Necessary conditions and non-existence results for autonomous nonconvex variational problems

Giovanni Cupini[†], Marcello Guidorzi[§] and Cristina Marcelli[§]

[†]Dipartimento di Matematica "U. Dini" - Università di Firenze
Viale Morgagni 67/A, 50134 Firenze (Italy) - email: cupini@math.unifi.it

[§]Dipartimento di Scienze Matematiche - Università Politecnica delle Marche
Via Brecce Bianche, 60131 Ancona (Italy)
email: marcelli@dipmat.univpm.it; guidorzi@dm.unife.it

Dedicated to Arrigo Cellina in honor of his 65th birthday

Abstract. Classical one-dimensional, autonomous Lagrange problems are considered. In absence of any smoothness, convexity or coercivity condition on the energy density, we prove a DuBois-Reymond type necessary condition, expressed as a differential inclusion involving the subdifferential of Convex Analysis. As a consequence, a non-existence result is obtained.

1. Introduction

Let $I \subseteq \mathbb{R}$ be a given interval, bounded or unbounded, and let $\alpha, \beta \in I$ be fixed. Consider the class

$$\Omega := \{v \in W^{1,1}(a,b) \ : \ v(a) = \alpha, \ v(b) = \beta, \ v(x) \in I\}.$$

In this paper we address our attention to the study of the autonomous variational problem

$$\text{minimize} \left\{ F(v) = \int_a^b f(v(x), v'(x)) \, \mathrm{d}t \ : \ v \in \Omega \right\}, \qquad (P)$$

where $f : I \times \mathbb{R} \to [0, +\infty)$ is a lower semicontinuous non negative function satisfying some further properties concerning the restriction $f(\cdot, 0)$ (see conditions (H1), (H2), (H3)). We do not assume any smoothness, convexity or coercivity condition on the Lagrangian f.

Recently various optimality conditions for nonconvex variational problems have been obtained by using advanced tools of variational analysis and generalized differentiation (see the monograph [11] for a survey and analysis of the results).

The first aim of the present paper is to obtain a DuBois-Reymond type necessary condition (from now on simply (DBR)) which in our nonsmooth setting has the form of a differential inclusion

$$f(u(x), u'(x)) - c \in u'(x)\partial f(u(x), u'(x)) \quad \text{a.e. in } (a, b).$$

A result on this matter was obtained for convex Lagrangian $f : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ by Ambrosio-Ascenzi-Buttazzo in [1], without any regularity condition (just Borel measurability), where the subdifferential is that of Convex Analysis. As it is well known, in a convex setting it coincides with the Clarke's one (see [4]) and its further generalizations (see [9], [10]).

On the contrary, the Clarke's subdifferential for nonconvex functions contains the usual one of the Convex Analysis. For instance (e.g. $f(s, z) = \sqrt{|z|}$), it may happen that the subdifferential of Convex Analysis is a singleton whereas the other ones are non-degenerate intervals. Moreover, it has a global nature (contrary to the other ones having a local nature). Finally, notice that the subdifferential of Convex Analysis can be empty if $f(s, \cdot)$ does not admit support straightlines; but if it is non-empty for some z_0 then, as a consequence, $f(s, z_0) = f^{**}(s, z_0)$.

Therefore, in our non-convex framework, it assumes a certain relevance to prove a DuBois-Reymond condition involving the subdifferential of Convex Analysis, since it is sharper than analogous ones involving other kinds of subdifferential and mainly since it implies that f and f^{**} coincide along the minimizer, which is a relaxation result.
So, our first result goes in this direction: we obtain a (DBR) condition expressed by means of the subdifferential of Convex Analysis, even if the integrand is not convex. More precisely, we prove (see Theorem 4.4) that if (P) is solvable then there exists a minimizer u such that $\partial f(u(x), u'(x)) \neq \emptyset$ for a.e. $x \in (a, b)$ and the (DBR) condition holds with a constant

$$c \leq \min_{s \in [\min u, \max u]} f(s, 0).$$

The limitation on the constant c has a relevance too, since it allows us to derive a non-existence result for problem (P) (see Proposition 5.1), which states that if

$$\operatorname{ess\,sup}_{s \in [\alpha, \beta]} \liminf_{|z| \to +\infty} \inf\{f^{**}(s, z) - z\partial f^{**}(s, z)\} \ > \ \min\{f(\alpha, 0), f(\beta, 0)\}$$

then problem (P) does not admit minimum. Such a (non-existence) condition is rather close to the negation of a sufficient condition for the existence of the minimum obtained by Clarke in [5] in the convex, but noncoercive, framework (see Remark 5.3 for the details). Moreover, it assumes a particularly simple form in the case of affine-type Lagrangian $f(s, z) := \phi(s) + \psi(s)h(z)$; in fact, for instance, when ϕ and ψ are increasing it becomes

$$\phi(\beta) + \psi(\beta)\ell \ > \ \phi(\alpha) + \psi(\alpha)h(0),$$

where $\ell := \liminf_{|z| \to \infty} \inf (h^{**}(z) - z\partial h^{**}(z))$ (see Proposition 5.2). Note, moreover, that if $\ell = -\infty$ and $\inf \psi > 0$, then f satisfies the growth condition, weaker then superlinearity, considered by Cellina-Ferriero in [3] for the existence of the minimum (see also [2] for a result on Lipschitz regularity of the minimizers).

Finally, we wish just to mention that the (DBR) condition with a suitable limitation on the constant c becomes also a sufficient condition for the existence of the minimum to problem (P), as we will show in a forthcoming paper (see [6]).

As regards our approach, we use a (DBR) condition (involving the sudifferential of Convex Analysis) obtained by Marcelli in [8] for autonomous non-convex variational problems

with constraints on the derivatives ($v'(x) \geq 0$ a.e.). In such a framework, in [8] it was proved that the (DBR) condition (with a suitable limitation on the constant c) is necessary and sufficient for the optimality of a trajectory u (see Theorem 4.2 below for a precise statement).

In order to apply this result, firstly we need to prove some monotonicity property of the minimizer of the free problem (P). More in detail, under our assumptions (H1)-(H3) on the restriction $f(\cdot, 0)$, we show that if (P) is solvable then there exists a minimizer u which oscillates at most once (see Theorem 3.1). We will refer to u satisfying this property saying that it satisfies a maximum/minimum principle (not to be confused with the well known Weierstrass-Pontryagin maximum principle), since its restriction to any subinterval of $[a, b]$ assumes the maximum/minimum value in correspondence of one of the endpoints.

The maximum principle for minimizers was already known (see [7]) when the Lagrangian $f(s, z)$ is increasing with respect to s and $f(s, 0) < f(s, z)$ for every $z \neq 0$. Here we show that actually the weaker inequality $f(s, 0) \leq f(s, z)$ suffices (see condition (H4) and Lemma 2.1). In this way we include situations where $f(s, 0)$ is a non-proper minimum for $f(s, z)$, arising for instance when dealing with convex envelopes f^{**} which are constantly null in a neighborhood of 0 (see Remarks 3.3 and 3.4 for further comments).

The quoted property of the minimizer allows us to associate to our original problem a suitable constrained one having a (monotone) minimizer. Therefore applying the result in [8], we obtain the (DBR) condition for the free problem.

2. Notations and preliminary results

As mentioned in Introduction, our approach for dealing with necessary conditions for the optimality of problem (P) is based on the reduction to a suitable constrained problem, in order to apply a necessary condition proved in [8].

Our main aim is to establish a DuBois-Reymond condition (from now on briefly (DBR)), which in our general setting of nonsmooth analysis assumes the form of a differential inclusion:

$$\text{(DBR)} \qquad f(v_0(x), v_0'(x)) - c \in v_0'(x)\, \partial f(v_0(x), v_0'(x)) \qquad \text{a.e. in } (a, b),$$

where $\partial f(s, z)$ denotes the subdifferential in the sense of convex analysis, that is

$$\partial f(s, z) := \{\alpha \in \mathbb{R} \ : \ f(s, w) - f(s, z) \geq \alpha(w - z) \text{ for every } w \in \mathbb{R}\}.$$

As usual, f^{**} is the convex envelope of f with respect to the second variable, i.e. fixed $s \in I$, $f^{**}(s, \cdot)$ is the largest convex function lower than $f(s, \cdot)$. We will not assume any coercivity or convexity condition.

Throughout the paper we will assume

- $f : I \times \mathbb{R} \to [0, +\infty)$ lower semicontinuous;
- $f(\cdot, 0)$ continuous;
- $f(s, \cdot)$ continuous at $z = 0$ for every $s \in I$.

We list here the properties to which we will refer to in the sequel.

(H1) there exists a continuous selection of $\partial f^{**}(\cdot, 0)$;

(H2) $f(s, 0) = f^{**}(s, 0)$ for every $s \in I$;

(H3) there exists a value $k \in [\inf I, \sup I]$ such that $f(\cdot, 0)$ is monotone decreasing in $(\inf I, k)$ and monotone increasing in $(k, \sup I)$;

(H4) $f(s, 0) = \min\limits_{z \in \mathbb{R}} f(s, z)$, for every $s \in I$.

Of course, if $f(\cdot, 0)$ is monotone in I then it satisfies condition (H3), in fact it suffices to take $k = \inf I$ if it is increasing or $k = \sup I$ if it is decreasing.

As regards condition (H4), notice that it implies both (H1) and (H2), but the vice versa is trivially false. Nevertheless, the following lemma states that when (H1) and (H2) are assumed, then, without loss of generality, we can assume that also (H4) holds true.

Lemma 2.1. *Let $f : I \times \mathbb{R} \to [0, +\infty)$ satisfy (H1) and (H2). Then there exists $\tilde{f} : I \times \mathbb{R} \to [0, +\infty)$ lower semicontinuous, with $s \mapsto \tilde{f}(s, 0)$ continuous, satisfying the following properties*

(a) $f(s, 0) = \tilde{f}(s, 0) = (\tilde{f})^{**}(s, 0)$ *for every $s \in I$;*

(b) $\tilde{f}(s, 0) = \min\limits_{z \in \mathbb{R}} \tilde{f}(s, z)$ *for every $s \in I$;*

(c) $v_0 \in \Omega$ *satisfies the (DBR) condition relatively to function f if and only if it satisfies the (DBR) condition relatively to $\tilde{f}$ with the same constant c, i.e.*

$$f(v_0(x), v_0'(x)) - c \in v_0'(x)\, \partial f(v_0(x), v_0'(x)) \iff \tilde{f}(v_0(x), v_0'(x)) - c \in v_0'(x)\, \partial \tilde{f}(v_0(x), v_0'(x))$$

(d) *there exists $k \in \mathbb{R}$ such that for every $u \in \Omega$ we get*

$$F(u) = \tilde{F}(u) + k.$$

where $\tilde{F}(u)$ stands for $\int_a^b \tilde{f}(u(x), u'(x))\, \mathrm{d}x$.

Proof. By (H1) we define $\tilde{f} : I \times \mathbb{R} \to [0, +\infty)$ as

$$\tilde{f}(s, z) := f(s, z) - g(s)z,$$

where g is a continuous selection of $\partial f^{**}(\cdot, 0)$. Notice that since $f(s, z) \geq 0$, from the definition of $g(s)$ and (H2) then $\tilde{f}(s, z) \geq 0$ too. In fact,

$$\tilde{f}(s, z) = f(s, z) - g(s)z \geq f^{**}(s, z) - g(s)z \geq f^{**}(s, 0) = f(s, 0) \geq 0. \qquad (2.1)$$

Moreover, the lower semicontinuity of $\tilde{f}$ and the continuity of $\tilde{f}(\cdot, 0)$ are a straightforward consequence of the corresponding properties on f and g.

The equalities in (a) are an immediate consequence of the definition of $\tilde{f}$ and of the following relation

$$(\tilde{f})^{**}(s, z) = f^{**}(s, z) - g(s)z,$$

whereas property (b) follows from (2.1) since $f(s, 0) = \tilde{f}(s, 0)$. To prove (c) it suffices to remark that

$$\partial \tilde{f}(s, z) = \partial f(s, z) - g(s).$$

As for property (d) notice that

$$\tilde{F}(u) = F(u) - \int_a^b g(u(x))u'(x) \, \mathrm{d}x = F(u) - \int_\alpha^\beta g(s) \, \mathrm{d}s$$

for every $u \in \Omega$. $\hfill\square$

Remark 2.2. If f^{**} is continuous and $f^{**}(s, \cdot)$ is differentiable at $z = 0$ for every $s \in I$, then (H1) is satisfied. Indeed in this case $\partial f^{**}(s, 0)$ is a singleton, say $\{g(s)\}$, with $g(s)$ continuous. In fact, if $\liminf_{s \to s_0} g(s) < g(s_0)$ for some $s_0 \in I$, then there exists a real number $\alpha > 0$ and a sequence $(s_n)_n$ converging to s_0 such that $g(s_n) < g(s_0) - \alpha$ for every n. Therefore,

$$f^{**}(s_n, z) - f^{**}(s_n, 0) \geq g(s_n)z > (g(s_0) - \alpha)z \quad \text{for every } z < 0,$$

so by the continuity of f^{**} we deduce

$$f^{**}(s_0, z) - f^{**}(s_0, 0) \geq (g(s_0) - \alpha)z \quad \text{for every } z < 0.$$

On the other hand,

$$f^{**}(s_0, z) - f^{**}(s_0, 0) \geq g(s_0)z > (g(s_0) - \alpha)z \quad \text{for every } z > 0,$$

then $g(s_0) - \alpha \in \partial f^{**}(s_0, 0)$, in contradiction with the differentiability of $f^{**}(s_0, 0)$. The proof of the upper-semicontinuity of g is analogous.

Remark 2.3. Condition (H1) is trivially satisfied if the integrand has the type-affine structure $f(s, z) = \phi(s) + \psi(s)h(z)$, provided that ψ is continuous.

3. The monotonicity property of the minimizers

In this section we investigate the monotonicity property of the minimizers of problem (P), which is the key tool for our approach. Our goal is to show that under the assumptions (H1)-(H3) if (P) is solvable then there exists a minimizer which has at most one oscillation, that is one can split the interval $[a, b]$ into two subintervals (one of them possibly degenerate) in which the minimizer is first decreasing and then increasing or vice versa. Moreover, if the value k in condition (H3) satisfies $\min\{\alpha, \beta\} \leq k \leq \max\{\alpha, \beta\}$, then the minimizer is monotone.

From now on, we will say that a function $u \in \Omega$ satisfies the *maximum principle* if

(M) there exists $x_0 \in [a, b]$ such that u is decreasing in $[a, x_0]$ and increasing in $[x_0, b]$

and the *minimum principle* if

(m) there exists $x_0 \in [a, b]$ such that u is increasing in $[a, x_0]$ and decreasing in $[x_0, b]$.

We define

$$\Omega_M = \{u \in \Omega : u \text{ satisfies } (M)\}, \quad \Omega_m := \{u \in \Omega : u \text{ satisfies } (m)\}, \quad \Omega^* := \Omega_M \cup \Omega_m.$$

To justify the expression *maximum principle* we observe that any function in Ω_M has the remarkable property that any restriction on a subinterval of $[a, b]$ assumes its maximum value in correspondence of one of the endpoints.

Theorem 3.1. [*Monotonicity property*] *Let* $f : I \times \mathbb{R} \to [0, +\infty)$ *satisfy (H1) – (H3). Then*

$$\inf_{\Omega} F = \inf_{\Omega^*} F.$$

Moreover, if (P) *has a solution, then there exists a (possibly different) solution to* (P) *belonging to* Ω^*.

Finally, if $\min\{\alpha, \beta\} \le k \le \max\{\alpha, \beta\}$, *then the class* Ω^* *can be replaced by the subclass of monotone functions.*

The proof of this result needs the following lemma, whose proof is postponed at the end of this section.

Lemma 3.2. *Let* $u : [c, d] \to \mathbb{R}$ *be an absolutely continuous function such that* $u(c) \le u(x)$ *for every* $x \in [c, d]$. *Then, there exists an increasing absolutely continuous function* $w : [c, d] \to \mathbb{R}$ *such that*

$$w(x) \le u(x) \ \ \text{for every } x, \quad w(c) = u(c), \quad w(d) = u(d)$$

and finally

$$w(x) = u(x), \quad w'(x) = u'(x) \qquad \text{for a.e. } x \text{ such that there exists } w'(x) > 0.$$

Similarly, if $u(x) \ge u(d)$ *for every* $x \in [c, d]$, *there exists a decreasing absolutely continuous function* $w : [c, d] \to \mathbb{R}$ *such that* $w(x) \le u(x)$ *for every* x, $w(c) = u(c)$, $w(d) = u(d)$ *and finally* $w(x) = u(x)$, $w'(x) = u'(x)$ *for a.e.* x *such that there exists* $w'(x) < 0$.

Proof of Theorem 3.1. Without loss of generality we can assume that f satisfies assumption (H4), too. In fact, if not, let $\tilde{f}$ be as in Lemma 2.1 and use (d) in Lemma 2.1. Suppose by contradiction that $\inf_{\Omega} F < \inf_{\Omega^*} F$. Then there exists $u \in \Omega \setminus \Omega^*$ such that $F(u) < \inf_{\Omega^*} F$.

We split the proof into various cases.

Case 1. Let $k \le \min u$, which implies that $f(\cdot, 0)$ is increasing in $[\min u, +\infty)$.

Let $x_0 := \max\{x : u(x) = \min u\}$. By applying Lemma 3.2 in the intervals $[a, x_0]$ and $[x_0, b]$, we deduce the existence of an absolutely continuous function w such that $w(a) = u(a)$, $w(b) = u(b)$, $w(x_0) = u(x_0)$, $w(x) \le u(x)$ in $[a, b]$, w is decreasing in $[a, x_0]$ and increasing in $[x_0, b]$. Moreover, $w(x) = u(x)$ and $w'(x) = u'(x)$ for a.e. x such that there exists $w'(x) \ne 0$. Hence, $w \in \Omega_M \subset \Omega^*$ and $\min w = \min u$. Thus, if A denotes the set $\{x : w'(x) = 0\}$, by the monotonicity of $f(\cdot, 0)$ on $[\min w, +\infty)$ and (H4) we get

$$
\begin{aligned}
F(w) = \int_a^b f(w(x), w'(x)) \, \mathrm{d}x &= \int_A f(w(x), 0) \, \mathrm{d}x + \int_{[a,b] \setminus A} f(w(x), w'(x)) \, \mathrm{d}x \\
&\le \int_A f(u(x), 0) \, \mathrm{d}x + \int_{[a,b] \setminus A} f(u(x), u'(x)) \, \mathrm{d}x \\
&\le \int_A f(u(x), u'(x)) \, \mathrm{d}x + \int_{[a,b] \setminus A} f(u(x), u'(x)) \, \mathrm{d}x = F(u),
\end{aligned}
$$

which is an absurd.

Now, suppose that $u \in \Omega$ is a solution to (P). Then, reasoning as above, we get the existence of $w \in \Omega_M \subset \Omega^*$ such that w is a solution.

Finally, we stress that $w(x) \in [\min u, \max\{\alpha, \beta\}]$ for every $x \in [a, b]$ and if $\min u = \min\{\alpha, \beta\}$ then w is monotone.

Case 2. Let $k \geq \max u$, which implies that $f(\cdot, 0)$ is decreasing in $[0, \max u]$.

We omit the proof of this step which is quite similar to that of the previous one. In fact, in this situation it suffices to replace Ω_M with Ω_m and give a version of Lemma 3.2 with the statement $w(x) \geq u(x)$ and define $x_0 := \max\{x : u(x) = \max u\}$. In particular, if $u \in \Omega$ is a solution to (P) then we get the existence of $w \in \Omega_m \subset \Omega^*$ such that w is a solution satisfying $w(x) \in [\min\{\alpha, \beta\}, \max u]$ for every x. Moreover, if $\max u = \max\{\alpha, \beta\}$ then w is monotone.

Case 3. Let $\min u < k < \max u$, with $k \leq \min\{\alpha, \beta\}$.

Consider the function $\hat{u} := \max\{u(x), k\}$. Since $\min\{\alpha, \beta\} \geq k$, then $\hat{u} \in \Omega$. Moreover, denoted by $B := \{x : \hat{u}(x) > k\}$, then B is an open set and $\hat{u}'(x) = 0$ for a.e. $x \notin B$. So, being by (H3) and (H4)

$$f(k, 0) \leq f(s, 0) \leq f(s, z) \qquad \text{for every } (s, z) \in [0, +\infty) \times \mathbb{R},$$

we deduce

$$F(\hat{u}) = \int_B f(u(x), u'(x)) \, \mathrm{d}x + \int_B f(k, 0) \, \mathrm{d}x \leq \int_a^b f(u(x), u'(x)) \, \mathrm{d}x = F(u) < \inf_{\Omega^*} F.$$

Therefore, since $\min \hat{u} \geq k$, we can apply what proved in Case 1 to obtain the conclusion.

Moreover, observe that if $k = \min\{\alpha, \beta\}$ then the function w given by the application of Case 1 is monotone.

Case 4. Let $\min u < k < \max u$, with $\max\{\alpha, \beta\} \leq k$.

The proof of this case is quite similar to that of Case 3, (applying Case 2 instead Case 1).

Case 5. Let $\min u < k < \max u$, with $\min\{\alpha, \beta\} < k < \max\{\alpha, \beta\}$.

Set

$$H^+ := \{x \in (a, b) : u(x) > k\} \quad \text{and} \quad H^- := \{x \in (a, b) : u(x) < k\}.$$

Of course, H^+ and H^- are non-empty open sets and there exist at most countable many disjoint intervals (c_i, d_i), (γ_j, δ_j), $i, j = 1, 2, \ldots$, such that

$$H^+ = \cup_{i=1,2,\ldots}(c_i, d_i) \quad \text{and} \quad H^- = \cup_{j=1,2,\ldots}(\gamma_j, \delta_j).$$

Let us define $\hat{u} : [a, b] \to \mathbb{R}$ by

$$\hat{u}(x) := \begin{cases} k & \text{if } x \in (c_i, d_i) \text{ for some } i, \text{ with } [c_i, d_i] \subset (a, b) \\ k & \text{if } x \in (\gamma_j, \delta_j) \text{ for some } j, \text{ with } [\gamma_j, \delta_j] \subset (a, b) \\ u(x) & \text{otherwise.} \end{cases}$$

Of course, $\hat{u} \in \Omega$ and it satisfies $F(\hat{u}) \leq F(u)$. Indeed, by (H3) and (H4)

$$f(k, 0) \leq f(u(x), 0) \leq f(u(x), u'(x)) \qquad \text{for every } x \in [a, b].$$

Thus, defined $B := \{x : \hat{u}(x) \neq u(x)\}$, we have

$$\int_a^b f(\hat{u}(x), \hat{u}'(x)) \, \mathrm{d}x = \int_B f(k, 0) \, \mathrm{d}x + \int_{[a,b]\setminus B} f(u(x), u'(x)) \, \mathrm{d}x \leq F(u).$$

Now let us prove the existence of a monotone function $w \in \Omega$ such that $F(w) \leq F(\hat{u})$. To this end, we assume now $\alpha < \beta$ (the proof in the opposite case is analogous), so that $\alpha < k < \beta$.

We claim that there exists a point $x_0 \in (a, b)$ such that $\hat{u}(x) \leq k$ for every $x \leq x_0$ and $\hat{u}(x) \geq k$ for every $x \geq x_0$.

Indeed, fix $x_0 \in (a, b)$ such that $u(x_0) = k$. Then $\hat{u}(x_0) = k$ too. If $\hat{u}(x) > k$ for some $x < x_0$, then by $\alpha < k$ we get $a < x$ and, by definition of $\hat{u}$, $u(x) = \hat{u}(x) > k$. Therefore, $x \in H^+$, so that there exists i such that $x \in (c_i, d_i)$, $a < c_i < x < d_i \leq x_0 < b$ and $u(c_i) = u(d_i) = k$. By the very definition of $\hat{u}$ this implies $\hat{u}(y) = k$ for every $y \in [c_i, d_i]$, which is an absurd.

Now, applying Case 2 to the function $\hat{u}$ with (a, b) replaced by (a, x_0) and β replaced by k, being $\max_{x \in [a, x_0]} \hat{u}(x) = \max\{\alpha, k\} = k$, we get the existence of an increasing function $w_1 : [a, x_0] \to \mathbb{R}$, satisfying the boundary conditions $w_1(a) = \alpha$, $w_1(x_0) = k$, such that

$$\int_a^{x_0} f(w_1(x), w_1'(x)) \, \mathrm{d}x \leq \int_a^{x_0} f(\hat{u}(x), \hat{u}'(x)) \, \mathrm{d}x.$$

Similarly, applying Case 1 with (a, b) replaced by (x_0, b), and α replaced by k, being $\min_{x \in [x_0, b]} \hat{u}(x) = \min\{k, \beta\} = k$, we get the existence of an increasing function $w_2 : [x_0, b] \to \mathbb{R}$, satisfying the boundary conditions $w_1(x_0) = k$, $w_1(b) = \beta$, such that

$$\int_{x_0}^b f(w_2(x), w_2'(x)) \, \mathrm{d}x \leq \int_{x_0}^b f(\hat{u}(x), \hat{u}'(x)) \, \mathrm{d}x.$$

Finally, gluing the functions w_1, w_2 we obtain an increasing function $w \in \Omega$ such that $F(w) \leq F(\hat{u})$ and this concludes the proof. $\square$

Remark 3.3. In view of the proof of the previous theorem, it is immediate to see that it holds true also for Lagrangian depending on the independent variable x too, that is $f = f(x, s, z)$, provided that $f(x, \cdot, \cdot)$ satisfies (H3) for every $x \in [a, b]$ and $f(x, s, 0) \leq f(x, s, z)$ for every $z \in \mathbb{R}$. In this case condition (H4) has to be explicitly assumed since Lemma 2.1 does not hold for non-autonomous problems.

Remark 3.4. If $f(\cdot, 0)$ is strictly monotone in $(\inf I, k)$ and in $(k, \sup I)$ or if $f(s, 0) < f(s, z)$ for every $z \neq 0$, then any possible minimizer belongs to the class Ω^*, i.e. it satisfies either the maximum or the minimum principle.

Proof of Lemma 3.2. We write the proof in the case $u(c) \leq u(x)$ for every $x \in [c, d]$, since in the other case the scheme of the proof is the same.

Put

$$A := \{v : [c, d] \to \mathbb{R} : v \text{ is increasing}, v(x) \leq u(x) \text{ for } x \in (c, d), v(c) = u(c), v(d) = u(d)\},$$

and for every $x \in [c, d]$ define $w : [c, d] \to \mathbb{R}$,

$$w(x) := \sup\{v(x) : v \in A\}.$$

Note that the function $\hat{v}(x) :\equiv u(c)$ for $x \in [c, d)$, $\hat{v}(x) = u(d)$ for $x = d$ belongs to A, so w is well-defined. Moreover, it is immediate to verify that w is monotone increasing, since if $x_1 < x_2$ then $v(x_1) \leq v(x_2)$ for every $v \in A$, hence $w(x_1) \leq w(x_2)$. So, $w \in A$.

From now on, the proof will proceed by steps.

Step 1. *w is continuous.*

Since w is increasing, then $w(x^-) \leq w(x^+)$ at every $x \in (c, d)$. Assume by contradiction that $w(\tilde{x}^-) < w(\tilde{x}^+)$ for some $\tilde{x} \in (c, d)$. Choose an intermediate value $y \in (w(\tilde{x}^-), w(\tilde{x}^+))$. Since u is continuous and $u(\tilde{x}) = u(\tilde{x}^+) \geq w(\tilde{x}^+) > y$, we get $u(x) > y > w(x)$ in $(\tilde{x} - \rho, \tilde{x})$ for some $\rho > 0$ sufficiently small. So, if we consider the function $\tilde{v}(x) :\equiv y$ for $x \in (\tilde{x} - \rho, \tilde{x})$ and $\tilde{v}(x) = w(x)$ elsewhere, we have that $\tilde{v} \in A$ with $\tilde{v}(x) > w(x)$ in $(\tilde{x} - \rho, \tilde{x})$, in contradiction with the definition of w.

The continuity at $x = d$ can be proved in a similar way. Finally, note that $w(c) = u(c) = u(c^+) \geq w(c^+) \geq w(c)$, so w is continuous at $x = c$ too.

Step 2. *If there exists $w'(x) > 0$ then $w(x) = u(x)$.*

In order to show this, assume by contradiction the existence of $\bar{x}$ with $w'(\bar{x}) > 0$ and $w(\bar{x}) < u(\bar{x})$. Of course, $\bar{x} \in (c, d)$ and we infer that for some $\delta > 0$ small enough we have $w(x) < w(\bar{x}) < u(x)$ in $[\bar{x} - \delta, \bar{x}]$. Hence, put $\bar{v}(x) := w(\bar{x})$ for $x \in [\bar{x} - \delta, \bar{x}]$ and $\bar{v}(x) = w(x)$ elsewhere, we have that $\bar{v} \in A$, but $\bar{v}(x) > w(x)$ in $(\bar{x} - \delta, \bar{x})$, again in contradiction with the definition of w.

Step 3. *If $w(\xi) < u(\xi)$ there exist $\xi_1 < \xi < \xi_2$ such that $w(\xi_1) = u(\xi_1) = w(\xi_2) = u(\xi_2)$.*

Indeed, note that for every $x \in (c, d)$ such that $w(x) < u(x)$ we have that w is constant in a left neighborhood of x (it suffices to repeat the same argument of Step 2). So, fixed $\xi \in (c, d)$ such that $w(\xi) < u(\xi)$, set

$$\xi_1 := \inf\{x : w(t) < u(t) \text{ for every } t \in [x, \xi]\}, \ \xi_2 := \sup\{x : w(t) < u(t) \text{ for every } t \in [\xi, x]\}.$$

Of course $w(\xi_1) = u(\xi_1)$ and $w(\xi_2) = u(\xi_2)$ (since $w(c) = u(c)$ and $w(d) = u(d)$). Moreover, by virtue of what just observed, $w(x)$ is constant in a left neighborhood of every point $x \in (\xi_1, \xi_2)$. Hence, by the continuity of w we infer that w is constant in $[\xi_1, \xi_2]$.

Step 4. *For every $[\alpha, \beta] \subset [c, d]$ there exists $[\alpha', \beta'] \subset [\alpha, \beta]$ such that $w(\beta) - w(\alpha) = |u(\beta') - u(\alpha')|$.*

In order to show this, let us consider the nontrivial case $w(\beta) > w(\alpha)$. Note that if $w(\alpha) < u(\alpha)$ then by virtue of Step 3 there exists $\alpha' > \alpha$ such that w is constant in $[\alpha, \alpha']$ and $w(\alpha') = u(\alpha')$. Since $w(\beta) > w(\alpha)$, we get $\alpha' < \beta$. Similarly, if $w(\beta) < u(\beta)$ there exists $\beta' < \beta$ such that w is constant in $[\beta', \beta]$ with $u(\beta') = w(\beta')$. Of course, since $w(\beta) > w(\alpha)$, we get $\alpha' < \beta'$. Therefore, if we denote again by α' the value α in the case $w(\alpha) = u(\alpha)$ and similarly for β', we obtain $0 < w(\beta) - w(\alpha) = u(\beta') - u(\alpha') = |u(\beta') - u(\alpha')|$.

Step 5. *w is absolutely continuous.*

Since u is absolutely continuous, for every $\epsilon > 0$ there exists a positive real $\delta = \delta(\epsilon) > 0$ such that for every finite collection $\{(x_k, y_k), k = 1, \ldots, n\}$ of nonoverlapping intervals with $\sum_{k=1}^{n}(y_k - x_k) < \delta$, we have $\sum_{k=1}^{n}|u(y_k) - u(x_k)| < \epsilon$.

Fixed a family $\{(\alpha_k, \beta_k), k = 1, \ldots, n\}$ of nonoverlapping intervals with $\sum_{k=1}^{n}(\beta_k - \alpha_k) < \delta$, by what proved in Step 4 we have

$$\sum_{k=1}^{n}|w(\beta_k) - w(\alpha_k)| = \sum_{k=1}^{n}|u(\beta_k') - u(\alpha_k')| < \epsilon$$

since $\sum_{k=1}^{n}(\beta_k' - \alpha_k') \leq \sum_{k=1}^{n}(\beta_k - \alpha_k) < \delta$, then w is absolutely continuous.

Step 6. $w'(x) = u'(x)$ for a.e. x such that $w'(x) > 0$.

Let us fix a point $x \in [c, d]$ such that there exist $u'(x)$, $w'(x)$ and $w'(x) > 0$. Then, by Step 2 we have $w(x) = u(x)$ and so

$$u'(x) = \lim_{\xi \to x^+} \frac{u(\xi) - u(x)}{\xi - x} = \lim_{\xi \to x^+} \frac{u(\xi) - w(x)}{\xi - x} \geq \lim_{\xi \to x^+} \frac{w(\xi) - w(x)}{\xi - x} = w'(x),$$

$$u'(x) = \lim_{\xi \to x^-} \frac{u(\xi) - u(x)}{\xi - x} = \lim_{\xi \to x^-} \frac{u(\xi) - w(x)}{\xi - x} \leq \lim_{\xi \to x^-} \frac{w(\xi) - w(x)}{\xi - x} = w'(x)$$

hence $w'(x) = u'(x)$ and this concludes the proof. $\qquad\square$

4. DuBois-Reymond necessary condition

The main result in this section is the following DuBois-Reymond necessary condition for minimizers of problem (P), expressed by a differential inclusion involving the subdifferential of Convex Analysis.

Theorem 4.1. *Let $f : I \times \mathbb{R} \to [0, +\infty)$ satisfy (H4). Let $u \in \Omega^*$ be a solution to (P). Then $\partial f(u(x), u'(x)) \neq \emptyset$ for a.e. $x \in (a, b)$ and there exists a constant $c \leq \mu$ such that*

$$f(u(x), u'(x)) - c \in u'(x)\, \partial f(u(x), u'(x)) \qquad a.e. \ in \ (a, b),$$

where

$$\mu = \min_{s \in [\min u, \max\{\alpha, \beta\}]} f(s, 0) \qquad if \ u \in \Omega_M$$

and

$$\mu = \min_{s \in [\min\{\alpha, \beta\}, \max u]} f(s, 0) \qquad if \ u \in \Omega_m.$$

Moreover, if $u'(x) = 0$ in a set of positive measure, then $c = \mu$.

To prove the above theorem, we use an analogous result obtained in [8] for constrained variational problems, which asserts that in the presence of constraints on the derivatives the (DBR) condition is necessary and sufficient for the optimality of a trajectory u_0.

In its statement we adopt the following notation: for any function $h : [0, +\infty) \to \mathbb{R}$

$$\partial_+ h(z_0) := \{k \in \mathbb{R} \ : \ h(z) - h(z_0) \geq k(z - z_0) \ \text{ for every } z > 0\}, \quad z_0 \geq 0.$$

Theorem 4.2. ([8, Theorem 7]). *Let* $\alpha \leq \beta$ *and let* $f : I \times [0, +\infty) \to [0, +\infty)$ *be lower-semicontinuous. Consider the minimization problem*

$$(P^+) \qquad \text{minimize } \int_a^b f(u(x), u'(x)) \ \mathrm{d}x , \qquad \text{for } u \in \Omega^+$$

with

$$\Omega^+ := \{u \in W^{1,1}(a, b) : u(a) = \alpha, u(b) = \beta, u'(x) \geq 0 \ \text{ for a.e. } x \in (a, b)\}.$$

Then $u_0 \in \Omega^+$ *is a minimizer of (P^+) if and only if the following two properties hold:*

(i) $\partial_+ f(u_0(x), u_0'(x)) \neq \emptyset$ *for a.e.* $x \in (a, b)$ *satisfying* $u_0'(x) > 0$;

(ii) *there exists a constant* $c \leq \nu$, $\nu := \min\limits_{s \in [\alpha, \beta]} f(s, 0)$, *such that*

$$f(u_0(x), u_0'(x)) - c \in u_0'(x) \partial_+ f(u_0(x), u_0'(x)) \qquad \text{a.e. in } (a, b). \tag{4.1}$$

Moreover, if (4.1) holds true and $u_0'(x) = 0$ *in a set having positive measure, then* $c = \nu$ *(with the position* $0 \cdot \emptyset = 0$).

To link the (DBR) conditions for constrained and non-constrained problems, first we need to establish a relation between $\partial f(s, \cdot)$ and $\partial_+ f(s, \cdot)$. The following lemma answers to this question.

Lemma 4.3. *Let* $h : \mathbb{R} \to \mathbb{R}$ *be a continuous function at 0 such that* $h(0) = \min h(z)$. *Then, for every* $z_0 > 0$ *we have*

$$\partial_+ h(z_0) = \partial h(z_0).$$

Similarly, for every $z_0 < 0$ *we have*

$$\partial_- h(z_0) = \partial h(z_0),$$

where

$$\partial_- h(z) := \{k \in \mathbb{R} : h(z) - h(z_0) \geq k(z - z_0) \ \text{ for every } \ z < 0\}.$$

Proof. Of course, $\partial h(z_0) \subseteq \partial_+ h(z_0)$; so let us prove the reversed inclusion. To this aim, first note that by the continuity of h at 0 we get

$$h(0) - h(z_0) \geq -k z_0 \text{ for every } k \in \partial_+ h(z_0).$$

Hence, by $h(0) = \min h(z)$ we have

$$h(0) \geq h(z_0) - k z_0 \geq h(0) - k z_0$$

implying $k \geq 0$ since $z_0 > 0$. Thus, for every $z \leq 0$ we have

$$h(z) - h(z_0) \geq h(0) - h(z_0) \geq -k z_0 \geq k(z - z_0) \qquad \text{for every } k \in \partial_+ h(z_0)$$

that is $\partial_+ h(z_0) \subseteq \partial h(z_0)$.

The proof for the case $z_0 < 0$ is analogous. □

We turn now to prove our main result.

Proof of Theorem 4.1. We only give the proof for $u \in \Omega_M$, the case $u \in \Omega_m$ being similar.

Set

$$x_0 := \max\{x \in [a,b] \ : \ u(x) = \min u\}.$$

Of course, u is decreasing in $[a, x_0]$ and increasing in $[x_0, b]$. Put

$$\hat{\Omega} := \{w \in W^{1,1}(a,b) \ : \ w(a) = \alpha, \ w(b) = \beta + 2(\alpha - u(x_0)), \ w'(x) \geq 0 \text{ a.e. in } (a,b)\}$$

and define the function $\hat{f} : [\alpha, \beta + 2(\alpha - u(x_0))] \times [0, +\infty) \to [0, +\infty)$ by

$$\hat{f}(s,z) := \begin{cases} f(2\alpha - s, -z) & \text{if } \alpha \leq s < 2\alpha - u(x_0) \\ \\ f(s - 2(\alpha - u(x_0)), z) & \text{if } 2\alpha - u(x_0) < s \leq \beta + 2(\alpha - u(x_0)) \end{cases} \tag{4.2}$$

and extended by lower semicontinuity at $(2\alpha - u(x_0), z)$, i.e.

$$\hat{f}(2\alpha - u(x_0), z) = \min\{\liminf_{(s,\zeta) \to ((2\alpha - u(x_0))^-, z)} \hat{f}(s, \zeta) \ , \ \liminf_{(s,\zeta) \to ((2\alpha - u(x_0))^+, z)} \hat{f}(s, \zeta)\}.$$

By the lower semicontinuity of f, $\hat{f}$ is lower semicontinuous too.

We claim that $\hat{f}(\cdot, 0)$ is continuous. To prove this it is enough to prove that

$$\hat{f}(2\alpha - u(x_0), 0) = f(u(x_0), 0), \tag{4.3}$$

and the claim immediately follows by (4.2) and the continuity of $f(\cdot, 0)$.

The definition of $\hat{f}$ and the lower semicontinuity of f easily imply

$$\hat{f}(2\alpha - u(x_0), 0) \geq f(u(x_0), 0).$$

To obtain the reverse inequality use the continuity of $f(2\alpha - s, z)$ and of $f(s - 2(\alpha - u(x_0)), z)$ at $z = 0$ and the continuity of $f(\cdot, 0)$ so that

$$\hat{f}(2\alpha - u(x_0), 0) \leq$$
$$\leq \min\{\liminf_{s \to (2\alpha - u(x_0))^-} f(2\alpha - s, 0) \ , \ \liminf_{s \to (2\alpha - u(x_0))^+} f(s - 2(\alpha - u(x_0)), 0)\} = f(u(x_0), 0).$$

Let us now consider the function $\hat{u} : [a,b] \to [\alpha, \beta + 2(\alpha - u(x_0))]$,

$$\hat{u}(x) := \begin{cases} 2\alpha - u(x) & \text{if } a \leq x \leq x_0 \\ \\ u(x) + 2(\alpha - u(x_0)) & \text{if } x_0 < x \leq b. \end{cases} \tag{4.4}$$

It is easy to see that $\hat{u} \in \hat{\Omega}$.

Now, we split the proof into three steps.

Step 1. Let us prove that

$$\hat{f}(\hat{u}(x), \hat{u}'(x)) = f(u(x), u'(x)) \qquad \text{for a.e. } x \in [a,b]. \tag{4.5}$$

To this end, first note that (4.5) holds for a.e. $x \notin [x_1, x_0]$, where $x_1 := \min\{x : u(x) = \min u\}$. Indeed, if $x \in [a, x_1)$ then $u(x) \in (u(x_0), \alpha]$ and hence $\hat{u}(x) = 2\alpha - u(x) \in [\alpha, 2\alpha - u(x_0))$. Thus we get

$$\hat{f}(\hat{u}(x), \hat{u}'(x)) = \hat{f}(2\alpha - u(x), -u'(x)) = f(u(x), u'(x)) \quad \text{for a.e. } x \in [a, x_1).$$

Analogously, if $x \in (x_0, b]$ then $u(x) \in (u(x_0), \beta]$ and so $\hat{u}(x) = u(x) + 2(\alpha - u(x_0)) \in (2\alpha - u(x_0), \beta + 2(\alpha - u(x_0))]$. Thus we have

$$\hat{f}(\hat{u}(x), \hat{u}'(x)) = \hat{f}(u(x) + 2(\alpha - u(x_0)), u'(x)) = f(u(x), u'(x)) \quad \text{for a.e. } x \in (x_0, b].$$

Then, if $x_1 = x_0$ (4.5) holds. If instead $x_1 < x_0$, since $u \in \Omega_M$ then $u(x) = u(x_0)$ for each $x \in (x_1, x_0)$. Therefore, for such values of x we get $\hat{u}(x) = 2\alpha - u(x_0)$ and $\hat{u}'(x) = 0$. Thus, by (4.3) we get that

$$\hat{f}(\hat{u}(x), \hat{u}'(x)) = \hat{f}(2\alpha - u(x_0), 0) = f(u(x_0), 0) = f(u(x), u'(x)) \quad \text{for every } x \in (x_1, x_0)$$

and (4.5) follows.

Step 2. The function $\hat{u}$ is a solution to the problem

$$\text{minimize} \left\{ \int_a^b \hat{f}(v(x), v'(x)) \, dx \, : \, v \in \hat{\Omega} \right\}. \tag{$\hat{P}$}$$

Indeed, for every $w \in \hat{\Omega}$, let us consider the function $v_w \in \Omega$ defined by

$$v_w(x) := \begin{cases} 2\alpha - w(x) & \text{if } a \le x \le y_w^0 \\[2mm] w(x) - 2(\alpha - u(x_0)) & \text{if } y_w^0 < x \le b, \end{cases}$$

with $y_w^0 = \max\{x \in [a, b] : w(x) = 2\alpha - u(x_0)\}$. Note that $v_{\hat{u}} = u$, indeed from the definition of x_0 it follows $y_{\hat{u}}^0 = x_0$.

Let us prove that

$$\hat{f}(w(x), w'(x)) = f(v_w(x), v_w'(x)) \quad \text{for every } w \in \hat{\Omega} \text{ and for a.e. } x. \tag{4.6}$$

In fact, since w is increasing, then v_w is decreasing in $[a, y_w^0]$ and increasing in $[y_w^0, b]$. Moreover, define

$$y_w^1 = \min\{x \in [a, b] : w(x) = 2\alpha - u(x_0)\}.$$

Recalling that $v_w(x) = u(x_0)$ for every $x \in [y_w^1, y_w^0]$, if $x \in [a, y_w^1)$ then $w(x) = 2\alpha - v_w(x) \in [\alpha, 2\alpha - u(x_0))$ and so

$$\hat{f}(w(x), w'(x)) = f(2\alpha - w(x), -w'(x)) = f(v_w(x), v_w'(x)) \quad \text{for a.e. } x \in [a, y_w^1);$$

and if $x \in (y_w^0, b]$ then $w(x) = v_w(x) + 2(\alpha - u(x_0)) \in (2\alpha - u(x_0), \beta + 2(\alpha - u(x_0))]$ and so

$$\hat{f}(w(x), w'(x)) = f(w(x) - 2(\alpha - u(x_0)), w'(x)) = f(v_w(x), v_w'(x)) \quad \text{for a.e. } x \in (y_w^0, b].$$

Thus, if $y_w^1 = y_w^0$ then (4.6) holds true. If instead $y_w^1 < y_w^0$ then

$$\hat{f}(w(x), w'(x)) = \hat{f}(2\alpha - u(x_0), 0) = f(u(x_0), 0) = f(v_w(x), v_w'(x)) \quad \text{for a.e.} x \in [y_w^1, y_w^0]$$

and (4.6) follows. Therefore, from (4.5), (4.6) and the optimality of u, for every $w \in \hat{\Omega}$ we have

$$
\begin{aligned}
\int_a^b \hat{f}(\hat{u}(x), \hat{u}'(x)) \, \mathrm{d}x &= \int_a^b f(u(x), u'(x)) \, \mathrm{d}x \\
&\leq \int_a^b f(v_w(x), v_w'(x)) \, \mathrm{d}x = \int_a^b \hat{f}(w(x), w'(x)) \, \mathrm{d}x, \quad (4.7)
\end{aligned}
$$

i.e. $\hat{u}$ is a solution to $(\hat{P})$.

Step 3. In this step we conclude the proof. Let

$$
\mu^* = \min_{s \in [\alpha, \beta + 2\alpha - u(x_0)]} \hat{f}(s, 0).
$$

Such a constant is well-defined, since $s \mapsto \hat{f}(s, 0)$ is continuous.

Put $A := \{x \in [a, b] : \hat{u}'(x) > 0\}$ and $B := \{x \in [a, b] : \hat{u}'(x) = 0\}$. From Step 2 we may apply Theorem 4.2 to $\hat{f}$ and $\hat{u}$ (which is increasing). We get that $\partial_+ \hat{f}(\hat{u}(x), \hat{u}'(x)) \neq \emptyset$ for a.e. $x \in A$ and there exists a constant $c \leq \mu^*$ such that

$$
\hat{f}(\hat{u}(x), \hat{u}'(x)) - c \in \hat{u}'(x) \partial_+ \hat{f}(\hat{u}(x), \hat{u}'(x)) \qquad \text{a.e. in } A. \quad (4.8)
$$

Moreover, when $|B| > 0$ we have $c = \mu^*$ and $\hat{f}(\hat{u}(x), 0) = \mu^*$ for a.e. $x \in B$.

Now observe that for a.e. $x \in A$ we have

$$
\partial_+ \hat{f}(\hat{u}(x), \hat{u}'(x)) = \begin{cases} -\partial_- f(2\alpha - \hat{u}(x), -\hat{u}'(x)) & \text{if } \hat{u}(x) \in [\alpha, 2\alpha - u(x_0)) \\ \\ \partial_+ f(\hat{u}(x) - 2(\alpha - u(x_0)), \hat{u}'(x)) & \text{if } \hat{u}(x) \in (2\alpha - u(x_0), \beta + 2(\alpha - u(x_0))] \end{cases}
$$

that is by (4.4),

$$
\partial_+ \hat{f}(\hat{u}(x), \hat{u}'(x)) = \begin{cases} -\partial_- f(u(x), u'(x)) & \text{if } x \in [a, x_1] \\ \\ \partial_+ f(u(x), u'(x)) & \text{if } x \in (x_0, b], \end{cases}
$$

with $x_1 := \min\{x : u(x) = \min u\}$.

Therefore, recalling that u is decreasing in $[a, x_0]$ and increasing in $[x_0, b]$, by virtue of Lemma 4.3, we get

$$
\partial_+ \hat{f}(\hat{u}(x), \hat{u}'(x)) = \operatorname{sgn}(u'(x)) \, \partial f(u(x), u'(x)) \quad \text{whenever } u'(x) \neq 0, \text{ a.e.}
$$

then $\hat{u}'(x) \, \partial_+ \hat{f}(\hat{u}(x), \hat{u}'(x)) = u'(x) \, \partial f(u(x), u'(x))$ for a.e. $x \in A$.

Moreover, since by (H4) $0 \in \partial f(s, 0)$ for every $s \in I$, from (4.5) and (4.8) we deduce

$$
f(u(x), u'(x)) - c \in u'(x) \partial f(u(x), u'(x)) \quad \text{for a.e. } x \in [a, b],
$$

with $c = \mu^*$ if $|B| > 0$. Moreover notice that since $0 \in \partial f(s, 0)$ for every $s \geq 0$, then $\partial f(u(x), u'(x)) \neq \emptyset$ for a.e. $x \in (a, b)$.

Finally, since

$$
\begin{aligned}
\mu^* &= \min\{ \min_{s \in [\alpha, 2\alpha - u(x_0)]} \hat{f}(s, 0), \min_{s \in [2\alpha - u(x_0), \beta + 2(\alpha - u(x_0))]} \hat{f}(s, 0)\} = \\
&= \min\{ \min_{s \in [u(x_0), \alpha]} f(s, 0), \min_{s \in [u(x_0), \beta]} f(s, 0)\} = \mu,
\end{aligned}
$$

the conclusion follows. $\qquad\square$

As a consequence of Lemma 2.1, Theorem 3.1 and Theorem 4.1, the following necessary condition holds.

Theorem 4.4. *Let $f : I \times \mathbb{R} \to [0, +\infty)$ satisfy (H1) – (H3). If (P) is solvable then there exists a solution $u \in \Omega^*$ such that $\partial f(u(x), u'(x)) \neq \emptyset$ for a.e. $x \in (a,b)$ and the following DuBois-Reymond condition holds*

$$f(u(x), u'(x)) - c \in u'(x) \ \partial f(u(x), u'(x)) \qquad a.e. \ in \ (a,b)$$

for some constant $c \leq \min\limits_{s \in [\min u, \max u]} f(s, 0)$.
Moreover, if $u'(x) = 0$ in a set having positive measure then $c = \min\limits_{s \in [\min u, \max u]} f(s, 0)$.

Remark 4.5. It is easy to check that the limitation of the constant c in condition (DBR) is

$$\begin{aligned} c &\leq f(\min u, 0) && \text{if } k \leq \min u \\ c &\leq f(\max u, 0) && \text{if } k \geq \max u \\ c &\leq f(k, 0) && \text{if } \min u < k < \max u, \end{aligned}$$

where k is as in (H3).

Remark 4.6. According to Remark 3.4, if $f(s, 0) < f(s, z)$ for every $z \neq 0$ or $f(\cdot, 0)$ is strictly monotone in $(\inf I, k)$ and in $(k, \sup I)$ then the (DBR) condition holds for any possible minimizer to problem (P).

As an immediate consequence of condition (DBR) the following result holds.

Corollary 4.7. *Let $f : I \times \mathbb{R} \to [0, +\infty)$ satisfy (H1)–(H3) and assume that (P) is solvable. Then there exists a minimizer $u \in \Omega^*$ satisfying condition (DBR) and such that*

(a) $f(u(x), u'(x)) = f^{**}(u(x), u'(x))$, for a.e. x in (a,b),
(b) $\partial f(u(x), u'(x)) = \partial f^{**}(u(x), u'(x))$ for a.e. x in (a,b).

In particular,

$$f^{**}(u(x), u'(x)) - c \in u'(x) \ \partial f^{**}(u(x), u'(x)) \quad \text{for a.e. } x \in (a,b),$$

for some constant $c \leq \min\limits_{s \in [\min u, \max u]} f(s, 0)$.

5. Non-existence results

In this section we emphasize the importance of the limitation on the constant c in the (DBR) necessary condition, by deriving some non-existence result for problem (P). For the sake of simplicity, in the sequel we take $\alpha \leq \beta$ (in the opposite case, it suffices to invert α and β in all the statements).

Proposition 5.1. *Let $f : I \times \mathbb{R} \to [0, +\infty)$ satisfy conditions (H1)-(H3). Moreover assume that*

$$\operatorname*{ess\,sup}_{s \in [\alpha, \beta]} \ \liminf_{|z| \to +\infty} \ \inf\{f^{**}(s, z) - z \partial f^{**}(s, z)\} \ > \min\{f(\alpha, 0), f(\beta, 0)\}. \qquad (5.1)$$

Then, problem (P) does not admit solution.

Proof. By Lemma 2.1 we assume (H4) without loss of generality. First observe that by the convexity of f^{**} we have that the function $g_s(z) := \inf\{f^{**}(s,z) - z\partial f^{**}(s,z)\}$ is increasing in $(-\infty, 0)$ and decreasing in $(0, +\infty)$, for every $s \in I$. Hence,

$$\liminf_{|z| \to +\infty} g_s(z) = \inf_{z \in \mathbb{R}} g_s(z).$$

So, by (5.1) there exists a set $H \subset [\alpha, \beta]$ having positive measure such that

$$\inf\{f^{**}(s,z) - z\partial f^{**}(s,z)\} \ > \min\{f(\alpha,0), f(\beta,0)\} \quad \text{for every } s \in H,\ z \in \mathbb{R}. \quad (5.2)$$

Assume by contradiction that (P) admits a solution. Then, by Theorem 4.4 and Corollary 4.7 there exists an optimal trajectory u satisfying the (DBR) condition relatively to f^{**} for some constant $c \le \min_{s \in [\min u, \max u]} f(s,0) \le \min\{f(\alpha,0), f(\beta,0)\}$. Put $A := \{x \in [a,b] : u(x) \in H\}$, we have $\text{meas}(A) > 0$ since u is absolutely continuous. Hence

$$c \in f^{**}(u(x), u'(x)) - u'(x)\partial f^{**}(u(x), u'(x)) \quad \text{for some } x \in A$$

and this implies

$$\inf\{f^{**}(u(x), u'(x)) - u'(x)\partial f^{**}(u(x), u'(x))\} \le \min\{f(\alpha,0), f(\beta,0)\} \quad \text{for some } x \in A$$

in contradiction with (5.2). $\qquad \square$

The previous result finds simple immediate applications to the case of integrands having the affine structure

$$f(s,z) = \phi(s) + \psi(s)h(z).$$

Set

$$\ell := \liminf_{|z| \to +\infty} \inf\{h^{**}(z) - z\partial h^{**}(z)\}.$$

The following result holds.

Proposition 5.2. *Let $\phi, \psi : I \to [0, +\infty)$ be continuous and increasing functions. Let $h : \mathbb{R} \to \mathbb{R}$ be lower semicontinuous, continuous at 0, non-negative and such that $h(0) = h^{**}(0)$. Suppose moreover that $\ell > -\infty$.*
Then, if

$$\phi(\beta) + \psi(\beta)\ell \ > \ \phi(\alpha) + \psi(\alpha)h(0) \qquad (5.3)$$

problem (P) has no solution.

Proof. First note that assumptions (H1)–(H3) are trivially satisfied (see also Remark 2.3).

If (5.3) holds, by the continuity of ϕ and ψ we have $\phi(s) + \ell\psi(s) > \phi(\alpha) + \psi(\alpha)h(0)$ for every s in a neighborhood of β. Then since $\ell = \inf_{z \in \mathbb{R}} \inf\{h^{**}(z) - z\partial h^{**}(z)\}$,

$$\begin{aligned} \inf_{z \in \mathbb{R}} \inf\{f^{**}(s,z) - z\partial f^{**}(s,z)\} &= \phi(s) + \psi(s)\inf_{z \in \mathbb{R}}\inf\{h^{**}(z) - z\partial h^{**}(z)\} \\ &= \phi(s) + \ell\psi(s) \end{aligned}$$

we have that (5.1) holds and by Proposition 5.1 we deduce the conclusion. $\qquad \square$

Remark 5.3. The reverse of inequality (5.1) is quite close to the formula appearing in the last of the two assumptions used by F.H. Clarke to prove the existence of Lipschitz minimizers of convex, non-coercive variational problems, see assumption (H2), formula $(*)$ in [5]. In fact, for instance, in the case of type-affine Lagrangian with the same notations and properties of Proposition 5.2, $(*)$ can be rewritten as

$$A := \lim_{M \to +\infty} \sup_{s \in I, |z| > M} \{\phi(s) + \psi(s)(h^{**}(z) - z\partial h^{**}(z))\} <$$

$$< \inf_{s \in I, |z| < k} \{\phi(s) + \psi(s)(h^{**}(z) - z\partial h^{**}(z))\} =: B$$

for a suitable positive k.

Now, $B \leq \phi(\alpha) + \psi(\alpha)h(0)$ and

$$A \geq \lim_{M \to +\infty} \sup_{|z| > M} \{\phi(\beta) + \psi(\beta)(h^{**}(z) - z\partial h^{**}(z))\} \geq \phi(\beta) + \psi(\beta)\ell.$$

So, if (5.3) is satisfied, then Clarke's sufficient condition $(*)$ does not hold. Conversely, if condition $(*)$ holds, then assumption (5.3) is not satisfied.

Example 1. Let $f(s,z) := \psi(s)\left(\sqrt{1+z^2} + k\right)$, with ψ continuous, increasing and non-negative. In this case it is easy to check that $\ell = k$, so if $k\psi(\beta) > (k+1)\psi(\alpha)$, problem (P) does not admit solution.

Example 2. Let $f(s,z) := \phi(s) + \psi(s)|z|$, with ϕ, ψ continuous, increasing and $\phi(s), \psi(s) \geq 0$. In this case $\ell = h(0) = 0$, so if ϕ is not constant then problem (P) has no solution.

Of course, the previous examples can be reviewed taking a generic function $h(z)$ whose convex envelope $h^{**}(z)$ coincides with one of the functions appearing in the definition of f.

Proposition 5.2 can be extended in the following way.

Proposition 5.4. *Let ϕ, ψ, h be as in Proposition 5.2 and let $\ell > -\infty$. If a constant $m_0 \in [0, \alpha]$ satisfies*

$$\phi(m_0) + h(0)\psi(m_0) < \phi(\beta) + \ell\psi(\beta)$$

and (P) admits a solution u, then $\min u > m_0$.

Proof. Assume by contradiction that $u \in \Omega$ is a minimizer to (P) with $\min u \leq m_0$. Then by Theorem 3.1 there exists a (possibly different) minimizer, say v, which satisfies the maximum principle and the (DBR) condition for some constant c. Moreover, by the proof of Theorem 3.1 we may choose v in such a way that $\min v \leq \min u$. Thus, we have that

$$c \leq f(\min v, 0) \leq f(m_0, 0) = \phi(m_0) + h(0)\psi(m_0) < \phi(\beta) + \ell\psi(\beta).$$

By the continuity of ϕ, ψ we get

$$c < \phi(s) + \ell\psi(s) \leq \phi(s) + \psi(s)\inf\{h^{**}(z) - z\partial h^{**}(z)\} \quad \text{for every } z \in \mathbb{R} \text{ and } s \text{ close to } \beta$$

in contradiction with (DBR). $\qquad\square$

Remark 5.5. Analogous non-existence results can be stated assuming ϕ, ψ decreasing instead of increasing. Indeed, in this case is suffices to replace α with β in condition (5.3). Moreover, in Proposition 5.4 we have to take $m_0 \geq \beta$ and the condition $\min u > m_0$ has to be replaced by $\max u < m_0$.

References

[1] L. Ambrosio, O. Ascenzi, G. Buttazzo: Lipschitz regularity for minimizers of integral functionals with highly discontinuous integrands, *J. Math. Anal. Appl.* **142** (1989), 301-316.

[2] A. Cellina: The classical problem of the calculus of vairations in the autonomous case: relaxation and lipschitzianity of solutions, *Trans. Amer. Math. Soc.* **356** (1), (2003), 415-426.

[3] A. Cellina, A. Ferriero: Existence of Lipschitzian solutions to the classical problem of the calculus of variations in the autonomous case, *Ann. I. H. Poincaré* **20** (2003), 911-919.

[4] F.H. Clarke: *Optimization and nonsmooth analysis,* Wiley-Interscience, New York,1983.

[5] F.H. Clarke: An Indirect Method in the Calculus of Variations, *Trans. Amer. Math. Soc.* **336** (1993), 655-673.

[6] G. Cupini, M. Guidorzi, C. Marcelli: Existence results for free autonomous nonconvex noncoercive variational problems via constrained problems, in preparation.

[7] O.A. Ladyzhenskaya, N.N. Ural'tseva: *Linear and Quasilinear Elliptic Equations,* Mathematics in Science and Engineering 46, Academic Press, New York - London, 1968.

[8] C. Marcelli: Necessary and sufficient conditions for optimality of nonconvex, noncoercive autonomous variational problems with constraints, *Trans. Amer. Math. Soc.,* to appear.

[9] B.S. Mordukhovich: Generalized differential calculus for nonsmooth and set valued mappings, *J. Math. Anal. Appl.* **183** (1994), 250-288.

[10] B.S. Mordukhovich: Discrete approximations and refined Euler Lagrange conditions for nonconvex differential inclusions, *Siam J. Control Optim.,* **33** (1995), 882-915.

[11] B.S. Mordukhovich: *Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications,* Grundlehren Series (Fundamental Principles of Mathematical Sciences), 330 and 331, Springer, 2006.