perturbed set E' in (2.20) satisfies

$$\begin{aligned}
\mathcal{L}^2((E')^r\setminus E^r) - \mathcal{L}^2(E'\setminus E) \ &= \ r\int_{-\delta}^{\delta}\bigg(\frac{\big(\phi'(x)\big)^2}{2} + \phi'(t)\frac{x}{\rho}\bigg)\big(1+\mathcal{O}(\delta)\big)\,dx \\
&= \ r(1+\mathcal{O}(\delta))\int_{-\delta}^{\delta}\bigg[\frac{(\delta-|x|)^2}{2a^2} - \frac{|x|(|\delta-|x|)}{a\rho}\bigg]\,dx \\
&= \ r\big(1+o(1)\big)\bigg[\frac{\delta^3}{3a^2} + \frac{\delta^3}{3a\rho}\bigg] \ = \ r\big(1+o(1)\big)\bigg(\frac{1}{a}+\frac{1}{\rho}\bigg)\frac{\delta^3}{3a} \\
&= \ r\big(1+o(1)\big)\bigg(\frac{1}{a}+\frac{1}{\rho}\bigg)\mathcal{L}^2(E'\setminus E).
\end{aligned}$$

In particular, by letting $a \to +\infty$, we obtain the following proposition:

Proposition 2.1 *If E is a convex set such that there is a Lebesgue point for the curvature $\kappa = \frac{1}{\rho}$, then for every $\varepsilon > 0$, there is a set $E' \supset E$ such that*

$$\bigg|\mathcal{L}^2((E')^r\setminus E^r) - \Big(1+\frac{r}{\rho}\Big)\mathcal{L}^2(E'\setminus E)\bigg| \ < \ \varepsilon\,\mathcal{L}^2(E'\setminus E). \tag{2.23}$$

Next, we study what happens when we remove a set E' from a convex set E, so that the difference $E \setminus E'$ is still convex. Consider a point $\bar{t}$ such that

$$\liminf_{t\to\bar{t}} \ \frac{\theta(t)-\theta(\bar{t})}{t-\bar{t}} \ \geq \ \frac{1}{\rho}.$$

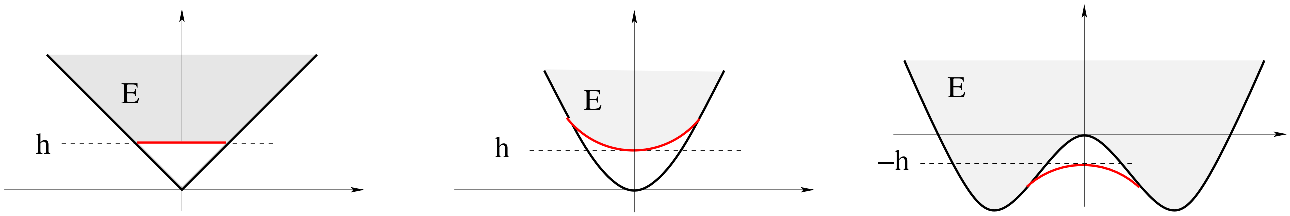

Figure 5: Left and center: the perturbations of the convex set E considered at (2.24) and at (2.27). Right: the perturbation of the semiconvex set E considered at (2.31).

We first consider the case where $\bar{t}$ is a corner point. Up to a change of coordinates (see Fig. 5), the boundary ∂E is thus the graph of a function $x_2 = f(x_1)$, with

$$\lim_{x_1 \to 0\pm} f'(x_1) \;=\; \pm a, \qquad a > 0.$$

We then define (see Fig. 5, left)

$$E'(h)) \;=\; E \cap \{x_2 \geq h\}. \tag{2.24}$$

Using (2.2), one obtains

$$\begin{aligned}
\mathcal{L}^2(E \setminus E'(h)) \;&=\; \frac{h^2}{a}\big(1 + o(1)\big),\\
\mathcal{L}^2(E^r \setminus (E'(h))^r) - \mathcal{L}^2(E \setminus E'(h)) \;&=\; r\big(\mathcal{H}^1(\partial E) - \mathcal{H}^1(\partial E'(h))\big)\\
&=\; r\big(\sqrt{1+a^2} - 1\big)\frac{2h}{a}\big(1 + o(1)\big)\\
&=\; \frac{2r}{h}\Big(\sqrt{1+a^2} - 1\Big)\mathcal{L}^2(E \setminus E'(h))\big(1 + o(1)\big).
\end{aligned} \tag{2.25}$$

A similar computation can be done in the case

$$\lim_{x_1 \to 0} f'(x_1) = 0, \qquad \lim_{x_1 \to 0} \frac{f'(x_1)}{x_1} = +\infty.$$

In this case, indeed, for $h \ll 1$ one has $|f'(x_1)| > k|x_1|$, $f(x_1) < h$, with $k \gg 1$ arbitrarily large. Defining again $E'(h) = E \cap \{x_2 \geq h\}$, one finds

$$\begin{aligned}
&\mathcal{L}^2(E^r \setminus (E'(h))^r) - \mathcal{L}^2(E \setminus E'(h)) \;=\; r\Big(\mathcal{H}^1(\partial E) - \mathcal{H}^1(\partial E'(h))\Big)\\
&\quad=\; r\int_{f(x_1)<h} \big(\sqrt{1 + (f'(x_1))^2} - 1\big)\,dx_1 \;\geq\; \frac{r}{2}(1+o(1))\int_{f(x_1)<h} (f'(x_1))^2 dx_1\\
&\quad\geq \frac{rk}{2}\int_{f(x_1)<h} x_1 f'(x_1) \;=\; \frac{rk}{2}\mathcal{L}^2(E \setminus E).
\end{aligned} \tag{2.26}$$

Finally, assume that

$$\lim_{x \to 0\pm} \frac{f'(x)}{x} = \frac{1}{\rho},$$

so that $f(x_1) = \frac{x_1^2}{2\rho}\big(1 + o(1)\big)$. Choosing $a > \rho$ and defining (see Fig. 5, right)

$$E'(h) = E \cap \left\{x_2 \geq g(x_1) \doteq h + \frac{(x_1)^2}{2a}\right\}, \tag{2.27}$$

we now obtain

$$\mathcal{L}^2(E\setminus E'(h)) \;=\; \int_{f<g} x_1\big(f'(x_1)-g'(x_1)\big)dx_1 \;=\; \big(1+o(1)\big)\left(\frac{1}{\rho}-\frac{1}{a}\right)\int_{f<g}(x_1)^2dx_1,$$

$$\begin{aligned}
\mathcal{L}^2(E^r\setminus(E'(h))^r)-\mathcal{L}^2(E\setminus E'(h)) &= r\big(\mathcal{H}^1(\partial E)-\mathcal{H}^1(\partial E'(h))\big)\\
&= r\int_{f<g}\left(\sqrt{1+\big(f'(x_1)\big)^2}-\sqrt{1+\big(g'(x_1)\big)^2}\right)dx_1\\
&= \frac{r}{2}\big(1+o(1)\big)\int_{f<g}\big((f'(x_1))^2-(g'(x_1))^2\big)dx_1\\
&= \frac{r}{2}\big(1+o(1)\big)\left(\frac{1}{\rho^2}-\frac{1}{a^2}\right)\int_{f<g}x_1^2\,dx_1\\
&= \frac{r}{2}\big(1+o(1)\big)\left(\frac{1}{\rho}+\frac{1}{a}\right)\mathcal{L}^2(E\setminus E'(h)).
\end{aligned}$$

For every $\varepsilon>0$, letting $a\nearrow\rho$ we obtain that, for $h>0$ small enough,

$$\left|\mathcal{L}^2(E^r\setminus(E'(h))^r)-\left(1+\frac{r}{\rho}\right)\mathcal{L}^2(E\setminus E'(h))\right| \;<\; \varepsilon\,\mathcal{L}^2(E\setminus E'(h)). \tag{2.28}$$

We summarize the results into the following proposition:

Proposition 2.2 *If E is a convex set, and there is a point where the inner radius of curvature is 0, then for every $\rho>0$ there is a perturbation $E'\subset E$ such that*

$$\mathcal{L}^2(E^r\setminus(E')^r)-\mathcal{L}^2(E\setminus E') \;>\; \frac{r}{\rho}\mathcal{L}^2(E\setminus E'). \tag{2.29}$$

If at a boundary point $x\in\partial E$ the radius of curvature is $\rho>0$, then for every $\varepsilon>0$ there is a set $E'\subset E$ such that

$$\left|\mathcal{L}^2\big(E^r\setminus(E')^r\big)-\left(1+\frac{r}{\rho}\right)\mathcal{L}^2(E\setminus E')\right| \;<\; \varepsilon\,\mathcal{L}^2(E\setminus E'). \tag{2.30}$$

2.4 Local perturbations of r-semiconvex sets.

Similar computations can be done locally for a semiconvex set E. To fix ideas, consider a boundary point $\bar{x}=\gamma(\bar{t})\in\partial E$. As before, denote by $e^{i\theta(t)}=\mathbf{n}(t)$ the set of outer normals and assume that $\bar{t}$ is a Lebesgue point of θ, with $\dot{\theta}(\bar{t})=-\frac{1}{\rho}$. Writing ∂E locally as the graph of a semiconvex function $x_2=f(x_1)$ with $\ddot{f}(0)=-1/\rho$, we choose $a>\rho$ and replace E with the slightly larger sets

$$E'(h) \;\doteq\; E\cup\left\{x_2\geq g(x_1)\doteq -h-\frac{x_1^2}{2a}\right\}. \tag{2.31}$$

Similarly to the previous cases, we compute

$$\begin{aligned}
\mathcal{L}^2((E'(h))^r \setminus E^r) - \mathcal{L}^2(E'(h) \setminus E) &\le r\big(\mathcal{H}^1(\partial E'(h)) - \mathcal{H}^1(\partial E)\big)\\
&= -r \int_{f<g} \left(\sqrt{1+(f'(x_1))^2} - \sqrt{1+(g'(x_1))^2}\right) dx_1\\
&= -\frac{r}{2}\big(1+o(1)\big) \int_{f<g} \big((f'(x_1))^2 - (g'(x_1))^2\big) dx_1\\
&= -\frac{r}{2}\big(1+o(1)\big) \left(\frac{1}{\rho^2} - \frac{1}{a^2}\right) \int_{f<g} (x_1)^2 dx_1\\
&= -\frac{r}{2}\big(1+o(1)\big) \left(\frac{1}{\rho} + \frac{1}{a}\right) \mathcal{L}^2(E'(h) \setminus E).
\end{aligned}$$

The first inequality is due to the fact that (2.17) holds only locally, and in general there can be point in E^r belonging to more than one optimal ray. Letting $a \nearrow \rho$ we obtain the following lemma:

Proposition 2.3 *Let E be a semiconcave set and x a point with outer curvature ρ. Then for every $\varepsilon > 0$ there exists a set $E' \supset E$ such that*

$$\left| \mathcal{L}^2((E')^r \setminus E^r) - \left(1 - \frac{r}{\rho}\right) \mathcal{L}^2(E' \setminus E) \right| < \varepsilon \mathcal{L}^2(E' \setminus E). \tag{2.32}$$

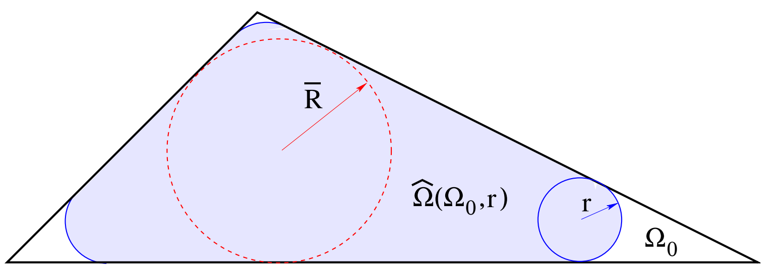

Figure 6: The maximum inner radius $\overline{R}$ defined at (3.1), and the set $\widehat{\Omega}(\Omega_0, \rho)$ (shaded region), in the case where Ω_0 is a triangle.

3 A one-step minimization problem

Given a bounded convex closed set $\Omega_0 \subset \mathbb{R}^2$ with nonempty interior, its inner radius is defined by setting

$$\overline{R} = \overline{R}(\Omega_0) \doteq \max\big\{ r > 0\,;\ \ \Omega_0 \text{ contains an open ball } B_r(x) \text{ of radius } r \big\}. \tag{3.1}$$

As shown in Fig. 6, for $\rho \in]0, \overline{R}]$ we also consider the open set

$$\widehat{\Omega}(\Omega_0, \rho) \doteq \bigcup_{B_\rho(x) \subseteq \Omega_0} B_\rho(x)\,. \tag{3.2}$$

In this section, given a constant $0 < a \le \mathcal{L}^2(\Omega_0)$, and a radius $r > 0$, we will study the following one-step minimization problem:

$$\text{minimize:} \quad \mathcal{L}^2(\Omega^r), \qquad \text{subject to} \quad \Omega \subseteq \Omega_0, \quad \mathcal{L}^2(\Omega) = a. \tag{3.3}$$

In other words, among all sets of fixed area a contained inside Ω_0, we seek one that minimizes the area of its r-neighborhood. We will show that the optimal solutions to (3.3) do not depend on the radius $r > 0$. Namely, a set $\widetilde{\Omega}$ is optimal if and only if it solves the corresponding minimization problem for the perimeter:

$$\text{minimize:} \quad \mathcal{H}^1(\partial\Omega), \qquad \text{subject to} \quad \Omega \subseteq \Omega_0, \quad \mathcal{L}^2(\Omega) = a. \tag{3.4}$$

The existence and various properties of optimal solutions to (3.4) have been established in [16]. To show the equivalence of the two problems (3.3) and (3.4), the key step is to prove that the optimal solutions of (3.3) are convex. This is the content of the following theorem.

Theorem 3.1 *Consider a compact convex set $\Omega_0 \subset \mathbb{R}^2$ and let $0 < a \leq |\Omega_0|$. Then there exists a set*

$$\Omega \;=\; \arg\min\big\{\mathcal{L}^2(\Omega^r)\,;\;\; \Omega \textit{ is a closed subset of } \Omega_0 \textit{ with area } \;\mathcal{L}^2(\Omega) = a\big\}. \tag{3.5}$$

Moreover, every such minimizer is convex.

The proof will be achieved in several steps. The existence of a minimizer follows from a standard compactness argument. However, its convexity requires a careful analysis. First we prove that every connected component of an optimal set Ω must be convex. Then we show that an optimal set can have at most finitely many components. Finally, we will prove that every optimal set Ω is connected.

3.1 Existence of a minimizer.

We prove here that the optimization problem (3.5) has a solution. Let $(\Omega_n)_{n\geq 1}$ be a minimizing sequence of compact sets, such that

$$\Omega_n \;\subseteq \Omega_0, \qquad \mathcal{L}^2(\Omega_n) \geq a, \qquad \lim_{n\to\infty} \mathcal{L}^2(\Omega_n^r) \;=\; m \;\doteq \inf\big\{\mathcal{L}^2(\Omega^r)\,;\;\; \Omega \subseteq \Omega_0,\; \mathcal{L}^2(\Omega) = a\big\}. \tag{3.6}$$

By possibly replacing Ω_n with the larger set $(\Omega_n^r)^{-r}$, which is still contained inside Ω_0, we can assume that Ω_n^r and Ω_n are in duality.

We use the following lemma.

Lemma 3.1 *If Ω_n, Ω_n^r, $n \in \mathbb{N}$, are a family of sets in duality, with $\Omega_n \subset \operatorname{clos} B_R(0)$ for a fixed closed ball $\operatorname{clos} B_R(0)$, then there exists a subsequence $\Omega_{n_k}, \Omega_{n_k}^r$, $k \in \mathbb{N}$, and a set Ω such that*

$$\Omega_{n_k} \to \Omega, \qquad\qquad \operatorname{clos} B_{R+r}(0) \setminus \Omega_{n_k}^r \to \operatorname{clos} B_{R+r}(0) \setminus \Omega^r,$$

w.r.t. the Hausdorff distance of compact sets and w.r.t. the $\mathbf{L}^1$-distance, respectively. In particular, the limit sets are in duality.

Proof. 1. By possibly taking a subsequence, we obtain the convergence

$$\Omega_{n_k} \underset{\text{Hausdorff}}{\longrightarrow} \Omega_{\text{Haus}}, \qquad\qquad \Omega_{n_k} \underset{\mathbf{L}^1}{\longrightarrow} \Omega_{L^1},$$

$$\operatorname{clos} B_{R+r}(0) \setminus \Omega^r_{n_k} \underset{\text{Hausdorff}}{\longrightarrow} \operatorname{clos} B_{R+r}(0) \setminus \Omega'_{\text{Haus}}, \qquad\qquad \Omega^r_{n_k} \underset{\mathbf{L}^1}{\longrightarrow} \Omega'_{L^1},$$

for some limit sets Ω_{Haus}, Ω_{L^1}, Ω'_{Haus}, and Ω'_{L^1}, all contained in the closed ball $\operatorname{clos} B_{R+r}(0)$. Indeed, this follows immediately from the finite perimeter estimate (Lemma 2.2) and the compactness of the family of compact subsets of the compact set $\operatorname{clos} B_{R+r}(0)$.

2. It remains to prove that

$$\Omega'_{\text{Haus}} = \Omega^r_{\text{Haus}}, \qquad \Omega_{\text{Haus}} = \Omega_{L^1}, \qquad \Omega^r_{\text{Haus}} = \Omega'_{L^1},$$

where the last two equalities should be intended as $\mathbf{L}^1$-equivalence of the characteristic functions.

First of all, we have that if $y \in \Omega^r_{\text{Haus}}$, then there exists a point $x \in \Omega_{\text{Haus}}$ such that $|x - y| = r - \delta r < r$ for some $\delta r > 0$. Then by Hausdorff convergence, there exists $K \gg 1$ such that for all $k \geq K$ it holds $d(\Omega_{n_k}, \Omega_{\text{Haus}}) < \delta r/2$. Then if $x_{n_k} \in \Omega_{n_k} \cap B_{\delta r/2}(x)$ it holds

$$|y - x_{n_k}| \leq |y - x| + |x - x_{n_k}| < r - \frac{\delta r}{2}.$$

We thus conclude that $y \in \Omega^r_{n_k}$ for all k large enough. Hence $\Omega^r_{\text{Haus}} \subset \Omega'_{\text{Haus}}$.

A similar argument shows that, if $y \notin \operatorname{clos} \Omega^r_{\text{Haus}}$, then $y \notin \Omega^r_{n_k}$ for all k large enough. Moreover, for every $\varepsilon > 0$ we obtain

$$(\Omega^r_{\text{Haus}})^{-\varepsilon} \subset \Omega^r_{n_k} \subset \Omega^{r+\varepsilon}_{\text{Haus}} \tag{3.7}$$

for all k suitably large. Since Ω^r_{Haus} is open, we conclude that

$$\Omega'_{\text{Haus}} = \Omega^r_{\text{Haus}}.$$

Next, since the boundary of Ω^r_{Haus} is rectifiable, we have

$$\mathcal{L}^2\big(\Omega^{r+\varepsilon}_{\text{Haus}} \setminus (\Omega^r_{\text{Haus}})^{-\varepsilon}\big) \;=\; \mathcal{L}^2\big(\partial\Omega^r_{\text{Haus}} + B_\varepsilon(0)\big) \;=\; (2\varepsilon + o(\varepsilon))\mathcal{H}^1(\Omega^r_{\text{Haus}}),$$

and from (3.7) we get that the symmetric difference satisfies

$$\lim_{k\to\infty} \mathcal{L}^2\big(\Omega^r_{n_k} \Delta \Omega^r_{\text{Haus}}\big) = 0,$$

i.e. $\Omega'_{L^1} = \Omega^r_{\text{Haus}}$.

Reversing the analysis, i.e. considering $\Omega_{n_k} = (\Omega^r_{n_k})^{-r}$, we obtain that for every $\varepsilon > 0$ there exists $K > 0$ such that, for $k \geq K$,

$$(\Omega^r_{\text{Haus}})^{-r-\varepsilon} \;\subset\; (\Omega^r_{n_k})^{-r} \;\subset\; (\Omega^r_{\text{Haus}})^{-r+\varepsilon}.$$

Being $(\Omega^r_{n_k})^{-r} = \Omega_{n_k}$, we obtain up to negligible sets

$$\operatorname{int} \Omega_{\text{Haus}} \;\subset\; \Omega_{L^1} \;\subset\; (\Omega^r_{\text{Haus}})^{-r} \;=\; \Omega_{\text{Haus}}.$$

This yields $\Omega_{L^1} = \Omega_{\text{Haus}}$, because $\mathcal{L}^2(\partial\Omega_{\text{Haus}}) = 0$. □

Applying this result to the sequence introduced at (3.6), we obtain a subset $\Omega = \lim_k \Omega_{n_k}$ such that $\Omega^r = \lim_k \Omega^r_{n_k}$ and

$$\mathcal{L}^2(\Omega) \;=\; \lim_{k\to\infty} \mathcal{L}^2(\Omega_{n_k}) \;\geq\; a, \qquad\qquad \mathcal{L}^2(\Omega^r) \;=\; \lim_{k\to\infty} \mathcal{L}^2(\Omega^r_{n_k}) \;=\; m.$$

It remains to show that $\mathcal{L}^2(\Omega) = a$. If on the contrary $\mathcal{L}^2(\Omega) > a$, we fix a unit vector $\mathbf{e}_1 \in \mathbb{R}^2$ and consider the smaller sets

$$\Omega(\lambda) \;\doteq\; \big\{x \in \Omega\,;\;\; x\cdot\mathbf{e}_1 \;\leq\; \lambda\big\}. \tag{3.8}$$

For a suitable $\lambda \in \mathbb{R}$, one has $\mathcal{L}^2(\Omega(\lambda)) = a$. But this would imply $\mathcal{L}^2(\Omega(\lambda)^r) < \mathcal{L}^2(\Omega^r) = m$, reaching a contradiction.

We collect the results of this section into the following proposition.

Proposition 3.1 *Every minimizer Ω of (3.5) satisfies*

$$\Omega = (\Omega^r)^{-r} \quad \textit{and} \qquad \Omega = \mathrm{clos}(\mathrm{int}\,\Omega).$$

Proof. The first identity was already proved in Lemma 3.1. The second identity can be proved as follows. Since $\mathcal{L}^2(\partial\Omega) = 0$, then $\mathcal{L}^2\big(\mathrm{clos}(\mathrm{int}\,\Omega)\big) = a$. Moreover, for all $x \in \partial(\mathrm{int}\,\Omega)$ there is $y \in \partial\Omega^r$ such that $|x-y| = r$, being $\partial(\mathrm{int}\,\Omega) \subset \partial\Omega$. This yields

$$\mathrm{clos}(\mathrm{int}\,\Omega) = ((\mathrm{clos}(\mathrm{int}\,\Omega))^r)^{-r}.$$

If $\Omega \supsetneq \mathrm{clos}\,(\mathrm{int}\,\Omega)$, then $\Omega^r \supsetneq (\mathrm{clos}(\mathrm{int}\,\Omega))^r$. Since these two sets are open, we would have $\mathcal{L}^2(\Omega^r) \geq \mathcal{L}^2((\mathrm{clos}(\mathrm{int}\,\Omega))^r)$, contradicting the optimality of Ω_r. $\square$

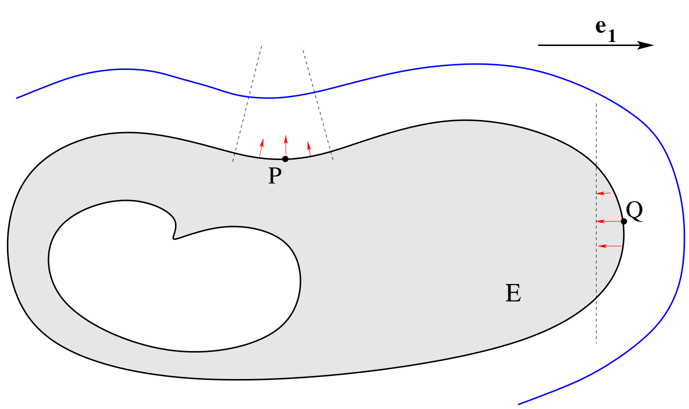

Figure 7: Proving Lemma 3.3. If the set E is not convex, we can enlarge it in a neighborhood of the point x^* where the curvature of the boundary is negative. At the same time, we shrink it in a neighborhood of the exposed point y. This yields a perturbed set E_ε with same area as E, but with $\mathcal{L}^2(E^r_\varepsilon) < \mathcal{L}^2(E^r)$. We remark that, in general, the points y and x^* may belong to distinct connected components of E.

3.2 Convexity of the optimal set.

Aim of this section is to prove that every connected component of a minimizer is a compact convex set. Because of Proposition 3.1, it is enough to study the connected components with positive measure.

Lemma 3.2 *Let Ω be any compact set and let $\Omega(\lambda)$ be as in (3.8). If $\mathcal{L}^2(\Omega(\lambda)) < \mathcal{L}^2(\Omega)$, then*

$$\mathcal{L}^2(\Omega^r \setminus \Omega(\lambda)^r) \ \geq \ \mathcal{L}^2(\Omega \setminus \Omega(\lambda)). \tag{3.9}$$

Proof. The inequality (3.9) is an immediate consequence of the inclusions

$$\big(\Omega \setminus \Omega(\lambda)\big) + r\mathbf{e}_1 \ \subseteq \ \Omega^r \cap \{x\,;\ x\cdot\mathbf{e}_1 \geq \lambda + r\} \ \subseteq \ \Omega^r \setminus \Omega(\lambda)^r.$$

□

Lemma 3.3 *Let $\Omega \subseteq \Omega_0$ be a minimizer for (3.5). Then every connected component of* $\operatorname{int}\Omega$ *is convex.*

Proof. Assume, on the contrary, that the optimal set Ω has a connected component E which is not convex. Since Ω is r-semiconvex, according to Lemma 2.1 there is a point $x^* \in \partial E$ where the boundary has negative curvature. We will derive a contradiction showing that Ω is not optimal. As shown in Fig. 7, by slightly enlarging the set E near x^* and shrinking the set Ω at some other point y along its boundary, we can keep constant the area $\mathcal{L}^2(\Omega)$, but decrease the area $\mathcal{L}^2(\Omega^r)$ of the r-neighborhood.

1. To construct these perturbations, as stated in Lemma 2.1 we can find a boundary point $x^* = \gamma_k(t^*)$ where the derivative $D\theta_k(t^*)$ exists and is strictly negative, say

$$D\theta_k(t^*) \ = \ -\frac{1}{\rho}\,,$$

with $\rho > 0$. Constructing the slightly larger sets $E'(h)$ as in (2.31), the change in the area of the r-neighborhoods $(E'(h))^r$ is bounded above by (2.32).

2. Next, for any $h > 0$ small enough, choose $\lambda = \lambda_h$ so that the set $\Omega(\lambda)$ in (3.8) satisfies

$$\mathcal{L}^2(\Omega \setminus \Omega(\lambda_h)) \ = \ \mathcal{L}^2(E'(h) \setminus E). \tag{3.10}$$

Then define the perturbed set

$$\Omega_h \ \doteq \ (\Omega \cup E'(h)) \cap \{x\,;\ x\cdot\mathbf{e}_1 \leq \lambda_h\}. \tag{3.11}$$

The above definition implies $\mathcal{L}^2(\Omega_h) = \mathcal{L}^2(\Omega)$ for every $h > 0$ small enough. Moreover, combining (2.28) with (3.9) we obtain

$$\begin{aligned}
\mathcal{L}^2(\Omega_h^r) - \mathcal{L}^2(\Omega^r) \ &\leq \ \mathcal{L}^2((E'(h))^r \setminus E^r) - \mathcal{L}^2(\Omega^r \setminus \Omega(\Lambda_h)^r) \\
&\leq \ \left(1 - \frac{r}{\rho} + o(1)\right)\mathcal{L}^2(E(h) \setminus E) - \mathcal{L}^2(\Omega \setminus \Omega(\Lambda_h)) \\
&= \ \left(1 - \frac{r}{\rho} + o(1)\right)\mathcal{L}^2(E(h) \setminus E) - \mathcal{L}^2(E(h) \setminus E) \\
&= \ \left(-\frac{r}{\rho} + o(1)\right)\frac{h^3}{3} \ < \ 0
\end{aligned}$$

for all $h > 0$ small enough. This contradicts the optimality of Ω, proving the lemma. □

We observe that the same proof can be adapted to the case of a connected component with 0 measure. Indeed in this case one observes that every connected set of finite length is covered by a closed curve. We will not need this fact, because we will prove in Lemma 3.9 that there are only finitely many components.

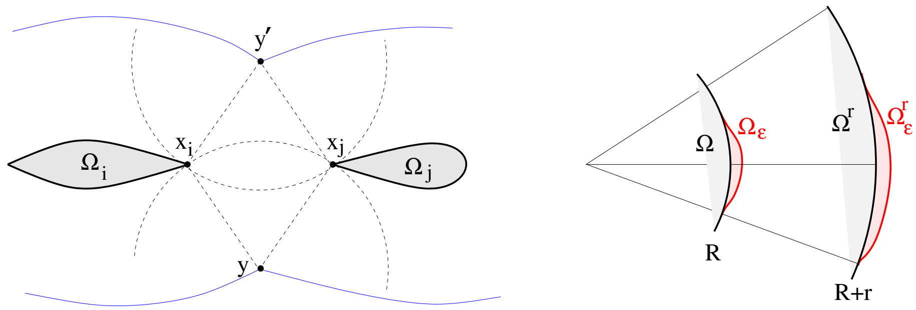

Figure 8: Left: the points x_i, x_j, y, y' considered in Lemma 3.4. Right: A slight perturbation of the set Ω. If R is the constant curvature radius, then the increase in the area of the neighborhood Ω_r satisfies $\mathcal{L}^2(\Omega^r_\varepsilon) - \mathcal{L}^2(\Omega^r) \approx \frac{R+r}{R}\big(\mathcal{L}^2(\Omega_\varepsilon) - \mathcal{L}^2(\Omega)\big)$.

To prove Theorem 3.1 it remains to prove that the optimal set Ω is connected. As an intermediate step, we will show that Ω has at most finitely many connected components.

Let $\{\Omega_i\}_{i\in\mathbb{N}}$ be the connected components of $\operatorname{int}\Omega$.

We will use the following lemmas, whose proofs are elementary.

Lemma 3.4 *Let Ω be an optimal set, and let Ω_i, Ω_j be distinct connected components of* $\operatorname{int}\Omega$. *If $(\operatorname{clos}\Omega_i)^r \cap (\operatorname{clos}\Omega_j)^r \neq \emptyset$, then there exists points y, y', such that*

$$\partial\Omega_i^r \cap \partial\Omega_j^r \;=\; \{y, y'\}.$$

Moreover, there exists points $x_i \in \Omega_i$, $x_j \in \Omega_j$ such that

$$(\operatorname{clos}\Omega_i)^r \cap (\operatorname{clos}\Omega_j)^r \;=\; B_r(x_i) \cap B_r(x_j), \qquad \{x_i, x_j\} \;=\; \partial B_r(y) \cap \partial B_r(y'). \tag{3.12}$$

Finally, for every $x \in \mathbb{R}^2$

$$\sharp\big\{i : x \in \Omega_i^r\big\} \le 2. \tag{3.13}$$

Proof. It is clear that the boundaries of the two open convex sets $\partial\Omega_i^r, \partial\Omega_j^r$ intersect exactly at two points y, y' (see Fig. 8, left). Let $x_i, x_i', \in \partial\Omega_i$ and $x_j, x_j' \in \partial\Omega_j$ be points such that

$$|y - x_i| \;=\; |y - x_j| \;=\; |y' - x_i'| \;=\; |y' - x_j'| \;=\; r.$$

We claim that $x_i = x_i'$ and $x_j = x_j'$. Indeed, if $x_i \neq x_i'$, consider the arc along the boundary of Ω_i with endpoints x_i, x_i'. This arc has positive length and thus contains at least one point $x \in \partial\Omega_i$ where the unit outer normal $\mathbf{n}(x)$ is unique. By optimality, the point $x + r\mathbf{n}(x)$ cannot lie inside the open set Ω^r, otherwise we could enlarge the set Ω in a neighborhood of x, without changing Ω^r. On the other hand, the above construction implies $x + r\mathbf{n}(x) \in \Omega_j^r$, yielding a contradictions. A similar argument yields $x_j = x_j'$.

The remaining identities (3.12) are clear.

The last estimate follows by the following consideration of elementary geometry. If (3.13) is false, then there is a point x such that (up to relabeling)

$$x \in \Omega_1^r \cap \Omega_2^r \cap \Omega_3^r,$$

then we are in the situation of Fig. 9: in particular we can assume that x is the intersection w of the symmetry axis of the sides of the triangle $\{x_1, x_2, x_3\}$. The hexagon $\{x_1, y_2, x_2, y_2, x_3, y_2\}$ must be convex, and then its angles satisfy

$$\sum_{i=1,2,3} \angle(y_{i-1} x_i y_i) + \sum_{i=1,2,3} \angle(x_i y_i x_{x+1}) = 4\pi, \qquad x_4 = x_1, y_0 = y_3. \tag{3.14}$$

Here and in the following, by $\angle(xyz)$ we denote the angle formed at y by the two segments xy and yz. Since $w \in \cap_{i=1,2,3} B_r(x_i)$, we obtain

$$\angle(y_{i-1} w y_i) \ > \ \angle(y_{i-1} x_i y_i), \qquad y_0 = y_3,$$

and then

$$\sum_{i=1,2,3} \angle(y_{i-1} x_i y_i) \ < \ \sum_{i=1,2,3} \angle(x_i w x_{i+1}) = 2\pi, \qquad y_0 = y_3.$$

In a similar way, let z be the intersection of the axis of symmetry of the triangle $\{y_1, y_2, y_3\}$: being z equidistant from y_i, $i = 1, 2, 3$, we deduce that if $|z - y_i| < r$ then

$$\sum_{i=1,2,3} \angle(x_i y_i x_{i+1}) \ < \ \sum_{i=1,2,3} \angle(x_i w x_{i+1}) = 2\pi, \qquad x_4 = x_1.$$

This however contradicts (3.14). Hence the set Ω cannot be in duality or optimal: indeed, if it is in duality, the set containing z and not covered by the three balls $B_r(y_i)$, $i = 1, 2, 3$, is not convex, and the only points belonging to $\partial\Omega^r$ at distance r from this set are the points y_1, y_2, y_3. □

In the following, given a distance $\delta > 0$, we shall say that two components Ω_i, Ω_j are δ-**related** if there exists a point y such that

$$d(y, \Omega_i) \ = \ d(y, \Omega_j) \ = \ d(y, \Omega) \ < \ \delta. \tag{3.15}$$

By the previous analysis, this can happen only if there are points $x \in \partial\Omega_i$, $x' \in \partial\Omega_j$ such that, calling $\theta = \big|\mathbf{n}(x_i)\big|$, one has

$$r \cos \frac{\theta}{2} \ < \ \delta. \tag{3.16}$$

Corollary 3.1 *For each connected component Ω_i there are at most 2 sets that are $(r/2)$-related to Ω_i, and at most 3 sets Ω_j which are $(r/\sqrt{2})$-related to Ω_i.*

Proof. The boundary $\partial\Omega_i$ can contain at most two points where $\big|\mathbf{n}(x)\big| > 2\pi/3$. Taking $\delta = r \cos \frac{\pi}{3} = \frac{r}{2}$ by (3.16) we obtain the first assertion.

Similarly, there can be at most 3 points where $\big|\mathbf{n}(x)\big| > \pi/2$. Taking $\delta = r \cos \frac{\pi}{4} = \frac{r}{\sqrt{2}}$, by (3.16) we obtain the second assertion. □

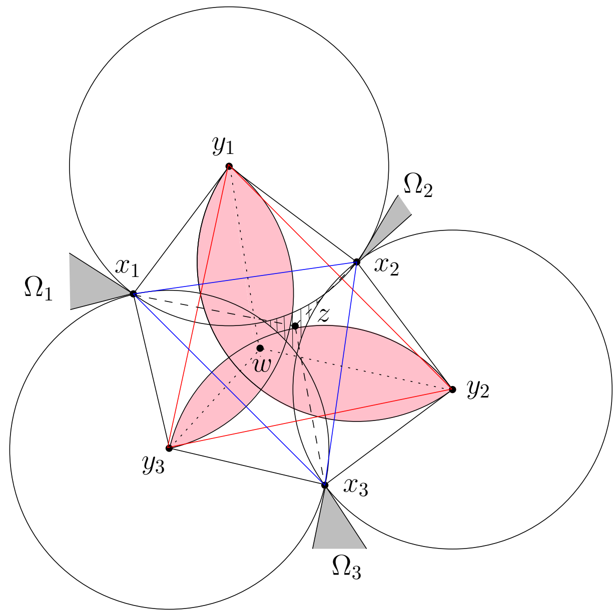

Figure 9: An illustration of the last statement in Lemma 3.4. If we assume that $w \in \cap_{i=1,2,3} B_r(x_i)$, then the point z, intersection of the axis of symmetry of the sides of the triangle $\{y_1, y_2, y_3\}$ cannot belong to any of the balls $B_r(y_i)$, $i = 1, 2, 3$.

Lemma 3.5 *Let Ω_i be a connected component of* $\operatorname{int}\Omega$*, where Ω is an optimal set. Let*

$$\theta_{max} \doteq \max_{x \in \partial\Omega_i} \big|\mathbf{n}(x)\big| < \pi$$

be the maximal angle at corner points of $\partial\Omega_i$. Then all other components of Ω have a strictly positive distance from Ω_i. Namely

$$(\operatorname{clos}\Omega_i)^\rho \cap \big(\Omega \setminus \Omega_i\big) = \emptyset, \qquad \textit{with} \quad \rho = r \sin\left(\frac{\pi - \theta_{max}}{2}\right). \tag{3.17}$$

Proof. Consider the set of boundary points admitting a single outer normal:

$$S_i \doteq \big\{x \in \partial\Omega_i\,;\ \mathbf{n}(x) \text{ is a singleton}\big\}. \tag{3.18}$$

Being Ω semiconvex and $y = x + rn(x)$ the only point in $\partial\Omega^r$ at distance r from $x \in S_i$, the open ball $B_r\big(x + r\mathbf{n}(x)\big)$ cannot intersect Ω. As shown in Fig. 12, left, it now suffices to observe that

$$\Omega_i \cup \bigcup_{x \in S_i} B_r\big(x + r\mathbf{n}(x)\big) \supseteq (\operatorname{clos}\Omega_i)^\rho,$$

where ρ is the radius at (3.17). □

In particular, $\operatorname{clos}\Omega_i$ coincides with the connected components of Ω with positive measure.

From the definition (3.18) it follows that $\partial\Omega_i \setminus S_i$ is a set of corner points, admitting multiple outer normals; hence it is countable. A further property of points $x \in S_i$ is now described.

Lemma 3.6 *For every point $x \in S_i$ there exists $\delta = \delta(x) > 0$ such that*

$$\big\{x + \rho\mathbf{n}(x)\,;\ r - \delta < \rho < r\big\} \subset (\operatorname{clos}\Omega_i)^r \setminus \operatorname{clos}(\Omega \setminus \operatorname{clos}\Omega_i)^r. \tag{3.19}$$

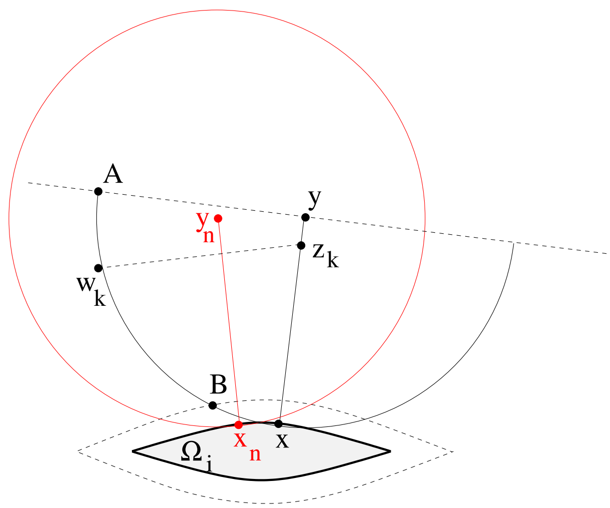

Figure 10: Proving Lemma 3.6. Here the arc of circumference AB along the circumference centered at $y = x + r\mathbf{n}(x)$ with radius r is entirely contained in the interior of the ball $B_r(y_n)$, for n large enough. Therefore, for k large, the point w_k cannot lie inside Ω.

Proof. With reference to Fig. 10, consider a sequence of boundary points $x_n \in S_i$ with $x_n \to x$. Let $y = x + r\mathbf{n}(x)$ and $y_n = x_n + r\mathbf{n}(x_n)$. Assume that the conclusion of the Lemma fails. Then there exists an increasing sequence $c_k \to r-$, and sequences of points z_k, w_k, $k \geq 1$, such that

$$z_k = x + c_k\mathbf{n}(x), \qquad w_k \in \Omega \setminus \operatorname{clos}\Omega_i\,, \qquad |w_k - z_k| \leq r.$$

Since $z_k \to y$, by possibly taking a subsequence we conclude that $w_k \to \overline{w} \in \Omega$, with $|\overline{w} - y| = r$.

By Lemma 3.5, every point w_k has uniformly positive distance from Ω_i. Hence the limit point $\overline{w}$ must lie on an arc AB of the circumference $\partial B_r(y)$ of length $< \pi r/2$. However, this is impossible because such arc is entirely contained in the open ball $B_r(y_n)$, for $n \geq 1$ large enough. □

Lemma 3.7 *Assume that the interior* $\operatorname{int}\Omega$ *of an optimal set* Ω *has infinitely many connected components. Then there exists a sequence of components* Ω_k *such that*

(i) $\operatorname{diam}(\Omega_k) \to 0$*;*

(ii) each set Ω_k *contains two corner points* $x_k, x'_k \in \partial\Omega_k$*, where the sets of outer normals* $\mathbf{n}(x_k)$*,* $\mathbf{n}(x'_k)$ *satisfy*

$$\big|\mathbf{n}(x_k)\big| \;\to\; \pi, \qquad \big|\mathbf{n}(x'_k)\big| \;\to\; \pi, \qquad \textit{as} \quad k \to \infty; \tag{3.20}$$

- *for* $k \gg 1$*, writing* $\partial\Omega_k \setminus \{x_k, x_{k'}\}$ *as the union of the two Lipschitz arcs* $(\partial\Omega_k)^+, (\partial\Omega_k)^-$*, only one of the two arcs may have nonempty intersection with* $\partial\Omega_0$*.*

Proof. 1. As shown in Fig. 12, left, consider any convex component $\Omega_i \subset \operatorname{int}\Omega$. Assume that, at each point $x \in \partial\Omega_i$, the set $\mathbf{n}(x) \subset \mathbb{S}^1$ of outer normals covers an angle $\leq \theta$. Then, by the duality relation $\Omega = (\Omega^r)^{-r}$, every other component Ω_j must have distance from $\Omega \backslash (\operatorname{clos}\Omega_i)$ at least

$$\delta \;\geq\; |x' - x| \;=\; 2r\,\cos\frac{\theta}{2}\,.$$

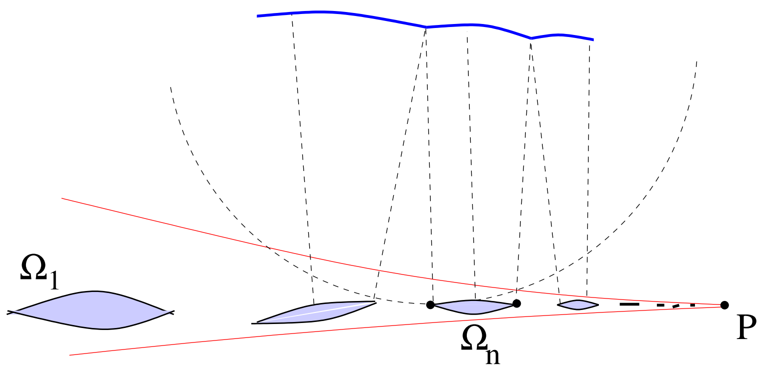

Figure 11: The situation considered in Lemma 3.7.

2. If $\mathrm{int}\,\Omega$ contains infinitely many components, since each one of them has positive Lebesgue measure, there can be only countably many of them. Since Ω is bounded, by taking a subsequence $(\Omega_k)_{k\geq 1}$ we can assume that their barycenters b_k converge to some limit point $\bar{x}$. By the previous step, there is a sequence of points $x_k \in \Omega_k$ where the sets of unit normal vectors have 1-dimensional measures $\big|\mathbf{n}(x_k)\big| \to \pi$. As shown in Fig. 12, right, call $\mathbf{n}_k = e^{i\theta_k}$ the central unit normal at x_k, so that the entire set of unit normals has the representation

$$\mathbf{n}(x_k) \;=\; \Big\{e^{i\theta}\,;\;\;\theta_k - \alpha_k \leq \theta \leq \theta_k + \alpha_k\Big\}\,,$$

for suitable angles $\alpha_k < \pi/2$. By possibly taking a further subsequence, we can assume $\mathbf{n}_k \to \overline{\mathbf{n}}$.

3. Next, we claim that each set $\partial\Omega_k$ must also contain a second point x'_k, such that the sets of unit normal vectors $\mathbf{n}(x'_k)$ also satisfy $\big|\mathbf{n}(x'_k)\big| \to \pi$.

Indeed, if the claim did not hold, we could find $\delta > 0$ such that for each $k \geq 1$, the half circle

$$\Big\{y \in \mathbb{R}^2\,;\;\;|y - x_k| \leq \delta,\;\;(y - x_k)\cdot\mathbf{n}(x_k) \leq 0\Big\}$$

does not intersect any other connected component besides Ω_k. This would exclude the existence of a limit point $\overline{x}$, providing a contradiction. This establishes part (ii) of the statement.

Notice that, from the convergence $\big|\mathbf{n}(x_k)\big| \to \pi$, it follows that the central unit normals $\mathbf{n}'_k$ at x'_n satisfy $\mathbf{n}'_k \to -\overline{\mathbf{n}}$, as $k \to \infty$.

If (iii) is false, then the limit point $\bar{x}$ belongs to $\partial\Omega_0$, and the suppporting cone $\mathbb{R}\mathbf{n}(\bar{x})$ of Ω_0 would have an opening of π, which is impossible by convexity if $\mathcal{L}^2(\Omega_0) > 0$.

4. To complete the proof, assume that (i) fails. By possibly taking a subsequence, we can assume that $\mathrm{diam}(\Omega_k) \;=\; |x_k - x'_k| \;\geq\; \delta \;>\; 0$ for every $k \geq 1$. Taking further subsequences, we obtain the convergence

$$x_k \to \overline{x}, \qquad x'_k \to \overline{x}', \qquad\qquad |\overline{x} - \overline{x}'| \geq \delta.$$

This yields an obvious contradiction with the duality assumption: $\Omega = (\Omega^r)^{-r}$. $\square$

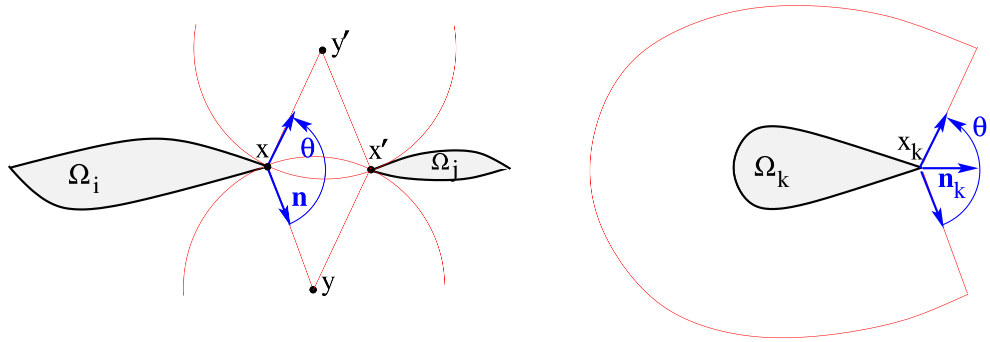

Figure 12: Left: The minimum distance between the component Ω_i and any other component Ω_j is bounded below in terms of the angle θ. Right: the central unit normal vector $\mathbf{n}_k$ at the point x_k. If the set $\partial\Omega_k$ does not contain a second point x'_k where the set of outer normals has size $\left|\mathbf{n}(x'_k)\right| \approx \pi$, then there is a large region to the left of Ω_k which cannot intersect any other connected component.

In connection with the optimization problem (3.5), the next lemma provides necessary conditions for a set $\Omega \subseteq \Omega_0$ to be optimal. Namely, every component Ω_i of $\operatorname{int}\Omega$ must have the same curvature radius R, at all points in the interior of Ω_0, with the exception of countably many corner points described in Lemma 3.4. More precisely, define the sets of boundary points

$$\tilde{S}_i \doteq \left\{x \in \partial\Omega_i\,;\quad x + \mathbf{n}(x)r \subset \partial\Omega^r\right\}. \tag{3.21}$$

Lemma 3.8 *Let Ω be an optimal set for the problem (3.5), and $\operatorname{int}\Omega = \cup_i\Omega_i$. Then the curvature of its boundary is constant in the set $S = (\cup_i\tilde{S}_i) \setminus \partial\Omega_0$, and hence $\tilde{S}_i = S_i$, where S_i is defined in (3.18).*

Proof. The identity $S_i = \tilde{S}_i$ follows is all the points in $\tilde{S}_i$ have the same curvature, a conditions which implies that $\mathbf{n}(x)$ is singleton.

This statement follows by removing a small area near a point $x' \in S$ where the boundary has a smaller curvature radius R', and adding the same area near a point $x'' \in S$ with larger curvature radius R''. This can be done as long as x'' lies in the interior of Ω_0.

More precisely, let Ω' be the set obtained from Ω by removing a small region near x', as in (2.24), (2.27). In view of Lemma 3.6, most of the points removed from $(\operatorname{clos}\Omega_i)^r$ do not lie in the set $(\Omega \setminus \operatorname{clos}\Omega_i)^r$.

By (2.28), the change in the area of the r-neighborhood is estimated by

$$\mathcal{L}^2(\Omega^r \setminus (\Omega')^r) \;=\; \left(1 + \frac{r}{R'} + o(1)\right)\mathcal{L}^2(\Omega \setminus \Omega').$$

Next, let Ω'' be the set obtained from Ω by adding a small region near x'', as in (2.20). By (2.23), the change in the area of the r-neighborhood is estimated by

$$\mathcal{L}^2((\Omega'')^r \setminus \Omega^r) \;=\; \left(1 + \frac{r}{R''} + o(1)\right)\mathcal{L}^2(\Omega'' \setminus \Omega).$$

By simultaneously performing the two modifications, we obtain a new set $\widetilde{\Omega} \subset \Omega_0$, with $\mathcal{L}^2(\widetilde{\Omega}) = \mathcal{L}^2(\Omega)$ but $\mathcal{L}^2(\widetilde{\Omega}^r) < \mathcal{L}^2(\Omega^r)$, against the optimality of Ω. $\square$

In the following, we shall denote by R be the constant curvature radius, outside the corner points, as in Lemma 3.8.

To prove that the configuration considered in Lemma 3.7 is not optimal, we study what happens if we remove one of the sets Ω_i.

As shown in Fig. 11, let $x_i, x_i' \in \partial\Omega_i$ be the corner points where the set of outer normals is large, say $\big|\mathbf{n}(x_i)\big| > \pi - \varepsilon_0$, $\big|\mathbf{n}(x_i')\big| > \pi - \varepsilon_0$. By convexity, the boundary $\partial\Omega_i$ is the union of two parts, above and below the segment $[x_i, x_i']$, only one of which may have nonempty intersection with $\partial\Omega_0$ by Lemma 3.7. To fix ideas, we assume that the upper boundary lies in the interior of Ω_0. The lower boundary may have nonzero intersection with $\partial\Omega_0$.

Call $S_i^+ \subset \partial\Omega_i$ the set of points on the upper boundary having a unique outer normal, and consider the set of points projecting into S_i^+, namely

$$Y_i^+ \ \doteq\ \Big\{y \in \mathbb{R}^2\,;\ \ d(y, \Omega_i) = |y - x| \ \text{ for some } x \in S_i^+\Big\}.$$

We estimate how many of these points will not be in $(\Omega \setminus \mathrm{clos}\,\Omega_i)^r$. For this purpose, for each $x \in S_i^+$, consider the segments

$$\ell(x) \ \doteq\ \big\{x + c\mathbf{n}(x)\,;\ c \in \mathbb{R}\big\} \cap \Omega_i\,, \qquad\qquad L(x) \ \doteq\ \ell(x) + r\,\mathbf{n}(x). \tag{3.22}$$

We observe that, for every $x \in S_i^+$, $\big|L(x)\big| = \big|\ell(x)\big|$ and moreover

$$\ell(x) \ \subset\ \Omega_i\,, \qquad\qquad L(x) \ \subset\ \Omega^r \setminus (\Omega \setminus \mathrm{clos}\,\Omega_i)^r. \tag{3.23}$$

Indeed (see Fig. 13), let $x_{i-1}' \in \partial\Omega_{i-1}$ be the corner point opposite to x_i as in Lemma 3.7. Similarly, let $x_{x+i} \in \partial\Omega_{i+1}$ be the corner point opposite to x_i'. We then have the identity

$$Y_i^+ \cap (\Omega \setminus \mathrm{clos}\,\Omega_i)^r \ =\ Y_i^+ \cap \big(B_r(x_{i-1}') \cup B_r(x_{i+1})\big).$$

Since $L(x) \cap \big(B_r(x_{i-1}') \cup B_r(x_{i+1})\big) = \emptyset$, this yields the second relation in (3.23).

By Lemma 3.8, at every point $x \in S_i$, the curvature of $\partial\Omega_i$ is constantly equal to R.

Being the curve $\partial\Omega_i$ convex, the map

$$S_i^+ \times \mathbb{R} \ni x, c \mapsto T(x, c) = x + c(x) \in \mathbb{R}^2$$

is BV on the rectifiable set $S_i^+ \times \mathbb{R}$, and its a.c. Jacobian is

$$J(x, c) = \left|1 + \frac{c}{R}\right|.$$

This allows us to compute by the area formula

$$\mathcal{L}^2(\Omega_i) \ \leq\ \int_{\Omega_i} \mathcal{H}^0(T^{-1}(y))\mathcal{L}^2(dy) = \int_{S_i^+ \times \mathbb{R}^- \cap T^{-1}(\Omega_i)} J(x, c)\mathcal{H}^1(dx)\mathcal{L}^1(dc) \leq \int_{S_i^+} |\ell(x)|\mathcal{H}^1(dx), \tag{3.24}$$

where we observe that, being the set convex, $\Omega_i \subset T(S_i^+ \times \mathbb{R}^-)$. Similarly

$$
\begin{aligned}
\mathcal{L}^2\big((\Omega \setminus \operatorname{clos}\Omega_i)^r \cap Y_i^+\big) &= \mathcal{H}^2\big(S_i^+ \times \mathbb{R}^+ \cap T^{-1}((\Omega \setminus \operatorname{clos}\Omega_i)^r \cap Y_i^+)\big) \\
&\geq \int_{S_i^+} \left[\int_{r-|L(x)|}^{r} \left(1 + \frac{z}{R}\right) dz\right] \mathcal{H}^1(dx) \\
&= \int_{S_i^+} \left(r + \frac{r^2}{R} - (r - L(x)) - \frac{(r-L(x))^2}{R}\right) \mathcal{H}^1(dx) \qquad (3.25)\\
&= \int_{S_i^+} \left(\frac{R+r}{R} - \frac{|\ell(x)|}{2R}\right) |\ell(x)|\, dx.
\end{aligned}
$$

Combining the above inequalities, we conclude

$$
\frac{\mathcal{L}^2\big((\Omega \setminus \operatorname{clos}\Omega_i)^r \cap Y_i^+\big)}{\mathcal{L}^2(\Omega_i)} \;\geq\; 1 + \frac{r}{R} - o(1), \tag{3.26}
$$

where $o(1)$ is a quantity that approaches zero as $\sup_x |\ell(x)| \to 0$.

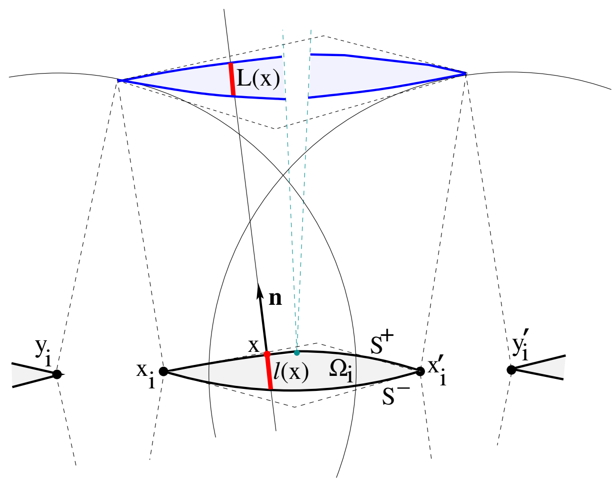

Figure 13: The configuration considered at (3.24)-(3.25). Here $S_i^+ \subset \partial\Omega_i$ is the upper part of the boundary of Ω_i, excluding the points with multiple outer normals. Removing the set $\operatorname{clos}\Omega_i$, all points in the upper shaded region are no longer in the set $(\Omega \setminus \operatorname{clos}\Omega_i)^r$. These are points $y \in Y_i^+$ which project into S_i^+, but whose distance from x'_{i-1} and from x_{i+1} (and from all other components of Ω) is $> r$.

A similar estimate can be performed for the set of points $y \in (\Omega \setminus \operatorname{clos}\Omega_i)^r \cap Y_i^-$ which project onto the lower boundary $S_i^- \subset \partial\Omega_i$. However, if this lower boundary touches $\partial\Omega_0$, we have no lower bound on its curvature. With the same computations as above can only obtain the weaker estimate

$$
\frac{\mathcal{L}^2((\Omega \setminus \Omega_i)^r \cap Y_i^-)}{\mathcal{L}^2(\Omega_i)} \;\geq\; 1. \tag{3.27}
$$

This enough to conclude the non-optimality of Ω. Indeed, when Ω_i is small enough, (3.26) and (3.27) together yield

$$
\frac{\mathcal{L}^2((\Omega \setminus \operatorname{clos}\Omega_i)^r}{\mathcal{L}^2(\Omega_i)} \;\geq\; \frac{3}{2} + \frac{r}{R}\,,
$$

providing a contradiction. We thus have

Lemma 3.9 *The optimal set Ω can have at most finitely many connected components.*

Indeed, if there are infinitely many components, then there exists an accumulation point and thus we are in the situation of Lemma 3.7 for arbitrary small sets: deleting one of these very small components and transferring its mass along the boundary of another set Ω_j with curvature R, the total area $\mathcal{L}^2(\Omega^r)$ will decrease.

In particular, the connected components of Ω are the closure of the connected components Ω_i, $i = 1, \ldots, N$: with a slight abuse of notation, we will use the notation Ω_i for the components of Ω.

3.3 Completion of the proof of Theorem 3.1.

It now remains to prove that a set $\Omega = \cup_{i=1}^N \Omega_i$, with a finite number $N \geq 2$ of connected components, is not optimal. W.l.o.g., we can assume that $\Omega_0 = \operatorname{conv} \Omega$. We will show that it is possible to rigidly move each component, so that

$$\Omega_i(t) \;=\; t(z - z_i) + \Omega_i, \tag{3.28}$$

in such a way that the area of the r-neighborhood

$$\mathcal{L}^2(\Omega(t)^r) \;=\; \sum_i \mathcal{L}^2(\Omega_i(t)^r) - \sum_{i \neq j} \mathcal{L}^2\big(\Omega_i(t)^r \cap \Omega_j(t)^r\big)$$

is strictly decreasing. Here the points $z \in \Omega_0$ and $z_i \in \Omega_i$ must be carefully chosen, so that all the sets $\Omega_i(t)$ remain inside Ω_0, for $t \in [0, \varepsilon]$ with $\varepsilon > 0$ small enough. In the above formula we have used (3.13).

If $\Omega = \cup_i \Omega_i$ is an optimal set, the time derivative of the area $\mathcal{L}^2(\Omega(t)^r)$ is computed as follows. Call $\mathcal{A}$ the set of all couples (i, j) such that $i \neq j$ and

$$d_{ij} \;\doteq\; \operatorname{dist}(\Omega_i, \Omega_j) \;=\; |x_i - x_j| \;<\; 2r.$$

Here x_i, x_j are the points considered in Lemma 3.7. At time $t = 0$, assuming that $z_i \in \Omega_i$ for all $i = 1, \ldots, N$, the time derivative is computed by

$$\frac{d}{dt}\mathcal{L}^2\bigg(\bigcup_i \Omega_i(t)^r\bigg)_{t=0} \;=\; \sum_{(i,j) \in \mathcal{A}} 2\sqrt{r^2 - \frac{d_{ij}^2}{4}}\; (x_i - x_j) \cdot (z_j - z_i) \;\leq\; 0. \tag{3.29}$$

We can always assume that $\Omega^r = \cup_i \Omega_i^r$ is connected, otherwise the non-optimality is trivial: indeed, with the same ideas here below, otherwise there is a component of Ω^r which can be moved freely inside Ω_0 until it superimpose to another component of Ω^r. Notice that this assumption implies that, for every i, there exists $j \neq i$ such that $\Omega_i^r \cap \Omega_j^r \neq \emptyset$. As a consequence, we can assume that the inequality (3.29) is strict.

If Ω has at least two components, the first lemma rules out the existence of an Ω_i which cannot be translated inside Ω_0 (see Fig. 14).

Lemma 3.10 *Let $\Omega \subset \Omega_0$ be an optimal set. If Ω has more than one component, then for every component Ω_i the set $\partial\Omega_i \cap \partial\Omega_0$ is a connected arc whose normal vectors span an angle $< \pi$.*

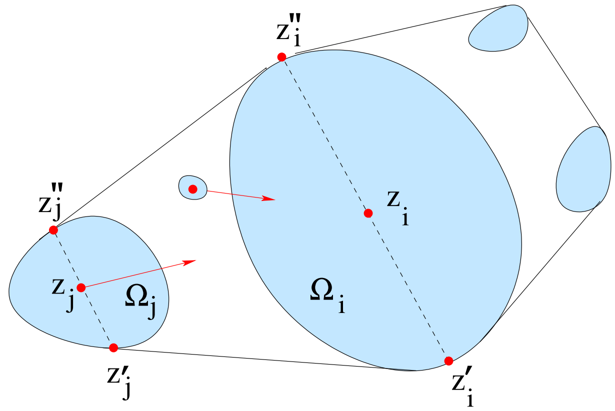

Figure 14: Proving that a set $\Omega = \bigcup_i \Omega_i \subset \Omega_0$ with finitely many connected components cannot be optimal. The case considered in Lemma 3.10.

Proof. 1. For a component Ω_i such that $\partial\Omega_i \cap \partial\Omega_0 = \emptyset$, the spanned angle is zero and the conclusion is trivial.

2. Next, assume that there is a component Ω_i such that $\partial\Omega_0 \setminus \partial\Omega_i$ contains an arc $\overset{\frown}{z_i' z_i''}$ such that the angle spanned by $\mathbf{n}(x)$, $x \in \overset{\frown}{z_i' z_i''}$, is $\leq \pi$. We consider only the components Ω_j contained in the part of $\Omega_0 \setminus \Omega_i$ whose boundary contains the arc $\overset{\frown}{z_i' z_i''}$. There must be at least one such set, because $\Omega_0 = \operatorname{conv} \Omega$. To obtain a contradiction, we shall move only these sets.

If $\partial\Omega_j \cap \partial\Omega_0 = \overset{\frown}{z_j' z_j''}$, we consider the point $z_j = \frac{z_j' + z_j''}{2} \in \Omega_j$. Otherwise, if $\partial\Omega_j \cap \partial\Omega_0 = \emptyset$, we take the barycenter: $z_j = \fint_{\Omega_j} x\,dx$. In addition, we set $z = z_i = \frac{z_i' + z_i''}{2}$. Defining $\Omega_i(t) = \Omega_i$ while $\Omega_j(t) = \Omega_j + t(z - z_j)$, we check that $\Omega_j(t) \subset \Omega_0$ for $t > 0$ small. Moreover, (3.29) is satisfied as a strict inequality. Hence Ω is not optimal. □

If Ω contains more than one component, by Lemma 3.10, for every component Ω_i the intersection $\partial\Omega_i \cap \partial\Omega_0$ either is empty, or is a connected arc spanning an angle $< \pi$. The next lemma rules out this remaining possibility.

Lemma 3.11 *If $\Omega \subset \Omega_0$ has more that one component, and every component intersects $\partial\Omega_0$ in an arc of opening $< \pi$ (or does not touch $\partial\Omega_0$ at all), then Ω is not optimal.*

Proof. For every component Ω_i, by assumption the intersection $\partial\Omega_i \cap \partial\Omega_0$ is a connected arc $\overset{\frown}{z_i' z_i''}$ of opening $< \pi$, or it is empty (see Fig. 15).

Fix a unit vector $\mathbf{e} \in \mathbb{S}^1$, and choose two points $z_1, z_2 \in \partial\Omega_0$ such that $\mathbf{e} \in \mathbf{n}(z_1)$, $\mathbf{e} \in -\mathbf{n}(z_2)$. Define $z = \frac{z_1 + z_2}{2}$. Then, for every component Ω_i, if $z \in \Omega_i$ we define $z_i \doteq z$. On the other hand, if $z \notin \Omega_i$, we consider two cases.

- If $\partial\Omega_i \cap \partial\Omega_0 = \overset{\frown}{z_i' z_i''}$, we take the mid-point: $z_i \doteq \frac{z_i' + z_i''}{2}$.
- If $\partial\Omega_i \cap \partial\Omega_0 = \emptyset$, we take the barycenter: $z_i = \fint_{\Omega_i} x\,dx$.

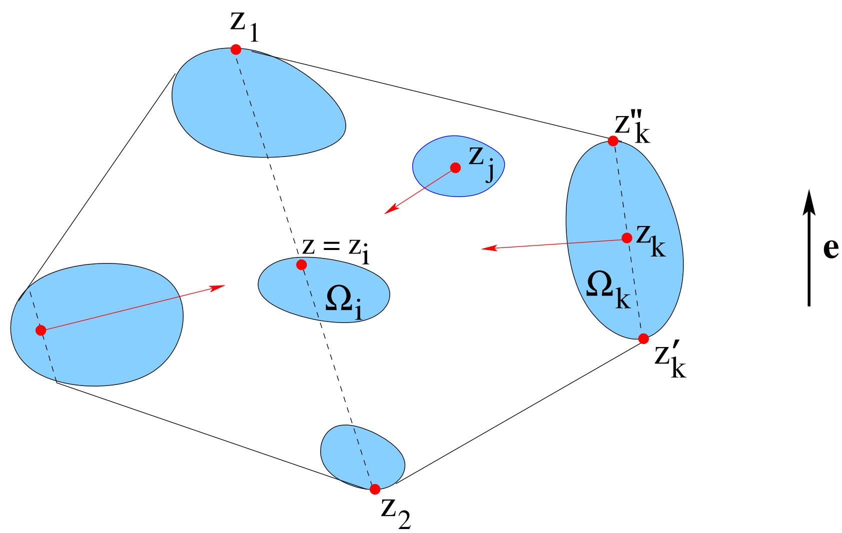

Figure 15: The configuration considered in Lemma 3.11.

By elementary geometry, we check that, for $t > 0$ small, all the sets $\Omega_i(t)$ defined at (3.28) remain inside Ω_0. Again, the relation (3.29) is satisfied as a strict inequality. Hence Ω is not optimal. □

Combining the two above lemmas, we achieve the proof of Theorem 3.1.

4 Further properties of the optimal set

Having proved that the optimal set for the problem (3.3) is convex, by the area formula

$$\mathcal{L}^2(\Omega^r) \;=\; \mathcal{L}^2(\Omega) + r\,\mathcal{H}^1(\partial\Omega) + \pi r^2$$

it is clear that the same set Ω is an optimal set for the constrained isoperimetric problem (3.4). This second problem has been studied in [16]. Recalling the definition of the sets $\widehat{\Omega}$ at (3.2), we collect here the main results:

Theorem 4.1 *Let $\Omega_0 \subset \mathbb{R}^2$ be a compact, convex set. Then, for every $0 < a \leq \mathcal{L}^2(\Omega_0)$ the constrained minimization problem (3.4) has a solution. Moreover, the following holds.*

(i) The optimal set $\widetilde{\Omega} = \widetilde{\Omega}(\Omega_0, a)$ is convex. Moreover its boundary $\partial\widetilde{\Omega}$ has curvature κ which is constant and maximal along each connected arc in $\partial\widetilde{\Omega} \setminus \partial\Omega_0$.

(ii) Conversely, any convex set $\widetilde{\Omega} \subseteq \Omega_0$, with $\mathcal{L}^2(\widetilde{\Omega}) = a$ and such that the curvature is maximal and constant along $\partial\widetilde{\Omega} \setminus \partial\Omega_0$, is an optimal solution to (3.3).

(iii) When $0 < a \leq \pi\overline{R}^2$, with $\overline{R}$ being the inner radius in (3.1), the optimal solutions are precisely the balls $B_\rho(x) \subseteq \Omega_0$ with radius $\rho = \sqrt{a/\pi}$.

(iv) When $\pi\overline{R}^2 < a \leq \mathcal{L}^2(\widehat{\Omega}(\Omega_0, \overline{R}))$, any optimal solution is contained in $\widehat{\Omega}(\Omega_0, \overline{R})$, and coincides with the convex closure of two balls of radius $\overline{R}$.

(v) For $\mathcal{L}^2(\widehat{\Omega}(\Omega_0,\overline{R})) \le a \le \mathcal{L}^2(\Omega_0)$, the optimal solution is unique. Indeed, there exists a unique $\rho \in]0,\overline{R}]$ such that

$$\widetilde{\Omega}(\Omega_0,a) \;=\; \widehat{\Omega}(\Omega_0,\rho).$$

(vi) For $0 < a < \mathcal{L}^2(\Omega_0)$, one has

$$\frac{d}{da}\mathcal{H}^1(\partial\widetilde{\Omega}(\Omega_0,a)) \;=\; \kappa, \tag{4.1}$$

where κ is the maximal curvature of the boundary $\partial\widetilde{\Omega}$.

(vii) The map $a \mapsto \mathcal{H}^1(\partial\widetilde{\Omega}(\Omega_0,a))$ is monotone increasing.

Proof. **1.** The existence and the convexity of solutions were the main results proved in Theorem 3.31 of [16], together with the properties stated in (i) and (vii). Parts (iii), (iv), (v) are proved in Theorem 3.32 of [16].

2. To prove (ii), let $\Omega \subset \Omega_0$ be such that $\mathcal{L}^2(\Omega) = a$ and the curvature κ is constant and maximal in the set $\partial\Omega \setminus \partial\Omega_0$. Any relatively open connected component of $\partial\Omega \setminus \partial\Omega_0$ is thus an arc of a circle of radius $\rho = 1/\kappa > 0$.

Being the of every point of the boundary $\kappa(x) \le 1/\rho$, it follows that Ω has the internal ball property. Namely, for every $x \in \Omega$ there exists a ball $B_\rho(y)$ such that $x \in \operatorname{clos} B_\rho(y) \subset \Omega$. Therefore, Ω is the closure of a union of balls of radius ρ.

We use the following observation. Let Ω be a compact convex set set whose boundary has curvature $\kappa(x) \le 1/\rho$ at every point $x \in \partial\Omega$. If the boundary is tangent to a ball $B_\rho(y)$ at two points $z_1, z_2 \in \partial B_\rho(y)$, then Ω must contain the arc $\widehat{z_1z_2} \subset \partial B_\rho(y)$ with minimal length (in case where both arcs have equal length $\pi\rho$, it must contain at least one of the two arcs). If Ω contains an arc with length strictly greater than $\pi\rho$ (i.e., spanning an angle $> \pi$), then it follows that Ω itself is a ball of radius ρ (case (iii) of the statement).
If an arc $\widehat{z_1z_2}$ has length exactly $\pi\rho$ (i.e., it spans an angle $= \pi$), then the set Ω must be the convex hull of two balls of radius ρ. Namely: $\Omega = \operatorname{clos}\Big(\bigcup_{\alpha\in[0,1]} B_\rho\big((1-\alpha)x_1 + \alpha x_2\big)\Big)$ for some x_1, x_2. Here we can take x_1 to be the center of the ball whose boundary contains the arc $\widehat{z_1z_2}$, while x_2 is the center of the inner ball tangent to $\partial\Omega$ at the furthest point from x_1. Hence $\rho = \overline{R}$. This corresponds to case (iv) of the statement.

If every arc of radius ρ contained in $\partial\Omega$ has length $< \pi\rho$ (i.e., it spans an angle $< \pi$), we claim that $\Omega = \widehat{\Omega}(\Omega_0,\rho)$ (case (v) of the statement). In other words, it is impossible to enlarge Ω by adding other balls $B_\rho(y)$ of radius ρ contained in Ω_0. To prove the claim, two cases are considered:

- If the ball $B_\rho(y)$ we add does not intersect Ω, then by convexity one arc must be $\ge \pi$ (which contradicts the assumptions here), otherwise there are two points with distance $> 2\rho$ connected by an arc of radius ρ, which is impossible.

- On the other hand, if $B_\rho(y) \cap \partial\Omega \neq \emptyset$, then the intersection must be an arc $\widehat{z_1z_2}$ of curvature ρ. But since the opening of this arc is $< \pi$, $B_\rho(y)$ must be already in Ω; otherwise $B_\rho(y)$ is not a subset of Ω_0.

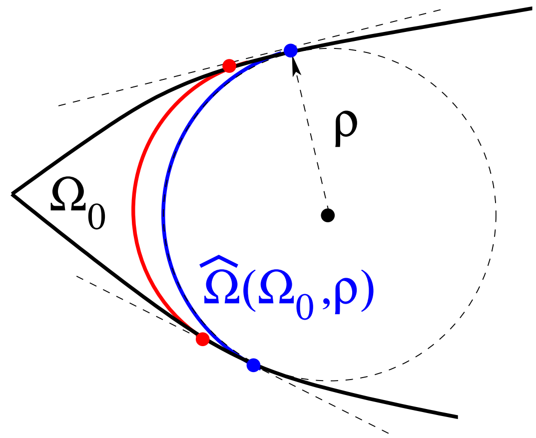

Figure 16: Computing the increase in the area of in $\widehat{\Omega}(\Omega_0, \rho)$, as the radius ρ is reduced.

3. To prove (vi), let θ_i be the angle covered by the free arc $L_i \subset \partial\Omega \cap \Omega_0$ of maximal curvature, $i \geq 1$. Since by definition

$$\partial\Omega \setminus \partial\Omega_0 \ = \ \bigcup_i L_i, \qquad \theta_i = \frac{\mathcal{H}^1(L_i)}{\rho},$$

it is enough to study the variation of area and perimeter about a single arc L_i, spanning an angle θ_i, which has constant curvature $\kappa = 1/\rho$. Moreover, the only non elementary case is when the opening of the arc is $< \pi$. In this case we will use the dependence w.r.t. the maximal radius of curvature ρ of (3.2) for $0 < \rho < \overline{R}$.
Up to a rigid change of coordinates, about every arc L_i we are in the situation shown in Fig. 16. If $\mathcal{H}^1(L_i(\rho))$ is the length of the arc $\widehat{z_1 z_2} \subset \partial\widehat{\Omega}(\Omega_0, \rho)$, one gets

$$\begin{aligned} \mathcal{H}^1(L_i(\rho)) \ < \ \mathcal{H}^1(L_i(\rho - h)) \ &= \ (\rho - h)\theta_i + 2h \tan\tfrac{\theta_i}{2} + o(h) \\ &= \ \mathcal{H}^1(L_i(\rho)) + h\left(2\tan\tfrac{\theta_i}{2} - \theta_i\right) + o(h) \\ &\leq \ \mathcal{H}^1(L_i(\rho)) + h\left(2\tan\tfrac{\theta_i}{2} - \theta_i\right), \end{aligned} \tag{4.2}$$

where $\theta_i = \mathcal{H}^1(L_i)/\rho$. Here we observed that $o(h) \leq 0$ by convexity. Note that the case of maximal growth occurs when $\partial\widehat{\Omega}(\Omega_0, \rho)$ coincides with the tangent cone. In particular, the map $\rho \mapsto \mathcal{H}^1(L_i(\rho))$ is right differentiable with derivative bounded by

$$2\tan\left(\frac{\theta_i}{2}\right) - \theta_i.$$

The above arguments yield the estimate

$$\mathcal{H}^1(\partial\widehat{\Omega}(\Omega_0, \rho)) \ < \ \mathcal{H}^1(\partial\widehat{\Omega}(\Omega_0, \rho - h)) \ \leq \ \mathcal{H}^1(\partial\widehat{\Omega}(\Omega_0, \rho)) + h\sum_i \left(2\tan\frac{\theta_i}{2} - \theta_i\right).$$

Therefore the function $\rho \mapsto \mathcal{H}^1(\partial\widehat{\Omega}(\Omega_0, \rho))$ is Lipschitz for $\rho < \overline{R}$ and strictly decreasing w.r.t.

ρ. For every $k \geq 1$, taking the derivative for a.e. $0 < \rho < \overline{R}$ we obtain

$$\sum_{i \leq k} \left(2 \tan \frac{\theta_i}{2} - \theta_i \right) = \sum_{i \leq k} - \frac{d}{d\rho} \mathcal{H}^1(L_i(\rho))$$

$$\leq \; - \frac{d}{d\rho} \mathcal{H}^1(\partial \widehat{\Omega}(\Omega_0, \rho)) \; \leq \; \sum_i \left(2 \tan \frac{\theta_i}{2} - \theta_i \right).$$

Since the series is convergent, for a.e. ρ we conclude that

$$- \frac{d}{d\rho} \mathcal{H}^1(\partial \widehat{\Omega}(\Omega_0, \rho)) \; = \; \sum_i \left(2 \tan \frac{\theta_i}{2} - \theta_i \right).$$

The same computation can be done for the area: the variation of area $A_i(\rho)$ inside each region of opening θ_i is computed by

$$\mathcal{L}^2(A_i(\rho)) \; < \; \mathcal{L}^2(A_i(\rho - h)) \; = \; \mathcal{L}^2(A_i(\rho)) + \big(\rho^2 - (\rho - h)^2\big) \left(\tan \frac{\theta_i}{2} - \frac{\theta_i}{2} \right) + o(h) \tag{4.3}$$

with $o(h) \leq 0$. Arguing as before, for a.e. $0 < \rho < \overline{R}$ we obtain

$$- \frac{d}{d\rho} \mathcal{L}^2(\widehat{\Omega}(\Omega_0, \rho)) \; = \; \rho \sum_i \left(2 \tan \frac{\theta_i}{2} - \theta_i \right) \; = \; - \rho \frac{d}{d\rho} \mathcal{H}^1(\partial \widehat{\Omega}(\Omega_0, \rho)).$$

This yields (4.1), for a.e. ρ. The final observation is that $\rho \mapsto \mathcal{L}^2(\widehat{\Omega}(\Omega_0, \rho))$ is strictly decreasing and continuous, as seen by (4.3). Therefore the inverse map

$$\mathcal{L}^2(\widehat{\Omega}(\Omega_0, \rho)) \mapsto \rho$$

is continuous, thus implying that (4.1) holds for all $\rho \in \,]0, \overline{R}[$. □

Remark 4.1 We observe that the minimizer $\widetilde{\Omega} = \widetilde{\Omega}(\Omega_0, a)$ is not uniquely determined when

$$a \; < \; \mathcal{L}^2(\widehat{\Omega}(\Omega_0, \overline{R})). \tag{4.4}$$

As shown in Fig. 17, one can remove this ambiguity and single out a unique set $\widetilde{\Omega}$ by considering the barycenter $\mathbf{b}$ of $\widehat{\Omega}(\Omega_0, \overline{R})$, and imposing that the barycenter of $\widetilde{\Omega}$ also coincide with $\mathbf{b}$.

More precisely, for a suitable unit vector $\mathbf{e}$ and $\ell^* \geq 0$, we have the representation

$$\widehat{\Omega}(\Omega_0, \overline{R}) \; = \; \bigcup_{|h| \leq \ell^*} B_{\overline{R}}(\mathbf{b} + h\mathbf{e}).$$

We can then define

$$\widetilde{\Omega}(\Omega_0, a) \; \doteq \; \begin{cases} B_{\sqrt{a/\pi}}(\mathbf{b}) & \text{if} \quad 0 \leq a \leq \pi \overline{R}^2, \\ \displaystyle\bigcup_{|h| \leq \ell} B_{\overline{R}}(\mathbf{b} + h\mathbf{e}) & \begin{array}{l} \text{if} \quad \pi \overline{R}^2 < a < \mathcal{L}^2(\widehat{\Omega}(\Omega_0, \overline{R})), \\ \text{choosing } \ell \text{ so that } 2\overline{R}\ell + \pi \overline{R}^2 = a. \end{array} \end{cases} \tag{4.5}$$

On the other hand, when $\mathcal{L}^2(\widehat{\Omega}(\Omega_0, \overline{R})) \; \leq \; a \; \leq \; \mathcal{L}^2(\Omega_0)$, the optimal set $\widetilde{\Omega}(\Omega_0, a)$ is already uniquely determined.

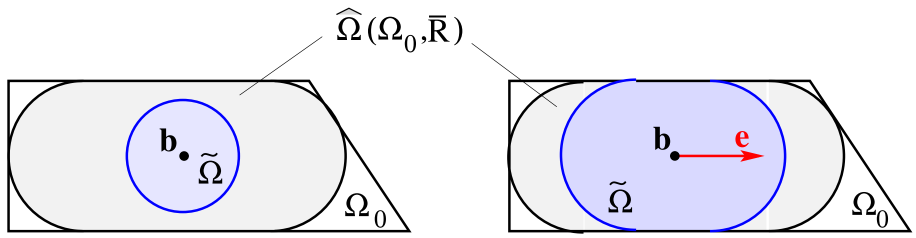

Figure 17: Optimal sets $\widetilde{\Omega} = \widetilde{\Omega}(\Omega_0, a)$ in the cases considered at (iii) and at (iv) of Theorem 3.1, respectively. Here Ω_0 is a trapezoid. Note that in both of these cases the optimal sets are not unique. Uniqueness can be achieved by imposing that the barycenter $\mathbf{b}$ of $\widetilde{\Omega}$ coincide with the barycenter of $\widehat{\Omega}(\Omega_0, \overline{R})$.

Lemma 4.1 *With the definition (4.5), the following holds:*

(1) $a_1 < a_2$ *implies* $\widetilde{\Omega}(\Omega_0, a_1) \subset \widetilde{\Omega}(\Omega_0, a_2)$.

(2) $\widetilde{\Omega}^r(\Omega_0, a) \;=\; \widetilde{\Omega}\Big(\Omega_0^r,\, \mathcal{L}^2(\widetilde{\Omega}^r(\Omega_0, a))\Big)$.

Proof. Part (1) of the lemma is obvious. To prove part (2), we first observe that, if $a \leq \mathcal{L}^2(\widehat{\Omega}(\Omega_0, \overline{R}))$, the result is a simple consequence of the identity

$$\widehat{\Omega}(\Omega_0^r, \overline{R}+r) \;=\; \widehat{\Omega}^r(\Omega_0, \overline{R}).$$

To handle the remaining case where $a > \mathcal{L}^2(\widehat{\Omega}(\Omega_0, \overline{R}))$, we will prove that the set $\widetilde{\Omega}^r(\Omega_0, a)$, i.e., the neighborhood of radius r around $\widetilde{\Omega}(\Omega_0, a)$, provides an optimal solution to the problem

$$\text{minimize:} \quad \mathcal{L}^2(\Omega^r), \qquad \text{subject to} \quad \Omega \subseteq \Omega_0^r, \quad \mathcal{L}^2(\Omega) = \mathcal{L}^2(\widetilde{\Omega}^r(\Omega_0, a)). \tag{4.6}$$

Toward this goal, we claim that $\widetilde{\Omega}^r(\Omega_0, a)$ satisfies the conditions stated in part (ii) of Lemma 2.1. Indeed, consider any point $x \in \partial\widetilde{\Omega}(\Omega_0, a)$, and call $\mathbf{n}(x)$ the unit outer normal. Then the curvature of $\widetilde{\Omega}^r(\Omega, a)$ at the boundary point $x + r\mathbf{n}(x) \in \partial\widetilde{\Omega}^r(\Omega_0, a)$ is computed by

$$\kappa\Big(\widetilde{\Omega}^r(\Omega_0, a),\, x + r\mathbf{n}(x)\Big) \;=\; \frac{\kappa\big(\widetilde{\Omega}(\Omega_0, a), x\big)}{1 + r\kappa\big(\widetilde{\Omega}(\Omega_0, a), x\big)}. \tag{4.7}$$

Since by assumption the curvature $\kappa\big(\widetilde{\Omega}(\Omega_0, a), x\big)$ is constant and maximal at points $x \in \partial\widetilde{\Omega}(\Omega_0, a) \setminus \Omega_0$, by (4.7) the curvature $\kappa\Big(\widetilde{\Omega}^r(\Omega_0, a),\, x + r\mathbf{n}(x)\Big)$ is constant and maximal at points $x + r\mathbf{n}(x) \in \partial\widetilde{\Omega}^r(\Omega_0, a) \setminus \Omega_0^r$. Using (ii) in Lemma 2.1, we thus obtain the optimality of the set $\widetilde{\Omega}^r(\Omega_0, a)$.

Finally, we notice the implication

$$a \,>\, \mathcal{L}^2(\widehat{\Omega}(\Omega_0, \overline{R})) \qquad \Longrightarrow \qquad \mathcal{L}^2(\widetilde{\Omega}^r(\Omega_0, a)) \,>\, \mathcal{L}^2(\widehat{\Omega}(\Omega_0^r, \overline{R}+r)).$$

This implies that the optimal solution on (4.6) is unique, completing the proof of part (2) of the present lemma. $\square$

5 The optimal strategy, in continuum time

We study the optimization problems **(MTP)**, **(OP)** on the entire plane, assuming that the initial set $\Omega_0 \subset \mathbb{R}^2$ is convex.

According to Definition 1.1, we say that the multifunction $t \mapsto \Omega(t) \subset \mathbb{R}^2$ with compact values is **admissible** if the following holds:

- *For every time $t \geq 0$ and $\delta > 0$,*

$$\Omega(t+\delta) \subset \Omega^\delta(t), \qquad \lim_{\delta \to 0+} \frac{\mathcal{L}^2(\Omega^\delta(t)) - \mathcal{L}^2(\Omega(t+\delta))}{\delta} = M. \tag{5.1}$$

Assuming that all sets $\Omega(t)$ have perimeter with finite length, one has

$$\begin{aligned} \frac{d^+}{dt}\mathcal{L}^2(\Omega(t)) &= \lim_{\delta \to 0+} \frac{\mathcal{L}^2(\Omega(t+\delta)) - \mathcal{L}^2(\Omega(t))}{\delta} \\ &= \lim_{\delta \to 0+} \frac{\mathcal{L}^2(\Omega^\delta(t)) - \mathcal{L}^2(\Omega(t))}{\delta} - M \;=\; \mathcal{H}^1(\partial\Omega(t)) - M. \end{aligned}$$

For a given time interval $[0,T]$, we are interested in finding the evolutions which minimize the terminal area $\mathcal{L}^2(\Omega(T))$. Assuming that the initial set $\Omega(0) = \Omega_0$ is convex, we will show that these evolutions also minimize the areas $\mathcal{L}^2(\Omega(t))$ of all intermediate sets, for $t \in [0,T]$, respectively.

Let a bounded, open convex set Ω_0 and a constant $M > 0$ be given. Using the notation introduced in Lemma 2.1 and in Lemma 4.1, we define an admissible strategy by setting

$$A(t) = \widetilde{\Omega}\big(\Omega_0^t, a(t)\big), \tag{5.2}$$

where the area function $a(t) = \mathcal{L}^2(A(t))$ satisfies

$$\frac{d}{dt}a(t) = \mathcal{H}^1\big(\partial\widetilde{\Omega}\big(\Omega_0^t, a(t)\big)\big) - M, \qquad a(0) = \mathcal{L}^2(\Omega_0). \tag{5.3}$$

This strategy is defined on a time interval $t \in [0, T^*]$, where

$$T^* \doteq \sup\big\{t > 0\,;\; a(t) > 0\big\} \in \;]0, +\infty] \tag{5.4}$$

is the first time when the area vanishes. We will show that this is indeed the optimal strategy.

Theorem 5.1 *Let a bounded, open convex set Ω_0 and a constant $M > 0$ be given. Then the set valued map $A(\cdot)$ introduced at (5.2)–(5.4) is a well defined, admissible strategy. For any other admissible strategy $\Omega(\cdot)$, at every time $t \in [0, T^*]$ the areas satisfy*

$$\mathcal{L}^2(A(t)) \leq \mathcal{L}^2(\Omega(t)). \tag{5.5}$$

As a consequence, one has

(i) Setting $A(t) = \emptyset$ for $t > T^*$, the map $t \mapsto A(t)$ provides a solution to the optimization problem **(OP)**.

(ii) The minimum time problem **(MTP)** *is solvable if and only if $T^* < +\infty$. In this case, the map $A(\cdot)$ provides an optimal solution.*

Proof. 1. We begin by showing that the function $a(t)$ is well defined. The proof of Theorem 4.1 shows that the map

$$(t,a) \ \mapsto \ \mathcal{H}^1(\partial\widetilde{\Omega}(\Omega_0^t, a))$$

is continuous w.r.t. both variables and monotone increasing w.r.t. a. It is also locally Lipschitz continuous w.r.t. a as long as $0 < \mathcal{L}^2(\widetilde{\Omega}(\Omega_0^t, a)) < \mathcal{L}^2(\Omega_0^t)$. By Peano's theorem, a solution to the ODE (5.3) thus exists.

To prove uniqueness, consider two solutions, say $a(t) \leq a'(t)$. Observing that the maximum curvature of the boundary of Ω_0^t is $\leq 1/t$, using (4.1) we obtain the bound

$$\begin{aligned}\frac{d}{dt}\big(a'(t) - a(t)\big) \ &= \mathcal{H}^1(\partial\widetilde{\Omega}(\Omega_0^t, a'(t))) - \mathcal{H}^1(\partial\widetilde{\Omega}(\Omega_0^t, a(t)))\\ &= \int_{a(t)}^{a'(t)} \frac{d}{da}\mathcal{H}^1(\partial\widetilde{\Omega}(\Omega_0^t, a))\, da \ \leq \ \frac{a'(t) - a(t)}{t}\,,\end{aligned} \tag{5.6}$$

valid for $t > 0$ sufficiently small. Therefore for $0 < s \leq t$

$$a'(t) - a(t) \ \leq \ \frac{t}{s}\big(a'(s) - a(s)\big). \tag{5.7}$$

Since at $t = 0$ one has

$$\lim_{\delta\to 0+} \frac{\mathcal{L}^2(\Omega_0^\delta) - a(\delta)}{\delta} \ = \ \lim_{\delta\to 0+} \frac{\mathcal{L}^2(\Omega_0^\delta) - a'(\delta)}{\delta} \ = \ M,$$

we conclude that

$$\lim_{s\to 0} \frac{a'(s) - a(s)}{s} \ = \ 0.$$

Letting $s \to 0$ in (5.7), this yields the uniqueness of the solution.

By the definition of the area function at (5.3), it follows that $t \mapsto A(t)$ is an admissible strategy.

3. We now prove the inequalities (5.5), showing that the strategy (5.2) is optimal. Given any other admissible strategy $t \mapsto \Omega(t)$, consider the sets

$$Z(t) \ \doteq \ \widetilde{\Omega}\Big(\Omega_0^t,\, \mathcal{L}^2(\Omega(t))\Big).$$

The time derivative of the function $z(t) \doteq \mathcal{L}^2(Z(t)) \ = \ \mathcal{L}^2(\Omega(t))$ is computed by

$$\begin{aligned}\frac{d^+}{dt}z(t) \ &= \ \liminf_{\delta\to 0+} \frac{\mathcal{L}^2(Z(t+\delta)) - \mathcal{L}^2(Z(t))}{\delta}\\ &= \ \lim_{\delta\to 0+} \frac{\mathcal{L}^2(\Omega(t+\delta)) - \mathcal{L}^2(\Omega^\delta(t))}{\delta} + \liminf_{\delta\to 0+} \frac{\mathcal{L}^2(\Omega^\delta(t)) - \mathcal{L}^2(Z(t))}{\delta}\\ &\geq \ -M + \liminf_{\delta\to 0+} \frac{|Z^\delta(t)| - \mathcal{L}^2(Z(t))}{\delta}\\ &= \ |\partial Z(t)| - M.\end{aligned}$$

Therefore, $z(t) = \mathcal{L}^2(\Omega(t))$ satisfies the differential inequality

$$\frac{d^+}{dt} z(t) \;\geq\; \left|\partial\widetilde{\Omega}(\Omega_0^t, z(t))\right| - M.$$

Following the same line as in (5.6), a comparison with the optimal solution $a(t) = \mathcal{L}^2(A(t))$ now yields

$$\frac{d^+}{dt}(z(t) - a(t)) \;=\; \liminf_{\delta\to 0+} \frac{(z(r+\delta) - a(t+\delta)) - (z(t) - a(t))}{\delta} \;\geq\; \frac{z(t) - a(t)}{t},$$

Since the map $t \mapsto z(t) - a(t)$ is continuous, we deduce

$$z(t) - a(t) \;\geq\; \frac{t}{s}(z(s) - a(s)). \tag{5.8}$$

On the other hand, since both strategies are admissible, at the initial time we have

$$\lim_{\delta\to 0+} \frac{z(\delta) - a(\delta)}{\delta} \;=\; 0. \tag{5.9}$$

Letting $s \to 0$ in (5.8) and using (5.9), for every $t \in [0, T^*]$ we thus obtain

$$z(t) \;\geq\; a(t),$$

proving (5.5).

The two statements (i)-(ii) are now an immediate consequence of (5.5). □

Remark 5.1 The optimal strategy $A(t)$ introduced at (5.2)-(5.3) is uniquely determined for all $t \in [0, T^*]$, thanks to the formulas (4.5) which remove any ambiguity also in the case (4.4) where $a > 0$ is very small. In general, however, other optimal solutions can exist.

Calling $R(t)$ the inner radius of the set Ω_0^t, if $a(t) < \left|\widehat{\Omega}(\Omega_0^t, R(t))\right|$, according to (iv) and (v) in Theorem 4.1, the minimization problem

$$\text{minimize:} \qquad \mathcal{H}^1(\partial\Omega) \qquad \text{subject to} \quad \Omega \subseteq \Omega_0^t\,, \quad \mathcal{L}^2(\Omega) = a(t), \tag{5.10}$$

has multiple solutions. One can thus construct a different optimal strategy, say $t \mapsto A'(t) \subset \Omega_0^t$, where each set $A'(t)$ is a translation of the corresponding set $A(t)$.

6 Large time behavior

In this last section we study the large time behavior of the optimal strategy $A(t)$.

Proposition 6.1 *Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded, open convex set. Then there exists a constant $M_0 > 0$ such that the following holds.*

(i) For $0 < M < M_0$, the optimal strategy $A(t)$ defined at (5.2)–(5.4) satisfies

$$\lim_{t\to+\infty} \mathcal{L}^2(A(t)) \;=\; +\infty. \tag{6.1}$$

(ii) For $M = M_0$, after some time $T^\dagger \geq 0$ the set $A(t)$ becomes a ball:

$$A(t) \;=\; B_{M/2\pi}(\bar{x}) \qquad\qquad \textit{for all } t \geq T^\dagger. \tag{6.2}$$

(iii) For $M > M_0$, the area $\mathcal{L}^2(A(t))$ shrinks to zero at a finite time $T^* < +\infty$. In this case, there exists a time $T^\dagger < T^*$ such that the set $A(t)$ is a ball for all $T^\dagger \leq t < T^*$.

Proof. 1. Since our solutions now depend on the choice of M, the notation $a_M(t) = \mathcal{L}^2(A_M(t))$ will be used. We consider the set of all solutions a_M of (5.3) which remain uniformly bounded for all times $t \geq 0$, and define

$$M_0 \;=\; \inf\left\{ M > 0\,; \quad \sup_{t>0}\; a_M(t) < +\infty \right\}. \tag{6.3}$$

From the equation (5.3) it immediately follows

$$M' < M \qquad \Longrightarrow \qquad a_M(t) \;\leq\; a_{M'}(t) \qquad\qquad \text{for all } t \geq 0. \tag{6.4}$$

2. To prove (i), assume $M < M_0$. By definition, $\sup_{t>0} a_M(t) = +\infty$. By the isoperimetric inequality, any set Ω with area $\mathcal{L}^2(\Omega) = a$ has perimeter $\mathcal{H}^1(\Omega) \geq 2\sqrt{\pi a}$. Hence from (5.3) it follows

$$\frac{d}{dt} a_M(t) \;\geq\; 2\sqrt{\pi a_M(t)} - M.$$

In particular, if at some time $\tau > 0$ one has $2\sqrt{\pi a_M(\tau)} > M$, then the area function $t \mapsto a_M(t)$ is monotone increasing, and approaches infinity as $t \to +\infty$. This proves part (i).

3. To prove (iii), assume $M > M_0$. Let $M_0 < M' < M$. By assumption, the solution $a_{M'}(t) \geq a_M(t)$ is also uniformly bounded. The difference between these two solutions of (5.3) satisfies

$$\frac{d}{dt}\big(a_{M'}(t) - a_M(t)\big) \;\geq\; M - M'.$$

At all times $t \geq 0$ where $a_M(t)$ is defined, this implies

$$a_M(t) \;\leq\; \left(\sup_{\tau>0}\; a_{M'}(\tau) \right) - (M - M')t. \tag{6.5}$$

Since the right hand side of (6.5) becomes negative for t large, by continuity there exists a time T^* such that $\mathcal{L}^2(A_M(T^*)) = a_M(T^*) = 0$. This proves the first statement in (iii).

Next, call $R_0 \geq 0$ the inner radius of the convex set Ω_0, i.e. the radius of the largest ball contained inside Ω_0. Then the inner radius of Ω_0^t is $R(t) = R_0 + t$. By the definition (5.2), when the area satisfies $a_M(t) = \mathcal{L}^2(A_M(t)) \leq \pi R^2(t)$, the set $A_M(t)$ becomes a ball. This is certainly true when t is sufficiently close to T^*, because

$$\lim_{t \to T^*-} a_M(t) \;=\; 0, \qquad\qquad \lim_{t \to T^*-} R(t) \;=\; R_0 + T^*.$$

This establishes the last statement in (iii).

4. Finally, consider the case $M = M_0$. We claim that the corresponding solution $t \mapsto a_M(t)$ remains uniformly positive, and uniformly bounded.

Indeed, assume that at some time $\tau > 0$ one has $2\sqrt{\pi a_{M_0}(\tau)} > M_0$. By continuity, there exists $\varepsilon > 0$ such that $2\sqrt{\pi a_{M'}(\tau)} > M'$ for every $M' \in [M_0, M_0 + \varepsilon]$ as well. As remarked in step **2**, this implies $a_{M'}(t) \to +\infty$ as $t \to +\infty$, reaching a contradiction with the definition of M_0.

On the other hand, assume that $\liminf_{t\to\infty} a_{M_0}(t) = 0$. In this case, following the argument in step **3**, we can find a time τ such that $A_{M_0}(\tau)$ is a ball of radius $R < \frac{M}{2\pi}$. By continuity there exists $\varepsilon > 0$ such that, for every $M' \in [M_0 - \varepsilon, M_0]$, the set $A_{M'}(\tau)$ is also a ball of radius $R < \frac{M'}{2\pi}$. If this happens, then the area $A_{M'}(\tau)$ shrinks to zero in finite time. Again, this yields a contradiction with the definition of M_0.

It remains to prove that, for t sufficiently large, $A_{M_0}(t)$ is a ball. Toward this goal we observe that the inner radius of Ω_0^t is $R(t) > t$. Hence, when $a = a_{M_0}(t) < \pi t^2$, the solution to the optimization problem (5.10) is a ball. Since $a_{M_0}(t)$ remains bounded, this inequality is true (and hence $A_{M_0}(t)$ is a ball) for all times t large enough. It is now clear that the radius of this ball must be $R = \frac{M}{2\pi}$. Otherwise, the solution of (5.3) will tend to infinity, or else become zero in finite time. $\square$

Remark 6.1 When the initial set Ω_0 is a ball of radius R, one has $M_0 = 2\pi R = 2\sqrt{\pi \mathcal{L}^2(\Omega_0)}$. For a general convex set Ω_0, one has

$$M_0 \ \le\ 2\sqrt{\pi \mathcal{L}^2(\Omega_0)}\,.$$

Indeed, this threshold is smaller than the threshold for the ball, since the perimeter remains larger for the same area.

In the case where Ω_0 is a square, for various values of M the optimal strategy $A_M(t)$ has been studied in Example 8.1 of [7]. In particular, for the unit square the value of M_0 can be computed explicitly:

$$M_0 \ =\ \frac{4-\pi}{1-\ln 2} \ \approx\ 2.797 \ <\ 2\sqrt{\pi}.$$

In this case, for $M = M_0$, the optimal set $A(t)$ becomes a ball at time

$$T^\dagger = \frac{1}{4(1-\ln 2)} - \frac{1}{2}.$$

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References

[1] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems.* Clarendon Press, Oxford, 2000.

[2] S. Anita, V. Capasso, and G. Dimitriu, Regional control for a spatially structured malaria model. *Math. Meth. Appl. Sci.* **42** (2019), 2909–2933.

[3] S. Anita, V. Capasso, and A. M. Mosneagu, Global eradication for spatially structured populations by regional control. *Discr. Cont. Dyn. Syst.*, Series B, **24** (2019), 2511–2533.

[4] A. Bressan, Differential inclusions and the control of forest fires, *J. Differential Equations* (special volume in honor of A. Cellina and J. Yorke), **243** (2007), 179–207.

[5] A. Bressan, Dynamic blocking problems for a model of fire propagation. In *Advances in Applied Mathematics, Modeling, and Computational Science,* pp. 11–40. R. Melnik and I. Kotsireas editors. Fields Institute Communications, Springer, New York, 2013.

[6] A. Bressan, M. T. Chiri, and N. Salehi, On the optimal control of propagation fronts, *Math. Models Methods Appl. Sci.*, **32** (2022), 1109–1140.

[7] A. Bressan, M. T. Chiri, and N. Salehi, Optimal control of moving sets. *J. Differential Equations* **361** (2023), 97–137.

[8] A. Bressan, M. Mazzola, and K. T. Nguyen, Approximation of sweeping processes and controllability for a set valued evolution, *SIAM J. Control Optim.* **57** (2019), 2487–2514.

[9] A. Bressan and D. Zhang, Control problems for a class of set valued evolutions, *Set-Valued Var. Anal.* **20** (2012), 581–601.

[10] R. M. Colombo and N. Pogodaev, On the control of moving sets: Positive and negative confinement results, *SIAM J. Control Optim.* **51** (2013), 380–401.

[11] R. M. Colombo, T. Lorenz and N. Pogodaev, On the modeling of moving populations through set evolution equations. *Discrete Contin. Dyn. Syst.* **35** (2015), 73–98.

[12] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions.* CRC Press, 1991.

[13] A. Gray, *Tubes.* Second edition. Progress in Mathematics, 221. Birkhäuser, Basel, 2004.

[14] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems.* Cambridge University Press, 2012.

[15] D. Ruiz-Balet and E. Zuazua, Control under constraints for multi-dimensional reaction-diffusion monostable and bistable equations. *J. Math. Pures Appl.* **143** (2020) 345–375.

[16] E. Stredulinsky and W. P. Ziemer, Area minimizing sets subject to a volume constraint in a convex set. *J. Geometric Analysis* **7** (1997), 653–677.