INVARIANT MEASURES FOR GENERALIZED RANDOM DYNAMICAL SYSTEMS

ARCADY PONOSOV AND EUGENE STEPANOV

ABSTRACT. The results proved in this paper provide existence of an invariant measure for non-Carathéodory random dynamical systems typically coming from stochastic differential equations with nonregular solution flows. An example of such an equation is a stochastic differential equation with time delay in the diffusion.

CONTENTS

1. INTRODUCTION

The classical notion of a random dynamical system has proved to be very fruitful in many applications to stochastic differential equations (see, e.g. [2] and references therein). Roughly speaking, a random dynamical system (RDS) amounts to a combination of a measure preserving dynamical system with a smooth (or just continuous) dynamical system, typically generated by a differential or difference equation [2]. However, there are some natural examples of differential equations, especially in infinite dimensional spaces, which do not produce smooth or even continuous dynamical systems and cannot therefore be regarded as RDS in the

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above classical sense. The most prominent example of this ilk are stochastic delay equations that give rise to RDS only in very special cases (see e.g. [8]).

In this paper we introduce a more general notion of a RDS (called further "generalized random dynamical systems") which covers both "classical" RDS in the sense of [2] and RDS coming from stochastic delay differential equations. The basic idea is as follows. The core of the definition of the classical smooth (or continuous) RDS is constituted by two objects: the shift operator generating the measure preserving dynamical system and a Carathéodory function, which provides an analogue of the smooth deterministic dynamical system in the stochastic case. The classical cocycle condition is then imposed on the composition of the shift with the respective Nemystkiĭ operator generated by this Carathéodory function [2]. In our generalization the compositions of Nemystkiĭ operators generated by Carathéodory functions with shifts (such objects are referred to usually as nonlinear shift operators) are substituted by a more general object, namely, by so-called atomic operators introduced in [5]. Such operators roughly speaking amount to compositions of *local* operators with shifts and in general are not reducible to nonlinear shifts.

In this paper we show that this idea is well adapted to extend the classical notion of RDS also to a much wider class of more general stochastic equations including stochastic functional differential equations and equations in infinite-dimensional spaces. We then extend the fundamental result ensuring the existence of invariant measures for RDS with a compact phase space (Theorem 1.5.10 in [2]) to generalized RDS. The respective extension which is the main result of the paper is formulated at the end of Section 6 (Theorem 6.2 and Corollary 6.1).

2. Notation and preliminaries

In this section we describe the basic notions used in the paper.

Given two probability spaces $(\Omega_1, \Sigma_1, \mathbf{P}_1)$ and $(\Omega_2, \Sigma_2, \mathbf{P}_2)$, we say that a map $F\colon \Sigma_1 \to \Sigma_2$ is a *σ-homomorphism*, if $F(\Omega_1) = \Omega_2$, $F(\Omega_1 \backslash e) = \Omega_2 \backslash F(e)$ whenever $e \in \Sigma_1$ and

$$F\left(\bigsqcup_{i=1}^{\infty} e_i\right) = \bigsqcup_{i=1}^{\infty} F(e_i)$$

for any pairwise disjoint collection of $\mathbf{P}_1$-measurable sets $\{e_i\}_{i=1}^{\infty}$, where $\sqcup$ stands for the disjoint union. The σ-homomorphisms considered below will silently be assumed to be *nullset preserving*, that is, $\mathbf{P}_2(F(e_1)) = 0$ when $\mathbf{P}_1(e_1) = 0$. Thus F naturally induces a σ-homomorphism $\tilde{F}\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ between the respective *measure algebrae* $\tilde{\Sigma}_i := \Sigma_i/\Sigma_i^0$, where $\Sigma_i^0 := \mathbf{P}_i^{-1}(0)$, $i = 1, 2$. The respective induced σ-homomorphism is defined by the relationship

$$\tilde{F}(\tilde{e}_1) := [F(e_1)],$$

where $\tilde{e}_i = [e_i]$ stands for the equivalence class of the set $e_i \in \Sigma_i$ in the respective measure algebra $\tilde{\Sigma}_i$, $i = 1, 2$. It is worth remarking that for complete measure spaces the inverse implication is valid, namely, every σ-homomorphism $\tilde{F}\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ between the respective measure algebrae is induced according to the above formula by some nullset-preserving sigma-homomorphism $F\colon \Sigma_1 \to \Sigma_2$.

In what follows we will only use a particular case of this definition where $\Omega_1 = \Omega_2 := \Omega$, $\mathbf{P}_1 = \mathbf{P}_2 := \mathbf{P}$ and Σ_i are subalgebras of a bigger σ-algebra Σ. Moreover, we assume $\mathbf{P}$ to be a complete probability measure. In other words, in the sequel we are supposed given *a complete probability space* $(\Omega, \Sigma, \mathbf{P})$.

If X is a Polish space, then we denote by $L^0(\Omega, \Sigma, \mathbf{P}; X)$ the space of (classes of P-a.s. equal) X-valued measurable functions defined on Ω endowed with the topology of convergence in measure. We recall that this is a complete metric space. If X is a separable Banach space we, as usual, denote the Lebesgue spaces of p-integrable functions $x\colon \Omega \to X$ by $L^p(\Omega, \Sigma, \mathbf{P}; X)$, $p > 0$.

3. Local and atomic operators

Here we present the main notions we will be dealing with in the sequel, namely, those of local and atomic operators. Let $\mathcal{X}_i \subset L^0(\Omega, \Sigma_i, \mathbf{P}; X_i)$, $i = 1, 2$. We start with the following definition due to I. Shragin [14].

Definition 3.1. *An operator $T\colon \mathcal{X}_1 \to \mathcal{X}_2$ is called* **local**, *if for any $e_1 \in \Sigma_1$ and $x, y \in \mathcal{X}_1$ the equality $x\,|_{e_1} = y\,|_{e_1}$ a. s. implies $T(x)\,|_{e_1} = T(y)\,|_{e_1}$ a. s.*

We give two most important examples of local operators one usually deals with in applications.

Example 3.1. Let X_1 and X_2 be separable metric spaces, Σ be a complete σ-algebra on Ω, $\Sigma_1 \subset \Sigma_2 \subset \Sigma$, and, finally, $f\colon \Omega \times X_1 \to X_2$ be a sup-measurable function (i.e. $f(\cdot, x(\cdot))$ is Σ_2-measurable whenever $x(\cdot)$ is Σ_1-measurable). Then the Nemytskiĭ (superposition) operator $N_f\colon L^0(\Omega, \Sigma_1, \mathbf{P}; X_1) \to L^0(\Omega, \Sigma_2, \mathbf{P}; X_2)$, defined by

$$(N_f x)(\omega) := f(\omega, x(\omega))$$

is local. If $f\colon \Omega \times X_1 \to X_2$ is a Carathéodory function (i.e. $f(\omega, \cdot)$ is continuous for P-almost every $\omega \in \Omega$ and $f(\cdot, x)$ is Σ_2-measurable for all $x \in X_1$), then the Nemytskiĭ operator N_f becomes continuous in measure (i.e. in the topology of convergence in L^0).

Example 3.2. Let $W(t)$, $t \geq 0$ be the standard m-dimensional Brownian motion and $\Sigma_t := \sigma\{W(u) : 0 \leq u \leq t\}$ be the filtration (an increasing family of σ-algebras) generated by the stochastic process $W(t)$.

Setting $X := L^2[c, d]$, we define a local operator

$$N : \ L^0(\Omega, \Sigma_c, \mathbf{P}; X) \to L^0(\Omega, \Sigma_d, \mathbf{P}; X)$$

to be the stochastic Itô integral

$$(Nx)(\omega) := \int_c^d 1_{[c,\cdot]}(s) x(s, \omega) dW_s(\omega).$$

The stochastic integral cannot be represented by a Nemytskiĭ operator generated by a Carathéodory function (otherwise, the stochastic integral could have been, by the Riesz representation theorem, reduced to the ordinary Lebesgue-Stieltjes integral, which is not the case).

We pass now to the more general notion of an atomic operator introduced in [5].

Definition 3.2. *An operator $T\colon \mathcal{X}_1 \to \mathcal{X}_2$ is called* **atomic**, *if there is a σ-homomorphism $F\colon \Sigma_1 \to \Sigma_2$ such that for any $e_1 \in \Sigma_1$, $x, y \in \mathcal{X}_1$, the equality $x\,|_{e_1} = y\,|_{e_1}$ a. s. implies $T(x)\,\big|_{F(e_1)} = T(y)\,\big|_{F(e_1)}$ a. s.*

Clearly, local operators are atomic operators with $F = id$. It is worth mentioning the in [6] another equivalent definition of an atomic operator is proposed that is intrinsic in the sense that it does not depend on the knowledge of a particular σ-homomoprhism F.

To simplify description of atomic operators we shall formulate an important representation theorem. Here and in the sequel we use the notation $\mathbf{1}_e$ for the indicator function of a set e.

Theorem 3.1. *Let X_i be two separable Banach spaces. Then for every atomic operator*

$$T: \ L^0(\Omega_1, \Sigma_1, \mathbf{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbf{P}_2; X_2)$$

there exists a local operator

$$N: \ L^0(\Omega_2, \Sigma_2, \mathbf{P}_2; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbf{P}_2; X_2)$$

such that

$$T = N \circ T_F, \tag{1}$$

where T_F: $L^0(\Omega_1, \Sigma_1, \mathbf{P}; X_1) \to L^0(\Omega_2, F\Sigma_2, \mathbf{P}; X_1)$ is the generalized shift operator associated with σ-homomorphism F: $\Sigma_1 \to \Sigma_2$ defined as the unique continuous linear operator satisfying

$$T_F(\mathbf{1}_{e_1} z) = \mathbf{1}_{F(e_1)} z \tag{2}$$

for all $e_1 \in \Sigma_1$ and $z \in X_1$. The operator T is continuous (in measure), if and only if so is the restriction of the operator N to the subspace $L^0(\Omega_2, F\Sigma_1, \mathbf{P}_2; X_1) \subset L^0(\Omega_2, \Sigma_2, \mathbf{P}_2; X_1)$, where F: $\Sigma_1 \to \Sigma_2$ is the σ-homomorphism corresponding to T. Moreover, if $(\Omega_1, \Sigma_1, P_1)$ is a standard measure space, then there is a (Σ_1, Σ_2)-measurable function g: $\Omega_2 \to \Omega_1$ satisfying

$$\mathbf{P}_2(g^{-1}(e_1)) = 0 \ \textit{whenever} \ e_1 \in \Sigma_1, \, \mathbf{P}_1(e_1) = 0, \tag{3}$$

so that $T = N \circ T_g$ where the operator T_g is the usual shift operator defined by $(T_g x_1)(\omega) := x_1(g\omega)$, while $g\omega$ stands for $g(\omega)$.

The proof of this result can be found in [5].

Now we come up with two important examples of atomic operators. The first is a nonlinear weighted shift, the second is a shifted Itô integral. More examples will be presented in the sequel.

Example 3.3. Let $f: \ \Omega \times X_1 \to X_2$ satisfy the assumptions of the example 3.1, and the shift $g: \Omega \to \Omega$ satisfy the condition (3). Then the nonlinear weighted shift, given by $(Tx)(\omega) = f(w, x(g\omega))$, is an atomic operator from $L^0(\Omega, \Sigma_1, \mathbf{P}; X_1)$ to $L^0(\Omega, g^{-1}(\Sigma_2), \mathbf{P}; X_2)$. If, in addition, the generating function f is Carathéodory, then the operator T will be continuous in measure.

In the above example for the local operator N in the representation of T given by the above theorem 3.1 one has $N = N_f$. There are however continuous in measure operators for which the respective local operator N is not generated by any Carathéodory function as the following example shows.

Example 3.4. Consider the standard Brownian motion $W(t)$, described in the example 3.2. In addition, we will use tha *Wiener shift* $\theta(a, \cdot)$ described below in Subsection 4.1. Making use of the notation from example 3.2, we put

$$g := \theta_{-1}(a, \cdot): \ \Omega \to \Omega, \quad \text{ where } \ a := d - c.$$

This mapping induces an isomorphism of the σ-subalgebras Σ_c and $\Sigma_d = g(\Sigma_c)$. Letting again $X := L^2[c, d]$, we define the operator

$$T: \ L^0(\Omega, \Sigma_d, \mathbf{P}; X) \to L^0(\Omega, \Sigma_d, \mathbf{P}; X)$$

as the stochastic Itô integral with respect to the Brownian motion

$$(Tx)(\omega) := \int_c^d 1_{[c,\cdot]}(s)x(s, g\omega)dW(s, \omega).$$

Note that we shifted the Σ_d-measurable integrand $x(t, \omega)$ with the help of g. In this way the stochastic process $x(s, g\omega)$ becomes Σ_c-measurable, so that the stochastic integral is well defined. The operator T is atomic since it is a composition of the stochastic integral (which is local) and the shift T_g. However, the stochastic integral cannot be represented by a Nemytskiĭ operator generated by a Carathéodory function.

4. The generalized cocycle property

To any (deterministic or stochastic) differential equation satisfying the assumption of existence and uniqueness of solutions, one can associate the solution flow $U(t, s)$ defined as the shift along the trajectories: $x(s) \mapsto x(t)$. If $s = 0$ one usually simplifies the notation by putting $U(t) := U(t, 0)$.

It is customary to assume that the operators $(\omega, x) \mapsto U(t, s)x$ in the case of stochastic differential equations satisfy the Carathéodory condition with respect to (ω, x). This assumption is often referred to as regularity of the solution flow (see [8]-[10]). This means that for any $t \geq s$ there exists a Carathéodory function $u_{t,s}(\omega, x)$ such that

$$(U(t, s)x)(\omega) = u_{t,s}(\omega, x(\omega)) \text{ a. s.}$$

for each Σ_s-measurable x. However, some important classes of stochastic dynamical systems (especially infinite dimensional ones) do not generate regular solution flows (in [8]-[10] such systems and the respective solution flows are called *singular*). Singular systems are hard to handle since some fundamental properties valid for regular flows, are not true in general for singular flows, for instance, cocycle property and the existence of the Lyapunov exponents in the linear case. That is why in the classical monograph [2] one studies regular flows only, calling them *random dynamical systems*. Singular flows fall therefore outside of this concept, although they are not at all rare in applications. Moreover, in some sense singular systems constitute a generic subset in the set of all random dynamical systems (see Remark 4.1).

In this section we introduce a concept of a *generalized random dynamical system* based on *the generalized cocycle property*, which holds for both regular and singular solution flows.

4.1. **The usual cocycle property.**

For deterministic differential equations with time independent coefficients the solution flow $U(t)$ has the semigroup property

$$U(t + s) = U(t) \circ U(s), \quad (t \geq s).$$

In the case of A-periodic coefficients this property becomes

$$U(t + A) = U(t) \circ U(A).$$

Considering stochastic differential equations, define the canonical Wiener shift $\theta : \mathbf{R} \times \Omega \to \Omega$ on the Wiener space Ω by setting

$$\theta(a, \omega)(u) := \omega(a + u) - \omega(a), \ u, a \in \mathbf{R}, \ \omega \in \Omega. \tag{4}$$

This map is invertible, measure preserving and a.s. satisfies

$$W(t,\theta(a,\omega)) - W(u,\theta(a,\omega)) = W(t+a,\omega) - W(u+a,\omega) \tag{5}$$

for any a, t, u (see [2] or [10] for details). Moreover, it is straightforward that

$$\theta(a,\Sigma_t) = \Sigma_{t-a} \tag{6}$$

up to a $\mathbf{P}$-nullset.

The formula (4) gives also

$$W(t,\theta(a,\omega)) - W(u,\theta(a,\omega)) = W(t+a,\omega) - W(u+a,\omega) \tag{7}$$

which can be viewed as a trajectory based definition of a stochastic process with stationary increments.

In the case when an **ordinary** stochastic differential equation has time independent coefficients and satisfies some additional regularity assumptions (like smoothness of the coefficients), for its solution flow $U(t,\omega)$ one has the so-called "cocycle property" which reads as follows:

$$U(t+s,\omega) = U(t,\theta(s,\omega)) \circ U(s,\omega) \tag{8}$$

almost surely for all $t \geq s \geq 0$. If coefficients are A-periodic, we obtain a discrete version of the cocycle property (8):

$$U(t+A,\omega) = U(t,\theta(A,\omega)) \circ U(A,\omega). \tag{9}$$

The aim of what follows is to generalize these formulas to the case of stochastic functional differential equations (10).

4.2. **Stochastic functional differential equations.** In this paper we do not intend to study most general stochastic differential equations (SFDE). The aim is rather to use some particular SFDEs to describe the generalized cocycle property and by this means to illustrate the idea of the generalized stochastic dynamical system. For technical reasons, we assume the coefficients of the equation to be periodic with a period $A > 0$. Of course, the results we present remain valid if the coefficients are time independent, i.e. A-periodic for any $A > 0$.

The following notation is used in what follows: $C := C([-r,0],\mathbf{R}^d)$, $x_t(s) := x(t+s)$, while $s \in [-r,0]$ and $t \geq 0$. We study the following stochastic functional differential equation (see [8] for the detailed definitions):

$$dx(t) = H(t,x_t)dt + G(t,x_t)dW(t), \tag{10}$$

where $t > 0$, with the initial condition

$$x_0 = \varphi \in C. \tag{11}$$

Here $W(t)$, $t \geq 0$ is the m-dimensional Brownian motion, and $H : [0,\infty) \times C \to \mathbf{R}^d$, $G : [0,\infty) \times C \to \mathbf{R}^{d\times m}$ are two jointly continuous, A-periodic and globally Lipschitz in the second variable functionals, i.e.

$$|H(t,y_1) - H(t,y_2)| + |G(t,y_1) - G(t,y_2)| \leq L\|y_1 - y_2\|_C$$

for all $t \in [0,\infty)$ and $y_1, y_2 \in C$. Applying the contraction principle one yields (see [8]) existence, uniqueness and continuous dependence of solutions on initial function $\varphi \in C$ (in fact, we can assume the initial function φ to be random, more exactly, Σ_0-measurable, see for instance [12]). Each solution satisfies the usual adaptedness property with respect to the filtration $\Sigma_t := \sigma\{W(u) : \ 0 \le u \le t\}$. This gives rise to a two-parameter family of continuous mappings

$$U(\tau,\sigma) : \ L^2(\Omega, \Sigma_\sigma; C) \to L^2(\Omega, \Sigma_\tau; C), \quad \tau \ge \sigma,$$

defined by

$$U(\tau,\sigma)(\varphi) := {}^{\varphi}x^{\sigma}_{\tau}, \quad \varphi \in L^2(\Omega, \Sigma_\sigma; C), \tag{12}$$

where ${}^{\varphi}x^{\sigma}(t)$ satisfies

$${}^{\varphi}x^{\sigma}(t) = \begin{cases} \varphi(0) + \int_\sigma^t H(u, {}^{\varphi}x^{\sigma}_u)du + \int_\sigma^t G(u, {}^{\varphi}x^{\sigma}_u)dW(u), & t > \sigma, \\ \varphi(t-\sigma), & \sigma - r \le t \le \sigma. \end{cases} \tag{13}$$

Clearly,

$$U(\tau,\sigma) \circ U^{-1}(\sigma) = U(\tau), \quad \sigma \le \tau, \tag{14}$$

where $U(r) := U(r, 0)$ (see [8, Theorem II (2.2), p. 40] for details).

Our first result in this section justifies the property of locality for the operator $U(\tau,\sigma)$.

Theorem 4.1. *Assume that $H : [0,\infty) \times C \to \mathbf{R}^d$, $G : [0,\infty) \times C \to \mathbf{R}^{d\times m}$ are jointly continuous and globally Lipschitz in the second variable functionals. Then the solution flow $U(\tau,\sigma)$, defined in (12)-(13), satisfies the property of locality, which in this case reads as follows: for any $\varphi, \psi \in L^2(\Omega, \Sigma_\sigma; C)$ and $e \in \Sigma_\sigma$ the equality $\varphi|_e = \psi|_e$ a. s. implies $(U(\tau,\sigma)\varphi)|_e = (U(\tau,\sigma)\psi)|_e$ a. s.*

The PROOF of this result can be found in [6]. We do not provide it here in order not to overburden the reader with technical details.

Remark 4.1. *It is worth mentioning that in fact the property of locality is normally implied by the uniqueness property of solutions. In particular, the result analogous to the above Theorem 4.1 holds true also for solution flows $U(\tau,\sigma)$ associated with many more different stochastic differential equations, including infinite dimensional ones. It is also quite important to point out that the solution flows described in this theorem can be non-Carathéodory as it has been shown by S.-E. A. Mohammed (see e.g.* [8] *where such equations and flows are called singular in contrast to the Carathéodory flows that are called regular). S.-E. A. Mohammed described also some important classes of regular and singular flows. Roughly speaking, one obtains a Carathéodory operator $U(t,s)$, and hence a regular flow, if the diffusion G contains no time lags. A switching from regular to singular flows typically occurs when there are delays in the diffusion G.*

Let us consider an example. The stochastic delay equation

$$dx(t) = ax(t - h_1)dt + bx(t - h_2)dW(t) \tag{15}$$

is regular, if and only if $bh_2 = 0$ (see [8]*). This means that singular equations constitute a generic subset of the set of all equations of the form (15).*

The precise definition of the regular flow was described in the beginning of this section. It says that for any τ, σ the function $F_{\tau,\sigma}(\omega, x) := U(\tau,\sigma,\omega)x$ should be Carathéodory, thus generating a local operator. However, the class of local operators is wider, and the singular flows are examples of non-Carathéodory local operators. As it was mentioned before, the difference between regular and singular flows is crucial in many studies. For example, the usual cocycle property (8) does not *hold for singular flows. In the next subsection we describe the generalized cocycle property which is valid for both regular and singular flows.*

We conclude this subsection with a simple, yet important observation that the operators $U(\tau,\sigma)$ can be continuously extended to the space $L^0(\Omega,\Sigma_\sigma,\mathbf{P};X)$. In what follows we will use the same notation for the extended operators.

Corollary 4.1. *Under the assumptions of Theorem 4.1 the operators (12) can be uniquely extended to continuous operators*

$$U(\tau,\sigma):\ L^0(\Omega,\Sigma_\sigma;C)\to L^0(\Omega,\Sigma_\tau;C),\quad \tau\geq\sigma. \tag{16}$$

PROOF: Let $f_\nu\to f_0$, as $\nu\to\infty$, where $f_\nu\in L^0(\Omega,\Sigma_\sigma;C)$. Then for any $m\in\mathbf{N}$ there exists a compact subset $K_m\subset C$ such that $\mathbf{P}(\Omega_{m,\nu})\geq 1-1/m$, where

$$\Omega_{m,\nu}\equiv\{\omega:\ f_\nu(\omega)\in K_m\},\qquad\qquad \nu\in\mathbf{N}.$$

Clearly, $f_\nu\mathbf{1}_{m,\nu}\to f_0\mathbf{1}_{m,0}$ in L^2 as $\nu\to\infty$ for each $m\in\mathbf{N}$, where $\mathbf{1}_{m,\nu}:=\mathbf{1}_{\Omega_{m,\nu}}$, $\nu\in\mathbf{N}\cup\{0\}$. One has also $\Omega_{m,\nu}\in\Sigma_\sigma$, and

$$\mathbf{P}\left(\bigcup_m\Omega_{m,\nu}\right)=1 \tag{17}$$

for all ν.

Since the operator $U(\tau,\sigma)$ is continuous (with respect to the L^2 convergence) and local, we get

$$U(\tau,\sigma)(f_\nu\mathbf{1}_{m,\nu})\to U(\tau,\sigma)(f_0\mathbf{1}_{m,0})$$

in the L^2-topology for each $m\in\mathbf{N}$ as $\nu\to\infty$. Fix an arbitrary $\varepsilon>0$ and a $\sigma>0$, and choose $m\in\mathbf{N}$ so that $2/m<\varepsilon/2$ and a $k\in\mathbf{N}$ so that

$$\mathbf{P}(|U(\tau,\sigma)(f_\nu\mathbf{1}_{m,\nu})-U(\tau,\sigma)(f_0\mathbf{1}_{m,0})|)\geq\sigma)<\varepsilon/2$$

for all $\nu\geq k$. Since

$$\mathbf{P}(f_\nu\mathbf{1}_{m,\nu}\neq f_\nu)\leq 1/m \text{ and } \mathbf{P}(f_0\mathbf{1}_{m,0}\neq f_0)\leq 1/m$$

and in view of locality of $U(\tau,\sigma)$ we have

$$\begin{array}{rcl}\mathbf{P}(U(\tau,\sigma)(f_\nu\mathbf{1}_{m,\nu})\neq U(\tau,\sigma)(f_\nu)) & \leq & 1/m \text{ and}\\ \mathbf{P}(U(\tau,\sigma)(f_0\mathbf{1}_{m,0})\neq U(\tau,\sigma)(f_0)) & \leq & 1/m\end{array}$$

for all $\nu\in\mathbf{N}$. Therefore,

$$\begin{aligned}\mathbf{P}(|U(\tau,\sigma)(f_\nu)-U(\tau,\sigma)(f_0)|)\geq\sigma)\leq&\\ \mathbf{P}(|U(\tau,\sigma)(f_\nu\mathbf{1}_{m,\nu})-U(\tau,\sigma)(f_0\mathbf{1}_{m,0})|)&\geq\sigma)\\ +\mathbf{P}(U(\tau,\sigma)(f_\nu\mathbf{1}_{m,\nu})\neq U(\tau,\sigma)(f_\nu))&\\ +\mathbf{P}(U(\tau,\sigma)(f_0\mathbf{1}_{m,0})\neq U(\tau,\sigma)(f_0))&\\ \leq 2/m+\varepsilon/2\leq\varepsilon,&\end{aligned}$$

which means, since $\varepsilon>0$ and $\sigma>0$ are arbitrary, that

$$U(\tau,\sigma)(f_\nu)\to U(\tau,\sigma)(f_0)$$

in the L^0-topology. □

4.3. **How to generalize the cocycle property.** In this subsection we concentrate on SFDE (10) equipped with the initial condition (11). The coefficients of this equation are assumed to be A-periodic, which enables us to simplify our previous notation by putting $g = \theta^{-1}(A, \cdot)$. Te result we formulate will clearly be applicable also to the case of stationary coefficients (which can be considered A-periodic for every $A > 0$). The formula (7) becomes in this case

$$W(t,\omega) - W(u,\omega) = W(t+A, g\omega) - W(u+A, g\omega) \tag{18}$$

almost surely for all t. This can be extended to stochastic integrals with respect to the Brownian motion (so-called "Protter's formula"):

$$T_g\left(\int_{t_0}^{t} \Phi(v)dW(v+A)\right) = \int_{t_0}^{t} T_g(\Phi(v))dW(v) \tag{19}$$

a.s. for any Σ_t-adapted stochastic process $\Phi = \Phi(v,\omega)$ that can be integrated. For the details in the case when the integrating process has stationary increments (i.e. if it is a so-called 'helix') see e.g. [13], while in the paper [3] the corresponding result for the case of periodic increments is discussed in details.

Note that in the formula (19) the operator T_g acts in the random variable, namely, $(T_g y)(u,\omega) = (T_g)(y(u,\cdot))(\omega)$. From the definition of the shift g it is also evident that the shift operator T_g provides an isometry

$$T_g: \ L^2(\Omega, \Sigma_{\sigma+A}; C) \to L^2(\Omega, \Sigma_\sigma; C), \quad \sigma \geq 0.$$

The result below proven in [6] describes the generalized cocycle property for stochastic functional differential equations with periodic coefficients.

Theorem 4.2. *Assume that $H : [0,\infty) \times C \to \mathbf{R}^d$, $G : [0,\infty) \times C \to \mathbf{R}^{d\times m}$ are jointly continuous, A-periodic and globally Lipschitz in the second variable functionals. Then the solution flow $U(\tau,\sigma)$, defined in (12)-(13), satisfies the generalized cocycle property*

$$T_g \circ U(\tau + A, \sigma + A) \circ T_g^{-1} = U(\tau,\sigma), \quad (\tau \geq \sigma \geq 0). \tag{20}$$

Remark 4.2. *It is straightforward to verify that in (20) one can, without loss of generality, put $\sigma = 0$. The resulting expression will be equivalent to (20) as follows immediately from (14).*

The theorem 4.2 describes a property which holds for both singular and regular solution flows associated with SFDEs with periodic coefficients. The following immediate observation adjusts this result to the case of time independent coefficients.

Remark 4.3. *Assume that $H : C \to \mathbf{R}^d$, $G : C \to \mathbf{R}^{d\times m}$ are jointly continuous and globally Lipschitz functionals. Then one can rewrite the generalized cocycle property as follows:*

$$U(\tau+\sigma) = T_{\theta(\sigma,\cdot)} U(\tau) T^{-1}_{\theta(\sigma,\cdot)} U(\sigma), \quad (\tau \geq \sigma \geq 0). \tag{21}$$

It is important to observe that whenever $U(\tau)$ is Carathéodory, then the generalized cocycle property (20) reduces to the usual cocycle property (8). For this purpose consider the general case of A-periodic coefficients since it covers also the stationary case. Assume that $U(\tau,\sigma)$ is regular, i.e. that for every τ,σ the function

$F_{\tau,\sigma}(\omega, x) := U(\tau, \sigma, \omega)x$ is Carathéodory. In this case we may use nonrandom $x \in \mathbf{R}^d$. Letting $\sigma = 0$ we get that a.s.

$$\begin{aligned} U(\tau,\omega)x &= (T_g \circ U(\tau + A, A, \omega) \circ T_g^{-1})x(\omega) = U(\tau + A, A, g\omega)x \\ &= (U(\tau + A, g\omega) \circ U^{-1}(A, g\omega))x. \end{aligned}$$

Recalling that $\theta(A, \cdot) = g^{-1}$ and multiplying by U yields the cocycle property (9) for the periodic case. If the coefficients are time independent, we obtain (8) in a similar way.

Let us again emphasize that without assuming $U(\tau, \omega)x$ to be Carathéodory in (ω, x) we would not be able to obtain the standard cocycle property (9) from its generalized version (20). Theorem 4.2 describes therefore how one should adjust the standard cocycle property for the case of singular (i.e. non Carathéodory) solution flow. Moreover, a careful analysis of the its proof from [6] shows that the generalized cocycle property (20) is determined by the fundamental properties of the solutions (existence, uniqueness, continuous dependence on initial data), while being the standard cocycle (regular flow) depends, as it was shown by S.-E. A. Mohammed, on a particular form of the coefficients F and G. Especially G influences the property of regularity in an essential way (see Remark 4.1).

4.4. **Generalized random dynamical systems.** The concept of a random dynamical system is widely used in stochastic analysis (see e.g. [2] and references therein). However, as it was remarked before, this notion does not cover some solution flows naturally arising in applications. The results in the previous subsection justifies a more general notion of a random dynamical system which covers, for instance, SFDEs satisfying the hypotheses listed in Theorem 4.2 (or in Remark 4.3).

Let $\mathcal{A}$ be a sub-semigroup of the additive group $\mathbf{R}$ of all real numbers, Σ_t $(t \in \mathcal{A})$ be a family of complete σ-algebrae (a filtration), while X be a Polish space. In addition, we assume to be given a commutative family of measure preserving maps $\theta(a, \cdot)$ $(a \in \mathcal{A})$ satisfying (see [2]):

$$\theta^{-1}(a, \Sigma_\tau) = \Sigma_{\tau - a}.$$

Definition 4.1. *A generalized random dynamical system is a family of continuous (with respect to the L^0-topologies), local operators*

$$U(\tau) : L^0(\Omega, \Sigma_0, \mathbf{P}; X) \to L^0(\Omega, \Sigma_\tau, \mathbf{P}; X) \quad (\tau \in \mathcal{A}) \tag{22}$$

satisfying the generalized cocycle property (21) for any $\tau, \sigma \in \mathcal{A}, \tau \geq \sigma$.

For $\mathcal{A} = \mathbf{R}$ or $\mathcal{A} = \mathbf{R}^+$ one obtains the case of stationary solution flows, while $\mathcal{A} = A\mathbf{Z}$ or $A\mathbf{Z}^+$ corresponds to the case of A-periodic solution flows as it will be proven in Theorem 5.1.

Remark 4.4. *In this paper we do not intend to give a detailed justification of Definition 4.1, rather than that following from Theorem 4.2 and Remark 4.3. However, it is worth emphasizing that, roughly speaking, the property (22) is valid for any stochastic differential equations (finite or infinite dimensional), which satisfies the properties of existence, uniqueness and continuous dependence on initial data in the L^0-topology. A general argument, close to that used in Corollary 4.1 can also be used to replace the L^0-continuity in the definition of the generalized cocycle by the L^p-continuity for any other number $p > 0$. Here we do not discuss this problem in details.*

In the next section we use the generalized cocycle property to study the stochastic periodic problem.

5. Periodic and stationary in law solutions

This section is again of illustrative nature. Here we study one particular problem of practical importance in order to show how generalized random dynamical systems can be handled. The question we pose is as follows: given a SFDE (10), how can one find its strong periodic in law (if the coefficients are periodic), or stationary in law (if the coefficients are time independent) solutions? We show that the periodic problem can be reduced to the problem of finding fixed points for a certain atomic operator called in the sequel the monodromy operator related to (10). However, we do not consider the method in all its generality, just trying to describe the basic idea of how to construct the monodromy operator and to formulate the fixed point problem. As it is common in stochastic analysis, the fixed point problem can be formulated in two different ways. Either we can try to solve the fixed point problem in the initial domain L^0 (in an attempt to find what is called "strong solutions"), or we can extend the initial domain by completing it in a weaker topology. The second way leads to *measure solutions* and, finally, to weak solutions.

In the last section we will look at the problem of existence of periodic and stationary measure solutions in the case when the phase space X is compact. The case of a non-compact X, as well as the other problems of practical importance, like existence of strong and weak solutions, will be considered in forthcoming papers.

5.1. Strong periodic solutions.

We start with some definitions.

Definition 5.1. *A solution $x(t)$ of the SFDE (10) with A-periodic coefficients is called a strong periodic in law solution with the period A if*

$$x(t+A,\omega) = x(t,\theta(A,\omega)) \ \ a.\ s. \tag{23}$$

for all $t \in [-r,\infty)$. Here θ is the Brownian shift from (7). If the formula (23) holds for every $A > 0$, the respective solution will be called stationary in law solution of (10).

The formula (23) ensures that every deterministic characteristics of the stochastic process $x(t)$ (like expectation, distributions etc.) will be A-periodic. On the other hand, it is unrealistic to find a periodic solution in a proper sense (i.e. without the Brownian shift θ), since

$$\Delta_h W(t+A) \neq \Delta_h W(t) \text{ a.s.},$$

where $\Delta_h W(t) = W(t+h) - W(t)$. In fact, the increments of the Brownian motion $W(t)$ are not a.s. periodic but only periodic (with any period, i.e. stationary) in distribution.

Definition 5.2. *The monodromy operator associated with the SFDE (10) with periodic coefficients is defined by*

$$T = T_g \circ U(A), \tag{24}$$

where $g := \theta^{-1}(A,\cdot)$.

Obviously, $T\colon L^2(\Omega,\mathcal{F}_0;C)\to L^2(\Omega,\mathcal{F}_0;C)$ as $T_g: L^2(\Omega,\mathcal{F}_A;C)\to L^2(\Omega,\mathcal{F}_0;C)$. From discussions in the previous sections it follows immediately that the operator T is atomic.

In the theorem below we use a (simplified) notation from (13), namely, we set

$$^{\varphi}x^0(t) := {}^{\varphi}x(t) \quad\text{and}\quad {}^{\varphi}x_t^0 := {}^{\varphi}x_t,$$

where φ is the initial function (at $\sigma=0$) for the solution ${}^{\varphi}x(t)$.

Theorem 5.1. *Assume that $H:[0,\infty)\times C\to\mathbf{R}^d$, $G:[0,\infty)\times C\to\mathbf{R}^{d\times m}$ are jointly continuous, globally Lipschitz in the second variable and A-periodic in t functionals. Then the following statements are equivalent:*

1) *${}^{\varphi}x(t)$ is a strong A-periodic in law solution to (10);*
2) *$\varphi\in L^0(\Omega,\mathcal{F}_0;C)$ is a fixed point of the monodromy operator T associated with (10).*

PROOF: We rewrite (23) as follows:

$$x_{t+A}(\omega) = x_t(\theta(A,\omega)) \text{ a. s.} \tag{25}$$

for all $t\in[0,\infty)$. Assume first that ${}^{\varphi}x(t)$ is a strong A-periodic in law solution. Setting $t=0$ and using $g=\theta^{-1}(A,\cdot)$ we obtain $T_g({}^{\varphi}x_A)={}^{\varphi}x_0=\varphi$ so that

$$T\varphi = (T_g\circ U(A))\varphi = T_g({}^{\varphi}x_A) = \varphi.$$

Conversely, if $\varphi\in L^2(\Omega,\mathcal{F}_0;C)$ is a fixed point of the monodromy operator T, then using the generalized cocycle property (20) we get

$$\begin{array}{rcllcl}
T_g({}^{\varphi}x_{t+A}) &=& (T_g\circ U(t+A))\varphi && & \\
&=& (T_g\circ U(t+A,A)\circ U(A))\varphi && & \\
&=& (T_g\circ U(t+A,A)\circ T_g^{-1})\circ(T_g\circ U(A))\varphi &=& U(t)T\varphi \\
&=& U(t)\varphi && = & {}^{\varphi}x_t
\end{array}$$

a.s. for any $t\in[0,\infty)$. Hence the solution ${}^{\varphi}x_t$ is A-periodic in law. □

Remark 5.1. *This result is only based on the generalized cocycle property and therefore can be viewed as a property of any generalized random dynamical system. For instance, one can replace the standard Brownian shift in (7) by another measure preserving shift θ defined on Ω (or its extension) and satisfying (6) for some σ-algebrae $\mathcal{F}_t$, not necessarily generated by $W(t)$. In addition, the Brownian motion $W(t)$ should preserve the property of being a Brownian motion on the extended probability space and again satisfy (5). Such a situation occurs, for instance, when $\mathcal{F}_t$ is generated by a pair $(W(t),V(t))$, where $V(t)$ is another stochastic process independent of $W(t)$.*

In the case of non-Brownian shift, a solution $x(t)$ to (10) satisfying (23) with the new θ will be a particular case of a weak A-periodic in law solution. The corresponding monodromy operator T can be called the weak monodromy operator associated with the shift θ, while its fixed points are usually referred to as weak fixed points. Theorem 5.1 reads therefore as follows: ${}^{\varphi}x(t)$ is a weak A-periodic solution to (10), if and only if φ is a weak fixed point of the corresponding monodromy operator T (which of course is still atomic).

Remark 5.2. *If we look for stationary solutions, then we have to verify that $\varphi\in L^2(\Omega,\mathcal{F}_0;C)$ is a common fixed point for all the operators $T^A := T_{\theta^{-1}(A,\cdot)}\circ U(A,0)$ for all $A>0$.*

Remark 5.3. *Theorem 5.1 is of course valid for any Lebesgue space $L^p(\Omega, \mathcal{F}_0; C)$ with an arbitrary $p > 0$ provided that the assumptions on the coefficients ensure solvability of the stochastic equation (i.e. existence of the flow) in this space. Moreover, using the property of locality and following the proof of Corollary 4.1 one can show that for any $p > 0$ the monodromy operator can be continuously extended from the initial domain $L^p(\Omega, \mathcal{F}_0; C)$ to the space $L^0(\Omega, \mathcal{F}_0; C)$.*

5.2. **Measure solutions.** Measure solutions are widely used in the theory of stochastic differential equations as a generalization of the notion of a strong solution. This generalization is of importance when strong solutions do not exist. We study here periodic and stationary measure solutions, also known as invariant measures for the solution flow. The classical result based on the Markov-Kakutani theorem (see e.g. [2]) says that any regular (i.e. Carathéodory) random dynamical system on a compact metric space has at least one invariant measure. Our aim is to extend this result to the case of generalized (non-Carathéodory) random dynamical systems on a compact metric space. Mathematically, it means that we use local and atomic operators instead of the Nemytskiĭ operators and nonlinear weighted shifts, respectively. In addition, we have to replace the relationship (23), which defines strong solutions, by another one defined on an extended domain consisting of the so-called Young measures. This extension as well as the definition of an invariant measure (measure solution) can be done almost automatically when the flow is regular (see Subsection 6.1), but implementing a similar idea in the singular flow case courses much more troubles.

Postponing the formal construction of the extended solution flow until the next section, we give now an informal description of a periodic measure solution.

Let X be a Polish space, $\mathcal{M}(X)$ be the associated space of all signed measures on X. We assume $L^0(\Omega, \Sigma, \mathbf{P}; X)$ to be (topologically) identified with a subspace of the space $L^\infty(\Omega, \Sigma, \mathbf{P}; \mathcal{M}(X))$ via the embedding $\alpha(\cdot) \mapsto \delta_{\alpha(\cdot)}$, where δ_c is the Young measure measure (i.e. its marginal is $\mathbf{P}$) supported on the graph of c. If no strong periodic solution exists, we may hope to find approximating solutions φ_n satisfying $T\varphi_n \approx \varphi_n$ in the sense of a weaker topology. Assuming that φ_n converge to a random measure μ in this topology and that the operator T can be continuously extended to the closure of the space L^0 in this topology, we can regard the limit measure μ as a measure solution of the operator equation $T\varphi = \varphi$. As the relation $T\varphi_n \approx \varphi_n$ means "φ_n is nearly periodic in law", the limit measure should be periodic in law.

In the next section we will formalize this idea for the case when X is a compact metric space.

6. Young spaces and atomic operators

Throughout this section we assume X to be a compact metric space, and $\mathcal{B}_X$ to be its Borel σ-algebra. The main result states that atomic operators can be extended to the space of Young measures associated with X. The result is crucial for the existence theorem proved in the last section.

We will use the notation from [2]:

$\mathcal{M}(X)$ (resp. $\mathcal{P}(X)$) is the Banach space of signed (resp. probability) measures on $(X,\ \mathcal{B}_X)$ of finite total variation (i.e. $\mathcal{M}(X) = C(X)'$);

$L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{M}(X))$ (resp. $L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{P}(X))$) is the space of $\mathbf{P}$-essentially bounded $\mathcal{M}(X)$-valued (resp. $\mathcal{P}(X)$-valued) measurable (with respect to the $*$-weak topology on $\mathcal{M}(X)$) functions.

Clearly,

$$L^1(\Omega,\Sigma,\mathbf{P};C(X))' = L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{M}(X)). \tag{26}$$

The set $\mathcal{M}(\Omega,\Sigma,\mathbf{P};X)$ consists of all signed measures μ of finite variation on $(\Omega\times X,\ \Sigma\otimes\mathcal{B}_X)$, for which the marginal measure μ' on $(\Omega,\ \Sigma)$, defined by $\mu'(A) = \mu(A\times X)$ for every $A\in\Sigma$, is absolutely continuous with respect to $\mathbf{P}$.

The Young space $\mathcal{Y}(\Omega,\Sigma,\mathbf{P};X)$ consists of probability measures with marginal $\mathbf{P}$ on $(\Omega,\ \Sigma)$, that is $\mu(A\times X) = \mathbf{P}A$ for any $A\in\Sigma$.

Each $\mu\in\mathcal{M}(\Omega,\Sigma,\mathbf{P};X)$ can be factorized with respect to $\mathbf{P}$:

$$\mu(A) = \int_\Omega\int_X 1_A(\omega,x)\mu_\omega(dx)\mathbf{P}(d\omega), \tag{27}$$

where μ_ω (called the *sample measure* or the *conditional measure* of μ with respect to $\mathbf{P}$) satisfies two conditions:

(1) for all $B\in\mathcal{B}_{\mathcal{X}}$, the map $\omega\mapsto\mu_\omega(B)$ is Σ-measurable,
(2) for $\mathbf{P}$-almost all $\omega\in\Omega$, the map $B\mapsto\mu_\omega(B)$ is a signed measure of finite variation on $(X,\ \mathcal{B}_X)$. If, in addition, $\mu\in\mathcal{Y}(\Omega,\Sigma,\mathbf{P};X)$, then the sample measure becomes a probability measure for almost all $\omega\in\Omega$.

One may symbolically rewrite (27) as

$$\mu(d\omega,dx) = \mu_\omega(dx)\mathbf{P}(d\omega). \tag{28}$$

Replacing μ by its sample measure, one can identify the space $\mathcal{M}(\Omega,\Sigma,\mathbf{P};X)$ with $L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{M}(X))$ and $\mathcal{Y}(\Omega,\Sigma,\mathbf{P};X)$ with $L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{P}(X))$.

The following useful result (Theorem 1.5.10a from [2]) follows from Alaoglu's theorem.

Lemma 6.1. *For a compact metric space X*

$$\mathcal{Y}(\Omega,\Sigma,\mathbf{P};X)\cong L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{P}(X)) \tag{29}$$

is a convex compact subset of $\mathcal{M}(\Omega,\Sigma,\mathbf{P};X)\cong L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{M}(X))$.

We will need also the following straightforward result.

Lemma 6.2. *The space $L^0(\Omega,\Sigma,\mathbf{P};X)$ can be embedded into $L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{M}(X))$ as a (not closed) topological subspace, if one identifies each $\alpha\in L^0(\Omega,\Sigma,\mathbf{P};X)$ with the respective sample measure $\delta_{\alpha(\omega)}$. In other words, the associated functional μ_α is given in this case by*

$$\mu_\alpha(f) = \int_\Omega f(\omega,\alpha(\omega))\mathbf{P}(d\omega). \tag{30}$$

The above embedding lemma identifies $L^0(\Omega,\Sigma,\mathbf{P};X)$ with an everywhere dense subspace of

$$L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{M}(X))\simeq\mathcal{Y}(\Omega,\Sigma,\mathbf{P};X).$$

In order to be able to study measure solutions, which are elements of the space $\mathcal{Y}(\Omega,\Sigma,\mathbf{P};X)$, we first have to extend the operator T from $L^0(\Omega,\Sigma,\mathbf{P};X)$ to $\mathcal{Y}(\Omega,\Sigma,\mathbf{P};X)$.

In the case of a nonlinear weighted shift (in other words, in the case of the random dynamical systems in the sense of L. Arnold [2]) this extension procedure is quite simple and explicit. It is described in the next subsection.

6.1. **Extension of a nonlinear weighted shift.** The nonlinear weighted shift T is a superposition of a Nemytskiĭ operator N_f generated by a Carathédory function f, and a shift g, i. e. $T = N_f \circ T_g$.

Treating the factors separately, we extend the operator

$$T_g : \ L^0(\Omega, \Sigma, \mathbf{P}; X) \to L^0(\Omega, g^{-1}(\Sigma), \mathbf{P}; X)$$

to a continuous (in the $*$-weak topology of measures) operator

$$\hat{T}_g : \ \mathcal{Y}(\Omega, \Sigma, \mathbf{P}; X) \to \mathcal{Y}(\Omega, g^{-1}(\Sigma), \mathbf{P}; X)$$

by setting

$$(\hat{T}_g \mu_\omega)(dx) = \mu_{g\omega}(dx),$$

where we use the sample measure representation.

Since the Nemytskiĭ operator

$$N_f : \ L^0(\Omega, g^{-1}(\Sigma), \mathbf{P}; X) \to L^0(\Omega, \Sigma, \mathbf{P}; X)$$

is generated by a Carathéodory function f, the operator N extends to the space of Young measures with the help of the following explicit formula

$$\hat{N}_f \nu = f_{\#} \nu, \tag{31}$$

where $(f_{\#}\nu)(A) := \nu(f^{-1}(A))$ for all $A \in \Sigma \otimes \mathcal{B}_X$. As f is Carathéodory, the operator $\hat{N}_f$ easily seen to be continuous in the $*$-weak topology of measures.

The composition of $\hat{N}_f$ with $\hat{T}_g$ gives a continuous extension $\hat{T}$ of the nonlinear weighted shift $T = N_f \circ T_g$. Since X is compact, the Young space $\mathcal{Y}(\Omega, \Sigma, \mathbf{P}; X)$ will be compact too, and this will guarantee the existence of a fixed point of $\hat{T}$ and hence existence of a measure solution for T. It is a well-known procedure used in the case of regular random dynamical systems (see [2]).

Unfortunately, singular random dynamical systems does not fit into this scheme. The main reason is that continuous local operators can be non-Carathéodory, so that the direct extension formula (31) cannot be applied.

In the next subsection we construct the required extension of an arbitrary continuous atomic operator. For this purpose we recall that $\mathcal{Y}(\Omega, \Sigma, \mathbf{P}; X)$ can be naturally associated with a subset of $L^\infty(\Omega, \Sigma, \mathbf{P}; \mathcal{M}(X))$ (i.e. to each Young measure μ one can associate a unique Borel family of disintegrated probability measures $\{\mu_\omega\}_{\omega\in\Omega}$), while there is a natural duality between $L^1(\Omega, \Sigma, \mathbf{P}; C(X))$ and $L^\infty(\Omega, \Sigma, \mathbf{P}; \mathcal{M}(X))$ given by

$$\langle f, \mu \rangle = \int_\Omega \int_X f(\omega, x) \mu_\omega(dx) \mathbf{P}(d\omega), \tag{32}$$

where $\mu \in L^\infty(\Omega, \Sigma, \mathbf{P}; \mathcal{M}(X))$ and $f \in L^1(\Omega, \Sigma, \mathbf{P}; C(X))$.

6.2. **Extension of a general atomic operator.** Assume that an atomic operator

$$T : \ L^0(\Omega, \Sigma, \mathbf{P}; X) \to L^0(\Omega, \Sigma, \mathbf{P}; X)$$

is given. Recall that we may identify $L^0(\Omega, \Sigma, \mathbf{P}; X)$ with a subset of $\mathcal{Y}(\Omega, \Sigma, \mathbf{P}; X)$. We are looking for a continuous operator

$$\hat{T} : \ \mathcal{Y}(\Omega, \Sigma, \mathbf{P}; X) \to \mathcal{Y}(\Omega, \Sigma, \mathbf{P}; X)$$

the restriction of which to $L^0(\Omega, \Sigma, \mathbf{P}; X)$ coincides with T.

The basic idea of our construction is to show that any continuous in measure nonlinear atomic operator T: $L^0(\Omega,\Sigma,\mathbf{P};X)\to L^0(\Omega,\Sigma,\mathbf{P};X)$ gives rise to a *linear* bounded operator $\tilde{T}: L^1(\Omega,\Sigma,\mathbf{P};C(X))\to L^1(\Omega,\Sigma,\mathbf{P};C(X))$. The adjoint operator $\tilde{T}'$ will be thus weakly continuous linear operator acting in the space $L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{M}(X))$. The operator $\tilde{T}$ will be constructed in such a way that $\tilde{T}'$ will coincide with T when restricted to the (topological) subspace $L^0(\Omega,\Sigma,\mathbf{P};X)$. If it is the case, then we can set $\hat{T}$ to be the restriction of $\tilde{T}'$ to the set of Young measures $\mathcal{Y}(\Omega,\Sigma,\mathbf{P};X)$. The basic idea of our construction is to use the representation $T=N\circ T_F$ from Theorem 3.1 which enables us to construct $\tilde{N}$ and $\tilde{T_F}$ separately and to put $\tilde{T}=\tilde{T_f}\circ\tilde{N}$.

We first describe the construction of $\tilde{N}$.

6.2.1. *Constructing* $\tilde{N}$. Let

$$N:\ L^0(\Omega,F\Sigma,\mathbf{P};X)\to L^0(\Omega,\Sigma,\mathbf{P};X) \tag{33}$$

be a continuous in measure local operator. We are going first to define a linear bounded operator

$$\tilde{N}:\ L^1(\Omega,\Sigma,\mathbf{P};C(X))\to L^1(\Omega,F\Sigma,\mathbf{P};C(X)). \tag{34}$$

Let $f\in L^1(\Omega,\Sigma,\mathbf{P};C(X))$ be arbitrary. This function can be identified with a Carathéodory function $f:\Omega\times X\to\mathbf{R}$, for which

$$\sup_{x\in X}|f(\cdot,x)|\in L^1(\Omega,\Sigma,\mathbf{P};\mathbf{R}). \tag{35}$$

The operator

$$(h_f)(\cdot)=\mathbf{E}\{f(\cdot,(N\alpha)(\cdot))\mid F\Sigma\}, \tag{36}$$

where $\mathbf{E}\{\cdot\mid F\Sigma\}$ denotes the conditional expectation with respect to $F\Sigma$, will be local and continuous as an operator from $L^0(\Omega,F\Sigma,\mathbf{P};X)$ to $L^1(\Omega,\Sigma,\mathbf{P};X)$. Indeed, as $N\alpha\in L^0(\Omega,\Sigma,\mathbf{P};X)$ and X is compact, and due to (35)

$$|f(\cdot,(N\alpha)(\cdot))|\le\sup_{x\in X}|f(\cdot,x)|\in L^1(\Omega,\Sigma,\mathbf{P};X),$$

so that $f(\cdot,(N\alpha)(\cdot))\in L^1(\Omega,\Sigma,\mathbf{P};X)$, and the operator h_f is well defined.

To show locality of h_f, we pick arbitrary $\alpha,\beta\in L^0(\Omega,F\Sigma,\mathbf{P};X)$ and $A\in F\Sigma$ such that $\alpha(\omega)=\beta(\omega)$ a.s. on A. As both N and the Nemytskiĭ operator generated by the function f are local, we have that

$$f(\omega,(N\alpha)(\omega))=f(\omega,(N\beta)(\omega))$$

a. s. on A, or in other words,

$$\mathbf{1}_A(f\circ N)\alpha=\mathbf{1}_A(f\circ N)\beta$$

a.e. Hence

$$\begin{aligned}\mathbf{1}_A\mathbf{E}\{(f\circ N)\alpha\mid F\Sigma\}&=\mathbf{E}\{\mathbf{1}_A(f\circ N)\alpha\mid F\Sigma\}=\mathbf{E}\{\mathbf{1}_A(f\circ N)\beta\mid F\Sigma\}\\&=\mathbf{1}_A\mathbf{E}\{(f\circ N)\beta\mid F\Sigma\}\end{aligned}$$

a.s., which proves locality of h_f.

To prove continuity of h_f, consider an arbitrary sequence

$$\{\alpha_n\}_{n=1}^\infty\subset L^0(\Omega,F\Sigma,\mathbf{P};X)$$

such that $\alpha_n\to\alpha_0$ in probability as $n\to\infty$, then $N\alpha_n\to N\alpha_0$ in $L^0(\Omega,\Sigma,\mathbf{P};X)$ in view of continuity of N. Therefore, $(f\circ N)\alpha_n\to(f\circ N)\alpha_0$ in $L^0(\Omega,\Sigma,\mathbf{P};X)$ as $n\to$

∞ since f, being a Carathéodory function, generates a continuous in probability operator. Due to (35) the latter convergence implies in fact also convergence in the space $L^1(\Omega, \Sigma, \mathbf{P}; X)$, which also gives

$$h_f(\alpha_n) \to h_f(\alpha_0) \text{ in the space } L^1(\Omega, F\Sigma, \mathbf{P}; X)$$

as $n \to \infty$, and hence proves the continuity of h_f.

Since $h_f : L^0(\Omega, F\Sigma, \mathbf{P}; X) \to L^1(\Omega, F\Sigma, \mathbf{P}; X)$ is local and continuous, then there exists a unique (up to a.s. equivalence for every $x \in X$) Carathéodory function $\gamma_f : \Omega \times X \to \mathbf{R}$, which is $F\Sigma$-measurable in ω for all x and represents the operator h_f:

$$(h_f\alpha)(\omega) = \gamma_f(\omega, \alpha(\omega)) \text{ a. s.}$$

for any $\alpha \in L^0(\Omega, F\Sigma, \mathbf{P}; X)$ (see [11]). Moreover, we claim that

$$\gamma_f \in L^1(\Omega, F\Sigma, \mathbf{P}; C(X)). \tag{37}$$

To prove (37), we put $\hat{\gamma}(\omega) = \max_{x\in X} |\gamma_f(\omega, x)|$, which is a $F\Sigma$-measurable a. s. finite function. Since the set

$$\{(\omega, x) : |\gamma_f(\omega, x)| = \hat{\gamma}(\omega)\}$$

is $F\Sigma \otimes \mathcal{B}_{\mathcal{X}}$-measurable, there exists a $F\Sigma$-measurable function $\beta : \Omega \to X$, for which $|\gamma_f(\omega, \beta(\omega))| = \hat{\gamma}(\omega)$ a. s. On the other hand, $h_f\beta \in L^1(\Omega, F\Sigma, \mathbf{P}; X)$. From this we see that

$$\max_{x\in X} |\gamma_f(\omega, x)| = |\gamma_f(\omega, \beta(\omega))| = |(h_f\beta)(\omega)| \text{ a. s.},$$

so that (37) is proven.

Summing up, we have shown that the operator $\tilde{N}$ defined by

$$\tilde{N}f = \gamma_f$$

acts from $L^1(\Omega, \Sigma, \mathbf{P}; C(X))$ to $L^1(\Omega, F\Sigma, \mathbf{P}; C(X))$. This operator is obviously linear. Moreover, it is bounded since

$$\max_{x\in X} |\gamma_f(\cdot, x)| = |(h_f\beta)(\cdot)| = |\mathbf{E}\{f(\cdot, (N\beta)(\cdot)) \mid F\Sigma\}| \le \mathbf{E}\{\max_{x\in X} |f(\cdot, x)| \mid F\Sigma\}$$

and $\|\tilde{N}f\|_{L^1} \le \|f\|_{L^1}$.

6.2.2. *Constructing* $\tilde{T}_F$. A generalized shift operator is given by

$$T_F : L^0(\Omega, \Sigma, \mathbf{P}; X) \to L^0(\Omega, F\Sigma, \mathbf{P}; X), \tag{38}$$

where $F : \Sigma \to \Sigma$ is a σ-homomorphism. To define the operator

$$\tilde{T}_F : L^1(\Omega, F\Sigma, \mathbf{P}; C(X)) \to L^1(\Omega, \Sigma, \mathbf{P}; \mathcal{C}(X)) \tag{39}$$

we introduce the Radon derivative

$$\psi := \frac{dP_F}{dP} \in L^1(\Omega, \Sigma, \mathbf{P}; \mathbf{R}), \tag{40}$$

where $P_F(A) := P(FA)$. Clearly,

$$\mathbf{E}\psi\gamma = \mathbf{E}T_F(\gamma) \tag{41}$$

for any $\gamma \in L^\infty(\Omega, \Sigma, \mathbf{P}; \mathbf{R})$, where $\mathbf{E}$ stands for the expectation.

Let us take arbitrary $B_1, B_2 \in \Sigma$, for which $FB_i = A$, $i = 1, 2$. We verify that

$$\psi \mathbf{1}_{B_1} = \psi \mathbf{1}_{B_2} \text{ a. s.} \tag{42}$$

In fact, setting $\beta = \operatorname{sign}(\psi(\mathbf{1}_{B_1} - \mathbf{1}_{B_2}))$ we obtain

$$\mathbf{E}\left(|\psi(\mathbf{1}_{B_1} - \mathbf{1}_{B_2})|\right) = \mathbf{E}\left(\psi(\mathbf{1}_{B_1} - \mathbf{1}_{B_2})\beta\right) = \mathbf{E}\left(T_F((\mathbf{1}_{B_1} - \mathbf{1}_{B_2})\beta)\right) = 0$$

which proves (42).

The equality (42) justifies the following definition: given $A \in F\Sigma$ we set

$$S_F(\mathbf{1}_A) := \psi \mathbf{1}_B,$$

if $FB = A$. This mapping has two important properties

$$\mathbf{E}\left(|S_F(\mathbf{1}_A)|\right) \leq \mathbf{E}\mathbf{1}_A \tag{43}$$

and

$$\operatorname{supp} S_F(\mathbf{1}_{A_1}) \cap \operatorname{supp} S_F(\mathbf{1}_{A_2}) = \emptyset, \tag{44}$$

if $A_1 \cap A_2 = \emptyset$, where $\operatorname{supp} u := \{u \neq 0\}$ and the equalities are understood up to $\mathbf{P}$-negligible sets. The relationship (43) follows immediately from the calculation

$$\mathbf{E}\left(|\psi \mathbf{1}_B|\right) = \mathbf{E}\left(\psi \mathbf{1}_B\right) = \mathbf{E}\mathbf{1}_A,$$

where $FB = A$. To prove (44) we first note that $\operatorname{supp} S_F(\mathbf{1}_{A_i}) \subset B_i$, where $FB_i = A_i$, $i = 1, 2$. One has also

$$\mathbf{E}\left(|S_F(\mathbf{1}_{A_1})\mathbf{1}_{B_2}|\right) = \mathbf{E}\mathbf{1}_{A_1}\mathbf{1}_{A_2} = 0.$$

Therefore, $S_F(\mathbf{1}_{A_1})\mathbf{1}_{B_2} = 0$ a. s. and, thus,

$$\operatorname{supp} S_F(\mathbf{1}_{A_1}) \cup B_2 = \emptyset.$$

Since $\operatorname{supp} S_F(\mathbf{1}_{A_2}) \subset B_2$, we obtain (44).

Now we are ready to define the operator $\tilde{T}_F$. Considering first the case

$$g(\omega, x) = \sum_{i=1}^{\infty} \mathbf{1}_{A_i}(\omega) g_i(x),$$

where A_i are disjoint sets in $F\Sigma$ and $g_i \colon X \to \mathbf{R}$, $i \in \mathbf{N}$, are arbitrary continuous functions, we set

$$\tilde{T}_F(g) := \sum_{i=1}^{\infty} S_F(\mathbf{1}_{A_i}) g_i.$$

Then

$$\|\tilde{T}_F(g)\|_{L^1(C)} = \sum_{i=1}^{\infty} \|S_F(\mathbf{1}_{A_i})\|_{L^1} \|g_i\|_C \leq \sum_{i=1}^{\infty} \|\mathbf{1}_{A_i}\|_{L^1} \|g_i\|_C = \|g\|_{L^1(C)}.$$

Since in view of separability of $C(X)$ the set of such g is dense in $L^1(\Omega, \Sigma, \mathbf{P}; C(X))$, then there exists the unique bounded linear extension of the above defined linear operator $\tilde{T}_F$ to the whole space $L^1(\Omega, \Sigma, \mathbf{P}; C(X))$.

6.2.3. *General extension theorem.* Summarizing the two previous paragraphs we define $\tilde{T}$ by setting

$$\tilde{T} = \tilde{T}_F \circ \tilde{N}. \tag{45}$$

Clearly, in this way $\tilde{T} : L^1(\Omega, \Sigma, \mathbf{P}; C(X)) \to L^1(\Omega, \Sigma, \mathbf{P}; C(X))$ is a bounded linear operator.

The main result of the section is the following extension theorem.

Theorem 6.1. *Let X be a compact metric space and $L^0(\Omega, \Sigma, \mathbf{P}; X)$ be (topologically) identified with a subspace of the space $L^\infty(\Omega, \Sigma, \mathbf{P}; \mathcal{M}(X)) \cong \mathcal{M}(\Omega, \Sigma, \mathbf{P}; X)$. Then the operator $\tilde{T}' = \tilde{N}' \circ \tilde{T}'_F : \mathcal{M}(\Omega, \Sigma, \mathbf{P}; X) \to \mathcal{M}(\Omega, \Sigma, \mathbf{P}; X)$ is continuous (in the $*$-weak topology of measures) and its restriction to the subspace $L^0(\Omega, \Sigma, \mathbf{P}; X)$ coincides with T.*

PROOF: We will show that

$$\tilde{N}' : \mathcal{M}(\Omega, F\Sigma, \mathbf{P}; X) \to \mathcal{M}(\Omega, \Sigma, \mathbf{P}; X)$$

and

$$\tilde{T}'_F : \mathcal{M}(\Omega, \Sigma, \mathbf{P}; X) \to \mathcal{M}(\Omega, F\Sigma, \mathbf{P}; X)$$

extend the operators

$$N : L^0(\Omega, F\Sigma, \mathbf{P}; X) \to L^0(\Omega, \Sigma, \mathbf{P}; X)$$

and

$$T_F : L^0(\Omega, \Sigma, \mathbf{P}; X) \to L^0(\Omega, F\Sigma, \mathbf{P}; X),$$

respectively. The property of their continuity is trivial.

For an arbitrary $f \in L^1(\Omega, \Sigma, \mathbf{P}; C(X))$ one has

$$\begin{array}{rclcl} \langle f, \tilde{N}'\delta_\alpha\rangle & = & \langle \tilde{N}f, \delta_\alpha\rangle & = & \langle \gamma_f, \delta_\alpha\rangle \\ & = & \mathbf{E}h_f\alpha & = & \mathbf{E}f(\cdot, (N\alpha)(\cdot)) \\ & & & = & \langle f, \delta_{N\alpha}\rangle, \end{array}$$

where γ_f is a Carathéodory function used in the definition of the operator $\tilde{N}$. Thus, we have verified that $\tilde{N}'(\delta_\alpha) = \delta_{N\alpha}$.

To prove the desired property of the operator $\tilde{T}'_F$ we take a Carathéodory function g: $\Omega \times X \to \mathbf{R}$ of the form $g(\omega, x) = \sum \mathbf{1}_{A_i}(\omega) g_i(x)$, where A_i are disjoint sets in $F\Sigma$ and $g_i : X \to \mathbf{R}$ are continuous. Then, applying (41) for B_i, for which $FB_i = A_i$, we have

$$\begin{array}{rclcl} \langle g, \tilde{T}'_F\delta_\alpha\rangle & = & \langle \tilde{T}_F g, \delta_\alpha\rangle & = & \langle \sum S_F(\mathbf{1}_{A_i}) g_i, \delta_\alpha\rangle \\ & = & \mathbf{E}\left(\sum S_F(\mathbf{1}_{A_i}) g_i \circ \alpha\right) & = & \sum_{i=1}^\infty \mathbf{E}\left(S_F(\mathbf{1}_{A_i}) g_i \circ \alpha\right) \\ & = & \sum_{i=1}^\infty \mathbf{E}\left(\psi(\mathbf{1}_{B_i}) g_i \circ \alpha\right) & = & \sum_{i=1}^\infty \mathbf{E}\left((\mathbf{1}_{A_i}) g_i \circ T_F\alpha\right) \\ & = & \mathbf{E}\sum_{i=1}^\infty \left((\mathbf{1}_{A_i}) g_i \circ T_F\alpha\right) & = & \mathbf{E}g(\cdot, (T_F\alpha)(\cdot)). \end{array})$$

Thus, $\langle g, \tilde{T}'_F\delta_\alpha\rangle = \langle g, T_F\alpha\rangle$ for any g defined above. Since the set of such g is dense in $L^1(\Omega, F\Sigma, \mathbf{P}; C(X))$ and the both functionals are bounded, we obtain that $\tilde{T}'_F(\delta_\alpha) = \delta_{T_F\alpha}$.

The restriction of the operator $\tilde{T}'$ to the set of Young measures will be denoted by $\hat{T}$. Clearly, $\hat{T}$ is continuous (with respect to $*$-weak topology of measures) and affine. Since $\tilde{T}'(\delta_\alpha) = \delta_{T\alpha}$ and the set of all random Young measures of the form δ_α is dense in $L^\infty(\Omega, \Sigma, \mathbf{P}; \mathcal{M}(X))$ in the $*$-weak topology, then we have also that $\hat{T}(\mathcal{Y}(\Omega, \Sigma, \mathbf{P}; X)) \subset \mathcal{Y}(\Omega, \Sigma, \mathbf{P}; X)$.

Remark 6.1. *Theorem 6.1 can naturally be generalized to the case when* $T: L^0(\Omega, \Sigma, \mathbf{P}; X_1) \to L^0(\Omega, \Sigma, \mathbf{P}; X_2)$, *where* X_1 *and* X_2 *are different compact metric spaces. Indeed, in this case we can put* $X = X_1 \times X_2$ *and redefine the Nemytskiĭ operator* N *in the representation (1) by setting*

$$(N_1\alpha)(\omega) := \{x_1\} \times (N(\pi\alpha))(\omega) \in X_1 \times X_2,$$

where $x_1 \in X_1$ *is an arbitrarily chosen point (independent of* α*) and* $\pi(x_1, x_2) := x_1$. *It is then straightforward that the operator* $T_1 := N_1 \circ T_f$ *defined in this way satisfies all the assumptions of Theorem 6.1.*

6.3. **Existence of periodic and stationary measure solutions.** Now we are in a position to define formally the notion of a stationary and periodic measure solutions to SFDEs as invariant measures for the respective generalized random dynamical systems.

Definition 6.1. *A one parameter family of Young measures* $\{\mu(t, \cdot)\}_{t\geq 0}$ *is called* A*-periodic in law, if*

$$\hat{T}_{\theta(A,\cdot)}\mu(t, \cdot) = \mu(t + A, \cdot) \tag{46}$$

for all $t \geq 0$, *where the shift* θ *is described in the definition of the generalized dynamical system (for instance,* θ *is the standard Wiener shift).*

Clearly, this definition is consistent with that of a strong periodic solution in the sense that, if one has $\mu(t, \cdot) = \delta_{x(t,\cdot)}$ for each $t \geq 0$, then the process x is strong periodic in the sense of (23). Also, if the equality (46) holds for any positive A, one obtains Young measures which are stationary in law.

Now we describe how one finds measure solutions. Roughly speaking we exploit the operator $\hat{T}$ instead of the operator T from Theorem 5.1.

Definition 6.2. *Let* $U(\tau)$, $\tau \in \mathcal{A}$, *be a generalized stochastic dynamical system (i.e. a family of local, continuous mappings (22) on a compact metric space* X, *satisfying the generalized cocycle property (21)). A measure* $\mu_0 \in \mathcal{Y}(\Omega, \Sigma_0, \mathbf{P}; X)$ *is called invariant for the family* $U(\tau)$, *if it satisfies*

$$\hat{U}(\tau)\mu_0 = \hat{T}_{\theta(\tau,\cdot)}\mu_0, \quad \tau \in \mathcal{A}. \tag{47}$$

If $U(\tau)$ *is generated by some stochastic differential equation, then* $U(\tau)\mu_0$ *is referred to as the respective measure solution (i.e. if* $\mathcal{A} = \mathbf{R}^+$, *then this is a stationary measure solution, while if* $\mathcal{A} = A\mathbf{N}$, *then this is a periodic in law measure solution).*

Now we are able to formulate the main result of this paper which follows from Theorem 6.1 via the well-known Markov-Kakutani fixed point theorem [7, p. 456].

Theorem 6.2. *Let* X *be a compact metric space and* $L^0(\Omega, \Sigma, \mathbf{P}; X)$ *be (topologically) identified with a subspace of the space* $L^\infty(\Omega, \Sigma, \mathbf{P}; \mathcal{M}(\Omega, \Sigma, \mathbf{P}; X)$ *by* $\alpha(\cdot) \mapsto \delta_{\alpha(\cdot)}$. *Let*

$$T: \ L^0(\Omega, \Sigma, \mathbf{P}; X) \to L^0(\Omega, \Sigma, \mathbf{P}; X)$$

be a continuous atomic operator. Then the equation $Tx = x$ *has at least one measure solution (invariant measure)* $\mu \in \mathcal{Y}(\Omega, \Sigma, \mathbf{P}; X)$, *i.e. such that* $\hat{T}\mu = \mu$.

As an immediate corollary we obtain the following result.

Corollary 6.1. *Let X be a compact metric space and $L^0(\Omega,\Sigma,\mathbf{P};X)$ be (topologically) identified with a subspace of the space $L^\infty(\Omega,\Sigma,\mathbf{P};\mathcal{M}(\Omega,\Sigma,\mathbf{P};X)$ by means of $\alpha(\cdot)\mapsto\delta_{\alpha(\cdot)}$. Let*

$$U(\tau):\ L^0(\Omega,\Sigma_0,\mathbf{P};X)\to L^0(\Omega,\Sigma_\tau,\mathbf{P};X)\quad(\tau\in\mathcal{A})$$

be a generalized random dynamical system. Then the family $U(\tau)$ admits at least one invariant measure μ_0.

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E-mail address: `matap@imf.nlh.no`

E-mail address: `stepanov@cibs.sns.it`

Institutt for matematiske fag, postboks 5035, N-1432 Ås-NLH, Norway

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy