On the n-dimensional Dirichlet problem for isometric maps

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Abstract

We exhibit *explicit* Lipschitz maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ which have almost everywhere orthogonal gradient and are equal to zero on the boundary of a cube. We solve the problem by induction on the dimension n.

1 Introduction

We consider in the general $n-$dimensional case ($n > 1$) the *nonlinear system of pde's*

$$Du^t Du = I\,, \tag{1}$$

where Du^t denotes the *transpose* matrix of the gradient Du of a map $u : \mathbb{R}^n \to \mathbb{R}^n$, while I is the *identity matrix.* A map u satisfying (1) is said to be an *isometric map* or *rigid map* and its gradient is an *orthogonal matrix*; briefly as usual we write $Du \in O(n)$.

To the system (1) we associate the homogeneous boundary condition $u = 0$ on the boundary of a bounded open set of $\mathbb{R}^n$. The Dirichlet problem that we obtain is critical; i.e., it is incompatible with classical solutions. In fact any isometric map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ *of class* C^1 on an open connected set Ω of $\mathbb{R}^n$ is *affine* by the classical Liouville theorem, and it therefore cannot be equal to zero on its boundary $\partial\Omega$. Even more: since its invertibility, it cannot be equal to zero in more than a single point. We can then consider *Lipschitz continuous* maps $u : \mathbb{R}^n \to \mathbb{R}^n$, satisfying the system (1) almost everywhere; then, if u is equal to zero at the boundary $\partial\Omega$ it must be not differentiable at *any* neighbourhood of *any* boundary point, thus presenting a *fractal behaviour* at the boundary.

In this paper we find an *explicit* Lipschitz solutions to the differential problem

$$\begin{cases} Du(x) \in O(n) & \text{a.e. } x \in Q \\ u(x) = 0 & x \in \partial Q\,, \end{cases}$$

where $Q = (0,1)^n$ is the unit cube and $O\,(n)$ stands, as said above, for the set of *orthogonal matrices* in $\mathbb{R}^{n\times n}$.

The study of *differential inclusions* of the form

$$\begin{cases} Du(x) \in E & \text{a.e. } x \in \Omega \\ u(x) = u_0(x) & x \in \partial\Omega\,, \end{cases}$$

where $E \subset \mathbb{R}^{N\times n}$, $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and u_0 is a given map, has received considerable attention. In the vectorial case $n, N \geq 2$, general theories of existence have been developed either via *the Baire category method* (see Dacorogna–Marcellini [3], [4], [5]) or via *the convex integration method by Gromov* (see Müller–Sverak [9]). These methods are purely existential and do not give a way of constructing explicit solutions. In parallel, for some special problems mostly related to the case when E is the set of orthogonal matrices, some solutions were provided in a constructive way. This started with the work of Cellina–Perrotta [1] when $n = N = 3$ and $u_0 = 0$, Dacorogna–Marcellini–Paolini [6], [7] when $n = N = 2$ or $n = N = 3$ and Iwaniec–Verchota–Vogel [8] for $n = N = 2$. In this context there are also some related unpublished arguments by R. D. James for $n = N = 2$. In [7] the connection between this problem with *isometric immersions* and *origami* has been made. Moreover in [7] we also dealt with inhomogeneous linear boundary data.

In the present article we give a self contained and purely analytical construction *in any dimension.* Despite its generality our proof is shorter than the existing ones which were, however, restricted to the cases $n = 2, 3$. We first solve the problem by induction on the dimension in the half space $(0, \infty) \times \mathbb{R}^{n-1}$. We then get the solution to our problem by composing the solution in the half space with a map that sends the whole boundary of the unit cube in $\mathbb{R}^n$ to one of its faces. We should point out that our construction in fact solves the problem in a more precise way: instead of considering matrices in the whole of $O(n)$, we use only a finite number of them, namely *permutation matrices* whose non zero entries are ± 1.

2 The fundamental brick

Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \min\{t, 1 - t\}.$$

Then define $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h(x, y) = (h^1(x, y), h^2(x, y)) = \begin{cases} (x, f(y)) & \text{if } x \leq y, \\ (y, f(x)) & \text{if } x \geq y. \end{cases} \tag{2}$$

Finally we define a map $\phi_n\colon \mathbb{R}^n \to \mathbb{R}^n$, for $n = 2, 3, \cdots$, by induction on n

$$\begin{cases} \phi_2(x_1, x_2) = h(x_1, x_2) \\ \phi_{n+1}(x_1, x_2, \cdots, x_{n+1}) \\ = \big(\phi_n(h^1(x_1 - n + 1, x_{n+1}) + n - 1, x_2, \cdots, x_n), h^2(x_1 - n + 1, x_{n+1})\big). \end{cases}$$

More in details, ϕ_{n+1} can be written *as a composition* of the following maps

$$\begin{aligned}
(x_1, \cdots, x_{n+1}) &\mapsto (x_1 - n + 1, x_2, \cdots, x_{n+1}) \\
(y_1, \cdots, y_{n+1}) &\mapsto (h^1(y_1, y_{n+1}), y_2, \cdots, y_n, h^2(y_1, y_{n+1})) \\
(z_1, \cdots, z_{n+1}) &\mapsto (z_1 + n - 1, z_2, \cdots, z_{n+1}) \\
(w_1, \cdots, w_{n+1}) &\mapsto (\phi_n(w_1, \cdots, w_n), w_{n+1}).
\end{aligned}$$

We recall that $u\colon \mathbb{R}^n \to \mathbb{R}^n$ is called a *rigid map*, or equivalently an *isometric map*, if it is Lipschitz continuous and $Du(x) \in O(n)$ for almost every $x \in \mathbb{R}^n$; i.e., if u satisfies (1) for almost every $x \in \mathbb{R}^n$.

Theorem 1 (properties of ϕ_n) *The map $\phi_n\colon \mathbb{R}^n \to \mathbb{R}^n$, for every $n = 2, 3, \cdots$, satisfies the following properties*

(i) ϕ_n is a piecewise affine rigid map;

(ii) if $x_2, \cdots, x_{n+1} \in [0, 1]$ then

$$\phi_n(0, x_2, \cdots, x_n) = (0, f(x_2), \cdots, f(x_n));$$

(iii) on the cube $[n-1, n] \times [0, 1]^{n-1}$ the map ϕ_n is affine.

Proof. We will use the following properties of the map h defined in (2):

1) h is a piecewise affine rigid map;

2) if $y \geq 0$ and $x \leq 0$ then $h(x, y) = (x, f(y))$;

3) if $x \geq 1$ and $y \in [0, 1]$ then $h(x, y) = (y, 1 - x)$.

We prove the theorem by induction on n. In the case $n = 2$ the claims are direct consequences of the properties of h. We assume now that the theorem holds true for n, and we prove it for $n + 1$.

Claim (i) is a consequence of the fact that the composition of piecewise affine rigid maps is again a piecewise affine rigid map.

To prove (ii) we compute, for any $x_2, \cdots, x_{n+1} \in [0, 1]$,

$$\begin{aligned}
&\phi_{n+1}(0, x_2, \cdots, x_{n+1}) \\
&= (\phi_n(h^1(1-n, x_{n+1}) + n - 1, x_2, \cdots, x_n), h^2(1-n, x_{n+1}))\,;
\end{aligned}$$

since $1-n \leq 0 \leq x_{n+1}$, by property 2 of the function h we have $h(1-n, x_{n+1}) = (1-n, f(x_{n+1}))$, hence we continue

$$= (\phi_n(0, x_2, \cdots, x_n), f(x_{n+1}))$$

and by the induction hypothesis

$$= (0, f(x_2), \cdots, f(x_n), f(x_{n+1}))$$

which is the claim.

Let us conclude by proving (iii). Let $x_1 \in [n, n+1]$ and $x_2, \cdots, x_{n+1} \in [0, 1]$. We have

$$\begin{aligned}
&\phi_{n+1}(x_1, x_2, \cdots, x_{n+1}) \\
&= (\phi_n(h^1(x_1 - n + 1, x_{n+1}) + n - 1, x_2, \cdots, x_n), h^2(x_1 - n + 1, x_{n+1}));
\end{aligned}$$

since $x_1 - n + 1 \geq 1$ and $x_{n+1} \in [0, 1]$, by the property 3 of h we find that $h(x_1 - n + 1, x_{n+1}) = (x_{n+1}, n - x_1)$, hence

$$= (\phi_n(x_{n+1} + n - 1, x_2, \cdots, x_n), n - x_1);$$

since now $x_{n+1} + n - 1 \in [n - 1, n]$, by the induction hypothesis we conclude that ϕ_{n+1} is affine on this region. ■

3 The pyramid construction

Let us start with some notations. Let f be as in the previous section. For every $x = (x_1, \cdots, x_n)$ we order the real numbers $f(x_1), \cdots, f(x_n)$ so that

$$f(x_{i_1}) \leq f(x_{i_2}) \leq \cdots \leq f(x_{i_n}).$$

We then define $v\colon [0, 1]^n \to \mathbb{R}^n$ as

$$v(x) = (f(x_{i_1}), f(x_{i_2}), \cdots, f(x_{i_n})).$$

Note that for $v(x) = \big(v^1(x), \cdots, v^n(x)\big)$ we have

$$v^1(x) = \min_{i=1,\cdots,n} \{f(x_i)\}, \quad v^n(x) = \max_{i=1,\cdots,n} \{f(x_i)\}$$

$$v^k(x) = \max_{i_1,\cdots,i_{k-1}} \left[\min_{i \neq i_1,\cdots,i_{k-1}} \{f(x_i)\} \right], \ k = 2, \cdots, n-1,$$

in particular, when $n = 3$,

$$v^2(x) = \max\left[\min\{f(x_1), f(x_2)\}, \min\{f(x_1), f(x_3)\}, \min\{f(x_2), f(x_3)\}\right].$$

Theorem 2 (pyramid construction) *Let $Q = (0, 1)^n \subset \mathbb{R}^n$. The map $v\colon \bar{Q} \to \mathbb{R}^n$ defined above, has the following properties*

(i) v is a piecewise affine rigid map;

(ii) $v(Q) \subset (0, 1/2]^n \subset \{x \in \mathbb{R}^n : x_1 > 0\}$;

(iii) $v(\partial Q) \subset \{x \in \mathbb{R}^n : x_1 = 0\}$, meaning that $v^1 = 0$ on ∂Q.

Proof. The map v is constructed as the composition of piecewise affine rigid maps, so it is piecewise affine rigid. The second property is a consequence of the fact that if $x_1, \cdots, x_n \in (0, 1)$ then $f(x_1), \cdots, f(x_n) \in (0, 1/2]$. If we take $x \in \partial Q$ we know that at least one component x_k of x is equal to either 0 or 1. So $f(x_k) = 0$. Since $f(x_j) \geq 0$ for every $x_j \in [0, 1]$ we conclude that $f(x_k) = 0$ is the first component of $v(x)$. ■

4 The solutions to the Dirichlet problem

Now we are going to construct a locally piecewise rigid map $w\colon [0,+\infty) \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ with zero boundary condition.

First we consider the *zigzag* function $F\colon \mathbb{R} \to \mathbb{R}$ which is defined by the conditions

$$\begin{cases} F(t) = 2f(t/2) = \min\{t, 2-t\}, & \text{when } t \in [0,2], \\ F(t) = F(t+2), & \text{for every } t \in \mathbb{R}. \end{cases}$$

We also consider the affine map $x \mapsto Jx + a$, with $J \in O(n)$, $a \in \mathbb{R}^n$, such that (as mentioned in Theorem 1)

$$\phi_n(x) = Jx + a \qquad \text{when } x_1 \in [n-1, n] \text{ and } x_2, \cdots, x_n \in [0,1]\,.$$

Define, for $k \in \mathbb{Z}$, the vector $b_k \in \mathbb{R}^n$ as

$$b_k = \sum_{j=k}^{+\infty} \frac{J^{-j}}{2^{j+1}} a'\,,$$

where $a' = (n-1, 0, \cdots, 0) + J^{-1}a$.

Let $H = (0,+\infty) \times \mathbb{R}^{n-1}$. Given $x \in H$ there exists $k \in \mathbb{Z}$ such that

$$(n-1)2^{-k} \le x_1 < (n-1)2^{1-k}\,.$$

Then, for such a point x, we define

$$w(x_1, \cdots, x_n) = 2^{-k} J^{-k} \phi_n(2^k x_1 - n + 1, F(2^k x_2), \cdots, F(2^k x_n)) + b_k\,,$$

where ϕ_n is the map considered in Theorem 1, while for $x_1 = 0$ we define

$$w(0, x_2, \cdots, x_n) = 0 \qquad \text{for all } x_2, \cdots, x_n \in \mathbb{R}\,.$$

Theorem 3 (solution in the half space) *Let $H = (0,+\infty) \times \mathbb{R}^{n-1}$. The map $w\colon \overline{H} \to \mathbb{R}^n$ is locally piecewise affine in H and it is rigid on $\overline{H}$. Moreover $w(\partial H) = 0$.*

Proof. We first want to check the continuity of w on the planes $x_1 = (n-1)2^{-k}$, for every $k \in \mathbb{Z}$. So let x be a point on such a plane and let us check that

$$\begin{aligned} & w(x_1, x_2, \cdots, x_n) \\ & \quad = 2^{-k-1} J^{-k-1} \phi_n(2^{k+1} x_1 - n + 1, F(2^{k+1} x_2), \cdots, F(2^{k+1} x_n)) + b_{k+1}. \end{aligned} \tag{3}$$

With the substitution $x_1 = (n-1)2^{-k}$ in the definition of w, the left hand side of (3) becomes

$$2^{-k} J^{-k} \phi_n(0, F(2^k x_2), \cdots, F(2^k x_n)) + b_k\,;$$

by Theorem 1, since $F(t) \in [0,1]$ for all t,

$$= 2^{-k} J^{-k}(0, f(F(2^k x_2)), \cdots, f(F(2^k x_n))) + b_k$$

and by the identity $f(F(t)) = F(2t)/2$

$$= 2^{-k-1} J^{-k}(0, F(2^{k+1} x_2), \cdots, F(2^{k+1} x_n)) + b_k \, .$$

While the right hand side of (3) is, for $x_1 = (n-1)2^{-k}$, equal to

$$2^{-k-1} J^{-k-1} \phi_n(n-1, F(2^{k+1} x_2), \cdots, F(2^{k+1} x_n)) + b_{k+1} \, ;$$

since $F(t) \in [0,1]$ for every $t \in \mathbb{R}$, by Theorem 1 we can replace ϕ_n with the affine map $Jx + a$, and get

$$\begin{aligned}
&= 2^{-k-1} J^{-k-1} [J(n-1, F(2^{k+1} x_2), \cdots, F(2^{k+1} x_n)) + a] + b_{k+1} \\
&= 2^{-k-1} J^{-k} (n-1, F(2^{k+1} x_2), \cdots, F(2^{k+1} x_n)) + 2^{-k-1} J^{-k-1} a + b_{k+1} \\
&= 2^{-k-1} J^{-k} (0, F(2^{k+1} x_2), \cdots, F(2^{k+1} x_n)) \\
&\quad + 2^{-k-1} J^{-k} (n-1, 0, \cdots, 0) + 2^{-k-1} J^{-k-1} a + b_{k+1} \\
&= 2^{-k-1} J^{-k} (0, F(2^{k+1} x_2), \cdots, F(2^{k+1} x_n)) \\
&\quad + 2^{-k-1} J^{-k} a' + b_{k+1} \, ;
\end{aligned}$$

by recalling the definition of b_k, we obtain, as desired

$$= 2^{-k-1} J^{-k}(0, F(2^{k+1} x_2), \cdots, F(2^{k+1} x_n)) + b_k \, .$$

So the map w on $H = (0, +\infty) \times \mathbb{R}^{n-1}$ is locally piecewise affine and rigid. We now inspect the boundary values of w. Take any $k \in \mathbb{Z}$, and $i_2, \cdots, i_n \in \mathbb{Z}$. We have

$$w(2^{-k}(n-1), 2^{-k} i_1, \cdots, 2^{-k} i_n) = 2^{-k} J^{-k} \phi_n(0, F(i_1), \cdots, F(i_n)) + b_k \, ;$$

by Theorem 1 we get

$$= 2^{-k} J^{-k}(0, f(F(i_1)), \cdots, f(F(i_n))) + b_k \, ;$$

now notice that $F(i_k)$ is either 0 or 1 hence $f(F(i_k)) = 0$, so we find

$$= b_k \, .$$

Now since $b_k \to 0$ as $k \to +\infty$ and w is Lipschitz continuous, we conclude that $w \to 0$ at every point of ∂H and hence is continuous on the whole set $\overline{H}$. ■

Theorem 4 (solution in the cube) *Let $Q = (0,1)^n$, w be as above and v as in Section 3. The map $u = w \circ v : \overline{Q} \to \mathbb{R}^n$ is locally piecewise affine in Q and it is rigid on $\overline{Q}$. Moreover $u(\partial Q) = 0$.*

Proof. The map w of Theorem 3 is a Lipschitz solution to the Dirichlet problem

$$\begin{cases} Dw \in O(n) & \text{a.e. in } H \\ w = 0 & \text{on } \partial H \end{cases}$$

where H is the half space of $\mathbb{R}^n$. Since $u = w \circ v$, we clearly have that u is rigid and so $Du \in O(n)$ a.e. Moreover since $v(\partial Q) \subset \partial H$ and $w(\partial H) = 0$, we get the condition $u(\partial Q) = 0$. ■

Notice that we have solved a more precise problem, namely

$$Du\,(x) \in \Pi\,(n) \subset O\,(n)$$

where $\Pi\,(n)$ is the set of *permutation matrices* whose non zero entries are ± 1. In particular we have used at most $n!2^n$ different matrices in the construction of w and v.

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