CURVED THIN-FILM LIMITS OF CHIRAL DIRICHLET ENERGIES

GIOVANNI DI FRATTA AND VALERIY SLASTIKOV

ABSTRACT. We investigate the curved thin-film limit of a family of perturbed Dirichlet energies in the space of H^1 Sobolev maps defined in a tubular neighborhood of an $(n-1)$-dimensional submanifold N of $\mathbb{R}^n$ and with values in an $(m-1)$-dimensional submanifold M of $\mathbb{R}^m$. The perturbation K that we consider is represented by a matrix-valued function defined on M and with values in $\mathbb{R}^{m \times n}$. Under natural regularity hypotheses on N, M, and K, we show that the family of these energies converges, in the sense of Γ-convergence, to an energy functional on N of an unexpected form, which is of particular interest in the theory of magnetic skyrmions. As a byproduct of our results, we get that in the curved thin-film limit, antisymmetric exchange interactions also manifest under an anisotropic term whose specific shape depends both on the curvature of the thin film and the curvature of the target manifold. Various types of antisymmetric exchange interactions in the variational theory of micromagnetism are a source of inspiration and motivation for our work.

1. INTRODUCTION AND MOTIVATION

In this paper, we investigate the curved thin-film limit of a perturbed Dirichlet energy of the form

$$\mathcal{G}_\varepsilon\left(\boldsymbol{v}\right) := \frac{1}{2\varepsilon} \int_{\Omega_\varepsilon} |D\boldsymbol{v}(x) + \mathsf{K}\left(\boldsymbol{v}(x)\right)|^2 \, \mathrm{d}x \tag{1}$$

defined on H^1 Sobolev maps $\boldsymbol{v} : \Omega_\varepsilon \to M$, where for every $\varepsilon > 0$ the domain $\Omega_\varepsilon \subseteq \mathbb{R}^n$ is the ε-tubular neighborhood of an $(n-1)$-dimensional submanifold N of $\mathbb{R}^n$, and M is an $(m-1)$-dimensional submanifold of $\mathbb{R}^m$. The perturbation K is represented by a matrix-valued function defined on M and with values in $\mathbb{R}^{m \times n}$ (m rows, n columns). We show that under natural regularity hypotheses on N, M, and K, the family $(\mathcal{G}_\varepsilon)$ converges, in the sense of Γ-convergence, to an energy functional on N, which strongly depends both on N and M and reveals remarkable physical implications to relevant systems, for instance, to magnetic materials.

Our analysis shows that in the curved thin-film regime, when general base and target manifolds are considered, generic antisymmetric exchange interactions manifest themselves under an additional anisotropy term whose specific shape depends both on the curvature of the thin film and the curvature of the target manifold.

A specific form of (1), namely the one emerging when N is a *planar* surface in $\mathbb{R}^2 \times \{0\}$, $M := \mathbb{S}^2$, and K is the antisymmetric matrix $\mathsf{K}\left(\boldsymbol{v}\right) = \bullet \times \boldsymbol{v}$, was recently addressed in the framework of micromagnetics [13]. This specific choice of N, M, and K, describes a micromagnetic energy functional (1) that accounts for antisymmetric interactions known as *bulk Dzyaloshinskii–Moriya interactions* (DMI), whose presence explains the emergence of chiral spin textures known as magnetic skyrmions [22, 42]. Even in this particular case, the

derived limiting model in [13] revealed intriguing physics: a portion of the isotropic bulk DMI contributes to the shape anisotropy originating from the magnetostatic self-energy. Our extension unveils additional mechanisms even in the planar setting. Namely, what happens when the target manifold is not really $\mathbb{S}^2$, the DMI contribution is anisotropic, or there are temperature variations in the ferromagnetic media (see Subsection 1.2).

Given the importance of the energy $\mathcal{G}_\varepsilon$ in the analysis of antisymmetric interactions, we will also refer to the energy (1) as the *chiral Dirichlet energy*, even when K is not antisymmetric.

1.1. Outline.
To adequately explain the ramifications of our findings, we first review the physical framework that led us to the investigation of (1). In Subsection 1.3, we sum up earlier research on the subject. In Section 2, we describe the rigorous setting of the problem and detail the contributions of the present work. Proofs are given in Section 3. In Section 4, we look at several applications of our results to the variational theory of micromagnetism.

1.2. Physical context.
In single-crystal ferromagnets, the observable magnetization states correspond to the local minimizers of the micromagnetic energy functional, which, after normalization, reads as [8, 30]

$$\mathcal{G}_{\mathrm{sym}}(\boldsymbol{m}) := \mathcal{H}_{\mathrm{ex}}(\boldsymbol{m}) + \mathcal{W}(\boldsymbol{m}) := \frac{1}{2}\int_\Omega |\nabla \boldsymbol{m}(x)|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_{\boldsymbol{m}}(x)|^2\,\mathrm{d}x, \qquad (2)$$

with $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$. The nonlocal term $\mathcal{W}(\boldsymbol{m})$ represents the *magnetostatic self-energy* and describes the energy due to the demagnetizing field $\nabla u_{\boldsymbol{m}}$ generated by $\boldsymbol{m}\chi_\Omega$, where $\boldsymbol{m}\chi_\Omega$ denotes the extension of $\boldsymbol{m}$ by zero in $\mathbb{R}^3 \setminus \Omega$. The scalar potential $u_{\boldsymbol{m}}$ can be characterized as the unique solution in $H^1(\mathbb{R}^3)$ of the Poisson equation [7, 16, 43]:

$$-\Delta u_{\boldsymbol{m}} = \mathrm{div}\,\boldsymbol{m}\chi_\Omega \quad \text{in } \mathcal{D}'\left(\mathbb{R}^3\right). \qquad (3)$$

The *exchange energy* $\mathcal{H}_{\mathrm{ex}}$ penalizes nonuniformities in the orientation of the magnetization. Up to a constant term, $\mathcal{H}_{\mathrm{ex}}$ can be considered as the very-short-range limit of a family of nonlocal Heisenberg models of the type

$$\mathcal{H}_{\mathrm{ex}}^\varepsilon(\boldsymbol{m}) := -\int_{\Omega\times\Omega} \omega_\varepsilon(|y-x|)\boldsymbol{m}(x)\cdot\boldsymbol{m}(y)\mathrm{d}x\mathrm{d}y, \qquad (4)$$

with ω_ε being a family of interaction kernels which tends to concentrate around the origin (cf. [6]).

The interaction energy $\mathcal{H}_{\mathrm{ex}}^\varepsilon$ is physically *symmetric* in the *lattice points* x, y where the magnetic dipoles $\boldsymbol{m}(x)$, $\boldsymbol{m}(y)$ are located and, mathematically, this is captured in the commutativity of the dot product in (4). However, magnetic materials with low crystallographic symmetry can exhibit a weak *antisymmetric* exchange interaction [19, 38]: a relativistic effect caused by the spin-orbit coupling among neighbor magnetic spins. This type of interaction was initially predicted and investigated by Dzyaloshinskii [19] and Moriya [38]. For this reason, antisymmetric exchange interactions are usually referred to as Dzyaloshinskii–Moriya interactions (DMI).

The DMI induces a spin canting of the magnetic moments, whereas symmetric exchange interactions favor parallel-aligned spins. Mimicking the expression of the DMI energy associated with a discrete distribution of magnetic moments (cf. [11, p. 862]), one can assume that for a continuous distribution of magnetic moments $\boldsymbol{m} \in L^2(\Omega, \mathbb{S}^2)$, after normalization, the

overall antisymmetric interactions among magnetic moments is described by the Hamiltonian function

$$\mathcal{H}_{\mathrm{DM}}^{\varepsilon}(\boldsymbol{m}) := \int_{\Omega \times \Omega} \boldsymbol{d}_{\varepsilon}(y - x) \cdot (\boldsymbol{m}(x) \times \boldsymbol{m}(y)) \,\mathrm{d}x \mathrm{d}y. \tag{5}$$

The Dzyaloshinskii vector $\boldsymbol{d}_{\varepsilon}$ is an axial vector that, other than from the relative distance between the magnetic moments $\boldsymbol{m}(x)$ and $\boldsymbol{m}(y)$, also depends on the symmetry group of the crystal lattice; its precise form has to be determined following Moriya's rules [38].

Under appropriate assumptions on the ε-decay rate (and concentration) of the family $(\boldsymbol{d}_{\varepsilon})_{\varepsilon > 0}$ in (5), one can obtain the very-short-range limiting model of *antisymmetric* exchange interactions

$$\mathcal{H}_{\mathrm{DM}}(\boldsymbol{m}) = \sum_{i=1}^{3} \int_{\Omega} \partial_i \boldsymbol{m}(x) \cdot (\boldsymbol{d}_i \times \boldsymbol{m}(x)) \,\mathrm{d}x \tag{6}$$

with $(\boldsymbol{d}_i)_{i=1}^{3}$ being suitable constant vectors that depend on the symmetry of the crystal lattice. The energy $\mathcal{H}_{\mathrm{DM}}$ in (6) is, nowadays, the primary term used in the variational theory of micromagnetics to explain the emergence of magnetic skyrmions.

Overall, in the presence of *both* symmetric and antisymmetric interactions, the Heisenberg interaction energy, up to a constant factor, can be expressed under the general form:

$$\mathcal{H}_{\Omega}(\boldsymbol{m}) := \frac{1}{2} \sum_{i=1}^{3} \int_{\Omega} |\partial_{\tau_i(\xi)} \boldsymbol{m}(x)|^2 + 2\,\partial_i \boldsymbol{m}(x) \cdot (\boldsymbol{d}_i \times \boldsymbol{m}(x)) \,\mathrm{d}x. \tag{7}$$

Therefore, if we denote by J the linear operator mapping the standard basis $(\boldsymbol{e}_i)_{i=1}^{3}$ into the vectors $(\boldsymbol{d}_i)_{i=1}^{3}$, and denote by $\mathsf{M}(\boldsymbol{m}) := \bullet \times \boldsymbol{m}$ the $\boldsymbol{m}$-dependent *antisymmetric* linear operator representing the action of the cross-product with $\boldsymbol{m}$, then (7) can be rewritten as

$$\begin{aligned}
\mathcal{H}_{\Omega}(\boldsymbol{m}) &= \frac{1}{2} \sum_{i=1}^{3} \int_{\Omega} |\partial_i \boldsymbol{m}(x) + \mathsf{M}(\boldsymbol{m}(x))\,\mathsf{J}\boldsymbol{e}_i|^2 - \frac{1}{2} \sum_{i=1}^{3} \int_{\Omega} |\,\mathsf{M}(\boldsymbol{m}(x))\,\mathsf{J}\boldsymbol{e}_i|^2 \,\mathrm{d}x \\
&= \frac{1}{2} \int_{\Omega} |D\boldsymbol{m}(x) + \mathsf{M}(\boldsymbol{m}(x))\,\mathsf{J}|^2 - \frac{1}{2} \int_{\Omega} |\,\mathsf{M}(\boldsymbol{m}(x))\,\mathsf{J}|^2 \,\mathrm{d}x \\
&= \frac{1}{2} \int_{\Omega} |D\boldsymbol{m}(x) + \mathsf{K}(\boldsymbol{m}(x))|^2 - \frac{1}{2} \int_{\Omega} |\,\mathsf{K}(\boldsymbol{m}(x))|^2 \,\mathrm{d}x,
\end{aligned} \tag{8}$$

where we denoted by $\mathsf{K}(\boldsymbol{m}) := \mathsf{M}(\boldsymbol{m})\,\mathsf{J}$ the product of the $\boldsymbol{m}$-dependent antisymmetric linear operator K with the constant matrix J.

It is worth noting that the second term in (8) reflects a Γ-continuous term whose analysis in the thin-film regime is straightforward. A similar remark applies to the magnetostatic self-energy $\mathcal{W}$ investigated in [16, 15], where it is shown that in the curved thin-film limit, the magnetostatic self-energy $\mathcal{W}$ localizes to an easy-surface anisotropy in the direction of the normal to the base manifold. Overall, it is sufficient to study energy (1) to comprehend how DMI affects magnetization behavior in curved thin layers.

1.3. State of the art. The investigations of Dirichlet-type energy functionals between manifolds is a subject with a long history. Their minimization problem naturally arises in differential geometry for studying unit-speed geodesics and minimal surfaces [20, 44]. However, it also appears in nonlinear field theories where certain forms of interactions are assimilated through the ideas of elastic deformations. Outstanding examples are the elastic free energy in the Oseen-Frank theory of nematic liquid crystals [3, 48], the exchange energy in the variational theory of micromagnetism [8, 25], and string theory and M-theory [1, 4]. The first-order

minimality conditions of Dirichlet energy produce harmonic map equations, whose systematic treatment gained impetus after the seminal work of Eells and Sampson [21], who showed that in specific geometric contexts, arbitrary maps could be deformed into harmonic maps. The overall subject is so vast that it would be impossible here even to scratch the surface of the literature on the topic; therefore, we refer to [20, 44, 32] and references therein for further details.

Here, instead, we give an overview of the literature intimately related to our investigations in the curved thin film regime. In the variational theory of micromagnetics, which consists of a nonlocal addendum to the theory of $\mathbb{S}^2$-valued harmonic maps, the story dates back to the seminal paper [24], where the authors showed that in thin films, the nonlocal effects of the so-called demagnetizing field operator localize to an easy-surface anisotropy term. Later, various static and dynamic reduced theories for thin-film micromagnetics has been established under different scaling regimes in [2, 14, 26, 27, 28, 34, 35, 39, 40].

Ferromagnetic systems in the shape of *curved* thin films are currently of interest due to their capability to host *spontaneous* skyrmion solutions, i.e., chiral spin textures that carry a non-trivial topological degree, even in materials where the antisymmetric spin-orbit coupling mechanism (in the guise of DMI) can be neglected. The evidence of these states sheds light on the role of the geometry in magnetism: chiral spin-textures can be stabilized by curvature effects only, in contrast to the planar case where DMI is required. In addition to fundamental reasons, the interest in these geometries is triggered by recent advances in the fabrication of magnetic spherical hollow nanoparticles, which lead to artificial materials with unexpected characteristics and numerous applications ranging from logic devices to biomedicine (see, e.g., the topical review [46] and [33]). From the mathematical point of view, the dimension reduction problems in non-planar thin magnetic layers have been studied in [10, 17, 16, 15, 36, 37, 45].

Dimension reduction results taking into account also antisymmetric exchange interactions have been reported in the regime of planar thin film and for bulk DMI, i.e., when in (7) one considers $\Omega_\varepsilon \subseteq \mathbb{R}^3$ of the type $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ with ω an open subset of $\mathbb{R}^2$, and the classical $\mathbb{S}^2$-valued magnetization $\boldsymbol{m} \in H^1(\Omega_\varepsilon, \mathbb{S}^2)$, see [13, Theorem 1]. The surprising consequence of this result is that a portion of the bulk DMI energy contributes an additional shape anisotropy term $\kappa^2 \int_\omega (\boldsymbol{m}(\sigma) \cdot \boldsymbol{e}_3)^2$ in the limiting energy, to enhance the shape anisotropy of the thin film, see Subsection 4.1 for extended discussion.

In this paper, we remove the rigid conditions of $\boldsymbol{m}$ being $\mathbb{S}^2$-valued, the thin film being planar, and treat a general perturbation K. It allows us to treat an interesting case of anisotropic DMI [9, 23] and other engaging physical scenarios, such as constrained Q-tensor theories of nematic liquid crystals as well their Oseen-Frank counterpart in the presence of anisotropic bulk potentials [3]. It can also be applied to various limiting Ginzburg–Landau-type models [5].

An important contribution of our result to the physics of magnetic materials comes from the fact that it uncovers yet another effective anisotropy term (apart from one reported in [13]) coming from the interplay between a ferromagnetic layer, target manifold for magnetization vector, and a specific form of DMI. It reveals, for example, what happens when one considers a ferromagnet in which the temperature is not necessarily uniform or DMI is anisotropic. We refer to Section 4 for various interesting examples in micromagnetics.

2. CONTRIBUTIONS OF THE PRESENT WORK

2.1. Notation and setup. The main result of this paper concerns the variational characterization of the asymptotic behavior of a rescaled version of (1) in the limit for $\varepsilon \to 0$. To state our results in a precise way, we need to set up the framework and introduce notation.

We denote by N and M closed (i.e., compact, without boundary, and connected) hypersurfaces of class C^2, respectively, of $\mathbb{R}^n$ and $\mathbb{R}^m$. It is well-known that as a consequence of the Jordan-Brouwer separation theorem, N and M are orientable [31]. The normal fields associated with the choice of orientations of N and M will be denoted, respectively, by $\boldsymbol{n}_N : N \to \mathbb{S}^{n-1}$, and $\boldsymbol{n}_M : M \to \mathbb{S}^{m-1}$.

To make the reading more comfortable, we consistently denote by ξ a generic point on N and by σ a generic point on the target space M. Also, we denote by $T_\xi N$ and $T_\sigma M$, respectively, the tangent space to N at ξ and the tangent space to M at σ, and we use the notation TN and TM to denote the corresponding tangent bundles.

Our hypotheses on N and M assure that they both admit a tubular neighborhood (of uniform thickness). We recall the definition as it also allows us to fix notation. Given a closed C^2-hypersurface $S \subseteq \mathbb{R}^n$ (for which Jordan-Brouwer separation theorem holds), we can denote by $d_S : \mathbb{R}^n \to \mathbb{R}$ the *signed distance* from S, defined by

$$d_S(x) = \begin{cases} d(x,S) & \text{if } x \in S_+, \\ -d(x,S) & \text{if } x \in S_-, \end{cases}$$

where we denoted by $d(x,S)$ the *Euclidean distance* of x from S, by S_+ the outer (unbounded) component of $\mathbb{R}^n \setminus S$, and by S_- the interior one. We say that the open set

$$O_\delta := \{x \in \mathbb{R}^n : d_S(x) < \delta\}$$

is a *tubular neighborhood* of S of uniform thickness $\delta > 0$ if the following property holds (cf. [18]): with $\mathcal{S} := S \times I$, $I := (-1,1)$, one has that for every $0 < \varepsilon < \delta$, the map

$$\psi_\varepsilon : (\xi,s) \in \mathcal{S} \mapsto \xi + \varepsilon s \boldsymbol{n}_S(\xi) \in O_\varepsilon \tag{9}$$

is a C^1-diffeomorphism of $\mathcal{S}$ onto O_ε. In this case, the nearest point projection map

$$\pi_S : O_\varepsilon \to S, \tag{10}$$

which maps any $x \in O_\varepsilon$ onto the unique $\pi_S(x) \in S$ such that $x = \pi_S(x) + d_S(x)\boldsymbol{n}_S(\pi_S(x))$, is a map of class C^1 and, therefore, so is $d_S(x) = (x - \pi_S(x)) \cdot \boldsymbol{n}_S(\pi_S(x))$. Moreover, one has

$$\nabla d_S(\pi_S(x)) = \boldsymbol{n}_S(\pi_S(x)).$$

In what follows, we denote by $\Omega_\delta := \{x \in \mathbb{R}^n : d_N(x) < \delta\}$ a tubular neighborhood of N of thickness δ and by O_δ a tubular neighborhood of M. Also, we set $\mathcal{N} := N \times I$, so that, from (9), $\Omega_\delta \equiv \psi_\delta(\mathcal{N})$. Moreover, to shorten the notation, for any $\delta > 0$ we set $I_\delta := (-\delta, \delta)$ and $I_{\delta_+} := (0, \delta)$.

For every $\xi \in N$ the symbols $\boldsymbol{\tau}_1(\xi), \boldsymbol{\tau}_2(\xi), \ldots, \boldsymbol{\tau}_{n-1}(\xi)$ are used to denote an orthonormal basis of $T_\xi N$ made by its principal directions, i.e., an orthonormal basis consisting of eigenvectors of the shape operator of N (cf., e.g., [18]). We then write $\kappa_1(\xi), \kappa_2(\xi), \ldots, \kappa_{n-1}(\xi)$ for the principal curvatures at $\xi \in N$. Note that, for any $x \in \Omega_\delta$ the frame

$$(\boldsymbol{\tau}_1(\xi), \boldsymbol{\tau}_2(\xi), \ldots, \boldsymbol{\tau}_{n-1}(\xi), \boldsymbol{n}_N(\xi)) \quad \text{with} \quad \xi := \pi_N(x), \tag{11}$$

constitutes an orthonormal basis of $T_{\pi_N(x)}\Omega_\delta$ that depends only on N. Note that we do not specify the symbol N in the notation related to the orthonormal basis of $T_\xi N$. This creates no confusion because we will not use orthonormal bases of $T_\sigma M$.

For $0 < \varepsilon < \delta$, we denote by $\sqrt{\mathfrak{g}_\varepsilon(\xi, s)}$ the metric factor which relates the volume form on Ω_ε at $\xi + \varepsilon s \boldsymbol{n}_N(\xi)$ to the volume form on $\mathcal{N}$ at (ξ, s). Also, we denote by $(\mathfrak{h}_{1,\varepsilon}(\xi, s))_{i=1}^{n-1}$ the metric coefficients which connect the tangential part of the gradient at $\xi + \varepsilon s \boldsymbol{n}_N(\xi) \in \Omega_\varepsilon$ to the tangential gradient on N at (ξ, s).

A direct computation (see Lemma 1 in Section 3) shows that

$$\sqrt{\mathfrak{g}_\varepsilon(\xi, s)} := \Pi_{i=1}^{n-1}(1 + \varepsilon s \kappa_i(\xi)), \quad \mathfrak{h}_{i,\varepsilon}(\xi, s) := \frac{1}{1 + \varepsilon s \kappa_i(\xi)} \quad (i = 1, 2, \ldots, n-1). \tag{12}$$

In what follows, without loss of generality, we shall always assume that the thickness δ is sufficiently small so that for every $\varepsilon \in I_{\delta+}$ there holds

$$c_\mathcal{N}^{-1} \leqslant \sqrt{\mathfrak{g}_\varepsilon(\xi, s)} \leqslant c_\mathcal{N}, \quad c_\mathcal{N}^{-1} \leqslant \mathfrak{h}_{i,\varepsilon}(\xi, s) \leqslant c_\mathcal{N}, \tag{13}$$

for some positive constant $c_\mathcal{N} > 0$.

Also, we shall denote by $H^1(\mathcal{N}, \mathbb{R}^m)$ the Sobolev space of vector-valued functions defined on $\mathcal{N}$ (see, e.g., [49]) endowed with the norm

$$\|\boldsymbol{u}\|_{H^1(\mathcal{N},M)}^2 := \int_\mathcal{N} |\boldsymbol{u}(\xi, s)|^2 \, \mathrm{d}\xi \mathrm{d}s + \int_\mathcal{N} |\nabla_\xi \boldsymbol{u}(\xi, s)|^2 + |\partial_s \boldsymbol{u}(\xi, s)|^2 \, \mathrm{d}\xi \mathrm{d}s. \tag{14}$$

Here, ∇_ξ is the tangential gradient of $\boldsymbol{u}$ on N, and $|\nabla_\xi \boldsymbol{u}(\xi, s)|^2 = \sum_{i=1}^{n-1} |\partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{u}(\xi, s)|^2$.

We write $H^1(\mathcal{N}, M)$ for the subset of $H^1(\mathcal{N}, \mathbb{R}^m)$ made by vector-valued functions with values in M, and we use the notation $H^1(\mathcal{N}, TM)$ when we want to emphasize that the target manifold is the tangent bundle of M.

2.2. The chiral Dirichlet energy. Let N and M be closed and smooth (C^2) hypersurfaces of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For every $0 < \varepsilon < \delta$ we consider the (n-dimensional) tubular neighborhood $\Omega_\varepsilon = \{x \in \mathbb{R}^n : d_N(x) < \varepsilon\}$, and consider the family of energy functionals defined for every $\boldsymbol{v} \in H^1(\Omega_\varepsilon, M)$ by

$$\mathcal{G}_\varepsilon(\boldsymbol{v}) = \frac{1}{2\varepsilon} \int_{\Omega_\varepsilon} |D\boldsymbol{v}(x) + \mathsf{K}(\boldsymbol{v}(x))|^2 \, \mathrm{d}x, \tag{15}$$

with $|\cdot|$ being the Euclidean norm on $\mathbb{R}^{m \times n}$ (m rows, n columns) and where

$$\mathsf{K} : \sigma \in M \mapsto \mathsf{K}(\sigma) \in \mathbb{R}^{m \times n}$$

is a Lipschitz continuous function, i.e., there exists $c_\mathsf{K} > 0$ such that

$$|\mathsf{K}(\sigma_1) - \mathsf{K}(\sigma_2)|_{n \times m} \leqslant c_\mathsf{K} |\sigma_1 - \sigma_2|_m \quad \forall \sigma_1, \sigma_2 \in M. \tag{16}$$

Remark 1. Since we are assuming M to be bounded, (16) implies that for any $\sigma_0 \in M$ there holds $|\mathsf{K}(\sigma)| \leqslant |\mathsf{K}(\sigma_0)| + c_\mathsf{K} \operatorname{diam}(M)$, i.e., that the image of K is bounded. In what follows, to simplify the constants, we will assume that a Lipschitz constant c_K is chosen big enough so that there holds

$$|\mathsf{K}(\sigma)| \leqslant c_\mathsf{K} \quad \forall \sigma \in M. \tag{17}$$

For any $\varepsilon \in I_{\delta+}$, the existence of at least a minimizer for $\mathcal{G}_\varepsilon$ in $H^1(\Omega_\varepsilon, M)$ is a simple application of the direct method of the calculus of variations. We are interested in the asymptotic behavior of the family of minimizers of $(\mathcal{G}_\varepsilon)_{\varepsilon \in I_{\delta+}}$ as $\varepsilon \to 0$.

Let us introduce the following functional defined on $H^1(\mathcal{N}, M)$, which can be thought of as the pull-back of $\mathcal{G}_\varepsilon$ on the product manifold $\mathcal{N} := N \times I$, $I := (-1, 1)$:

$$\mathcal{E}_{\mathcal{N}}^\varepsilon (\boldsymbol{u}_\varepsilon) := \frac{1}{2} \int_{\mathcal{N}} \sum_{i=1}^{n-1} \left| \mathfrak{h}_{i,\varepsilon}(\xi, s) \partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{u}_\varepsilon(\xi, s) + \mathsf{K}\left(\boldsymbol{u}_\varepsilon(\xi, s)\right) \boldsymbol{\tau}_i(\xi) \right|^2 \sqrt{\mathfrak{g}_\varepsilon(\xi, s)} \mathrm{d}\xi \mathrm{d}s$$

$$+ \frac{1}{2} \int_{\mathcal{N}} \left| \frac{1}{\varepsilon} \partial_s \boldsymbol{u}_\varepsilon(\xi, s) + \mathsf{K}\left(\boldsymbol{u}_\varepsilon(\xi, s)\right) \boldsymbol{n}_N(\xi) \right|^2 \sqrt{\mathfrak{g}_\varepsilon(\xi, s)} \mathrm{d}\xi \mathrm{d}s. \qquad (18)$$

Our main result is stated in the following statement.

Theorem 1. *For any $\varepsilon \in I_{\delta+}$, the minimization problem for $\mathcal{G}_\varepsilon$ in $H^1(\Omega_\varepsilon, M)$ is equivalent to the minimization in $H^1(\mathcal{N}, M)$ of the functional $\mathcal{E}_{\mathcal{N}}^\varepsilon$ defined by* (18) *in the sense that a configuration $\boldsymbol{v}_\varepsilon \in H^1(\Omega_\varepsilon, M)$ minimizes $\mathcal{G}_\varepsilon$ if and only if $\boldsymbol{u}_\varepsilon := \boldsymbol{v}_\varepsilon \circ \psi_\varepsilon \in H^1(\mathcal{N}, M)$ minimizes $\mathcal{E}_{\mathcal{N}}^\varepsilon$.*

The family $(\mathcal{E}_{\mathcal{N}}^\varepsilon)_{\varepsilon \in I_{\delta+}}$ is equicoercive in the weak topology of $H^1(\mathcal{N}, M)$ and the Γ-limit $\mathcal{E}_{\mathcal{N}} := \Gamma\text{-}\lim_{\varepsilon \to 0} \mathcal{E}_{\mathcal{N}}^\varepsilon$ is defined for every $\boldsymbol{u} \in H^1(\mathcal{N}, M)$ by

$$\mathcal{E}_{\mathcal{N}}(\boldsymbol{u}) = \begin{cases} \displaystyle\sum_{i=1}^{n-1} \int_N \left| \partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{u}(\xi) + \mathsf{K}(\boldsymbol{u}(\xi)) \boldsymbol{\tau}_i(\xi) \right|^2 \\ \qquad\qquad + \displaystyle\int_N \left(\mathsf{K}\left(\boldsymbol{u}(\xi)\right) \boldsymbol{n}_N(\xi) \cdot \boldsymbol{n}_M\left(\boldsymbol{u}(\xi)\right) \right)^2 \mathrm{d}\xi & \text{if } \partial_s \boldsymbol{u} = 0, \\ +\infty & \text{otherwise.} \end{cases} \qquad (19)$$

Moreover,

$$\min_{H^1(\Omega_\varepsilon, M)} \mathcal{G}_\varepsilon = \min_{H^1(\mathcal{N}, M)} \mathcal{E}_{\mathcal{N}}^\varepsilon = \min_{H^1(\mathcal{N}, M)} \mathcal{E}_{\mathcal{N}} + o(1), \qquad (20)$$

and if $(\boldsymbol{u}_\varepsilon)_{\varepsilon \in I_{\delta+}}$ is a minimizing family for $(\mathcal{E}_{\mathcal{N}}^\varepsilon)_{\varepsilon \in I_{\delta+}}$ then $(\boldsymbol{u}_\varepsilon)_{\varepsilon \in I_{\delta+}}$ converges, strongly in $H^1(\mathcal{N}, M)$, to a minimizer of $\mathcal{E}_{\mathcal{N}}$.

Remark 2. The arguments we present to prove Theorem 1 extend with minor modifications to the analysis of chiral Dirichlet energies of the form

$$\tilde{\mathcal{G}}_\varepsilon (\boldsymbol{v}) := \frac{1}{2\varepsilon} \int_{\Omega_\varepsilon} |A(x) D\boldsymbol{v}(x) + \mathsf{K}(x, \boldsymbol{v}(x))|^2 \, \mathrm{d}x, \qquad (21)$$

in which the operator K in (15) depends on $x \in \Omega_\varepsilon$ as well as $\sigma \in M$, (16) and (17) are assumed to hold uniformly in $x \in \Omega_\varepsilon$, and with the tensor $A \in L^\infty\left(\Omega_\varepsilon, \mathbb{R}^{m \times m})\right)$ uniformly elliptic, i.e., such that for every $x \in \Omega_\varepsilon$ and every $y \in \mathbb{R}^m \setminus \{0\}$, there holds

$$\Lambda_A |y|^2 \geqslant A(x) y \cdot y \geqslant \lambda_A \cdot |y|^2, \qquad (22)$$

for positive constants $\Lambda_A, \lambda_A > 0$ that do not depend on $x \in \Omega_\varepsilon$. For that, one also assumes that both $\mathsf{K}(\cdot, \boldsymbol{v})$ and $A(\cdot)$ are uniformly continuous in the normal direction, i.e., in terms of a modulus of continuity ϖ_{K} and ϖ_A, such that $|\mathsf{K}(x, \sigma) - \mathsf{K}(\pi_N(x), \sigma)| \leqslant \varpi_{\mathsf{K}}(|x - \pi_N(x)|)$ and $|A(x) - A(\pi_N(x))| \leqslant \varpi_A(|x - \pi_N(x)|)$ for every $x \in \Omega_\varepsilon$. To make the reading more comfortable, we give the proof by focusing on the energy $\mathcal{G}_\varepsilon$ in Theorem 1. Nevertheless, the generalized energy (21) can be important in applications; therefore, we give details about the curved thin-film limit in this case. In stating the result, we will denote by $\tilde{\mathcal{E}}_{\mathcal{N}}^\varepsilon$ the pull-back of $\tilde{\mathcal{G}}_\varepsilon$ from $H^1(\Omega_\varepsilon, M)$ to $H^1(\mathcal{N}, M)$ that one can obtain following the same steps that led to the pull-back (18).

Theorem 2. *For any $\varepsilon \in I_{\delta+}$, the minimization problem for $\tilde{\mathcal{G}}_\varepsilon$ in $H^1(\Omega_\varepsilon, M)$ is equivalent to the minimization in $H^1(\mathcal{N}, M)$ of its pull-back $\tilde{\mathcal{E}}_{\mathcal{N}}^\varepsilon$, in the sense that a configuration $\boldsymbol{v}_\varepsilon \in H^1(\Omega_\varepsilon, M)$ minimizes $\tilde{\mathcal{G}}_\varepsilon$ if and only if $\boldsymbol{u}_\varepsilon := \boldsymbol{v}_\varepsilon \circ \psi_\varepsilon \in H^1(\mathcal{N}, M)$ minimizes $\tilde{\mathcal{E}}_{\mathcal{N}}^\varepsilon$.*

The family $(\tilde{\mathcal{E}}_{\mathcal{N}}^{\varepsilon})_{\varepsilon \in I_{\delta+}}$ is equicoercive in the weak topology of $H^1(\mathcal{N}, M)$ and the Γ-limit $\tilde{\mathcal{E}}_{\mathcal{N}} := \Gamma\text{-}\lim_{\varepsilon \to 0} \tilde{\mathcal{E}}_{\mathcal{N}}^{\varepsilon}$ is defined for every $\boldsymbol{u} \in H^1(\mathcal{N}, M)$ by

$$
\tilde{\mathcal{E}}_{\mathcal{N}}(\boldsymbol{u}) = \begin{cases} \sum_{i=1}^{n-1} \int_N \left| A(\xi) \partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{u}(\xi) + \mathsf{K}(\xi, \boldsymbol{u}(\xi)) \, \boldsymbol{\tau}_i(\xi) \right|^2 \mathrm{d}\xi \\ \qquad + \int_N \left(\dfrac{A^{-1}(\xi)\mathsf{K}(\xi, \boldsymbol{u}(\xi)) \, \boldsymbol{n}_N(\xi) \cdot \boldsymbol{n}_M(\boldsymbol{u}(\xi))}{A^{-1}(\xi)\boldsymbol{n}_M(\boldsymbol{u}(\xi)) \cdot \boldsymbol{n}_M(\boldsymbol{u}(\xi))} \right)^2 \mathrm{d}\xi \qquad \textit{if } \partial_s \boldsymbol{u} = 0, \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{otherwise.} \end{cases}
$$
(23)

Moreover, if $(\boldsymbol{u}_{\varepsilon})_{\varepsilon \in I_{\delta+}}$ is a minimizing family for $(\tilde{\mathcal{E}}_{\mathcal{N}}^{\varepsilon})_{\varepsilon \in I_{\delta+}}$ then $(\boldsymbol{u}_{\varepsilon})_{\varepsilon \in I_{\delta+}}$ converges, strongly in $H^1(\mathcal{N}, M)$, to a minimizer of $\tilde{\mathcal{E}}_{\mathcal{N}}$.

Remark 3. To make the analysis more pleasant to read, we assumed N and M to be closed hypersurface of class C^2. However, as it will be transparent from the proofs, all the results hold as soon as N and M are bounded hypersurfaces (without boundary) of class C^2 that admit a tubular neighborhood (of uniform thickness). Therefore, the range of hypersurfaces N and M included in our analysis is broad. Indeed, since any closed hypersurface of class C^2 admits a tubular neighborhood (of uniform thickness) [18, Prop. 1, p. 113], our analysis holds for classical convex surfaces like spheres, ellipsoids, planar surfaces, as well as for non-convex ones like the torus. The analysis also extends to the class of bounded surfaces that are diffeomorphic to an open subset of a compact surface; typical examples are the finite cylinder and the graph of a C^2 function.

The proof of Theorem 1 is subdivided into four steps. In Subsection 3.1, we show that for any $\varepsilon \in I_{\delta+}$ and any $\boldsymbol{v}_{\varepsilon} \in H^1(\Omega_{\varepsilon}, M)$ the equality $\mathcal{G}_{\varepsilon}(\boldsymbol{v}_{\varepsilon}) = \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{v}_{\varepsilon} \circ \psi_{\varepsilon})$ holds, where ψ_{ε} stands for the diffeomorphism of $\mathcal{N}$ onto Ω_{ε} given by (9). In Subsection 3.2, we show that the family $(\mathcal{E}_{\mathcal{N}}^{\varepsilon})_{\varepsilon \in I_{\delta+}}$ is equicoercive in the weak topology of $H^1(\mathcal{N}, M)$. The identification of the Γ-limit $\mathcal{E}_{\mathcal{N}}$ is given in Subsection 3.3. Finally, in Section 4, we consider several instances of our main result and remark on the importance they can have in the micromagnetic community, both as a source of new mathematical problems and new models of interest for the physical community.

3. The curved thin-film limit: proof of Theorem 1

3.1. The equivalence of $\mathcal{G}_{\varepsilon}$ and $\mathcal{E}_{\mathcal{N}}^{\varepsilon}$. In this section, we prove the first part of Theorem 1, namely that once introduced, for any $\varepsilon \in I_{\delta}$, the diffeomorphism of $\mathcal{N}$ onto Ω_{ε} given by $\psi_{\varepsilon} : (\xi, s) \in \mathcal{N} \mapsto \xi + \varepsilon s \boldsymbol{n}_N(\xi) \in \Omega_{\varepsilon}$, one has $\mathcal{G}_{\varepsilon}(\boldsymbol{v}_{\varepsilon}) = \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{v}_{\varepsilon} \circ \psi_{\varepsilon})$ for any $\boldsymbol{v}_{\varepsilon} \in H^1(\Omega_{\varepsilon}, M)$ and, therefore, $\boldsymbol{v}_{\varepsilon}$ minimizes $\mathcal{G}_{\varepsilon}$ if and only if $\boldsymbol{u}_{\varepsilon}(\xi, s) := \boldsymbol{v}_{\varepsilon}(\psi_{\varepsilon}(\xi, s))$ minimizes $\mathcal{E}_{\mathcal{N}}^{\varepsilon}$. For that, we need the following result.

Lemma 1. *Let $0 < \varepsilon < \delta$ and $s \in I$. Set $N_{\varepsilon s} := \{x \in \Omega_{\varepsilon} : d(x, N) = \varepsilon s\}$. The following assertions hold:*

   i. The metric factor which relates the volume form on $N_{\varepsilon s}$ to the volume form on N is given by

$$
\sqrt{\mathfrak{g}_{\varepsilon}(\xi, s)} := \Pi_{i=1}^{n-1}\left(1 + \varepsilon s \kappa_i(\xi)\right).
$$
(24)

   In particular, when $n = 3$, one has

$$
\sqrt{\mathfrak{g}_{\varepsilon}(\xi, s)} := \left| 1 + 2\left(\varepsilon s\right) H(\xi) + \left(\varepsilon s\right)^2 G(\xi) \right|,
$$
(25)

where $H(\xi)$ and $G(\xi)$ are, respectively, the mean and Gaussian curvature at $\xi \in N$.

ii. The metric coefficients $\mathfrak{h}_{1,\varepsilon}(\xi, s), \mathfrak{h}_{2,\varepsilon}(\xi, s), \ldots, \mathfrak{h}_{n-1,\varepsilon}(\xi, s)$ which connect the tangential gradient on $N_{\varepsilon s}$ to the tangential gradient on N are given by

$$\mathfrak{h}_{i,\varepsilon}(\xi, s) := \frac{1}{1 + \varepsilon s \kappa_i(\xi)} \quad (i = 1, 2, \ldots, n-1). \tag{26}$$

In other words, if $\boldsymbol{u}_\varepsilon(\xi, s) := \boldsymbol{v}_\varepsilon(\psi_\varepsilon(\xi, s))$, then for every $i = 1, \ldots, n-1$, there holds

$$\left(\partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{v}_\varepsilon\right) \circ \psi_\varepsilon(\xi, s) = \mathfrak{h}_{i,\varepsilon}(\xi, s) \partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{u}_\varepsilon(\xi, s).$$

iii. The metric coefficient $\mathfrak{h}_n(\xi, s)$, which connects the normal derivative on $N_{\varepsilon s}$ to the s-derivative on I is given by $\mathfrak{h}_{n,\varepsilon}(\xi, s) = 1/\varepsilon$. In other words, if $\boldsymbol{u}_\varepsilon(\xi, s) := \boldsymbol{v}_\varepsilon(\psi_\varepsilon(\xi, s))$ then

$$\left(\partial_{\boldsymbol{n}_N(\xi)} \boldsymbol{v}_\varepsilon\right) \circ \psi_\varepsilon(\xi, s) := \frac{1}{\varepsilon} \partial_s \boldsymbol{u}_\varepsilon(\xi, s). \tag{27}$$

Proof. We use the language of differential forms as it simplifies the argument.

i. For every $\varepsilon \in I_\delta$ the n-dimensional thin domain Ω_ε is diffeomorphic to the product manifold $N \times I$ via the (positively oriented) map $\psi_\varepsilon : (\xi, s) \in N \times I \mapsto \xi + \varepsilon s \boldsymbol{n}_N(\xi) \in \Omega_\varepsilon$. The tangent map $d\psi_\varepsilon(\xi, s)$ at the point (ξ, s) is the linear map from $T_{(\xi, s)}(N \times I) \approx T_\xi N \oplus \mathbb{R}$ into $\mathbb{R}^m$ defined for every $\boldsymbol{\mu} := (\boldsymbol{\tau}, t) \in T_\xi N \oplus \mathbb{R}$ by

$$d\psi_\varepsilon(\xi, s)\boldsymbol{\mu} = (\boldsymbol{\tau} + \varepsilon s \partial_{\boldsymbol{\tau}} \boldsymbol{n}_N(\xi)) + \varepsilon t \boldsymbol{n}_N(\xi). \tag{28}$$

Now, consider the orthonormal basis of $T_\xi N \oplus \mathbb{R}$ given by

$$(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n) = (\boldsymbol{\tau}_1 \oplus 0, \ldots, \boldsymbol{\tau}_{n-1} \oplus 0, \boldsymbol{0} \oplus 1), \tag{29}$$

where the unit tangent vectors $(\boldsymbol{\tau}_i)_{i=1}^n$ are along the principal directions of N at ξ, so that

$$\partial_{\boldsymbol{\tau}_i} \boldsymbol{n}(\xi) = \kappa_i(\xi) \boldsymbol{\tau}_i(\xi) \quad \text{for } i = 1, \ldots, n-1, \tag{30}$$

with κ_i the i-th principal curvatures of N at ξ. We then have

$$d\psi_\varepsilon(\xi, s)\boldsymbol{\mu}_i = (1 + \varepsilon s \kappa_i) \boldsymbol{\tau}_i \quad \text{for } i = 1, \ldots, n-1, \tag{31}$$

$$d\psi_\varepsilon(\xi, s)\boldsymbol{\mu}_n = \varepsilon \boldsymbol{n}_N. \tag{32}$$

We denote by $\mathrm{d}x := dx_1 \wedge \cdots \wedge dx_n$ the volume form on $\mathbb{R}^n$ and by $(d\mu_i)_{i=1}^n$ the basis of $(T_\xi N \oplus \mathbb{R})^*$ dual to $(\boldsymbol{\mu}_i)_{i=1}^n$. Given that both $\mathrm{d}x$ and $d\mu_1 \wedge \cdots \wedge d\mu_n$ are forms of maximal degree n, there exists a function $\lambda(\xi, s)$ such that $(\psi_\varepsilon^* \mathrm{d}x)(\xi, s) = \lambda(\xi, s)(d\mu_1 \wedge \cdots \wedge d\mu_n)$. A simple computation then gives

$$\lambda = (\psi_\varepsilon^* \mathrm{d}x)(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n) \tag{33}$$

$$= (\mathrm{d}x)((1 + \varepsilon s \kappa_1) \boldsymbol{\tau}_1, \ldots, (1 + \varepsilon s \kappa_{n-1}) \boldsymbol{\tau}_{n-1}, \varepsilon \boldsymbol{n}_N) \tag{34}$$

$$= \varepsilon \Pi_{i=1}^{n-1} |1 + \varepsilon s \kappa_{n-1}|, \tag{35}$$

and we can always assume that ε is sufficiently small so that

$$\varepsilon \Pi_{i=1}^{n-1} |1 + \varepsilon s \kappa_{n-1}| = \varepsilon \Pi_{i=1}^{n-1} (1 + \varepsilon s \kappa_{n-1}). \tag{36}$$

Overall we get that

$$\frac{1}{\varepsilon} (\psi_\varepsilon^* \mathrm{d}x)(\xi, s) = \sqrt{\mathfrak{g}_\varepsilon(\xi, s)}(d\mu_1 \wedge \cdots \wedge d\mu_n) \tag{37}$$

with $\sqrt{\mathfrak{g}_\varepsilon(\xi, s)}$ given by (24). This proves *i*.

(*ii*) The expression of the metric coefficients $\mathfrak{h}_{i,\varepsilon}$ is a simple application of chain rule and (31). Indeed, for every $i = 1, \ldots, n-1$ we have

$$d\boldsymbol{u}_\varepsilon(\xi, s)\,(\boldsymbol{\mu}_i) = d\boldsymbol{v}_\varepsilon\,(\psi_\varepsilon(\xi, s))\,d\psi_\varepsilon(\xi, s)\boldsymbol{\mu}_i = (1 + \varepsilon s \kappa_i)\,d\boldsymbol{v}_\varepsilon\,(\psi_\varepsilon(\xi, s))\,\boldsymbol{\tau}_i \tag{38}$$

from which the relations in (26) follow.

(*iii*) To compute the metric coefficients $\mathfrak{h}_{n,\varepsilon}$, we proceed as in *ii* using (32). We have

$$d\boldsymbol{u}_\varepsilon(\xi, s)\,(\boldsymbol{\mu}_n) = \varepsilon d\boldsymbol{v}_\varepsilon(\psi_\varepsilon(\xi, s))\boldsymbol{n}_N \tag{39}$$

from which (27) follows. $\qquad\qquad\square$

The previous Lemma 1 allows us to fix the domain of integration and gives an equivalent representation of $\mathcal{G}_\varepsilon$ from an integral functional defined on Ω_ε to an integral functional on $\mathcal{N} = N \times I$.

By coarea formula and (24) from Lemma 1 we get that

$$\mathcal{G}_\varepsilon\,(\boldsymbol{v}_\varepsilon) = \frac{1}{2\varepsilon} \int_{\Omega_\varepsilon} |D\boldsymbol{v}_\varepsilon(x) + \mathsf{K}\,(\boldsymbol{v}_\varepsilon(x))|^2\,\mathrm{d}x \tag{40}$$

$$= \frac{1}{2\varepsilon} \int_I \int_{N_{\varepsilon s}} |D\boldsymbol{v}_\varepsilon(y) + \mathsf{K}\,(\boldsymbol{v}_\varepsilon(y))|^2\,\mathrm{d}\mathcal{H}^{n-1}(y)\mathrm{d}s$$

$$= \frac{1}{2} \int_{\mathcal{N}} |D\boldsymbol{v}_\varepsilon \circ \psi_\varepsilon(\xi, s) + \mathsf{K}\,((\boldsymbol{v}_\varepsilon \circ \psi_\varepsilon)\,(\xi, s))|^2\,\sqrt{\mathfrak{g}_\varepsilon\,(\xi, s)}\mathrm{d}\mathcal{H}^{n-1}(\xi)\mathrm{d}s. \tag{41}$$

Next, projecting the gradient onto the orthonormal (moving) frame induced by N, i.e., on

$$(\boldsymbol{\tau}_1(\xi), \boldsymbol{\tau}_2(\xi), \ldots, \boldsymbol{\tau}_{n-1}(\xi), \boldsymbol{n}_N(\xi)) \tag{42}$$

we get that for any $x \in \Omega_\varepsilon$ there holds

$$|D\boldsymbol{v}_\varepsilon(x) + \mathsf{K}\,(\boldsymbol{v}_\varepsilon(x))|^2 = \sum_{i=1}^{n-1} |\partial_{\boldsymbol{\tau}_i(\xi)}\boldsymbol{v}_\varepsilon(x) + \mathsf{K}\,(\boldsymbol{v}_\varepsilon(x))\,\boldsymbol{\tau}_i(\xi)|^2 \tag{43}$$

$$+ |\partial_{\boldsymbol{n}_N(\xi)}\boldsymbol{v}_\varepsilon(x) + \mathsf{K}\,(\boldsymbol{v}_\varepsilon(x))\,\boldsymbol{n}_N(\xi)|^2\,, \tag{44}$$

with $\xi = \pi_N(x)$ the nearest point projection of x on N defined by (10). Therefore, from (26) and (27) we have that

$$|D\boldsymbol{v}_\varepsilon \circ \psi_\varepsilon(\xi, s) + \mathsf{K}\,((\boldsymbol{v}_\varepsilon \circ \psi_\varepsilon)\,(\xi, s))|^2 = \sum_{i=1}^{n-1} |\mathfrak{h}_{i,\varepsilon}(\xi, s)\partial_{\boldsymbol{\tau}_i(\xi)}\boldsymbol{u}_\varepsilon(\xi, s) + \mathsf{K}\,(\boldsymbol{u}_\varepsilon(\xi, s))\,\boldsymbol{\tau}_i(\xi)|^2$$

$$+ \left|\frac{1}{\varepsilon}\partial_s \boldsymbol{u}_\varepsilon(\xi, s) + \mathsf{K}\,(\boldsymbol{u}_\varepsilon(\xi, s))\,\boldsymbol{n}_N(\xi)\right|^2. \tag{45}$$

Combining (41) and (45), we infer the equality of $\mathcal{G}_\varepsilon\,(\boldsymbol{v}_\varepsilon)$ and $\mathcal{E}_N^\varepsilon(\boldsymbol{u}_\varepsilon)$. Note that the previous computation also shows that $\boldsymbol{v}_\varepsilon \in H^1(\Omega_\varepsilon, M)$ if, and only if, $\boldsymbol{u}_\varepsilon \in H^1(\mathcal{N}, M)$. Finally, since the superposition operator $\boldsymbol{v}_\varepsilon \in H^1(\Omega_\varepsilon, M) \mapsto (\boldsymbol{v}_\varepsilon \circ \psi_\varepsilon) \in H^1(\mathcal{N}, M)$ is surjective, we get that

$$\inf_{\boldsymbol{v}_\varepsilon \in H^1(\Omega_\varepsilon, M)} \mathcal{G}_\varepsilon\,(\boldsymbol{v}_\varepsilon) = \inf_{\boldsymbol{u}_\varepsilon \in H^1(\mathcal{N}, M)} \mathcal{E}_N^\varepsilon\,(\boldsymbol{u}_\varepsilon). \tag{46}$$

This concludes the proof of the first part of Theorem 1.

3.2. Compactness. We now show that the family $(\mathcal{E}_{\mathcal{N}}^{\varepsilon})_{\varepsilon \in I_{\delta+}}$ is equicoercive in the weak topology of $H^1(\mathcal{N}, M)$. This means, by definition, that there exists a nonempty and weakly compact set $K \subseteq H^1(\mathcal{N}, M)$ such that

$$\min_{H^1(\mathcal{N},M)} \mathcal{E}_{\mathcal{N}}^{\varepsilon} = \min_{K} \mathcal{E}_{\mathcal{N}}^{\varepsilon} \quad \forall \varepsilon \in I_{\delta+}. \tag{47}$$

The equicoercivity of $(\mathcal{E}_{\mathcal{N}}^{\varepsilon})_{\varepsilon \in I_{\delta+}}$ will assure that we can rely on the fundamental theorem of Γ-convergence concerning the variational convergence of minimum problems (see, e.g., [12]). To show (47), we observe that for any constant in space $\sigma \in H^1(\mathcal{N}, M)$ we have

$$\min_{\boldsymbol{u} \in H^1(\mathcal{N},M)} \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{u}) \leqslant \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\sigma) = \frac{1}{2} \int_{\mathcal{N}} |\mathsf{K}(\sigma)|^2 \sqrt{\mathfrak{g}_{\varepsilon}(\xi, s)} \mathrm{d}\xi \mathrm{d}s. \tag{48}$$

Taking into account (13) and (17), we end up with

$$\min_{\boldsymbol{u} \in H^1(\mathcal{N},M)} \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{u}) \leqslant \frac{c_{\mathsf{K}}^2}{2} \int_{\mathcal{N}} \sqrt{\mathfrak{g}_{\varepsilon}(\xi, s)} \mathrm{d}\xi \mathrm{d}s \leqslant c_{\mathsf{K}}^2 \cdot c_{\mathcal{N}} \cdot |N|, \tag{49}$$

Therefore, for every $\varepsilon \in I_{\delta+}$, the minimizers of $(\mathcal{E}_{\mathcal{N}}^{\varepsilon})_{\varepsilon \in I_{\delta+}}$ are in

$$K(\mathcal{N}, M) := \bigcup_{\varepsilon \in I_{\delta+}} \left\{ \boldsymbol{u} \in H^1(\mathcal{N}, M) : \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{u}) \leqslant c_{\mathsf{K}}^2 \cdot c_{\mathcal{N}} \cdot |N| \right\}. \tag{50}$$

Also, since the principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$ are bounded on N, whenever the radius $\delta \in \mathbb{R}_+$ of the tubular neighborhood Ω_{δ} is sufficiently small, we have that $\inf_{(\xi,s) \in \mathcal{N}} \mathfrak{h}_{i,\varepsilon}(\xi, s) \geqslant c_{\mathcal{N}}$ for every $\varepsilon \in I_{\delta+}$ (cf. (13)). Therefore, from Young inequality, we get

$$\left| \mathfrak{h}_{i,\varepsilon}(\xi, s) \partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{u}_{\varepsilon}(\xi, s) + \mathsf{K}(\boldsymbol{u}_{\varepsilon}(\xi, s)) \, \boldsymbol{\tau}_i(\xi) \right|^2 \geqslant \frac{1}{2c_{\mathcal{N}}^2} \left| \partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{u}_{\varepsilon}(\xi, s) \right|^2 - \left| \mathsf{K}(\boldsymbol{u}_{\varepsilon}(\xi, s)) \, \boldsymbol{\tau}_i(\xi) \right|^2, \tag{51}$$

and

$$\left| \frac{1}{\varepsilon} \partial_s \boldsymbol{u}_{\varepsilon}(\xi, s) + \mathsf{K}(\boldsymbol{u}_{\varepsilon}(\xi, s)) \, \boldsymbol{n}_N(\xi) \right|^2 \geqslant \frac{1}{2\varepsilon^2} \left| \partial_s \boldsymbol{u}_{\varepsilon}(\xi, s) \right|^2 - \left| \mathsf{K}(\boldsymbol{u}_{\varepsilon}(\xi, s)) \, \boldsymbol{n}_N(\xi) \right|^2. \tag{52}$$

Therefore

$$\sum_{i=1}^{n-1} \frac{1}{2c_{\mathcal{N}}^2} |\partial_{\boldsymbol{\tau}_i} \boldsymbol{u}_{\varepsilon}|^2 + \frac{1}{2\varepsilon^2} |\partial_s \boldsymbol{u}_{\varepsilon}|^2 \leqslant +c_{\mathcal{N}} \sum_{i=1}^{n-1} |\mathfrak{h}_{i,\varepsilon} \partial_{\boldsymbol{\tau}_i} \boldsymbol{u}_{\varepsilon} + \mathsf{K}(\boldsymbol{u}_{\varepsilon}) \, \boldsymbol{\tau}_i|^2 \sqrt{\mathfrak{g}_{\varepsilon}}$$
$$+ c_{\mathcal{N}} \left| \frac{1}{\varepsilon} \partial_s \boldsymbol{u}_{\varepsilon} + \mathsf{K}(\boldsymbol{u}_{\varepsilon}) \, \boldsymbol{n}_N \right|^2 \sqrt{\mathfrak{g}_{\varepsilon}} + c_{\mathcal{N}} c_{\mathsf{K}}^2. \tag{53}$$

It follows that if $\boldsymbol{u} \in K(\mathcal{N}, M)$, then

$$\frac{1}{2c_{\mathcal{N}}^2} \int_{\mathcal{N}} |\nabla_{\xi} \boldsymbol{u}_{\varepsilon}(\xi, s)|^2 \, \mathrm{d}\xi \mathrm{d}s + \frac{1}{2\varepsilon^2} \int_{\mathcal{N}} |\partial_s \boldsymbol{u}_{\varepsilon}(\xi, s)|^2 \, \mathrm{d}\xi \mathrm{d}s \leqslant 2c_{\mathcal{N}} \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{u}) + c_{\mathcal{N}} c_{\mathsf{K}}^2$$
$$\leqslant c_{\mathsf{K}}^2 \left(2c_{\mathcal{N}}^2 |N| + c_{\mathcal{N}} \right), \tag{54}$$

and, in particular,

$$\|\boldsymbol{u}\|_{H^1(\mathcal{N},M)}^2 \leqslant c_{\mathsf{K},N}^2 \tag{55}$$

for some positive constant $c_{\mathsf{K},N}$ that does not depend on ε. In other words, the set $K(\mathcal{N}, M)$ is contained in the bounded subset $H_b^1(\mathcal{N}, M)$ of $H^1(\mathcal{N}, \mathbb{R}^m)$ given by the intersection of $H^1(\mathcal{N}, M)$ with the ball of $H^1(\mathcal{N}, \mathbb{R}^m)$ centered at the origin and of radius $c_{\mathsf{K},N}$. Thus, for any $\varepsilon \in I_{\delta+}$

$$\min_{\boldsymbol{u} \in H^1(\mathcal{N},M)} \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{u}) = \min_{\boldsymbol{u} \in H_b^1(\mathcal{N},M)} \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{u}). \tag{56}$$

To prove that $H_b^1(\mathcal{N}, M)$ is weakly compact, it is sufficient to show that the set $H_b^1(\mathcal{N}, M)$ is weakly closed. To this end, we simply observe that if $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ is a sequence in $H_b^1(\mathcal{N}, M)$ such that $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}_0$ weakly in $H^1(\mathcal{N}, \mathbb{R}^m)$, by Rellich-Kondrachov theorem, $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}_0$ strongly in $L^2(\mathcal{N}, \mathbb{R}^m)$, and therefore, up to the extraction of an a.e. pointwise converging subsequence, we get that the $\boldsymbol{u}_0$ still takes values in M. Indeed, the distance function from the boundary is a continuous function and, therefore, if $d_M(\boldsymbol{u}_n) = 0$, then also $d_M(\boldsymbol{u}_0) = 0$. Thus $\boldsymbol{u}_0(\xi, s) \in M$ for a.e. $(\xi, s) \in \mathcal{N}$ and this concludes the equicoerciveness proof.

3.3. The identification of the Γ-limit. In this section, we compute $\mathcal{E}_{\mathcal{N}} := \Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{E}_{\mathcal{N}}^\varepsilon$. We first establish a compactness result (cf. Proposition 1). Afterward, in Proposition 2, we prove the Γ-limsup inequality, i.e., the existence of a recovery sequence, and the Γ-liminf inequality for $(\mathcal{E}_{\mathcal{N}}^\varepsilon)_{\varepsilon\in I_{\delta+}}$, i.e., that for any family $(\boldsymbol{u}_\varepsilon)_{\varepsilon\in I_{\delta+}}$ weakly convergent to some $\boldsymbol{u}_0 \in H^1(\mathcal{N}, M)$ we have $\mathcal{E}_{\mathcal{N}}(\boldsymbol{u}_0) \leqslant \liminf_{\varepsilon\to 0}\mathcal{E}_{\mathcal{N}}^\varepsilon(\boldsymbol{u}_\varepsilon)$. With no loss of generality, for the computation of Γ-liminf we can assume that $\liminf_{\varepsilon\to 0}\mathcal{E}_{\mathcal{N}}^\varepsilon(\boldsymbol{u}_\varepsilon) < +\infty$.

Proposition 1. *Assume that $(\boldsymbol{u}_\varepsilon)_{\varepsilon\in I_{\delta+}}$ is a family in $H^1(\mathcal{N}, M)$ such that*

$$\liminf_{\varepsilon\to 0}\mathcal{E}_{\mathcal{N}}^\varepsilon(\boldsymbol{u}_\varepsilon) < +\infty.$$

Then there exist elements $\boldsymbol{u}_0 \in H^1(\mathcal{N}, M)$ and $\boldsymbol{d}_0 \in L^2(\mathcal{N}, \mathbb{R}^m)$ such that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0 \quad \text{weakly in } H^1(\mathcal{N}, M), \tag{57}$$

$$\frac{1}{\varepsilon}\partial_s\boldsymbol{u}_\varepsilon(\xi, s) \to \boldsymbol{d}_0 \quad \text{weakly in } L^2(\mathcal{N}, \mathbb{R}^m). \tag{58}$$

Moreover, $\boldsymbol{u}_0$ is 0-homogeneous along the normal to N, i.e., it is an element $\boldsymbol{u}_0$ of the form

$$\boldsymbol{u}_0(\xi, s) = \widetilde{\boldsymbol{u}}_0(\xi), \quad \text{for a.e. } (\xi, s) \in \mathcal{N} \tag{59}$$

for some $\widetilde{\boldsymbol{u}}_0 \in H^1(N, M)$, and $\boldsymbol{d}_0$ is tangent to M at $\boldsymbol{u}_0$, i.e.,

$$\boldsymbol{d}_0(\xi, s) \cdot \boldsymbol{n}_M(\boldsymbol{u}_0(\xi)) = 0 \quad \text{for a.e. } (\xi, s) \in \mathcal{N}. \tag{60}$$

Proof. Using (13), (18), and (52), we obtain

$$+\infty > \liminf_{\varepsilon\to 0}\mathcal{E}_{\mathcal{N}}^\varepsilon(\boldsymbol{u}_\varepsilon) \geqslant \liminf_{\varepsilon\to 0}\frac{1}{2}\int_{\mathcal{N}}\left|\frac{1}{\varepsilon}\partial_s\boldsymbol{u}_\varepsilon(\xi, s) + \mathsf{K}(\boldsymbol{u}_\varepsilon(\xi, s))\,\boldsymbol{n}_N(\xi)\right|^2\sqrt{\mathfrak{g}_\varepsilon(\xi, s)}\mathrm{d}\xi\mathrm{d}s,$$

$$\geqslant \liminf_{\varepsilon\to 0}\frac{c_{\mathcal{N}}^{-1}}{4\varepsilon^2}\int_{\mathcal{N}}|\partial_s\boldsymbol{u}_\varepsilon(\xi, s)|^2\,\mathrm{d}\xi\mathrm{d}s - \frac{1}{2}c_{\mathcal{N}}^{-1}c_{\mathsf{K}}^2. \tag{61}$$

From the previous estimate, (57) and (58) immediately follows. Also, by the lower semicontinuity of the norm, we get that $0 = \lim_{\varepsilon\to 0}\|\partial_s\boldsymbol{u}_\varepsilon\|_{L^2(\mathcal{N})} = \|\partial_s\boldsymbol{u}_0\|_{L^2(\mathcal{N})}$. Since we also have $\partial_s\boldsymbol{u}_\varepsilon \rightharpoonup \partial_s\boldsymbol{u}_0$ in $\mathcal{D}'(\mathcal{N})$, overall, we get that

$$\partial_s\boldsymbol{u}_\varepsilon \to \partial_s\boldsymbol{u}_0(\xi, s) \text{ strongly in } L^2(\mathcal{N}, \mathbb{R}^m), \quad \partial_s\boldsymbol{u}_0(\xi, s) = 0 \text{ a.e. in } \mathcal{N}. \tag{62}$$

This proves (59).

To show (60), we denote by d_M the signed distance from M, which in our hypotheses on M is such that $\nabla d_M(\sigma) = \boldsymbol{n}_M(\sigma)$. We then have $0 = \partial_s(d_M(\boldsymbol{u}_\varepsilon)) = \boldsymbol{n}_M(\boldsymbol{u}_\varepsilon) \cdot \partial_s\boldsymbol{u}_\varepsilon$ with $\varepsilon^{-1}\partial_s\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{d}_0$ weakly in $L^2(\mathcal{N}, \mathbb{R}^m)$ and $\boldsymbol{n}_M(\boldsymbol{u}_\varepsilon) \to \boldsymbol{n}_M(\boldsymbol{u}_0)$ strongly in $L^2(M, \mathbb{R}^m)$. This shows (60). $\qquad\square$

Proposition 2. *The family* $(\mathcal{E}_{\mathcal{N}}^{\varepsilon})_{\varepsilon \in I_{\delta+}}$ *of exchange energy on* $\mathcal{N}$, Γ-*converges, with respect to the weak topology of* $H^1(\mathcal{N}, M)$, *to the functional* (19), *i.e., to the functional* $\mathcal{E}_{\mathcal{N}}$ *defined on* $H^1(\mathcal{N}, M)$ *by*

$$\mathcal{E}_{\mathcal{N}}(\boldsymbol{u}) = \int_N |\nabla_{\xi} \boldsymbol{u}(\xi) + \mathsf{K}_{\xi}(\boldsymbol{u}(\xi))|^2 + (\mathsf{K}(\boldsymbol{u}(\xi))\boldsymbol{n}_N(\xi) \cdot \boldsymbol{n}_M(\boldsymbol{u}(\xi)))^2 \,\mathrm{d}\xi \tag{63}$$

if $\boldsymbol{u}$ *does not depend on the* s-*variable, and* $\mathcal{E}_{\mathcal{N}}(\boldsymbol{u}) = +\infty$ *otherwise.*

Proof. We split the proof into two steps. We start by addressing the Γ-$\liminf$ inequality for $(\mathcal{E}_{\mathcal{N}}^{\varepsilon})_{\varepsilon \in I_{\delta+}}$.

Step 1 [Γ-liminf inequality]. By Proposition 1, we have that for the identification of the Γ-limit of $(\mathcal{E}_{\mathcal{N}}^{\varepsilon})_{\varepsilon \in I_{\delta}}$ it is sufficient to consider those families of $H^1(\mathcal{N}, M)$ functions that weakly converge to a 0-homogeneous element $\boldsymbol{u}_0 \in H^1(\mathcal{N}, M)$, i.e., to an element $\boldsymbol{u}_0$ of the form $\boldsymbol{u}_0(\xi, s) = \chi_I(s) \widetilde{\boldsymbol{u}}_0(\xi)$ for some $\widetilde{\boldsymbol{u}}_0 \in H^1(N, M)$, i.e., not depending on the s variable. In the following, with a slight abuse of notation, we write $\boldsymbol{u}_0(\xi)$ instead of $\widetilde{\boldsymbol{u}}_0(\xi)$.

Taking into account the lower semicontinuity of the norm, for any $\boldsymbol{u}_{\varepsilon} \rightharpoonup \boldsymbol{u}_0$ in $H^1(\mathcal{N}, M)$, with $\boldsymbol{u}_0$ of the type (59), we get (cf. (18))

$$\liminf_{\varepsilon \to 0} \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{u}_{\varepsilon}) \geqslant \sum_{i=1}^{n-1} \int_N |\partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{u}_0(\xi) + \mathsf{K}(\boldsymbol{u}_0(\xi)) \,\boldsymbol{\tau}_i(\xi)|^2 \,\mathrm{d}\xi$$
$$+ \frac{1}{2} \int_{N \times I} |\boldsymbol{d}_0(\xi, s) + \mathsf{K}(\boldsymbol{u}_0(\xi)) \,\boldsymbol{n}_N(\xi)|^2 \,\mathrm{d}\xi \mathrm{d}s.$$

To shorten the notation, it is convenient to introduce the following notation. For every $\sigma \in M$ we denote by $\mathsf{K}_{\xi}(\sigma)$ the restriction of $\mathsf{K}(\sigma)$ to the tangent space of N at ξ represented by the matrix $\mathsf{K}_{\xi}(\sigma) = (\mathsf{K}(\sigma) \,\boldsymbol{\tau}_1(\xi), \ldots, \mathsf{K}(\sigma) \,\boldsymbol{\tau}_{n-1}(\xi)) \in \mathbb{R}^{(n-1) \times m}$. The previous estimate then reads as

$$\liminf_{\varepsilon \to 0} \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{u}_{\varepsilon}) \geqslant \int_N |\nabla_{\xi} \boldsymbol{u}_0(\xi) + \mathsf{K}_{\xi}(\boldsymbol{u}_0(\xi))|^2 \,\mathrm{d}\xi + \mathsf{R}[\boldsymbol{d}_0] \tag{64}$$

with

$$\mathsf{R}[\boldsymbol{d}_0] := \frac{1}{2} \int_{N \times I} |\boldsymbol{d}_0(\xi, s) + \mathsf{K}(\boldsymbol{u}_0(\xi)) \,\boldsymbol{n}_N(\xi)|^2 \,\mathrm{d}\xi \mathrm{d}s. \tag{65}$$

Next, we minimize (pointwise) the integrand of $\mathsf{R}[\boldsymbol{d}_0]$ *under the constraint that* $\boldsymbol{d}_0(\xi, s) \cdot \boldsymbol{n}_M(\boldsymbol{u}_0(\xi)) = 0$, i.e., we look for

$$\min_{\boldsymbol{d}_0 \cdot \boldsymbol{n}_M(\boldsymbol{u}_0) = 0} |\boldsymbol{d}_0 + \mathsf{K}(\boldsymbol{u}_0)\boldsymbol{n}_N|^2. \tag{66}$$

This leads to the Euler–Lagrange equation $\boldsymbol{d}_0 + \mathsf{K}(\boldsymbol{u}_0)\boldsymbol{n}_N = (\mathsf{K}(\boldsymbol{u}_0)\boldsymbol{n}_N \cdot \boldsymbol{n}_M(\boldsymbol{u}_0)) \,\boldsymbol{n}_M(\boldsymbol{u}_0)$ which, in particular, shows that for every $(\xi, s) \in \mathcal{N}$ the minimal vector $\boldsymbol{d}_0(\xi, s)$ does not depend on the s variable and is given by

$$\boldsymbol{d}_0(\xi) := ([\boldsymbol{n}_M(\boldsymbol{u}_0) \otimes \boldsymbol{n}_M(\boldsymbol{u}_0)] - \mathsf{I}) \,\mathsf{K}(\boldsymbol{u}_0)\boldsymbol{n}_N, \tag{67}$$

with I denoting the identity operator.

The previous relation leads to the following expression of the minimal energy density

$$|\boldsymbol{d}_0(\xi, s) + \mathsf{K}(\boldsymbol{u}_0(\xi))\boldsymbol{n}_N(\xi)|^2 = \left(\left(\mathsf{K}^{\top}(\boldsymbol{u}_0(\xi))\boldsymbol{n}_M(\boldsymbol{u}_0(\xi)) \right) \cdot \boldsymbol{n}_N(\xi) \right)^2 \tag{68}$$

and, therefore, from (64), we infer that

$$
\liminf_{\varepsilon \to 0} \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{u}_{\varepsilon}) \geqslant \int_{N} |\nabla_{\xi} \boldsymbol{u}_0(\xi) + \mathsf{K}_{\xi}(\boldsymbol{u}_0(\xi))|^2 \, \mathrm{d}\xi + \int_{N} (\mathsf{K}(\boldsymbol{u}_0(\xi)) \, \boldsymbol{n}_N(\xi) \cdot \boldsymbol{n}_M(\boldsymbol{u}_0(\xi)))^2 \, \mathrm{d}\xi.
$$
(69)

This concludes the proof of the Γ-$\liminf$ inequality.

Step 2 [Γ-limsup inequality]. We now address the existence of a recovery sequence, i.e., that the lower bound (69) is indeed optimal. To this end, it is sufficient to show that if $\boldsymbol{u}_0 \in H^1(\mathcal{N}, M)$ is independent of the s-variable, then there exists a sequence $(\boldsymbol{u}_{\varepsilon}^*)_{\varepsilon \in I_{\delta+}}$ of functions in $H^1(\mathcal{N}, M)$ such that $\boldsymbol{u}_{\varepsilon}^* \to \boldsymbol{u}_0$ strongly in $H^1(\mathcal{N}, M)$. Indeed, if this is the case, then

$$
\lim_{\varepsilon \to 0} \mathcal{E}_{\mathcal{N}}^{\varepsilon}(\boldsymbol{u}_{\varepsilon}^*) = \mathcal{E}_{\mathcal{N}}(\boldsymbol{u}_0).
$$
(70)

For $\boldsymbol{u}_0 \in H^1(\mathcal{N}, M)$ and independent of the s-variable, we claim that

$$
\boldsymbol{u}_{\varepsilon}^*(\xi, s) := \pi_M(\boldsymbol{u}_0(\xi) + \varepsilon s \boldsymbol{d}_0(\xi))
$$
(71)

with $\boldsymbol{d}_0$ defined by (67), gives the recovery sequence. Here π_M is the nearest point projection on M defined in (10) and, therefore $\boldsymbol{u}_{\varepsilon}^*(\xi, s)$ is the projection on M of a suitable (and small) perturbation of $\boldsymbol{u}_0$ along the tangent space of M at $\boldsymbol{u}_0(\xi)$.

We point out that $\boldsymbol{u}_{\varepsilon}^*$ is well-defined because for almost every $(\xi, s) \in \mathcal{N}$ we have that

$$
\begin{aligned}
|d_M(\boldsymbol{u}_0(\xi) + \varepsilon s \boldsymbol{d}_0(\xi))| &= |d(\boldsymbol{u}_0(\xi) + \varepsilon s \boldsymbol{d}_0(\xi), M)| \\
&\leqslant |d(\boldsymbol{u}_0(\xi) + \varepsilon s \boldsymbol{d}_0(\xi), \boldsymbol{u}_0(\xi))| \\
&\leqslant \varepsilon |\boldsymbol{d}_0(\xi)|,
\end{aligned}
$$
(72)

with $|\boldsymbol{d}_0(\xi)|$ bounded, uniformly in $\xi \in N$, by some constant which depends only on c_{K}. Therefore, for ε sufficiently small, $\boldsymbol{u}_0(\xi) + \varepsilon s \boldsymbol{d}_0(\xi)$ is in a tubular neighborhood of N where the unique nearest point projection π is defined.

To evaluate $D\boldsymbol{u}_{\varepsilon}^*$ we proceed as follows. For any y in a tubular neighborhood O_{δ} of M we have

$$
y = \pi_M(y) + d_M(y)(\boldsymbol{n}_M \circ \pi_M)(y).
$$
(73)

Therefore, if we set $\tilde{\boldsymbol{n}}_M(y) := (\boldsymbol{n}_M \circ \pi_M)(y)$, given that $\tilde{\boldsymbol{n}}_M = \tilde{\boldsymbol{n}}_M \circ \pi_M$, passing to the tangent maps in the previous relation, we have that $\mathsf{I} = D\pi_M(y) + \tilde{\boldsymbol{n}}_M(y) \otimes \tilde{\boldsymbol{n}}_M(y) + d_M(y)D\tilde{\boldsymbol{n}}_M(y)D\pi_M(y)$ with I the identity operator. Hence, the following relation holds

$$
\mathsf{I} - \tilde{\boldsymbol{n}}_M(y) \otimes \tilde{\boldsymbol{n}}_M(y) = [\mathsf{I} + d_M(y)D\tilde{\boldsymbol{n}}_M(y)] \, D\pi_M(y),
$$
(74)

from which we get that

$$
D\pi_M(y) = [\mathsf{I} + d_M(y)D\tilde{\boldsymbol{n}}_M(y)]^{-1} [\mathsf{I} - \tilde{\boldsymbol{n}}_M(y) \otimes \tilde{\boldsymbol{n}}_M(y)]
$$
(75)

which is continuous in $y \in O_{\delta}$ because of $d_M(y)$ being small. Therefore, with $\boldsymbol{\eta}_{\varepsilon}(\xi, s) := \boldsymbol{u}_0(\xi) + \varepsilon s \boldsymbol{d}_0(\xi)$, we have that

$$
\partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{u}_{\varepsilon}^*(\xi, s) = D\pi_M(\boldsymbol{\eta}_{\varepsilon}(\xi, s)) \, \partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{\eta}_{\varepsilon}(\xi, s),
$$
(76)

$$
\partial_s \boldsymbol{u}_{\varepsilon}^*(\xi, s) = \varepsilon D\pi_M(\boldsymbol{\eta}_{\varepsilon}(\xi, s)) \, \boldsymbol{d}_0(\xi).
$$
(77)

But then, since for every $\sigma \in M$ the map $D\pi_M(\sigma)$ acts as the identity on the tangent space $T_\sigma M$ of M at σ, taking into account (75), we get the following relations

$$\boldsymbol{u}_\varepsilon^*(\xi, s) \to \boldsymbol{u}_0(\xi) \qquad \text{strongly in } L^2(\mathcal{N}, M), \tag{78}$$

$$\nabla_\xi \boldsymbol{u}_\varepsilon^*(\xi, s) \to \nabla_\xi \boldsymbol{u}_0(\xi) \qquad \text{strongly in } L^2\left(\mathcal{N}, (TM)^{n-1}\right), \tag{79}$$

$$\partial_s \boldsymbol{u}_\varepsilon^*(\xi, s) \to 0 \qquad \text{strongly in } L^2(\mathcal{N}, TM), \tag{80}$$

$$\varepsilon^{-1} \partial_s \boldsymbol{u}_\varepsilon^*(\xi, s) \to \boldsymbol{d}_0(\xi) \qquad \text{strongly in } L^2(\mathcal{N}, TM). \tag{81}$$

In particular, we have that $\boldsymbol{u}_\varepsilon^*(\xi, s) \to \boldsymbol{u}_0(\xi)$ strongly in $H^1(\mathcal{N}, M)$ and, therefore, (70) holds. $\qquad \square$

3.4. Strong convergence of minimizers. Assertion (20), as well as the *weak* convergence in $H^1(\mathcal{N}, M)$ of a minimizing family for $(\mathcal{E}_\mathcal{N}^\varepsilon)_{\varepsilon \in I_{\delta+}}$ to a minimum point of $\mathcal{E}_\mathcal{N}$, is nothing but the fundamental theorem of Γ-convergence concerning the variational convergence of minimum problems (cf. [12]), which holds under the equicoercivity result we proved in Subsection 3.2. Therefore, it remains to prove that the convergence is (up to a subsequence) *strong* in $H^1(\mathcal{N}, M)$.

To this end, we observe that, by assumptions, there exists $\boldsymbol{u}_0 \in H^1(\mathcal{N}, M)$, not depending on the s-variable, such that $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}_0$ weakly in $H^1(\mathcal{N}, M)$ and $\partial_s \boldsymbol{u}_\varepsilon \to 0$ strongly in $L^2(\mathcal{N}, TM)$. By the Γ-liminf inequality, the strong convergence of $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0$ in $L^2(\mathcal{N}, M)$, and the minimality of $\boldsymbol{u}_\varepsilon$, we get that

$$\mathcal{E}_\mathcal{N}(\boldsymbol{u}_0) \leqslant \liminf_{\varepsilon \to 0} \mathcal{E}_\mathcal{N}^\varepsilon(\boldsymbol{u}_\varepsilon) \leqslant \limsup_{\varepsilon \to 0} \mathcal{E}_\mathcal{N}^\varepsilon(\boldsymbol{u}_\varepsilon) \leqslant \lim_{\varepsilon \to 0} \mathcal{E}_\mathcal{N}^\varepsilon(\boldsymbol{u}_\varepsilon^*) = \mathcal{E}_\mathcal{N}(\boldsymbol{u}_0), \tag{82}$$

where $(\boldsymbol{u}_\varepsilon^*)_{\varepsilon \in I_{\delta+}}$ is the family built from $\boldsymbol{u}_0$ as in (71). Therefore, we conclude that if $(\boldsymbol{u}_\varepsilon)_{\varepsilon \in I_{\delta+}}$ is a minimizing family associated with $(\mathcal{E}_\mathcal{N}^\varepsilon)_{\varepsilon \in I_{\delta+}}$, then $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}_0$ weakly in $H^1(\mathcal{N}, M)$ for some $\boldsymbol{u}_0$ which does not depend on the s-variable and, moreover, $\lim_{\varepsilon \to 0} \mathcal{E}_\mathcal{N}^\varepsilon(\boldsymbol{u}_\varepsilon) = \mathcal{E}_\mathcal{N}(\boldsymbol{u}_0)$. But this last relation, in expanded form, is equivalent to the relation

$$\frac{1}{2} \int_\mathcal{N} |\nabla_\xi \boldsymbol{u}_0(\xi) + \mathsf{K}_\xi(\boldsymbol{u}_0(\xi))|^2 \, \mathrm{d}\xi \mathrm{d}s = \lim_{\varepsilon \to 0} \frac{1}{2} \int_\mathcal{N} |\nabla_\xi \boldsymbol{u}_\varepsilon(\xi, s) + \mathsf{K}_\xi(\boldsymbol{u}_0(\xi))|^2 \, \mathrm{d}\xi \mathrm{d}s. \tag{83}$$

From (83) we deduce the convergence of the norms $\|\boldsymbol{u}_\varepsilon\|_{H^1(\mathcal{N}, M)} \to \|\boldsymbol{u}_0\|_{H^1(\mathcal{N}, M)}$ which together with the weak convergence $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}_0$ in $H^1(\mathcal{N}, M)$, assures strong convergence in $H^1(\mathcal{N}, M)$.

4. APPLICATIONS TO MICROMAGNETICS

Because of the growing interest in spintronics applications, magnetic skyrmions are currently the focus of considerable research activity ranging from mathematics to physics and materials science. These chiral structures can be found in a wide range of experimental conditions. This section demonstrates how our curved thin film analysis can account for various situations that might arise when ferromagnetic crystals lack inversion symmetry, and the Dzyaloshinskii-Moriya interaction (DMI) can twist the otherwise uniform ferromagnetic state.

Generally speaking, due to material defects or anisotropy in saturation magnetization [47], the loss of the $\mathbb{S}^2$-valued constraint is possible. As we show below (cf. (86)), even in the planar setting, extra contributions to those reported in [13] must be taken into consideration every time the magnetization vector is not precisely $\mathbb{S}^2$-valued. A specific instance of our findings provides a model for curved thin films in the presence of bulk DMI.

In addition, we analyze the curved thin-film limit associated with interfacial DMI. This effect occurs when two ferromagnetic materials with different crystal structures are separated by an interface.

Finally, we consider the case when the temperature of the ferromagnet is not uniform and, as a result, the (temperature-dependent) saturation magnetization cannot be assumed constant. We show how to use Theorem 2 to cover this situation.

4.1. Isotropic Bulk DMI. Isotropic bulk DMI arises when Dzyaloshinskii vectors $(\boldsymbol{d}_i)_{i=1}^3$ in (6) are the elements of the standard basis $(\boldsymbol{e}_i)_{i=1}^3$ of $\mathbb{R}^3$. In this circumstance, the perturbation K takes the form $\mathsf{K}(\boldsymbol{m}) = \kappa(\bullet \times \boldsymbol{m})$, $\kappa \in \mathbb{R}$. Theorem 1 from [13] is about this setting in the particular case where N is a planar surface and $M = \mathbb{S}^2$. More generally, when M, N are two-dimensional surfaces in $\mathbb{R}^3$, the energy functional (1) associated with the isotropic bulk DMI reads as

$$\mathcal{G}_\varepsilon(\boldsymbol{m}) := \frac{1}{2\varepsilon} \sum_{i=1}^3 \int_{\Omega_\varepsilon} |\mathfrak{d}_i \boldsymbol{m}(x)|^2 \, \mathrm{d}x, \tag{84}$$

and is defined for every $\boldsymbol{m} \in H^1(\Omega_\varepsilon, M)$. Here, for $i = 1, 2, 3$, the quantities $\mathfrak{d}_i \boldsymbol{m} := \partial_i \boldsymbol{m} + \kappa(\boldsymbol{e}_i \times \boldsymbol{m})$ are usually referred to as *helical derivatives*. The curved thin-film limit, as identified in Theorem 1, is given by

$$\mathcal{E}_N(\boldsymbol{m}) := \int_N |\nabla_\xi \boldsymbol{m}(\xi) + \mathsf{K}_\xi(\boldsymbol{m}(\xi))|^2 + (\mathsf{K}(\boldsymbol{m}(\xi))\boldsymbol{n}_N(\xi) \cdot \boldsymbol{n}_M(\boldsymbol{m}(\xi)))^2 \, \mathrm{d}\xi$$

$$= \sum_{i=1}^2 \int_N |\partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{m}(\xi) + \kappa(\boldsymbol{\tau}_i(\xi) \times \boldsymbol{m}(\xi))|^2 \, \mathrm{d}\xi$$

$$+ \kappa^2 \int_N ((\boldsymbol{n}_M(\boldsymbol{m}(\xi)) \times \boldsymbol{m}(\xi)) \cdot \boldsymbol{n}_N(\xi))^2 \, \mathrm{d}\xi. \tag{85}$$

When $N \subseteq \mathbb{R}^2 \times \{0\}$ is a planar surface (cf. Remark 3), one has $\boldsymbol{n}_N(\xi) = \boldsymbol{e}_3$ and taking the tangential derivatives in the direction of the standard basis of $\mathbb{R}^2$, one obtains from (85) that

$$\mathcal{E}_N(\boldsymbol{m}) = \sum_{i=1}^2 \int_N |\mathfrak{d}_i \boldsymbol{m}(\xi)|^2 \, \mathrm{d}\xi + \kappa^2 \int_N ((\boldsymbol{m}(\xi) \times \boldsymbol{n}_M(\boldsymbol{m}(\xi))) \cdot \boldsymbol{e}_3)^2 \, \mathrm{d}\xi. \tag{86}$$

Compared to Theorem 1 from [13], we observe here that if $\boldsymbol{m}$ is $\mathbb{S}^2$-valued, i.e., if $M = \mathbb{S}^2$, then $\boldsymbol{m}(\xi) \times \boldsymbol{n}_{\mathbb{S}^2}(\boldsymbol{m}(\xi)) = \boldsymbol{m}(\xi) \times \boldsymbol{m}(\xi) = 0$, and the limiting energy, as we already know, reduces to

$$\mathcal{E}_N(\boldsymbol{m}) = \sum_{i=1}^2 \int_N |\mathfrak{d}_i \boldsymbol{m}(\xi)|^2 \, \mathrm{d}\xi. \tag{87}$$

In other words, the second term in (86) vanishes for $\mathbb{S}^2$-valued vector fields defined on a planar surface. Actually, the second term in (86) also vanishes when $\boldsymbol{m}$ is defined on a closed surface N provided that $\boldsymbol{m}$ is still $\mathbb{S}^2$-valued. However, and this is an important point, when for physical reasons $\boldsymbol{m}$ cannot be considered $\mathbb{S}^2$-valued (e.g., because of anisotropy in the saturation magnetization), then a correction must be taken into account, and this is expressed by the second term in (86).

4.2. Isotropic interfacial DMI. Let M, N be two-dimensional surfaces in $\mathbb{R}^3$. We are interested in the case where the interfacial DMI occurs in the direction of the normal to N. For that, we observe that in terms of the moving frame

$$(\boldsymbol{\tau}_1(\xi), \boldsymbol{\tau}_2(\xi), \boldsymbol{\tau}_3(\xi) := \boldsymbol{n}_N(\xi)), \tag{88}$$

the expression of $\mathcal{H}_{\mathrm{DM}}$ in (6) can be written under the form

$$\mathcal{H}_{\mathrm{DM}}\left(\boldsymbol{m}\right) = \sum_{i=1}^{3} \int_{\Omega} \partial_{\tau_i(\xi)} \boldsymbol{m}(x) \cdot \left(\boldsymbol{d}_i(\xi) \times \boldsymbol{m}(x)\right) \mathrm{d}x \tag{89}$$

with $\xi := \pi_N(x)$ and the Dzyaloshinskii vectors $(\boldsymbol{d}_i(\xi))_{i=1}^{3}$ depending on ξ. Hence, up to an additive term, uninfluential for our purposes, the expression in (7) can be rewritten under the form

$$\mathcal{H}_{\Omega}\left(\boldsymbol{m}\right) = \frac{1}{2} \sum_{i=1}^{3} \int_{\Omega} |\partial_{\tau_i(\xi)} \boldsymbol{m}(x) + \boldsymbol{d}_i(\xi) \times \boldsymbol{m}(x)|^2 \mathrm{d}x \tag{90}$$

$$= \frac{1}{2} \sum_{i=1}^{3} \int_{\Omega} |\mathrm{D}\boldsymbol{m}(x)T(\xi) + \mathsf{M}\left(\xi, \boldsymbol{m}(x)\right)|^2 \mathrm{d}x\,, \tag{91}$$

with $T(\xi)$ the orthogonal operator mapping the standard basis of $\mathbb{R}^3$ into the moving frame $(\boldsymbol{\tau}_1(\xi), \boldsymbol{\tau}_2(\xi), \boldsymbol{\tau}_3(\xi) := \boldsymbol{n}_N(\xi))$ and $\mathsf{M}\left(\xi, \boldsymbol{m}\right) = (\boldsymbol{d}_1(\xi) \times \boldsymbol{m}, \boldsymbol{d}_2(\xi) \times \boldsymbol{m}, \boldsymbol{d}_3(\xi) \times \boldsymbol{m})$. In the presence of curved interfacial DMI, the Dzyaloshinskii vectors have the form

$$\boldsymbol{d}_i(\xi) := \kappa\left(\boldsymbol{n}_N(\xi) \times \boldsymbol{\tau}_i(\xi)\right) \tag{92}$$

so that (note that $\boldsymbol{d}_3(\xi) = \boldsymbol{0}$) for $i = 1, 2$, there holds

$$\boldsymbol{d}_i(\xi) \times \boldsymbol{m}(x) = \kappa\left[(\boldsymbol{m}(x) \cdot \boldsymbol{n}_N(\xi)) \boldsymbol{\tau}_i(\xi) - (\boldsymbol{m}(x) \cdot \boldsymbol{\tau}_i(\xi))\boldsymbol{n}_N(\xi)\right]. \tag{93}$$

Equivalently, we have $\mathsf{M}\left(\xi, \boldsymbol{m}\right) = \mathsf{K}\left(\xi, \boldsymbol{m}\right) \circ T(\xi)$ with $\mathsf{K}\left(\xi, \boldsymbol{m}\right)$ being the matrix-valued function

$$\mathsf{K}\left(\xi, \boldsymbol{m}\right) := \kappa\left[(\bullet \otimes \boldsymbol{n}_N(\xi)) - (\boldsymbol{n}_N(\xi) \otimes \bullet)\right] \boldsymbol{m}. \tag{94}$$

Overall, from (91) and (94) and the invariance of the norm under the orthogonal group, we get that

$$\mathcal{H}_{\Omega}\left(\boldsymbol{m}\right) = \frac{1}{2} \sum_{i=1}^{3} \int_{\Omega} |\mathrm{D}\boldsymbol{m}(x) + \mathsf{K}\left(\xi, \boldsymbol{m}\right)|^2 \mathrm{d}x. \tag{95}$$

Coming back to the notation used for our asymptotic regime, the energy functional of interest has the form ($\xi = \pi_N(x)$)

$$\tilde{\mathcal{G}}_{\varepsilon}\left(\boldsymbol{m}\right) := \frac{1}{2\varepsilon} \sum_{i=1}^{3} \int_{\Omega_{\varepsilon}} |\mathrm{D}\boldsymbol{m}(x) + \mathsf{K}\left(\xi, \boldsymbol{m}\right)|^2 \mathrm{d}x \tag{96}$$

and, again, it is defined for every $\boldsymbol{m} \in H^1(\Omega_{\varepsilon}, M)$. Note that now $\mathsf{K}\left(\xi, \boldsymbol{m}\right)$ depends also on ξ, but the analysis we performed to establish Theorem 1 extends to this setting (see Remark 2 and Theorem 2). The curved thin-film limit, as identified in Theorem 2, reads as

$$\tilde{\mathcal{E}}_N\left(\boldsymbol{m}\right) = \sum_{i=1}^{2} \int_N |\partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{m}(\xi) + \mathsf{K}\left(\xi, \boldsymbol{m}(\xi)\right) \boldsymbol{\tau}_i(\xi)|^2 \mathrm{d}\xi$$

$$+ \int_N \left(\mathsf{K}\left(\xi, \boldsymbol{m}(\xi)\right) \boldsymbol{n}_N(\xi) \cdot \boldsymbol{n}_M\left(\boldsymbol{m}(\xi)\right)\right)^2 \mathrm{d}\xi \tag{97}$$

$$= \sum_{i=1}^{2} \int_N |\partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{m}(\xi) + \kappa\left[(\boldsymbol{\tau}_i(\xi) \otimes \boldsymbol{n}_N(\xi)) - (\boldsymbol{n}_N(\xi) \otimes \boldsymbol{\tau}_i(\xi))\right] \boldsymbol{m}(\xi)|^2 \mathrm{d}\xi. \tag{98}$$

In fact, the second term in (97) vanishes because of $\mathsf{K}\left(\xi, \boldsymbol{m}(\xi)\right) \boldsymbol{n}_N(\xi) = 0$.

When $N \subseteq \mathbb{R}^2 \times \{0\}$ is a flat surface (cf. Remark 3), one has $\boldsymbol{n}_N(\xi) = \boldsymbol{e}_3$. By taking the tangential derivatives in the direction of the standard basis of $\mathbb{R}^2$, we infer from (98) that

$$\tilde{\mathcal{E}}_N(\boldsymbol{m}) = \sum_{i=1}^{2} \int_N |\partial_i \boldsymbol{m}(\xi) + \kappa \left[(\boldsymbol{e}_i \otimes \boldsymbol{e}_3) - (\boldsymbol{e}_3 \otimes \boldsymbol{e}_i) \right] \boldsymbol{m}|^2 \, \mathrm{d}\xi$$

$$= \int_N |\nabla_\xi \boldsymbol{m}|^2 \, \mathrm{d}\xi + 2\kappa \int_N m_3 \, \mathrm{div}_\xi \, \boldsymbol{m} - \boldsymbol{m} \cdot \nabla_\xi m_3 \, \mathrm{d}\xi + \kappa^2 \int_N 1 + m_3^2(\xi) \, \mathrm{d}\xi. \quad (99)$$

The above formula is the classical expression for the interfacial DMI, which is used in the analysis of magnetic skyrmions in planar thin films [41]. We deduced (99) from (98) and, therefore, from Theorem 2. It should be noted, however, that in the planar setting, invoking Theorem 1 directly rather than its generalized counterpart stated in Theorem 2 is sufficient to determine the thin-film limit. In fact, when the surface is flat, the perturbation K does not depend on ξ.

4.3. Anisotropic DMI.

For completeness, we report the general case of anisotropic DMI where the Dzyaloshinskii vectors $(\boldsymbol{d}_i)_{i=1}^3$ are in general position. With M, N being two-dimensional surfaces in $\mathbb{R}^3$, the energy functional we are interested in reads as

$$\mathcal{G}_\varepsilon(\boldsymbol{m}) = \frac{1}{2\varepsilon} \int_{\Omega_\varepsilon} |\mathrm{D}\boldsymbol{m}(x) + \mathsf{K}(\boldsymbol{m}(x))|^2 \, \mathrm{d}x = \frac{1}{2\varepsilon} \sum_{i=1}^3 \int_{\Omega_\varepsilon} |\partial_i \boldsymbol{m}(x) + \boldsymbol{d}_i \times \boldsymbol{m}(x)|^2 \, \mathrm{d}x, \quad (100)$$

with $\mathsf{K}(\boldsymbol{m}) = \mathsf{M}(\boldsymbol{m}) \mathsf{J}$. Here, as in (8), J represents the linear operator that maps the standard basis $(\boldsymbol{e}_i)_{i=1}^3$ into the vectors $(\boldsymbol{d}_i)_{i=1}^3$, while $\mathsf{M}(\boldsymbol{m}) := \bullet \times \boldsymbol{m}$ represents the $\boldsymbol{m}$-dependent *antisymmetric* linear operator acting as the cross-product with $\boldsymbol{m}$.

By Theorem 1, we infer that in the curved thin-film limit, anisotropic DMI contributions result in the limiting energy functional

$$\tilde{\mathcal{E}}_N(\boldsymbol{m}) = \sum_{i=1}^{2} \int_N |\partial_{\boldsymbol{\tau}_i(\xi)} \boldsymbol{m}(\xi) + \tilde{\boldsymbol{d}}_i(\xi) \times \boldsymbol{m}(\xi)|^2 + \int_N (\mathsf{K}(\boldsymbol{m}(\xi)) \boldsymbol{n}_N(\xi) \cdot \boldsymbol{n}_M(\boldsymbol{m}(\xi)))^2 \, \mathrm{d}\xi \quad (101)$$

with $\tilde{\boldsymbol{d}}_i(\xi) := \mathsf{J}\boldsymbol{\tau}_i(\xi)$. Observe that when $M = \mathbb{S}^2$, the second term in (101) vanishes. Indeed, we have

$$\mathsf{K}(\boldsymbol{m}(\xi)) \boldsymbol{n}_N(\xi) \cdot \boldsymbol{n}_M(\boldsymbol{m}(\xi)) = [\mathsf{M}(\boldsymbol{m}) \mathsf{J}] \boldsymbol{n}_N(\xi) \cdot \boldsymbol{m}(\xi) = 0, \quad (102)$$

because the image of $\mathsf{M}(\boldsymbol{m}) \mathsf{J}$ is orthogonal to $\boldsymbol{m}$.

4.4. The nonuniform temperature setting.

We discuss here the case when the temperature of the ferromagnet is not uniform and, as a result, the (temperature-dependent) saturation magnetization cannot be assumed constant. To that purpose, we recall that in the variational theory of micromagnetism (cf. [30, 8]), systemized by Brown in the 60s, exchange interactions are described through the order parameter $\boldsymbol{M}$, called the *magnetization vector*. The magnetization vector $\boldsymbol{M}$ of a rigid ferromagnetic body, filling a domain $\Omega \subseteq \mathbb{R}^3$, is a vector field satisfying the so-called *saturation constraint*, that is, $|\boldsymbol{M}| = M_s$ in Ω for a positive constant $M_s \geqslant 0$. The *saturation magnetization* $M_s := M_s(T)$ is temperature dependent and vanishes when T exceeds the so-called *Curie temperature* T_c, whose specific value depends on the crystal type. Mean field theory predicts that the value of M_s can be considered almost constant in Ω when $T \ll T_c$ (see [25, 29]). As a result, it is customary to express the magnetization in the form $\boldsymbol{M} := M_s \boldsymbol{m}$, where $\boldsymbol{m} : \Omega \to \mathbb{S}^2$ is now a vector field taking values in the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$; we did the same when introducing the micromagnetic energy functionals in (2) and (7).

Returning to our asymptotic regime, let M, N be two-dimensional surfaces in $\mathbb{R}^3$. In general, when the prescribed temperature profile is not uniform in Ω_ε, M_s is a function defined on Ω_ε, and the family of energy functionals we are interested in has the form (cf. with (21))

$$\tilde{\mathcal{G}}_\varepsilon\left(\boldsymbol{m}\right) = \frac{1}{2\varepsilon}\sum_{i=1}^{3}\int_{\Omega_\varepsilon}\left|\partial_i\left(M_s(x)\boldsymbol{m}(x)\right) + M_s(x)\boldsymbol{d}_i \times \boldsymbol{m}(x)\right|^2\mathrm{d}x. \tag{103}$$

In what follows, we assume that M_s is 0-homogeneous along the normal direction, i.e., that $M_s(x) = M_s(\xi)$, $\xi := \pi_N(x)$, and that M_s is regular enough. We can rearrange $\mathcal{G}$ in the following way, which is compatible with Theorem 2 (see also (21)):

$$\tilde{\mathcal{G}}_\varepsilon\left(\boldsymbol{m}\right) = \frac{1}{2\varepsilon}\int_{\Omega_\varepsilon}\left|M_s(x)\partial_i\boldsymbol{m}(x) + \partial_i M_s(x)\boldsymbol{m}(x) + \boldsymbol{d}_i \times \boldsymbol{m}(x)\right|^2$$

$$= \frac{1}{2\varepsilon}\int_{\Omega_\varepsilon}\left|A(x)\mathrm{D}\boldsymbol{m}(x) + \mathsf{K}\left(x, \boldsymbol{m}(x)\right)\right|^2\mathrm{d}x \tag{104}$$

with $A(x) = M_s(x)I$ and $\mathsf{K}\left(x, \boldsymbol{m}\right) = \boldsymbol{m}\otimes\nabla M_s(x) + \mathsf{M}\left(\boldsymbol{m}\right)\mathsf{J}$. Here, as in (8), J represents the linear operator that maps the standard basis $(\boldsymbol{e}_i)_{i=1}^3$ into the vectors $(\boldsymbol{d}_i)_{i=1}^3$, whereas $\mathsf{M}\left(\boldsymbol{m}\right) := \bullet \times \boldsymbol{m}$ represents the $\boldsymbol{m}$-dependent *antisymmetric* linear operator acting as the cross-product with $\boldsymbol{m}$. More explicitly, in matrix form, we have

$$\mathsf{K}\left(x, \boldsymbol{m}\right) = \left(\partial_1 M_s(x)\boldsymbol{m}, \partial_2 M_s(x)\boldsymbol{m}, \partial_3 M_s(x)\boldsymbol{m}\right) + \left(\boldsymbol{d}_1 \times \boldsymbol{m}, \boldsymbol{d}_2 \times \boldsymbol{m}, \boldsymbol{d}_3 \times \boldsymbol{m}\right). \tag{105}$$

Invoking Theorem 2, we infer that the curved thin-film limit reads as

$$\tilde{\mathcal{E}}_\mathcal{N}\left(\boldsymbol{m}\right) = \sum_{i=1}^{2}\int_N\left|M_s(\xi)\partial_{\boldsymbol{\tau}_i(\xi)}\boldsymbol{m}(\xi) + \mathsf{K}\left(\xi, \boldsymbol{m}(\xi)\right)\boldsymbol{\tau}_i(\xi)\right|^2\mathrm{d}\xi$$

$$+ \int_N\left(\mathsf{K}\left(\xi, \boldsymbol{m}(\xi)\right)\boldsymbol{n}_N(\xi)\cdot\boldsymbol{n}_M\left(\boldsymbol{m}(\xi)\right)\right)^2\mathrm{d}\xi \tag{106}$$

Observe that when $M = \mathbb{S}^2$, we have

$$\mathsf{K}\left(\xi, \boldsymbol{m}(\xi)\right)\boldsymbol{n}_N(\xi)\cdot\boldsymbol{n}_M(\boldsymbol{m}(\xi)) = \left[\boldsymbol{m}\otimes\nabla_\xi M_s(\xi) + \mathsf{M}\left(\boldsymbol{m}\right)\mathsf{J}\right]\boldsymbol{n}_N(\xi)\cdot\boldsymbol{m}(\xi) = 0, \tag{107}$$

because the image of $\mathsf{M}\left(\boldsymbol{m}\right)\mathsf{J}$ is orthogonal to $\boldsymbol{m}$ and $\nabla_\xi M_s(\xi)$ is tangent at $\xi \in N$. Hence, in the classical micromagnetic setting $(M = \mathbb{S}^2)$, we get that the limiting model reads as

$$\tilde{\mathcal{E}}_\mathcal{N}\left(\boldsymbol{m}\right) = \sum_{i=1}^{2}\int_N\left|\partial_{\boldsymbol{\tau}_i(\xi)}\left(M_s(\xi)\boldsymbol{m}(\xi)\right) + \tilde{\boldsymbol{d}}_i(\xi) \times \boldsymbol{m}(\xi)\right|^2\mathrm{d}\xi, \tag{108}$$

with $\tilde{\boldsymbol{d}}_i(\xi) := \mathsf{J}\boldsymbol{\tau}_i(\xi)$.

Acknowledgments

References

[1] S. Albeverio, J. Jost, S. Paycha, and S. Scarlatti, *A Mathematical Introduction to String Theory*, Cambridge University Press, jul 1997.

[2] J.-F. Babadjian, G. Di Fratta, I. Fonseca, G. Francfort, M. Lewicka, and C. Muratov, *The mathematics of thin structures*, Quarterly of Applied Mathematics, (2022).

[3] J. M. Ball, *Mathematics and liquid crystals*, Molecular Crystals and Liquid Crystals, 647 (2017), pp. 1–27.

[4] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory*, Cambridge University Press, dec 2006.

[5] F. Bethuel, H. Brezis, and F. Helein, *Ginzburg-Landau Vortices*, Springer International Publishing, 2017.

[6] J. Bourgain, H. Brezis, and P. Mironescu, *Another look at Sobolev spaces*, in Optimal control and partial differential equations, IOS, Amsterdam, 2001, pp. 439–455.

[7] W. F. Brown, *Magnetostatic principles in ferromagnetism*, North-Holland Publishing Company, New York, 1962.

[8] ———, *Micromagnetics*, Interscience Publishers, London, 1963.

[9] L. Camosi, S. Rohart, O. Fruchart, S. Pizzini, M. Belmeguenai, Y. Roussigné, A. Stashkevich, S. M. Cherif, L. Ranno, M. de Santis, and J. Vogel, *Anisotropic Dzyaloshinskii-Moriya interaction in ultrathin epitaxial Au/Co/W(110)*, Physical Review B, 95 (2017), p. 214422.

[10] G. Carbou, *Thin layers in micromagnetism*, Mathematical Models and Methods in Applied Sciences, 11 (2001), pp. 1529–1546.

[11] J. M. D. Coey and S. S. Parkin, eds., *Handbook of Magnetism and Magnetic Materials*, Springer International Publishing, 2021.

[12] G. Dal Maso, *Introduction to Γ-convergence*, Birkhäuser Basel, 1993.

[13] E. Davoli, G. Di Fratta, D. Praetorius, and M. Ruggeri, *Micromagnetics of thin films in the presence of Dzyaloshinskii-Moriya interaction*, Mathematical Models and Methods in Applied Sciences, 32 (2022), pp. 911–939.

[14] A. DeSimone, R. V. Kohn, S. Müller, and F. Otto, *A reduced theory for thin-film micromagnetics*, Communications on Pure and Applied Mathematics, 55 (2002), pp. 1408–1460.

[15] G. Di Fratta, *Micromagnetics of curved thin films*, Zeitschrift für angewandte Mathematik und Physik, 71 (2020).

[16] G. Di Fratta, C. B. Muratov, F. N. Rybakov, and V. V. Slastikov, *Variational Principles of Micromagnetics Revisited*, SIAM Journal on Mathematical Analysis, 52 (2020), pp. 3580–3599.

[17] G. Di Fratta, V. Slastikov, and A. Zarnescu, *On a Sharp Poincaré-Type Inequality on the 2-Sphere and its Application in Micromagnetics*, SIAM Journal on Mathematical Analysis, 51 (2019), pp. 3373–3387.

[18] M. P. do Carmo, *Differential geometry of curves & surfaces*, Dover Publications, Inc., Mineola, NY, 2016. Revised & updated second edition of [MR0394451].

[19] I. Dzyaloshinsky, *A thermodynamic theory of weak ferromagnetism of antiferromagnetics*, Journal of Physics and Chemistry of Solids, 4 (1958), pp. 241–255.

[20] J. Eells and L. Lemaire, *Two reports on harmonic maps*, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[21] J. Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, American Journal of Mathematics, 86 (1964), pp. 109–160.

[22] A. Fert, N. Reyren, and V. Cros, *Magnetic skyrmions: advances in physics and potential applications*, Nature Reviews Materials, 2 (2017).

[23] Y. Ga, Q. Cui, Y. Zhu, D. Yu, L. Wang, J. Liang, and H. Yang, *Anisotropic Dzyaloshinskii-Moriya interaction protected by D2d crystal symmetry in two-dimensional ternary compounds*, npj Computational Materials, 8 (2022).

[24] G. Gioia and R. D. James, *Micromagnetics of very thin films*, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 453 (1997), pp. 213–223.

[25] A. Hubert and R. Schäfer, *Magnetic Domains*, Springer Berlin Heidelberg, 1998.

[26] H. Knüpfer, C. B. Muratov, and F. Nolte, *Magnetic domains in thin ferromagnetic films with strong perpendicular anisotropy*, Archive for Rational Mechanics and Analysis, 232 (2019), pp. 727–761.

[27] R. V. Kohn and V. V. Slastikov, *Another thin-film limit of micromagnetics*, Archive for Rational Mechanics and Analysis, 178 (2005), pp. 227–245.

[28] ———, *Effective dynamics for ferromagnetic thin films: a rigorous justification*, Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences, 461 (2005), pp. 143–154.

[29] M. D. Kuz'min, *Shape of Temperature Dependence of Spontaneous Magnetization of Ferromagnets: Quantitative Analysis*, Physical Review Letters, 94 (2005), p. 107204.

[30] L. Landau and E. Lifshitz, *On the theory of the dispersion of magnetic permeability in ferromagnetic bodies*, in Perspectives in Theoretical Physics, Elsevier, 1992, pp. 51–65.

[31] E. L. Lima, *The Jordan-Brouwer separation theorem for smooth hypersurfaces*, American Mathematical Monthly, 95 (1988), pp. 39–42.

[32] F. Lin and C. Wang, *The analysis of harmonic maps and their heat flows*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[33] D. Makarov and D. D. Sheka, eds., *Curvilinear Micromagnetism*, Springer International Publishing, 2022.

[34] C. Melcher, *Thin-film limits for Landau-Lifshitz-Gilbert equations*, SIAM Journal on Mathematical Analysis, 42 (2010), pp. 519–537.

[35] ———, *Chiral skyrmions in the plane*, Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences, 470 (2014), p. 20140394.

[36] C. Melcher and Z. N. Sakellaris, *Curvature-stabilized skyrmions with angular momentum*, Letters in Mathematical Physics, 109 (2019), pp. 2291–2304.

[37] M. Morini and V. Slastikov, *Reduced models for ferromagnetic thin films with periodic surface roughness*, Journal of Nonlinear Science, 28 (2018), pp. 513–542.

[38] T. Moriya, *Anisotropic Superexchange Interaction and Weak Ferromagnetism*, Physical Review, 120 (1960), pp. 91–98.

[39] R. Moser, *Boundary vortices for thin ferromagnetic films*, Archive for Rational Mechanics and Analysis, 174 (2004), pp. 267–300.

[40] ———, *Moving boundary vortices for a thin-film limit in micromagnetics*, Communications on Pure and Applied Mathematics, 58 (2005), pp. 701–721.

[41] C. B. Muratov and V. V. Slastikov, *Domain structure of ultrathin ferromagnetic elements in the presence of Dzyaloshinskii-Moriya interaction*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 473 (2017), p. 20160666.

[42] N. Nagaosa and Y. Tokura, *Topological properties and dynamics of magnetic skyrmions*, Nature Nanotechnology, 8 (2013), pp. 899–911. doi: 10.1038/nnano.2013.243.

[43] D. Praetorius, *Analysis of the operator $\Delta^{-1} \mathrm{div}$ arising in magnetic models*, Zeitschrift für Analysis und ihre Anwendungen, 23 (2004), pp. 589–605.

[44] R. Schoen and S.-T. Yau, *Lectures on harmonic maps*, Conference Proceedings and Lecture Notes in Geometry and Topology, II, International Press, Cambridge, MA, 1997.

[45] V. Slastikov, *Micromagnetics of thin shells*, Mathematical Models and Methods in Applied Sciences, 15 (2005), pp. 1469–1487.

[46] R. Streubel, P. Fischer, F. Kronast, V. P. Kravchuk, D. D. Sheka, Y. Gaididei, O. G. Schmidt, and D. Makarov, *Magnetism in curved geometries*, Journal of Physics D: Applied Physics, 49 (2016), p. 363001.

[47] M. Venkatesan, C. B. Fitzgerald, and J. M. D. Coey, *Unexpected magnetism in a dielectric oxide*, Nature, 430 (2004), pp. 630–630.

[48] E. G. Virga, *Variational Theories for Liquid Crystals*, Chapman and Hall/CRC, dec 2018.

[49] J. Wloka, B. Rowley, and B. Lawruk, *Boundary value problems for elliptic systems*, Cambridge University Press, Cambridge, 1995.

Giovanni Di Fratta, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Via Cintia, Complesso Monte S. Angelo, 80126 Napoli, Italy

Valeriy Slastikov, School of Mathematics, University of Bristol, Bristol BS8 1UG, United Kingdom