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MORSE-SARD THEOREM IN SOBOLEV SPACES

OPTIMAL TRANSPORT PROBLEMS AND APPLICATIONS

Version 1.0

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Contents

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This is my PhD thesis. It differs from the version which I defended the 28/04/2001 in some details. Namely I corrected some misprintings and I update some references because in the meantime some preprints where published. Still there are errors around. I plan to correct the misprinting that I will find and to update the references at least once for year. So the versione number will tell if you have the last one.

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Introduction

This thesis is divided in two parts in which two different problems are considered. The first part is composed of only one chapter (the first) in which we study the Morse-Sard theorem in Sobolev spaces. The second one is composed by chapter 2 to 6 and is about the Monge-Kantorovich optimal transport problem and its application to a shape optimization problem.

The Morse-Sard theorem is a classical theorem and it was reconsidered and refined in the period 1987-1995 by various mathematicians. The study of this problem in the setting of Sobolev spaces was proposed to me in the 1997 by G.Buttazzo and G.Alberti together with a two dimensional proof made a couple of years before by G.Alberti and J.Maly.

In the summer of the 1998 G.Buttazzo and G.Bouchitté proposed to me the study of the Monge-Kantorovich problem together with an application to shape optimization. In the following two years the interest for the Monge-Kantorovich problem grew enormously and now there are various introductory papers and lecture notes ([**29**], [**5**], [**36**]). As the known theory on this topic is already well written in various places we think useless to write long chapters about it and we will just report the part of the theory close to our contributions.

- **Chapter 1** This chapter is the content of a paper with the same title which will appear on the *Indiana University Mathematics Journal.*
- **Chapter 2** This chapter contains a short introduction to the Monge-Kantorovich problem and some preliminary result on various methods we will use in the rest of the thesis.
- **Chapter 3** This chapter contains the introduction of the shape optimization problems we consider and the proof of the existence of solutions and of the relationships between these problems and the problem introduced in chapter 2. All sections but the last constitute the content of the preprint *On a constrained shape optimization problem* which will appear on *Communications in Applied Analysis.*
- **Chapter 4** This chapter contains some regularity results for the solution of the shape optimization problems we consider and for the so called Monge transport density introduced in chapter 2. This is the content of the paper *Regularity results for*

Monge transport density and for solutions of some shape optimization problem written in collaboration with A.Pratelli, will appear on Calculus of Variations and P.D.E..

• **Chapter 5** This chapter is the content of the paper *Topological Equivalence of Some Variational Problems Involving Distances* written in collaboration with G.Buttazzo and I. Fragalà and that will appear on *Discrete and Continuous Dynamical Systems.* We study the equivalence of the Γ- convergence of some variational problems involving distances. In the class of problem we consider there are two formulations of the Monge-Kantorovich problem.

• **Chapter 6** This chapter is the content of the paper *A p-Laplacian approximation of some shape optimization problems* which is still in preparation (In collaboration with G.Bouchitté and G.Buttazzo). We study the approximation of the shape optimization problems introduced in the previous chapters by means of Dirichlet problems of $p-$Laplacian type. This approximation has been used by F.Golay and P.Seppecher [**39**] to design numerical solutions of the problems.

CHAPTER 1

Morse–Sard Theorem in Sobolev Spaces

1.1. History and Introduction

Let Ω be an open subset of $\mathbb{R}^n$. Consider a differentiable map $u : \Omega \to \mathbb{R}^m$. For many application in differential topology, dynamical systems and degree theory, it is important to study the "size" of the set of critical values of u. Usually the word "size" we just used is intended in the sense of some measure (e.g. Hausdorff measure, Lebesgue measure, entropy measure.). The Morse-Sard Theorem is concerned exactly about the size of such set. To be precise we will indicate by C_u the set of the critical points of u (i.e. the set of points $x \in \Omega$ such that $\nabla u(x)$ is not of maximum rank) and by V_u the set $u(C_u)$ which is by definition the set of the critical values of u.

In this chapter we will prove that if $u \in W^{k,p}_{loc}(\Omega, \mathbb{R}^m)$ for $k = n - m + 1$, $n < p$ then the set of the critical value of u has m-measure zero.

As we are dealing with a very classical theorem we find suitable to give an account with discussed bibliography of what is already known about the finite dimensional Morse-Sard theorem. Along the chapter we will make the suitable comparisons.

1.1.1. C^k functions. The history of the first and classical version of the Morse-Sard theorem begins before 1935 with an unpublished paper by M. Morse and A. Sard. The question at that time was slightly less general, for instance in the fundamental paper [**52**] was:

If $k \geq 0$, $n \geq 1$ and $u \in C^k(\Omega, \mathbb{R})$ does u transforms its critical set into a set of (linear) measure zero ?

At that time it was already known, thanks to the famous counterexample published by Whitney in 1935 [**65**] that the answer is in general negative. Indeed in [**65**] Whitney gave an example of C^1 function from $I^2 \subset \mathbb{R}^2$ to $I \subset \mathbb{R}$ non constant on a connected set of critical points. Of course such a connected set cannot be rectifiable, and Whitney observed in the same paper that it must be so because of a previous theorem published by Whyburn in 1929 [**66**]. Then, to the original question Whitney added the new one:

How far from rectifiable has to be the critical set?

In their unpublished paper M. Morse and A. Sard proved that if $k \geq \frac{(n-3)^3}{16} + n$ then the answer to the first question is positive.

In [**52**] A. P. Morse studied the behavior of a regular function near its zero set, and then applied this study to the derivative of a function u obtaining the following results about the behavior of u near its critical set.

THEOREM 1.1.1 (Morse Criticality Theorem). *Let Ω be an open subset of $\mathbb{R}^n$, let $k \geq 1$ and let $B \subset \Omega$. Then there exist sets $B_0, B_1, \ldots$ having the following properties:*

(1) $B = \cup_{i=0}^{\infty} B_i$;
(2) B_0 *is countable;*
(3) *for $i > 0$ the set B_i is bounded and has no isolated points;*
(4) *for every $y' \in B_i$ $(i > 0)$ and for each function $u \in C^k(\Omega, \mathbb{R})$ whose critical set includes B,*

$$\lim_{\substack{y \to y' \\ y \in B_i}} \frac{u(y) - u(y')}{|y - y'|^k} = 0. \tag{1.1.1}$$

From this theorem Morse deduced:

THEOREM 1.1.2. *Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in C^n(\Omega, \mathbb{R})$. Then u transforms its critical set in a set of linear measure zero.*

In consideration of Whitney's example, the previous theorem is optimal for what concerns the case of scalar functions (i.e. $m = 1$).

The general optimal result was to be obtained by A. Sard in 1942 [**59**] and is heavily based on the first of the two theorems by A. P. Morse we just have reported; the result is divided in two parts, namely the case $m \geq n$ and the case $m < n$.

THEOREM 1.1.3 (Sard). *Let $\Omega \subset \mathbb{R}^n$, let $m \geq n$ be an integer and let $u \in C^1(\Omega, \mathbb{R}^m)$. Then the critical values of u are an $m-$null set in $\mathbb{R}^m$.*

THEOREM 1.1.4 (Sard). *Let $\Omega \subset \mathbb{R}^n$, let $0 < m < n$ be an integer and let $u \in C^k(\Omega, \mathbb{R}^m)$. Let A be the set of critical points of rank r of u in Ω (i.e. rank$(\nabla u(x)) = r$). Then $u(A)$ is an m-null set if $k \geq \frac{m-r}{n-r}$.*

Substituting r by $m - 1$ in the previous theorem one gets:

THEOREM 1.1.5 (Sard). *Let $\Omega \subset \mathbb{R}^n$, let $0 < m < n$ be an integer and let $u \in C^k(\Omega, \mathbb{R}^m)$. Then the set of critical values of u is an m-null set provided $k \geq n - m + 1$.*

From that time, many peoples (starting with the same A. Sard) have been involved in studying infinite dimensional versions of these theorems, and many others in producing counterexamples to prove the optimality of the results. Among them we quote: R. Kaufman [**42**] for a very general counterexample and E.L.Grinberg [**40**] for a very elegant one. Other papers devoted their attention to the precise understanding of how small is the set of critical values of u . The main results from this point of view are due to Federer [**31**] and Yomdim [**67**].

1.1.2. $C^{k,\alpha}$ functions. In 1986 Norton [**54**] studied an answer to the question of Whitney we reported in the previous subsection, discovering that there exists an arbitrarily unrectifiable (i.e. with arbitrarily small Hausdorff dimension) critical set whose image is nonnul. Moreover Hölder spaces appeared in Norton's paper as the natural spaces to extend the Morse-Sard theorem. In his paper Norton obtained a refined version of the Morse criticality theorem and an extension of the Morse-Sard theorem.

For a positive and real number t we will use the notation C^t for the Hölder space $C^{[t],t-[t]}$, and $C^{k,\alpha+}$ for the space of C^k functions f whose k-th derivative $\nabla^k f$ satisfies a strong α-Hölder condition: for every compact neighborhood U there exists a continuous nonnegative monotone function $\varepsilon : \mathbb{R} \to \mathbb{R}$ such that $\varepsilon(0) = 0$ and

$$|\nabla^k f(x) - \nabla^k f(y)| \leq \varepsilon(|x-y|)|x-y|^\alpha \quad \forall x, y \in U.$$

THEOREM 1.1.6 (Norton). *Let n, m be two positive integers, $n > m$, and define $\nu = \frac{n}{m}$. If E is an n-null set of rank 0 for $f : \mathbb{R}^n \to \mathbb{R}^m$ and either*

(a) *$\nu \in \mathbb{N} \cup \{0\}$ and $f \in C^{\nu-1,1}$, or*

(b) *$f \in C^\nu$,*

then $f(E)$ is m-null.

Using this last theorem together with his improved version of the Morse criticality theorem Norton was able to fully extend the Sard theorem to the class of $C^{k,\alpha+}$ functions (see [**54**] th. 1 and th.2). Here we prefer to report theorem (1.1.6) because is closer to our problem. We remark that this is a simplified version of Norton's theorems.

As the reader can see, without the assumption $\mathcal{H}^n(E) = 0$ theorem 1.1.6 would be an extension of the classical Morse-Sard theorem to the class of $C^{k,\alpha}$ functions. We wanted to emphasize the assumption "*If E is an n-null set*" in the previous theorem because n-null sets will play a fundamental role in this chapter.

Some years later in 1992-1993 S.M.Bates [**7**],[**8**],[**9**] was able to prove that theorem 1.1.6 holds without the assumption $\mathcal{H}^n(E) = 0$.

THEOREM 1.1.7 (Bates). *Let n, m, r be positive integers, $n > m > r$, and define $s = \frac{n-r}{m-r}$. If E is a set of rank r for $f : \mathbb{R}^n \to \mathbb{R}^m$ and either*

(a) *$s \in \mathbb{N} \cup \{0\}$ and $f \in C^{s-1,1}$, or*

(b) *$f \in C^s$,*

then $f(E)$ is m-null.

Substituting $s = m - 1$ in the previous theorem one gets:

THEOREM 1.1.8 (Bates). *Let n, m be positive integers with $n > m$ and $k = n-m+1$. If $f \in C^{k-1,1}(\mathbb{R}^n, \mathbb{R}^m)$ then the set of critical values of f has m-measure zero.*

Bates also proved that this last result is optimal by the following theorem.

THEOREM 1.1.9 (Bates). *For n, m, r, s as in theorem 1.1.7, there exists a map $f : \mathbb{R}^n \to \mathbb{R}^m$ contained in C^t for all real $t < s$ and a subset E of rank r for f such that $f(E)$ contains an open set.*

Theorem 1.1.8 shows that the assumptions of the classical Morse-Sard theorem can actually be relaxed. Theorem 1.1.9 prove that in this setting (i.e. for C^k or $C^{k,\alpha}$ functions) nothing more can be done. Recently G.Comte [**18**] completed this picture giving an optimal estimate of the size of the set of critical values under the assumptions of Bates's theorems.

Remark: In another paper [**55**] Norton extended the Morse-Sard theorem to the class of $C^{k,Zygmund}$ functions. The Zygmund class is the class of functions $f : \mathbb{R}^n \to \mathbb{R}^m$ continuous and such that for every compact set $K \subset \mathbb{R}^n$, there exists a constant C such that: if $x, x+h, x-h \in K$ then

$$|f(x+h) + f(x-h) - 2f(x)| \leq C|h|.$$

The class of $C^{k,Zygmund}$ functions is the class of C^k functions whose k-th derivative belongs to the Zygmund class. It is important to remark that this last theorem imply as particular cases both the previous (the classical and the Hölder case).

1.1.3. Sobolev functions. The main theorem of this chapter is the following:

THEOREM 1.1.10. *Let n, m be two integers with $m < n$, let $k = n - m + 1$, $n < p$ and let $u \in W^{k,p}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$. Then the set of critical values of u has zero m-measure.*

Remarks 1) As u is defined only a.e. it is important to specify that we refer to the continuous representative of u. Moreover thanks to the assumptions of theorem 1.1.10 $u \in C^1$ and then the critical set of u is well defined.

2) Thanks to the classical imbedding theorems for Sobolev spaces u is in $C^{k-1,\alpha}$ for some $\alpha = \alpha(p)$, we emphasize that such α is smaller than the one in Bates's theorems 1.1.7, 1.1.8. This is the effect of the existence of another weak derivative summable enough.

3) The classical examples show that we cannot take a smaller k in the assumptions of theorem 1.4.1. For $n > 3$ taking a smaller p one would still have the continuity of the gradient, we will be concerned about that issue in the last section of the chapter.

Here we describe the strategy of the proof.
It is well known (we will report this result in section 1.2) that given a function $u \in W^{k,p}(\Omega, \mathbb{R}^m)$, for every $\varepsilon > 0$ there exists a set F_ε and a function $u_\varepsilon \in C^k(\Omega, \mathbb{R}^m)$ such

that u and u_ε coincide on F_ε together with all their derivatives and the measure of $\Omega \setminus F_\varepsilon$ is smaller then ε.
Moreover theorem (1.1.4) implies that the set of critical values of u_ε has zero m-measure.
Let now C_u be the critical set of u. We can write C_u in the following way:

$$C_u = \bigcup_n (C_u \cap F_{\frac{1}{n}}) \cup N$$

where the measure of N is zero.
Thanks to the classical Morse-Sard theorem

$$\mathcal{H}^m(u(C_u \cap F_{\frac{1}{n}})) = \mathcal{H}^m(u_{\frac{1}{n}}(C_u \cap F_{\frac{1}{n}})) = 0,$$

then the Morse-Sard theorem in $W^{k,p}$ will be proved if we prove that $\mathcal{H}^m(u(N)) = 0$.

At this point it can be useful the following definition

Definition 1.1.11. *We say that a function $u \in C^1(\Omega, \mathbb{R}^m)$ has the N_0-property if*

$$E \subset C_u, \quad \mathcal{H}^n(E) = 0 \Rightarrow \mathcal{H}^m(u(E)) = 0 \tag{1.1.2}$$

and then what we actually proved is :

Theorem 1.1.12. *Let $n > m$ two positive integers and let $k = n - m + 1$ and let $u \in W^{k,p}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$. Then V_u is m-null if and only if u has the N_0-property.*

The rest of the chapter will be devoted to prove that every map in a suitable Sobolev space satisfies N_0 property. More precisely in section 3 we will give some general estimates of the oscillation of a Sobolev function near its zero set. To do this we will decompose the critical set of a function $u \in W^{k,p}$ in a countable number of subsets on each of which suitable estimate of the oscillation of u holds.
Finally in section 4 we will conclude the proof.
In section 5 we will briefly survey the case $m \geq n$ which is deeply different and already well studied in literature.
Section 6 will contains some last considerations.

Remark In order to define the critical set of u we assumed that u is C^1. It is possible to define the critical set also for functions which are of class C^1 only up to a "small" set, in a suitable sense. We will consider this possibility in the last section.

1.2. Notation and preliminaries

Along the chapter we denote by $|B|$ the Lebesgue measure of a measurable set B while $\mathcal{H}^m_d(B)$ will be its m-dimensional Hausdorff measure with respect to the distance d. If d is not indicated the Euclidean distance will be intended. A δ will be used to denote the approximate m-dimensional Hausdorff measure $\mathcal{H}^m_{d,\delta}$ of size δ, then in the

Euclidean case the symbol will be $\mathcal{H}^m_\delta$, we will be careful to avoid denoting by δ some distance. A classical result[1] [**60**] state that $\mathcal{H}^n$ coincide with the Lebesgue measure in R^n. We say that a set is m-null if it has zero $\mathcal{H}^m$ measure. We will denote by $D^k u$ the k-th distributional derivative of the function u, while by $\nabla^k u$ we will denote the derivative in any other sense.

As it was already said in the introduction we are interested in the following property:

Definition 1.2.1. *We say that a function $u : \Omega \to \mathbb{R}$ as the Lusin property of order k if for each $\varepsilon > 0$ there exists a closed set F_ε and a function $u_\varepsilon \in C^k(\mathbb{R}^n)$ such that*

(1) $|\Omega \setminus F_\varepsilon| \le \varepsilon$;
(2) $D^\alpha u(x) = \nabla^\alpha u_\varepsilon(x)$ *for* $x \in F_\varepsilon$ *and* $|\alpha| \le k$;

this property is strictly related to the existence of an approximate Taylor polynomial of order k for u, where approximate as to be intended in the sense we are going to specify.

Let $A \subset \mathbb{R}^n$ be a measurable set and let $x \in \mathbb{R}^n$, the upper density of A at x is defined by:

$$\theta^*(A, x) = \overline{\lim_{r\to 0}} \frac{|B(x,r) \cap A|}{|B(x,r)|}. \tag{1.2.1}$$

Analogously we can define the lower density of A at x and if the upper and the lower density coincide we can define the density of a set at a point.

Definition 1.2.2. *Let $u : \Omega \to \mathbb{R}^n$ a Borel function. We write:*

$$z = \operatorname*{ap\text{-}lim}_{y\to x} u(y) \tag{1.2.2}$$

iff for any $\varepsilon > 0$ the set

$$E_\varepsilon := \{y \in \Omega : |u(y) - z| > \varepsilon\} \tag{1.2.3}$$

has density zero at x.

It is easy to see that if Ω has a positive upper density at x then if the approximate limit (1.2.2) exists it is unique.

Consider now a measurable set A and a point x such that $\theta^*(A, x) > 0$.

Definition 1.2.3. *We say that a measurable function $u : A \to \mathbb{R}$ is approximately differentiable at x if there exist a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\operatorname*{ap\text{-}lim}_{\substack{y\to x\\ y\in A}} \frac{|u(y) - u(x) - L(y-x)|}{|y-x|} = 0. \tag{1.2.4}$$

[1] Due to A.Sard.

In this case we say that L is the approximate differential of u at x and we write $L = \nabla u(x)$. Using an higher order approximating polynomial $p_k(y-x)$ instead of $L(y-x)$ one can give the definition of approximate Taylor development of any order.

It turns out that for a function u, having a Lusin property of order k is equivalent to having an approximate Taylor polynomial of degree k. In the case $k=1$ (i.e. Lusin property of order 1 is equivalent to approximate differentiability) this is due to Federer and a proof can be found in [**37**] th. 7. The statement in the higher order case can be found in [**45**] th. 1 while the proof is in [**46**]. Moreover Sobolev functions and maps enjoy both this good property as said in the next theorem which can be found in [**46**] or [**68**].

THEOREM 1.2.4. *Every function in $W^{k,1}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ has the Lusin property of order k.*

We now give some technical lemmas . The proofs we omit are straightforward. Let $D^m = \{x \in \mathbb{R}^m \,||x| \leq 1\}$.

DEFINITION 1.2.5. *We say that a subset C of R^n is a C^1-embedded topological disk of dimension m if there exists an injective map $\varphi \in C^1(D^m, \mathbb{R}^n)$ such that $C = \varphi(D^m)$ and $\nabla\varphi$ has maximal rank at every point of D^m.*

LEMMA 1.2.6. *Let C a C^1-embedded topological disk. Then there exists a constant $\sigma(C)$ such that for every $x, y \in C$ there exists a C^1 path from x to y of length less than or equal of $\sigma(C)|x-y|$.*

LEMMA 1.2.7. *Let $D \subset \mathbb{R}^n$ be a C^1-embedded topological disk of dimension γ_i equipped with a geodesic distance d equivalent to the Euclidean distance. Let $\varphi : D \to \mathbb{R}^{n+k}$ a C^1 bijective map whose gradient as rank γ_i at every point. Then $\varphi(D)$ is also a C^1-embedded topological disk. Moreover if we equip $\varphi(D)$ with the image distance d_φ (which is geodesic and equivalent to the Euclidean one)*

$$\varphi : (D, d) \to (\varphi(D), d_\varphi)$$

is isometric.

Let us give a last simple remark about the implicit function theorem for Sobolev maps.

Remark Let $1 < k,\ n < p$, and let $u \in W^{k,p}(\Omega, \mathbb{R}^n)$. If the Jacobian of u at x_0 is non singular we can apply to u the implicit functions theorem for C^{k-1} functions and find a local inverse u^{-1} of u. We have:

$$\nabla(u^{-1}) = \mathrm{inv} \circ \nabla u \circ u^{-1} \tag{1.2.5}$$

where inv : $GL(n) \to GL(n)$ is the inversion mapping. We have: $\nabla u \in W^{k-1,p}$, inv $\in C^\infty$ and $u^{-1} \in C^{k-1}$ then their composition is in $W^{k-1,p}$ in a ball around $u(x_0)$ sufficiently small. Then the local inverse of u is also a $W^{k,p}$-map. If $k = 1$ we cannot do the same and we need a more advanced result [**32**], but we will use this remark just in the case $k > 1$.

1.3. The Morse criticality theorem

DEFINITION 1.3.1. *We denote by $\mathfrak{D}^\alpha u$ the vector whose components are all the distributional derivative of u of order greater then 1 and smaller then α (1 and α included).*

THEOREM 1.3.2 (Morse Criticality Theorem). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $C \subset \Omega$ a subset, $p > n$ a real number and $k > 0$ a positive integer, then there exists two sequences of sets $\{A_i\}_{i=0}^{+\infty}$, $\{B_i\}_{i=1}^{+\infty}$ and positive constants $\{C_i\}$ such that the following holds:*

- $\boxed{1}$ *A_0 is countable,*
- $\boxed{2}$ *B_i is a C^1-embedded compact disk of dimension $\gamma_i \leq n$ equipped with a geodesic distance d_i equivalent to the Euclidean one, and $A_i \subset B_i$*
- $\boxed{3}$ *$C = \cup_{i=0}^{+\infty} A_i$*
- $\boxed{4}$ *$\forall v \in W^{k,p}(\Omega, \mathbb{R})$, verifying $v \equiv 0$ on C, $\forall y \in B_i$ and $x \in A_i$ the following estimate holds:*

$$|v(y) - v(x)| \leq C_i d_i(y,x)^{\alpha(k,\gamma_i)-\frac{\gamma_i}{p}} \left(\int_{B_{d_i(x,y)}(x)} |\mathfrak{D}^{\alpha(k,\gamma_i)} v(z)|^p d\mathcal{H}^{\gamma_i}_{d_i} \right)^{\frac{1}{p}}, \tag{1.3.1}$$

where $k - (n - \gamma_i) \leq \alpha(k, \gamma_i) \leq k$ is integer.

DEFINITION 1.3.3. *We will call the sequences $\{A_i\}_{i=0}^{+\infty}$, $\{B_i\}_{i=1}^{+\infty}$ of theorem (1.3.2), a Morse decomposition of the set C.*

PROOF. The theorem will be proved by induction on the couple n and k.

Step 1: $k = 1$.

Let $x \in C$, and let $y \in B(x, r) \subset \Omega$, by the Morrey-Sobolev inequality we get:

$$|v(x) - v(y)| \leq cr^{1-\frac{n}{p}} (\int_{B(x,r)} |Dv(z)|^p dz)^{\frac{1}{p}}. \tag{1.3.2}$$

So we can choose $A_i = B_i \bigcap C$, $\{B_i\}$ being just some covering of C made of balls and d_i the Euclidean distance. (We remark that inequality (1.3.2) was used by Morrey in [**51**] to prove some embedding theorem for Sobolev spaces).

Step 2: $n = 1,\ k > 1$ (the case $n = k = 1$ was already proved).

Set $C = C^* \cup (C \setminus C^*)$, where C^* is the set of condensation points of C, and take $C \setminus C^* \subset A_0$. On C^* we have $v \equiv ... \equiv v^{k-1} \equiv 0$ moreover for each $x_0 \in C^*$:

$$\begin{aligned} |v(y) - v(x_0)| &\leq |v'(\xi)||y - x_0| = \\ &= |v'(\xi) - v'(x_0)||y - x_0| \leq \\ &\leq |v''(\xi_1)||y - x_0|^2 \leq \\ &\leq ... \leq \\ &\leq |v^{k-1}(\xi_{k-2}) - v^{k-1}(x_0)||y - x_0|^{k-1}. \end{aligned}$$

then, using the decomposition obtained for $k = 1$ and using the inequality already established for v^{k-1} in $B(x_0, r) \cap C^*$ we get for $y \in B_i$ and $x \in A_i$

$$|v(y) - v(x)| \leq |y - x|(C_i|y - x|^{k-1-\frac{1}{p}}(\int_{B_{|y-x|}(x)} |v^k(x)|dx)^{\frac{1}{p}}) \tag{1.3.3}$$

so that also in this case we can choose $A_i = B_i \bigcap C$, $\{B_i\}$ being just some covering of C^* made by closed bounded intervals equipped with the Euclidean distance.

Step 3: $n \in \mathbb{N},\ k > 1$ (we remark that in this case v is at least of class C^1).
Set $C = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{B} = C \setminus \mathcal{A}$ and $\mathcal{A} := \{x \in C | \nabla G(x) = 0 \textit{ whenever } G \equiv 0 \textit{ on } C \textit{ and } G \in W^{k,p}\}$. We will decompose separately $\mathcal{A}$ and $\mathcal{B}$. On $\mathcal{A}$ we will use the induction on the order of (weak) differentiability k while on $\mathcal{B}$ we will locally reduce the dimension.
We already know that there exists a decomposition for which the suitable estimate holds for function in $W^{k-1,p}$, consider then such decomposition of $\mathcal{A}$: $\{A_i\}$, $\{B_i\}$, and the respective constants $\{C_i\}$.
Let $y \in B_i$ and $x_0 \in A_i$, we have:

$$|v(y) - v(x_0)| \leq |\nabla v(\xi) - \nabla v(x_0)| d_i(y, x_0) \leq$$

$$\leq C_i d_i(y, x_0) \sum_{j=1}^{n} |\frac{\partial v}{\partial x_j}(\xi) - \frac{\partial v}{\partial x_j}(x_0)| \leq$$

$$\leq \tilde{C}_i d_i(y, x_0) \sum_{j=1}^{n} d_i(y, x_0)^{\alpha(k-1,\gamma_i)-\frac{\gamma_i}{p}} (\int_{B_{d_i(y,x_0)}(x_0)} |\mathfrak{D}^{\alpha(k-1,\gamma_i)} \frac{\partial v}{\partial x_j}(z)|^p d\mathcal{H}^{\gamma_i}_{d_i})^{\frac{1}{p}}. \tag{1.3.4}$$

In the first inequality we used the mean-value theorem and the fact that $\nabla v(x_0) = 0$. Moreover we used the fact that $d_i(\xi, x_0) \leq d_i(y, x_0)$. Simple calculations complete the proof in this case.

Let now $x_0 \in \mathcal{B}$; and let $B(x_0, r)$ a ball around x_0 for which we will choose a suitable r during the proof, there exist $G \in W^{k,p}(\Omega)$ such that $G \equiv 0$ on C but $\nabla G(x_0) \neq 0$.

Applying the implicit function theorem for functions of class C^{k-1} to such G we can say that there exists a C^{k-1} function $\varphi : B_{n-l}(0,r_1) \to \mathbb{R}^n$ such that $\varphi(0) = x_0$, $\mathcal{B} \cap B(x_0,r) \subset \varphi(B_{n-l}(0,r_1))$ and $c|z-w| \leq |\varphi(z)-\varphi(w)| \leq C|z-w|$. Moreover (up to a little restriction of the ball) $v \circ \varphi \in W^{k-1,p}(B_{n-l}(0,r_1),\mathbb{R})$ (because both φ and v are in C^{k-1}).
Let $\{\hat{A}_i\},\{\hat{B}_i\}$, the decomposition of $\varphi^{-1}(\mathcal{B}\cap B(x_0,r))$ that we found for $n-l$ and $k-1$. Let us choose $B_i := \varphi(\hat{B}_i)$, d_i the image distance of of $\hat{d}_i$ through φ and $A_i := \varphi(\hat{A}_i)$, let $x \in A_i$, $y \in B_i$:

$$\begin{aligned}
|v(x)-v(y)| &= |v(\varphi(z))-v(\varphi(w))| \leq \\
&\leq C_i \hat{d}_i(z,w)^{\alpha(k-1,\gamma_i)-\frac{\gamma_i}{p}} \Big(\int_{B_{\hat{d}_i(z,w)}(z)} |\mathfrak{D}^{\alpha(k-1,\gamma_i)}(v\circ\varphi)(\omega)|^p d\mathcal{H}_{\hat{d}_i}{}^{\gamma_i}\Big)^{\frac{1}{p}} \leq \\
&\leq K_i d_i(x,y)^{\alpha(k-1,\gamma_i)-\frac{\gamma_i}{p}} \Big(\int_{B_{d_i(x,y)}(x)} |\mathfrak{D}^{\alpha(k-1,\gamma_i)} v(\omega)|^p d\mathcal{H}_{d_i}{}^{\gamma_i}\Big)^{\frac{1}{p}}.
\end{aligned}$$

The last inequality depends on the fact that φ is isometric (see lemma 1.2.7) and has bounded derivative of any order smaller then $k-1$. To conclude the proof it is enough to cover $\mathcal{B}$ with a suitable family of balls with the properties of $B(x_0,r)$, and to consider the decomposition given by the union of the decompositions obtained inside each ball. □

Remarks:

1)The constant C_i of inequality (1.3.1) depends on the curvature of B_i and on the constants of the Poincaré inequality for lower dimensions.

2)We remark that the decomposition depends on k. This is a difference with the classical case of C^k functions.

1.4. The Morse-Sard theorem

THEOREM 1.4.1. *Let $n > m$ be two positive integers, let $k := n-m+1$, and let $u \in W^{k,p}_{loc}(\mathbb{R}^n,\mathbb{R}^m)$ for some $p > n$. Then u has the N_0 property.*

PROOF. (of theorem 1.4.1.) Let $u \in W^{k,p}_{loc}(\mathbb{R}^n,\mathbb{R}^m)$ and let $N \subset C_u$ be a subset of n-measure zero of the critical set of u. We want to prove that $\mathcal{H}^m(u(N)) = 0$.

Step 1 Without loss of generality we can suppose that $\nabla u = 0$ on N. In fact, let R_i the set where $\mathrm{rank}\nabla u = i$ and let $p \in R_i$, then in a small neighborhood V of p, u can be written as follows:

$$u(y_1,...,y_n) = (y_1,...,y_i,g(y_1,...,y_n)),$$

(assume $\det(\frac{\partial(u_1,...u_i)}{\partial(x_1,...x_i)} \neq 0)$ and choose the coordinate chart $(u_1,...,u_i,x_{i+1},...,x_n)$). In these coordinates ,

$$\nabla u(y) = \begin{pmatrix} Id_i & 0 \\ * & \nabla(g||y_1,...y_i) \end{pmatrix},$$

where Id_i is the identity matrix on $\mathbb{R}^i$ and $g||y_1,...y_i : \mathbb{R}^{n-i} \to \mathbb{R}^{m-i}$ is defined by

$(y_{n-i},...y_n) \to g(y_1,...,y_n)$, thanks to the slicing theorem for Sobolev functions for $\mathcal{H}^i$-a.e. $(y_1,...,y_i)$ the map $g||y_1,...,y_i$ is in $W^{k,p}(\mathbb{R}^{n-i},\mathbb{R}^{m-i})$ and $\nabla(g||y_1,...y_i)$ as rank 0 for all $y \in \mathbb{R}^{n-i}$. Applying the theorem to $g||y_1,...y_i$ one obtain that $\mathcal{H}^i$-a.a. slice of $u(V \cap R_i)$ as $(n-i)$-dimensional measure zero. An application of Fubini's theorem conclude the reduction to the case $\nabla u = 0$.

Step 2 Consider a Morse decomposition (see definition 1.3.3) $\{A_i\},\{B_i\}$ for functions $W^{k-1,p}$ of C_u and let $\sigma(B_i)$ the constant of lemma 1.2.6 relative to B_i.
We have $N = \cup_0^\infty(N \cap A_i)$, then it is enough to prove that $u(N \cap A_i)$ is m-null. Moreover $N \cap A_0$ is countable, therefore we will consider $i > 0$. Let us set $N \cap A_i = N_i$. For each i we will choose a suitable covering of N_i. The only distinctions we need to make is on the dimension γ_i of B_i.

Starting from estimate (1.3.1) we will obtain a general estimate for $x \in N_i$ and $y \in B_i$.

Case $\gamma_i > m > 1$ Let $x \in N_i$ and $y \in B_i$ arguing as in the first inequality of (1.3.4) and applying theorem 1.3.2 to the components of ∇u we obtain the estimate

$$|u(y) - u(x)| \leq C_i d_i(y,x)^{\alpha_i+1-\frac{\gamma_i}{p}} (\int_{B_{d_i(x,y)}(x)} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}_{d_i}{}^{\gamma_i})^{\frac{1}{p}}$$

where α_i is taken from theorem 1.3.2 and then $\alpha_i + 1 \geq \gamma_i - m + 1$. We can rewrite the previous estimate as follows

$$|u(y) - u(x)| \leq C_i d_i(y,x)^{\alpha_i+1-\frac{\gamma_i+1}{p}} (d_i(x,y) \int_{B_{d_i(y,x)}(x)} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}_{d_i}{}^{\gamma_i})^{\frac{1}{p}}$$

and then

$$|u(y) - u(x)|^m \leq C_i d_i(y,x)^{(\alpha_i+1-\frac{\gamma_i+1}{p})m} (d_i(x,y) \int_{B_{d_i(y,x)}(x)} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}_{d_i}{}^{\gamma_i})^{\frac{m}{p}}.$$

Applying now Young's inequality with exponents $\frac{p}{m}$ and $\frac{p}{p-m}$ we obtain:

$$|u(y) - u(x)|^m \leq C_i (d_i(y,x)^{(\alpha_i+1-\frac{\gamma_i+1}{p})\frac{mp}{p-m}} + d_i(x,y) \int_{B_{d_i(y,x)}(x)} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}_{d_i}{}^{\gamma_i}).$$

Here we remark that $(\alpha_i + 1 - \frac{\gamma_i+1}{p})\frac{mp}{p-m} > \gamma_i$ under the conditions $\gamma_i > m$ and $m > 1$ so that we will need to discuss separately the case $m = 1$.

For small $\varepsilon, \delta > 0$ select a sequence of balls (for the intrinsic distance d_i) $B(x_j, r_j)$ covering N_i with $x_j \in N_i$, $r_j < \delta$ such that

$$\sum_j (2r_j)^{\gamma_i} \leq \mathcal{H}^{\gamma_i}_{d_i}(N_i) + \varepsilon$$

and

$$\sum_j \int_{B_{r_j}(x_j)} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}^{\gamma_i}_{d_i} \leq K_i \int_{B_i} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}^{\gamma_i}_{d_i} + \varepsilon.$$

Then $u(N_i)$ is covered by the balls centered at $u(x_j)$ and of radius equal to the diameter of $u(B(x_j, r_j))$, it follows the estimate

$$\mathcal{H}^m_{\delta_1}(u(N_i)) \leq$$

$$\leq C_i \sum_j r_j^{(\alpha_i + 1 - \frac{\gamma_i + 1}{p})\frac{mp}{p-m}} + C_i \delta \sum_j \int_{B_{r_j}(x_j) \cap B_i} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}^{\gamma_i} \leq$$

$$\leq C_i \delta^{(\alpha_i + 1 - \frac{\gamma_i + 1}{p})\frac{mp}{p-m} - \gamma_i} \mathcal{H}^{\gamma_i}(N_i) + K C_i \delta \int_{B_i} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}^{\gamma_i} + o(\varepsilon), \tag{1.4.1}$$

and the last term tends to 0 when ε and δ go to 0.

Case $\gamma_i \leq m$ In this case we can apply the classical Morse-Sard theorem. In fact the Morse-Sard theorem is local and it can be applied on manifolds too. Then it is enough to remark that u is a C^1 map and N_i is contained in the critical set of u restricted to the C^1 manifold B_i and then $u(N_i)$ is $m-$null.

Case $m = 1$ Here $k = n$ and then u is at least C^{k-1+} (see the introduction for the definition). If $\gamma_i \leq n - 1$ it is enough to apply lemma 1 of [**54**] to obtain that the image on N_i is $\mathcal{H}^1$-null. Then we just need to consider the case $\gamma_i = n$ which imply $\mathcal{H}^{\gamma_i}_{d_i}(N_i) = 0$. We start again from the estimate

$$|u(y) - u(x)| \leq C_i d_i(y,x)^{\alpha_i + 1 - \frac{\gamma_i}{p}} \left(\int_{B_{d_i(x,y)}(x)} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}_{d_i}{}^{\gamma_i}\right)^{\frac{1}{p}}.$$

And we apply the Young inequality with exponents p and $\frac{p}{p-1}$ to obtain:

$$|u(y) - u(x)| \leq C_i \left(d_i(y,x)^{(\alpha_i + 1 - \frac{\gamma_i}{p})\frac{p}{p-1}} + \int_{B_{d_i(y,x)}(x)} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}_{d_i}{}^{\gamma_i}\right).$$

Here we remark that $(\alpha_i + 1 - \frac{\gamma_i}{p})\frac{p}{p-1} = \gamma_i$.

For small $\varepsilon, \delta > 0$ select a sequence of balls (for the intrinsic distance d_i) $B(x_j, r_j)$ covering N_i with $x_j \in N_i$, $r_j < \delta$ such that

$$\sum_j (2r_j)^{\gamma_i} \leq \varepsilon$$

and

$$\sum_j \int_{B_{r_j}(x_j)} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}^{\gamma_i}_{d_i} \leq \varepsilon,$$

this choice is permitted by the fact that N_i has zero n-measure.

Then, as in the first case, $u(N_i)$ is covered by the balls centered at $u(x_j)$ and of radius equal to the diameter of $u(B(x_j, r_j))$, it follows the estimate

$$\mathcal{H}^m_{\delta_1}(u(N_i)) \leq$$

$$\leq C_i \sum_j r_j^{\gamma_i} + C_i \sum_j \int_{B_{r_j}(x_j)} |\mathfrak{D}^{\alpha_i} u(z)|^p d\mathcal{H}^{\gamma_i}_{d_i} \leq 2\varepsilon,$$

letting ε and δ go to 0 the proof is concluded. □

1.5. Area formula and Morse-Sard Theorem

In this section we will briefly consider the case $u : \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq n$. A theorem of Varberg (see [**63**]) imply that in this case the image of the set $\{x \mid u$ is differentiable at x and $\nabla u(x) = 0\}$ has always image $n-$null provided u is measurable. It is well known that Sobolev functions are almost everywhere differentiable, then Varberg's theorem apply here. But the question we have in mind is: if $C \subset \Omega$ is such that $\nabla u = 0$ a.e. in C is it true that $\mathcal{H}^n(u(C)) = 0$? In the classical case the Morse Sard theorem holds for C^1-maps, then by what we have previously shown about approximate differentiability and Lusin's type property of Sobolev functions, it is enough to consider $u \in W^{1,p}$ and to choose p such that the N property holds.
The theory of area formula (see [**37**]) provides us the following result.

Definition 1.5.1 (Lusin's N-property). *Let $u : \mathbb{R}^n \to \mathbb{R}^m$ a Borel map, we say that u satisfies the Lusin's N-property iff*

$$E \subset \mathbb{R}^n, \ |E| = 0 \Rightarrow \mathcal{H}^n(u(E)) = 0. \tag{1.5.1}$$

Theorem 1.5.2. *Let $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ and let $n < p$. Then u satisfies the Lusin's N-property.*

Thanks to this last theorem it does not matters the fact that Du is defined just up to a set of measure zero and we can say that the Morse-Sard theorem holds in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ for $n < p$.
It is well know that if $p = n$ the N-property fails ([**58**], [**49**]). Recently in ([**43**]) the authors proved that a continuous map in $W^{1,n}$ whose gradient is in the Lorentz space $L^{1,n}$ have the N-property.
From this results and what we said it follow a that the Morse-Sard theorem hold for functions in $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ whose gradient is in $L^{1,n}$.

Apparently the N property is much stronger then the N_0 property, in [**49**] the authors give an example of a continuous map $u \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ whose Jacobian determinant Ju is zero almost everywhere and that maps the segment $\{te_1 : \ 0 \leq t \leq 1\}$ into the solid cube $Q = [0,1]^n$. In that example the set $\{te_1 : \ 0 \leq t \leq 1\}$ is contained in the set of non differentiability of the map. Moreover it is known that a map in $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ such that $Ju \geq c > 0$ has the N-property. Finer results of this type are contained in [**50**]. All these results and example seems to say that there is a relationship stronger than the obvious implication between the N_0 and the N property.

1.6. Further remarks

In all but section 5 of this chapter, we worked with functions with continuous gradient. This allowed us to define C_u and its image through u. In section 5 we considered maps in $W^{1,p}$ whose gradient is defined up to a set of measure 0. In this case the N-property was of greet help because if from one side the critical set of u is defined up to a set N_u of measure 0, from the other side we can forget about N_u because u has the N-property. The same difficulty appears dealing for example with $u \in W^{2,2}_{loc}(\mathbb{R}^2, \mathbb{R})$; in this case in order to give a statement of the theorem one would need to prove that u has the N-property, and this would give the proof of the theorem at once. An other possibility is given by the use of capacities instead of Hausdorff measures: in fact it can be proved that a function in $W^{2,2}_{loc}(\mathbb{R}^2, \mathbb{R})$ is defined up to a set of capacity zero.
Finally we remark that in the case $W^{3,p}(\mathbb{R}^3, \mathbb{R})$ we could take a smaller exponent p without loosing the continuity of the gradient. Also in this case we don't know if the theorem remains true.

CHAPTER 2

A short introduction to Monge problem and some preliminary results

In this chapter we briefly describe the Monge–Kantorovich problem, and a small part of the theory which grew around it. Moreover we provide some preliminary results which will be used in the following chapters.

2.1. The Monge-Kantorovich Problem

Let $\Omega \subset \mathbb{R}^N$ be an open, convex, bounded, set and let $f^+, f^- \in \mathcal{M}^+(\Omega)$ be two positive measures of equal total mass. The Monge problem consists in transporting the mass f^+ on the mass f^- minimizing a certain cost functional. A Borel map $s : \Omega \to \Omega$ is a transport of f^+ to f^- if $s_\sharp f^+ = f^-$. By $s_\sharp f^+$ we denote the image measure (or push-forward) of f^+ through s defined by $s_\sharp f^+(B) = f^+(s^{-1}(B))$ for each Borel set B. Denote by $\mathcal{A}(f^+, f^-)$ the set of maps transporting f^+ to f^-. In the 1780's Monge proposed (of course in different terms) the problem of minimizing the total transportation work to bring f^+ onto f^-. With the notation above, if we measure the work to transport x in $s(x)$ by the Euclidean distance $|x - s(x)|$, then the total work turns out to be $\int_\Omega |x - s(x)| df^+(x)$, so that we are faced to the following minimization problem

$$\min_{s \in \mathcal{A}(f^+, f^-)} \int_\Omega |x - s(x)| df^+(x). \tag{2.1.1}$$

Other costs, quite well studied in literature because of the wide number of applications, are of the type

$$\int_\Omega \phi(|x - s(x)|) df^+(x), \tag{2.1.2}$$

with ϕ a positive strictly convex or strictly concave function. For example, several papers are concerned with the case $\phi(t) = t^\alpha$. We refer to [**35**] for complete surveys on these cases.

Remark 2.1.1. In all what we said we used only the metric structure of Ω, then everything could be formulated on a metric space. Here we prefer to deal with the euclidean case to avoid too many distinctions in the assumptions of the theorems we will report.

Notations and Assumptions:
1) For any of the costs we just considered in (2.1.2) the integrand of the cost wil be denoted by $c(x,y)$; in the previous case is then $c(x,y) = \phi(|x-y|)$.
2) As f^+ and f^- have the same total mass and since neither the formulation of the problem nor the existence of the solution depend on their mass, we will assume both of them to be probability measures, moreover we will set $f := f^+ - f^-$.

Even if quite simple in its formulation, Monge problem revealed to be quite hard. Indeed the problem of existence of an optimal transport has been open until now, and for certain aspects is still not well understood. Indeed a conjecture says that if f^+ and f^- do not concentrate on sets of dimension less than or equal to $N-1$ then there exists at least a minimizer of (2.1.1). In the 1979 Sudakov claimed to have a proof of this conjecture [**62**]. Unfortunately has been found not completely correct (see [**2**], [**44**]) in a decomposition theorem for measures he used. More recently three proofs of existence appeared ([**5**], [**28**], [**17**]) but all of them require f^+ and f^- to be absolutely continuous with respect to the Lebesgue measure.

A first simplification of the problem was proposed by Kantorovich in the '40s. The basic idea of Kantorovich was to relax problem (2.1.1) in the space $\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})$ and to consider the set

$$\mathcal{P}(f^+,f^-) := \{\gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}) |\ \pi^1_\sharp\gamma = f^+,\ \pi^2_\sharp\gamma = f^-\}$$

trying to minimize

$$\min_{\mathcal{P}(f^+,f^-)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\gamma(x,y). \tag{2.1.3}$$

The relationship between problems (2.1.1) and (2.1.3) is not evident at a first glance and it is given by the fact that if $s \in \mathcal{A}(f^+,f^-)$ then $(id \times s)_\sharp f^+ \in \mathcal{P}(f^+,f^-)$. The elements of $\mathcal{A}(f^+,f^-)$ will be called transports while elements of $\mathcal{P}(f^+,f^-)$ will be called plans (or plannings) of transport. Before going in more details let us give some examples.

The first example is one dimensional and shows (in a trivial way) that the minimum for (2.1.3) can be strictly smaller than the minimum for problem (2.1.1).

EXAMPLE 2.1.2. Let $f^+ = \delta_0$ and let $f^- = \frac{1}{2}(\delta_{-1} + \delta_1)$. It is easy to see that the unique optimal planning in this case is given by

$$\gamma = \frac{1}{2}\delta_0 \times \delta_1 + \frac{1}{2}\delta_0 \times \delta_{-1}.$$

On the contrary the set of optimal transports is empty and the inf in (2.1.1) is $+\infty$.

The next example is two dimensional and shows as the previous case a gap between the two problems. In this case the class $\mathcal{A}(f^+,f^-)$ is not empty.

EXAMPLE 2.1.3. Let $f^+ = \delta_{(0,-1)} + \frac{1}{2}(\delta_{(0,\frac{1}{2})} + \delta_{(0,\frac{3}{2})})$ and $f^- = \delta_{(1,1)} + \frac{1}{2}(\delta_{(1,-\frac{1}{2})} + \delta_{(1,-\frac{3}{2})})$. It is easy to see that the optimal plan in this case is given by:

$$\gamma = \delta_{(0,-1)} \times \frac{1}{2}(\delta_{(1,-\frac{1}{2})} + \delta_{(1,-\frac{3}{2})}) + \frac{1}{2}(\delta_{(0,\frac{1}{2})} + \delta_{(0,\frac{3}{2})}) \times \delta_{(1,1)}$$

which is not a graph. However the class of the transports is not empty as it contains at least $s : \mathbb{R}^2 \to \mathbb{R}^2$ defined by: $s(x,y) = (x+1,-y)$ which is optimal in $\mathcal{A}(f^+,f^-)$. A simple comparison between the costs now shows the gap.

In both the previous examples f^+ shows some atoms. This is not incidental, indeed the next theorem which appear on [**5**] and [**36**] (with different proofs) shows that the gap between the two problems does not appear for distributed measures.

THEOREM 2.1.4. *Assume that the pointwise cost $c(\cdot,\cdot)$ is continuous, then provided f^+ has no atom the set*

$$\{(id \times s)_\sharp f^+ |\ s \in \mathcal{A}(f^+,f^-)\}$$

is dense in $\mathcal{P}(f^+,f^-)$ for the weak convergence of measures.*

REMARK 2.1.5 (The Monge-Kantorovich-Wasserstein distance). *It is simple to verify that $\mathfrak{D}(f^+,f^-)$:=(2.1.3) defines a distance on the set of probability measure on Ω. This distance, very useful in various contests, is called Monge-Kantorovich-Wasserstein distance or Wasserstein*[1] *distance and induces a topology equivalent to the weak* topology on the set where it is defined. In the next chapter we will show an interesting relationship with some shape optimization problems.*

The reason for Kantorovich to introduce problem (2.1.3) was that being the new cost functional linear with respect to the variable γ and being the class $\mathcal{P}(f^+,f^-)$ weakly* compact, the existence of a minimizer can be obtained straightforward.

THEOREM 2.1.6. *If $c(x,y)$ is a lower semicontinuous function then there exists an optimal planning γ for problem (2.1.3).*

The counterpart of the simplification in proving the existence of an optimal planning is the difficulty of proving that there exists an optimal planning γ which is generated by a transport s (i.e. $\gamma = (id \times s)_\sharp f^+$). The next theorem gives a geometric criterion for this to happen, as it was proved in [**5**].

THEOREM 2.1.7. *Any transport plan γ concentrated on a γ-measurable graph Γ is induced by a transport map.*

[1] In many contests the name Wasserstein distance is referred to the case in which the integrand in (2.1.3) is $|x-y|^2$ and the name Monge's distance refers to the case $|x-y|$.

Another feature of the Kantorovich approach to Monge problem is the duality. Several problems where shown to be dual to problem (2.1.3). In particular here we want to refer to two of them. The first one has been the main way to obtain solutions of Monge problem in the case of strictly convex or concave cost functions, the second one will be a bridge connecting the Monge problem with the shape optimization problems we will introduce in the next chapter. The duality between $\mathcal{M}(\overline{\Omega} \times \overline{\Omega})$ and the space of the continuous and bounded functions $C_b(\Omega \times \Omega)$ leads to linear problems defined on the space of continuous functions, with constraints .

THEOREM 2.1.8. *Problem (2.1.3) is dual to*

$$\sup_{(v,w)\in C_b(\Omega)\times C_b(\Omega)} \{\int_\Omega v(x)df^+(x) + \int_\Omega w(y)df^-(y)|\ s.t.\ v(x)+w(y) \le |x-y|\}. \quad (2.1.4)$$

PROOF. The proof is a minor variation of the proof of theorem 3.3.6. □

THEOREM 2.1.9. *Problem (2.1.3) is dual to*

$$\sup\{\int_\Omega u(x)df^+(x) - \int_\Omega u(y)df^-(y)|\ u \in C(\Omega)\ s.t.\ u(x)-u(y) \le |x-y|\}. \quad (2.1.5)$$

PROOF. See theorem 3.3.6. □

One of the main properties which has been used to prove the existence of an optimal transport is the following property of optimal plans. The support of an optimal plan γ satisfies a property called c-cyclical monotonicity where $c = c(x, y)$ denotes, as already said, the pointwise cost function.

DEFINITION 2.1.10. *A set $S \subset \overline{\Omega} \times \overline{\Omega}$ is said c-cyclically monotone (or simply cyclically monotone if c is the Euclidean distance in a subset of $\mathbb{R}^N$) if and only if for every integer n, for all sets of pairs $(x_1, y_1),\ \ldots, (x_n, y_n) \in S$ and for any permutation σ in $\mathfrak{S}_n$*

$$c(x_1, y_1) + \cdots + c(x_n, y_n) \le c(x_1, y_{\sigma(1)}) + \cdots + c(x_n, y_{\sigma(n)}). \quad (2.1.6)$$

In the case $n = 2$, (2.1.6) becomes

$$c(x_1, y_1) + c(x_2, y_2) \le c(x_1, y_2) + c(x_2, y_1), \quad (2.1.7)$$

which holds whenever (x_1, y_1) and (x_2, y_2) belong to S. Even if the general property (2.1.6) is stronger than (2.1.7 and it is not difficult to construct a situation for which (2.1.7) holds but (2.1.6) does not, the proofs of chapter 4 will use only property 2.1.7.

Under very general assumptions on c (see [**57**], [**35**]) it is possible to prove the next theorem (which is now classical), that explains the main property of the support of an optimal plan:

THEOREM 2.1.11. *Let $\gamma \in \mathcal{M}^+(\overline{\Omega} \times \overline{\Omega})$ be a measure such that $\pi^1_{\#}\gamma = f^+$ and $\pi^2_{\#}\gamma = f^-$. Then γ is an optimal plan of transport if and only if spt(γ) is c-cyclically monotone.*

The amazing fact is that in the case $c(x, y) = \phi(|x - y|)$ for some positive, strictly convex ϕ the c-cyclical monotonicity is, roughly speaking, all what is needed in order to prove that γ is actually a graph [**35**]. It seems that in the linear case cyclical monotonicity is not enough and other properties come in play. We will use this property in chapter 4 to prove some regularity result for the solutions of some shape optimization problem which will be introduced in the next chapter and for the so called Monge transport density which we will introduce in a moment.

Let u be a 1-Lipschitz function which is a maximum point for the dual functional introduced in theorem 2.1.9. Then thanks to theorem 3.3.4 there are positive measures μ such that the pairs (μ, u) are solutions of the following system:

$$(WMK) = \begin{cases} -\text{div } \mu D_\mu u = f & on\ \Omega \\ |D_\mu u| = 1 & \mu - a.e.; \end{cases} \tag{2.1.8}$$

where the first equation is written in the sense of distributions and the sense of the tangential gradient D_μ is explained in the next section. A measure μ solution of (2.1.8) is called transport density while u is called Kantorovich potential. In chapter 4 we will prove some regularity result for transport densities. The following theorem contains a relationship between equation (2.1.8) and the Monge problem.

THEOREM 2.1.12. *Let (μ, u) be a solution of (2.1.8). Then spt$(\mu) \subset \overline{\Omega}$ and $\mu(\overline{\Omega}) =$ (2.1.3).*

The proof of this theorem will be a byproduct of another theorem of the next chapter. A much deeper result is contained in [**5**], theorems 4.2 and 5.1, which permits to construct an optimal plan starting from a solution of (2.1.8) passing through a solution of another equivalent problem. Here, in the next chapter, we will do the opposite passage, i.e. we will show how to construct a solution of (2.1.8) starting from an optimal plan γ. We will meet again equation (2.1.8) in the next chapter. Indeed equation (2.1.8) will appear again but μ in that case will be called an optimal shape.

2.2. Calculus of Variations with respect to a measure

Here we introduce some notions from [**13**] and in chapter 3 we will show that the same notions can be introduced on a manifold and applied there as well.

Given a positive measure $\mu \in \mathcal{M}^+(\Omega)$, we consider the space

$$X_\mu := \{\varphi \in L^1_\mu \mid \text{div } (\varphi\mu) \in \mathcal{M}(\Omega)\}.$$

In some sense X_μ can be seen as the space of tangent fields to μ. Then we define the *tangent space* to μ for μ-a.e. $x \in M$ as:

$$T_\mu(x) := \mu - ess \bigcup \{\varphi(x) | \ \varphi \in X_\mu\}.$$

The $\mu-$essential union is defined as a μ-measurable closed multi-function such that:

- $\varphi \in X_\mu \Rightarrow \varphi(x) \in T_\mu(x)$ for μ-a.a. $x \in \Omega$;
- among all the multi-functions with the previous property the $\mu-$essential union is minimal with respect to the set inclusion.

Properties and applications of this definition of tangent space to a measure have been explored in various papers, between them, we address to [**13**], [**14**], [**12**], [**34**]. Once we have the notion of tangent space to μ it is natural to define the notion of μ-*tangential gradient* of a function $u \in \mathcal{D}(\Omega)$ as:

$$D_\mu u(x) = P_\mu(x, Du(x)) \qquad \mu - a.e.$$

where we denoted by $P_\mu(x, \cdot)$ the orthogonal projection on $T_\mu(x)$. It is possible to show that the operator $D_\mu(x)$ is closable in L^p_μ and this takes to a suitable definition of Sobolev space with respect to μ:

DEFINITION 2.2.1. *The Sobolev space* $\mathrm{W}^{1,p}_\mu(\Omega)$ *is the completion of* $\mathcal{D}(\Omega)$ *with respect to the norm:*

$$\|u\|_{1,p} := \|u\|_{L^p_\mu} + \|D_\mu u\|_{L^p_\mu}.$$

An important property is the following generalization of the integration by part formula, which holds for any $v \in \mathrm{W}^{1,p}_\mu(\Omega)$ and $\varphi \in X_\mu$:

$$\int D_\mu v \cdot \varphi \ d\mu = -\langle \text{div } (\varphi\mu), v\rangle. \tag{2.2.1}$$

This notions are useful for various applications; the one we will be interested in is that if f belongs to the dual space of $(W^{1,p}_\mu(\Omega)$ then the relaxed energy of (4.1.1) is given by:

$$\overline{E}(\mu, u) = \frac{1}{2}\int_\Omega |D_\mu u|^2 \ d\mu - \langle f, u\rangle \qquad u \in \mathrm{W}^{1,p}_\mu(\Omega). \tag{2.2.2}$$

2.3. A duality theorem

We report here a reformulation useful in our settings of a classical duality theorem (see [**27**] proposition 2.3).

THEOREM 2.3.1. *Let* X, Y *be two Banach spaces and let* X', Y' *be respectively their dual spaces. Let* $f : X \to]-\infty, +\infty]$ *and* $g : Y \to]-\infty, +\infty]$ *be respectively a linear* *functional and a convex and l.s.c. function. Let $A : X \to Y$ be a linear operator such that $D(A)$ is dense in X. Assume that there exists $x_0 \in D(A)$ such that*

$$y \to g(Ax_0 + y) + f(x_0)$$

is continuous at $0 \in Y$ and that the infimum in (2.3.1) below is finite. Then

$$\inf_{D(A)} \{g(Ax) + f(x)\} = \sup_{Y'}\{-f^*(-A^*y^*) - g^*(y^*)\}. \tag{2.3.1}$$

Proof. Let $h(y) = \inf_{D(A)}\{g(Ax + y) + f(x)\}$, so that $h(0)$ is the quantity on the left side of equation (2.3.1). Let us calculate $h^*(y^*)$

$$h^*(y^*) = \sup_Y\{< y, y^* > -h(y)\} =$$

$$= \sup_Y\{< y, y^* > - \inf_{D(A)} \{g(Ax + y) + f(x)\}\} =$$

$$= \sup_Y \sup_{D(A)} \{< y, y^* > -g(Ax + y) - f(x)\} =$$

$$= \sup_{D(A)} \sup_Y\{< y, y^* > -g(Ax + y) - f(x)\} =$$

$$= \sup_{D(A)} \sup_Y\{< y + Ax, y^* > -g(Ax + y)- < Ax, y^* > -f(x)\} =$$

$$= \sup_{D(A)} \{g^*(y^*)- < x, A^*y^* > -f(x)\} =$$

$$= g^*(y^*) + f^*(-A^*y^*). \tag{2.3.2}$$

Then $h^{**}(0) = \sup_{Y'}\{-g^*(y^*) - f^*(-A^*y^*)\}$ which is the quantity on the right side of equation (2.3.1) The Fenchel inequality imply $h^{**}(0) \leq h(0)$; to prove the reverse inequality it is enough to prove that h is bounded in a neighborhood of 0 (in fact h is convex). As the function

$$y \to g(Ax_0 + y) + f(x_0)$$

is continuous at $0 \in Y$ and $h(0)$ is finite, there exists a neighborhood U on $0 \in Y$ and a constant M such that

$$g(Ax_0 + y) + f(x_0) \leq M \quad \forall y \in U.$$

Then for all $y \in U$

$$h(y) \leq g(Ax_0 + y) + f(x_0) \leq M.$$

□

2.4. Convergences of measures

In this section we report some definition of convergences of measures which will be useful in the last chapter. Moreover we give some basic theorem about these notions.

LEMMA 2.4.1. *Let $\{\mu_h\} \subset \mathcal{M}$, with $\mu_h \rightharpoonup \mu$. Let d be a positive integer and let $f_h \in (L^p_{\mu_h})^d$ with $\|f_h\|_{(L^p_{\mu_h})^d} \leq M$ for some $p \in [0, +\infty)$. Then the sequence $\{f_h \mu_h\}$ is bounded in $\mathcal{M}^d$, and the weak limit of any convergent subsequence is absolutely continuous with respect to μ, with density $f \in (L^p_{\mu_h})^d$. Moreover for every convex and lower semicontinuous function $\Phi : \mathbb{R}^d \to [0, +\infty)$ we have :*

$$\liminf_{h \to \infty} \int \Phi(f_h) d\mu_h \geq \int \Phi(f) d\mu.$$

DEFINITION 2.4.2. *Let $\mu_h, \mu \in \mathcal{M}^+$, with $\mu_h \to \mu$ and let $f_h, f : \mathbb{R}^N \to \mathbb{R}^d$ be respectively μ_h and μ integrable. We say that f_h weakly converges to f with respect to the pair (μ_h, μ) and we write $f_h \overset{w(\mu_h,\mu)}{\to} f$, if $f_h \mu_h \rightharpoonup f\mu$ in $\mathcal{M}^d$.*

When $f_h \overset{w(\mu_h,\mu)}{\to} f$ it is natural to consider the sequence of Young measures associated to f_h, $\nu_h \in Y(\mathbb{R}^N, \mu_h; \mathbb{R}^d)$ which describe the oscillation of f_h with respect to the measure μ_h, that is

$$< \nu_h, \varphi > := \int \varphi(x, f_h(x)) d\mu_h(x), \quad \forall \varphi \in C_0(\mathbb{R}^N \times R^d).$$

Since $\{\nu_h\}$ is a bounded sequence, up to subsequences it converges to some measure ν on the product space $\mathbb{R}^N \times \mathbb{R}^d$ whose marginal on the first factor is μ. The disintegration of ν yields then

$$< \nu, \varphi > := \int < \varphi(x, \cdot), \nu_x > d\mu(x), \quad \forall \varphi \in C_0(\mathbb{R}^N \times R^d).$$

In other words we will say that $\{f_h\}$ weakly converges to the family of probabilities (ν_x) in the sense of Young measures with respect to (μ_h, μ), and we will write $f_h \overset{Y(\mu_h,\mu)}{\to} (\nu_x)$ if and only if

$$\int \varphi(x, f_h(x)) d\mu_h(x) \to \int < \varphi(x, \cdot), \nu_x > d\mu(x), \quad \forall \varphi \in C_0(\mathbb{R}^N \times R^d). \qquad (2.4.1)$$

DEFINITION 2.4.3. *Let $\mu_h, \mu \in \mathcal{M}^+$, with $\mu_h \to \mu$ and let $f_h, f : \mathbb{R}^N \to \mathbb{R}^d$ be respectively μ_h and μ integrable. We say that f_h strongly converges to f with respect to the pair (μ_h, μ) and we write $f_h \overset{s(\mu_h,\mu)}{\to} f$, if*

$$\lim_{h \to \infty} \int \varphi(x, f_h(x)) d\mu_h = \int \varphi(x, f(x)) d\mu, \quad \forall \varphi \in C_0(\mathbb{R}^N \times \mathbb{R}^d).$$

PROPOSITION 2.4.4. *Let* $f_h \in (L^1_{\mu_h})^d$, *with* $f_h \overset{Y(\mu_h,\mu)}{\to} (\nu_x)$ *and assume that* $\mu_h \to \mu$ *tightly (i.e.* $< \mu_h, \psi > \to < \mu, \psi >$ *for each* ψ *continuous and bounded.). Then*
(i) for every nonnegative lower semicontinuous function φ *on* $\mathbb{R}^N \times \mathbb{R}^d$ *we have*

$$\liminf_{h\to\infty} \int \varphi(x, f_h(x)) d\mu_h(x) \geq \int < \varphi(x,\cdot), \nu_x > d\mu(x);$$

(ii) assume in addition that $\{f_h\}$ *is uniformly bounded in the* $(L^p_{\mu_h})^d)-$*norm, for some* $p > 1$*; then, for any continuous function* φ *on* $\mathbb{R}^N \times \mathbb{R}^d$ *satisfying a growth condition of the type* $|\varphi(x,z)| \leq C(1 + |z|^q)$ *with* $q \leq p$, *we have*

$$\lim_{h\to\infty} \int \varphi(x, f_h(x)) d\mu_h(x) = \int < \varphi(x,\cdot), \nu_x > d\mu(x).$$

(in particular $\int < |z|^p, \nu_x > d\mu(x) < \infty$.*)*

LEMMA 2.4.5. *Let* $\{\mu_h\} \subset \mathcal{M}^+$, *with* $\mu_h \rightharpoonup \mu$, *and let* $f_h \in (L^p_{\mu_h})^d$ *with* $\|f_h\|_{(L^p_{\mu_h})^d} \leq M$ *for some* $p > 1$. *Assume that* $f_h \overset{w(\mu_h,\mu)}{\to} f$ *and* $f_h \overset{Y(\mu_h,\mu)}{\to} (\nu_x)$. *Then* $[\nu_x] = f(x)$ *for* μ-*a.e.* x. *Moreover, if it exists a closed subset* C *of* $\mathbb{R}^d$ *such that, for every* h, $f_h(x) \in C$ *for* μ_h-*a.e.* x, *then* $spt(\nu_x) \subset C$ *for* $\mu-$*a.e.* x.

CHAPTER 3

Existence theorems for some shape optimization problems in a general setting

3.1. Introduction

The goal of this chapter is two-fold. The first aim is to prove an existence theorem for a classical shape optimization problem in a constrained or anisotropic version. Indeed we will assume that the design region is an imbedded Riemannian manifold (then an open subset of $\mathbb{R}^N$ equipped with a smooth anisotropic Riemannian distance is allowed). The second aim of the chapter is to exploit the method introduced in [**12**], [**13**], [**14**], to prove existence results for this kind of problems. This method do not requires any linear structure on the underlying design region while more classical method do. For different approachs to the shape optimization problems we refer to: [**47**], [**48**], [**53**].

Let us describe the problem: Assume that we are given an heat source f and a certain amount m of a conductor C (conductor means positive conductivity coefficient). Which is the best way to distribute C in a certain assigned region Ω? (Of course one need to explain what "best" means). Taking as a model for the distribution of C a nonnegative bounded function θ in Ω open subset of $\mathbb{R}^N$ such that $\int_\Omega \theta(x)dx = m$ (see [**3**]) the energy associated to a distribution of temperature u is given by:

$$E(\theta, u) = \int_\Omega |Du(x)|^2 \theta(x) dx - \int_\Omega f(x)u(x)dx. \tag{3.1.1}$$

For an assigned distribution of material $\theta(x)$ the equilibrium temperature u_θ realize:

$$E(\theta, u_\theta) = \min_u E(\theta, u) := \mathfrak{C}(\theta). \tag{3.1.2}$$

A distribution of material θ_1 is better then θ if

$$\mathfrak{C}(\theta_1) > \mathfrak{C}(\theta). \tag{3.1.3}$$

So we look for

$$\max_\theta \mathfrak{C}(\theta). \tag{3.1.4}$$

The use of functions as model for the distribution of C do not allows concentrate distributions, then instead of $\theta(x)dx$ we will deal with positive measures μ such that $\int d\mu = m$. Moreover using measures we can expect the problem to have a solution and usually this does not happens in the class of "distributed mass". Also f will be assumed to be a measure whose positive and negative part f^+ and f^- are finite, compactly supported and of equal total mass. To deal with such kind of distributions of mass we need to adapt the instruments for Calculus of Variations with respect to a measure developed in [**13**], [**14**] and introduced in the previous chapter. The use of Riemannian setting permits also to deal with situations in which there are preferred direction for heat propagation. In section 2 we will summarize some background notions on Sobolev spaces with respect to a measure on a manifold. In section 3 we will prove the equivalence of (3.2.3) with two problem related to the Monge-Kantorovich mass transportation problem (see [**28**], [**57**], [**12**]), and we will exploit this equivalence to obtain an existence result for (3.2.3) and an explicit formula for some solutions of it. It is in section 3 that the difference of geometry between the manifold and $\mathbb{R}^n$ will show up in a multiplicity of solutions with very different structures. In section 4 we will give an explicit and simple example of application and we will make some remarks. In section 5 we will introduce, in an open set of $\mathbb{R}^N$ the shape optimization problem which correspond to (3.1.4) in the vectorial case and which stems from a classical problem in elasticity.

3.2. Notation and preliminaries

Through all this chapter M will be a connected Riemannian manifold without boundary and not necessarily compact. The Riemannian metric on the tangent space at x will be denoted by $g_{ij}(x)$. By TM and T^*M we will denote the tangent and the cotangent bundle of M, $g^*_{ij}(x)$ will denote the dual metric of $g_{ij}(x)$. $D_M u$ denote the tangential gradient of u while $div_M v$ will be the tangential divergence (which is in some sense the adjoint of D_M) of v. We will often refer to M as a metric space, in this case it is intended that d is the geodesic distance on M. $\mathcal{M}(X, Y)$ will denote the space of Y-valued finite measure on X. When $Y = \mathbb{R}$ will be omitted. With these notations the energy associated to a smooth distribution of temperature u will be then:

$$E(\mu, u) = \frac{1}{2}\int_M |Du|^2_{g_{ij}}\, d\mu - < f, u > . \tag{3.2.1}$$

Then $\mathfrak{C}(\mu)$ is given by:

$$\mathfrak{C}(\mu) = \min_u E(\mu, u) \tag{3.2.2}$$

and we look for:

$$\max_{|\mu|=m} \mathfrak{C}(\mu). \tag{3.2.3}$$

3.2.1. Adapting the calculus of variations with respect to a measure to the manifold and relaxed energy. In order to find a suitable representation of the relaxed energy of (3.2.1) we adapt some notion from section 2.2. Let $\mu \in \mathcal{M}(M)$ we consider the space

$$X_\mu := \{\varphi \in L^1_\mu |\ div_M(\varphi\mu) \in \mathcal{M}(M)\}, \tag{3.2.4}$$

where div_M is intended in the sense of the distributions. In some sense X_μ is the space of tangent field to μ. Then we define the tangent space to μ for μ-a.e. $x \in M$ as:

$$T_\mu(x) := \mu - ess \bigcup \{\varphi(x) |\ \varphi \in X_\mu\}. \tag{3.2.5}$$

(We defined the μ essential union in the previous chapter). With this notion of tangent space to μ the μ-tangential gradient of a function u is:

$$D_\mu u(x) = P_\mu(x, D_M u(x)) \quad \mu - a.e.$$

where we denoted by $P_\mu(x, \cdot)$ the orthogonal projection on $T_\mu(x)$. It is possible to show that the operator D_μ is closable in L^p_μ and this take again to a suitable definition of Sobolev space with respect to μ:

DEFINITION 3.2.1. *The Sobolev space $W^{1,p}_\mu(M)$ is the completion of $\mathcal{D}(M)$ with respect to the norm:*

$$\|u\|_{1,p} := \|u\|_{L^p_\mu} + \|D_\mu u\|_{L^p_\mu}.$$

Using this notions one obtain that if f belongs to the dual space $(W^{1,p}_\mu(M))'$ the relaxed energy of (3.2.1) is given (see [**13**], [**56**]) by:

$$\frac{1}{2}\int_M |D_\mu u|^2_{g_{ij}} d\mu - < f, u > \qquad u \in W^{1,p}_\mu(M). \tag{3.2.6}$$

3.2.2. Metric formulation the Monge mass transfer problem. Given two measures f^+, f^- on M of equal total mass, the transport problem (2.1.1) here can be written as

$$\min_{\varphi \in \mathcal{A}(f^+, f^-)} \int_M d(x, \varphi(x)) d\mu_1. \tag{3.2.7}$$

where d is the distance on M induced by the metric. Then the weak formulation (2.1.3) can be written

$$\min_{\gamma \in \mathcal{P}(f^+, f^-)} \int_{M \times M} d(x, y) d\gamma(x, y). \tag{3.2.8}$$

3.2.3. Monge-Kantorovich equation. With the name Monge-Kantorovich equation, we refer to the system:

$$(MK) = \begin{cases} -div_M a(x)Du(x) = f & on\ M \\ |Du|_{g_{ij}} = 1 & a(x)dg_{ij} - a.e.. \end{cases} \tag{3.2.9}$$

The integration by parts formula

$$\int_\Omega div_M \phi d\sqrt{g_{ij}} = \int_{\partial\Omega} \phi \nu_{\partial\Omega} d\mathcal{H}^{n-1},$$

which holds for each regular open subset Ω of M end each vector field $\phi \in C^1(M, TM)$ permits to give a weak formulation of this equation:

$$(WMK) = \begin{cases} -div_M \mu D_\mu u(x) = f & on\ M \\ |D_\mu u| = 1 & \mu - a.e.. \end{cases} \tag{3.2.10}$$

We remark that on an open subset of $\mathbb{R}^N$ these equations coincide with (2.1.8).

3.3. Existence of optimal shapes

PROPOSITION 3.3.1. *Let M be a connected Riemannian manifold, then the following duality holds:*

$$\inf\{\frac{1}{2}\int_M |D_M u|^2_{g_{ij}} d\mu - < f, u >: \quad u \in C^\infty(M)\} \tag{3.3.1}$$

is equal to

$$\sup\{-\frac{1}{2}\int_M |\sigma|^2_{g^*_{ij}} d\mu : \sigma \in L^2_\mu(M, T^*M),\ -div_M \sigma\mu = f\}. \tag{3.3.2}$$

Moreover whenever (3.3.1) is finite (3.3.2) is a max.

PROOF. Apply theorem 2.3.1 with $X = L^2_\mu(M)$, $A = D_M$, $Y = L^2_\mu(M, TM)$. □

PROPOSITION 3.3.2. *Let M a connected Riemannian manifold, then the following duality holds:*

$$I(f, M) =: \min\{< -f, \phi >: \quad \phi \in Lip(M), \quad |D_M \phi|_{g_{ij}} \le 1\ a.e.\} \tag{3.3.3}$$

is equal to

$$\sup\{-\int_M d|\nu|_{g^*_{ij}} : \nu \in \mathcal{M}(M, T^*M),\ -div_M \nu = f\}. \tag{3.3.4}$$

PROOF. Apply theorem 2.3.1 with $X = C(M)$, $A = D_M$, $Y = C(M, TM)$. □

Remark We remark that the total variation of ν depends on the Riemannian metric. In local coordinates (3.3.4) would involve a convex functional of ν in the sense of [**38**]. We do not need to work in local coordinates because our approach is global.

THEOREM 3.3.3.

$$\mathfrak{C}(\mu) \leq \frac{1}{2m}(I(f,M))^2$$

PROOF. Let $\phi \in Lip_1(M)$

$$\frac{1}{2}\int_M |\sigma|^2_{g^*_{ij}} d\mu \geq \frac{1}{2m}(\int_M < \sigma, D_M\phi >_{T^*,T_*} d\mu)^2 = \frac{1}{2m}(< f, \varphi >)^2. \tag{3.3.5}$$

Then

$$\frac{1}{2}\int_M |\sigma|^2_{g^*_{ij}} d\mu \geq \frac{1}{2m}(I(f,M))^2. \tag{3.3.6}$$

Then using the definition of $\mathfrak{C}(\mu)$ and proposition (3.3.1) we obtain the following upper estimate

$$\mathfrak{C}(\mu) \leq \frac{1}{2m}(I(f,M))^2. \tag{3.3.7}$$

□

We will now prove that such estimate is optimal and establish an existence result for problem (3.2.3) and a relationship between the Wasserstein distance of f^+ and f^- and the value of $\mathfrak{C}(\mu)$, which extend to this setting a results of [**12**].

THEOREM 3.3.4. *There exists a pair (u,μ), $u \in Lip(M)$ and $\mu \in \mathcal{M}(M)$ solution of the problem:*

$$\begin{cases} -div_M \mu D_\mu u(x) = f & on\ M \\ |D_\mu u| = 1 & \mu - a.e.. \end{cases} \tag{3.3.8}$$

PROOF. Let u and ν be respectively solutions for problems (3.3.3) and (3.3.4). Applying proposition (3.3.2) we get:

$$-\int_M d|\nu|_{g^*_{ij}} =< -f, u > . \tag{3.3.9}$$

By Riesz representation theorem we may write $\nu = \theta\mu$ where $\mu = |\nu|$ and θ is an unitary cotangent vector defined μ-a.e. We get :

$$\int_M < \theta, D_\mu u >_{T^*,T_*} d\mu =< f, u >= \int_M d\mu, \tag{3.3.10}$$

then $\theta = D_\mu u$ μ-a.e.. □

We now connect the previous theorem with the shape optimization problem.

THEOREM 3.3.5. *Let (u,μ) a pair of solutions of (WMK) as obtained in theorem 3.3.4, define $\mu_0 = \frac{m}{I(f,M)}\mu$ then,*

$$C(\mu_0) = \frac{1}{2m}(I(f,M))^2.$$

PROOF. Let us choose $u_0 = \frac{I(f,M)}{m} u$ and $\sigma_0 = du_0$ then:

$$-\frac{1}{2}\int_M |du_0|^2_{g^*_{ij}} d\mu_0 = -\frac{1}{2}\int_M |D_M u_0|^2_{g_{ij}} d\mu_0 = -\frac{1}{2} < u_0, f >= -\frac{1}{2m} I(f)^2.$$

□

The following theorem establish a relationship between the optimal value of $\mathfrak{C}(\mu)$ for assigned heat source f and the Monge-Wasserstein distance of f^+ and f^-, $\mathfrak{D}(f^+, f^-)$.

THEOREM 3.3.6. $I(f, M) = \mathfrak{D}(f^+, f^-)$

PROOF. We can write $I(f, M)$ in the following way:

$$I(f, M) = \inf\{< -f, u > | \quad u(x) - u(y) \leq d(x, y)\}. \tag{3.3.11}$$

Let us consider $p \in C(M \times M)$ and

$$h(p) = \inf\{< -f, u > | \quad u(x) - u(y) + p(x, y) \leq d(x, y)\}. \tag{3.3.12}$$

(In particular $I(f, M) = h(0)$.)

$$G(p, u) = \begin{cases} < -f, u > & if\ u(x) - u(y) + p(x, y) \leq d(x, y) \\ +\infty & otherwise \end{cases} \tag{3.3.13}$$

G is convex, then $h(p) = \inf_u G(p, u)$ is convex too. Let us calculate $h^*(\gamma)$ for $\gamma \in \mathcal{M}(M \times M)$.

$$\begin{aligned} h^*(\gamma) &= \sup_p\{< \gamma, p > - \inf_u\{< -f, u > \mid u(x) - u(y) + p(x, y) \leq d(x, y)\}\} = \\ &= \sup_u\{< -f, u > + \sup_p\{< \gamma, p > \mid u(x) - u(y) + p(x, y) \leq d(x, y)\}\} = \\ &\sup_u\{< -f, u > + \int d(x, y) d\gamma(x, y) + < \gamma, u(y) - u(x) >\} = \\ &= \begin{cases} +\infty & if\ f \neq \pi^1(\gamma) - \pi^2(\gamma), \\ \int\int d(x, y) d\gamma(x, y) & if\ f = \pi^1(\gamma) - \pi^2(\gamma). \end{cases} \end{aligned} \tag{3.3.14}$$

Then $h^{**}(0) = \mathfrak{D}(f^+, f^-)$. Then we just need to prove that h is lower semicontinuous at 0, it will follows that $h^{**}(0) = h(0)$ which will conclude the proof. Let $\{p_n\}$ be a sequence such that $p_n \to 0$ uniformly and

$$\varepsilon_n + h(p_n) \geq < -f, u_n > \tag{3.3.15}$$

for u_n such that $u_n(x) - u_n(y) + p_n(x, y) \leq d(x, y)$, applying the Ascoli-Arzelá's theorem $u_{n_k} \to u$ uniformly, then:

$$\liminf_{k\to\infty} h(p_{n_k}) = \liminf_{k\to\infty}\{h(p_{n_k}) + \varepsilon_k\} \geq \liminf_{k\to\infty} < -f, u_{n_k} > \geq h(0). \tag{3.3.16}$$

□

REMARK 3.3.7. The Riemannian metric structure plays a fundamental role in equation (3.3.11), it would be not possible to make such a reformulation for a generic distance.

COROLLARY 3.3.8. *The optimal value of $\mathfrak{C}(\mu_0)$ for assigned heat source f is :*

$$\mathfrak{C}(\mu_0) = \frac{1}{2m}(\mathfrak{D}(f^+, f^-))^2. \tag{3.3.17}$$

In ([**12**]) the authors introduced a method to produce explicit solutions of (3.3.4) and then of the problem of optimal mass distribution with or without Dirichlet conditions, this method can be extended to manifolds. Notice that by to the non uniqueness of the minimal geodesic joining two points the problem is much more involved then in the case of convex subsets of the Euclidean spaces.

Let γ be a minimizer for (3.2.8), fix $\epsilon > 0$ and for each $(x, y) \in supp\gamma$ select an almost-minimal curve $\Gamma_\epsilon(x, y)$ (almost minimal means $length(\gamma_\epsilon(x, y)) \leq d(x, y) + \epsilon$) joining x and y in such a way that the mapping $(x, y) \to \Gamma_\epsilon(x, y)$ is a measurable multivalued map. Let us define $\nu_\epsilon \in \mathcal{M}(M, T^*M)$ as follow

$$< \nu_\epsilon, \phi >:= \int_{M\times M} \int_{\Gamma_\epsilon(x,y)} \phi \cdot \underline{t} d\mathcal{H}^1 d\gamma(x, y), \tag{3.3.18}$$

where $\underline{t}$ is the unitary tangent vector to Γ_ϵ. With ν_ϵ defined as in (3.3.18) it holds:

LEMMA 3.3.9.

$$-div_M \nu_\epsilon = f \tag{3.3.19}$$

PROOF. Let $\varphi \in \mathcal{D}(M)$, then

$$\begin{aligned} < \nu_\epsilon, D\varphi > &= \textstyle\int_{M\times M} \int_{\Gamma_\epsilon(x,y)} D_M\varphi \cdot \underline{t}\ d\mathcal{H}^1 d\gamma(x, y) = \\ &= \textstyle\int_{M\times M} \{\varphi(y) - \varphi(x)\} d\gamma(x, y) = \\ &= \textstyle\int_M \varphi df^+ - \int_M \varphi df^- = \int_M \varphi df. \end{aligned} \tag{3.3.20}$$

□

Moreover ν_ϵ has the following minimality property:

THEOREM 3.3.10. *ν_ϵ is a minimizing sequence for the problem*

$$\inf\{\int_M d|\nu|_{g^*_{ij}} : \nu \in \mathcal{M}(M, T^*M),\ -div_M \nu = f\}. \tag{3.3.21}$$

Moreover it is weakly compact in $\mathcal{M}(M)$ and any cluster point is a solution of (3.3.21).*

PROOF. Using theorem 3.3.6,

$$\sup_{|D\varphi|_{g_{ij}} \leq 1} < \varphi, f >= \int_{M\times M} d_M(y, x) d\gamma(y, x). \tag{3.3.22}$$

On the other hand, using equation (3.3.19) and the definition of total variation of ν_ϵ:

$$\sup_{|D\varphi|_{g_{ij}}\leq 1} < \nu_\epsilon, D\varphi > \leq \int d|\nu_\epsilon|_{g^*_{ij}} \leq$$
$$\leq \int_{M\times M} (d_M(y,x)+\epsilon)d\gamma(y,x) =$$
$$= \sup_{|D\varphi|_{g_{ij}}\leq 1} < \varphi, f > +\epsilon\gamma(M\times M). \quad (3.3.23)$$

□

The set $\{\nu_\epsilon\}_\epsilon$ is bounded in $\mathcal{M}(M)$ and then is weakly* compact. Consider a sequence ν_{ϵ_n} converging to ν, then ν is optimal for problem (3.3.4). In general ν is not given by a formula like (3.3.18) and the support of ν is contained in the closure of M.

REMARK 3.3.11. If M is geodesicaly complete (i.e. for each pairs of points $x, y \in M$ there exists a minimal geodesic connecting x and y) , then choosing $\Gamma(x,y)$ to be a geodesic between (x,y) we obtain, by formula (3.3.18) an optimal measure for (3.3.4).

Then to each optimal plan of transport γ it is associated at least a solution of the shape optimization problem. We now show that using the same method one can construct a big family of solutions associated to each γ. Some remark about this family will be in order in the next section. For simplicity we will now assume M geodesicaly complete. Let γ be an optimal plan of transport. For each pair $(x,y)\in supp\gamma$ select a family of minimal geodesics $\mathfrak{B}(x,y)$ in such a way that the mapping $(x,y)\to\mathfrak{B}(x,y)$ is measurable (see [**12**]). Consider a probability measure $\alpha_{(x,y)}$ on $\mathfrak{B}(x,y)$ (to be short we will write just α), and define $\nu\in\mathcal{M}(M,T^*M)$ as follows:

$$< \nu, \varphi > = \int_{M\times M}\int_{\mathfrak{B}}\int_{\Gamma(x,y)} \varphi\cdot \underline{t} d\mathcal{H}^1 \llcorner \Gamma(x,y) d\alpha(\Gamma) d\gamma. \quad (3.3.24)$$

With some minor variations on the previous proofs we obtain

LEMMA 3.3.12.

$$-div\nu = f \quad (3.3.25)$$

PROOF. Let $\varphi\in\mathcal{D}(M)$, then

$$\begin{aligned} < \nu, D\varphi > &= \textstyle\int_{M\times M}\int_{\mathfrak{B}}\int_{\Gamma(x,y)} D_M\varphi\cdot \underline{t} d\mathcal{H}^1 d\alpha d\gamma(x,y) = \\ &= \textstyle\int_{M\times M}\{\varphi(y)-\varphi(x)\}d\gamma(x,y) = \\ &= \textstyle\int_M \varphi df^+ - \int_M \varphi df^- = \int_M \varphi df. \end{aligned} \quad (3.3.26)$$

□

THEOREM 3.3.13. *ν minimize the total variation in the class of measures satisfying equation (3.3.19).*

PROOF. The proof is similar to that of theorem (3.3.10):

$$\sup_{|D\varphi|_{g_{ij}}\leq 1} < \varphi, f >= \int_{M\times M} d_M(y,x)d\gamma(y,x). \tag{3.3.27}$$

On the other hand, using equation (3.3.19) and the definition of total variation of ν:

$$\begin{aligned} \sup_{|D\varphi|_{g_{ij}}\leq 1} < \nu, D\varphi > & \leq \int d|\nu|_{g^*_{ij}} \leq \\ & \leq \int_{M\times M} d_M(y,x)d\gamma(y,x) = \\ & = \sup_{|D\varphi|_{g_{ij}}\leq 1} < \varphi, f > . \end{aligned} \tag{3.3.28}$$

$\square$

The solutions of the shape optimization problem we generate by this last method are extremely various, for dimension and for qualitative behaviour, as we will show in an example in the next section.

3.4. Some remarks and examples on the scalar case

If a pair (u,μ) is solution of $-div_M \mu D_\mu u = f$ then the tangent space $T_\mu(x)$ has dimension at least one (indeed $D_\mu u(x) \in T_\mu(x)$ and $|D_\mu u|_{g_{ij}} = 1$ $\mu-$a.e.). The Hausdorff dimension of a measure is bigger then the dimension of the tangent space then μ has Hausdorff dimension at least one. From figure (1),(2),(3) it is simple to see that on S^2 with $f = \delta_N - \delta_S$ one can build solutions μ whose dimension range between 1 and 2.

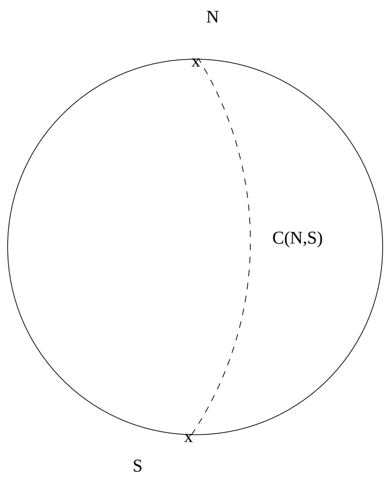

FIGURE 1. $f^+ = \delta_N$ and $f^- = \delta_S$; $\mu = \mathcal{H}^1 \llcorner C(N,S)$

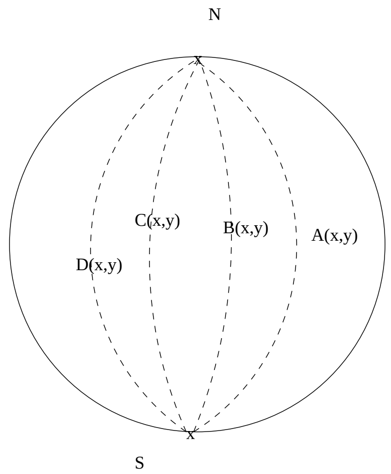

FIGURE 2. $f^+ = \delta_N$ and $f^- = \delta_S$; $\mu = \frac{1}{4}(\mathcal{H}^1 \llcorner A(N,S) + \mathcal{H}^1 \llcorner B(N,S) + \mathcal{H}^1 \llcorner C(N,S) + \mathcal{H}^1 \llcorner D(N,S))$

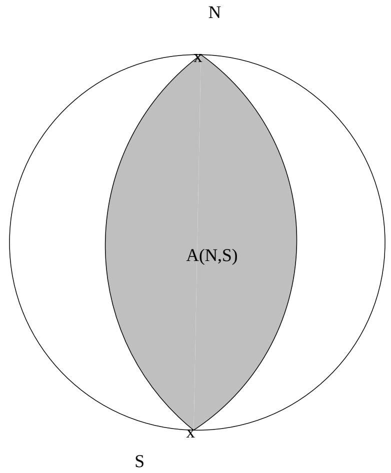

FIGURE 3. $f^+ = \delta_N$ and $f^- = \delta_S$; $\mu = \frac{1}{C}(\mathcal{H}^2 \llcorner A(N,S)$

The explicit formula for the solutions we found in the previous section permits to make a deep qualitative study of the distributions of mass. In the Euclidean setting we will use (chapter 4) this formula to obtain various regularity results for this distributions of mass. It has been recently proved ([**5**] that in a convex, open subset of $\mathbb{R}^N$ all the solution of (3.2.10) are given by formula (3.3.18) using a suitable optimal plan γ. As we have proved, on a manifold the situation is different.

3.5. The vectorial case

Here in this section we introduce a vectorial shape optimization problem classical in the framework of elasticity. This is the vectorial analogous of the problem we introduced at the beginning of this chapter. It still has a strong relationship with the transport problems and it is possible to make a discussion very similar to that of the previous sections of this chapter. However, as the only original contribution of this thesis on this problem is the approximation we will make in the last chapter, we will not go beyond an introduction. The content of this section is taken from [**12**] and we refer to that paper for a deeper discussion of the problem. In the last chapter we will find a suitable variational approximation which as been useful for numerical solution of this problem. The problem we describe consists in finding the best distribution of a given total mass in order to minimize the elastic compliance under the action of a given force field. As we did for heat sources and distributions of a conductor we will consider a force field $f \in \mathcal{M}(\Omega, \mathbb{R}^N)$. For a given displacement $u : \mathbb{R}^N \to \mathbb{R}^N$ we denote by $j(Du)$ the stored elastic energy density associated to u and we assume:

- j is convex;
- j is positively p-homogeneous, with $p > 1$
- $j(z) = j(z^{sym})$ where z^{sym} is the symmetric part of z;
- there exist two positive constant λ_1 and λ_2 such that
$$\lambda_1 |z^{sym}|^p \le j(z) \le \lambda_2 |z^{sym}|^p \quad \forall z \in \mathbb{R}^{N\times N}.$$

For instance, in the case of a homogeneous isotropic linearly elastic material, the function j is given by
$$j(z) = \beta |z^{sym}|^2 + \frac{\alpha}{2} |tr(z^{sym})|^2 \tag{3.5.1}$$
where α and β are the so called Lamé constants. Thus for a given mass distribution μ the stored elastic energy of a smooth displacement $u \in \mathcal{D}(\mathbb{R}^N; \mathbb{R}^N)$ is given by
$$\int j(Du)d\mu$$
and the total energy is given by
$$E(\mu, u) = \int j(Du)d\mu - < f, u > .$$
The standard notation in elasticity for $(Du)^{sym}$ is $e(u)$, so that we will write the stored energy functional as:
$$\int j(e(u))d\mu.$$
The problem above is a variational model which describes the behaviour of light structures, where the force due to their own weight can be neglected. Finally in order to

take in account possibly prescribed Dirichlet boundary conditions, we denote by $\mathcal{U}$ the set of smooth admissible displacements, which we assume to be a given convex cone of $\mathcal{D}(\mathbb{R}^N, \mathbb{R}^N)$. Therefore the infimum

$$\mathcal{E}(\mu) = \inf\{E(\mu, u) : \ u \in \mathcal{U}\} \tag{3.5.2}$$

can be considered as the energy associated to the mass distribution μ. The compliance $\mathfrak{C}(\mu)$ is then defined as

$$\mathfrak{C}(\mu) = -\mathcal{E}(\mu).$$

It must be noticed that we may have $\mathfrak{C}(\mu) = +\infty$ for some μ this happens for example if f is concentrated on a set of dimension smaller then $N-1$ and μ is absolutely continuous with respect to the Lebesgue measure. More precise characterization of the relationship between μ and f which imply that $\mathfrak{C}(\mu)$ is finite can be given in terms of the Sobolev spaces with respect to μ. However we are not interested in such singular measures as we want to minimize:

$$\min\{\mathfrak{C}(\mu) |\ \mu \in \mathcal{M}^+(\mathbb{R}^N), \ \int d\mu = m, \ spt(\mu) \subset K\} \tag{3.5.3}$$

where the total amount m of mass is prescribed, as well as the design region K. If we denote the polar cone of $\mathcal{U}$ by $\mathcal{U}^0$ (we recall that $\mathcal{U}^0$ is defined by: $\{T \in \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^N) : \ < T, u > \leq 0 \, \forall u \in \mathcal{U}\}$) by a standard duality argument we can write $\mathfrak{C}(\mu)$ in the following way:

$$\mathfrak{C}(\mu) = \inf\{\int j^*(\sigma) d\mu : \ \sigma \in L^{p'}_\mu(\mathbb{R}^N, \mathbb{R}^{N\times N}), \ f + \text{div } (\sigma\mu) \in \mathcal{U}^0\}.$$

By analogy with the scalar case we introduce the following problem

$$I(f, \mathcal{U}, K) = \sup\{< f, u > \mid u \in \mathcal{U}, \ j(Du) \leq 1/p \ on \ K\}.$$

We will now start a chain of inequalities similar to that of the scalar case but before that let us remark that since j is convex and positively $p-$homogeneous we can write

$$j(z) = \frac{1}{p}(\rho(z))^p$$

where ρ is convex and positively $1-$homogeneous. Then

$$I(f, \mathcal{U}, K) = \sup\{< f, u > \mid u \in \mathcal{U}, \ \rho(Du) \leq 1 \ on \ K\}. \tag{3.5.4}$$

Using the duality arguments introduced in the previous chapter:

$$I(f, \mathcal{U}, K) = \inf\{\int \rho^0(\lambda) : \ \lambda \in \mathcal{M}(\mathbb{R}^N, \mathbb{R}^{N\times N}), \ spt\lambda \subset K, \ f + \text{div } \lambda \in \mathcal{U}^0\} \tag{3.5.5}$$

where ρ^0 is the polar function associated to ρ:

$$\rho^0(z) = \sup\{z : \xi : \ \rho(\xi) \leq 1\}.$$

It is possible to prove the following theorem (see [**12**])

THEOREM 3.5.1. *Assume that $I(f,\mathcal{U},K)$ is finite. Then the following facts hold:*
i) The mass optimization problem (3.5.3) admits a solution μ and we have

$$\mathcal{C}(\mu) = \frac{(I(f,\mathcal{U},K))^{p'}}{p' m^{1/(p-1)}}$$

ii) If μ is a solution of the mass optimization problem (3.5.3) then one has

$$I(f,\mathcal{U},K) = \min\{\int \rho^0(\sigma)d\mu : \ spt\mu \subset K, \ \int d\mu = m, \\ \sigma \in L^1_\mu(\mathbb{R}^N, \mathbb{R}^{N\times N}), \ f + \text{div } (\sigma\mu) \in \mathcal{U}^0)\}, \quad (3.5.6)$$

where the optimal σ in (3.5.6) verifies

$$\rho^0(\sigma) = \frac{I(f,\mathcal{U},K)}{m} \quad \mu - \textit{almost everywhere.}$$

iii) Conversely, if λ is a solution of (3.5.5), then $\mu := \frac{m}{I(f,\mathcal{U},K)}\rho^0(\lambda)$ is optimal for (3.5.3).

A field σ solution of (3.5.6) is related to the gradient of the optimal displacement u (for 3.5.2) through a constitutive equation involving the notion of tangent space of μ. In order to characterize the solution (u,μ) of the two problems we need to introduce two more notions related to the tangent calculus with respect to a measure. The first one is the set of the "projections" of symmetric matrices:

$$M_\mu = \{P_\mu(x)EP_\mu(x) : \ E \in \mathbb{R}^{N\times N}_{sym}\}.$$

The second one is a sort of relaxation of j

$$j_\mu(x,z) = \inf\{j(z+\xi) : \ \xi \in (M_\mu(x))^\perp\}.$$

In conclusion, if we restrict our attention to the case of Dirichlet problem (i.e. $\mathcal{U} = \{u \in \mathcal{D}(\Omega,\mathbb{R}^N) : \ u = 0 \ on \ \Sigma\}$) in [**12**] the authors proved that a pair (u,μ) is a solution of (3.5.4) and (3.5.3) if there exists a matrix ξ such that the triple (ξ,u,μ) is solution of:

$$\begin{cases} 1) & u \in Lip_{1,\rho}(\Sigma), \\ 2) & u \text{ is a minimizer for (3.5.4)}, \\ 3) & \mu(\Sigma) = 0 \\ 4) & j_\mu(x, e_\mu(u)) = \frac{1}{p} \ \ \mu - a.e. \\ 5) & -div(\xi\mu) = f \ in \ \Omega\setminus\Sigma. \\ 6) & \xi \in \partial j_\mu(e_\mu(u)). \end{cases} \qquad (3.5.7)$$

Where is $Lip_{1,\rho}(\Sigma)$ the closure in the sup norm of the subclass of functions in $\mathcal{D}(\mathbb{R}^N,\mathbb{R}^N)$ such that $u = 0$ on Σ and $\rho(e(u)) \le 1$.

CHAPTER 4

Regularity properties for Monge Transport Density and for Solutions of some Shape Optimization Problem

4.1. Introduction

In this chapter we will study the regularity of the solutions of the scalar shape optimization problem we introduced in the previous chapter and for the Monge transport density we introduced in chapter 2. The setting will not be as general as that of chapter 3. In particular M will be replaced by an open, bounded, convex subset Ω of $\mathbb{R}^N$. More precisely the object of our attention will be the regularity (i.e. the dimension and the summability) of the measures μ which together with a Lipschitz function u are solution of the following system:

$$(WMK) = \begin{cases} -\text{div } \mu D_\mu u = f & on\ \Omega \\ |D_\mu u| = 1 & \mu - a.e.; \end{cases} \tag{2.1.8}$$

A measure μ solution of (2.1.8) has a double meaning. If multiplied by a suitable constant (see theorem 3.3.5) is a maximizer of (3.2.3) and then (with a little abuse of notations) we will call it optimal shape. At the same time, as we explained in the second chapter, μ is related to the Monge-Kantorovich problem and then will also be called transport density. It is the relationship with the transport problem which permit our approach to the regularity of μ. Indeed, formula (3.3.18) (revisited in this chapter in (4.4.1)) permits to associate to each optimal plan γ a solution μ of (2.1.8). Vice versa in [**5**] it as been proved that in $\mathbb{R}^N$ all the solutions of (2.1.8) are obtained in this way for a suitable γ. Property (2.1.7) of γ will reflect on regularity properties of μ. We think that it will be suitable (to have simpler and more consistent with the classical notations) to revisit in their simpler form some of the formulas and theorems we gave in the previous chapter, then some of the theorems of this chapter are just a particular case (but some time they have independent proof) of that of the previous chapter. Let us recall the main players of the problem. Being Ω, as we said, a bounded, convex, open subset of $\mathbb{R}^N$ we will still represent by f the heat source and m will be the assigned amount of conductor C . The model for the distribution of C is a nonnegative bounded measure μ in Ω such that $\int_\Omega d\mu = m$, and then the energy associated to a smooth distribution of temperature

$u \in \mathcal{D}(\Omega)$ became:

$$E(\mu, u) = \frac{1}{2}\int_\Omega |Du(x)|^2 d\mu - \langle f, u\rangle. \tag{4.1.1}$$

The quantity we associate to μ that we will try to maximize is

$$\mathfrak{E}(\mu) := \inf_u E(\mu, u).$$

In the previous chapter we have shown in a more general setting (and we will give a simplified proof in section 4.3) that the problem

$$\mathcal{E}(f) := \sup\left\{\mathfrak{E}(\mu) : \mu \in \mathcal{M}^+(\Omega), \int d\mu = m\right\}, \tag{4.1.2}$$

is related to

$$I(f) := \inf\left\{\langle -f, u\rangle : u \in Lip_1(\Omega)\right\}, \tag{4.1.3}$$

by the formula

$$\mathcal{E}(f) = -\frac{I(f)^2}{2m}, \tag{4.1.4}$$

and (4.1.3) is dual to

$$\sup\left\{-\int d|\nu| : \nu \in \mathcal{M}(\Omega, \mathbb{R}^N),\ -\mathrm{div}\ \nu = f\right\}; \tag{4.1.5}$$

As we already proved all these problems are related to 2.1.8 (see [**14**])

4.2. Preliminary

Here we briefly list some notations we use throughout the chapter, even if most of the symbols we use are standard.

$B(x,\rho)$	Ball in $\mathbb{R}^N$ of centre x and radius ρ
xy, $\overline{xy}$	Close segment in $\mathbb{R}^N$ and its length
D_μ	$\mu-$tangential gradient (see section 2.2)
$\mathrm{W}^{1,p}_\mu$	Sobolev space with respect to a measure (see section 2.2)
$\mathrm{Dim}\ \mu$	Dimension of the measure μ (see section 4.2.1)
$\theta^*_k(\mu, x)$	$k-$upper density of μ at x (see section 4.2.1)
$d_\mu(x)$	Pointwise dimension of μ at x (see section 4.2.1)
$\mathcal{L}$	Lebesgue measure
$\mathcal{H}^k$	$k-$dimensional Hausdorff measure
$\mu \llcorner B$	Restriction of the measure μ to the set B
$\mathcal{B}(\Omega)$	Borel subsets of Ω

$\mathrm{Lip}_1(\Omega)$	$1-$Lipschitz functions on $\Omega \subseteq \mathbb{R}^N$
$\mathcal{M}(\Omega),\ \mathcal{M}^+(\Omega)$	Spaces of the measures and of the positive measures on Ω
$\tau \leq \gamma$	We intend that $\tau(A) \leq \gamma(A)$ for all Borel sets $A \subseteq \Omega$
$\lvert\gamma\rvert$	Total variation of $\gamma \in \mathcal{M}(\Omega)$
$\lVert\gamma\rVert$	Norm of γ in the space $\mathcal{M}(\Omega)$, or total mass, i.e. $\lvert\gamma\rvert(\Omega)$
ω_k	Lebesgue measure of the ball in $\mathbb{R}^k$ of unit ray

4.2.1. Dimension of a measure. In this section we introduce a notion of "dimension" for measures belonging to $\mathcal{M}^+(\Omega)$, which we will use later:

DEFINITION 4.2.1. *The dimension of $\mu \in \mathcal{M}^+(\Omega)$ is defined by*

$$\mathrm{Dim}\ \mu := \sup\{k :\ \mu \ll \mathcal{H}^k\}.$$

where $\mathcal{H}^k$ denotes the $k-$dimensional Hausdorff measure.

Notice that if μ is made of pieces of different dimensions then Dim μ is the smallest of these.

In order to calculate the dimension of a measure it will be sometimes useful to give another representation of it and this will be done in the next proposition. First we need to introduce the notion of $k-$upper density of μ at x:

DEFINITION 4.2.2. *The $k-$upper density of μ at x is defined by*

$$\theta_k^*(\mu, x) := \overline{\lim_{\rho\to 0}} \frac{\mu(B(x,\rho))}{\omega_k \rho^k}.$$

A first useful result about θ_k^* is the following

THEOREM 4.2.3. *The following facts hold:*

a) $\theta_k^*(\mu, x) \leq t \quad \forall x \in B \in \mathcal{B}(\Omega) \Longrightarrow \mu \llcorner B \leq 2^k\ t\ \mathcal{H}^k \llcorner B$;
b) $\theta_k^*(\mu, x) \geq t \quad \forall x \in B \in \mathcal{B}(\Omega) \Longrightarrow \mu(B) \geq t\ \mathcal{H}^k(B)$;
c) $\theta_k^*(\mu, x) < +\infty$ *for* $\mathcal{H}^k - a.e.\ x \in \Omega$.

Parts a) and b) are two particular cases from Theorem 3.2 in [**61**]; c) can be obtained immediately from b).

Thanks to theorem 4.2.3 it is quite easy to obtain the following characterization of the dimension:

PROPOSITION 4.2.4. *Given $\mu \in \mathcal{M}^+(\Omega)$, the following three numbers are equal:*

$\mathrm{D}_1 = \sup\{k :\ \mu \ll \mathcal{H}^k\} = \mathrm{Dim}\ \mu$;
$\mathrm{D}_2 = \inf\{k :\ \exists B \subseteq \Omega,\ \mu(B) > 0,\ \mathcal{H}^k(B) = 0\}$;
$\mathrm{D}_3 = \sup\{k :\ \theta_k^*(\mu, x) < +\infty$ *for* $\mu - a.e.\ x \in \Omega\}$.

PROOF. Let us note that immediate consequences of the definitions are the following: $\mu \ll \mathcal{H}^k$ for all $k < \mathrm{D}_1$, $\exists B \subseteq \Omega$ such that $\mu(B) > 0$ and $\mathcal{H}^k(B) = 0$ for all $k > \mathrm{D}_2$ and $\theta^*_k(\mu, x) = 0$ $\mu-$a.e. for all $k < \mathrm{D}_3$.

Step 1: $\mathrm{D}_1 \geq \mathrm{D}_2$.

If $k < \mathrm{D}_2$, by definition $\mathcal{H}^k(B) = 0$ implies $\mu(B) = 0$; in other words, $\mu \ll \mathcal{H}^k$, and then $k \leq \mathrm{D}_1$. As $k < \mathrm{D}_2$ is arbitrary, the claimed inequality follows.

Step 2: $\mathrm{D}_2 \geq \mathrm{D}_3$.

Let $k > \mathrm{D}_2$, $B \subseteq \Omega$ a subset as in the definition and $B_i = \{x \in B : \ \theta^*_k(\mu, x) \leq i\}$; part a) of theorem 4.2.3 implies that $\mu(B_i) = 0$ for all i, and then $\theta^*_k(\mu, x) = +\infty$ $\mu-$a.e., then $k \geq \mathrm{D}_3$ and the claimed inequality follows.

Step 3: $\mathrm{D}_3 \geq \mathrm{D}_1$.

Let $k < \mathrm{D}_1$, thanks to part c) of theorem 4.2.3 we have $\theta^*_k(\mu, x) < +\infty$ $\mathcal{H}^k-$a.e., and then $\theta^*_k(\mu, x) < +\infty$ $\mu-$a.e., because $\mu \ll \mathcal{H}^k$. This assures that $k < \mathrm{D}_3$ and the claimed inequality follows. □

DEFINITION 4.2.5. *The pointwise dimension of μ at x is defined by*

$$d_\mu(x) = \sup\{k : \theta^*_k(\mu, x) < +\infty\}.$$

Thanks to proposition 4.2.4 the dimension of μ is the $\mu-\inf$ of the pointwise dimensions of μ.

We now prove some simple facts about the behavior of the dimension under the action of Lipschitz continuous functions, which we will need in section 4.4.3 to show that no nontrivial upper estimates for the dimension of μ can be given.

LEMMA 4.2.6. *Let $\mu \in \mathcal{M}^+(\Omega)$ and let $\varphi : \Omega \to \mathbb{R}^M$ be a Lipschitz continuous function, then* $\mathrm{Dim}\ \varphi_\#\mu \leq \mathrm{Dim}\ \mu$. *Moreover, if φ is bilipschitz then* $\mathrm{Dim}\ \varphi_\#\mu = \mathrm{Dim}\ \mu$.

PROOF. Using the definition of dimension of a measure we just need to prove that $\varphi_\#\mu \ll \mathcal{H}^k \Longrightarrow \mu \ll \mathcal{H}^k$. Then, let $\varphi_\#\mu \ll \mathcal{H}^k$ and let $A \in \mathcal{B}(\Omega)$ be a set such that $\mathcal{H}^k(A) = 0$: we have $\mu(A) \leq \mu(\varphi^{-1}(\varphi(A)) = \varphi_\#\mu(\varphi(A)) = 0$, where the last equality holds because $\mathcal{H}^k(\varphi(A)) = 0$. Finally, if φ is bilipschitz, the same argument gives also the other inequality, since $\varphi^{-1}_\#(\varphi_\#\mu) = \mu$. □

LEMMA 4.2.7. *Let $\mu \in \mathcal{M}^+(\Omega)$ and let $\varphi : \Omega \to \mathbb{R}^M$ be a Lipschitz map with the property that Ω can be covered ($\mu-$a.e.) by countable Borel sets A_n, $n \in \mathbb{N}$, such that φ is bilipschitz on each of the A_n: then* $\mathrm{Dim}\ \varphi_\#\mu = \mathrm{Dim}\ \mu$.

PROOF. We just need to prove that $\mathrm{Dim}\ \varphi_\#\mu \geq \mathrm{Dim}\ \mu$, and this inequality will follow if we prove that $\mu \ll \mathcal{H}^k \Longrightarrow \varphi_\#\mu \ll \mathcal{H}^k$; let then k be such that $\mu \ll \mathcal{H}^k$, and $A \in \mathcal{B}(\mathbb{R}^M)$ with $\mathcal{H}^k(A) = 0$. Thanks to the previous lemma and to the assumptions, we have

$$\varphi_{\#}\mu(A) = \mu(\varphi^{-1}(A)) \leq \sum_{n\in\mathbb{N}} \mu(\varphi^{-1}(A) \cap A_n) = 0$$

□

4.3. Transport set and connections between the different problems

In this section we begin our study of problem (2.1.8). As we said we will refer to μ indifferently as optimal shape or transport density with reference to the two possible points of view.

Let $u \in \mathrm{Lip}_1(\Omega)$: it is usual (see [**28**], [**62**]) to associate to u the so-called *transport set* as follows

$$T_u := \{z \in \Omega : \ \exists x \in spt(f^+) \ s.t. \ u(x) - |z-x| = u(z) \ and \\ \exists y \in spt(f^-) \ s.t. \ u(y) + |y-z| = u(z)\}.$$

If x, y and z are as in the definition of T_u then they are aligned, in fact (we recall $u \in \mathrm{Lip}_1(\Omega)$) we have

$$|x-y| \geq u(x) - u(y) = |x-z| + |z-y|.$$

Moreover the closed segment xy (which is often called *transport ray* with respect to u) is contained in the transport set.

Remark 4.3.1. *T_u is contained in the union of the segments joining $spt(f^+)$ and $spt(f^-)$.*

Remark 4.3.2. *T_u is a closed set.*

In the next lemma we will use a test function which first appeared in a paper by Janfalk [**41**] and was also used by Evans and Gangbo in [**28**]. Let v be defined as follows

$$v(z) = \begin{cases} u(z) \displaystyle\max_{w\in spt(f)} \frac{\xi + u(w)}{\xi + u(z) + |w-z|} & if \ u(z) \geq 0 \\ u(z) \displaystyle\max_{w\in spt(f)} \frac{\xi - u(w)}{\xi - u(z) + |w-z|} & if \ u(z) \leq 0, \end{cases} \tag{4.3.1}$$

where ξ is a constant such that $\xi \pm u > 0$ everywhere on Ω.

Lemma 4.3.3. *The function v is Lipschitz continuous and satisfies the following properties:*

a) *$v = u$ on $spt(f)$;*
b) *$|Dv| \leq 1$ a.e.;*

c) *If $x \notin T_u$ then there exist a ball B centered x and a constant $\delta \in (0,1)$ such that $|Dv| \leq 1-\delta$ on B.*

The proof of this lemma is simple but not short, and can be found in [**28**], page 19–22.

THEOREM 4.3.4. *Let (u,μ) be a solution of (2.1.8) and let T_u be the transport set related to u. Then $spt(\mu) \subseteq T_u$.*

PROOF. Let v be defined as in (4.3.1), $x_0 \notin T_u$ and B, δ as in property c) of lemma 4.3.3. Using the integration by parts formula and the estimate b) of lemma 4.3.3 we obtain:

$$\begin{aligned}\int_\Omega v\ df &= \int_\Omega D_\mu u \cdot Dv\ d\mu \leq \int_{\Omega\setminus B} |D_\mu u||Dv|\ d\mu + \int_B |D_\mu u||Dv|\ d\mu \\ &\leq \int_{\Omega\setminus B} |D_\mu u|\ d\mu + (1-\delta)\int_B |D_\mu u|\ d\mu.\end{aligned}$$

Then, since property a) of lemma 4.3.3 implies $\int_\Omega u\ df = \int_\Omega v\ df$, it follows

$$\begin{aligned}\delta \int_B |D_\mu u|\ d\mu &\leq \int_\Omega |D_\mu u|\ d\mu - \int_\Omega u\ df \\ &= \int_\Omega |D_\mu u|^2\ d\mu - \int_\Omega |D_\mu u|^2\ d\mu = 0,\end{aligned}$$

also using (2.2.1). As $\int_B |D_\mu u|\ d\mu = \mu(B)$ we conclude that $\mu(B) = 0$ and then $x_0 \notin spt(\mu)$. □

Thanks to remark 4.3.1 and theorem 4.3.8, a consequence of the previous theorem is the following result concerning the region occupied by optimal distributions of the conductor, once given the heat sources.

COROLLARY 4.3.5. *The optimal measures for problem (4.1.2) are supported in the union of the segments joining $spt(f^+)$ and $spt(f^-)$.*

REMARK 4.3.6. *The set of the segments joining $spt(f^+)$ and $spt(f^-)$ is clearly a subset of the convex envelope of $spt(f)$, where $f = f^+ - f^-$, and it can be strictly smaller. For example, if $spt(f^+)$ and $spt(f^-)$ are two concentric spherical surfaces, the first set is the annulus between the surfaces while the second is the whole sphere. It can also happen that the dimension of the first set is strictly smaller than that of the second, as in the next example.*

EXAMPLE 4.3.7. Let $ABCD$ be a square with sides of length l and define $f^+ = \delta_A + \delta_C$ and $f^- = \delta_B + \delta_D$. In this case each transport set is contained in the boundary of the square, whose dimension is 1, while the convex envelope of $spt(f)$ is the whole square, whose dimension is 2. Let us write now an explicit formula for the optimal shapes: denoted by $a = AB$, $b = BC$, $c = CD$ and $d = DA$ the sides of the square and fixed $0 \le \alpha \le 1$,

$$\mu := \alpha \left(\mathcal{H}^1 \llcorner a + \mathcal{H}^1 \llcorner c \right) + (1-\alpha) \left(\mathcal{H}^1 \llcorner b + \mathcal{H}^1 \llcorner d \right) \tag{4.3.2}$$

defines a solution of (2.1.8) together with any $u \in \mathrm{Lip}_1(\Omega)$ such that $u(A) = u(C) = l$ and $u(B) = u(D) = 0$. Vice versa for any solution (u, μ) of (2.1.8), the measure μ can be written as in (4.3.2) for a suitable α. To prove what stated we remark that the admissible plans of transport have support contained in $\{(A, B), (A, D), (C, B), (C, D)\}$, so it is easy to write explicitely each of them (note that they are all optimal) and then, thanks to the general formula (4.4.2), the optimal measures μ.

Let us finally recall (and give a proof with this notations) the connections between (4.1.2), (4.1.3), (4.1.5) and (2.1.8) with the following result, first given in [**14**]:

THEOREM 4.3.8. *Problem (4.1.2) has a solution and (4.1.4) holds. Moreover if* u *and* ν *are optimal for (4.1.3) and (4.1.5) then* $(u, |\nu|)$ *solves (2.1.8) and, conversely, if* (u, μ) *solves (2.1.8) then* u *and* $\mu D_\mu u$ *are optimal for (4.1.3) and (4.1.5) respectively. Finally, a measure* μ *solves (2.1.8) together with some* $u \in \mathrm{Lip}_1(\Omega)$ *if and only if* $-m\mu/I(f)$ *solves problem (4.1.2).*

PROOF. In the previous chapter we proved that

$$\begin{aligned} I(f) &= \sup \left\{ -\int d|\nu| : \nu \in \mathcal{M}(\Omega, \mathbb{R}^N),\ -\mathrm{div}\ \nu = f \right\}, \quad &(4.3.3) \\ \mathfrak{C}(\mu) &= \sup \left\{ -\frac{1}{2} \int |\sigma|^2\ d\mu : \sigma \in L^2_\mu(\Omega, \mathbb{R}^N),\ -\mathrm{div}\ (\sigma\mu) = f \right\} \quad &(4.3.4) \end{aligned}$$

and the extremals in (4.1.3), (4.1.5) and (4.3.4) are reached. Then we have

Step 1: $\mathcal{E}(f) \le -I(f)^2/(2m)$.

Let $\mu \in \mathcal{M}^+(\Omega)$ with $\int d\mu = m$, σ such that $-\mathrm{div}\ (\sigma\mu) = f$ and $u \in \mathrm{Lip}_1(\Omega)$; then

$$\langle f, u \rangle^2 = \left(\int \sigma Du\ d\mu \right)^2 \le \int |\sigma|^2\ d\mu \int |Du|^2\ d\mu \le m \int |\sigma|^2\ d\mu,$$

which implies

$$-\frac{1}{2} \int |\sigma|^2\ d\mu \le -\frac{\langle f, u \rangle^2}{2m} :$$

thanks to (4.3.4), taking the sup in the left hand side and the inf in the right hand side we obtain the claimed inequality.

Step 2: Let u and ν be optimal respectively for (4.1.3) and (4.1.5): thanks to (4.3.3) we have $-I(f) = \langle f, u \rangle = \int d|\nu|$. Define then $\mu = |\nu|$ and $\theta : \Omega \to \mathbb{R}^N$ such that $\nu = \theta\mu$ (and then $|\theta| = 1$ μ–a.e.). Using (2.2.1) we have then

$$\int d\mu = \int d|\nu| = \langle f, u \rangle = \langle -\mathrm{div}\ (\theta\mu), u \rangle = \int D_\mu u \cdot \theta\ d\mu,$$

which implies $|D_\mu u| = 1$ $\mu-$a.e. and $\theta = D_\mu u$, then (u,μ) solves (2.1.8). Define now $\bar\mu = -m\mu/I(f)$ and $\bar u = -I(f)u/m$: we have $\int d\bar\mu = m$ and $f = -\text{div } \nu = -\text{div } (\bar\mu D_{\bar\mu}\bar u)$ so that, thanks to (4.3.4) and using (2.2.1),

$$\mathfrak{C}(\bar\mu) \geq -\frac{1}{2}\int |D_{\bar\mu}\bar u|^2 \, d\bar\mu = -\frac{1}{2}\langle f, \bar u\rangle = -\frac{I(f)^2}{2m} :$$

thanks to the first step this gives the optimality of $\bar\mu$ for problem (4.1.2) and the validity of (4.1.4).

Step 3: Let us take (u,μ) solution of (2.1.8) and define $\nu = \mu D_\mu u$, then $-\text{div } \nu = f$: finally we have

$$I(f) \geq -\int d|\nu| = -\int d\mu = -\int |D_\mu u|^2 \, d\mu = \langle -f, u\rangle \geq I(f),$$

so that $I(f) = \langle -f, u\rangle = -\int d|\nu|$, which gives the stated optimality of u and ν for (4.1.3) and (4.1.5).

Step 4: Let μ be optimal for (4.1.2): then there exists $\sigma \in L^2_\mu(\Omega, \mathbb{R}^N)$ such that $-\text{div } (\sigma\mu) = f$ and $\mathcal{E}(f) = \mathfrak{C}(\mu) = -1/2\int |\sigma|^2 \, d\mu$. Let us define $\nu = \sigma\mu$ and note that

$$\int d|\nu| = \int |\sigma| \, d\mu \leq \sqrt{\int |\sigma|^2 \, d\mu}\sqrt{\int d\mu} = \sqrt{-2\mathcal{E}(f)}\sqrt{m} = \sqrt{\frac{I(f)^2}{m}}\sqrt{m} = -I(f),$$

then ν is optimal for (4.1.5) and $|\sigma|$ must be constant, whence $|\sigma| = -I(f)/m \quad \mu-$a.e.: this implies that $-I(f)\mu/m$ solves (2.1.8) together with any u optimal for (4.1.3). □

4.4. Dimension and summability of optimal measures

4.4.1. Definition of the optimal measures. In this section we will recall the formula (4.4.1), which was first introduced in [**12**], to write, starting from a solution γ of (3.2.8), a measure ν which is extremal for (4.1.5). Then, as proved in theorem 4.3.8, $\mu = |\nu|$ –which is given by formula (4.4.2)– solves (2.1.8) together with any u extremal for (4.1.3) and, up to a rescaling constant, μ realizes the sup in (4.1.2). As we said many time one can look at μ from two different point of view, moreover as seen in [**5**], [**56**] this measure μ is also related to an ODE version of the optimal transport problem introduced by Brenier and explained in [**10**] (see also [**5**]): for these reasons we will refer to the measure μ simply as *optimal measure*. Notice that μ will be defined starting from an optimal plan of transport γ, while the data of the problem are f^+ and f^-: in fact, in general different optimal plans can generate different optimal shapes, as it happens in example 4.3.7. In this section we will study the regularity properties of the measure μ in relation with the analogous properties of f^+ and f^-, but we will not use anything that comes from the particular choice of γ: then the properties we will find are owned by any μ solving (2.1.8).

From now on we consider a fixed optimal plan of transport γ from f^+ to f^-, and in this section we will call *transport ray* each close segment xy such that $(x,y) \in spt(\gamma)$; the relationship with the transport set and the transport rays defined in the previous section is the following: for each optimal plan of transport γ there exists a $1-$Lipschitz function u such that the transport rays in this setting are contained in T_u and are transport rays also in the sense of section 4.3 with respect to u. Moreover, given $x, y \in \Omega$, we denote by $\mathcal{H}^1_{xy}$ the one–dimensional Hausdorff measure on the segment xy. Finally we can define $\nu \in \mathcal{M}(\Omega, \mathbb{R}^N)$ as follows:

$$\langle \nu, \varphi \rangle := \int_{\Omega\times\Omega} \int_\Omega \varphi(z) \cdot \frac{x-y}{|x-y|} \, d\mathcal{H}^1_{xy}(z) \, d\gamma(x,y) \qquad \forall \varphi \in C_0(\Omega, \mathbb{R}^N) \tag{4.4.1}$$

Let us prove now the extremality property of ν:

PROPOSITION 4.4.1. *The measure ν defined by (4.4.1) is extremal for (4.1.5).*

PROOF. First we need to verify that $-\text{div } \nu = f$, and this holds since given any $\varphi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned} \langle -\text{div } \nu, \varphi \rangle = \langle \nu, D\varphi \rangle &= \int_{\Omega\times\Omega} \int_\Omega D\varphi(z) \cdot \frac{x-y}{|x-y|} \, d\mathcal{H}^1_{xy}(z) \, d\gamma(x,y) \\ &= \int_{\Omega\times\Omega} \varphi(x) - \varphi(y) \, d\gamma(x,y) \\ &= \int_\Omega \varphi \, df^+ - \int_\Omega \varphi \, df^- = \int_\Omega \varphi \, df = \langle f, \varphi \rangle. \end{aligned}$$

Now, using a standard duality theorem for functional (3.2.8) due to Kantorovich (that can be found in [**57**]), we know that

$$\sup_{|D\varphi|\le 1} \langle \varphi, f \rangle = \int_{\Omega\times\Omega} |y-x| \, d\gamma(x,y).$$

On the other hand, using (4.4.1) we obtain

$$\int_\Omega d|\nu| = \int_{\Omega\times\Omega} |y-x| \, d\gamma(x,y) = \sup_{|D\varphi|\le 1} \langle \varphi, f \rangle = -I(f),$$

that thanks to (4.3.3) gives the assert. □

We can apply now theorem 4.3.8 to obtain that $\nu = \mu D_\mu u$ for each u extremal for (4.1.3) and that μ is an optimal shape, where $\mu = |\nu|$ is the optimal measure we will study in this section. First it is useful to write an explicit formula for μ, which follows from (4.4.1):

$$\langle \mu, \varphi \rangle := \int_{\Omega\times\Omega} \left(\int_\Omega \varphi(z) \, d\mathcal{H}^1_{xy}(z) \right) d\gamma(x,y). \tag{4.4.2}$$

In particular from (4.4.2) it is possible to write the measure μ of a set $A \in \mathcal{B}(\Omega)$ as

$$\mu(A) = \int_{\Omega\times\Omega} l(xy \cap A)\ d\gamma(x,y), \tag{4.4.3}$$

where $l(xy \cap A) = \mathcal{H}^1_{xy}(A)$ is the length of the intersection between the segment xy and A. It is easy to note that if the mass is moved from f^+ to f^- following the plan of transport γ then $\mu(A)$ is the work done in the set A: for this reason the measure μ defined by (4.4.2) is called, as we said before transport density.

4.4.2. Some technical lemmas. In this section we report some technical results which we will need in sections 4.4.3 and 4.4.4. We will begin with two propositions, then we will give two technical lemmas. A first property which follows directly from the definition of μ is the following:

PROPOSITION 4.4.2. *$\mu-$a.e. point in Ω is contained in the interior of some transport ray.*

PROOF. As we said, this is a consequence of the definition of μ: let S be the subset of the points of Ω which are not contained in the interior of a transport ray. For every $(x,y) \in spt(\gamma)$ we have $xy \cap S \subseteq \{x,y\}$ and then $l(xy \cap S) = 0$ which, from (4.4.3), gives the claimed assertion. □

Let us recall now a simple but useful property of the optimal plans of transport:

PROPOSITION 4.4.3. *If $\gamma \in \mathcal{M}^+(\Omega\times\Omega)$ is an optimal plan of transport from $\pi^1_\#\gamma$ to $\pi^2_\#\gamma$ and $\tau \in \mathcal{M}^+(\Omega\times\Omega)$ is a measure such that $\tau \leq \gamma$, then τ is an optimal plan from $\pi^1_\#\tau$ to $\pi^2_\#\tau$. In particular $\gamma \llcorner B \times \Omega$, where B is any Borel subset of Ω, is optimal.*

PROOF. By contradiction, if τ were not optimal, it would exist $\tilde\tau \in \mathcal{M}^+(\Omega\times\Omega)$ with the same marginals as τ but less total work. In that case $\gamma - \tau + \tilde\tau$ would be a positive measure with the same marginals as γ and less total work, which contradicts the optimality of γ. □

The above proposition will be very useful in the proofs of the next results, when it will be convenient to divide Ω in subsets with assigned properties.

Lemmas 4.4.4 and 4.4.5 give two estimates about the location of the extreme points of the transport rays which meet an open ball centered at a fixed point $\underline{x}$ and of sufficiently small radius. In particular in lemma 4.4.4 $\underline{x}$ is an interior point of some transport ray and the estimate degenerates as $\underline{x}$ get closer to the extreme points of the ray. In lemma 4.4.5 $\underline{x}$ can be a generic point of the ray, even an extreme point, but stronger assumptions are needed in order to get the estimate.

The proofs of these two lemmas are entirely based on property (2.1.7). It will be convenient to reduce the possible configurations of the set of points we will use by moving them suitably. To make this, obviously, we are allowed only to move the points in such a way that (2.1.7) continues to hold. Then, given (x, y) and (x', y') in $spt(\gamma)$, we will call *admissible transformations* all the changes of the points such that the quantity

$$\overline{xy} + \overline{x'y'} - \overline{xy'} - \overline{x'y}$$

decreases. Such transformations maintain the validity of (2.1.7), and we will pay attention to use only these. For example, thanks to the triangular inequality it is easily seen that moving x to y or y to x on the line xy is admissible, and the same holds for x', y' and $x'y'$. The next two lemmas play a key role in the proofs of theorems 4.4.6 and 4.4.10 and of lemma 4.4.11.

LEMMA 4.4.4. *Let z be contained in the interior of a transport ray xy: there exists a constant K (which depends on z, x, y and Ω) and cones C_ε of volume less than $K\varepsilon^{N-1}$ such that, if ε is sufficiently small and another transport ray $x'y'$ meets $B(z, \varepsilon)$, then $x' \in C_\varepsilon$.*

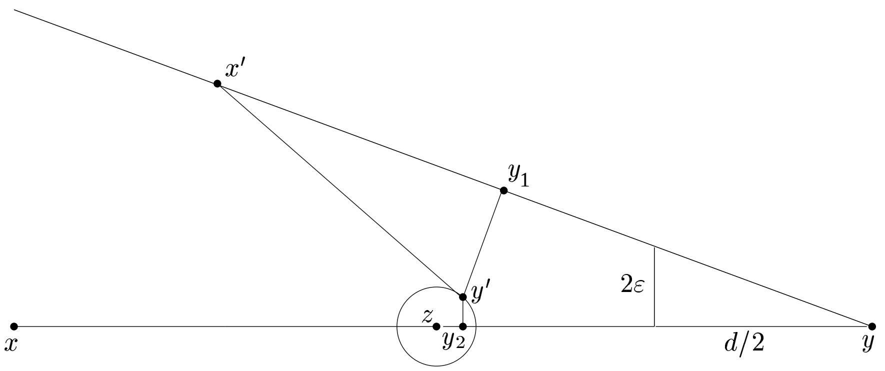

FIGURE 1. Lemma 4.4.4

PROOF. Up to an admissible transformation (bring x and y closer) we can assume that z is the middle point of xy. Let $2d = \overline{xy}$ and let C_ε be the cone with axis xy, vertex y and which is large 2ε at a distance $d/2$ from y, as in figure 1. This cone has volume less than $K\varepsilon^{N-1}$ because Ω is bounded.

<u>*Step 1:*</u> Two dimensional case.

Suppose that a transport ray $x'y'$ meets $B(z, \varepsilon)$ but $x' \notin C_\varepsilon$. Up to an admissible transformation (bring x' and y' closer) we can assume x' and y' are respectively on the boundary of the cone C_ε and of the ball $B(z, \varepsilon)$. We now denote (as in figure 1) by

y_1 and y_2 the orthogonal projections of y' on the lines $x'y$ and xy respectively. An immediate geometric argument shows that if $\varepsilon \ll 1$ then $\overline{y'y_1} \geq 2\varepsilon$ and then, using this fact and Pitagora's theorem, the following relations hold:

$$\begin{aligned} &\overline{xy} = \overline{xy_2} + \overline{y_2y}, && \overline{x'y_1} \leq \overline{x'y'}, \\ &\overline{xy'} \leq \sqrt{\overline{xy_2}^2 + \varepsilon^2}, && \overline{x'y} \leq \overline{x'y_1} + \overline{y_1y}, \\ &\overline{y_1y} \leq \sqrt{\overline{yy'}^2 - (2\varepsilon)^2}, && \overline{yy'} \leq \sqrt{\overline{y_2y}^2 + \varepsilon^2}. \end{aligned}$$

Inequality (2.1.7) assures

$$\overline{xy} + \overline{x'y'} \leq \overline{xy'} + \overline{x'y}$$

and then, substituting the previous relations and simplifying,

$$\overline{xy_2} + \overline{y_2y} \leq \sqrt{\overline{xy_2}^2 + \varepsilon^2} + \sqrt{\overline{y_2y}^2 - 3\varepsilon^2}.$$

Since if $\varepsilon \ll 1$ then $\overline{xy_2} \approx \overline{y_2y} \approx d$, it follows easily that this last inequality implies $\varepsilon \geq \overline{\varepsilon} > 0$ for some $\overline{\varepsilon}$ which depends on d.

Step 2: Higher dimensional case.

We can reduce the proof to the two–dimensional case. In fact, it is admissible to bring y' to the point of least distance from x' remaining on the $(N-2)-$dimensional sphere where the distances from x and y are fixed, because in this way $\overline{xy}$, $\overline{xy'}$ and $\overline{x'y}$ remain equal, while $\overline{x'y'}$ decreases. After this transformation the four points x, y, x' and y' are on a plane and then, using the first step, we know that if ε is sufficiently small x' is in the cone C_ε. This concludes the proof in the general case. □

Lemma 4.4.5. *There exists a constant K with the following property: if x belongs to a transport ray and $d(x, spt(f^-)) = \delta > 0$ there exist cylinders C_ε of radius $r_\varepsilon \leq K\varepsilon/\delta$ such that if ε is sufficiently small, $x' \in C_\varepsilon$ for every transport ray $x'y'$ that meets $B(x,\varepsilon)$.*

Proof. If $x \in spt(f^+)$, then there exists at least a point $y \in spt(f^-)$ such that $(x,y) \in spt(\gamma)$ or, in other words, xy is a transport ray. Let us fix y with this property and consider an other ray $x'y'$ that meets $B(x,\varepsilon)$, where $\varepsilon \leq \delta$: as y and y' are in $spt(f^-)$ we have $\overline{xy} \geq \delta$ and $\overline{xy'} \geq \delta$. Possibly bringing y and y' closer to x and x' (which is admissible) we can assume $y, y' \in \partial B(x,\delta)$. Let us consider now a system of coordinates $c_1, \ldots, c_N$ such that x is the origin, $y \equiv (\delta, 0, \ldots, 0)$, y' is on the plane $\{c_3 = \cdots = c_N = 0\}$ and $c_2(y') \leq 0$. In this coordinates, if we find a constant K such that

$$|c_i(x')| \leq K\varepsilon/\delta \tag{4.4.4}$$

holds for ever $i = 2, \ldots, N$, the thesis follows. In fact, the set of the points for which (4.4.4) holds with $i = 2, \ldots, N$ is contained in a cylinder of radius $r_\varepsilon = K\varepsilon/\delta$. For $i \geq 3$ the existence of a suitable constant K for which (4.4.4) holds for ε sufficiently small follows by similitude recalling that $c_i(y') = 0$, $x'y'$ meets $B(x, \varepsilon)$ and Ω is bounded. Moreover, the fact that $c_2(y') \leq 0$ implies by a similar argument that $c_2(x') \geq -K\varepsilon/\delta$. It remains only to prove

$$c_2(x') \leq K\varepsilon/\delta. \tag{4.4.5}$$

To this aim let us note that $\overline{xy} = \overline{xy'} = \delta$, and then (2.1.7) reads

$$\overline{x'y'} \leq \overline{x'y}. \tag{4.4.6}$$

It is admissible to project x' on the plane $\{c_3 = \cdots = c_N = 0\}$, because this does not change the validity of (4.4.6) nor $c_2(x')$: in this manner, we are led to a two–dimensional situation, which collects all the difficulty of the proof. It is convenient to consider separately the cases with different positions of x' and y'. For sake of shortness we will only sketch how the proof works in each case without making all the computations, that can be found in [**56**].

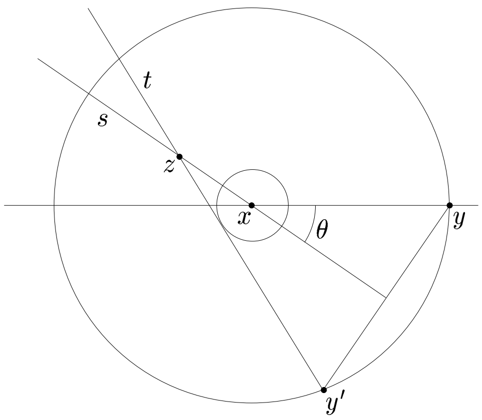

FIGURE 2. Case 4 of lemma 4.4.5

Case 1: $c_1(y') < 0,\ c_1(x') \leq 0$.
In this case the condition that $x'y'$ meets $B(x, \varepsilon)$ is very strong, as it can be imagined. In fact if ε is sufficiently small it can be easily proved that $c_2(x') \leq 3\varepsilon$, obtaining an estimate better than (4.4.5).
Case 2: $c_1(y') < 0,\ c_1(x') > 0$.
We can reduce this case to one of the other. In fact, changing x' and y' with the symmetrical points with respect to the hyperplane $\{c_2 = 0\}$ is an admissible transformation, because $\overline{xy}$, $\overline{xy'}$ and $\overline{x'y'}$ remain equal while $\overline{x'y}$ increases. This transformation does not change $c_2(x')$ and leads us to case 3 or 4.
Case 3: $c_1(y') \geq 0,\ c_2(y') \geq -5\varepsilon$.
In this case an argument similar to that used for (4.4.4) with $i \geq 3$ gives (4.4.5). *Case 4:*

$c_1(y') \geq 0,\ c_2(y') < -5\varepsilon$.
This case requires some involved calculations. The basic idea is simple: with reference to figure 2, we know that $x'y'$ must meet $B(x,\varepsilon)$, and then x' is over the line t. On the other hand (4.4.6) implies that x' is under the line s, the axis of yy'. Summarizing, $c_2(x') \leq c_2(z)$ and then (4.4.5) will follow from

$$c_2(z) \leq K\varepsilon\delta, \tag{4.4.7}$$

which gives in fact an estimate stronger than (4.4.5), and also stronger than the estimate found in the first case. Finally, to prove (4.4.7) one has to evaluate $c_2(z)$ with respect to ε and θ, recalling the hypothesis $c_2(y') < -5\varepsilon$: one needs some very simple but not so short calculation, and finally (4.4.7) follows for ε sufficiently small. □

4.4.3. Dimension of the optimal measure. This section is entirely devoted to discuss estimates for the dimension (in the sense of section 4.2.1) of μ: we will prove theorem 4.4.6, which gives a lower estimate, and show with an example that it is sharp, and then we will show that no upper estimate is possible. Let us begin with the lower estimate, which follows by a careful analysis from lemma 4.4.4.

Theorem 4.4.6. $\mathrm{Dim}\ \mu \geq \max\{\mathrm{Dim}\ f^+,\ \mathrm{Dim}\ f^-,\ 1\}$.

Proof. *Step 1:* $\mathrm{Dim}\ \mu \geq 1$.
From (4.4.3) it follows that for any $x \in \Omega$ and $\rho > 0$ we have $\mu(B(x,\rho)) \leq 2\rho\|\gamma\|$ which implies, from definition 4.2.2, that $\theta_1^*(\mu,x) \leq \|\gamma\|$ for any x. Then using proposition 4.2.4 we have the claim.

Let us suppose now that $\mathrm{Dim}\ f^+ \geq \mathrm{Dim}\ f^-$ (we can do it by the symmetry of the problem) and denote $k = \mathrm{Dim}\ f^+$. We have to prove that $\mathrm{Dim}\ \mu \geq k$.
Step 2: Assume for the moment that the following hypothesis holds:

$$\exists K:\ \forall x \in \Omega,\ \forall \rho > 0\ f^+(B(x,\rho)) \leq K\rho^k; \tag{4.4.8}$$

we will get rid of it in the next step.
If z is contained in the interior of some transport ray, let C_ρ be the cones defined by lemma 4.4.4. Since C_ρ has volume less the $K\rho^{N-1}$, it can be covered by M/ρ balls of ray ρ and this, thanks to (4.4.8), assures that

$$f^+(C_\rho) \leq \frac{M}{\rho}K\rho^k \leq MK\rho^{k-1}.$$

Then, if ρ is sufficiently small, using lemma (4.4.4) we have

$$\begin{aligned}\mu(B(z,\rho)) &= \int_{C_\rho\times\Omega} l(xy\cap B(z,\rho))\ d\gamma(x,y) \leq 2\rho\int_{C_\rho\times\Omega} d\gamma(x,y)\\ &= 2\rho f^+(C_\rho) \leq 2MK\rho^k.\end{aligned}$$

We then infer that $\theta_k^*(\mu, x) < +\infty$ for ever $z \in \Omega$ which is contained in the interior of some transport ray, and then from propositions 4.2.4 and 4.4.2 it follows that Dim $\mu \geq k$. The theorem is then proved under the additional assumption (4.4.8).

Step 3: General case.

Fix $\varepsilon > 0$ and define for any $m \in \mathbb{N}$

$$\Omega_{m,\varepsilon} = \{x \in \Omega : \forall \rho > 0,\ f^+(B(x,\rho)) \leq m\rho^{k-\varepsilon}\}.$$

Since $k - \varepsilon <$ Dim f^+, we have $\theta^*_{k-\varepsilon}(f^+, x) = 0$ f^+–a.e., as noted in the beginning of the proof of proposition 4.2.4, and then

$$\sup_{\rho>0} \frac{\mu(B(x,\rho))}{\rho^{k-\varepsilon}} < +\infty \qquad \text{for } \mu\text{–a.e. } x:$$

then, varying m, $\Omega_{m,\varepsilon}$ is an increasing sequence of subsets that fills Ω up to a set of null f^+ measure and then, as $m \to +\infty$, $f^+ \llcorner \Omega_{m,\varepsilon} \longrightarrow f^+$ in the strong convergence of $\mathcal{M}^+(\Omega)$. Defining $\gamma_{m,\varepsilon} = \gamma \llcorner \Omega_{m,\varepsilon} \times \Omega$ and recalling that $\pi^1_{\#}\gamma = f^+$ we infer that $\gamma_{m,\varepsilon} \longrightarrow \gamma$. Let us define now $\mu_{m,\varepsilon}$ from $\gamma_{m,\varepsilon}$ by (4.4.2). It is clear from the definition that $\mu_{m,\varepsilon} \longrightarrow \mu$, and thanks to proposition 4.4.3 we can apply the result of step 2 to $\Omega_{m,\varepsilon}$ to obtain Dim $\mu_{m,\varepsilon} \geq k - \varepsilon$: this, thanks to the definition of dimension and to the convergence of $\mu_{m,\varepsilon}$, implies Dim $\mu \geq k - \varepsilon$. Finally the generality of ε completes the proof. □

The next example shows that the lower estimate given in theorems 4.4.6 is sharp.

Example 4.4.7. Let S_1 and S_2 be two disjoint segments of length l lying on the same line, and let f^+ and f^- be the restrictions of the measure $\mathcal{H}^1$ to S_1 and S_2 respectively. Clearly an optimal transport is given by the translation, then it is simple to calculate μ, which is the measure on the line which density is l between the segments, goes to 0 linearly on the segments and is 0 out, and which is then one–dimensional.

On the other hand, it is not possible to give a non-trivial estimate from above of the dimension, as the following example shows:

Example 4.4.8. Let x_1, x_2, B_1 and B_2 be respectively two points and two balls of unit volume in Ω, and let $f^+ = \delta_{x_1} + \mathcal{L}\llcorner B_1$ and $f^- = \delta_{x_2} + \mathcal{L}\llcorner B_2$: if x_1 is near to B_2, x_2 is near to B_1 and x_1, B_2 are far from x_2, B_1, there is a unique optimal plan of transport, which distributes on B_2 the mass of x_1 and concentrates on x_2 the mass of B_1. In this case the dimension of μ is easily seen to be N, the maximum possible, while the dimensions of f^+ and f^- is 0, the least possible.

The lack of control in the previous example is due to the fact that the pointwise dimensions of f^+ and f^- are not constant. However, even if the pointwise dimensions are constant it is not possible to give general estimates of Dim $(f^+ \otimes f^-)$ in terms of Dim f^+ and Dim f^- (on this topic see: 3.2.23, 2.10.45, 2.10.29 of [**31**] and theorem 5.11 of [**30**]). As in the next example we show how to construct an optimal measure μ whose dimension is equal to Dim $(f^+ \otimes f^-) + 1$, it is not possible to give a non trivial estimate from above of Dim μ even if the pointwise dimensions of f^+ and f^- are constant.

EXAMPLE 4.4.9. Let $p,\ k$ be two positive integers such that $p+k+1 \leq N$, S be the unit sphere of the subspace of $\mathbb{R}^N$ given by the first $(p+1)$ coordinates and B be the unit ball of the subspace given by the coordinates from the $(p+2)-th$ to the $(p+k+1)-th$. Let f^+ and f^- be two probability measure on S and B respectively. By the symmetry of the problem it follows that any plan of transport is optimal. If we take, for example, $\gamma = f^+ \otimes f^-$, then we remark that μ is the push forward of $f^+ \otimes f^- \otimes \mathcal{L}^1 \llcorner [0,1]$ through the map $\varphi : \mathbb{R}^{p+1} \times \mathbb{R}^k \times [0,1] \to \mathbb{R}^N$ defined by $(x, y, t) = tx + (1-t)y$. Applying lemma 4.2.7 taking $A_n = \mathbb{R}^p \times \mathbb{R}^k \times (\frac{1}{n}, 1 - \frac{1}{n})$ we obtain the desired result.

4.4.4. Summability of the optimal measure. In this section we will investigate the summability of μ. Let us first note that for a measure the property of being absolutely continuous is a bit stronger than having dimension N, which is obviously the maximal possible dimension in Ω. From theorem 4.4.6 we already know that if f^+ has dimension N, then also the dimension of μ is N. The first result we will give will be a little step forward and it represents the connection between the study of the dimension and that of the summability of μ:

THEOREM 4.4.10. *If at least one between f^+ and f^- is absolutely continuous with respect to the Lebesgue measure, then μ is absolutely continuous with respect to the same measure.*

PROOF. *Step 1:* Let us first suppose that $f^+ \in L^\infty$.
If z is contained in the interior of some transport ray, a simple application of lemma 4.4.4 (similar to the one made in second step of theorem 4.4.6) gives a constant $K(z)$ such that $\mu(B(z,\varepsilon)) \leq 2K(z)\|f^+\|_{L^\infty}\varepsilon^N$ for ε sufficiently small, and then $\theta^*_N(\mu, z) < +\infty$. Since this holds for each z contained in the interior of some transport ray, from proposition 4.4.2 we have

$$\theta^*_N(\mu, z) < +\infty \quad \text{for } \mu\text{–a.e. } z. \tag{4.4.9}$$

If now $\mathcal{L}(B) = 0$ and we define $B_m = \{z \in B : \ \theta^*_N(\mu, z) \leq m\}$, part a) of theorem 4.2.3 implies

$$\mu(B_m) \leq 2^N\, m\, \mathcal{H}^N(B_m) \leq 2^N\, m\, \mathcal{H}^N(B) = 0,$$

which together with (4.4.9) assures $\mu(B) = 0$. Then $\mu \in L^1$.
Step 2: General case.

Let $\Omega_m = \{x \in \Omega : f^+(x) \le m\}$, $\gamma_m = \gamma \llcorner (\Omega_m \times \Omega)$ and let μ_m be defined from γ_m using (4.4.2). Thanks to proposition 4.4.3 and to the first step we can infer that $\mu_m \in L^1$. Arguing as in the proof of theorem 4.4.6, $\gamma_m \longrightarrow \gamma$ in $\mathcal{M}^+(\Omega \times \Omega)$, and then $\mu_m \longrightarrow \mu$ in $\mathcal{M}^+(\Omega)$, which gives $\mu \in L^1$ in the general case too. □

We can generalize this last result, studying what happens if f^+ (or f^-) belongs to some L^p with $p > 1$. The previous theorem shows that $\mu \in L^1$, but we can also prove that μ belongs to some L^q with $q > 1$. Let us begin with the case $p = +\infty$, the general result will then follow. In the sequel we will denote by α' the conjugate exponent of α, i.e. $1/\alpha + 1/\alpha' = 1$.

LEMMA 4.4.11. *If $f^+ \in L^\infty$ then $\mu \in L^p$ for all $p < (2n)'$. In particular*

$$\|\mu\|_{L^p} \le K \|f^+\|_{L^\infty}^{1/p'} \|f^+\|_{L^1}^{1/p}.$$

PROOF. Given $(x,y) \in spt(\gamma)$, let y_σ be the point on the segment xy such that $\overline{yy_\sigma} = \sigma$, or $y_\sigma = x$ if $\overline{xy} < \sigma$. Let us define the measure μ_r as follows:

$$\langle \mu_r, \varphi \rangle := \int_{\Omega \times \Omega} \left(\int_\Omega \varphi(z) \; d\mathcal{H}^1_{y_r y_{2r}}(z) \right) d\gamma(x,y). \tag{4.4.10}$$

Thanks to (4.4.2), it is clear that $\mu = \mu_R + \mu_{R/2} + \mu_{R/4} + \dots$, where R is the diameter of Ω. From theorem 4.4.10 we know that $\mu \in L^1$, and then $\mu_r \in L^1$ because $\mu_r \ll \mu$. It is useful to write the measure μ_r of a set A, which from (4.4.10) is

$$\mu_r(A) = \int_{\Omega \times \Omega} l(y_r y_{2r} \cap A) \; d\gamma(x,y),$$

where $l(xy \cap A)$ is understood as in (4.4.3).

<u>*Step 1:*</u> Let us fix $r > 0$ and begin with a very particular case: we assume that $spt(f^-) \subseteq Q$, where Q is an hypercube of side λ such that $\lambda\sqrt{N} \le r/2$ (note that $\lambda\sqrt{N}$ is the diameter of Q). From the definition, it follows that $spt(\mu_r)$ is contained in an annulus S with radii $r - \lambda\sqrt{N}/2$ and $2r + \lambda\sqrt{N}/2$ centered at the center of the hypercube, furthermore $l(y_r y_{2r} \cap S) \le r$ for all (x,y) and then

$$\|\mu_r\|_{L^1} = \mu_r(S) \le r f^-(Q). \tag{4.4.11}$$

On the other hand, the hypothesis $\lambda\sqrt{N} \le r/2$ assures that $d(S,Q) \ge r/2$ and then from lemma 4.4.5 it follows that

$$\|\mu_r\|_{L^\infty} \le \frac{C\|f^+\|_{L^\infty}}{r^{N-1}}. \tag{4.4.12}$$

It is well known that if $\varphi \in L^1 \cap L^\infty$ then $\varphi \in L^p$ and

$$\|\varphi\|_{L^p} \le \|\varphi\|_{L^1}^{1/p} \|\varphi\|_{L^\infty}^{1/p'},$$

then from (4.4.11) and (4.4.12) we infer that

$$\|\mu_r\|_{L^p} \le \left(rf^-(Q)\right)^{1/p}\left(\frac{C\|f^+\|_{L^\infty}}{r^{N-1}}\right)^{1/p'} \le \frac{K\|f^+\|_{L^\infty}^{1/p'} f^-(Q)^{1/p}}{r^{N-1-N/p}}.$$

<u>*Step 2:*</u> Let us cover $spt(f^-)$ with $N(r)$ disjoint hypercubes Q_i with sides as in step 1 and define $\gamma_i = \gamma \llcorner (\Omega \times Q_i)$. From proposition 4.4.3 it follows that γ_i is an optimal plan of transport from $f_i^+ = \pi^1_{\#}\gamma_i$ to $f_i^- = \pi^2_{\#}\gamma_i = f^- \llcorner Q_i$. Define $\mu_{r,i}$ from γ_i using (4.4.10): from the first step and the fact that $\|f_i^+\|_{L^\infty} \le \|f^+\|_{L^\infty}$ we obtain that $\mu_r = \mu_{r,1} + \mu_{r,2} + \cdots + \mu_{r,N(r)}$ and

$$\|\mu_{r,i}\|_{L^p} \le \frac{K\ \|f^+\|_\infty^{1/p'}\ f^-(Q_i)^{1/p}}{r^{N-1-N/p}}.$$

Applying the inequality between the arithmetic mean and the p-th mean, integrating, extracting the p-th root and using the fact that $f^+(\Omega) = f^-(\Omega)$, we obtain that

$$\|\mu_r\|_{L^p} \le N(r)^{1/p'}\ \frac{K\ \|f^+\|_\infty^{1/p'}}{r^{N-1-N/p}}\ f^+(\Omega)^{1/p}.$$

We recall $\mu = \mu_R + \mu_{R/2} + \mu_{R/4} + \ldots$ and we remark that, up to a geometric constant, $N(r) \le r^{-N}$, then it suffices to show that

$$\sum_{i\in\mathbb{N}} \left(\frac{R}{2^i}\right)^{-N+1+N/p-N/p'} < +\infty; \tag{4.4.13}$$

a simple calculation assures that (4.4.13) holds if and only if $p < (2n)'$, and this concludes the proof. $\square$

We now derive the general result:

THEOREM 4.4.12. *If $f^+ \in L^q$ then $\mu \in L^p$ for all $p < (2N)' \wedge (1 + (q-1)/2)$.*

PROOF. Let $p < (2N)' \wedge (1 + (q-1)/2)$ and let us write $\Omega = \bigcup_i \Omega_i$ up to a f^+-negligible set, where $\Omega_i = \{x : i-1 \le f^+(x) < i\}$. As in the previous lemma we can define $\gamma_i = \gamma \llcorner \Omega_i \times \Omega$ and μ_i from γ_i using (4.4.2). Observe that, thanks to proposition 4.4.3, γ_i is an optimal plan of transport. As $p < (2N)'$, we can then apply lemma 4.4.11 obtaining

$$\|\mu_i\|_{L^p} \le K i^{1/p'} f^+(\Omega_i)^{1/p} \le K i^{1/p'} (i \cdot |\Omega_i|)^{1/p} \le K i |\Omega_i|^{1/p}.$$

Denote $\lambda_i = |\Omega_i|$: it is sufficient to prove that $\sum_i i\lambda_i^{1/p} < +\infty$. Let us define

$$\rho_i := \begin{cases} \lambda_i & \text{if } \lambda_i \ge i^{p'(1-q)} \\ 0 & \text{if } \lambda_i < i^{p'(1-q)} \end{cases} \qquad \sigma_i := \begin{cases} 0 & \text{if } \lambda_i \ge i^{p'(1-q)} \\ \lambda_i & \text{if } \lambda_i < i^{p'(1-q)} \end{cases} :$$

it is clear that $\sum_i i\lambda_i^{1/p} = \sum_i i\rho_i^{1/p} + \sum_i i\sigma_i^{1/p}$. Since $p < 1 + (q-1)/2$, the fact that $\sigma_i < i^{p'(1-q)}$ implies, by a simple calculation, that $\sum_i i\sigma_i^{1/p} < +\infty$. Moreover $f^+ \in L^q$

easily implies $\sum i^q\lambda_i < +\infty$ and then, since by definition $i\rho_i^{1/p} \le i^q\lambda_i$, $\sum i\rho_i^{1/p} < +\infty$, that gives the assert. □

Now, a natural question that arises is the following: which is the greatest p such that $f^+ \in L^q \Longrightarrow \mu \in L^p$? The previous theorem gives a lower bound for this number, but it is easy to give also an upper bound. In fact, thanks to the next example, this number cannot be greater then q:

EXAMPLE 4.4.13. In $\mathbb{R}^2$ let $R_1 = [-2,-1]\times[0,1]$ and $R_2 = [1,2]\times[0,1]$. Moreover let $g : [0,1] \to \mathbb{R}$ be an absolutely continuous function, consider $a^+ : R_1 \to \mathbb{R}$ and $a^- : R_2 \to \mathbb{R}$ defined by $a^+(x,y) = a^-(-x,y) = g(y)$ and define $f^+ = a^+\mathcal{H}^2 \llcorner R_1$ and $f^- = a^-\mathcal{H}^2 \llcorner R_2$. Thanks to the symmetry of the problem the horizontal translation of f^+ on f^- is an optimal transport, then using formula (4.4.2) we can explicitely write the measure μ obtaining that it is absolutely continuous and supported in $[-2,2]\times[0,1]$ with density given by

$$v(x,y) = \begin{cases} g(y)(x+2) & \text{if } x \in [-2,-1] \\ g(y) & \text{if } x \in [-1,1] \\ g(y)(2-x) & \text{if } x \in [1,2]. \end{cases}$$

Then the summability of μ is exactly the same of f^+ and f^-.

An other property is that $f^+ \in L^\infty$ does not imply $\mu \in L^\infty$ (and in a similar way $f^+ \in L^p$ does not imply $\mu \in L^p$), as shows the next example:

EXAMPLE 4.4.14. Let f^- be the Dirac mass in 0 and f^+ be the restriction of the Lebesgue measure to an annulus centered in 0 and of unitary volume, then $f^+ \in L^\infty$. Obviously $\gamma = f^+ \otimes f^-$, which brings all the mass of f^+ in 0, is the unique plan of transport and then is optimal. It is then easy, using (4.4.3), to evaluate $\mu(x)$ for each Lebesgue point x for μ, which we know yet, thanks to theorem 4.4.10, to be absolutely continuous: it follows $\mu(x) \propto |x|^{1-n}$ for x in the interior of the annulus, and then $\mu \notin L^\infty$; more precisely, $\mu \in L^p$ if and only if $p < n/(n-1)$.

We want finally study what happens when both $f^+,\ f^- \in L^q$: we already know, by example 4.4.13, that μ need not to belong to any L^p with $p > q$, but we can expect some result stronger than theorem 4.4.12, due to the summability of both the measures f^+ and f^-. Starting from lemma 4.4.5 and with a careful analysis of the geometric situation, it was recently proved in [**56**] that if the supports of f^+ and f^- are disjoint then $\mu \in L^q$; instead we will work in the general case when this hypothesis is not assumed.

REMARK 4.4.15. *The hypothesis of disjoint supports plays an important role in many other situations, even because it is needed to use tools from PDE: this is why, for example, it is assumed in* [**28**], [**35**] *and* [**29**].

We start with the following lemma:

LEMMA 4.4.16. *Let* $\alpha_i \geq 0$ *and* $A_i \subseteq \Omega$, $i \in \mathbb{N}$. *Then*

$$\int \left(\sum_{i=1}^{\infty} \alpha_i \chi_{A_i}\right)^p dx \leq \sum_{i=1}^{\infty} \mathcal{L}(A_i) \left(\sum_{j=1}^{i} \alpha_j\right)^p$$

PROOF. The formula is immediate if the sums from 1 to ∞ are replaced by sums from 1 to 2; an easy induction argument implies the validity with sums from 1 to n, where n is a generic integer; finally, since $\alpha_i \geq 0$ $\forall i$, the passage to the limit when n goes to ∞ concludes the proof. □

We can prove now that if both f^+ and f^- are in L^∞ then also $\mu \in L^\infty$.

PROPOSITION 4.4.17. *If both* f^+ *and* f^- *are in* L^∞, *then* $\mu \in L^\infty$. *In particular* $\|\mu\|_{L^\infty} \leq K(\|f^+\|_{L^\infty} + \|f^-\|_{L^\infty})$, *where* K *is a geometric constant that depends only on the diameter* R *of* Ω.

In order to give the proof, we need first the following

Claim *There exists a constant* λ *such that if* z *is an interior point of the transport ray* xy, ε *is sufficiently small and another ray* $x'y'$ *of length less then* $2\overline{xy}$ *intersects* $B(z,\varepsilon)$, *then either* x' *or* y' *belong to the cylinder* C_ε *with axis* xy *and radius* $\lambda\varepsilon$.

PROOF. (Claim) Let us suppose $\varepsilon \ll 1$ and that there exists a transport ray $x'y'$ of length less than $2\overline{xy}$ that intersects $B(z,\varepsilon)$ but neither x' nor y' is in the cilinder of axis xy and ray $(\lambda+1)\varepsilon$: in order to prove the Claim we must find an upper bound for λ.

By symmetry, we can assume that $a = \overline{xz} \leq b = \overline{zy}$: let us move x' closer to y' (moving it on the line $x'y'$, as in the previous theorems) until the distance between x' and the line xy is $\lambda\varepsilon$. We fix now coordinates $\{c_1, \dots, c_N\}$ such that z is the origin, the segment xy is on the first axis (with $c_1(x) < 0$, $c_1(y) > 0$), x' is on the plane $\{c_3 = \dots = c_N = 0\}$ with $c_2(x') < 0$ (and then $c_2(x') = -\lambda\varepsilon$) and y' is on the plane $\{c_4 = \dots = c_N = 0\}$ with $c_3(y') \geq 0$. Since the distance between y' and the line xy is more then $(\lambda+1)\varepsilon$ and $x'y'$ intersects $B(z,\varepsilon)$, from Pitagora's theorem follows that, if $\lambda \geq 4$, $c_2(y') > \lambda\varepsilon$: bring then y' closer to x' until $c_2(y') = \lambda\varepsilon$. The situation is illustrated in figure 3.

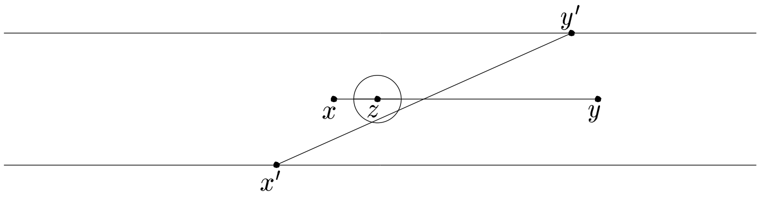

FIGURE 3. Geometry of the Claim

Let us define now $\delta = c_1(y')$ and $l = c_1(y') - c_1(x')$: an easy consequence of (2.1.7) is that $\delta,\ l \gg \varepsilon$ if $\varepsilon \ll 1$. The coordinates of the points are then

$$x \equiv (-a, 0, 0) \qquad\qquad y \equiv (b, 0, 0)$$
$$x' \equiv (\delta - l, -\lambda\varepsilon, 0) \qquad\qquad y' \equiv (\delta, \lambda\varepsilon, h\varepsilon),$$

writing only the first three coordinates (this in allowed because all the points are in the plane $\{c_4 = \cdots = c_N = 0\}$). The fact that $x'y'$ intersects $B(z,\varepsilon)$ gives $0 \le h \le \frac{2\lambda}{\lambda-1}$, and then

$$0 \le h \le 3 \tag{4.4.14}$$

if $\lambda \ge 3$. Let us write now the lengths of the segments:

$$\overline{xy} = a + b \qquad\qquad \overline{x'y} = b + l - \delta + \frac{\lambda^2\varepsilon^2}{2(b+l-\delta)} + o(\varepsilon^2) \tag{4.4.15}$$

$$\overline{x'y'} = l + \frac{(4\lambda^2 + h^2)\varepsilon^2}{2l} + o(\varepsilon^2) \qquad \overline{xy'} = a + \delta + \frac{(\lambda^2 + h^2)\varepsilon^2}{2(a+\delta)} + o(\varepsilon^2). \tag{4.4.16}$$

Finally the facts that $x'y'$ intersects $B(z,\varepsilon)$, $l \le \overline{x'y'} \le 2\overline{xy}$ and $b \ge a$ imply

$$\delta + \varepsilon \ge \frac{\lambda - 1}{2\lambda} l \qquad\qquad \delta - \varepsilon \le \frac{\lambda + 1}{2\lambda} l \qquad\qquad b \ge \frac{\overline{x'y'}}{4} \ge \frac{l}{4}. \tag{4.4.17}$$

Substituting (4.4.15) and (4.4.16) in the inequality $\overline{xy} + \overline{x'y'} \le \overline{xy'} + \overline{x'y}$ and making use of (4.4.14) and (4.4.17) the computation leads to the desired upper bound for λ. $\square$

PROOF. (Prop.) Let us define $\Lambda_i = \left\{(x,y) \in \Omega \times \Omega :\ R/2^{i+1} < \overline{xy} \le R/2^i\right\}$, where R is the diameter of Ω, and $\mu_i,\ i \in \mathbb{N}$, by

$$\mu_i(A) := \int_{\Lambda_i} l(xy \cap A)\ d\gamma(x,y). \tag{4.4.18}$$

Recalling (4.4.3) it is clear that $\mu = \sum_{i\in\mathbb{N}} \mu_i$. We already know that $\mu \in L^1$, and then $\mu_i \in L^1$ for each $i \in \mathbb{N}$: let us give then the last definition,

$$\Omega_i := \left\{z \in \Omega : \mu_i(z) > 0,\ \nexists (x,y) \in \Lambda_i \cap spt(\gamma)\ s.t.\ z \in xy\right\}.$$

From (4.4.18) it follows that $\mu_i(\Omega_i) = 0$, because $xy \cap \Omega_i = \varnothing$ for each $(x, y) \in \Lambda_i$, and then $\mathcal{L}(\Omega_i) = 0$: this implies $\mathcal{L}(\bigcup_i \Omega_i) = 0$ and then $\mu(\bigcup_i \Omega_i) = 0$.

Let $z \notin \bigcup_i \Omega_i$ be a Lebesgue point for μ such that $\mu(z) = \sum_i \mu_i(z) > 0$, and let j be the least integer such that $\mu_j(z) > 0$; then there exists a transport ray $uv \ni z$ such that $R/2^{j+1} < \overline{uv} \leq R/2^j$. If $(x, y) \in \Lambda_i$ and $i \geq j$, clearly $\overline{xy} \leq 2\overline{uv}$: then, using the Claim, if $r \ll 1$

$$\begin{aligned}\sum_{i=j}^{\infty} \mu_i(B(z,r)) &= \int_{\overline{xy} \leq R/2^j} l(B(z,r) \cap xy) d\gamma(x,y) \\ &\leq 2r \left(\int_{C_r \times \Omega} d\gamma(x,y) + \int_{\Omega \times C_r} d\gamma(x,y) \right) \leq 2r \left(f^+(C_r) + f^-(C_r) \right).\end{aligned}$$

The volume of the cone C_r is less than $R\omega_{N-1}(\lambda r)^{N-1}$, and then

$$\begin{aligned}\sum_{i=j}^{\infty} \mu_i(B(z,r)) &\leq 2rR\omega_{N-1}(\lambda r)^{N-1} \left(\|f^+\|_{L^\infty} + \|f^-\|_{L^\infty} \right) \\ &= 2R\omega_{N-1}\lambda^{N-1} \left(\|f^+\|_{L^\infty} + \|f^-\|_{L^\infty} \right) r^N,\end{aligned}$$

which gives

$$\mu(z) = \sum_{i=j}^{\infty} \mu_i(z) \leq 2R\omega_{N-1}\lambda^{N-1} \left(\|f^+\|_{L^\infty} + \|f^-\|_{L^\infty} \right) :$$

since this inequality holds for $\mu-$a.e. point z, this completes the proof. □

Once we solved the problem for $f^+,\ f^- \in L^\infty$, we can prove the general assert, which shows the situation when both $f^+,\ f^- \in L^p$:

THEOREM 4.4.18. *If both $f^+,\ f^- \in L^p$, then $\mu \in L^q$ for $1 \leq q < p$.*

PROOF. Let us define Ω_i^+, Ω_i^-, μ_{ij} and λ_{ij} with $i, j \in \mathbb{N}$ as follows

$$\Omega_0^\pm := \left\{ x \in \Omega : \ f^\pm(x) \leq 1 \right\}, \qquad \Omega_i^\pm := \left\{ x \in \Omega : \ 2^{i-1} < f^\pm(x) \leq 2^i \right\},$$

$$\mu_{ij} := \int_{\Omega_i^+ \times \Omega_j^-} l(xy \cap A)\ d\gamma(x,y), \qquad \lambda_{ij} := \gamma\left(\Omega_i^+ \times \Omega_j^- \right) :$$

then $\mu = \sum_{i,j} \mu_{ij}$, it is easy to give the bound

$$\|f^\pm\|_{L^p}^p \geq \sum_{i \geq 1} 2^{(i-1)p} \mathcal{L}(\Omega_i^\pm) \geq \sum_{i \geq 1} \frac{2^{i(p-1)} f^\pm(\Omega_i^\pm)}{2^p} \tag{4.4.19}$$

and we have also $\sum_j \lambda_{ij} = f^+(\Omega_i^+)$ and $\sum_i \lambda_{ij} = f^-(\Omega_j^-)$. From the definition it follows that $\|\mu_{ij}\|_{L^1} \leq R\lambda_{ij}$ and, since $\gamma \llcorner (\Omega_i \times \Omega_j)$ is an optimal plan of transport for the problem given by his marginals, thanks to proposition 4.4.17 we have $\|\mu_{ij}\|_{L^\infty} \leq K(2^i + 2^j)$. Then an upper bound for the L^q norm of μ is given if we suppose that

$\mu_{ij} = K(2^i + 2^j)$ on a set of Lebesgue measure $R\lambda_{ij}/\big(K(2^i + 2^j)\big)$ and that these sets intersect as much as possible. We want now make use of lemma 4.4.16, and then we have to give an order to $\mathbb{N} \times \mathbb{N}$: we do it by setting $(m,n) < (i,j)$ if $m \vee n < i \vee j$, or if $m \vee n = i \vee j$ and $m \wedge n < i \wedge j$, or if $m = j$, $n = i$ and $m < n$. This gives an isomorphism between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ and so allows us to use lemma 4.4.16, which gives

$$\|\mu\|_{L^q}^q \leq \sum_{i,j} \frac{R\lambda_{ij}}{K\left(2^i + 2^j\right)} \left(\sum\nolimits_{(m,n)\leq(i,j)} K\left(2^m + 2^n\right)\right)^q. \tag{4.4.20}$$

But if $i \geq j$ then

$$\sum\nolimits_{(m,n)\leq(i,j)} 2^m + 2^n \leq 2(i+1)\left(1 + 2 + 4 + \cdots + 2^i\right) \leq (i+1)\ 2^{i+2},$$

and then from (4.4.20) we have

$$\begin{aligned}
\|\mu\|_{L^q}^q &\leq RK^{q-1}\left(\sum\nolimits_{i\geq j} \frac{\lambda_{ij}\left((i+1)\ 2^{i+2}\right)^q}{2^i} + \sum\nolimits_{i\leq j} \frac{\lambda_{ij}\left((j+1)\ 2^{j+2}\right)^q}{2^j}\right)\\
&\leq 4^q RK^{q-1}\left(\sum\nolimits_{i\geq j} \lambda_{ij}\ (i+1)^q\ 2^{i(q-1)} + \sum\nolimits_{i\leq j} \lambda_{ij}\ (j+1)^q\ 2^{j(q-1)}\right)\\
&\leq \widetilde{K}\left(\sum\nolimits_i \Big((i+1)^q\ 2^{i(q-1)} \sum\nolimits_j \lambda_{ij}\Big) + \sum\nolimits_j \Big((j+1)^q\ 2^{j(q-1)} \sum\nolimits_i \lambda_{ij}\Big)\right)\\
&\leq \widetilde{K}\left(\sum\nolimits_i (i+1)^q\ 2^{i(q-1)} f^+(\Omega_i^+) + \sum\nolimits_j (j+1)^q\ 2^{j(q-1)} f^-(\Omega_j^-)\right).
\end{aligned}$$

By (4.4.19) and the fact that $q < p$, this proves the assert. $\square$

REMARK 4.4.19. *At the moment, we do not know if a better interpolation allows, from proposition 4.4.17, to prove theorem 4.4.18 even with $p = q$. In fact, we do not know a proof nor a counterexample which says if $f^+,\ f^- \in L^p \Longrightarrow \mu \in L^p$.*

CHAPTER 5

Topological Equivalence of Some Variational Problems Involving Distances

5.1. Introduction

The purpose of this chapter is to compare the asymptotic behaviour of different kinds of functionals involving distances on a given open subset Ω of $\mathbb{R}^N$. This issue was basically motivated by the fact that several variational problems can be described in terms of some distance functions, which can be suitably associated. This is for instance the case of the Kantorovich functional (2.1.3) which we introduced in chapter 2 for the case of the Euclidean distance but which can be considered in more general settings Roughly, for every distance d on $\mathbb{R}^N$ and any pair of probability measures f^+, f^- on $\mathbb{R}^N$, one deals with the following minimum problem:

$$\min\{\int_{\mathbb{R}^{2N}} d(x,y)d\mu(x,y), \quad \mu \in \mathcal{P}(f^+,f^-)\}, \tag{5.1.1}$$

where, we recall from chapter 2, $\mathcal{P}(f^+,f^-)$ is the class of positive measures on $\mathbb{R}^{2N}$ with marginals f^+ and f^- (i.e. $\mu(B\times\mathbb{R}^N)=f^+(B)$ and $\mu(\mathbb{R}^N\times B)=f^-(B)$).

Notation In this chapter we will be dealing with curves which are classically denoted by γ so the measure which play in some sense the role of the plans of transport will be denoted by μ. This is not a true change of notations because the functionals we consider here are defined for all positive measure and not just on sets of transports. But we wanted to precise it because the results of this chapter contains as particular case some results on Γ-convergence of transport functionals.

When d is the Euclidean distance, the infimum in (5.1.1) is equal, by a duality principle, to

$$\sup\{< f^+ - f^-, u >: \quad u \in \mathrm{Lip}(\mathbb{R}^N), \quad |\nabla u(x)| \le 1\}. \tag{5.1.2}$$

Notice that the Euclidean distance appears in (5.1.2) through the constraint on the modulus of the gradient. Actually more in general, for an arbitrary distance d, the equivalence between problems (5.1.1) and (5.1.2) still holds provided the constraint $|\nabla u(x)| \le 1$ is replaced by the requirement that $\nabla u(x)$ belongs to a suitable convex set

$K_d(x)$ associated with d (cf. Section 2). So we are again concerned with distances, now via indicators of convex sets of gradients (see [**19**]).
Besides the problems mentioned above, it is also quite natural to consider the length functional induced by the Finsler metric φ_d associated by differentiation to the given distance d (cf. Section 2). More precisely, for every x and y in $\mathbb{R}^N$, we consider the problem

$$\inf\{\int_0^1 \varphi_d(\gamma,\dot{\gamma})dt : \quad \gamma \in \mathrm{Lip}([0,1],\mathbb{R}^N),\ \gamma(0)=x,\ \gamma(1)=y\}\ . \tag{5.1.3}$$

The convergence of the variational problems (5.1.3) associated with a sequence of varying distances $\{d_n\}$, has been already studied in literature, especially for what concerns the homogenization of Riemannian and Finsler metrics (see [**1**], [**4**]).

In general, the most natural approach in order to study the dependence on the involved distance in all the variational problems quoted above, relies on Γ-convergence of the corresponding functionals (see Section 2 for a brief recall of this topic).
Our main result states the equivalence between the Γ-convergence of the functionals L_n, J_n, F_n associated to a sequence of distances $\{d_n\}$ respectively through

$$L_n(\gamma) = \int_0^1 \varphi_{d_n}(\gamma(t),\gamma'(t))dt, \tag{5.1.4}$$

if γ is a Lipschitz curve in Ω,

$$J_n(\mu) = \int_{\Omega\times\Omega} d_n(x,y)d\mu(x,y), \tag{5.1.5}$$

if μ is a positive measure on $\Omega\times\Omega$,

$$F_n(u) = \begin{cases} 0 & \text{if } \nabla u(x) \in K_n(x) \text{ for } \mathcal{L}^N\text{–a.e. } x \in \Omega\ , \\ +\infty & \text{otherwise}\ , \end{cases} \tag{5.1.6}$$

if u is a Lipschitz function on Ω.
All the convergences above actually turn out to reduce to the usual uniform convergence of d_n on compact subsets of $\mathbb{R}^{2N}$. Such equivalences are proved under the assumption that each d_n belongs to the class $\mathcal{D}(\Omega)$ of all geodesic distances on Ω (see Definition 5.2.1), which are in addition controlled from above and from below by fixed multiples of the Euclidean distance. In fact, the result becomes false when getting out of the class $\mathcal{D}(\Omega)$, as it can be easily shown by some examples. They are proposed in the final section of the paper, as well as a conjecture about the density of Riemannian metrics in the class of all Finsler ones.

The chapter is organized as follows: in Section 2 the basic notions and preliminaries are summed up; Section 3 is devoted to the statement and proof of the main result; Section 4 contains some further remarks and examples.

5.2. Notation and Preliminary Results

Let Ω be a given open subset of $\mathbb{R}^N$. For every $x, y \in \Omega$, we denote by $|x|$ and $x \cdot y$ respectively the Euclidean norm of x and the corresponding usual scalar product between x and y. The Lebesgue measure on Ω will be indicated by $\mathcal{L}^N$.
We shall consider distance functions d on Ω. To any such d, we may associate the length functional L_d defined by

$$L_d(\gamma) := \sup\left\{ \sum_{i=1}^{k-1} d(\gamma(t_i), \gamma(t_{i+1})) \ : \ t_i \in [0,1] \right\} , \tag{5.2.1}$$

where $\gamma : [0,1] \to \Omega$ is any curve, and $\{t_i\}$ runs over the class of finite families of points in $[0,1]$, with $t_1 = 0$, $t_k = 1$, and $t_i < t_{i+1}$.

DEFINITION 5.2.1. *We call geodesic distance on Ω any distance d such that*

$$d(x,y) = \inf\left\{ L_d(\gamma) \ : \ \gamma(0) = x \, , \, \gamma(1) = y \right\} , \qquad \forall \ (x,y) \in \Omega \times \Omega \ . \tag{5.2.2}$$

Given two positive constants c and C, we then consider the class $\mathcal{D}(\Omega)$ of all geodesic distances d on Ω such that

$$c|x-y| \le d(x,y) \le C|x-y| \ , \quad \forall \ (x,y) \in \Omega \times \Omega \ . \tag{5.2.3}$$

REMARK 5.2.2. It is easy to see that, for a general open set Ω, there exist distances d, even fulfilling (5.2.3), which are not of geodesic type. An interesting issue would be to characterize the domains Ω for which the class $\mathcal{D}(\Omega)$ is not empty. Notice indeed that we may have $\mathcal{D}(\Omega) = \emptyset$ if the domain Ω is disconnected or it has an irregular boundary.

We endow $\mathcal{D}(\Omega)$ with the uniform convergence on the compact subsets of $\Omega \times \Omega$. We shall prove in Section 3 (cf. proof of Theorem 5.3.1) that the class $\mathcal{D}(\Omega)$ is compact with respect to such topology. Moreover, the following criterion will be useful in order to check the convergence in $\mathcal{D}(\Omega)$.

PROPOSITION 5.2.3. *For a sequence $\{d_n\} \subset \mathcal{D}(\Omega)$ the following conditions are equivalent:*

i) $d_n \to d$ *uniformly on compact subsets of* $\Omega \times \Omega$*;*
ii) $\forall (x,y) \in \Omega \times \Omega \ , \ \exists \ (x_n, y_n) \to (x,y) \ : \ d_n(x_n, y_n) \to d(x,y)$;
iii) $\forall (x,y) \in \Omega \times \Omega \ , \ \forall \ (x_n, y_n) \to (x,y) \ : \ d_n(x_n, y_n) \to d(x,y)$.

PROOF. If $d_n \to d$ in $\mathcal{D}(\Omega)$, it is enough to take $(x_n, y_n) = (x, y)$ in ii). Assume now ii), let $(x, y) \in \Omega \times \Omega$ and let $(\tilde{x}_n, \tilde{y}_n)$ be a sequence converging to (x, y); we claim that $d_n(\tilde{x}_n, \tilde{y}_n) \to d(x, y)$ that is iii). Indeed, if (x_n, y_n) is a sequence as in ii), using also (5.2.3) we get

$$\begin{aligned} &\lim_{n\to\infty} |d_n(\tilde{x}_n, \tilde{y}_n) - d(x,y)| \leq \\ \leq \lim_{n\to\infty} &\big[|d_n(\tilde{x}_n, \tilde{y}_n) - d_n(x_n, y_n)| + |d_n(x_n, y_n) - d(x,y)|\big] \leq \\ &\leq C \lim_{n\to\infty} |(\tilde{x}_n, \tilde{y}_n) - (x_n, y_n)| = 0 \ . \end{aligned}$$

Finally, the implication iii)⇒i) is a well known fact. □

To any distance $d \in \mathcal{D}(\Omega)$, we may associate the function $\varphi_d : \Omega \times R^N \to [0, +\infty)$ defined by:

$$\varphi_d(x, z) := \overline{\lim_{t\to 0^+}} \frac{d(x, x+tz)}{t}. \tag{5.2.4}$$

In [**64**] it is shown that $\varphi_d(x, z)$ provides a Finsler metric (see for instance [**6**]), i.e. it is Borel measurable, and convex positively one-homogeneous in z. Moreover, for $\mathcal{L}^N$-a.e. $x \in \Omega$, the $\overline{\lim}$ at the right hand side of (5.2.4) is actually a limit, and by (5.2.3) we obtain

$$c|z| \leq \varphi_d(x, z) \leq C|z|. \tag{5.2.5}$$

The metric φ_d allows us to write in an integral form the length functional L_d.

PROPOSITION 5.2.4. *Let $d \in \mathcal{D}(\Omega)$. Then, for every Lipschitz curve $\gamma : [0,1] \to \Omega$, we have*

$$L_d(\gamma) = \int_0^1 \varphi_d\big(\gamma(t), \gamma'(t)\big)\, dt \ . \tag{5.2.6}$$

Moreover, for every $x, y \in \Omega$ we have

$$d(x,y) = \inf\Big\{ \int_0^1 \varphi\big(\gamma(t), \gamma'(t)\big)\, dt \ : \ \gamma \in \mathrm{Lip}([0,1], \mathbb{R}^N) \ , \ \gamma(0) = x \ , \ \gamma(1) = y \Big\} \tag{5.2.7}$$

PROOF. For a proof of (5.2.6), we refer to [**22**], Theorem 2.5. Equality (5.2.7) is obtained as a straightforward consequence of (5.2.6) and of the fact that every rectifiable curve admits a Lipschitz parametrization (see for instance [**15**]). □

The metric φ_d endows Ω with a natural metric structure via the distance

$$\delta_{\varphi_d}(x,y) := \sup\{u(x) - u(y) : \quad u \in \mathrm{Lip}(\Omega), \quad \nabla u(\cdot) \in K_d(\cdot) \ \ \mathcal{L}^N - \text{a.e. on } \Omega\}. \tag{5.2.8}$$

The sets $K_d(\cdot)$ appearing in (5.2.8) are the family of the (convex) unit balls of the polar function $\varphi_d^0(\cdot, z)$, namely

$$K_d(x) := \{z \in \mathbb{R}^N : \quad \varphi_d^0(x,z) \leq 1\}, \tag{5.2.9}$$

where $\varphi_d^0 : \Omega \times \mathbb{R}^N \to [0,+\infty)$ is defined by

$$\varphi_d^0(x,z^*) := \sup\{z^* \cdot z : \quad \varphi_d(x,z) \leq 1\}. \tag{5.2.10}$$

We notice that φ_d^0 provides a Finsler metric, and by (5.2.3) it satisfies:

$$\frac{1}{C}|z^*| \leq \varphi_d^0(x,z^*) \leq \frac{1}{c}|z^*|. \tag{5.2.11}$$

PROPOSITION 5.2.5. *For every $d \in \mathcal{D}(\Omega)$, we have $\delta_{\varphi_d} = d$.*

PROOF. It was proved by Venturini [**64**] that, for every given Finsler metric φ, the distance δ_φ induced by φ according to (5.2.8), satisfies

$$\delta_\varphi = \max\left\{\delta \text{ enjoying (5.2.3)} \ : \ \varphi_\delta \leq \varphi\right\} . \tag{5.2.12}$$

It is then obvious that, taking $\varphi = \varphi_d$, one has $\delta_{\varphi_d} \geq d$. To show the converse inequality, it is enough to prove that

$$\varphi_\delta \leq \varphi_d \Rightarrow \delta \leq d \ . \tag{5.2.13}$$

In fact, using Proposition 5.2.4, and the assumption that d is of geodesic type, the inequality $\varphi_\delta \leq \varphi_d$ yields

$$\begin{aligned}\delta(x,y) &\leq \inf\Big\{L_\delta(\gamma) \ : \ \gamma(0)=x \, , \, \gamma(1)=y\Big\} \\ &\leq \inf\Big\{L_d(\gamma) \ : \ \gamma(0)=x \, , \, \gamma(1)=y\Big\} = d(x,y)\end{aligned}$$

which concludes the proof. ☐

In the following it will be useful to reformulate the constraint $\nabla u \in K_d$ appearing in (5.2.8) in a slightly different way. To this aim, for every $d \in \mathcal{D}(\Omega)$ and any $u \in \mathrm{Lip}(\Omega)$, we define the inf-convolution between u and d, by

$$u \bigtriangledown d(x) := \inf_{y\in\Omega}\{u(y) + d(x,y)\}, \quad x \in \Omega. \tag{5.2.14}$$

Then, for u, d and K_d as above we have

PROPOSITION 5.2.6. *Let $d \in \mathcal{D}(\Omega)$. Then, for every Lipschitz function u on Ω, the following conditions are equivalent:*

(i) $\nabla u(\cdot) \in K_d(\cdot) \quad \mathcal{L}^N -$ a.e. on Ω;
(ii) $u(x) - u(y) \leq d(x,y) \quad \forall (x,y) \in \Omega \times \Omega$;
(iii) $u \bigtriangledown d = u$ on Ω.

Proof. We first notice that (ii) and (iii) are equivalent by the definition (5.2.14) of $u \bigtriangledown d$. Moreover, the implication (i) $\Rightarrow$ (ii) follows from (5.2.8) and Proposition 5.2.5. Conversely, assume that (ii) holds. For $\mathcal{L}^N$-a.e. $x \in \Omega$ and any $z \in \mathbb{R}^N$ we deduce

$$\nabla u(x) \cdot z = \lim_{t \to 0} \frac{u(x+tz) - u(x)}{t} \le \overline{\lim_{t \to 0}} \frac{d(x, x+tz)}{t} = \varphi_d(x, z) \ . \tag{5.2.15}$$

It follows by the definition of the dual metric $\varphi_d^0(x, \cdot)$ that $\varphi_d^0(x, \nabla u(x)) \le 1$ for $\mathcal{L}^N$ -a.e. $x \in \Omega$, hence (i) follows. $\square$

We finally recall the notion of Γ-convergence which will be widely employed in the next section. For more details on this tool we refer to the book [**20**]. Let X be a metric space, or more in general a topological space satisfying the first axiom of countability at a point x. We say that a sequence of functionals $F_n : X \to \overline{\mathbb{R}}$ Γ-converges to F at x if

$$\Gamma - \liminf F_n(x) = \Gamma - \overline{\lim} F_n(x), \tag{5.2.16}$$

where

$$\begin{cases} \Gamma - \liminf F_n(x) = \inf \big\{ \liminf F_n(x_n) : \quad x_n \to x \big\} \\ \Gamma - \overline{\lim} F_n(x) = \inf \big\{ \overline{\lim} F_n(x_n) : \quad x_n \to x \big\}. \end{cases} \tag{5.2.17}$$

We recall that if $F_n \to F$ on X, then F is lower semicontinuous, and it is coercive too provided the functionals F_n are equi-coercive. In this case, we have the crucial property of Γ-convergence, that is the sequence $\{\inf_X F_n\}$ converge to the minimum of F on X.

5.3. The Main Result

To any distance d in $\mathcal{D}(\Omega)$, we are going to associate some functionals defined respectively on the class $\mathcal{M}(\Omega)$ of all positive and finite Borel measures μ on $\Omega \times \Omega$, on the class $\mathcal{C}(\Omega)$ of all Lipschitz curves γ in Ω, and on the class $\mathrm{Lip}(\Omega)$ of all Lipschitz real functions u on Ω. We set:

$$J_d(\mu) = \int d(x,y) d\mu(x,y), \quad \mu \in \mathcal{M}(\Omega) \ ; \tag{5.3.1}$$

$$L_d(\gamma) = \int_0^1 \varphi_d(\gamma(t), \gamma'(t)) dt, \quad \gamma \in \mathcal{C}(\Omega) \ ; \tag{5.3.2}$$

$$F_d(u) = \begin{cases} 0 & \text{if } \nabla u(x) \in K_d(x) \text{ for } \mathcal{L}^N\text{–a.e. } x \in \Omega \,, \qquad u \in \mathrm{Lip}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{5.3.3}$$

We endow the class $\mathcal{M}(\Omega)$ with the topology of weak* convergence, the class $\mathcal{C}(\Omega)$ with the topology of the uniform convergence on the interval $[0, 1]$, and the space $\mathrm{Lip}(\Omega)$ with the topology of the uniform convergence on compact subsets of Ω. Then we can study the Γ-convergence of the functionals associated to a sequence of distances $\{d_n\}$ according to (5.3.1),(5.3.2), and (5.3.3). Our main result is the following.

THEOREM 5.3.1. *Let $\{d_n\}_{n\in\mathbb{N}}$ and d belong to $\mathcal{D}(\Omega)$. Let J_n, L_n, F_n, and J, L, F, be the functionals associated respectively to d_n and d, defined as in (5.3.1), (5.3.2), (5.3.3). Then the following conditions are equivalent:*

(i) $d_n \to d$ *in* $\mathcal{D}(\Omega)$;
(ii) $J_n \xrightarrow{\Gamma} J$ *on* $\mathcal{M}(\Omega)$;
(iii) $L_n \xrightarrow{\Gamma} L$ *on* $\mathcal{C}(\Omega)$;
(iv) $F_n \xrightarrow{\Gamma} F$ *on* $\mathrm{Lip}(\Omega)$.

PROOF. – (i) $\Rightarrow$ (ii). It is immediate.

– (i) $\Rightarrow$ (iii). For every fixed $\gamma \in \mathcal{C}(\Omega)$, we have to prove the following two claims:

$$\forall \gamma_n \to \gamma \text{ in } \mathcal{C}(\Omega) \ : \ L(\gamma) \le \liminf_{n\to\infty} L_n(\gamma_n), \tag{5.3.4}$$

$$\exists \gamma_n \to \gamma \text{ in } \mathcal{C}(\Omega) \ : \ L(\gamma) \ge \varlimsup_{n\to\infty} L_n(\gamma_n). \tag{5.3.5}$$

We begin by proving (5.3.4). Let $\gamma_n \to \gamma$ in $\mathcal{C}(\Omega)$. By the definition of $L(\gamma)$, for any $\delta > 0$ we can find a partition of the interval $[0,1]$, indexed over a finite set I_δ, such that

$$L(\gamma) \le \delta + \sum_{i\in I_\delta} d\big(\gamma(t_i), \gamma(t_{i+1})\big) \ . \tag{5.3.6}$$

Since $\{\gamma_n\}$ converges uniformly on $[0,1]$, we may assume that, for n large enough, $(\gamma_n(s), \gamma_n(t))$ belongs to a fixed compact set K of $\Omega\times\Omega$ for all $s,t \in [0,1]$. Then for every $i \in I_\delta$, we have

$$\begin{aligned} |d_n(\gamma_n(t_i), \gamma_n(t_{i+1})) - d(\gamma(t_i), \gamma(t_{i+1}))| \le \\ \le |d_n(\gamma_n(t_i), \gamma_n(t_{i+1})) - d\big(\gamma_n(t_i), \gamma_n(t_{i+1})\big)| \\ +|d(\gamma_n(t_i), \gamma_n(t_{i+1})) - d(\gamma(t_i), \gamma(t_{i+1}))| \\ \le \sup_K |d_n - d| + 2C \sup_{[0,1]} |\gamma_n - \gamma| := \epsilon_n. \end{aligned} \tag{5.3.7}$$

So we infer from (5.3.6) and from the definition of L_n that

$$L(\gamma) \le \delta + \sum_{i\in I_\delta} \big[d_n\big(\gamma_n(t_i), \gamma_n(t_{i+1})\big) + \epsilon_n\big] \le \delta + L_n(\gamma_n) + \varepsilon_n \mathrm{card}(I_\delta) \ .$$

Since by assumption ε_n tends to 0 as $n \to \infty$, passing to the lim inf as $n \to +\infty$, we get

$$L(\gamma) \le \liminf_{n\to\infty} L_n(\gamma_n) + \delta.$$

This yields (5.3.4) since $\delta > 0$ was arbitrary.

We prove now (5.3.5). Let $\gamma \in \mathcal{C}(\Omega)$, let K be a compact subset of $\Omega \times \Omega$ chosen as above, and let $k(n) \to \infty$ be a sequence such that

$$\lim_{n\to\infty} k(n) \sup_K |d_n - d| = 0.$$

For every $n \in \mathbb{N}$, let I_n be the partition of $[0,1]$ into $k(n)$ intervals of equal length, and denote by $\{t_n^i\}$, $i = 1, \ldots, k(n)+1$, the end-points of such intervals. Let γ_n be the curve whose restriction γ_n^i to the interval $[t_n^i, t_n^{i+1}]$ is defined by

$$\gamma_n^i(t_n^i) = \gamma(t_n^i)\ ,\quad \gamma_n^i(t_n^{i+1}) = \gamma(t_n^{i+1})\ ,\quad L_n\big(\gamma_n^i\big) \le d_n\big(\gamma(t_n^i), \gamma(t_n^{i+1})\big) + \frac{1}{2^{k(n)}}\ ; \tag{5.3.8}$$

in other words, γ_n^i is almost a minimal d_n-geodesic connecting $\gamma(t_n^i)$ to $\gamma(t_n^{i+1})$. We claim that the sequence $\{\gamma_n\}$ converges to γ in $\mathcal{C}(\Omega)$. Indeed, if for every $n \in \mathbb{N}$ and $t \in [0,1]$, $[t_n^-, t_n^+]$ denotes the interval of I_n containing t, we have

$$|\gamma_n(t) - \gamma(t)| \le |\gamma_n(t) - \gamma_n(t_n^+)| + |\gamma(t_n^+) - \gamma(t)| := (\mathrm{I})_n + (\mathrm{II})_n\ .$$

Now, by the uniform continuity of γ on $[0,1]$, the term $(\mathrm{II})_n$ tends to zero as $n \to +\infty$ uniformly with respect to t. The same holds for the term $(\mathrm{I})_n$ because, using (5.2.5) and (5.2.6), it can be estimated as follows:

$$\begin{aligned} c|\gamma_n(t) - \gamma_n(t_n^+)| &\le L_n(\gamma_n|_{(t,t_n^+)}) \le L_n(\gamma_n|_{(t_n^-,t_n^+)}) \le \\ &\le C|\gamma_n(t_n^-) - \gamma_n(t_n^+)| + \frac{1}{2^{k(n)}} = C|\gamma(t_n^-) - \gamma(t_n^+)| + \frac{1}{2^{k(n)}}, \end{aligned} \tag{5.3.9}$$

where $\gamma_n|_{(s,\tau)}$ denotes the restrictions of γ_n to the interval (s,τ). The sequence $\{\gamma_n\}$ satisfies the inequality required in (5.3.5). Indeed, using the definition of $L(\gamma)$ and (5.3.8), we infer

$$\begin{aligned} L(\gamma) &\ge \sum_{i=1}^{k(n)} d(\gamma(t_n^i), \gamma(t_n^{i+1})) = \\ &= \sum_{i=1}^{k(n)} d_n(\gamma(t_n^i), \gamma(t_n^{i+1})) + \sum_{i=1}^{k(n)} \left[d\big(\gamma(t_n^i), \gamma(t_n^{i+1})\big) - d_n(\gamma(t_n^i), \gamma(t_n^{i+1}))\right] \ge \\ &\ge L_n(\gamma_n) - \frac{k(n)}{2^{k(n)}} + \sum_{i=1}^{k(n)} \left[d\big(\gamma(t_n^i), \gamma(t_n^{i+1})\big) - d_n\big(\gamma(t_n^i), \gamma(t_n^{i+1})\big)\right]\ . \end{aligned} \tag{5.3.10}$$

To get the required inequality, it is then enough to pass to the $\overline{\lim}$ in (5.3.10), noticing that, by the choice of the sequence $k(n)$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{k(n)} \left|d\big(\gamma(t_n^i), \gamma(t_n^{i+1})\big) - d_n\big(\gamma(t_n^i), \gamma(t_n^{i+1})\big)\right| \le \lim_{n\to\infty} k(n) \sup_K |d_n - d| = 0\ .$$

– (i) ⇒ (iv). We have to prove the following two claims:

$$u_n \to u \text{ in Lip}(\Omega),\ \nabla u_n \in K_n\ \mathcal{L}^N\text{–a.e.} \ \Rightarrow\ \nabla u \in K\ \mathcal{L}^N\text{–a.e.} \tag{5.3.11}$$

$$u \in \text{Lip}(\Omega),\ \nabla u \in K\ \mathcal{L}^N\text{–a.e.} \ \Rightarrow$$
$$\exists\ u_n \to u \text{ in Lip}(\Omega),\ \text{s.t. } \nabla u_n \in K_n\ \mathcal{L}^N\text{–a.e.} \tag{5.3.12}$$

We begin by proving (5.3.11). Let $u_n \to u$ in $\text{Lip}(\Omega)$, with $\nabla u_n \in K_n\ \mathcal{L}^N$-a.e.; such condition implies by Proposition 5.2.6 that

$$u_n(x) - u_n(y) \le d_n(x,y) \ \ \forall (x,y) \in \Omega \times \Omega\ .$$

Since $u_n \to u$ in $\text{Lip}(\Omega)$, and $d_n \to d$ in $\mathcal{D}(\Omega)$, passing to the limit as $n \to +\infty$ in the above inequality, we obtain that the same relation is satisfied by u and d, that is

$$u(x) - u(y) \le d(x,y) \ \ \forall (x,y) \in \Omega \times \Omega\ .$$

Using again Proposition 5.2.6, this entails $\nabla u \in K\ \mathcal{L}^N$-a.e.
We pass now to prove (5.3.12). Let $u \in \text{Lip}(\Omega)$, with $\nabla u \in K\ \mathcal{L}^N$-a.e. We consider the sequence $u_n := u \bigtriangledown d_n$. By Proposition 5.2.6, in order to check that $u_n \to u$ in $\text{Lip}(\Omega)$, it is enough to show that $u_n \to u \bigtriangledown d$ in $\text{Lip}(\Omega)$.
We may assume that u is bounded from below on Ω since otherwise we approximate u by $u_k = u \vee (-k)$, take $u_{k,n} = u_k \bigtriangledown d_n$ and then use a diagonal argument. For every x belonging to a given compact subset K of Ω, by the definition of $u \bigtriangledown d_n$, we find $y_n \in \Omega$ such that

$$u(y_n) + d_n(x, y_n) \le u \bigtriangledown d_n(x) + \frac{1}{n}. \tag{5.3.13}$$

Since u has been assumed bounded from below and d_n satisfy (5.2.3), we obtain that the sequence $\{y_n\}$ belongs to a bounded set $B \subseteq \Omega$. Thus we find by (5.3.13)

$$u(x) - u_n(x) \le u(y_n) + d(x,y_n) - u(y_n) - d_n(x,y_n) + \frac{1}{n} = d(x,y_n) - d_n(x,y_n) + \frac{1}{n}.$$

On the other hand, since $u_n = u \bigtriangledown d_n \le u$, we obtain

$$\sup_K |u - u_n| \le \sup_{K \times B} |d - d_n| + \frac{1}{n}$$

which gives $u_n \to u$ in $\text{Lip}(\Omega)$ taking into account that $d_n \to d$ uniformly on bounded sets of $\Omega \times \Omega$.
It remains to prove that $\nabla u_n \in K_n\ \mathcal{L}^N$-a.e.. To this aim, it is enough to show that $(u \bigtriangledown d_n) \bigtriangledown d_n = u \bigtriangledown d_n$. The inequality $\le$ is obviously true. Conversely, using the triangular inequality for the distances d_n, we get

$$\begin{aligned}(u \bigtriangledown d_n) \bigtriangledown d_n(x) \quad &= \inf_{y\in\Omega}\left\{\inf_{z\in\Omega}\left[u(z)+d_n(y,z)\right]+d_n(x,y)\right\}\\ &= \inf_{y,z\in\Omega}\left[u(z)+d_n(y,z)+d_n(x,y)\right]\\ &\geq \inf_{z\in\Omega}\left[u(z)+d_n(x,z)\right]=u \bigtriangledown d_n(x).\end{aligned}$$

– (ii), or (iii), or (iv) $\Rightarrow$ (i). Each of these implications will follow easily from the correspondent converse already proved, coupled with the following

Claim: The class $\mathcal{D}(\Omega)$ is compact.

This claim is obtained as a consequence of the Ascoli-Arzelà Theorem and of the implication (i) $\Rightarrow$ (iii) already proved. Indeed, for every $d \in \mathcal{D}(\Omega)$, using the triangle inequality and the upper bound in (5.2.3), we get

$$|d(x,y)-d(x',y')| \leq d(x,x')+d(y,y') \leq C(|x-x'|+|y-y'|) \quad \forall (x,y),(x',y') \in \Omega\times\Omega,$$

so that d is $C-$Lipschitz continuous on $\Omega\times\Omega$. Therefore, from any sequence $\{d_n\} \subset \mathcal{D}(\Omega)$, we may extract a subsequence converging to some element d in $\mathcal{D}(\Omega)$, which clearly satisfies (5.2.3). The point is to establish whether d is still of geodesic type. To this aim we use the implication (i) $\Rightarrow$ (iii), which ensures that $L_n \stackrel{\Gamma}{\to} L$. In particular, we are going to prove that we have also the Γ-convergence of the the modified functionals $\widetilde{L}_n$ to the modified functional $\widetilde{L}$, where $\widetilde{L}_n$ and $\widetilde{L}$ are obtained by adding boundary conditions respectively to L_n and L as follows:

$$\widetilde{L}_n(\gamma) := \begin{cases} L_n(\gamma) & \text{if } \gamma(0)=x \text{ and } \gamma(1)=y\\ +\infty & \text{otherwise}\end{cases}$$

$$\widetilde{L}(\gamma) := \begin{cases} L(\gamma) & \text{if } \gamma(0)=x \text{ and } \gamma(1)=y\\ +\infty & \text{otherwise .}\end{cases}$$

It is immediate to verify that the inequality $\liminf_{n\to\infty}\widetilde{L}_n(\gamma_n) \geq \widetilde{L}(\gamma)$ holds whenever $\gamma_n \to \gamma$ in $\mathcal{C}(\Omega)$: indeed, if $\widetilde{L}(\gamma)=+\infty$, it must be $\widetilde{L}_n(\gamma_n)=+\infty$ for n large enough; in the case $\widetilde{L}(\gamma)<+\infty$, since $L_n \stackrel{\Gamma}{\to} L$, one gets

$$\liminf_{n\to\infty}\widetilde{L}_n(\gamma_n) \geq \liminf_{n\to\infty} L_n(\gamma_n) \geq L(\gamma)=\widetilde{L}(\gamma)\ .$$

It remains to prove that, for every $\gamma \in \mathcal{C}(\Omega)$, there exists an approximating sequence $\{\tilde{\gamma}_n\}$ satisfying $\overline{\lim\limits_{n\to\infty}}\ \widetilde{L}_n(\tilde{\gamma}_n) \leq \widetilde{L}(\gamma)$. Assume without loss of generality that $\widetilde{L}(\gamma)=L(\gamma)$, and take $\{\gamma_n\}$ such that $\gamma_n \to \gamma$ in $\mathcal{C}(\Omega)$ and $\lim_{n\to\infty} L_n(\gamma_n)=L(\gamma)$ (such a sequence exists since $L_n \stackrel{\Gamma}{\to} L$). We construct the optimal sequence $\{\tilde{\gamma}_n\}$ modifying $\{\gamma_n\}$ as follows:

$$\tilde{\gamma}_n(t) := \begin{cases} \text{an almost } d_n\text{-geodesic connecting } x \text{ and } \gamma_n(1/n) & \text{if } t \in [0, 1/n], \\ \gamma_n(t) & \text{if } \ t \in [1/n, 1-1/n], \\ \text{an almost } d_n\text{-geodesic connecting } \gamma_n(1-1/n) \text{ and } y & \text{if } t \in [1-1/n, 1]. \end{cases}$$

Let us now prove that $\tilde{\gamma}_n$ still converges to γ in $\mathcal{C}(\Omega)$. Fix ε arbitrarily small. If $t \in [1/n, 1-1/n]$ then the convergence of γ_n to γ in $\mathcal{C}(\Omega)$ yields, for n large enough

$$|\tilde{\gamma}_n(t) - \gamma(t)| = |\gamma_n(t) - \gamma(t)| \le \varepsilon.$$

If $t \in [0, 1/n]$ (the case $t \in [1-1/n, 1]$ is similar) then

$$|\tilde{\gamma}_n(t) - \gamma(t)| \le |\tilde{\gamma}_n(t) - x| + |x - \gamma(t)|. \tag{5.3.14}$$

For n large enough, the second term at the right side of (5.3.14) is less than ε because γ is a continuous curve; for the first term we have:

$$\begin{aligned} c|\tilde{\gamma}_n(t) - x| &\le d_n(\tilde{\gamma}_n(t), x) \le d_n\big(\tilde{\gamma}_n\big(\tfrac{1}{n}\big), x\big) + \varepsilon_n \le C|\tilde{\gamma}_n\big(\tfrac{1}{n}\big) - x| + \varepsilon_n \\ &= C|\gamma_n\big(\tfrac{1}{n}\big) - x| + \varepsilon_n \le C\big[|\gamma_n\big(\tfrac{1}{n}\big) - \gamma\big(\tfrac{1}{n}\big)| + |\gamma\big(\tfrac{1}{n}\big) - x|\big] + \varepsilon_n, \end{aligned} \tag{5.3.15}$$

and the last term tends to zero as $n \to \infty$, since $\gamma_n \to \gamma$ in $\mathcal{C}(\Omega)$, and γ is a continuous curve. Thus we have obtained that $\tilde{\gamma}_n \to \gamma$ in $\mathcal{C}(\Omega)$. It remains to show the inequality $\overline{\lim\limits_{n\to\infty}}\, \widetilde{L}_n(\tilde{\gamma}_n) \le \widetilde{L}(\gamma)$. We have:

$$\widetilde{L}_n(\tilde{\gamma}_n) \le d_n\big(x, \gamma_n(1/n)\big) + L_n(\gamma_n) + d_n\big(\gamma_n(1-1/n), y\big) + 2\varepsilon_n \ . \tag{5.3.16}$$

Notice now that, from (5.3.15), it follows in particular that :

$$\lim_{n\to\infty} d_n\big(x, \gamma_n(1/n)\big) = 0.$$

Similarly, one obtains that $\lim_{n\to\infty} d_n\big(\gamma_n(1-1/n), y\big) = 0$. Hence, passing to the $\overline{\lim}$ as $n \to +\infty$ in (5.3.16) gives

$$\overline{\lim_{n\to\infty}}\, \widetilde{L}_n(\tilde{\gamma}_n) \le \overline{\lim_{n\to\infty}}\, L_n(\gamma_n) = L(\gamma) = \widetilde{L}(\gamma) \ .$$

Thus, by the Γ-convergence of L_n to L, we deduce that

$$\inf_{\gamma}\{\widetilde{L}(\gamma)\} = \lim_{n\to+\infty} \inf_{\gamma}\{\widetilde{L}_n(\gamma)\} \tag{5.3.17}$$

Since d_n are geodesic distances, the right hand side is equal to the limit of $d_n(x, y)$, that is $d(x, y)$. Thus equation (5.3.17) means exactly that d is a geodesic distance, and our claim is proved.

If now we want to prove any of the implications (ii), or (iii), or (iv) $\Rightarrow$ (i), possibly passing to a subsequence we may assume that d_n converge to some element d' in $\mathcal{D}(\Omega)$. Since we already know that the converse implications hold, we have respectively $J_d = J_{d'}$, $L_d = L_{d'}$, and $M_d = M_{d'}$. If $J_d = J_{d'}$, then we obtain $d = d'$ taking $\mu(x, y) = \delta_x \otimes \delta_y$. If

$L_d = L_{d'}$, then we obtain $d = d'$ by (5.2.7). If $F_d = F_{d'}$, then we obtain $d = d'$ because the unit balls of φ_d^0 and $\varphi_{d'}^0$ coincide, so $\varphi_d = \varphi_{d'}$, and we are reduced to the previous case $L_d = L_{d'}$. □

REMARK 5.3.2. The equivalence between (ii) and (iv) connects the results of [**1**] with the results of [**19**], giving the equivalence between them for some particular integrands.

5.4. Examples and Further Remarks

We now discuss the validity of Theorem 5.3.1 for two examples of sequences which do not lie in $\mathcal{D}(\Omega)$. In both examples we assign the metric φ_n, the associated distances d_n being given by (5.2.7).

Example 1. Let us consider the metrics in $\mathbb{R}^2$ defined by

$$\varphi_n(x,z) := \begin{cases} |z| & \text{if } |x^2| \le \frac{1}{n}, \\ n|z| & \text{otherwise,} \end{cases}$$

where x^2 denotes the second component of the vector x. The corresponding sequence $\{d_n\}$ converges pointwise, but not uniformly, to the distance d associated to the metric

$$\varphi(x,z) := \begin{cases} |z| & \text{if } x^2 = 0 \text{ or if } z = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Nevertheless, for the limit functionals J, L, F associated to such d, the remaining three convergences of Theorem 5.3.1 are satisfied as shown below.

PROPOSITION 5.4.1. *For d_n as above, it holds:*

(i) $J_n \xrightarrow{\Gamma} J$;
(ii) $L_n \xrightarrow{\Gamma} L$;
(iii) $F_n \xrightarrow{\Gamma} F$.

PROOF. – (i) To prove the $\Gamma - \liminf$ inequality, let $\mu_n \to \mu$ in $\mathcal{M}(\Omega)$ and let us consider separately the cases $J(\mu) < +\infty$ and $J(\mu) = +\infty$. If $J(\mu) < +\infty$, we have $\operatorname{spt}\mu \subset P := \{(t,0,s,0),\ t,s \in \mathbb{R}\}$ and we obtain

$$\liminf_{n\to\infty} \int d_n(x,y)\,d\mu_n(x,y) \ge \liminf_{n\to\infty} \int |x-y|\,d\mu_n(x,y) \ge \int d(x,y)\,d\mu(x,y).$$

If $J(\mu) = +\infty$, there exist an open ball $B \subset \mathbb{R}^4 \setminus P$ and a positive number α such that $\mu(B) > \alpha$. We have

$$\liminf_{n\to\infty} \int d_n(x,y)d\mu_n(x,y) \geq \liminf_{n\to\infty} \int_B d_n(x,y)d\mu_n(x,y) =$$
$$= \lim_{n\to\infty} \int_B n|x-y|d\mu_n(x,y) = +\infty \ ;$$

where the last equality holds since the assumption $J(\mu) = +\infty$ implies that the measure μ cannot be concentrated on the diagonal of $\mathbb{R}^2 \times \mathbb{R}^2$. In order to prove the $\Gamma - \overline{\lim}$ inequality, it is enough to choose $\mu_n \equiv \mu$.

– (ii) To prove the $\Gamma - \liminf$ inequality, let $\gamma_n \to \gamma$ in $\mathcal{C}(\Omega)$ and let us consider separately the cases $L(\gamma) < +\infty$ and $L(\gamma) = +\infty$. If $L(\gamma) < +\infty$, by the lower semicontinuity of the length (see [**30**], Th. 3.18), we obtain

$$\liminf_{n\to\infty} L_n(\gamma_n) \geq \liminf_{n\to\infty} \int_0^1 |\gamma_n'|\, dt \geq \int_0^1 |\gamma'|\, dt = L(\gamma).$$

If $L(\gamma) = +\infty$, then we may assume that there exist an interval $I_\varepsilon = [t_0 - \varepsilon, t_0 + \varepsilon]$ and a positive number α such that $\gamma(t_0 - \varepsilon) \neq \gamma(t_0 + \varepsilon)$ and the second component γ^2 of γ satisfies $|\gamma^2(t)| > \alpha$ for each $t \in I_\varepsilon$. Then $|\gamma_n^2(t)| > \alpha$ for n large enough, and

$$\begin{aligned}
&\liminf_{n\to+\infty} L_n(\gamma_n) \geq \liminf_{n\to+\infty} \int_{I_\varepsilon} n|\gamma_n'(t)|dt \geq \liminf_{n\to+\infty} d_n(\gamma_n(t_0-\varepsilon), \gamma_n(t_0+\varepsilon)) \\
&= \liminf_{n\to+\infty} n|\gamma_n(t_0-\varepsilon) - \gamma_n(t_0+\varepsilon)| = \lim_{n\to+\infty} n|\gamma(t_0-\varepsilon) - \gamma(t_0+\varepsilon)| = +\infty \ . \quad (5.4.1)
\end{aligned}$$

To prove the $\Gamma - \overline{\lim}$ inequality, it is enough to choose $\gamma_n \equiv \gamma$.

– (iii) To prove the $\Gamma-$liminf inequality, let us consider a sequence $\{u_n\}$, uniformly converging to u, such that $F_n(u_n) = 0$. This implies in particular that the restriction of u_n to the x_1-axis is $1-$Lipschitz, so that u restricted to the same axis is $1-$Lipschitz too, which entails $F(u) = 0$. To prove the $\Gamma-$limsup inequality, assume that $F(u) = 0$ and let k be the Lipschitz constant of u. We construct the approximating sequence $\{u_n\}$ as follows: for $n > k$, we define

$$\tilde{u}_n(x) = \begin{cases} u(x^1, 0) & \text{if } |x^2| \leq \frac{1}{2n}, \\ u(x) & \text{if } |x^2| \geq \frac{1}{n}. \end{cases} \quad (5.4.2)$$

It is easy to check that $\tilde{u}_n$ is $k-$Lipschitz on its domain of definition. Now, if u_n is a $k-$Lipschitz extension of $\tilde{u}_n$ to $\mathbb{R}^2$, we have $u_n \to u$ in $\text{Lip}(\mathbb{R}^N)$, and $F_n(u_n) = 0$. $\square$

Example 2. For every $n \in \mathbb{N}$, let G_n be the grid in $\mathbb{R}^2$ given by $G_n := \Big(\mathbb{Z}/n \times \mathbb{R}\Big) \cup \Big(\mathbb{R} \times \mathbb{Z}/n\Big)$, and let us consider the metrics in $\mathbb{R}^2$ defined by $\varphi_n(x,z) = a_n(x)|z|$ where

$$a_n(x) := \begin{cases} 1 & \text{if } x \in G_n, \\ n & \text{otherwise.} \end{cases}$$

As in the previous example, the control from above on φ_n is missing, then we are out of the class $\mathcal{D}(\Omega)$. Moreover, again d_n do not converge uniformly, though their pointwise limit is the distance $d(x,y) = |x^1 - y^1| + |x^2 - y^2|$. Nevertheless, also in this case we can prove that the remaining three convergences of Theorem 5.3.1 are satisfied.

Proposition 5.4.2. *For d_n as above, it holds:*

(i) $J_n \xrightarrow{\Gamma} J$;
(ii) $L_n \xrightarrow{\Gamma} L$;
(iii) $F_n \xrightarrow{\Gamma} F$.

Proof. – (i) If $\mu_n \to \mu$ in $\mathcal{M}(\Omega)$, we have

$$\liminf_{n\to\infty} J_n(\mu_n) \geq \liminf_{n\to\infty} \int d(x,y)d\mu_n(x,y) \geq \int d(x,y)d\mu(x,y) = J(\mu).$$

To obtain the $\Gamma - \overline{\lim}$ inequality, since d_n agree with d on G_n, it is enough to take a sequence of measures μ_n converging to μ in $\mathcal{M}(\Omega)$, such that $\operatorname{spt}\mu_n \subseteq G_n$.

– (ii) The $\Gamma - \liminf$ inequality can be obtained by the same argument already used to prove Proposition 5.4.1 (ii). In order to prove the $\Gamma - \overline{\lim}$ inequality, let $\gamma \in \mathcal{C}(\Omega)$, and let $\{t_i^n\}$ be a partition of the interval $[0,1]$ such that every $\gamma([t_i^n, t_{i+1}^n])$ is contained into a square Q_i^n of side $1/n$ delimited by the grid G_n. We set $x_i^n := \gamma(t_i^n)$, and we define γ_n as follows: γ_n restricted to the interval $[t_i^n, t_{i+1}^n]$ is the parametrization with constant speed of the shortest part of boundary of Q_n^i joining x_i^n and x_{i+1}^n. We then have:

$$L_n(\gamma_n) \leq \sum_i d_n(x_i^n, x_{i+1}^n) = \sum_i d(x_i^n, x_{i+1}^n) \leq L(\gamma) \ . \tag{5.4.3}$$

– (iii) If u_n satisfy $F_n(u_n) = 0$ and converge to u in $\mathrm{Lip}(\mathbb{R}^n)$, then u turns out to be 1-Lipschitz with respect to d on the dense set $(\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Q})$. It follows that $F(u) = 0$. On the other hand, to prove the $\Gamma - \overline{\lim}$ inequality, it is enough to take $F(u) = 0$ and choose $u_n \equiv u$. □

Remark 5.4.3. We notice that the distances d_n of Example 2 are Riemannian distances, whereas their limit d is not (see also [**1**], [**4**]). It is natural now to ask whether the class of Riemannian distances is dense in $\mathcal{D}(\Omega)$. The equivalences of Theorem 5.3.1 togheter with the examples which appear in [**1**], [**4**] seems to push for a positive answer.

CHAPTER 6

A p-Laplacian Approximation for some Shape Optimization Problems

6.1. Introduction

The object of this chapter is the study of the asymptotic behaviour of the solutions of the p-Laplacian equation with measure right term when p go to ∞. Let Ω be an open, regular subset of $\mathbb{R}^N$ and let $\Sigma \subset \overline{\Omega}$ be a regular closed subset. The p-Laplacian equation with Dirichlet condition on Σ can be written as follows:

$$\begin{cases} -\text{div } (|Du_p|^{p-2}Du_p) = f & in\ \Omega \\ u_p = 0 & on\ \Sigma, \\ \frac{\partial u}{\partial n} = 0 & on\ \partial\Omega \setminus \Sigma. \end{cases} \tag{6.1.1}$$

Making use of the Γ-convergence and of the tangential calculus with respect to a measure we will show that the solutions of (6.1.1) converges to a minimizer of (4.1.3), in the case $\int df = 0$, $|Du_p|^{p-2}$ weakly* converges in the sense of measures to an optimal shape (or transport density) introduced in chapter 3, in the case $\int df \neq 0$ it converges to the analogous (see [**12**]). The same problem (but with regular right term) as been already studied in [**28**], [**41**], [**11**], and other papers. The main original contributions we give here is that in spite of the generality of the right term we obtain a quite strong convergence of the solutions and a "div-curl" type lemma for the measure $|Du_p|^{p-2}dx$ and the functions Du_p. Moreover we address to the question of the uniqueness of the limit which up to now is clear only in the case of f positive Lipschitz function. We also consider the vectorial analogous of this problem and we study the asymptotic behaviour of the energies

$$G_p(u) = \frac{1}{p}\int_\Omega \rho(e(u))^p dx - < f, u >, \tag{6.1.2}$$

where ρ is the convex positive one-homogeneous function we associate to the stored energy density (see the last section of chapter 3). The vectorial case is much more difficult of the scalar case, indeed the energies (6.1.2) permit to control only the symmetric part of the gradients.

Let us briefly summarize what has been proved in [**12**] and in chapter 3. A pair (u, μ) solution of (4.1.3) and (4.1.2) with $\mu \in \mathcal{M}(\overline{\Omega})$ and $u \in \text{Lip}_1(\Sigma)$ (we recall that

$\mathrm{Lip}_1(\Sigma)$ is the closure with respect to the uniform convergence of the set of the functions $u \in \mathcal{D}(\mathbb{R}^N, \mathbb{R})$ such that $|Du| \le 1$ and $u = 0$ on Σ) has been characterized as a solution of:

$$\begin{cases} 1) & |Du| \le 1 \;\; a.e., \\ 2) & u \text{ is a minimizer for } (6.3.2), \\ 3) & \mu(\Sigma) = 0, \;\; u = 0 \text{ on } \Sigma, \\ 4) & |D_\mu u| \le 1 \;\; \mu - a.e., \\ 5) & -\mathrm{div}\, (\mu D_\mu u) = f, \end{cases} \tag{6.1.3}$$

while in the vectorial case a pair (u, μ) is solution of (3.5.4, 3.5.3) if there exist a matrix ξ such that

$$\begin{cases} 1) & u \in Lip_{1,\rho}, \\ 2) & u \text{ is a minimizer for } (3.5.4), \\ 3) & \mu(\Sigma) = 0 \\ 4) & j_\mu(x, e_\mu(u)) = \frac{1}{p} \;\; \mu - a.e. \\ 5) & -div(\xi\mu) = f \; in \; \Omega \setminus \Sigma. \\ 6) & \xi \in \partial j_\mu(e_\mu(u)), \end{cases} \tag{3.5.7}$$

where $\mathrm{Lip}_{1,\rho}(\Sigma)$ has been defined at the end of chapter 3.

6.2. Preliminary Results

6.2.1. A characterization of $\mathrm{Lip}_{1,\rho}$ and $\mathrm{Lip}_1(\Sigma)$. In this subsection we give a characterization of $\mathrm{Lip}_{1,\rho}(\Sigma)$ and $\mathrm{Lip}_1(\Sigma)$ simple to handle. It will be useful the following space introduced in [**23**]:

$$U^\infty := \{u \in L^\infty(\Omega, \mathbb{R}^N) \mid e(u) \in L^\infty(\Omega, \mathbb{R}^{N\times N})\}. \tag{6.2.1}$$

Thanks to the Korn inequalities it turns out that U^∞ is the intersection of the spaces $\{u \in W^{1,q} \mid e(u) \in L^\infty(\Omega, \mathbb{R}^N)\}$ for $1 \le q \le \infty$. The main theorem of this section is the following

THEOREM 6.2.1. $\mathrm{Lip}_{1,\rho}(\Sigma) = \{u \in U^\infty(\Omega, \mathbb{R}^N) | \; u = 0 \; on \; \Sigma, \;\; \rho(e(u)) \le 1 \; a.e.\}$.

PROOF. **Step 1** Let $u \in U^\infty$ be such that $u = 0$ on Σ and $\rho(e(u)) \le 1$ a.e., then $u_n := (1 - \frac{1}{n})u$ is still in U^∞, $u_n = 0$ on Σ and $\rho(e(u_n)) \le 1 - \frac{1}{n}$. Moreover $u_n \to u$ uniformly. Then to prove ($\supseteq$) it is enough to prove that each function v in U^∞, $v = 0$ on Σ and $\rho(e(v)) \le c < 1$ can be approximate uniformly by a sequence v_n in $\mathcal{D}(\mathbb{R}^N, \mathbb{R}^N)$ such that $v_n = 0$ on Σ and $\rho(e(v_n)) \le 1$ in Ω.

Step 2 Let v in U^∞, $v = 0$ on Σ. Fix $\delta > 0$, we now show that there exists $v_\delta \in \mathcal{D}(\mathbb{R}^N, \mathbb{R}^N)$ such that:

$$|v - v_\delta| \leq \delta,$$

$$|\|\rho(e(v))\|_\infty - \|\rho(e(v_\delta))\|_\infty| \leq \delta,$$

$$v_\delta = 0 \quad \text{on } \Sigma,$$

it will follows, using step 1, that $\{u \in U^\infty(\Omega, \mathbb{R}^N)|\ u = 0\ on\ \Sigma,\ \rho(e(u)) \leq 1\ a.e.\} \subset \mathrm{Lip}_{1,\rho}(\Sigma)$.

Let $r > 0$ such that the open set

$$\Omega_0 = \{x \in \Omega|\ d(x, \Sigma) > \frac{1}{r}\}$$

verifies

$$\|\rho(e(v))\|_{L^\infty(\Omega_0)} \geq \|\rho(e(v))\|_{L^\infty(\Omega)} - \delta.$$

Moreover define the open sets Ω_j as follows:

$$\Omega_j = \{x \in \Omega|\ d(x, \Sigma) > \frac{1}{j + r}\}.$$

We have: $\Omega \setminus \Sigma = \bigcup_j \Omega_j$ and $\Omega_j \subset \overline{\Omega_j} \subset \Omega_{j+1}$. Then we can define $A_1 = \Omega_2$ and $A_j = \Omega_{j+1} \setminus \overline{\Omega}_{j-1}$. It is clear that the sets A_j constitute a covering of $\Omega \setminus \Sigma$, then we can consider a partition of unity ϕ_j related to this covering. Select now a sequence of mollifier g_{ε_j} with ε_j decreasing and such that:

$$\Omega_0 + B(0, \varepsilon_1) \subset A_1,$$

$$A_2 + B(0, \varepsilon_2) \subset \Omega_0^c \cap A_3,$$

$$A_j + B(0, \varepsilon_j) \subset A_{j-1} \cup A_j \cup A_{j+1}, \quad j \geq 3.$$

Since $v\phi_j$ is a continuous function $g_\varepsilon * \phi_j v \to \phi_j v$ uniformly on $\overline{\Omega}$. Then we can chose ε_j such that:

$$\|g_{\varepsilon_j} * (\phi_j v) - (\phi_j v)\|_\infty \leq \delta 2^{-j},$$

$$\|g_{\varepsilon_j} * \phi_j - \phi_j\|_\infty \leq \delta 2^{-j},$$

$$\rho(g_{\varepsilon_j} * (\nabla \phi_j \otimes v) - (\nabla \phi_j \otimes v)) \leq \delta 2^{-j},$$

where the last choice is allowed by the fact that $\rho(z) \leq k|z|$ for some constant k which depends on the stored energy density (see chapter 3). Moreover we can choose ε_1 so small that

$$|\|\rho(g_{\varepsilon_1} * \phi_1 e(v)\|_{L^\infty(\Omega_0)} - \|(\phi_1 e(v)))\|_{L^\infty(\Omega_0)}| \leq \delta.$$

Now we consider

$$v_\delta = \sum_1^\infty g_{\varepsilon_j} * (\phi_j v),$$

v_δ is in $\mathcal{D}(\mathbb{R}^N, \mathbb{R}^N)$ and it is equal to 0 on Σ (because each term is zero on Σ). Let us check that the suitable estimates hold for v_δ

$$|v - v_\delta| \leq \sum_1^\infty |g_{\varepsilon_j} * \phi_j v - \phi_j v| \leq \delta$$

moreover $e(v_\delta) = \sum_1^\infty g_{\varepsilon_j} * (\nabla\phi_j \otimes v + e(v)\phi_j)$ and since $\sum_1^\infty \nabla\phi_j = 0$ we obtain

$$e(v_\delta) = (\sum_1^\infty g_{\varepsilon_j} * (\nabla\phi_j \otimes v) - \nabla\phi_j \otimes v) + \sum_1^\infty g_{\varepsilon_j} * e(v)\phi_j).$$

Then since ρ is convex, sub-additive and 1-homogeneous we obtain

$$\begin{aligned} \rho(e(v_\delta)) &\leq \textstyle\sum_1^\infty \rho(g_{\varepsilon_j} * (\nabla\phi_j \otimes v) - \nabla\phi_j \otimes v) + \sum_1^\infty \rho(g_{\varepsilon_j} * e(v)\phi_j) \leq \\ &\leq \textstyle\sum \delta 2^{-j} + \delta + \rho(e(v)) \leq \\ &\leq 2\delta + \rho(e(v)). \end{aligned} \tag{6.2.2}$$

On the other hand if we consider a point $x \in \Omega_0$ then $v_\delta(x) = g_{\varepsilon_1} * \phi_1 v(x)$ and $e(u_\delta)(x) = (g_{\varepsilon_1} * \phi_1 e(u))(x)$. And then

$$\|\rho(e(u_\delta))\|_{L^\infty(\Omega_0)} \geq \|\rho(e(u))\|_{L^\infty(\Omega_0)} - 2\delta,$$

which conclude the second step.

Step 3 It remains to show that $\mathrm{Lip}_{1,\rho}(\Sigma) \subset \{u \in U^\infty(\Omega, \mathbb{R}^N) |\ u = 0\ on\ \Sigma,\ \rho(e(u)) \leq 1\ a.e.\}$. This is a consequence of the Korn inequalities in $W^{1,q}$ for $q < +\infty$. Indeed, let $u \in \mathrm{Lip}_{1,\rho}(\Sigma)$ and let u_n a sequence in $\mathcal{D}(\mathbb{R}^N, \mathbb{R}^N)$ converging to u uniformly, and such that $\rho(e(u_n)) \leq 1$ and $u_n = 0$ on Σ. Let $q < \infty$, using the Korn inequality and the growth conditions on ρ we have:

$$\int_\Omega |Du_n|^q dx \leq C_q \int_\Omega (\rho(e(u_n))^q dx \leq C_q|\Omega|,$$

then u_n is bounded in $W^{1,q}$, which imply $u \in W^{1,q}$ and $u_n \to u$ weakly in $W^{1,q}$. Moreover since ρ is convex and has linear growth

$$(\int_\Omega (\rho(e(u)))^q)^{1/q} \leq \liminf_{n\to\infty}(\int_\Omega (\rho(e(u_n)))^q)^{1/q} \leq |\Omega|^{1/q},$$

which imply, taking the limits for $q \to \infty$, that $\rho(e(u)) \leq 1$ a.e.. Then the proof is complete. $\square$

In the scalar case, the characterization of $\mathrm{Lip}_1(\Sigma)$ is simpler and it is the content of the next theorem.

THEOREM 6.2.2.

$$\mathrm{Lip}_1(\Sigma) = \{u \in W^{1,\infty}(\Omega) :\ u = 0\ on\ \Sigma,\ |Du| \leq 1\ a.e.\}$$

PROOF. The proof of $(\subseteq)$ is similar to the tirth step of the previous theorem. Let us then concentrate our attention on the proof of $(\supseteq)$. Let $u \in W^{1,\infty}(\Omega)$ be such that $u = 0$ *on* Σ and $|Du| \leq 1$ a.e., and let d_Σ denote the distance function from Σ. Define the following cut-off functions:

$$\alpha_{\varepsilon,t} := \begin{cases} 0 & if\ d_\Sigma(x) \leq \varepsilon, \\ \frac{(1-t)d_\varepsilon}{d_\varepsilon - \varepsilon} \frac{d_\Sigma(x)-\varepsilon}{d_\Sigma(x)} & if\ d_\Sigma(x) \in [\varepsilon, d_\varepsilon], \\ (1-t) & if\ d_\Sigma(x) \geq d_\varepsilon, \end{cases} \tag{6.2.3}$$

and consider $v_{\varepsilon,t} := \alpha_{\varepsilon,t} u$ it is just matter of calculations to verify that $v_{\varepsilon,t}$ is equal to zero in an ε-tubular neighborhood of Σ and that the norm of the gradient of $v_{\varepsilon,t}$ is less than 1 when the parameters $\varepsilon, t, d_\varepsilon$ are taken such that $\varepsilon \leq t d_\varepsilon$. Moreover $v_{\varepsilon,t}$ converges to u uniformly . Mollifying $v_{\varepsilon,t}$ one obtains the requested approximating sequence for u. □

6.3. Scalar case

For $p > N$ we define :

$$G_p(u) = \frac{1}{p} \int_\Omega |Du|^p dx - < f, u >, \quad u \in W^{1,p}(\Omega) \cap \{u = 0\ on \Sigma\}, \tag{6.3.1}$$

$$G(u) = \begin{cases} < -f, u > & if\ u \in Lip_1(\Sigma) \\ +\infty & otherwise. \end{cases} \tag{6.3.2}$$

In the sequel α_p, α will denote respectively the minimum value of G_p and G. It is a standard fact that the minimizers of G_p are solutions of

$$-\text{div}\ (|Du|^{p-2} Du) = f, \tag{6.3.3}$$

with zero Neumann boundary conditions on $\partial\Omega \setminus \Sigma$, zero Dirichlet condition on Σ and that $\alpha_p \leq 0$, $\alpha \leq 0$. The scope of this section is to prove the following result

THEOREM 6.3.1. *Let u_p be the minimum point of G_p. Then there exists $u \in Lip_1(\Sigma)$ and $\mu \in \mathcal{M}(\overline{\Omega})$ such that: as $p \to \infty$ (up to subsequences):*

(1) *$u_p \rightharpoonup u$ in $W^{1,q}(\Omega)$ for all $q \in [1, +\infty)$.*
(2) *$|Du_p|^{p-2} \rightharpoonup \mu$ in $\mathcal{M}(\overline{\Omega})$.*
(3) *$|Du_p|^{p-2} Du_p \rightharpoonup \mu D_\mu u$ in $\mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$.*
(4) *u is a minimizer for G and (u, μ) is a solution of equation (6.1.3).*

Moreover the strong convergence $Du_p \overset{s(\mu_p,\mu)}{\to} D_\mu u$ holds in the sense of definition (2.4.3).

Remark: As consequence of the previous theorem we have that μ suitably rescaled is solution of problem (4.1.2).

As the linear part of G_p does not influence the $\Gamma-$ convergence of the functionals we can restrict our attention to the nonlinear part. We define:

$$F_p(v) =: \frac{1}{p}\int_\Omega |Dv|^p dx, \quad v \in W^{1,p}(\Omega) \cap \{u = 0 \ on \ \Sigma\}. \tag{6.3.4}$$

THEOREM 6.3.2 (Compactness). *Assume that* $\{F_p(v_p)\} \le C$ *then for all* $1 \le q < \infty$ *the family* $\{Dv_p\}$ *is bounded in* $L^q(\Omega, \mathbb{R}^n)$. *In particular since* $\{v_p\}$ *is bounded in* $L^q(\Omega)$ *for some* q, *then it is precompact in* $C^0(\overline{\Omega})$.

PROOF. Let $p > q$ (which is eventually true for each fixed q); by Hölder inequality we obtain

$$\int_\Omega |Dv_p|^q dx \le |\Omega|^{1-\frac{q}{p}}(\int_\Omega |Dv_p|^p)^{\frac{q}{p}} \le |\Omega|^{1-\frac{q}{p}} p^{\frac{q}{p}} C^{\frac{q}{p}}, \tag{6.3.5}$$

which proves the boundedness of $\{Dv_p\}$ in $L^q(\Omega;\mathbb{R}^N)$. The last assertion is now a consequence of Rellich embedding theorem by taking $p > N$. □

THEOREM 6.3.3 (Γ-convergence). *The functionals* F_p $\Gamma-$*converges to*

$$F_\infty(v) = \begin{cases} 0 & if \ v \in Lip_1(\Sigma) \\ +\infty & otherwise \end{cases} \tag{6.3.6}$$

with respect to the uniform convergence of continuous functions.

PROOF. As the $\Gamma-$limsup inequality is evident we have to prove only the $\Gamma-$liminf inequality. Let $v_p \to v$ uniformly in Ω and assume without loss of generality that $\{F_p(v_p)\} \le C$. It is then clear that eventually $v \in W^{1,q}$ and that $v_p \rightharpoonup v$ in $W^{1,q}$, thanks to the semicontinuity of the norm and inequality (6.3.5):

$$(\int_\Omega |Dv|^q dx)^{\frac{1}{q}} \le |\Omega|^{\frac{1}{q}}. \tag{6.3.7}$$

Since q in the last inequality was arbitrary, we obtain that $Dv \in L^\infty$ and $\|Dv\|_\infty \le 1$. Moreover by definition $\{F_p(v_p)\} \le C$ imply $v_p = 0$ on Σ which in turns imply $v = 0$ on Σ. Thanks to theorem 6.2.2 $v \in \mathrm{Lip}_1(\Sigma)$ and then the proof is complete. □

PROPOSITION 6.3.4 (Boundedness). *Let* $\{u_p\}$ *the family of the minimum points of* G_p *then there exists a constant* C *such that :*

$$\int_\Omega |\nabla u_p|^p dx \le C. \tag{6.3.8}$$

PROOF. Using equation (6.3.3) we get the identity

$$\int_\Omega |\nabla u_p|^p dx =< f, u_p >, \tag{6.3.9}$$

and using the Poincaré inequality we get:

$$< f, u_p > \le \|f\|_{\mathcal{M}} \|u_p\|_\infty \le C\left(\int_\Omega |\nabla u_p|^p dx\right)^{\frac{1}{p}}. \tag{6.3.10}$$

It follows that

$$\left(\int_\Omega |\nabla u_p|^p dx\right)^{1-\frac{1}{p}} \le C. \tag{6.3.11}$$

□

COROLLARY 6.3.5 (Weak convergences). *The sequences* $\{|\nabla u_p|^{p-2}\}$ *and* $\{|\nabla u_p|^{p-2}\nabla u_p\}$ *are bounded respectively in* $L^1(\Omega)$ *and* $L^1(\Omega, \mathbb{R}^N)$ *and then weakly* compact in* $\mathcal{M}(\Omega)$ *and* $\mathcal{M}(\Omega, \mathbb{R}^N)$.

We will denote $|\nabla u_p|^{p-2}dx$ by μ_p and up to subsequences we can assume that

$$\mu_p \rightharpoonup \mu \tag{6.3.12}$$

weakly* in $\mathcal{M}$ and

$$\nabla u_p \mu_p \rightharpoonup \lambda \tag{6.3.13}$$

weakly* in $\mathcal{M}(\Omega, \mathbb{R}^N)$.

COROLLARY 6.3.6. *For each fixed* $q \in [1, \infty)$ *the family* $\{u_p\}$ *is bounded in* $W^{1,q}(\Omega)$.

As a consequence of this last corollary we can assume that up to subsequences:

$$u_p \to u \ \text{ uniformly on } \Omega, \tag{6.3.14}$$

and

$$\nabla u_p \rightharpoonup \nabla u \quad w - L^q, \ \forall q \in (1, +\infty). \tag{6.3.15}$$

It is simple to see that λ is solution of the equation:

$$-div\lambda = f \quad in\ \Omega \setminus \Sigma. \tag{6.3.16}$$

Moreover thanks to lemma 2.4.1, we can write

$$\lambda = \xi\mu \tag{6.3.17}$$

for a suitable $\xi \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$. We will now prove that $\xi = D_\mu u$. We start by proving that $|\xi| = 1$ μ-a.e. then $\xi = D_\mu u$ will follows by a general argument. Using theorems 6.3.2 and 6.3.3 we obtain:

$$\alpha_p \to \alpha := \max\{< f, u >: \quad u \in Lip_1(\Sigma)\}. \tag{6.3.18}$$

By a standard convex analysis arguments we may express α_p and α through dual problems:

$$\alpha_p = \max\{-\frac{1}{p'}\int_\Omega |\sigma|^{p'} dx : \quad \sigma \in L^{p'}(\Omega, \mathbb{R}^N), \ \ -div\sigma = f \ in\ \Omega \setminus \Sigma\}. \tag{6.3.19}$$

$$\alpha = -\min\{\int_\Omega d|\nu| : \quad -div\nu = f \ in \ \Omega \setminus \Sigma\}. \tag{6.3.20}$$

LEMMA 6.3.7 (Equipartition of Energy). *Let u_p be the minimum point of G_p, then the optimal σ for (6.3.19) is given by: $|\nabla u_p|^{p-2}\nabla u_p$. Moreover for u_p we have the following equipartition of the energy:*

$$\int_\Omega |\nabla u_p|^p dx =< f, u_p >= \frac{p}{1-p}\alpha_p.$$

PROOF. We have:

$$\alpha_p = \frac{1}{p}\int_\Omega |Du_p|^p dx - < f, u_p >, \tag{6.3.21}$$

then using equation (6.3.3),

$$\begin{aligned} \alpha_p &= \tfrac{1}{p}\textstyle\int_\Omega |Du_p|^p dx - \int_\Omega |Du_p|^p dx = \\ &= (\tfrac{1}{p} - 1)\textstyle\int_\Omega |Du_p|^p dx = \\ &= -\tfrac{1}{p'}\textstyle\int_\Omega ||Du_p|^{p-2}Du_p|^{p'} dx. \end{aligned} \tag{6.3.22}$$

□

Thanks to lemma 2.4.5, (still up to subsequences)

$$Du_p \overset{Y(\mu_p,\mu_\infty)}{\rightharpoonup} \nu_x, \tag{6.3.23}$$

and

$$\xi(x) = \int_{\mathbb{R}^N} z d\nu_x(z). \tag{6.3.24}$$

By proposition 6.3.4, the sequence $\{Du_p\}$ is uniformly bounded in $L^2_{\mu_p}(\Omega, \mathbb{R}^N)$ therefore by assertion (i) of proposition 2.4.4 the function

$$h(x,t) :=< |z|^t, \nu_x >, \tag{6.3.25}$$

is well defined on $[0,2]$ for $\mu-$a.e. $x \in \Omega$ and thanks to the inequality

$$\int_\Omega h(x,2)d\mu \le \liminf_{p\to\infty}\int_\Omega |Du_p|^2 d\mu_p = \liminf_{p\to\infty}\int_\Omega |Du_p|^p dx < +\infty. \tag{6.3.26}$$

$h(x,\cdot)$ is continuous on $[0,2]$ for μ-a.e. $x \in \Omega$. In addition by the assertion (ii) of proposition 2.4.4 for $s \in [0,2[$, the following equality holds

$$\liminf_{p\to\infty}\int_\Omega |Du_p|^{p-2+s} dx = \liminf_{p\to\infty}\int_\Omega |Du_p|^s d\mu_p = \int_\Omega h(x,s)d\mu. \tag{6.3.27}$$

We now deduce some property of the function h.

LEMMA 6.3.8 (Monotonicity). *Let $t_1 \le t_2 \in [0,2]$, then for $\mu-a.e.$ $x \in \Omega$ we have*

$$h(x,t_1) \le h(x,t_2). \tag{6.3.28}$$

PROOF. In order to obtain a pointwise inequality for h we need to localize some equality to open subsets ω of Ω. It follows from proposition 2.4.4 that for $t < 2$

$$\int_\omega h(x,t)d\mu = \lim_{p\to\infty} \int_\omega |Du_p|^{p-2+t}dx, \tag{6.3.29}$$

Using Hölder inequality and (6.3.27), we obtain that for every $0 \le t_1 < t_2 < 2$:

$$\begin{aligned}\int_\omega h(x,t_2)d\mu = \lim_{p\to\infty} \int_\omega |Du_p|^{p-2+t_2}dx \ge \\ \ge \lim_{p\to\infty} |\omega|^{1-\frac{p-2+t_2}{p-2+t_1}} (\int_\omega |Du_p|^{p-2+t_1}dx)^{\frac{p-2+t_2}{p-2+t_1}} \\ = \int_\omega h(x,t_1)d\mu.\end{aligned} \tag{6.3.30}$$

By localization we deduce (6.3.28) for $t_2 < 2$. We can extend it to $t_2 = 2$ using the continuity of $h(x,\cdot)$ on $[0,2]$. □

We will now show that h is constant with respect to the second variable in the interval $[1,2]$.

PROPOSITION 6.3.9 (Constancy). *For $\mu-$a.e. $x \in \Omega$ $h(x,\cdot)$ remains constantly equal to* 1 *on the interval* $[1,2]$.

PROOF. Passing to the limit in the equality of lemma 6.3.7 we get

$$\alpha = \lim_{p\to\infty} \int_\Omega |Du_p|^p dx.$$

By using (6.3.26), (6.3.27) and the monotonicity of h we get:

$$\begin{aligned}\alpha = \lim_{p\to\infty} \int_\Omega |Du_p|^p dx \ge \int_\Omega h(x,2)d\mu \ge \\ \ge \int_\Omega h(x,1)d\mu \ge \int_\Omega |\xi| d\mu \ge \alpha,\end{aligned} \tag{6.3.31}$$

where the last two inequalities follows from (6.3.24), (6.3.16) and (6.3.20). It follows that

$$\int_\Omega h(x,2)d\mu = \int_\Omega h(x,1)d\mu = \int_\Omega |\xi(x)| d\mu = \alpha.$$

In particular $h(x,s) = |\xi(x)|$ $\mu-$a.e. for any $s \in [1,2]$. It remains to prove that $|\xi| = 1$ μ-a.e.. Applying lemma (6.3.8) with $t_1 = 0$, we get $|\xi(x)| = h(x,2) \ge h(x,0) = 1$ μ-a.e., the converse inequality is deduced from Jensen inequality as follows:

$$|\xi(x)| = h(x,1) = h(x,2) =< |z|^2, \nu_x >\ge | < z, \nu_x > |^2 = |\xi(x)|^2. \tag{6.3.32}$$

□

COROLLARY 6.3.10 (Strong convergence of gradients). *For μ-a.e $x \in \Omega$ $\nu_x = \delta_{D_\mu u(x)}$*

Proof. Since $|\xi(x)| = 1$ μ-a.e., inequalities (6.3.32) become equalities then, using the strict convexity if $|z|^2$, we obtain that ν_x is the Dirac mass in $\xi(x)$ to complete the proof we have to show that $\xi(x) = D_\mu(x)$ μ-a.e. which can be done arguing as in [**14**] (or as in theorem 3.3.4) and exploiting the extremality relations between (4.1.3) and (4.1.5) $\square$

To complete the proof of theorem 6.3.1 it remains to show that $\mu(\Sigma) = 0$. With this aim, consider the measures $\tilde{\mu} := \mu \llcorner (\Omega \setminus \Sigma)$ and $\tilde{\lambda} := D_\mu u \tilde{\mu}$. It is simple to see that $-div\tilde{\lambda} = f$ in $\Omega \setminus \Sigma$, moreover $|\tilde{\mu}| \leq |\mu|$, the minimality of λ for (6.3.20), imply the equality between μ and $\tilde{\mu}$ then $\mu(\Sigma) = 0$.

6.4. Elasticity case

For $N < p$ we define

$$G_p(u) = \frac{1}{p}\int_\Omega \rho(e(u))^p dx - < f, u >, \quad u \in W^{1,p}(\Omega, \mathbb{R}^N) \cap \{u = 0 \ on \ \Sigma\}, \tag{6.4.1}$$

$$G(u) = \begin{cases} < -f, u > & if \ u \in Lip_{1,\rho}(\Sigma) \\ +\infty & otherwise. \end{cases} \tag{6.4.2}$$

By α_p and α we will denote respectively the minimum value of G_p and G. Through this section we will assume that α is finite, this assumption is needed to have existence of an optimal shape and then is not restrictive (see theorem 3.5.1). The dual functional of G_p is

$$D_p(\sigma) = \frac{1}{p'}\int_\Omega (\rho^0(\sigma))^{p'} dx.$$

Standard convex duality arguments yield

$$\alpha_p = -\min\{D_p(\sigma) \mid \sigma \in L^{p'}(\Omega, \mathbb{R}^{N\times N}), \ -div\sigma = f \ in \ \Omega \setminus \Sigma\}. \tag{6.4.3}$$

Let us study some relationship between the minimizers of G_p and the minimizers of D_p. We claim that the minimizer σ_p of D_p can be written as $\sigma_p = \xi_p(\rho(e(u)_p))^{p-2}$ where u_p is the minimizer of G_p and $\xi_p \in \partial(\rho(e(u_p)))$ indeed, as σ_p and u_p are minima we have:

$$\frac{1}{p'}\int_\Omega \rho^0(\sigma_p)^{p'} dx + \frac{1}{p}\int_\Omega \rho(e(u_p))^p dx =< f, u_p >= \int_\Omega \sigma_p : e(u_p) dx, \tag{6.4.4}$$

moreover from basic convex analysis we derive

$$\frac{1}{p'}\int_\Omega \rho^0(\sigma_p)^{p'} dx + \frac{1}{p}\int_\Omega \rho(e(u_p))^p dx \geq \int_\Omega \rho(e(u_p))\rho^0(\sigma_p) dx \geq \int_\Omega \sigma_p : e(u_p) dx. \tag{6.4.5}$$

It follows that

$$\frac{1}{p'}\rho^0(\sigma_p)^{p'} + \frac{1}{p}\rho(e(u_p))^p = \rho(e(u_p))\rho^0(\sigma_p) = \sigma_p : e(u_p), \tag{6.4.6}$$

then

$$\rho^0(\sigma_p) = \rho(e(u_p))^{p-1}. \tag{6.4.7}$$

Moreover if we write $\sigma_p = \xi_p \rho(e(u_p))^{p-2}$ we obtain $\rho^0(\xi_p) = \rho(e(u_p))$ and

$$\rho^0(\xi_p)\rho(e(u_p)) = \xi_p : e(u_p)$$

which means $\xi_p \in \partial(\rho(e(u_p)))$. Finally the minimality of σ_p imply that

$$-div(\xi_p \rho(e(u))^{p-2}) = f \quad in\ \Omega \setminus \Sigma. \tag{6.4.8}$$

The aim of this section is to prove the following:

THEOREM 6.4.1. *Let $\{u_p\}$ be the net of minimum points of G_p. Then up to subsequences there exists $u \in Lip_{1,\rho}(\Sigma)$ and $\mu \in \mathcal{M}(\overline{\Omega})$ such that:*

(1) *$u_p \to u$ uniformly.*
(2) *$\rho(e(u)_p)^{p-2} \rightharpoonup \mu$ in $\mathcal{M}(\overline{\Omega})$.*
(3) *$\xi_p \rho(e(u)_p)^{p-2} \rightharpoonup \xi\mu$ in $\mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$.*
(4) *u is a minimizer of (6.4.2) and (ξ, u, μ) is a solution of (3.5.7).*

Moreover the strong convergence $\xi_p \overset{s(\mu_p,\mu)}{\to} \xi$ holds in the sense of definition (2.4.3).

The proof will be divided in several lemmas and propositions.
As we did in the previous section we introduce the following auxiliary functional:

$$F_p(v) = \frac{1}{p}\int_\Omega (\rho(Dv))^p dx \quad v \in W^{1,p}(\Omega, \mathbb{R}^N) \cap \{v = 0 \ on\ \Sigma\},$$

THEOREM 6.4.2 (Compactness). *Assume that $\{F_p(v_p)\} \le C$ then for all $1 \le q < \infty$ the sequence $\{Dv_p\}$ is bounded in $L^q(\Omega, \mathbb{R}^n)$. In particular since $\{v_p\}$ is bounded in L^q for some q, then it is precompact in $C^0(\overline{\Omega}, \mathbb{R}^N)$*

PROOF. Let $p > q$ (which is eventually true for each fixed q); by Korn inequality, assumption d) on ρ and Hölder inequality we obtain

$$\int_\Omega |Dv_p|^q dx \le C_q \int_\Omega \rho(e(v_p))^q dx \le$$
$$\le C_q|\Omega|^{1-\frac{q}{p}}\Big(\int_\Omega \rho(e(v_p))^p\Big)^{\frac{q}{p}} \le |\Omega|^{1-\frac{q}{p}} p^{\frac{q}{p}} C^{\frac{q}{p}} C_q, \tag{6.4.9}$$

which proves the boundedness of $\{Dv_p\}$ in $L^q(\Omega; \mathbb{R}^N)$. The last part of the statement follows as in the scalar case by the Rellich embedding theorem. □

THEOREM 6.4.3 (Γ-covergence). *The functionals F_p $\Gamma-$converges to*

$$F_\infty(v) = \begin{cases} 0 & if\ v \in Lip_{1,\rho}(\Sigma) \\ +\infty & otherwise \end{cases} \tag{6.4.10}$$

with respect to the uniform convergence of continuous functions.

PROOF. As the $\Gamma-$limsup inequality is evident we have to prove only the $\Gamma-$liminf inequality. Let $v_p \to v$ uniformly in Ω and assume without loss of generality that $\{F_p(v_p)\} \leq C$. It is then clear that eventually $v_p \in W^{1,q}$ and that $v_p \rightharpoonup v$ in $W^{1,q}$, thanks to the semicontinuity of the functional

$$w \mapsto \int_\Omega (\rho(e(w)))^q dx$$

we obtain

$$(\int_\Omega (\rho(e(v)))^q dx)^{\frac{1}{q}} \leq |\Omega|^{\frac{1}{q}} C^{\frac{1}{q}}. \tag{6.4.11}$$

Since q in the last inequality was arbitrary we obtain that $e(v) \in L^\infty$ and $\rho(e(v)) \leq 1$. Moreover by definition $\{F_p(v_p)\} \leq C$ imply $v_p = 0$ on Σ which in turns imply $v = 0$ on Σ. Thanks to theorem 6.2.1 $v \in \mathrm{Lip}_{1,\rho}(\Sigma)$ and then the proof is complete. □

PROPOSITION 6.4.4 (Boundedness). *Let $\{u_p\}$ the net of the minimum points of G_p then there exists a constant C such that :*

$$\int_\Omega (\rho(e(u_p)))^p dx \leq C. \tag{6.4.12}$$

PROOF. Using equation (6.4.8) we get the identity

$$\int_\Omega (\rho(e(u_p)))^p dx =< f, u_p >, \tag{6.4.13}$$

and then, since we assumed that the minimum value α of G is finite that $F_p(u_p) \leq C$. Now using theorem 6.4.2:

$$\int_\Omega (\rho(e(u_p)))^p =< f, u_p >\leq \|f\|_{\mathcal{M}} \|u_p\|_\infty \leq C\|f\|_{\mathcal{M}}. \tag{6.4.14}$$

□

COROLLARY 6.4.5 (Weak convergences). *The sequences $\{\rho(e(u_p))^{p-2}\}$ and $\{\xi_p \rho(e(u_p))^{p-2}\}$ are bounded respectively in $L^1(\Omega)$ and $L^1(\Omega, \mathbb{R}^N)$ and then weakly* compact in $\mathcal{M}(\overline{\Omega})$ and $\mathcal{M}(\overline{\Omega}, \mathbb{R}^{N\times N})$.*

PROOF. This follows from the previous proposition and the growth condition on ρ. □

We will denote $\rho(e(u_p))^{p-2}dx$ by μ_p and up to subsequences we can assume that

$$\mu_p \rightharpoonup \mu \tag{6.4.15}$$

weakly* in $\mathcal{M}$ and

$$\xi_p \mu_p \rightharpoonup \lambda \tag{6.4.16}$$

weakly* in $\mathcal{M}(\Omega, \mathbb{R}^{N\times N})$.

COROLLARY 6.4.6. *For each fixed* $q \in [1, \infty)$ *the sequence* $\{u_p\}$ *of the minimum points of* G_p *is bounded in* $W^{1,q}(\Omega, \mathbb{R}^N)$ *and then is precompact in* $C(\Omega, \mathbb{R}^N)$.

As consequence of this last corollary we can assume that up to subsequences:

$$u_p \to u \quad uniformly\ on\ \Omega, \tag{6.4.17}$$

and

$$\nabla u_p \rightharpoonup \nabla u \quad w - L^q\ \forall q \in (1, +\infty). \tag{6.4.18}$$

It is simple to see that λ is solution of the equation:

$$-div\lambda = f, \quad in\ \Omega \setminus \Sigma. \tag{6.4.19}$$

Moreover thanks to lemma 2.4.1, we can write

$$\lambda = \xi\mu \tag{6.4.20}$$

for a suitable $\xi \in L^2_\mu(\Omega, \mathbb{R}^{N\times N})$. Using theorems (6.4.2) and (6.4.3) we obtain:

$$\alpha_p \to \alpha := \min\{< -f, u >: \quad u \in Lip_{1,\rho}(\Sigma)\}. \tag{6.4.21}$$

Moreover by a standard convex duality argument we may express α through the following dual problem

$$\alpha = -\min\{\int_\Omega \rho^0(\nu): \quad -div\nu = f \quad in\ \Omega \setminus \Sigma\}. \tag{6.4.22}$$

LEMMA 6.4.7 (Equipartition of energy). *Let* u_p *be the minimum point of* G_p*, then the optimal* σ_p *for* D_p *is given by:* $\xi_p \rho(u_p)^{p-2}$. *Moreover we have the following equipartition of the energy:*

$$\int_\Omega \rho(e(u_p))^p dx =< f, u_p >= \frac{p}{1-p}\alpha_p. \tag{6.4.23}$$

PROOF. The first part of this lemma is proved by the equations (6.4.7) and (6.4.8). By definition

$$\alpha_p = \frac{1}{p}\int_\Omega \rho(e(u_p))^p dx - < f, u_p > .$$

Then thanks to equations (6.4.7) and (6.4.8)

$$\begin{aligned} \alpha_p \quad &= \tfrac{1}{p}\textstyle\int_\Omega \rho(e(u_p))^p dx - < -div(\xi_p \rho(e(u_p))^{p-2}), u_p >= \\ &= \tfrac{1}{p}\textstyle\int_\Omega \rho(e(u_p))^p dx + \int_\Omega \xi_p : e(u_p)\rho(e(u_p))^{p-2} dx. = \\ &= (\tfrac{1}{p} - 1)\textstyle\int_\Omega \rho(e(u_p))^p dx. \end{aligned} \tag{6.4.24}$$

□

Thanks to lemma 2.4.5, up to subsequences

$$\xi_p \overset{Y(\mu_p,\mu)}{\rightharpoonup} \nu_x, \tag{6.4.25}$$

and

$$\xi(x) = \int_{\mathbb{R}^N} z d\nu_x(z). \tag{6.4.26}$$

By proposition 6.4.4 and the homogeneity of ρ, the sequence $\{\rho(e(u_p))\}$ is uniformly bounded in $L^2_{\mu_p}(\Omega, \mathbb{R})$ therefore by assertion (i) of proposition 2.4.4 the function

$$h(x,t) :=< \rho^0(z)^t, \nu_x >, \tag{6.4.27}$$

is well defined on $[0,2]$ for $\mu-$a.e. $x \in \Omega$ and thanks to the inequality

$$+\infty > \liminf_{p\to\infty} \int_\Omega \rho(e(u_p))^p dx = \liminf_{p\to\infty} \int_\Omega \rho(e(u_p))^2 d\mu_p \geq$$
$$\geq \liminf_{p\to\infty} \int_\Omega \rho^0(\xi_p)^2 d\mu_p \geq \int_\Omega h(x,2) d\mu \tag{6.4.28}$$

h is continuous on $[0,2]$ for μ-a.e. $x \in \Omega$. In addition by the assertion (ii) of proposition 2.4.4 for $s \in [0,2[$, the following equality holds

$$\liminf_{p\to\infty} \int_\Omega \rho(e(u_p))^{p-2+s} dx = \liminf_{p\to\infty} \int_\Omega \rho(e(u_p))^s d\mu_p =$$
$$= \liminf_{p\to\infty} \int_\Omega \rho^0(\xi_p)^s d\mu_p = \int_{\overline{\Omega}} h(x,s) d\mu. \tag{6.4.29}$$

We now deduce some property of h.

Lemma 6.4.8 (Monotonicity). *Let $t_1 \leq t_2 \in [0,2]$, then for $\mu-$a.e. $x \in \Omega$ we have*

$$h(x,t_1) \leq h(x,t_2). \tag{6.4.30}$$

Proof. It follows from proposition 2.4.4 that for $t < 2$ and for any Borel subset $\mathcal{A}$

$$\int_\mathcal{A} h(x,t) d\mu = \lim_{p\to\infty} \int_\mathcal{A} \rho(e(u_p))^{p-2+t} dx = \lim_{p\to\infty} \int_\mathcal{A} \rho^0(\xi_p)^t d\mu_p, \tag{6.4.31}$$

Using Hölder inequality and (6.4.29), we obtain that for every $0 \leq t_1 < t_2 < 2$:

$$\int_\mathcal{A} h(x,t_2) d\mu = \lim_{p\to\infty} \int_\mathcal{A} \rho(e(u_p))^{p-2+t_2} dx \geq$$
$$\lim_{p\to\infty} |\mathcal{A}|^{1-\frac{p-2+t_2}{p-2+t_1}} \left(\int_\mathcal{A} \rho(e(u_p))^{p-2+t_1} dx\right)^{\frac{p-2+t_2}{p-2+t_1}}$$
$$= \int_\mathcal{A} h(x,t_1) d\mu. \tag{6.4.32}$$

As in the scalar case, by localization we deduce (6.4.30) for $t_2 < 2$. We can extend it to $t_2 = 2$ using the continuity of $h(x,\cdot)$ on $[0,2]$. $\square$

We will now show that h is constant with respect to the second variable in the interval $[1,2]$.

PROPOSITION 6.4.9 (Constancy). *For $\mu-$a.e. $x \in \Omega$ $h(x,\cdot)$ is constantly equal to* 1 *on the interval* $[1,2]$.

PROOF. Passing to the limit in the equality of lemma 6.4.7 we get

$$-\alpha = \lim_{p\to\infty} \int_\Omega \rho(e(u_p))^p dx.$$

By using (6.4.28), (6.4.29) and the monotonicity of h we get:

$$\begin{aligned} \alpha \geq \lim_{p\to\infty} \int_\Omega \rho(Du_p)^p dx &\geq \int_\Omega h(x,2)d\mu \geq \\ &\geq \int_\Omega h(x,1)d\mu \geq \int_\Omega \rho^0(\xi)d\mu \geq \alpha, \end{aligned} \tag{6.4.33}$$

where the last two inequalities follows from (6.4.19), (6.4.20) and (6.4.22). It follows that

$$\int_\Omega h(x,2)d\mu = \int_\Omega h(x,1)d\mu = \int_\Omega \rho^0(\xi)d\mu = \alpha.$$

In particular $h(x,s) = \rho^0(\xi)$ $\mu-$a.e. for any $s \in [1,2]$. It remains to prove that $\rho^0(\xi) = 1$ μ-a.e.. Applying lemma (6.4.8) with $t_1 = 0$, we get $\rho^0(\xi) = h(x,2) \geq h(x,0) = 1$ μ-a.e., the converse inequality is deduced from Jensen inequality as follows:

$$\rho^0(\xi) = h(x,1) = h(x,2) =< \rho^0(z)^2, \nu_x >\geq |< \rho^0(z), \nu_x >|^2 = \rho^0(\xi)^2. \tag{6.4.34}$$

□

COROLLARY 6.4.10 (Strong convergence of symmetric gradients). *For μ-a.e $x \in \Omega$ $\nu_x = \delta_{\xi(x)}$*

PROOF. Since $\rho^0(\xi) = 1$ μ-a.e., Jensen inequality (6.4.34) become an equality then, using the strict convexity if $\rho^0(z)^2$, we obtain that ν_x is a Dirac mass and then, arguing as in the scalar case, we conclude the proof. □

To complete the proof of theorem 6.4.1 it remains to show that 3), 4) and 6) of (3.5.7) are satisfied. To prove that $\mu(\Sigma) = 0$, consider the measures $\tilde{\mu} := \mu \llcorner (\Omega \setminus \Sigma)$ and $\tilde{\lambda} := \xi\tilde{\mu}$. It is simple to see that $-div\tilde{\lambda} = f$ in $\Omega \setminus \Sigma$, moreover $|\tilde{\mu}| \leq |\mu|$, the minimality of λ for (6.3.20), imply the equality between μ and $\tilde{\mu}$ then $\mu(\Sigma) = 0$. 4) and 6) follow (as in [**12**] proposition 3.8) from the fact that u and $\xi\mu$ are minimizers respectively of G and its dual functional, then convex analysis arguments similar to that used at the beginning of this section apply.

6.5. Some final remarks

The approximations we considered in this chapter have been used for numerical approximations of the shape optimization problems we considered (see [**39**], [**12**]). Then our interest focus not just on the convergence of the subsequences of $\{|Du_p|^{p-2}\}$ and $\{\rho(e(u_p))^{p-2}\}$ but on the convergence of the whole sequence. It has been shown ([**5**]) that if f (the heat source or the external force) is absolutely continuous with respect to $\mathcal{L}^N$ then there exists essentially only one measure solution of the limit problem, which then give the desired result. In the case of singular f there is a "numerical evidence" (if any meaning on that) that the p-Laplacian approximation select only one solution of the limit problem. Let us give a final example on this fact:

EXAMPLE 6.5.1. In the scalar case let $f := \delta_0$ a Dirac mass centered at the origin. Let Ω be the unit ball $B(0,1)$ and let Σ be the boundary of Ω. In this case there is only one Lipschitz function u solution of the limit problem (6.1.3), indeed $u = 0$ on Σ and u has to grow with maximal slope on at least one segment joining 0 and Σ (∇u pointing inward), moreover u is a 1-Lipschitz function, then $u(x) = 1 - |x|$. On the other hand using the results of [**12**] it is possible to construct a wide family o measure μ which together with u are solutions of the limit problem as follows: let ν be a probability measure on Σ, define for each Borel set A

$$\mu(A) = \int_\Sigma \mathcal{H}^1(\{tx \mid t \in [0,1]\} \cap A) d\nu(x). \tag{6.5.1}$$

In [**12**] it has been proved that the measure μ associated by formula (6.5.1) to any probability measure ν on Σ is solution of the limit problem (6.1.3). However for symmetry reasons it is easy to see that the sequence $|Du_p|^{p-2}dx$ converges (without extracting any subsequences) to the measure μ associated to the probability measure ν uniformly distributed on Σ.

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