# Relaxed area of 0-homogeneous maps in the strict $BV$-convergence

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**Abstract**

We compute the relaxed Cartesian area for a general 0-homogeneous map of bounded variation, with respect to the strict $BV$-convergence. In particular, we show that the relaxed area is finite for this class of maps and we provide an integral representation formula.

**Key words:** Area functional, relaxation, strict convergence, total variation of the Jacobian, Plateau problem, tangential variation.

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## 1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and $v = (v_1, v_2) : \Omega \to \mathbb{R}^2$ be a map of class $C^1(\Omega; \mathbb{R}^2)$. The graph of $v$ is a Cartesian 2-manifold in $\Omega \times \mathbb{R}^2 \subset \mathbb{R}^4$ and its 2-dimensional Hausdorff measure $\mathcal{H}^2$ and is given by[1]

$$\mathcal{A}(v; \Omega) := \int_\Omega \sqrt{1 + |\nabla v_1|^2 + |\nabla v_2|^2 + (Jv)^2}dx, \tag{1.1}$$

where

$$Jv := \det \nabla v = \frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} - \frac{\partial v_1}{\partial x_2}\frac{\partial v_2}{\partial x_1} \tag{1.2}$$

is the Jacobian determinant of $v$. As opposite to the case when the map is scalar-valued, the area functional $\mathcal{A}(\cdot\,; \Omega)$ is not convex, but only polyconvex in $\nabla v$, and its growth is not linear, due to the presence of $\det\nabla v$.

It is interesting to extend the notion of area of a graph for singular maps. Following a well established tradition starting from [22] and generalized in [28] (see also [1, 14]), a typical way to proceed is by relaxation: in order to gain coercivity properties in some variational problems involving the area functional, a reasonable choice is to relax with respect to the $L^1$-convergence. In this way, we are allowing to define for every $u \in L^1(\Omega; \mathbb{R}^2)$

$$\overline{\mathcal{A}}_{L^1}(u; \Omega) := \inf \left\{ \liminf_{k\to+\infty} \mathcal{A}(v_k; \Omega) : (v_k) \subset C^1(\Omega; \mathbb{R}^2),\ v_k \xrightarrow{L^1} u \right\}. \tag{1.3}$$

It is not difficult to prove that if $\overline{\mathcal{A}}_{L^1}(u; \Omega) < +\infty$ then $u \in BV(\Omega; \mathbb{R}^2)$, i.e. the domain of $\overline{\mathcal{A}}_{L^1}(\cdot\,; \Omega)$ is contained in $BV(\Omega; \mathbb{R}^2)$. In truth, this inclusion holds strict: an example is provided by the map

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[1] Clearly, (1.1) is finite if $v \in W^{1,1}(\Omega; \mathbb{R}^2)$ and $Jv \in L^1(\Omega)$.

$u(x) = \frac{x}{|x|^{3/2}}$ in[2] $\Omega = B_1((1,0))$. To our best knowledge, a characterization of the domain of the $L^1$-relaxed area is still missing in the literature, as opposite to the case of scalar valued maps, where the domain is $BV(\Omega)$ and (1.3) can be represented as an integral [13,21]. Moreover, the analysis of (1.3) turns out to be very challenging, due to its non-local behaviour. Indeed, as conjectured in [14] and proved in [1], the set function $\overline{\mathcal{A}}_{L^1}(u;\cdot)$ is, in general, not subadditive: In two fundamental examples, the authors provide the existence of a map $u \in BV_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ and of three open sets $\Omega_1, \Omega_2, \Omega_3 \subset \mathbb{R}^2$ such that $\Omega_3 \subset \Omega_1 \cup \Omega_2$ and $\overline{\mathcal{A}}_{L^1}(u;\Omega_3) > \overline{\mathcal{A}}_{L^1}(u;\Omega_1) + \overline{\mathcal{A}}_{L^1}(u;\Omega_2)$. In the first example, denoting by $B_\ell$ the disk of $\mathbb{R}^2$ centered at 0 and of radius $\ell$, they consider the symmetric triple point map $u_T : B_\ell \to \{\alpha, \beta, \gamma\} \subset \mathbb{R}^2$, which sends three identical circular sectors of $B_\ell$ to the vertices $\alpha, \beta, \gamma$ of an equilateral triangle. In the second one, they show that the non-subaddivity arises even among Sobolev maps, like the vortex function $u_V : B_\ell \to \mathbb{S}^1 \subset \mathbb{R}^2$ defined by $u(x) = \frac{x}{|x|}$, $x \neq 0$. For an explicit computation of the value of $\overline{\mathcal{A}}_{L^1}(u_T;\Omega)$ and $\overline{\mathcal{A}}_{L^1}(u_V;\Omega)$ we refer to [8,27] and [6] (see also [7]), respectively. Moreover, for the analysis of the triple point map without symmetry assumptions, we refer to [5].
Although the $L^1$-topology induces some useful properties in Calculus of Variations, the previous examples show that we cannot avoid non-locality issues. An alternative approach is to choose a different topology in the relaxation, stronger than the $L^1$-topology, to put on the space $BV(\Omega;\mathbb{R}^2)$ in order to possibly get rid of non-local phenomena. Following [3,4,25], we choose the strict $BV$-convergence. We recall that for $u_k, u \in BV(\Omega;\mathbb{R}^2)$, we say that $u_k \to u$ strictly $BV(\Omega;\mathbb{R}^2)$ if $u_k \to u$ in $L^1(\Omega;\mathbb{R}^2)$ and $|Du_k|(\Omega) \to |Du|(\Omega)$, where $|\mu|(\Omega)$ stands for the total variation of a Radon measure $\mu$ on $\Omega$. So we are led to investigate for every $u \in BV(\Omega;\mathbb{R}^2)$

$$\overline{\mathcal{A}}_{BV}(u;\Omega) := \inf\left\{\liminf_{k\to+\infty} \mathcal{A}(v_k;\Omega) : (v_k) \subset C^1(\Omega;\mathbb{R}^2),\ v_k \to u \text{ strictly } BV(\Omega;\mathbb{R}^2)\right\}. \tag{1.4}$$

Another important functional, highly related to the area, is the total variation of the Jacobian determinant, which is classically defined for every $v \in C^1(\Omega;\mathbb{R}^2)$ by $TVJ(v;\Omega) := \int_\Omega |Jv|dx$ and extended to every $u \in BV(\Omega;\mathbb{R}^2)$ by relaxation

$$\overline{TVJ}_{BV}(u;\Omega) := \inf\left\{\liminf_{k\to+\infty} TVJ(v_k;\Omega) : (v_k) \subset C^1(\Omega;\mathbb{R}^2),\ v_k \to u \text{ strictly } BV(\Omega;\mathbb{R}^2)\right\}. \tag{1.5}$$

We refer to [9,10,15,16,24,26] for weak notion of area, total variation of the Jacobian determinant and related energies via relaxation with other different topologies. Moreover, we address to [17–19] for an approach to the study of graph of singular maps via Cartesian Currents.

In the present paper, we generalize at once the results in [3] about vortex-type maps and in [4] about piecewise constant 0-homogeneous maps, by considering general 0-homogeneous maps in $BV(B_\ell;\mathbb{R}^2)$.

**Definition 1.1.** *A map $u \in BV(B_\ell;\mathbb{R}^2)$ is 0-homogeneous if it is of the form*

$$u(x) = \gamma\left(\frac{x}{|x|}\right) \qquad a.e.\ x \in B_\ell \tag{1.6}$$

*for some $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$. In this case, we say that $u$ is the 0-homogeneous (or simply homogeneous) extension of $\gamma$ on $B_\ell$.*

[2]Notice that $u \in C^1(\Omega;\mathbb{R}^2) \cap W^{1,1}(\Omega;\mathbb{R}^2) \subset BV(\Omega;\mathbb{R}^2)$. Neverthless $Ju \notin L^1(\Omega;\mathbb{R}^2)$, giving $\overline{\mathcal{A}}_{L^1}(u;\Omega) = \mathcal{A}(u;\Omega) = +\infty$.

In order to ensure the consistency of Definition 1.1, we shall prove in Proposition 2.6 that the homogeneous extension of a map $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ belongs to $BV(B_\ell;\mathbb{R}^2)$. Notice that, according to Definition 1.1, the maps $u_V$ and $u_T$ are 0-homogeneous.
The fundamental idea in our analysis is to define a notion of area enclosed by the image of $\gamma$, in such a way it is compatible with the strict convergence. Precisely, we consider (compare [4]) the relaxation

$$\overline{P}(\gamma) := \inf\left\{\liminf_{n\to+\infty} P(\varphi_n) : \varphi_n \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2), \varphi_n \to \gamma \text{ strictly } BV(\mathbb{S}^1;\mathbb{R}^2)\right\} \tag{1.7}$$

of the (singular) Plateau problem

$$P(\varphi) = \inf\left\{\int_{B_1} |Jv|\, dx : \, v \in \mathrm{Lip}(B_1;\mathbb{R}^2), v \llcorner \partial B_1 = \varphi\right\} \tag{1.8}$$

associated to any $\varphi \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$. Our main result is the following:

**Theorem 1.2.** *Let $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ and $u$ as in Definition 1.1. Then*

$$\overline{\mathcal{A}}_{BV}(u;B_\ell) = \int_{B_\ell} \sqrt{1+|\nabla u|^2}dx + |D^s u|(B_\ell) + \overline{P}(\gamma), \tag{1.9}$$

*where $D^s u$ is the singular part of the measure $Du$.*

A crucial ingredient in the proof of Theorem 1.2 will be the computation of $\overline{TVJ}_{BV}(u;B_\ell)$. Indeed, finding the value of (1.4) corresponds to choose the most convenient way in terms of surface area to "fill the holes" in the graph of $u$, according to the approximation in the strict convergence. The same interpretation can be made for the functional (1.5), with the difference that it concerns the way of filling holes in the *image* of $u$. By adopting this point of view, due to the structure of the graph of a homogeneous map, it turns out that $\overline{TVJ}_{BV}(u;B_\ell)$ is the correct quantity to consider to fill the hole in the graph of $u$ upon the origin. In other words, in the case of homogeneous maps, the functional $\overline{TVJ}_{BV}$ represents a sort of (completely) vertical part of $\overline{\mathcal{A}}_{BV}$. In Theorem 3.5 we prove that $\overline{TVJ}_{BV}(u;B_\ell)$ can be expressed in terms of the relaxed Plateau problem (1.7). In turn, in Lemma 2.13 we shall see that $\overline{P}(\gamma)$ can be characterized as the area enclosed by the "completed map" $\widetilde{\gamma}$ which "fill the jumps" of $\gamma$ by means of segments, in other words $\overline{P}(\gamma) = P(\widetilde{\gamma})$. A precise construction of $\widetilde{\gamma}$ will be given in Lemma 2.10.
We point out that $\overline{\mathcal{A}}_{BV}(u;\cdot)$ is a measure for $u$ as in (1.6). However, to the best of our knowledge, it is not known if $\overline{\mathcal{A}}_{BV}(u;\cdot)$ is subadditive for a generic map $u \in BV(B_\ell;\mathbb{R}^2)$. Moreover, a complete characterization of the set $\mathrm{Dom}(\overline{\mathcal{A}}_{BV}(\cdot\,;B_\ell)) := \{u \in BV(B_\ell;\mathbb{R}^2) : \overline{\mathcal{A}}_{BV}(u;B_\ell) < +\infty\}$ is not yet available: from [4], we only know that $\mathrm{Dom}(\overline{\mathcal{A}}_{BV}(\cdot\,;B_\ell)) \subsetneq \mathrm{Dom}(\overline{\mathcal{A}}_{L^1}(\cdot\,;B_\ell)) \subsetneq BV(B_\ell;\mathbb{R}^2)$.

## 2 Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set. For any $u \in BV(\Omega;\mathbb{R}^2)$, we recall that the distributional derivative $Du$ is a finite Radon measure valued in $\mathbb{R}^{2\times 2}$. Denoting by $\mathscr{L}^2$ the Lebesgue measure of $\mathbb{R}^2$, by the Lebesgue decomposition theorem we have $Du = \nabla u \mathscr{L}^2 + D^s u$, where $\nabla u \in L^1(\Omega;\mathbb{R}^{2\times 2})$ and $D^s u \perp \mathscr{L}^2$. The symbol $|Du|(\Omega)$ stands for the total variation of $Du$ (see [2, Definition 3.4, pag. 119]) with $|\cdot|$ the Frobenius norm.

**Definition 2.1** (**Strict convergence**). *Let $u \in BV(\Omega;\mathbb{R}^2)$ and $(u_k) \subset BV(\Omega;\mathbb{R}^2)$. We say that $(u_k)$ converges to $u$ strictly $BV$, if*

$$u_k \xrightarrow{L^1} u \qquad \text{and} \qquad |Du_k|(\Omega) \to |Du|(\Omega).$$

Now, let $B_\ell$ be the disk of $\mathbb{R}^2$ centered at the origin of radius $\ell > 0$. If $u \in BV(B_\ell;\mathbb{R}^2)$, by Lebesgue differentiation theorem and Fubini theorem, for almost every $r < \ell$ the restriction $u\llcorner\partial B_r$ is well defined and independent of the representative of $u$, since it coincides with the trace of $u$ on $\mathcal{H}^1$-almost every point of $\partial B_r$. In particular, for almost every $r < \ell$, we can define the total variation of $u \llcorner \partial B_r$ as

$$|D(u\llcorner\partial B_r)|(\partial B_r) := \sup\left\{\int_0^{2\pi} \bar{u}(r,\theta)\cdot f'(\theta)d\theta; f \in C^1([0,2\pi];\overline{B}_1(0)), f(0) = f(2\pi), f'(0) = f'(2\pi)\right\} \tag{2.1}$$

which turns out to be finite (see Lemma 2.3), giving that $u\llcorner\partial B_r \in BV(\partial B_r;\mathbb{R}^2)$, for almost every $r < \ell$. Here $\bar{u}(r,\theta) := u(r\cos\theta, r\sin\theta)$ for every $r \in (0,\ell], \theta \in [0,2\pi]$.

We want to relate this quantity with the notion of *tangential total variation*.

**Definition 2.2** (**Tangential total variation in an annulus).** *For $x = (x_1,x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, set $\tau(x) = \frac{1}{|x|}(-x_2,x_1)$. Let $0 < \varepsilon < \ell$ and $A_{\varepsilon,\ell} := B_\ell \setminus \overline{B_\varepsilon}$ be an annulus around 0. We define the tangential total variation of $u \in BV(A_{\varepsilon,\ell};\mathbb{R}^2)$ as the total variation of the Radon measure $D_\tau u := Du\tau$, namely*

$$|D_\tau u|(A_{\varepsilon,\ell}) = |Du\tau|(A_{\varepsilon,\ell}) = \sup\Big\{-\int_{A_{\varepsilon,\ell}} u\cdot(\nabla g\tau)\;dx : g \in C^1_c(A_{\varepsilon,\ell};\overline{B}_1(0))\Big\}. \tag{2.2}$$

The first equality in (2.2) is a consequence of the definition of $D_\tau u$, while the second equality is justified as follows: since $\tau \in C^\infty(A_{\varepsilon,\ell};\mathbb{R}^2)$ satisfies $\mathrm{div}\tau = 0$ everywhere, for any $g = (g^1,g^2) \in C^1_c(A_{\varepsilon,\ell};\mathbb{R}^2)$ we have

$$\begin{aligned}-\int_{A_{\varepsilon,\ell}} u\cdot(\nabla g\tau)\;dx &= -\int_{A_{\varepsilon,\ell}} u^1\nabla g^1\cdot\tau\;dx - \int_{A_{\varepsilon,\ell}} u^2\nabla g^2\cdot\tau\;dx\\ &= -\int_{A_{\varepsilon,\ell}} u^1\mathrm{div}(g^1\tau)\;dx - \int_{A_{\varepsilon,\ell}} u^2\mathrm{div}(g^2\tau)\;dx\\ &= \int_{A_{\varepsilon,\ell}} g^1\tau\cdot dDu^1 + \int_{A_{\varepsilon,\ell}} g^2\tau\cdot dDu^2 = \int_{A_{\varepsilon,\ell}} g\cdot(dDu)\tau = \langle Du\tau, g\rangle.\end{aligned}$$

This computation shows that

$$|D_\tau u|(A_{\varepsilon,\ell}) \le |Du|(A_{\varepsilon,\ell}), \tag{2.3}$$

since $|\tau| \le 1$, and also that (2.2) is compatible with the case $u \in W^{1,1}(A_{\varepsilon,\ell};\mathbb{R}^2)$, where simply $|D_\tau u|(A_{\varepsilon,\ell}) = \int_{A_{\varepsilon,\ell}} |\nabla u\tau|\;dx$.
In [4], the following slicing result for the strict convergence is proven.

**Lemma 2.3** (**Inheritance of strict convergence to circumferences).** *Let $u \in BV(B_\ell;\mathbb{R}^2)$. Suppose that $(v_k) \subset C^1(B_\ell;\mathbb{R}^2)$ is a sequence converging to $u$ strictly $BV(B_\ell;\mathbb{R}^2)$. Then, for almost every $r \in (0,l)$, there exists a subsequence $(v_{k_h}) \subset (v_k)$, depending on $r$, such that*

$$v_{k_h}\llcorner\partial B_r \to u\llcorner\partial B_r \quad \textit{strictly } BV(\partial B_r;\mathbb{R}^2). \tag{2.4}$$

In the proof of Lemma 2.3 a useful Coarea-type formula is provided:

**Lemma 2.4.** *Let $u \in BV(B_\ell;\mathbb{R}^2)$. Then*

$$|D_\tau u|(A_{\varepsilon,\ell}) = \int_\varepsilon^\ell |D(u\llcorner\partial B_r)|(\partial B_r)\;dr. \tag{2.5}$$

This formula allows us to define a notion of tangential total variation for $u \in BV(B_\ell;\mathbb{R}^2)$ on the whole $B_\ell$, since the right hand side of (2.5) is monotone non-increasing and equibounded w.r.t. $\varepsilon$.

**Definition 2.5 (Tangential total variation in $B_\ell$).** *Let $\tau$ and $A_{\varepsilon,\ell}$ as in Definition 2.2. We define the tangential total variation of $u \in BV(B_\ell;\mathbb{R}^2)$ as*

$$|D_\tau u|(B_\ell) := \lim_{\varepsilon\to 0^+} |D_\tau u|(A_{\varepsilon,\ell}) = \int_0^\ell |D(u \llcorner \partial B_r)|(\partial B_r)\, dr. \tag{2.6}$$

**Proposition 2.6.** *Let $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ and $u$ be defined as in (1.6). Then $u \in BV(B_\ell;\mathbb{R}^2)$ and*

$$|Du|(B_\ell) = \ell|\dot\gamma|(\mathbb{S}^1). \tag{2.7}$$

*Moreover,*

$$\int_{B_\ell} |\nabla u| dx = \ell \int_{\mathbb{S}^1} |\dot\gamma^a| d\mathcal{H}^1, \qquad |D^s u|(B_\ell) = \ell|\dot\gamma^s|(\mathbb{S}^1). \tag{2.8}$$

*Proof.* Since $\bar u$ does not depend on $\rho$, by (2.1), we have $|D(u\llcorner\partial B_r)|(\partial B_r) = |\dot\gamma|(\mathbb{S}^1)$. So, thanks to (2.5), in order to prove (2.7) it is enough to show that the variation of $u$ is purely tangential, namely $|Du|(B_\ell) = |D_\tau u|(B_\ell)$. To this purpose we argue by approximation: Let $(\varphi_k) \subset C^\infty(\mathbb{S}^1;\mathbb{R}^2)$ be such that $\varphi_k \to \gamma$ strictly $BV(\mathbb{S}^1;\mathbb{R}^2)$ (e.g. a mollifying sequence) and set

$$u_k(x) := \varphi_k\left(\frac{x}{|x|}\right) \qquad \forall x \in B_\ell \setminus \{(0,0)\}.$$

Then $u_k \in W^{1,1}(B_\ell;\mathbb{R}^2)$. Indeed, for every $k \in \mathbb{N}$, in polar coordinates

$$\nabla u_k(\rho\cos\theta, \rho\sin\theta) = \frac{\dot{\bar\varphi}_k(\theta)}{\rho} \qquad \forall \rho \in (0,\ell]\, \forall\theta \in [0,2\pi], \tag{2.9}$$

so that

$$\int_{B_\ell} |\nabla u_k| dx = \int_0^\ell \int_0^{2\pi} \rho \frac{|\dot{\bar\varphi}_k(\theta)|}{\rho} d\theta d\rho = \ell \int_{\mathbb{S}^1} |\dot\varphi_k| d\mathcal{H}^1. \tag{2.10}$$

Moreover, $u_k$ converges to $u$ in $L^1(B_\ell;\mathbb{R}^2)$ since

$$\|u_k - u\|_{L^1(B_\ell;\mathbb{R}^2)} \le \|\varphi_k - \gamma\|_{L^1(\mathbb{S}^1;\mathbb{R}^2)} \to 0 \qquad \text{as } k \to +\infty.$$

By the lower semicontinuity property of the total variation and (2.10), we obtain

$$|Du|(B_\ell) \le \liminf_{k\to+\infty} \int_{B_\ell} |\nabla u_k| dx = \ell \lim_{k\to+\infty} \int_{\mathbb{S}^1} |\dot\varphi_k| d\mathcal{H}^1 = \ell|\dot\gamma|(\mathbb{S}^1),$$

where we used the strict convergence assumption on the sequence $(\varphi_k)$. The previous inequality ensures that $u \in BV(B_\ell;\mathbb{R}^2)$. On the other hand, by (2.3) and definition 2.5 we have

$$|Du|(B_\ell) \ge |D_\tau u|(B_\ell) = \int_0^\ell |D(u \llcorner \partial B_r)|(\partial B_r)\, dr = \ell|\dot\gamma|(\mathbb{S}^1),$$

so that $|Du|(B_\ell) = |D_\tau u|(B_\ell) = \ell|\dot\gamma|(\mathbb{S}^1)$. Finally, as in (2.9) we can write

$$\nabla u(\rho\cos\theta, \rho\sin\theta) = \frac{\dot{\bar\gamma}^a(\theta)}{\rho} \qquad \text{a.e. } \rho \in (0,\ell],\ \theta \in [0,2\pi], \tag{2.11}$$

so that

$$\int_{B_\ell} |\nabla u| dx = \ell \int_{\mathbb{S}^1} |\dot\gamma^a| d\mathcal{H}^1$$

and

$$|D^s u|(B_\ell) = |Du|(B_\ell) - \int_{B_\ell} |\nabla u| dx = \ell |\dot\gamma|(\mathbb{S}^1) - \ell \int_{\mathbb{S}^1} |\dot\gamma^a| dy = \ell |\dot\gamma^s|(\mathbb{S}^1).$$

□

## 2.1 Further properties in dimension 1

In [3, Proposition 2.4] the following is proved:

**Lemma 2.7.** *Let $(\gamma_k) \subset W^{1,1}([a,b];\mathbb{R}^2)$ be a sequence converging strictly $BV([a,b];\mathbb{R}^2)$ to $\gamma \in W^{1,1}([a,b];\mathbb{R}^2)$. Then $\gamma_k \to \gamma$ uniformly in $[a,b]$.*

The same result holds also in case $\gamma \in BV([a,b];\mathbb{R}^2)$, but only on those compact subsets of $[a,b]$ which do not intersect the jump set $J_\gamma$.

**Lemma 2.8.** *Let $(\gamma_k) \subset W^{1,1}([a,b];\mathbb{R}^2)$ be a sequence converging strictly $BV([a,b];\mathbb{R}^2)$ to $\gamma \in BV([a,b];\mathbb{R}^2)$. Then, for every compact subset $K \subset [a,b] \setminus J_\gamma$, we have that*

$$\gamma_k \to \gamma \quad \textit{uniformly in } K \quad \textit{as } k \to +\infty. \tag{2.12}$$

*Proof.* By contradiction, up to a not relabeled subsequence, we may suppose

$$\exists \delta > 0 \quad \exists (\tau_k) \subset K \quad \exists k_0 \in \mathbb{N}: \quad |\gamma_k(\tau_k) - \gamma(\tau_k)| > \delta \quad \forall k \geq k_0,$$

and there exists $\bar\tau \in K$ such that $\tau_k \to \bar\tau$ as $k \to +\infty$, since $K$ is compact. Now, consider an open interval $E \subset [a,b]$ such that[3] $\bar\tau \in E$, $\partial E \subset [a,b] \setminus J_\gamma$, and $|\dot\gamma|(E) < \frac{\delta}{4}$. Such an interval $E$ exists because $|\dot\gamma|(\{\bar\tau\}) = 0$. By hypothesis on strict convergence, since $|\dot\gamma|(\partial E) = 0$, we have

$$\lim_{k\to+\infty} \int_E |\dot\gamma_k| dt = |\dot\gamma|(E).$$

So, we can find an index $k_1 \in \mathbb{N}$ such that $k_1 \geq k_0$ and $\int_E |\dot\gamma_k| dt < \frac{\delta}{2}$, for every $k \geq k_1$. Moreover, there exists $k_2 \in \mathbb{N}$, $k_2 \geq k_1$, such that $\tau_k \in E$ for every $k \geq k_2$. Now fix $F \subset E$ such that $|F| = |E|$ and $\gamma \llcorner F$ can be identified with its natural continuous representative. Pick a point $z \in F$, then

$$\begin{aligned}
|\gamma_k(z) - \gamma(z)| &\geq -|\gamma_k(z) - \gamma_k(\tau_k)| + |\gamma_k(\tau_k) - \gamma(\tau_k)| - |\gamma(\tau_k) - \gamma(z)| \\
&\geq -\left| \int_{\tau_k}^z |\dot\gamma_k| dt \right| + \delta - |\dot\gamma|(E) \geq -\int_E |\dot\gamma_k| dt + \delta - \frac{\delta}{4} \\
&\geq -\frac{\delta}{2} + \frac{3}{4}\delta = \frac{\delta}{4}.
\end{aligned}$$

Therefore, $(\gamma_k)$ does not converge to $\gamma$ pointwise at any point of $F$, which leads to a contradiction with the fact that $\gamma_k \to \gamma$ in $L^1([a,b])$. So, (2.12) is proved. □

An immediate consequence of Lemma 2.8 is that the uniform convergence takes place on the full interval if $J_\gamma = \varnothing$. Precisely the following holds.

[3] If $\bar\tau = a$ or $\bar\tau = b$, $E$ is a semi-open interval.


Figure 1: The curve $\widetilde{\gamma}$ obtained from $\gamma$ by filling the jumps with segments.

**Corollary 2.9.** *Let $(\gamma_k) \subset W^{1,1}([a,b];\mathbb{R}^2)$ be a sequence converging strictly $BV([a,b];\mathbb{R}^2)$ to $\gamma \in C([a,b];\mathbb{R}^2) \cap BV([a,b];\mathbb{R}^2)$. Then,*

$$\gamma_k \to \gamma \quad \textit{uniformly} \;\; \textit{as } k \to +\infty.$$

This is clearly impossible if $J_\gamma$ is non-empty, but becomes true (up to extracting a subsequence) if we suitably reparametrize $\gamma_k$ and if instead of $\gamma$ we consider its "completed curve" $\widetilde{\gamma}$, obtained by filling the jumps with line segments (see Fig. 1). This is the content of [4, Lemma 2.7, Corollary 2.8], where the authors prove the result for $\gamma \in SBV([a,b];\mathbb{R}^2)$, which is allowed to jump on a finite number of points. Our goal is to provide a further improvement of this result, namely, when $\gamma$ is just a function of bounded variation.
To this purpose, suppose that $\gamma \in BV([a,b];\mathbb{R}^2)$. Then, it is well known that $J_\gamma$ is at most countable. So, let $\{t_i\}_{i\in\mathbb{N}}$ be an enumeration[4] of $J_\gamma$ and $\gamma^\pm(t_i)$ be the traces of $\gamma$ at $t_i$. We want to associate to $\gamma$ a unique continuous curve $\widetilde{\gamma}$ which "completes" the image of $\gamma$ by means of segments connecting $\gamma^-(t_i)$ to $\gamma^+(t_i)$. In particular, we require that $\widetilde{\gamma}$ has the same total variation $L$ of $\gamma$ and is compatible with the approximation via strict $BV$-convergence. Precisely we show the following result.

**Lemma 2.10.** *Suppose that $(\gamma_k) \subset W^{1,1}([a,b];\mathbb{R}^2)$ is a sequence converging strictly $BV([a,b];\mathbb{R}^2)$ to $\gamma \in BV([a,b];\mathbb{R}^2)$. Then there exist:*

*(a) a curve $\widetilde{\gamma} \in \mathrm{Lip}([a,b];\mathbb{R}^2)$,*

*(b) Lipschitz strictly increasing surjective reparametrizations $h_k : [a,b] \to [a,b]$ for any $k \in \mathbb{N}$, with $\sup_k \|\dot{h}_k\|_\infty < +\infty$,*

*such that*

$$\lim_{j\to+\infty} \gamma_k \circ h_k = \widetilde{\gamma} \qquad \textit{uniformly in } [a,b]. \tag{2.13}$$

*Moreover, $\widetilde{\gamma}$ does not depend on the approximating sequence $\gamma_k$, in the sense that if $(\eta_k) \subset W^{1,1}([a,b];\mathbb{R}^2)$ is another sequence converging strictly $BV([a,b];\mathbb{R}^2)$ to $\gamma$, then the corresponding $\widetilde{\eta} \in \mathrm{Lip}([a,b];\mathbb{R}^2)$ coincides with $\widetilde{\gamma}$.*

*Proof.* The lengths $L_k$ of $\gamma_k$ and $L$ of $\gamma$ are given by

$$L_k = \int_a^b |\dot{\gamma}_k|\, d\tau, \qquad L = |\dot{\gamma}|([a,b]).$$

[4]If the number of jumps is finite, then $\{t_i\}$ is definitively constant.

Since, by assumption, $\gamma_k \to \gamma$ strictly $BV([a,b];\mathbb{R}^2)$, we have that $L_k \to L$ as $k \to +\infty$. For every $k \in \mathbb{N}$, define

$$s_k : [a,b] \to [0,L], \qquad s_k(t) := \frac{L}{L_k + b - a} \int_a^t \Big( |\dot{\gamma}_k(\tau)| + 1 \Big)\, d\tau, \tag{2.14}$$

with Lipschitz inverse $\alpha_k := s_k^{-1} : [0,L] \to [a,b]$. Notice that

$$\dot{\alpha}_k(s) = \frac{1}{\dot{s}_k(\alpha_k(s))} = \frac{L_k + b - a}{L} \cdot \frac{1}{|\dot{\gamma}_k(\alpha_k(s))| + 1} \le \frac{L_k + b - a}{L} \le C \qquad \text{for a.e. } s \in [0,L], \tag{2.15}$$

for some constant $C > 0$ independent of $k$. Define

$$\bar{\gamma}_k : [0,L] \to \mathbb{R}^2, \qquad \bar{\gamma}_k(s) := \gamma_k(\alpha_k(s)) \qquad \forall s \in [0,L].$$

Since

$$\left| \frac{d\bar{\gamma}_k}{ds}(s) \right| \le \frac{|\dot{\gamma}_k(\alpha_k(s))|}{|\dot{s}_k(\alpha_k(s))|} \le \frac{L_k + b - a}{L} \le C \qquad \text{for a.e. } s \in [0,L],$$

the sequence $(\bar{\gamma}_k)$ is bounded in $W^{1,\infty}([0,L];\mathbb{R}^2)$. Thus, there exists a subsequence $(k_j) \subset (k)$ and $\bar{\gamma} \in W^{1,\infty}([0,L];\mathbb{R}^2)$ such that

$$\bar{\gamma}_{k_j} \rightharpoonup \bar{\gamma} \text{ weakly* in } W^{1,\infty}([0,L];\mathbb{R}^2) \text{ and uniformly in } [0,L]. \tag{2.16}$$

Then, we define $\widetilde{\gamma}$ and $h_k$ as the compositions of $\bar{\gamma}$ and $\alpha_k$, respectively, with an affine increasing diffeomorphism $\psi : [a,b] \to [0,L]$. In particular, by (2.15) we have

$$\sup_{k \in \mathbb{N}} \|\dot{h}_k\|_\infty < +\infty.$$

Now (2.16) reads as

$$\lim_{j \to +\infty} \gamma_{k_j} \circ h_{k_j} = \widetilde{\gamma} \text{ uniformly in } [a,b]. \tag{2.17}$$

Let us show the indipendence of $\bar{\gamma}$ (and consequently of $\widetilde{\gamma}$) from the sequence $\gamma_k$. Suppose that $\eta_k \in W^{1,1}([a,b];\mathbb{R}^2)$ converges to $\gamma$ strictly $BV([a,b];\mathbb{R}^2)$. Let $\sigma_k : [a,b] \to [0,L]$ be defined as $s_k$ with $\eta_k$ in place of $\gamma_k$ and $\beta_k := \sigma_k^{-1} : [0,L] \to [a,b]$ its (equi-)Lipschitz inverse. As before, we obtain that there exists $(k_h) \subset (k)$ and $\bar{\eta}$ such that

$$\bar{\eta}_{k_h} \rightharpoonup \bar{\eta} \text{ weakly* in } W^{1,\infty}([0,L];\mathbb{R}^2) \text{ and uniformly in } [0,L].$$

Observe that for any open interval $J \subseteq [0,L]$,

$$\int_J |\dot{\bar{\gamma}}| ds \le \liminf_{k \to +\infty} \int_J |\dot{\bar{\gamma}}_k| ds \le |J| \liminf_{k \to +\infty} \frac{L_k + b - a}{L} = \frac{L + b - a}{L} |J|,$$

and thus

$$|\dot{\bar{\gamma}}| \le 1 + \frac{b-a}{L} \text{ a.e. in } [0,L]. \tag{2.18}$$

Now, recalling that $J_\gamma = \{t_i\}_{i \in \mathbb{N}}$, fix $i \in \mathbb{N}$ and take any sequence $(t_{i,j}^\pm)_j \subset [a,b] \setminus J_\gamma$ such that $t_{i,j}^- \nearrow t_i$ and $t_{i,j}^+ \searrow t_i$ as $j \to +\infty$. By Lemma 2.8, for every $j \in \mathbb{N}$, $\gamma_k(t_{i,j}^\pm) \to \gamma(t_{i,j}^\pm)$ as $k \to +\infty$. On the other hand, by definition of $\gamma^\pm$, we have $\gamma(t_{i,j}^\pm) \to \gamma^\pm(t_i)$ as $j \to +\infty$. Therefore, by using a diagonal argument and by extracting a further (not relabeled) subsequence of $(k_j)$ if needed, we can assume that

$$\lim_{j \to +\infty} \gamma_{k_j}(t_{i,j}^\pm) = \gamma^\pm(t_i). \tag{2.19}$$

Setting

$$\begin{aligned} r^-_{i,j} :=& s_{k_j}(t^-_{i,j}) = \frac{L}{L_{k_j}+b-a}\int_a^{t^-_{i,j}} \big(|\dot\gamma_{k_j}|+1\big)\, d\tau,\\ r^+_{i,j} :=& s_{k_j}(t^+_{i,j}) = \frac{L}{L_{k_j}+b-a}\int_a^{t^+_{i,j}} \big(|\dot\gamma_{k_j}|+1\big)\, d\tau, \end{aligned} \tag{2.20}$$

we have

$$\begin{aligned} \lim_{j\to+\infty} r^-_{i,j} &= \frac{L}{L+b-a}\left[|\dot\gamma|([a,t_i))+t_i-a\right] =: s^-(t_i),\\ \lim_{j\to+\infty} r^+_{i,j} &= \frac{L}{L+b-a}\left[|\dot\gamma|([a,t_i])+t_i-a\right]\\ &= \frac{L}{L+b-a}\big[|\dot\gamma|([a,t_i))+|\gamma^+(t_i)-\gamma^-(t_i)|+t_i-a\big] =: s^+(t_i). \end{aligned} \tag{2.21}$$

As a consequence of (2.16), (2.19), and (2.21), we get

$$\bar\gamma(s^\pm(t_i)) \leftarrow \bar\gamma_{k_j}(r^\pm_{i,j}) = \gamma_{k_j}(\alpha_{k_j}(r^\pm_{i,j})) = \gamma_{k_j}(t^\pm_{i,j}) \to \gamma^\pm(t_i) \quad \text{as } j\to+\infty.$$

Therefore the curve $\bar\gamma$ maps the segment $[s^-(t_i), s^+(t_i)]$ into a curve joining $\gamma^-(t_i)$ and $\gamma^+(t_i)$. Now, since $s^+(t_i)-s^-(t_i) = \frac{L}{L+b-a}|\gamma^+(t_i)-\gamma^-(t_i)|$, from (2.18) we conclude that $\bar\gamma$ coincides with the $\big(1+\frac{b-a}{L}\big)$-speed parametrization $\ell_i$ of the segment joining $\gamma^-(t_i)$ and $\gamma^+(t_i)$ on $[s^-(t_i), s^+(t_i)]$. Hence we have shown that for every $i\in\mathbb{N}$

$$\gamma_{k_j}\circ\alpha_{k_j} \to \ell_i \text{ uniformly in } [s^-(t_i), s^+(t_i)] \text{ as } j\to+\infty.$$

An analogous conclusion holds also for $\eta_{k_h}$: indeed, let $\sigma_{k_h}(t^\pm_{i,h})$ be as in (2.20) but with $\eta_{k_h}$ in place of $\gamma_{k_j}$, then it is clear that $\sigma_{k_h}(t^\pm_{i,h}) \to s^\pm(t_i)$ as $h\to+\infty$ and so

$$\eta_{k_h}\circ\beta_{k_h} \to \ell_i \text{ uniformly in } [s^-(t_i), s^+(t_i)] \text{ as } h\to+\infty.$$

Therefore, $\bar\eta=\bar\gamma$ on $S=\cup_{i\in\mathbb{N}} S_i$, where $S_i := [s^-(t_i), s^+(t_i)]$. It remains to show that $\bar\eta=\bar\gamma$ on $[0,L]\setminus S$.
By (2.15), up to extract a not relabeled subsequence, we can assume that there exists $\alpha\in W^{1,\infty}([0,L])$ such that

$$\alpha_{k_j}\to\alpha \quad \text{uniformly in } [0,L] \text{ as } j\to+\infty \tag{2.22}$$

and, for the same reason, there exists $\beta\in W^{1,\infty}([0,L])$ such that

$$\beta_{k_h}\to\beta \quad \text{uniformly in } [0,L] \text{ as } h\to+\infty. \tag{2.23}$$

From Lemma 2.8, we deduce that $\bar\gamma=\gamma\circ\alpha$ on every compact subset $H\subset[0,L]\setminus S$. But, since $\alpha$ does not depend on the compact $H$, we deduce that $\bar\gamma=\gamma\circ\alpha$ on $[0,L]\setminus S$. In the same way, we infer that $\bar\eta=\gamma\circ\beta$ on $[0,L]\setminus S$. Let us show that $\alpha=\beta$ on $[0,L]\setminus S$. Indeed, notice that by definition of $s_k$,

$$s_k(t)\to s(t) := \frac{L}{L+b-a}(t-a+|\dot\gamma|([a,t])) \quad \forall t\in[a,b]\setminus J_\gamma.$$

The map $s:[a,b]\to[0,L]$ is strictly increasing with jumps at each point of $J_\gamma$. Notice that the traces of $s$ at every $t_i\in J_\gamma$ are exactly the numbers $s^\pm(t_i)$ in (2.21). We claim that $\alpha=s^{-1}$ on $[0,L]\setminus S$. Indeed, by (2.22) we have that for every $t\in[a,b]\setminus J_\gamma$

$$t=\alpha_{k_j}(s_{k_j}(t))\to\alpha(s(t)) \quad \text{as } j\to+\infty,$$

then $\alpha = s^{-1}$ on $s([a,b]\setminus J_\gamma) = [0,L]\setminus S$. In the same way, using (2.23) one can prove that $\beta = s^{-1}$ on $[0,L]\setminus S$.
Finally, it remains to show that (2.17) holds without passing to a subsequence. To this purpose, by applying (2.17) to any subsequence $(\gamma_{k_h})$ of $(\gamma_k)$, with reparametrizations $(h_{k_h}) \subset (h_k)$, we obtain that for a further subsequence $(k_{h_j}) \subset (k_h)$

$$\lim_{j\to+\infty} \gamma_{k_{h_j}} \circ h_{k_{h_j}} = \widetilde{\gamma} \text{ uniformly in } [a,b].$$

Since $\widetilde{\gamma}$ does not depend on the approximating sequence, we deduce (2.13), which concludes the proof.

**Remark 2.11.** From the previous proof, we deduce that the "completed" curve $\widetilde{\gamma}$ does not depend on the subsequence of the approximating sequence $\gamma_k$. Moreover, we do not need to discuss the dependence on the reparametrization $h_k$, because, for our purpose, we shall consider in the sequel a Plateau-type problem associated to $\gamma_k$ which is independent of the reparametrization of the curve.

□

## 2.2 Planar Plateau-type problem

In [4], the authors consider the following planar Plateau-type problem spanning a closed Lipschitz curve $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$ (see also [26] and [16]):

$$P(\varphi) := \inf\left\{\int_{B_1} |Jv|\,dx : \, v \in \mathrm{Lip}(B_1;\mathbb{R}^2), v \llcorner \partial B_1 = \varphi\right\} \tag{2.24}$$

and the corresponding relaxation problem for a general $BV$-map $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$:

$$\overline{P}(\gamma) := \inf\left\{\liminf_{n\to+\infty} P(\varphi_n) : \varphi_n \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2), \varphi_n \to \gamma \text{ strictly } BV(\mathbb{S}^1;\mathbb{R}^2)\right\}. \tag{2.25}$$

The authors of [4] show that, for $\varphi \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$, $P(\varphi)$ is invariant under rescaling of the integration domain, precisely if $r > 0$ and

$$\varphi_r(y) := \varphi\left(\frac{y}{r}\right) \quad y \in \partial B_r, \tag{2.26}$$

then

$$P(\varphi) = P_r(\varphi_r) := \inf\left\{\int_{B_r} |Jv|\,dx : \, v \in \mathrm{Lip}(B_r;\mathbb{R}^2), v \llcorner \partial B_r = \varphi_r\right\}. \tag{2.27}$$

Of course, we can consider also the rescaled version of (2.25) for $\gamma_r$:

$$\overline{P}_r(\gamma_r) := \inf\left\{\liminf_{n\to+\infty} P(\varphi_n) : \varphi_n \in \mathrm{Lip}(\partial B_r;\mathbb{R}^2), \varphi_n \to \gamma \text{ strictly } BV(\partial B_r;\mathbb{R}^2)\right\}. \tag{2.28}$$

Now we collect some useful properties of $P(\cdot)$ and $\overline{P}(\cdot)$. Without further specifying, all of these properties will be valid for $P_r(\cdot)$ and $\overline{P}_r(\cdot)$ as well.
First, $P(\cdot)$ is also invariant under reparametrization of the boundary datum, namely

$$P(\varphi) = P(\varphi \circ h) \qquad \forall h : \mathbb{S}^1 \to \mathbb{S}^1 \text{ Lipschitz homeomorphism.} \tag{2.29}$$

Moreover, the following continuity result for $P(\cdot)$ holds.

**Lemma 2.12** (**Continuity of** $P$)**.** *Let* $\varphi \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ *and suppose that* $(\varphi_k)_k \subset \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ *is such that*

$$\varphi_k \to \varphi \quad \textit{uniformly} \quad \textit{and} \quad \sup_k |\dot{\varphi}_k|(\mathbb{S}^1) < +\infty.$$

*Then* $P(\varphi_k) \to P(\varphi)$*.*

In [4, Lemma 2.14], the authors show that if $\gamma \in SBV(\mathbb{S}^1;\mathbb{R}^2)$ has a finite number of jump points, then $\overline{P}(\gamma) = P(\widetilde{\gamma})$, where $\widetilde{\gamma}$ is the Lipschitz curve[5] of Lemma 2.10 associated to $\gamma$. We want to extend this result to the case $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$.

**Lemma 2.13.** *Let* $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ *and* $\widetilde{\gamma} : \mathbb{S}^1 \to \mathbb{R}^2$ *be the corresponding Lipschitz curve of Lemma 2.10. Then*

$$\overline{P}(\gamma) = P(\widetilde{\gamma}). \tag{2.30}$$

*Proof.* Let $(\gamma_k)_k \subset \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ be a sequence converging strictly to $\gamma$. By Lemma 2.10 there are reparametrized maps $\widetilde{\gamma}_k := \gamma_k \circ h_k \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ of $\gamma_k$ such that $\widetilde{\gamma}_k \to \widetilde{\gamma}$ uniformly as $k \to +\infty$. Moreover, since by Lemma 2.10(b) the homeomorphism $h_k$ can be chosen with uniformly bounded Lipschitz constant, it follows that $\widetilde{\gamma}_k$ has uniformly bounded total variation. Hence it follows from Lemma 2.12 that $P(\widetilde{\gamma}_k) \to P(\widetilde{\gamma})$ as $k \to +\infty$. Thanks to (2.29), we have also $P(\gamma_k) \to P(\widetilde{\gamma})$ as $k \to +\infty$. Finally, since by Lemma 2.10 $\widetilde{\gamma}$ does not depend on the approximating sequence, we can repeat the previous argument for another sequence $(\eta_k) \subset \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ converging strictly to $\gamma$, obtaining that $P(\eta_k) \to P(\widetilde{\gamma})$. Therefore, we conclude $\overline{P}(\gamma) = P(\widetilde{\gamma})$. ☐

As a consequence of the argument in the proof of Lemma 2.13, we easily infer the following continuity property:

**Corollary 2.14.** *Let* $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ *and* $\widetilde{\gamma}$ *be as in Lemma 2.10, and assume that* $(\gamma_k)_k \subset \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ *is a sequence converging strictly to* $\gamma$*. Then*

$$\lim_{k\to+\infty} P(\gamma_k) = \overline{P}(\gamma) = P(\widetilde{\gamma}).$$

# 3 Relaxation results

In this section, we extend the results in [4, Sec.4] to homogeneous maps as in Definition 1.1.
To start with, it is worth to consider the case of homogeneous extension $u$ of a Lipschitz map $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$, namely

$$u(x) = \varphi\left(\frac{x}{|x|}\right) \qquad \forall x \in B_\ell \setminus \{(0,0)\}. \tag{3.1}$$

In this case, clearly $u \in W^{1,1}(B_\ell;\mathbb{R}^2)$ and $\int_{B_\ell} |\nabla u| dx = \ell \int_{\mathbb{S}^1} |\dot{\varphi}| d\mathcal{H}^1$. The following result extends the validity of [26, Thm.1] also for the relaxation with respect to the strict $BV$-convergence.

**Theorem 3.1.** *Suppose that* $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$ *is Lipschitz continuous and let* $u$ *be defined as in* (3.1)*. Then*

$$\overline{TVJ}_{BV}(u; B_\ell) = P(\varphi). \tag{3.2}$$

[5] $\mathbb{S}^1$ is identified with $[0, 2\pi]$.

*Proof.* Let us show the upper bound inequality. Following the proof of Theorem 1 in [26], for $k \geq 2$, a recovery sequence $v_k \in \mathrm{Lip}(B_\ell; \mathbb{R}^2)$ is given by

$$v_k(x) = \begin{cases} u(x) & \text{if } |x| > \ell/k, \\ (v)_{\frac{\ell}{k}}(x) & \text{if } |x| \leq \ell/k, \end{cases} \tag{3.3}$$

where $v \in \mathrm{Lip}(B_1; \mathbb{R}^2)$ is any map with $v = \varphi$ on $\partial B_1$ and $(v)_{\frac{\ell}{k}}(x) := v\left(\frac{k}{\ell}x\right)$ for $x \in B_{\frac{\ell}{k}}$. It is not difficult to see that $v_k \to u$ strongly in $W^{1,1}(B_\ell; \mathbb{R}^2)$ (and hence strictly $BV(B_\ell; \mathbb{R}^2)$). Moreover, by change of variable

$$\int_{B_\ell} |Jv_k| dx = \int_{B_{\frac{\ell}{k}}} |J(v)_{\frac{\ell}{k}}| dx = \int_{B_1} |Jv| dx \quad \forall k \in \mathbb{N}. \tag{3.4}$$

Finally, we get

$$\overline{TVJ}_{BV}(u; B_\ell) \leq \liminf_{k \to +\infty} \int_{B_\ell} |Jv_k| dx = \int_{B_1} |Jv| dx$$

for any $v \in \mathrm{Lip}(B_1; \mathbb{R}^2)$ such that $v = \varphi$ on $\partial B_1$, so we deduce that $\overline{TVJ}_{BV}(u; B_\ell) \leq P(\varphi)$.
Now let us prove the lower bound inequality. Assume that $v_k \in C^1(B_\ell; \mathbb{R}^2)$ is such that $v_k \to u$ strictly $BV(B_1; \mathbb{R}^2)$. Then for almost every $\rho < \ell$, there exists a subsequence $(v_{k_h})$ (depending on $\rho$) such that its restriction to $\partial B_\rho$ converges strictly $BV(\partial B_\rho; \mathbb{R}^2)$ to $u \llcorner \partial B_\rho$. So, fix $\varepsilon < 1$ and a not-relabeled subsequence of $(v_k)$ such that

$$v_k \llcorner \partial B_\varepsilon \to u \llcorner \partial B_\varepsilon \quad \text{strictly } BV(\partial B_\varepsilon; \mathbb{R}^2). \tag{3.5}$$

Now, define $w_k : B_\ell \to \mathbb{R}^2$ as

$$w_k(x) = \begin{cases} v_k(x) & \text{if } |x| \leq \varepsilon \\ \dfrac{\ell - |x|}{\ell - \varepsilon} v_k\left(\varepsilon \dfrac{x}{|x|}\right) + \dfrac{|x| - \varepsilon}{\ell - \varepsilon} u\left(\varepsilon \dfrac{x}{|x|}\right) & \text{if } \varepsilon \leq |x| \leq \ell. \end{cases} \tag{3.6}$$

Then $w_k$ is Lipschitz and $w = u$ on $\partial B_\ell$. Moreover, by (3.5), the convergence of $v_k$ to $u$ on $\partial B_\varepsilon$ is also uniform, so we have (see the proof of [3, Proposition 3.3, (3.29)])

$$\lim_{k \to +\infty} \int_{B_\ell \setminus B_\varepsilon} |Jw_k| dx = 0. \tag{3.7}$$

Finally, since $w_k = v_k$ in $B_\varepsilon$, by (3.7) we get

$$\liminf_{k \to +\infty} \int_{B_\ell} |Jv_k| dx \geq \liminf_{k \to +\infty} \int_{B_\varepsilon} |Jv_k| dx = \liminf_{k \to +\infty} \int_{B_\ell} |Jw_k| dx \geq P_\ell(u \llcorner \partial B_\ell) = P_\ell(\varphi_\ell) = P(\varphi), \tag{3.8}$$

where we used (2.27). We conclude by taking the infimum in the left hand side. □

**Corollary 3.2.** *Let $\varphi$ and $u$ as in Theorem 3.1. Then*

$$\overline{\mathcal{A}}_{BV}(u; B_\ell) = \int_{B_\ell} \sqrt{1 + |\nabla u|^2} dx + P(\varphi). \tag{3.9}$$

*Proof.* For the lower bound, suppose that $v_k \in C^1(B_\ell;\mathbb{R}^2)$ is such that $v_k \to u$ strictly $BV(B_\ell;\mathbb{R}^2)$. Now, let $\varepsilon < \ell$ such that (3.5) holds, and write $\mathcal{A}(v_k;B_\ell) = \mathcal{A}(v_k;B_\ell \setminus B_\varepsilon) + \mathcal{A}(v_k;B_\varepsilon) \geq \mathcal{A}(v_k;B_\ell \setminus B_\varepsilon) + \int_{B_\varepsilon} |Jv_k|dx$, so that, by [1, Theorem 3.7],

$$\begin{aligned}\lim_{k\to+\infty} \mathcal{A}(v_k;B_\ell) &\geq \liminf_{k\to+\infty} \mathcal{A}(v_k;B_\ell \setminus B_\varepsilon) + \liminf_{k\to+\infty} \int_{B_\epsilon} |Jv_k|dx \\ &\geq \int_{B_\ell\setminus B_\varepsilon} \sqrt{1+|\nabla u|^2}dx + \liminf_{k\to+\infty} \int_{B_\epsilon} |Jv_k|dx.\end{aligned}$$

We now apply (3.8) and next pass to the limit as $\varepsilon \to 0^+$ to get the lower bound in (3.9).

Concerning the proof of the upper bound for (3.9), consider the sequence $(v_k)$ defined in (3.3), which converges to $u$ in $W^{1,1}(B_\ell;\mathbb{R}^2)$. Then, upon extracting a subsequence such that $(\nabla v_k)$ converges almost everywhere to $\nabla u$, by (3.4) and dominated convergence we have, using the inequality $\sqrt{1+a^2+b^2+c^2} \leq \sqrt{1+a^2+b^2} + |c|$ for $a,b,c \in \mathbb{R}$,

$$\begin{aligned}\overline{\mathcal{A}}_{BV}(u;B_\ell) \leq \limsup_{k\to+\infty} \mathcal{A}(v_k;B_\ell) &\leq \lim_{k\to+\infty} \int_{B_\ell} \sqrt{1+|\nabla v_k|^2}dx + \lim_{k\to+\infty} \int_{B_\ell} |Jv_k|dx \\ &= \int_{B_\ell} \sqrt{1+|\nabla u|^2}dx + \int_{B_1} |Jv|dx,\end{aligned}$$

for any $v \in \mathrm{Lip}(B_1;\mathbb{R}^2)$ such that $v = \varphi$ on $\partial B_1$. Passing to the infimum on the right hand side we obtain the upper bound inequality in (3.9). □

**Remark 3.3.** We point out that the result of Corollary 3.2 is compatible with [3, Theorem 1.1], where $\varphi$ is valued in $\mathbb{S}^1$. Indeed, one can argue as in the proof of [26, Theorem 4] to show that $P(\varphi) = \pi|\deg(\varphi)|$ for any $\varphi \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{S}^1)$.

**Example 3.4** (The double eight curve)**.** A very interesting example is the homogeneous extension $u_8$ of the so called *double eight map* $\varphi_8 \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$, defined as $\varphi_8 = a\cdot b\cdot a^{-1}\cdot b^{-1}$, where $a, b$ are the loops in Fig. 2. This example was firstly considered by Malý [23] (see also [17], [16], [24], [26], [15]). Clearly, $\deg(\varphi_8) = 0$, however one can compute as in [26, Thm. 5] (see also [24, Thm. 1.2]) that

$$P(\varphi_8) = \inf\left\{\int_{B_1} |Jv|dx;\ v \in \mathrm{Lip}(B_1;\mathbb{R}^2) : v \llcorner \partial B_1 = \varphi_8\right\} = 2\min\{|D_1|,|D_2|\}.$$

Differently from the case of maps valued in $\mathbb{S}^1$, it is not possible to associate to $u_8$ a Cartesian current (with underlying map $u_8$) whose mass coincides with $\overline{\mathcal{A}}_{BV}(u_8;B_\ell)$ (see also [25]). The reason is that the graph of $u_8$, regarded as a current, is *already* a Cartesian current, even if the origin is a non-removable singularity for $u_8$. Finally, an interesting problem would be the study of $\overline{\mathcal{A}}_{L^1}(u_8;B_\ell)$: since the obstruction generated by $\varphi_8$ has a topological nature, we conjecture that, for $\ell$ sufficiently large, $\overline{\mathcal{A}}_{L^1}(u_8;B_\ell) = \overline{\mathcal{A}}_{BV}(u_8;B_\ell)$.

Now, we treat the case $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$. We recall that, by Proposition 2.6, its homogeneouos extension $u$ is still $BV(B_\ell;\mathbb{R}^2)$.

**Theorem 3.5.** *Let $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ and $u$ as in* (1.1)*. Let $\widetilde{\gamma} : \mathbb{S}^1 \to \mathbb{R}^2$ be as in Lemma 2.10. Then*

$$\overline{TVJ}_{BV}(u;B_\ell) = \overline{P}(\gamma) = P(\widetilde{\gamma}). \tag{3.10}$$

*Proof.* In order to show the upper bound inequality, consider a Lipschitz sequence $\varphi_k : \mathbb{S}^1 \to \mathbb{R}^2$ converging to $\gamma$ strictly $BV(\mathbb{S}^1;\mathbb{R}^2)$ (e.g. a mollifying sequence). Then, by Lemma 2.10, there

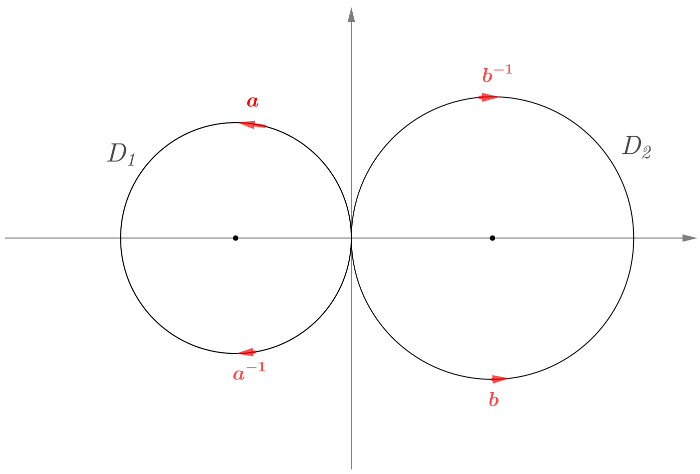


Figure 2: The double eight curve $\varphi_8$.

exists a equi-Lipschitz reparameterization $\widetilde{\varphi}_k$ of $\varphi_k$ that converges to $\widetilde{\gamma}$ uniformly (up to extracting a subsequence). For $k \in \mathbb{N}$, consider the map

$$u_k(x) = \varphi_k \left( \frac{x}{|x|} \right) \qquad \forall x \in B_\ell \setminus \{(0,0)\}. \tag{3.11}$$

In the proof of Proposition 2.6 we proved that $u_k \in W^{1,1}(B_\ell; \mathbb{R}^2)$ and $u_k \to u$ strictly $BV(B_\ell; \mathbb{R}^2)$, since

$$\begin{aligned} &\|u_k - u\|_{L^1(B_1;\mathbb{R}^2)} \leq \|\varphi_k - \gamma\|_{L^1(\mathbb{S}^1;\mathbb{R}^2)} \to 0, \\ &\int_{B_\ell} |\nabla u_k| dx = \ell \int_{\mathbb{S}^1} |\dot{\varphi}_k| d\mathcal{H}^1 \to \ell |\dot{\gamma}|(\mathbb{S}^1) = |Du|(B_\ell). \end{aligned}$$

Now, by lower semicontinuity of $\overline{TVJ}_{BV}(\cdot\,; B_\ell)$, Theorem 3.1, (2.29), and Lemma 2.12, we have

$$\overline{TVJ}_{BV}(u; B_\ell) \leq \liminf_{k \to +\infty} \overline{TVJ}_{BV}(u_k; B_\ell) = \liminf_{k \to +\infty} P(\varphi_k) = \liminf_{k \to +\infty} P(\widetilde{\varphi}_k) = P(\widetilde{\gamma}).$$

Let us prove the lower bound inequality. Assume that $v_k \in C^1(B_\ell; \mathbb{R}^2)$ is such that $v_k \to u$ strictly $BV(B_\ell; \mathbb{R}^2)$ and

$$\lim_{k \to +\infty} \int_{B_\ell} |Jv_k| dx = \overline{TVJ}_{BV}(u; B_\ell).$$

We use Lemma 2.3 to fix $\varepsilon < \ell$ and a subsequence $(v_{k_j}) \subset (v_k)$ such that $v_{k_j} \llcorner \partial B_\varepsilon \to u \llcorner \partial B_\varepsilon$ strictly $BV(\partial B_\varepsilon; \mathbb{R}^2)$. According to (2.26), we have $u \llcorner \partial B_\varepsilon = \gamma_\varepsilon$. So, let $\widetilde{\gamma}_\varepsilon$ be the Lipschitz curve of Lemma 2.10 associated[6] to $\gamma_\varepsilon$. Using Corollary 2.14 and (2.27), we conclude

$$\overline{TVJ}_{BV}(u; B_\ell) \geq \liminf_{j \to +\infty} \int_{B_\varepsilon} |Jv_{k_j}| dx \geq \liminf_{j \to +\infty} P_\varepsilon(v_{k_j} \llcorner \partial B_\varepsilon) = \overline{P}_\varepsilon(\gamma_\varepsilon) = P_\varepsilon(\widetilde{\gamma}_\varepsilon) = P(\widetilde{\gamma}). \tag{3.12}$$

$\square$

**Remark 3.6.** Setting $\widetilde{u}(x) := \widetilde{\gamma}\left(\frac{x}{|x|}\right)$, then $\widetilde{u} \in W^{1,1}(B_\ell; \mathbb{R}^2)$. So, by Theorem 3.1 and Theorem 3.5, we have

$$\overline{TVJ}_{BV}(\widetilde{u}; B_\ell) = \overline{TVJ}_{BV}(u; B_\ell). \tag{3.13}$$

[6] We identify $\partial B_\varepsilon$ with $[0, 2\pi\varepsilon]$.

We are in the position to prove Theorem 1.2.

*Proof of Theorem 1.2.* For the lower bound, suppose that $v_k \in C^1(B_\ell;\mathbb{R}^2)$ is such that $v_k \to u$ strictly $BV(B_\ell;\mathbb{R}^2)$. Now, let $\varepsilon < \ell$ such that (3.5) holds, and write $\mathcal{A}(v_k;B_\ell) = \mathcal{A}(v_k;B_\ell \setminus B_\varepsilon) + \mathcal{A}(v_k;B_\varepsilon) \geq \mathcal{A}(v_k;B_\ell \setminus B_\varepsilon) + \int_{B_\varepsilon} |Jv_k|dx$, so that, by [1, Theorem 3.7],

$$\begin{aligned}\lim_{k\to+\infty} \mathcal{A}(v_k;B_\ell) &\geq \liminf_{k\to+\infty} \mathcal{A}(v_k;B_\ell \setminus B_\varepsilon) + \liminf_{k\to+\infty} \int_{B_\epsilon} |Jv_k|dx \\ &\geq \int_{B_\ell\setminus B_\varepsilon} \sqrt{1+|\nabla u|^2}dx + |D^s u|(B_\ell \setminus B_\varepsilon) + \liminf_{k\to+\infty} \int_{B_\epsilon} |Jv_k|dx.\end{aligned}$$

We now apply (3.8) and next pass to the limit as $\varepsilon \to 0^+$ to get the lower bound in (1.9).

Concerning the proof of the upper bound for (1.9), consider the sequence $(u_k) \subset W^{1,1}(B_\ell;\mathbb{R}^2)$ defined in (3.11), which converges to $u$ strictly $BV(B_\ell;\mathbb{R}^2)$. Let us prove that

$$\lim_{k\to+\infty} \int_{B_\ell} \sqrt{1+|\nabla u_k|^2}dx = \int_{B_\ell} \sqrt{1+|\nabla u|^2}dx + |D^s u|(B_\ell). \tag{3.14}$$

In polar coordinates, we get

$$\int_{B_\ell} \sqrt{1+|\nabla u_k|^2}dx = \int_0^\ell \int_0^{2\pi} \rho\sqrt{1+\frac{|\dot{\varphi}_k(\theta)|^2}{\rho^2}}d\theta d\rho.$$

For a fixed $\rho \in (0,\ell)$, consider $f_\rho(\xi) = \rho\sqrt{1+\frac{|\xi|^2}{\rho^2}}$, $\xi \in \mathbb{R}^2$. Then, $f_\rho$ is convex on $\mathbb{R}^2$. Now, if $\mu \in \mathcal{M}([0,2\pi];\mathbb{R}^2)$, one can consider the measure $f_\rho(\mu) \in \mathcal{M}^+([0,2\pi])$ defined as[7]

$$f_\rho(\mu)(A) = \int_A \rho\sqrt{1+\frac{|a(\theta)|^2}{\rho^2}}d\theta + |\mu^s|(A),$$

for any Borel set $A \subseteq [0,2\pi]$, where $\mu^a = a\mathscr{L}^2$ for some $a \in L^1([0,2\pi])$. By [20, Theorem 4], $f_\rho(\cdot)$ is continuous w.r.t. the approximation by convolution. In particular, choosing $\mu := \dot{\bar{\gamma}} \in \mathcal{M}([0,2\pi];\mathbb{R}^2)$ and $A = [0,2\pi]$, for every $\rho \in (0,\ell)$ we have

$$\begin{aligned}\lim_{k\to+\infty} f_\rho(\dot{\varphi}_k)([0,2\pi]) &= \lim_{k\to+\infty} \int_0^{2\pi} \rho\sqrt{1+\frac{|\dot{\varphi}_k(\theta)|^2}{\rho^2}}d\theta \\ &= \int_0^{2\pi} \rho\sqrt{1+\frac{|\dot{\bar{\gamma}}^a(\theta)|^2}{\rho^2}}d\theta + |\dot{\gamma}^s|(\mathbb{S}^1) \\ &= f_\rho(\dot{\bar{\gamma}})([0,2\pi]).\end{aligned}$$

Integrating in $(0,\ell)$, by dominated convergence we infer

$$\begin{aligned}\lim_{k\to+\infty} \int_{B_\ell} \sqrt{1+|\nabla u_k|^2}dx &= \lim_{k\to+\infty} \int_0^\ell \int_0^{2\pi} \rho\sqrt{1+\frac{|\dot{\varphi}_k(\theta)|^2}{\rho^2}}d\theta d\rho \\ &= \int_0^\ell \int_0^{2\pi} \rho\sqrt{1+\frac{|\dot{\bar{\gamma}}^a(\theta)|^2}{\rho^2}}d\theta d\rho + \ell|\dot{\gamma}^s|(\mathbb{S}^1) \\ &= \int_{B_\ell} \sqrt{1+|\nabla u|^2}dx + |D^s u|(B_\ell),\end{aligned}$$

[7] See Theorem 2' in [20]: notice that $f_\rho^* = |\cdot|$ for every $\rho \in (0,\ell)$, where $f_\rho^*$ is the recession function associated to $f_\rho$.

where we used (2.11) and (2.8). Therefore, we obtain (3.14).
Finally, by lower semicontinuity of $\overline{\mathcal{A}}_{BV}(\cdot, B_\ell)$ and by Corollary 3.2, we conclude

$$\begin{aligned}\overline{\mathcal{A}}_{BV}(u; B_\ell) &\leq \liminf_{k\to+\infty} \overline{\mathcal{A}}_{BV}(u_k; B_\ell) = \lim_{k\to+\infty} \left[\int_{B_\ell} \sqrt{1+|\nabla u_k|^2}dx + P(\varphi_k)\right] \\ &= \int_{B_\ell} \sqrt{1+|\nabla u|^2}dx + |D^s u|(B_\ell) + \overline{P}(\gamma).\end{aligned}$$

□

**Remark 3.7.** We notice that, as a function of the set variable, $\overline{TVJ}_{BV}(u, \cdot)$ is a (finite) measure. Precisely, for every open set $A \subset B_\ell$

$$\overline{TVJ}_{BV}(u; A) = \overline{P}(\gamma)\delta_0(A).$$

Indeed, if $0 \in A$ then $B_\varepsilon \subset A$ for some $\varepsilon \in (0, \ell)$ and we can argue as in (3.12). On the other hand, suppose that $0 \notin A$ and consider $u_k$ as in (3.11). Then, $u_k \llcorner A \in \mathrm{Lip}(A; \mathbb{R}^2)$ and converges strictly $BV(A; \mathbb{R}^2)$ to $u \llcorner A$. Since the image of $u_k$ has zero Lebesgue measure, by lower semicontinuity of $\overline{TVJ}_{BV}(\cdot\,; A)$, we get that $\overline{TVJ}_{BV}(u; A) = 0$.
In the same way, one can prove that for every open set $A \subset B_\ell$

$$\overline{\mathcal{A}}_{BV}(u; A) = \int_A \sqrt{1+|\nabla u|^2}dx + |D^s u|(A) + \overline{P}(\gamma)\delta_0(A).$$

Therefore, also $\overline{\mathcal{A}}_{BV}(u; \cdot)$ is a measure and (1.9) is an integral representation.

**Remark 3.8 (On the Plateau problem** (2.24)**).** Let $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$ be Lipschitz. From [11, Theorem 1.3], there exists a least area mapping $v \in W^{1,p}(B_1; \mathbb{R}^2)$, for some $p > 2$, spanning $\varphi$, i.e. realizing the infimum of the total variation of the Jacobian determinant in the class of Sobolev maps in $W^{1,p}(B_1; \mathbb{R}^2)$ whose trace on $\partial B_1$ is $\varphi$. In truth, one can prove that the least area mapping is Lipschitz, so that the Plateau problem (2.24) attains a minimum. The proof is a consequence of results contained in [12]: interestingly, it seems that one needs to pass through a more general metric result, concerning spaces with upper curvature bounds.

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