On the closability of differential operators

Giovanni Alberti, David Bate, Andrea Marchese

Abstract. We discuss the closability of directional derivative operators with respect to a general Radon measure μ on $\mathbb{R}^d$; our main theorem completely characterizes the vectorfields for which the corresponding operator is closable from the space of Lipschitz functions $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p(\mu)$, for $1 \le p \le \infty$. We also consider certain classes of multilinear differential operators.
We then discuss the closability of the same operators from $L^q(\mu)$ to $L^p(\mu)$; we give necessary conditions and sufficient conditions for closability, but we do not have an exact characterization. As a corollary we obtain that classical differential operators such as gradient, divergence and jacobian determinant are closable from $L^q(\mu)$ to $L^p(\mu)$ only if μ is absolutely continuous with respect to the Lebesgue measure.
Lastly, we rephrase our results on multilinear operators in terms of metric currents.

Keywords: closable operators, directional derivative operators, Lipschitz functions, Sobolev spaces, normal currents, metric currents.

MSC (2010): 26B05, 49Q15, 26A27.

1. Introduction

One way of defining the Sobolev Spaces $W_0^{1,p}(\Omega)$ for an open set Ω in $\mathbb{R}^d$ is taking the completion of the space $C_c^1(\Omega)$ of functions of class C^1 with compact support on Ω with respect to the Sobolev norm $\|\cdot\|_{W^{1,p}}$.

This construction can be made more precise as follows: we consider the graph of the gradient operator $\nabla : C_c^1(\Omega) \to C_c^0(\Omega; \mathbb{R}^d)$ as a subset of the product space $L^p(\Omega) \times L^p(\Omega; \mathbb{R}^d)$, we take its closure Γ, and we show that Γ is still a graph, that is, for every $u \in L^p(\Omega)$ there exists *at most* one $v \in L^p(\Omega; \mathbb{R}^d)$ such that $(u, v) \in \Gamma$. We then consider the operator whose graph is Γ: the domain is the Sobolev space $W_0^{1,p}(\Omega)$ and the operator is the gradient for Sobolev functions.[1]

Note that the essential ingredient in this construction is that the closure of the graph of the gradient is still a graph. The extension of this construction to more general operators leads to the following abstract definition:

Closable operators. Given X, Y topological spaces, D subset of X, and a map $T : D \to Y$, we denote by Γ the closure of the graph $\{(x, T(x)) \colon x \in D\}$ in $X \times Y$, and we say T is *closable* (from X to Y) if Γ is also a graph, that is, for every $x \in X$ there exists *at most* one $y \in Y$ such that $(x, y) \in \Gamma$.

---

[1] Moreover $\|u\|_{W^{1,p}}$ is the norm of $(u, \nabla u)$ in the product space $L^p(\Omega) \times L^p(\Omega; \mathbb{R}^d)$.

In this paper we study the closability of certain first-order differential operators. The spaces X, D and Y will be always linear spaces of functions on $\mathbb{R}^d$, and we focus in particular on directional derivative operators.

More precisely:

Functions spaces. Through this paper X is either of the following spaces:

- $L^q(\mu)$ where μ is a Radon measure on $\mathbb{R}^d$ and $1 \le q \le \infty$;
- the space $\mathrm{Lip}(\mathbb{R}^d)$ of all Lipschitz functions $u : \mathbb{R}^d \to \mathbb{R}$, endowed with the weak* topology of $W^{1,\infty}$; in particular a sequence (u_n) converges to u in $\mathrm{Lip}(\mathbb{R}^d)$ if and only if $u_n \to u$ uniformly and the Lipschitz constants $\mathrm{Lip}(u_n)$ are uniformly bounded.

Moreover D is always the space $C^1_c(\mathbb{R}^d)$, and Y is $L^p(\mu)$ with $1 \le p \le \infty$.
We write L^p_w for the L^p space endowed with the weak topology, except that, with a slight abuse of notation, both L^∞ and L^∞_w stand for L^∞ endowed with the weak* topology (as dual of L^1).

Directional derivative operator. Let v be a vector field on $\mathbb{R}^d$ which is Borel measurable. We denote by T_v directional derivative operator on $C^1_c(\mathbb{R}^d)$ associated to v, that is,

$$T_v u := \frac{\partial u}{\partial v} \quad \text{for every } u \in C^1_c(\mathbb{R}^d). \tag{1.1}$$

Next theorem is our main result. The statement involves the notion of decomposability bundle $V(\mu, \cdot)$ of a measure μ: the precise definition is given in §2.2; for the time being it suffices to know that $V(\mu, x)$ is a linear subspace of $\mathbb{R}^d$ for every $x \in \mathbb{R}^d$.

1.1. Theorem. *Let μ be a Radon measure, let v and T_v be as above, and assume that $v \in L^p(\mu)$ for some $p \in [1, \infty]$.*

(i) *If $v(x) \in V(\mu, x)$ for μ-a.e. x, then every function $u \in \mathrm{Lip}(\mathbb{R}^d)$ is differentiable at μ-a.e. $x \in \mathbb{R}^d$ in the direction $v(x)$, and the linear operator $\widetilde{T}_v : \mathrm{Lip}(\mathbb{R}^d) \to L^p_w(\mu)$ defined by*

$$\widetilde{T}_v u(x) := \frac{\partial u}{\partial v}(x) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d \tag{1.2}$$

is a continuous extension of T_v.
It follows that T_v is closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p_w(\mu)$.

(ii) *Conversely, if $\{x \colon v(x) \notin V(\mu, x)\}$ has positive μ-measure, then T_v, viewed as an operator from $C^1_c(\mathbb{R}^d)$ to $L^p_w(\mu)$, is not continuous at any $u \in C^1_c(\mathbb{R}^d)$. More precisely, for every $\varepsilon > 0$ there exist a sequence (u_n) in $C^1_c(\mathbb{R}^d)$ such that*
- *$u_n \to u$ uniformly;*
- *$\mathrm{Lip}(u_n) \le \mathrm{Lip}(u) + \varepsilon$ for every n;*
- *$T_v u_n \to w$ in $L^p(\mu)$ for some $w \ne T_v u$.*

It follows that T_v is not closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p(\mu)$.

1.2. Remarks. The following are immediate consequences of Theorem 1.1.

(i) If $v(x) \in V(\mu, x)$ for μ-a.e. x, then $\tilde{T}_v$ is continuous also as an operator from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p(\mu)$. It follows that T_v is closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p(\mu)$.

(ii) If $\{x \colon v(x) \notin V(\mu, x)\}$ has positive μ-measure, then T_v is discontinuous everywhere also as an operator from $C^1_c(\mathbb{R}^d)$ to $L^p(\mu)$. Moreover T_v is not closable also from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p_w(\mu)$ nor from $L^q(\mu)$ to $L^p(\mu)$, for any $1 \leq q \leq \infty$. The latter follows from the fact that one can choose the functions u_n so that $u_n - u$ is compactly supported, see also Remark 4.2.

From Theorem 1.1 and Remarks 1.2, we deduce the following corollaries:

1.3. Corollary. *For $1 \leq p \leq \infty$, the gradient operator is closable from $\mathrm{Lip}(\mathbb{R}^d)$ to $L^p(\mu)$ (resp. $L^p_w(\mu)$) if and only if μ is absolutely continuous with respect to the Lebesgue measure ($\mu \ll \mathscr{L}^d$). The same holds for the divergence and the Jacobian determinant.*[2]

1.4. Corollary. *The gradient operator is a closable from $L^q(\mu)$ to $L^p(\mu)$ for some $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ only if $\mu \ll \mathscr{L}^d$.*

1.5. Remarks. (i) These results are naturally connected to the definition of Sobolev spaces in weighted Euclidean spaces and have applications in the representation of low-dimensional elastic structures, see [7]. Theorem 1.1 leads to a proof of the chain rule for BV function of [4] which can be adapted to finite dimensional RCD spaces, see [8]. Corollary 1.4 answers a question posed by Fukushima, see [6, Section 2.6] and [10].

(ii) We remind that the condition $\mu \ll \mathscr{L}^d$ alone is not sufficient for the closability of the gradient operator from $L^p(\mu)$ to $L^p(\mu)$. In [3, Theorem 2.2], the (absolutely continuous) measures μ on $\mathbb{R}$ with this property for $p = 2$ are characterized, see also [12, Theorem 3.1.6]. The question on the closability from $L^p(\mu)$ to $L^p(\mu)$ of directional derivative operators is also interesting. The condition given in point (i) of Theorem 1.1 remains necessary but it is no longer sufficient. In §4 we give some sufficient conditions on μ. We don't know if these conditions are also necessary.

(iii) In §5 we discuss the closability of some more general multilinear operators than the jacobian and we rephrase these results in terms of Ambrosio-Kirchheim metric currents in $\mathbb{R}^d$.

Acknowledgements. This research was initiated during visits of D.B. and A.M. at the Mathematics Department in Pisa. The visits of A.M. were partly supported by INdAM-GNAMPA. The research of D.B. is supported by the European Union's Horizon 2020 Research and Innovation Programme, grant agreement no. 948021. The research of G.A. and A.M. has been partially supported by the Italian Ministry of University and Research via the PRIN projects 2017BTM7SN and 2017TEXA3H, respectively.

[2] The Jacobian determinant of $u \in C^1_c(\mathbb{R}^d; \mathbb{R}^d)$ is $Ju := \det(\nabla u)$.

2. Notation and preliminary results

2.1. **Classical currents.** Recall that a *k-dimensional current* T in $\mathbb{R}^d$ is a continuous linear functional on the space of smooth and compactly supported differential k-forms on $\mathbb{R}^d$. The *boundary* of T, ∂T, is the $k-1$-current defined via $\langle \partial T, \omega\rangle := \langle T, d\omega\rangle$ for every smooth and compactly supported $k-1$-form ω. The *mass* of T, denoted by $\mathbb{M}(T)$, is the supremum of $\langle T, \omega\rangle$ over all k-forms ω such that $|\omega| \leq 1$ everywhere. A current T is called *normal* if both T and ∂T have finite mass.

By the Radon–Nikodým theorem, a k-dimensional current T with finite mass can be written in the form $T = \tau\mu$ where μ is a finite positive measure and τ is a k-vector field in $L^1(\mu)$. In particular, the action of T on a smooth and compactly supported k-form ω is given by

$$\langle T, \omega\rangle = \int_{\mathbb{R}^d} \langle \omega(x), \tau(x)\rangle d\mu(x) \ .$$

Given a Lipschitz curve $\gamma : [0,1] \to \mathbb{R}^d$, we denote by $[\gamma]$ the associated current, that is the current defined by

$$\langle [\gamma], \omega\rangle = \int_0^1 \langle \omega(\gamma(t)), \gamma'(t)\rangle dt \ ,$$

for every smooth and compactly supported 1-form ω. More information on currents can be found in [11].

2.2. **Decomposability bundle.** We recall the definition of decomposability bundle of a Radon measure, see [1, §2.6] and its relation with the differentiability properties of Lipschitz functions. We prefer to give a definition which is different from the original one, but it is equivalent and easier to state, see [1, §6.1 and Theorem 6.4].

2.3. Definition (Decomposability bundle). Given a Radon measure μ on $\mathbb{R}^d$ its *decomposability bundle* is a Borel map $V(\mu, \cdot)$ on $\mathbb{R}^d$ and taking values in the set $Gr := \bigcup_{0\leq k\leq d} Gr(k, d)$ (where $Gr(k, d)$ denotes the Grassmannian of k-dimensional vector subspaces of $\mathbb{R}^d$) defined as follows. A vector $v \in \mathbb{R}^d$ belongs to $V(\mu, x)$ if and only if there exists 1-dimensional normal current N with $\partial N = 0$ such that

$$\lim_{r\to 0} \frac{\mathbb{M}((N - v\mu) \llcorner B(x, r))}{\mu(B(x, r))} = 0.$$

2.4. Definition. For $1 < m \leq d$ we will denote $V_m(\mu, \cdot)$ the map corresponding to $V(\mu, \cdot)$ obtained by replacing 1-dimensional normal currents with m-dimensional ones and substituting $Gr(k, d)$ with $Gr(k, \Lambda_m(\mathbb{R}^d))$, where $\Lambda_m(\mathbb{R}^d)$ denotes the space of m-vectors in $\mathbb{R}^d$.

2.5. Theorem (see Theorem 1.1 of [1]). *Let μ be a positive Radon measure on $\mathbb{R}^d$ and let $V(\mu, \cdot)$ be the decomposability bundle of μ. Then the following statements hold:*

(i) Every Lipschitz function f on $\mathbb{R}^d$ is differentiable at μ-a.e. x with respect to the linear subspace $V(\mu, x)$. That is, there exists a linear function from $V(\mu, x)$ to $\mathbb{R}$, denoted by $dVf(x)$, such that

$$f(x+h) = f(x) + \langle dVf(x), h\rangle + o(|h|) \quad \text{for } h \in V(\mu, x).$$

(ii) The previous statement is optimal in the sense that there exists a Lipschitz function f on $\mathbb{R}^d$ such that for μ-a.e. x and every $v \notin V(\mu, x)$ the derivative of f at x in the direction v does not exist.

We conclude by recalling the converse of Rademacher's theorem. Its combination with Theorem 2.5 ensures that the decomposability bundle of a singular measure μ is a proper subspace μ-a.e.

2.6. Theorem (see Theorem 1.14 of [9]). *Let μ be a positive Radon measure on $\mathbb{R}^d$ such that every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable μ-a.e. Then $\mu \ll \mathscr{L}^d$.*

2.7. Corollary. *Let μ be a positive Radon measure on $\mathbb{R}^d$ such that $V(\mu, x) = \mathbb{R}^d$ for μ-a.e. x. Then $\mu \ll \mathscr{L}^d$.*

3. Closability of operators from Lip to L^p

Proof of Theorem 1.1: case $p = \infty$. (i) By [1, Theorem 6.3] there exists a normal 1-current $N = \tilde{v}\tilde{\mu}$ on $\mathbb{R}^d$ with $\partial N = 0$ such that $\tilde{v} \in L^\infty(\tilde{\mu})$ and $\tilde{v}$ and $\tilde{\mu}$ are extensions of v and μ in the following sense: $\tilde{\mu} = \mu + \sigma$ with $\sigma \perp \mu$ and $\tilde{v}(x) = v(x)$ for μ-a.e. x. Let u_n be a sequence such that $u_n \to u$ in $\mathrm{Lip}(\mathbb{R}^d)$. Using [1, Proposition 5.13] and the fact that $\partial N = 0$, we have

$$\partial(u_n N) = -d_{\tilde{v}} u_n \tilde{\mu}, \quad \text{and} \quad \partial(uN) = -d_{\tilde{v}} u \tilde{\mu}. \tag{3.1}$$

The uniform convergence $u_n \to u$ implies that $\mathbb{M}(u_n N - uN) \to 0$ and therefore the convergence is also in the sense of currents, and the same holds for their boundaries, that is,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f d_{\tilde{v}} u_n d\tilde{\mu} = \int_{\mathbb{R}^d} f d_{\tilde{v}} u d\tilde{\mu}, \quad \text{for every } f \in C_c^\infty(\mathbb{R}^d). \tag{3.2}$$

The density of $C_c^\infty(\mathbb{R}^d)$ in $L^1(\tilde{\mu})$ implies that (3.2) holds for every $f \in L^1(\tilde{\mu})$ and therefore $d_{\tilde{v}} u_n \to d_{\tilde{v}} u$ in $L^\infty_{w^*}(\tilde{\mu})$. In particular $d_v u_n \to d_v u = \langle d_V u, v\rangle$ in $L^\infty_{w^*}(\mu)$.

(ii) For every $w \in \mathbb{R}^d$ and $0 < \theta < \frac{\pi}{2}$, let $C(w, \theta)$ be the cone

$$C(w, \theta) = \{x \in \mathbb{R}^d : x \cdot w \geq |x| \cos\theta\}.$$

By assumption, there exists $w \in \mathbb{R}^d$ and $0 < \alpha < \beta < \frac{\pi}{2}$ such that the Borel set

$$E := \{x \in \mathbb{R}^d : v(x) \in C(w, \alpha) \text{ and } V(\mu, x) \cap C(w, \beta) = \{0\}\},$$

has positive μ-measure. By [1, Lemma 7.5], there exists a compact set $F \subset E$ with $\mu(F) > 0$ and F is $C(w, \beta)$-null, in the sense of [1, §4.11]. By [1, Lemma

4.12], for every $n = 1, 2, \dots$ there exists $f_n : \mathbb{R}^d \to \mathbb{R}$ smooth such that for every $x \in \mathbb{R}^d$:

(a) $0 \le f_n(x) \le \frac{1}{n}$;
(b) $0 \le d_w f_n(x) \le 1$ and $d_w f_n(x) \equiv 1$ on F;
(c) $|d_W f_n(x)| \le \frac{1}{\tan\beta}$ where $W = w^\perp$, d_W is the restriction of the differential d to W and $|\cdot|$ is the operator norm.

By (b) and (c) there exists $L > 0$ such that every f_n is L-Lipschitz and moreover there is a constant $C > 0$ depending on α and β such that $d_{v(x)} f_n(x) \ge C$ for every $x \in F$. Clearly the functions $u_n := u + L^{-1}\varepsilon f_n$ satisfy the requirements. □

Before proving Theorem 1.1 for $p < \infty$ we state the following corollary of Mazur's lemma.

3.1. Proposition. *Let $1 \le p < \infty$ and $T : C^1_c(\mathbb{R}^d) \to L^p(\mu)$ be a bounded linear operator. Let u_n be a sequence of functions in $C^1_c(\mathbb{R}^d)$ such that $u_n \to u$ uniformly and $Tu_n \to w$ in $L^p_w(\mu)$. Then there is a sequence $\tilde u_n$ of convex combinations of the elements of u_n such that $\tilde u_n \to u$ uniformly and $T\tilde u_n \to w$ strongly in $L^p(\mu)$.*

Proof. For every $m \in \mathbb{N}$ we consider the set A_m which is the w-closed convex hull of the set $\{Tu_m, Tu_{m+1}, \dots\}$. By Mazur's Lemma the set A_m coincides with the (strongly)-closed convex hull of the set $\{Tu_m, Tu_{m+1}, \dots\}$. Since by assumption $w \in A_m$ for every m, then for every m there exists a sequence $(w^m_n)_{n\in\mathbb{N}}$ of convex combinations of the elements of the set $\{Tu_m, Tu_{m+1}, \dots\}$ such that $w^m_n \to w$ (strongly). For every m and n we denote by $\tilde u^m_n$ the convex combination of the elements of the set $\{u_m, u_{m+1}, \dots\}$ with the same coefficients as those used to obtain w^m_n. Clearly the diagonal sequence $\tilde u_n := \tilde u^n_n$ has the desired properties. In particular the fact that $\tilde u^n_n \to u$ uniformly as $n \to \infty$ follows from the fact that the diameter of the convex hull of the set $\{u_m, u_{m+1}, \dots\}$ (with respect to the uniform norm) tends to zero as $m \to \infty$. □

In what follows we denote by p' the conjugate Hölder exponent of p. Moreover, for every $v \in \mathbb{R}^d$ we denote

$$\hat v := \begin{cases} \frac{v}{|v|} & \text{if } v \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Proof of Theorem 1.1: case $p < \infty$. Towards a proof of (i), we observe that the condition $v(x) \in V(\mu, x)$ is μ-a.e equivalent to the condition $\hat v(x) \in V(\mu, x)$. Therefore if $v(x) \in V(\mu, x)$ for μ-a.e. x, by the case $p = \infty$, $T_{\hat v}$ can be extended to the continuous operator $\tilde T_{\hat v} : \mathrm{Lip}(\mathbb{R}^d) \to L^\infty_{w^*}(\mu)$ given by $\tilde T_{\hat v} = \langle d_V u(x), \hat v(x)\rangle$. It follows that the operator $\hat T_v : \mathrm{Lip}(\mathbb{R}^d) \to L^p_w(\mu)$ given by

$$\hat T_v(u) := |v| \tilde T_{\hat v} u = |v| \langle d_V u(x)\ , \hat v(x)\rangle = \langle d_V u(x)\ , v(x)\rangle$$

is a continuous extension of T_v. Indeed, assume $u_n \to u$ in Lip and take $\phi \in L^{p'}(\mu)$. Then we have

$$\langle \hat{T}_v u_n, \phi \rangle = \langle \tilde{T}_{\hat{v}} u_n, |v|\phi \rangle.$$

Since $|v|\phi \in L^1(\mu)$, by the continuity of $\tilde{T}_{\hat{v}} : \mathrm{Lip}(\mathbb{R}^d) \to L^\infty_{w^*}(\mu)$ we have

$$\lim_{n\to\infty} \langle \tilde{T}_{\hat{v}} u_n, |v|\phi \rangle = \langle \tilde{T}_{\hat{v}} u, |v|\phi \rangle = \langle \hat{T}_v u, \phi \rangle.$$

Towards a proof of (ii), the possibility to replace $L^\infty_{w^*}(\mu)$ with $L^p_w(\mu)$ in the third bullet point can be proved with the same sequence u_n used in the case $p = \infty$. In order to replace $L^p_w(\mu)$ with $L^p(\mu)$, fix $\varepsilon > 0$ and a sequence u_n of functions in $C^1_c(\mathbb{R}^d)$ such that

- $u_n \to u$ uniformly as $n \to \infty$;
- $\mathrm{Lip}(u_n) \le \mathrm{Lip}(u) + \varepsilon$ for every $n \in \mathbb{N}$;
- $T_v u_n \to w$ in $L^p_w(\mu)$ with $w \ne T_v u$.

Applying Proposition 3.1 to such sequence we obtain a new sequence satisfying the required properties. □

Proof of Corollary 1.3. The closability of the gradient operator (as well as the closability of the divergence and the Jacobian determinant) implies the closability of all the partial derivatives operators. By point (ii) of Theorem 1.1, this implies that $V(\mu, x) = \mathbb{R}^d$ for μ-a.e. x. We deduce from Corollary 2.7 that $\mu \ll \mathscr{L}^d$. □

Proof of Corollary 1.4. This follows immediately from Remark 1.2 (ii) and Corollary 2.7. □

4. Closability of operators from L^q to L^p

4.1. Theorem. *Let μ be a Radon measure and let $v \in L^p(\mu, \mathbb{R}^d)$. Let $T_v : C^1_c(\mathbb{R}^d) \to L^p(\mu)$ be the directional derivative operator defined in (1.1). Assume there exists a real valued function α such that*

- *$\alpha \ne 0$ μ-a.e.;*
- *$\alpha \in L^{p'}(\mu)$ and $\alpha v \in L^{q'}(\mu)$;*
- *$N := \alpha v \mu$ is a normal 1-current.*

Then T_v is closable from $L^q(\mu)$ to $L^p(\mu)$.

Proof of Theorem 4.1. Assume by contradiction that T_v is not closable from $L^q(\mu)$ to $L^p(\mu)$. Since T_v is linear, this is equivalent to assuming that T_v is not closable "at $u = 0$", that is, there is a sequences u_n such that $u_n \to 0$ in $L^q(\mu)$ and $T_v u_n \to w \ne 0 \in L^p(\mu)$. Denote $\tau := \alpha v$ and let $\hat{\tau}$ be defined as in (3.3).

It follows from Smirnov's theorem, see [16], that N can be written as

(a) $N = \int_0^1 [\gamma_s] ds$, where $\gamma_s : [0, L_s] \to \mathbb{R}^d$ are 1-Lipschitz, nontrivial, and parametrized with constant speed;
(b) $\|N\| = \int_0^1 \|[\gamma_s]\| ds$;
(c) $\frac{\gamma_s'(t)}{|\gamma_s'(t)|} = \hat{\tau}(\gamma_s(t))$ for a.e. s and a.e. t.

The existence of such decomposition can be found in [14, Theorem 3.1] and in particular the validity of (c) can be proved as in [1, Theorem 5.5 (ii)].

Since $u_n \to 0$ in $L^q(\mu)$ and $|\tau| \in L^{q'}(\mu)$, then $u_n|\tau| \to 0$ in $L^1(\mu)$ or equivalently $u_n \to 0$ in $L^1(|\tau|\mu) = L^1(\|N\|)$. Hence by (b), denoting $\mu_s := \|[\gamma_s]\|$ for every $s \in [0,1]$, we have, up to subsequences,

$$u_n \to 0 \quad \text{in } L^1(\mu_s) \text{ for a.e. } s \in [0,1]. \tag{4.1}$$

Since $T_v u_n \to w$ in $L^p(\mu)$ and $\alpha \in L^{p'}(\mu)$, then $\alpha T_v u_n \to \alpha w$ in $L^1(\mu)$ or equivalently $\frac{\partial u_n}{\partial \hat\tau} \to \operatorname{sign}(\alpha)\frac{w}{|v|}$ in $L^1(|\tau|\mu)$. Hence by (b) we have, up to subsequences,

$$\frac{\partial u_n}{\partial \hat\tau} \to \operatorname{sign}(\alpha)\frac{w}{|v|} \quad \text{in } L^1(\mu_s) \text{ for a.e. } s \in [0,1]. \tag{4.2}$$

By (4.1) and property (a) for a.e. $s \in [0,1]$, up to subsequences,

$$u_n \circ \gamma_s \to 0 \quad \text{in } L^1([0, L_s]). \tag{4.3}$$

Moreover by property (c), for every n and for a.e. s, t, up to subsequences,

$$(u_n \circ \gamma_s)'(t) = \frac{\partial u_n}{\partial \hat\tau}(\gamma_s(t))|\gamma_s'(t)|, \tag{4.4}$$

so that, by (4.2) and (4.3), for a.e. s we have that $(\operatorname{sign}(\alpha)\frac{w}{|v|})\circ\gamma_s|\gamma_s'| = 0$. Since $\alpha \neq 0$ μ-a.e., this contradicts the fact that $w \neq 0 \in L^p(\mu)$. $\square$

4.2. Remark. Theorem 1.1, and Theorem 4.1 can be localized, namely they hold true also when the function spaces are replaced by their local versions. This follows from the fact that, by a standard cutoff argument, the functions f_n in the proof of Theorem 1.1 (ii) can be chosen to be compactly supported.

5. Closability of multilinear operators and metric currents

In this section we extend point (i) of Theorem 1.1 to the setting of multilinear operators and we rephrase some of the results of the paper in terms of metric currents. Our remarks are mostly consequences of the fact that the *continuity* property in the definition of metric current is equivalent to the closability of the operator J_v of Theorem 5.1 for a suitable k-vectorfield v and a measure μ.

5.1. Theorem. *Let $v \in L^\infty(\mu, \Lambda_k(\mathbb{R}^d))$ and let $J_v : C^1_c(\mathbb{R}^d, \mathbb{R}^k) \to L^\infty(\mu)$ be the multilinear differential operator*

$$J_v : (u_1, \dots, u_k) \mapsto \langle du_1 \wedge \cdots \wedge du_k; v\rangle.$$

If $v(x) \in V_k(\mu, x)$ for μ-a.e. x, then J_v can be extended to a continuous operator $\tilde J_v : \operatorname{Lip}(\mathbb{R}^d, \mathbb{R}^k) \to L^\infty_{w^*}(\mu)$ and more precisely

$$\tilde J_v(u_1, \dots, u_k)(x) = \langle d_V u_1(x) \wedge \cdots \wedge d_V u_k(x); v(x)\rangle, \quad \textit{for } \mu\textit{-a.e. } x$$

where $V = V(\mu, \cdot)$. It follows that J_v is closable from $\operatorname{Lip}(\mathbb{R}^d, \mathbb{R}^k)$ to $L^\infty_{w^*}(\mu)$.

Proof. The theorem can be proved verbatim as Theorem 1.1 (i), replacing the 1-current N with a k-current with the same properties, whose existence is proved by combining Theorems 1.1 and 1.2 of [2]. $\square$

We do not know whether the condition $v(x) \in V_k(\mu, x)$ for μ-a.e. x is necessary for the closability of J_v, see Remark 5.6.

5.2. **Metric currents.** Let (X, d) be a complete metric space. We denote $\mathscr{D}^k(X) := \mathrm{Lip}_b(X, \mathbb{R}) \times \mathrm{Lip}(X, \mathbb{R})^k$, where $\mathrm{Lip}_b(X, \mathbb{R})$ is the space of bounded Lipschitz functions on X.

5.3. Definition (Metric current). A multilinear functional $T : \mathscr{D}^k(X) \to \mathbb{R}$ is said to be a k-dimensional *metric current* if

(i) *continuity*: for every $f \in \mathrm{Lip}_b(X, \mathbb{R})$, $(\pi_1^n)_{n\in\mathbb{N}}, \ldots, (\pi_k^n)_{n\in\mathbb{N}} \subset \mathrm{Lip}(X, \mathbb{R})$ converging pointwise to $\pi_1, \ldots, \pi_k$ with $\mathrm{Lip}(\pi_i^n) \le C$ for every n

$$T(f, \pi_1^n, \ldots, \pi_k^n) \to T(f, \pi_1, \ldots, \pi_k);$$

(ii) *locality*: if there exists $i \in \{1, \ldots, k\}$ such that $\pi_i \equiv c$ on a neighbourhood of *supp*f then $T(f, \pi_1, \ldots, \pi_k) = 0$;

(iii) *finite mass*: there exists a finite Radon measure μ such that

$$|T(f, \pi_1, \ldots, \pi_k)| \le \mathrm{Lip}(\pi_1) \cdots \mathrm{Lip}(\pi_k) \int_X |f| d\mu.$$

More information on metric currents can be found in [5] and [13].

We will now focus on the choice $X = \mathbb{R}^d$, endowed with the Euclidean distance. We recall that to every k-dimensional metric current T on $\mathbb{R}^d$ with compact support one can associate a "classical" k-dimensional current $\tilde{T}$ as follows, see [5, Theorem 11.1]. Denoting $\Lambda(k, d)$ the set of multi-indices $\alpha = (1 \le \alpha_1 < \cdots < \alpha_k \le d)$ of length k in $\mathbb{R}^d$, one requires that for every smooth and compactly supported k-form

$$\omega = \sum_{\alpha\in\Lambda(k,d)} \omega_\alpha dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_k}$$

it holds

$$\langle \tilde{T}, \omega \rangle = \sum_{\alpha\in\Lambda(k,d)} T(\omega_\alpha, x_{\alpha_1}, \ldots, x_{\alpha_k}). \tag{5.1}$$

Conversely, to every flat chain T with finite mass and compact support one can associate a metric current $\hat{T}$ in such a way that the two maps are one the inverse of the other, when restricted to normal currents, see also [13, Theorem 5.5].

5.4. Proposition. *Let T be a k-dimensional metric current on $\mathbb{R}^d$ with compact support. Then there exists a k-vector field τ and a positive measure μ with* *span*$(\tau(x)) \subset V(\mu,x)$ *for* μ*-a.e.* $x \in \mathbb{R}^d$ *such that, for every* $(f,\pi_1,\dots,\pi_k) \in \mathscr{D}^k(\mathbb{R}^d)$,

$$T(f,\pi_1,\dots,\pi_k) = \int_{\mathbb{R}^d} f\langle d_V\pi_1 \wedge \cdots \wedge d_V\pi_k, \tau\rangle d\mu. \tag{5.2}$$

Proof. Consider the positive measure μ and the unit k-vector field τ such that $\tilde{T} = \tau\mu$. Assume by contradiction that there exists a Borel set E with $\mu(E) > 0$ and a Borel vector field $w : E \to \mathbb{R}^d$ with $w(x) \in \mathrm{span}(\tau(x))$ such that $w(x) \notin V(\mu,x)$ for every $x \in E$.

By [1, Proposition 5.9 (v)] there exist $\alpha_1,\dots,\alpha_{k-1}$ and a set $E' \subset E$ with $\mu(E') > 0$ such that

$$v(x) := \tau(x) \llcorner dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_{k-1}} \notin V(\mu,x),$$

for every $x \in E'$. Now consider the 1-dimensional metric current $S := T \llcorner dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_{k-1}}$ (see [5, Definition 2.5]). Since $\tilde{S} = \tilde{T} \llcorner dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_{k-1}}$, then we can write $\tilde{S} = (\tau \llcorner dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_{k-1}})\mu$. Choose any α_k different from $\alpha_1,\dots,\alpha_{k-1}$ and let $u_n \to x_{\alpha_k}$ be as in Theorem 1.1 (ii) for the vector field v. For every $f \in \mathrm{Lip}_b(\mathbb{R}^d)$ we have

$$\begin{aligned} S(f,x_{\alpha_k}) &= T(f,x_{\alpha_1},\dots,x_{\alpha_k}) = \langle \tilde{T}, f dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_k}\rangle \\ &= \int_{\mathbb{R}^d} \langle f dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_k}, \tau\rangle d\mu = \int_{\mathbb{R}^d} f\langle dx_{\alpha_k}, v(x)\rangle d\mu \end{aligned} \tag{5.3}$$

and similarly

$$S(f,u_n) = T(f,x_{\alpha_1},\dots,x_{\alpha_{k-1}},u_n) = \int_{\mathbb{R}^d} f\langle du_n, v(x)\rangle d\mu. \tag{5.4}$$

By the density of bounded Lipschitz functions in $L^1(\mu)$, the validity of (5.3) and (5.4) may be extended to $f \in L^1(\mu)$. The continuity of metric currents, see Definition 5.3 (i), implies that $\lim_{n\to\infty} S(f,u_n) = S(f,x_\alpha)$. However, Theorem 1.1(ii) implies that there exists $f \in L^1(\mu)$ such that the limit of the RHS of (5.4) is different from the the RHS of (5.3), which is a contradiction.

The validity of

$$T(f,\pi_1,\dots,\pi_k) = \langle \tilde{T}, f \wedge d\pi_1 \wedge \ldots \wedge d\pi_k)$$

for smooth $f,\pi_1,\dots,\pi_k$ is shown in [13, Theorem 5.5]. Consequently, (5.2) holds in this case, with d_V replaced by the full derivative. Its extension to $(f,\pi_1,\dots,\pi_k) \in \mathscr{D}^k(\mathbb{R}^d)$ follows from [1, Corollary 8.3]. □

5.5. Proposition. *Let* $T = \tau\mu$ *be a classical* k*-dimensional current on* $\mathbb{R}^d$ *with compact support with span*$(\tau(x)) \subset V(\mu,x)$ *for* μ*-a.e.* $x \in \mathbb{R}^d$*. Then the multilinear functional* $\hat{\hat{T}}$ *defined, for every* $(f,\pi_1,\dots,\pi_k) \in \mathscr{D}^k(\mathbb{R}^d)$*, by*

$$\hat{\hat{T}}(f,\pi_1,\dots,\pi_k) = \int_{\mathbb{R}^d} f\langle d_V\pi_1 \wedge \cdots \wedge d_V\pi_k, \tau\rangle d\mu \tag{5.5}$$

is separately continuous in each variable. If additionally $\tau(x) \in V_k(\mu, x)$ for μ-a.e. $x \in \mathbb{R}^d$, then T is a classical flat chain and therefore $\hat{\hat{T}}$ is a metric current and coincides with $\hat{T}$.

Proof. The continuity of $\hat{\hat{T}}$ in each variable is an immediate consequence of Theorem 1.1 (i). If $\tau(x) \in V_k(\mu, x)$, then [2, Theorem 1.2] implies that T is a flat chain, and hence by [2, Theorem 1.2] there exists a (classical) k-dimensional normal current N and a Borel set E such that $T = N \llcorner E$. By Proposition 5.4 the metric current $\hat{N}$ can be written as in (5.2) for a k-vector field σ in place of τ and a positive measure ν in place of μ, so that, since $\tilde{\hat{N}} = N$ then $\sigma\nu = \tau\mu$ as vector measures. Since $\hat{T} = \hat{N} \llcorner E$, then it follows that $\hat{T} = \hat{\hat{T}}$. $\square$

5.6. Remark. The Ambrosio-Kirchheim flat chain conjecture can be equivalently reformulated as follows. If T is a metric current such that $\tilde{T} = \tau\mu$ then $\tau \in V_k(\mu, \cdot)$.

The following corollary is a known fact, see [15, Theorem 1.6] and [9, Theorem 1.15]. We simply remark that it follows from the previous results.

5.7. Corollary. *Let T be either a 1-dimensional or a d-dimensional metric current on $\mathbb{R}^d$. Then $\tilde{T}$ is a flat chain.*

Proof. For $k = 1$, the result follows from Proposition 5.4 and Proposition 5.5. For $k = d$, the result follows additionally from Corollary 2.7. $\square$

References

[1] Giovanni Alberti and Andrea Marchese, *On the differentiability of Lipschitz functions with respect to measures in the Euclidean space*, Geom. Funct. Anal. **26** (2016), no. 1, 1–66.

[2] Giovanni Alberti and Andrea Marchese, *On the structure of flat chains with finite mass*, 2022.

[3] Sergio Albeverio and Michael Röckner, *Classical dirichlet forms on topological vector spaces — closability and a cameron-martin formula,*, J. Funct. Anal. **88** (1990), no. 2, 395–436.

[4] Luigi Ambrosio and Gianni Dal Maso, *A general chain rule for distributional derivatives*, Proc. Amer. Math. Soc. **108** (1990), no. 3, 691–702.

[5] Luigi Ambrosio and Bernd Kirchheim, *Currents in metric spaces*, Acta Math. **185** (2000), no. 1, 1–80.

[6] Vladimir I. Bogachev, *Differentiable measures and the malliavin calculus*, Mathematical surveys and monographs, American Mathematical Soc., 2010.

[7] Guy Bouchitte, Giuseppe Buttazzo, and Pierre Seppecher, *Energies with respect to a measure and applications to low dimensional structures*, Calc. Var. Partial Differential Equations **5** (1997), 37–54.

[8] Camillo Brena and Nicola Gigli, *About the general chain rule for functions of bounded variations*, To appear.

[9] Guido De Philippis and Filip Rindler, *On the structure of $\mathcal{A}$-free measures and applications*, Ann. of Math. (2) **184** (2016), no. 3, 1017–1039.

[10] Simone Di Marino, Danka Lučić, and Enrico Pasqualetto, *A short proof of the infinitesimal hilbertianity of the weighted euclidean space*, Comptes Rendus. Mathématique **358** (2020), no. 7, 817–825.

[11] Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[12] Masatoshi Fukushima, Yoichi Oshima, and Masayoshi Takeda, *Dirichlet forms and symmetric markov processes*, Berlin, New York: De Gruyter, 2010.

[13] Urs Lang, *Local currents in metric spaces*, J. Geom. Anal. **21** (2011), 683–742.

[14] Emanuele Paolini and Eugene Stepanov, *Structure of metric cycles and normal one-dimensional currents*, J. Funct. Anal. **264** (2013), no. 6, 1269–1295.

[15] Andrea Schioppa, *Metric currents and Alberti representations*, J. Funct. Anal. **271** (2016), no. 11, 3007–3081.

[16] Stanislav K. Smirnov, *Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows*, Algebra i Analiz **5** (1993), no. 4, 206–238.

G.A.
Dipartimento di Matematica, Università di Pisa, largo Pontecorvo 5, 56127 Pisa, Italy
e-mail: `giovanni.alberti@unipi.it`

D.B.
Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK
e-mail: `david.bate@warwick.ac.uk`

A.M.
Dipartimento di Matematica, Università di Trento, via Sommarive 14, 38123 Povo, Italy
e-mail: `andrea.marchese@unitn.it`