Boundary value problems and obstacle problem for elastic bodies with free cracks

Fabrizio Colombo [1] *and Franco Tomarelli* [2]

Contents

[1] Dipartimento di Matematica "Francesco Brioschi"– Politecnico di Milano – Piazza Leonardo da Vinci 32 – 20133 – Milano – Italia
[2] Dipartimento di Matematica "Francesco Brioschi"- Politecnico di Milano – Piazza Leonardo da Vinci 32 – 20133 – Milano – Italia

Introduction

We study the minimization of a linear elastic deformation energy coupled with damage energy under various boundary conditions and/or unilateral constraint. The damage is due to cracks at mesoscopic scale. The presence/absence and shape of cracks is not "a priori" prescribed but it is free and depends on a variational principle.

In this paper we prove existence of equilibrium for an elastic body which may undergo damage provided compatibility and safe load conditions are fulfilled. The functional space in which we set our problems is the space *SBD* of vector fields with special bounded deformations ([1],[6]), which is a suitable subspace of *BD* ([22],[24]): the choice is entailed by the structure of the energies under consideration. Our setting allows discontinuous deformations: the region of fracture is the set where $\mathbf{v}$ is discontinuous.

More precisely we assume that the stored energy for an elastic body with free cracks undergoing small deformations is described by the following functional (see [17],[18],[19]):

$$\begin{aligned} F(\mathbf{v}) \;=\; & \int_\Omega \left(\mu|\mathcal{E}(\mathbf{v})|^2 + \frac{\lambda}{2}|Tr\,\mathcal{E}(\mathbf{v})|^2 \right) d\mathbf{x} + \\ & + \alpha\mathcal{H}^{n-1}(J_\mathbf{v}) \;+\; \gamma \int_{J_\mathbf{v}} |[\mathbf{v}] \odot \nu_\mathbf{v}|\, d\mathcal{H}^{n-1} \end{aligned} \tag{0.1}$$

where $\Omega \subset \mathbf{R}^n,\ n = 2, 3$ is the reference configuration of the body, α and γ are strictly positive real constants, the constants λ and μ are the Lamé coefficients of the material, $\mu > 0,\ 2\mu + n\lambda > 0$, $\mathbf{v} : \Omega \subset \mathbf{R}^n \to \mathbf{R}^n$ is a displacement vector field with special bounded deformation, say $\mathbf{v} \in SBD(\Omega)$, $\mathcal{E}(\mathbf{v})$ is the absolutely continuous part of the linear strain tensor, $\mathcal{E}(\mathbf{v}) = \frac{d}{d\mathcal{L}^n}\mathbf{e}(\mathbf{v})$, $\mathbf{e}(\mathbf{v}) = \mathrm{sym}\, D\mathbf{v}$, $\mathcal{H}^{n-1}$ denotes the $(n-1)$ dimensional Hausdorff measure, $J_\mathbf{v}$ is the jump set of $\mathbf{v}$ (the set of points $\mathbf{x}$ where $\mathbf{v}$ has two different one sided Lebesgue limits $\mathbf{v}^+$, $\mathbf{v}^-$ with respect to a suitable direction $\nu_\mathbf{v} = \nu_\mathbf{v}(\mathbf{x})$), while $[\mathbf{v}] = \mathbf{v}^+ - \mathbf{v}^-$ and $\odot$ denotes the symmetric tensor product.

In (0.1) the first term (the volume integral) represents the elastic energy in undamaged regions; the second one is a surface energy (area of material surfaces where damage occurs); the third one describes a weak resistance of the material to compression or crack opening (after crack process has started), is related to Barenblatt model of damage ([4],[5],[8]) and allows the analysis of the model in presence of nontrivial load, even without artificial confinement of the body. For the study of functional (0.1) with $\gamma = 0$ we refer to [14], [15], [6]. We study variational problems for the (non convex) stored energy (0.1) when the body is subject to prescribed volume dead load $\mathbf{f} \in L^p(\Omega, \mathbf{R}^n)$, with $p \geq n$ under Dirichlet or Neumann boundary conditions and/or unilateral constraint on the deformations. Necessary conditions and sufficient conditions are shown under mechanical compatibility of the load (balance and vanishing moments), smallness assumptions on loads (safe load conditions) and geometric compatibility between loads and obstacles. The precise conditions are given by Theorems 2.1, 3.1, 4.1. The proofs use tools from geometric measure theory ([2]) and theory of recession functional for noncoercive problems ([3],[9]).

The result for the obstacle problem (Theorem 4.1) is new. Some particular cases (respectively cantilever and beam with homogeneous Neumann boundary condition) of the boundary value problems considered in Theorems 2.1, 3.1 were studied in [18] and [19]: these cases proved useful in deriving models of elastic-plastic plate and beam, as variational limit of functional of type (0.1) when each term is suitably weighted with respect to the thickness of the approximating 3d body ([10],[11],[12],[18],[19],[20]).

1. Functional framework and preliminary results

The space of vector fields with bounded deformation BD is the natural framework for the study of functionals with linear growth in the symmetrized gradient. Obstacle problems often lead to a lack of coerciveness. In this section we recall the basic definitions and results, about the space BD and some tools of the theory of recession functionals ([3]).

For a given set $U \subset \mathbf{R}^n$ we denote respectively by $\overline{U}$, ∂U, and $\mathrm{co}\, U$ the topological closure, the topological boundary and the convex hull of U; we denote by $\mathcal{H}^{n-1}(U)$ its $(n-1)$ dimensional Hausdorff measure and by $\mathcal{L}^n(U)$ (or shortly $|U|$) its Lebesgue outer measure. $B_\rho(\mathbf{x})$ is the open ball $\{\mathbf{y} \in \mathbf{R}^n; |\mathbf{y} - \mathbf{x}| < \rho\}$, and $B_\rho = B_\rho(\mathbf{0})$. If Ω ,Ω' are open subsets in $\mathbf{R}^n$, by $\Omega \subset\subset \Omega'$ we mean that $\overline{\Omega}$ is compact and $\overline{\Omega} \subset \Omega'$.

Assume $\Omega \subset \mathbf{R}^n,\ n=2,3$, is a Lipschitz open set. If Y is a finite dimensional space, we denote by $L^p(\Omega;Y)$ the space of p integrable functions with respect to the Lebesgue measure with value in Y. Let $\mathcal{M}(\Omega,Y)$ be the space of the bounded measures on Ω with values in Y ($\mathcal{M}(\Omega)$ when $Y=\mathbf{R}$) and let $|\cdot|_{T(\Omega)}$ be the total variation of a measure in $\mathcal{M}(\Omega,Y)$, i.e.

$$|\mu|_{T(\Omega)} = \int_\Omega |\mu| = \sup\Big\{ \int_\Omega \sum_{ij} \phi_{ij} d\mu_{ij} \ :\ \phi_{ij} \in C_0^0(\Omega),\ \ \sum_{ij} \phi_{ij}^2 \le 1, \ \text{ in }\ \Omega \Big\}.$$

If $A \subset \Omega$ is an open set then $|\mu|_{T(A)}$ is defined in the same way with $\phi_{ij} \in C_0^0(A)$ and we define a Borel measure $|\mu|$, by setting for every Borel set $B \subset \Omega$

$$|\mu|(B) = |\mu|_{T(B)} = \inf\left\{|\mu|_{T(A)};\ B\subset A,\ A \text{ open }\right\}.$$

For $\mathbf{v} \in L^1(\Omega,\mathbf{R}^n)$ the set of *Lebesgue points* $\Omega_\mathbf{v}$ is the set of $\mathbf{x}\in\Omega$ s.t. there is $\widetilde{\mathbf{v}}(\mathbf{x}) \in \mathbf{R}^n$ with $\lim_{\varrho\to 0_+} \int_{B_\varrho(\mathbf{x})} |\mathbf{v}(\mathbf{y}) - \widetilde{\mathbf{v}}(\mathbf{y})| dy \,/\, |B_\varrho| = 0$.

The *Lebesgue discontinuity set* $S_\mathbf{v}$ is the complement of Lebesgue points: $S_\mathbf{v} = \Omega \setminus \Omega_\mathbf{v}$. We say that $\mathbf{v}$ has *one-sided limits* $\mathbf{v}^+(\mathbf{x}), \mathbf{v}^-(\mathbf{x})$ at $\mathbf{x}\in\Omega$ with respect to a suitable direction $\nu_\mathbf{v}(\mathbf{x}) \in \{\mathbf{x}\in\mathbf{R}^n : |\mathbf{x}|=1\}$ if

$$\lim_{\rho\to 0^+} \rho^{-n} \int_{\{\mathbf{y}\in B_\rho(\mathbf{x});\ (\mathbf{y}-\mathbf{x})\cdot\nu>0\}} |\mathbf{v}(\mathbf{y}) - \mathbf{v}^+(\mathbf{x})| d\mathbf{y} = 0,$$

$$\lim_{\rho\to 0^+} \rho^{-n} \int_{\{\mathbf{y}\in B_\rho(\mathbf{x});\ (\mathbf{y}-\mathbf{x})\cdot\nu<0\}} |\mathbf{v}(\mathbf{y}) - \mathbf{v}^-(\mathbf{x})| d\mathbf{y} = 0.$$

The *jump set* $J_\mathbf{v}$ of $\mathbf{v}$ is the subset of points $\mathbf{x}$ in $S_\mathbf{v}$ where $\mathbf{v}$ has one-sided limits $\mathbf{v}^+(\mathbf{x}), \mathbf{v}^+(\mathbf{x})$ with respect to $\nu_\mathbf{v}(\mathbf{x})$ and $\mathbf{v}^+(\mathbf{x}) \ne \mathbf{v}^+(\mathbf{x})$.

In the framework of linearized elasticity Ω represents the unstressed reference configuration of an elastic body, we denote by $\mathbf{v}$, $\mathbf{e}(\mathbf{v})$, and $\mathcal{E}(\mathbf{v})$, respectively, the displacement vector field, the linearized strain tensor and its absolutely continuous part:

$$\mathbf{v} \ :\ \Omega \ \to\ \mathbf{R}^n\,,$$

$$\mathbf{e}(\mathbf{v}) = \frac{1}{2}(D\mathbf{v} + (D\mathbf{v})^T),$$

$$\mathcal{E}(\mathbf{v}) = \frac{d\mathbf{e}(\mathbf{v})}{d\mathcal{L}^n}, \qquad \mathbf{e}^a(\mathbf{v}) = \mathcal{E}(\mathbf{v}) d\mathcal{L}^n, \qquad \operatorname{div}\mathbf{v} = \operatorname{Tr}\mathcal{E}(\mathbf{v}) = \nabla\cdot\mathbf{v}\,,$$

here $D\mathbf{v} = \{D_j v_i\}$, $(i = 1, ..., k,\ j = 1, ..., m)$ denotes the distributional derivatives of $\mathbf{v}$; $\nabla\mathbf{v} = \frac{dD\mathbf{v}}{d\mathcal{L}^n}$ denotes its absolutely continuous part and $[\mathbf{v}] \odot \nu_{\mathbf{v}} = \mathrm{sym}(\mathbf{v} \otimes \nu_{\mathbf{v}})$.

The space of functions with bounded deformation. We say that a vector field $\mathbf{v} : \Omega \to \mathbf{R}^n$ has bounded deformation if $\mathbf{v}$ belongs to $L^1(\Omega, \mathbf{R}^n)$ and its symmetrized distributional gradient $\mathbf{e}(\mathbf{v})$ is a Radon measure:

$$BD(\Omega) = \Big\{ \mathbf{v} \in L^1(\Omega, \mathbf{R}^n)\ :\ \mathbf{e}(\mathbf{v}) \in \mathcal{M}(\Omega, M_{n,n}) \Big\}. \tag{1.1}$$

The space BD is endowed with the norm

$$\|\mathbf{v}\|_{BD(\Omega)} = \|\mathbf{v}\|_{L^1(\Omega,\mathbf{R}^n)} + \int_\Omega |\mathbf{e}(\mathbf{v})|.$$

We list the main properties of functions with bounded deformation.
The linear strain tensor $\mathbf{e}(\mathbf{v})$ has the following decomposition

$$\mathbf{e}(\mathbf{v}) = \mathbf{e}^a(\mathbf{v}) + \mathbf{e}^s(\mathbf{v}) = \mathbf{e}^a(\mathbf{v}) + \mathbf{e}^j(\mathbf{v}) + \mathbf{e}^c(\mathbf{v}), \tag{1.2}$$

where $\mathbf{e}^a(\mathbf{v}) = \mathcal{E}(\mathbf{v})d\mathbf{x}$ and $\mathbf{e}^s(\mathbf{v})$ are, respectively, the absolutely continuous and the singular part of $\mathbf{e}(\mathbf{v})$ with respect to $\mathcal{L}^n$, while $\mathbf{e}^j(\mathbf{v})$ and $\mathbf{e}^c(\mathbf{v})$ are respectively the restriction of $\mathbf{e}^s(\mathbf{v})$ to $J_{\mathbf{v}}$ and to its complement; $\mathbf{e}^j(\mathbf{v})$ and $\mathbf{e}^c(\mathbf{v})$ are called the *jump part* and the *Cantor part* of $\mathbf{e}^s(\mathbf{v})$.
The jump set $J_{\mathbf{v}}$ is $\mathcal{L}^n$ negligible, countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable, and

$$\mathbf{e}^j(\mathbf{v}) = (\mathbf{v}^+(x) - \mathbf{v}^-(x)) \odot \nu_{\mathbf{v}}\, \mathcal{H}^{n-1} \llcorner J_{\mathbf{v}} \qquad \mathcal{H}^{n-1}\text{a.e. in } J_{\mathbf{v}}\,. \tag{1.3}$$

We denote by R the set of *rigid body motions*: $\mathbf{v}(\mathbf{x})$ belongs to R iff $\mathbf{v} = A\mathbf{x} + \mathbf{b}$ with A skew-symmetric matrix.
If ϕ is a continuous seminorm on $BD(\Omega)$ which is a norm on R, then

$$\phi(\mathbf{v}) + |\mathbf{e}(\mathbf{v})|_{T(\Omega)} \tag{1.4}$$

is a norm on $BD(\Omega)$ equivalent to $\|\mathbf{v}\|_{BD(\Omega)}$.
For every connected Lipschitz open set Ω, and every continuous linear map $\mathcal{R} : BD(\Omega) \to R$ which leaves fixed the elements of R, there is a constant $c_1 = c_1(\Omega, \mathcal{R})$ such that ([22])

$$\|\mathbf{v} - \mathcal{R}(\mathbf{v})\|_{L^{n/(n-1)}(\Omega)} \le c_1(\Omega, \mathcal{R}) |\mathbf{e}(\mathbf{v})|_{T(\Omega)}, \qquad \forall \mathbf{v} \in BD(\Omega). \tag{1.5}$$

We make one choice of the "projection" $\mathcal{R}$ and we leave it unchanged along all the paper.

For every connected Lipschitz open set Ω there is a constant c_2, dependent only on Ω s.t. ([22]) a *Korn-Poincaré inequality* holds:

$$\|\mathbf{v}\|_{L^{n/(n-1)}(\Omega)} \leq c_2 \, |e(\mathbf{v})|_{T(\overline{\Omega})} \qquad \forall \mathbf{v} \in BD(\mathbf{R}^n) : \operatorname{spt} \mathbf{v} \subset \overline{\Omega}. \tag{1.6}$$

Embedding. $BD(\Omega) \subset L^s(\Omega)$ for all $s \in [1, n/(n-1)]$. The embedding is compact if $s \in [1, n/(n-1))$.

The space $BD(\Omega)$ is dual of a separable Banach space ([24]), hence there is a weak* topology on $BD(\Omega)$, such that closed balls are sequentially compact in the w^* $BD(\Omega)$ topology.

Remark 1.1 - A sequence $\mathbf{v}_k \overset{w^* BD(\Omega)}{\longrightarrow} \mathbf{v}$ iff

$$\mathbf{v}_k \to \mathbf{v} \quad \text{in} \quad L^1(\Omega)^n \quad \text{and} \tag{1.7}$$

$$\mathbf{e}(\mathbf{v}_k) \to \mathbf{e}(\mathbf{v}) \quad w^* \quad \text{in } \mathcal{M}(\Omega, \mathbf{R}^n). \tag{1.8}$$

Trace at the boundary. For any Lipschitz open set Ω there is a linear, continuous and surjective operator $\Gamma : BD(\Omega) \to L^1(\partial\Omega, \mathbf{R}^n)$ s.t. $\Gamma(\mathbf{v}) = \mathbf{v}|_{\partial\Omega}$ for all $\mathbf{v}$ in $BD(\Omega) \cap C(\overline{\Omega})^n$ and there is $c_3 = c_3(\Omega)$ such that

$$\|\Gamma(\mathbf{v})\|_{L^1(\partial\Omega, \mathbf{R}^n)} \leq c_3 \|\mathbf{v}\|_{BD(\Omega)}. \tag{1.9}$$

The trace operator Γ is continuous in the norm topology, but it is not continuous in the weak* topology of $BD(\Omega)$.

Vector fields with special bounded deformation. $SBD(\Omega)$ is the subspace of $BD(\Omega)$ where the Cantor part of $\mathbf{e}(\mathbf{v})$ is zero, i.e.:

$$SBD(\Omega) = \{\mathbf{v} \in BD(\Omega), \quad s.t. \quad \mathbf{e}^c(\mathbf{v}) = 0\}. \tag{1.10}$$

Now we list some tools from the theory of *recession functionals* developed in [3], [9]. For a different, but related perspective, we refer to [7].

Definition 1.2. *([3] Remark 3.17) Given a topological vector space (X,σ) and a functional*
$G : X \to (-\infty,+\infty]$, the sequential recession functional $G_\infty(x)$ of G is

$$(1.11)\qquad G_\infty(x) \;=\; \inf\left\{\liminf_{k\to\infty}\frac{1}{\lambda_k}G(x_0+\lambda_k x_k);\;\lambda_k\to\infty,\quad x_k\to x \text{ in } \sigma\right\},$$

where $x, x_0 \in X$ and $\{\lambda_k\}_k$, $\{x_k\}_k$ are sequences.

The definition was introduced in [3] with the more detailed notation $G^{seq}_{\infty,\sigma}(x)$ to distinguish it from the topological recession functional $G_{\infty,\sigma}(x)$. Both $G^{seq}_{\infty,\sigma}$ and $G_{\infty,\sigma}$ are extensions of the classic recession functional $G^\infty(x)$ of convex analysis. Here we use only the sequential recession functional (and denote it by G_∞) in a non convex context with the choice $\sigma =$weak ${}^*BD(\Omega)$ topology.

Definition 1.3. *For any convex σ closed subset K of a topological vector space (X,σ) we define the recession cone K^∞ of K as the σ closed convex cone*

$$K^\infty \;=\; \bigcap_{\lambda>0}\lambda^{-1}(K-k_0)$$

where k_0 is an element of K.

Definition 1.4. *For any subset K of a topological vector space (X,σ) we define the set K_∞ of sequentially unbounded directions in K:*

$$K_\infty = \left\{x\in X : \exists\lambda_k\in\mathbf{R},\ x_k\in X \text{ with } \lambda_k\to+\infty,\ x_k\xrightarrow{\sigma}x,\ x_0+\lambda_k x_k\in K\ \forall k\right\}$$

where x_0 is an element of K.

Lemma 1.5. *If $K\subset(X,\sigma)$, then (Lemma 2.11, Remark 2.17 of [3])*

$$(\chi_K)_\infty = \chi_{K_\infty}\,,$$

$$K_\infty = K^\infty \qquad \forall K \text{ convex, sequentially } \sigma \text{ closed}\,.$$

Theorem 1.6. *([3] Proposition 3.1, Remark 3.7) If $G : (X,\sigma)\to(-\infty,+\infty]$ is a proper functional, then the inequality*

$$(1.12)\qquad G_\infty(x)\geq 0$$

is a necessary condition for

$$(1.13)\qquad \inf\,\{G(x):\ x\in X\}\ >\ -\infty.$$

An enforcement of the above necessary condition leads to the existence of minimum for non coercive problems as stated below.

Theorem 1.7. *([3] Proposition 3.9) Assume X is the dual of a separable Banach space, σ denotes the weak* topology in X and $G : X \to]-\infty, +\infty]$ is a proper functional verifying:*
(i) semicontinuity: *G is sequentially σ -l.s.c. and proper;*
(ii) compactness: *for all sequences $\lambda_k \to +\infty$ and all sequences $x_k \xrightarrow{\sigma} x$, a uniform bound $G(\lambda_k x_k) \leq C < \infty$ entails $x_k \to x$ strongly;*
(iii) compatibility*: $G_\infty(x) \geq 0,\ \forall x \in X$ (necessary condition) and*

$$\forall z \in \ker G_\infty \text{ there exists } \mu > 0 \text{ such that } G(x - \mu\, z) \leq G(x), \quad \forall x \in X.$$

Then G achieves a finite minimum on X.

Eventually we recall the following theorem of compactness and lower semicontinuity which will be used to verify the hypotheses in Theorem 1.7.

Theorem 1.8. *(Theorems 2.1, 2.2 [18], Lemma 2.3 [19]) Assume $\Omega \subset \mathbf{R}^n$ is a connected Lipschitz open set, $n \geq 2$, Q a positive definite quadratic form on $M_{n,n}$ (symmetric square matrices), $a > 0$, $b > 0$ and $\{\mathbf{z}_k\}_k$ a sequence in $\mathrm{SBD}(\Omega)$ such that*

$$\sup_{k\in\mathbf{N}} \Big\{ \int_\Omega Q(\mathcal{E}(\mathbf{z}_k))\, d\mathbf{x} + \int_{J_{\mathbf{z}_k}} (a + b|(\mathbf{z}_k^+ - \mathbf{z}_k^-) \odot \nu_{\mathbf{z}_k}|)\, d\mathcal{H}^{n-1} \Big\} < +\infty, \tag{1.14}$$

$$\mathcal{R}(\mathbf{z}_k) = \mathbf{0} \quad \forall k. \tag{1.15}$$

Then there is a function $\mathbf{z} \in \mathrm{SBD}(\Omega)$ with $\mathcal{R}(\mathbf{z}) = \mathbf{0}$ and a subsequence such that, without relabelling,

$$\mathbf{z}_k \to \mathbf{z}, \quad \text{strongly in} \quad L^1(\Omega, \mathbf{R}^n), \tag{1.16}$$

$$\mathcal{E}(\mathbf{z}_k) \rightharpoonup \mathcal{E}(\mathbf{z}), \quad \text{weakly in} \quad L^1(\Omega, M_{n,n}), \tag{1.17}$$

$$\mathbf{e}^j(\mathbf{z}_k) \rightharpoonup \mathbf{e}^j(\mathbf{z}), \quad \text{weakly}^* \text{ in} \quad \mathcal{M}(\Omega, M_{n,n}), \tag{1.18}$$

$$\mathcal{H}^{n-1}(J_{\mathbf{z}}) \leq \liminf_k \mathcal{H}^{n-1}(J_{\mathbf{z}_k}), \tag{1.19}$$

$$(1.20)\qquad \int_\Omega Q(\mathcal{E}(\mathbf{z}))\,d\mathbf{x} \le \liminf_k \int_\Omega Q(\mathcal{E}(\mathbf{z}_k))\,d\mathbf{x},$$

$$(1.21)\qquad \int_{J_{\mathbf{z}}} |(\mathbf{z}^+ - \mathbf{z}^-)\odot \nu_{\mathbf{z}}|\,d\mathcal{H}^{n-1} \le \liminf_k \int_{J_{\mathbf{z}_k}} |(\mathbf{z}_k^+ - \mathbf{z}_k^-)\odot \nu_{\mathbf{z}_k}|\,d\mathcal{H}^{n-1}.$$

The hypothesis (1.15) can be dropped by replacing (1.14) with

$$(1.22)\qquad \begin{aligned} \sup_{k\in\mathbf{N}} \Big\{ \int_\Omega Q(\mathcal{E}(\mathbf{z}_k))\,d\mathbf{x} + \int_{J_{\mathbf{z}_k}} (a + b|(\mathbf{z}_k^+ - \mathbf{z}_k^-)\odot \nu_{\mathbf{z}_k}|)\,d\mathcal{H}^{n-1} + \\ + \int_{\partial\Omega} |\mathbf{z}_k|\,d\mathcal{H}^{n-1} \Big\} \ < \ \infty. \end{aligned}$$

In the case $n = 3$ *the relation* $\mathcal{R}(\mathbf{z}_k) = \mathbf{0}$, $\forall k$ *can be replaced by*

$$(1.23)\qquad \int_\Omega \mathbf{z}_k\,d\mathbf{x} = \int_\Omega \mathbf{z}_k \times (\mathbf{x} - \mathbf{P})\,d\mathbf{x} = \mathbf{0}, \quad \textit{for a fixed } \mathbf{P} \textit{ independent of } k.$$

2. Neumann problem in linear elasticity with free cracks

Let us consider an elastic body whose reference configuration is Ω s.t.

$$(2.1)\qquad \Omega \subset \mathbf{R}^n,\ n = 2, 3, \quad \text{non empty connected Lipschitz open set.}$$

The body may undergo small deformation and damage: say the deformations may be discontinuous. We assume that the stored energy due to a displacement field $\mathbf{v}$ in $SBD(\Omega)$ is given by

$$(2.2)\qquad \begin{aligned} F(\mathbf{v}) \ = \ \int_\Omega \left(\mu|\mathcal{E}(\mathbf{v})|^2 + \frac{\lambda}{2}|\mathrm{Tr}\,\mathcal{E}(\mathbf{v})|^2 \right) d\mathbf{x} + \\ + \ \alpha\mathcal{H}^{n-1}(J_{\mathbf{v}}) \ + \ \gamma \int_{J_{\mathbf{v}}} |[\mathbf{v}]\odot \nu_{\mathbf{v}}|\,d\mathcal{H}^{n-1} \end{aligned}$$

where $\mathcal{E}(\mathbf{v}) = \frac{d}{d\mathcal{L}^n}\mathbf{e}(\mathbf{v})$, $\mathbf{e}(\mathbf{v}) = \mathrm{sym}\,D\mathbf{v}$, and λ, μ are the Lamé constants of the material, with

$$(2.3)\qquad \mu > 0, \qquad 2\mu + n\lambda > 0, \qquad \alpha > 0, \qquad \gamma > 0.$$

The body is loaded by a dead force field L, with volume density $\mathbf{f}$ such that

$$(2.4)\qquad \mathbf{f} \in L^p(\Omega, \mathbf{R}^n), \quad p \ge n,$$

then the load energy associated to the displacement $\mathbf{v}$ is expressed by

$$L(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}. \tag{2.5}$$

The total energy G^N associated to the displacement field $\mathbf{v}$ is

$$G^N(\mathbf{v}) \;=\; F(\mathbf{v}) - L(\mathbf{v}) \qquad \forall \mathbf{v}\in SBD(\Omega). \tag{2.6}$$

We want to minimize energy G^N over $\mathbf{v}$ in $SBD(\Omega)$.

Theorem 2.1. *Assume (2.1)-(2.6) and*

$$L(\mathbf{z}) \;=\; 0 \qquad \forall \mathbf{z}\in R \qquad\qquad (\textbf{compatibility}) \tag{2.7}$$

$$\|\mathbf{f}\|_{L^p(\Omega)} \;<\; \frac{\gamma}{c_1\,|\Omega|^{\frac{1}{n}-\frac{1}{p}}} \qquad\qquad (\textbf{safe load}) \tag{2.8}$$

where $c_1 = c_1(\Omega, R)$ is defined in (1.5) and R denotes the rigid body motions. Then the functional $G^N(\mathbf{v})$ achieves a finite minimum over $\mathbf{v}$ in $SBD(\Omega)$.

The compatibility condition (2.7) is a necessary condition for finiteness of infimum of G^N over $SBD(\Omega)$.

Proof – Theorem 1.6 entails that property (2.7) is a necessary condition since

$$G^N(\mathbf{z}) = \;-L(\mathbf{z}) \qquad \forall \mathbf{z}\in R\ .$$

We show that functional (2.6) achieves a finite minimum under assumptions (2.1)-(2.8). By Hölder inequality and (1.5) we get

$$\begin{aligned} |L(\mathbf{v})| \;&=\; |L(\mathbf{v}-\mathrm{R}\mathbf{v})| \;\le\; \|\mathbf{f}\|_{L^p(\Omega)}\,\|\mathbf{v}-\mathcal{R}\mathbf{v}\|_{L^{p'}(\Omega)} \\ &\le\; \|\mathbf{f}\|_{L^p}\,|\Omega|^{\frac{1}{p'}-\frac{1}{n'}}\,c_1(\Omega,\mathcal{R})\,|\mathbf{e}(\mathbf{v})|_{T(\Omega)}. \end{aligned} \tag{2.9}$$

By setting

$$Q(A) \;=\; \mu|A|^2 + \frac{\lambda}{2}(\mathrm{Tr}A)^2\,, \tag{2.10}$$

inequality (2.3) entails

$$Q(A) \;\ge\; q|A|^2 \quad \text{where } q = q_n(\lambda,\mu) = \min(\mu, \mu + \frac{n}{2}\lambda) > 0, \;\; n = 2, 3. \tag{2.11}$$

Then, by Schwarz inequality and $t^2 \geq \delta st - \delta^2 s^2/4$ for $t, s \in \mathbf{R}$, $\delta > 0$, by choosing $s = |\Omega|$, we get

$$
\begin{aligned}
\int_\Omega Q(\mathcal{E}(\mathbf{v}))\, d\mathbf{x} \quad &\geq \; q \int_\Omega |\mathcal{E}(\mathbf{v})|^2\, d\mathbf{x} \;\geq\; q\,|\Omega|^{-1} \left(\int_\Omega |\mathcal{E}(\mathbf{v})|\, d\mathbf{x} \right)^2 \\
&\geq \; \delta\, q \int_\Omega |\mathcal{E}(\mathbf{v})|\, d\mathbf{x} - \frac{\delta^2\, q\, |\Omega|}{4}
\end{aligned}
\tag{2.12}
$$

by summarizing, with any choice $\delta \geq \gamma/q$,

$$
\begin{aligned}
& G^N(\mathbf{v}) \;\geq \\
& \geq \delta\, q \int_\Omega |\mathcal{E}(\mathbf{v})|\, d\mathbf{x} + \gamma \int_{J_\mathbf{v}} |[\mathbf{v}] \odot \nu_\mathbf{v}|\, d\mathcal{H}^{\,n-1} - L(\mathbf{v}) - \frac{\gamma^2\, |\Omega|}{4q} \geq \\
& \geq \left(\gamma - c_1 |\Omega|^{\frac{1}{n} - \frac{1}{p}} \|\mathbf{f}\|_{L^p(\Omega)} \right) |e(\mathbf{v})|_{T(\Omega)} - \frac{\gamma^2\, |\Omega|}{4q}
\end{aligned}
\tag{2.13}
$$

hence, by (2.8), G^N is bounded from below and, if $\{\mathbf{v}_k\}_{k\in\mathbf{N}}$ is a minimizing sequence and $\mathbf{u}_k = \mathbf{v}_k - \mathcal{R}\,\mathbf{v}_k$, then $G^N(\mathbf{v}_k) = G^N(\mathbf{u}_k)$, $\|\mathbf{u}_k\|_{BD(\Omega)} \leq C$. Thanks to Theorem 1.8 with (1.14),(1.15) we get, up to subsequences,

$$
\begin{aligned}
& \mathbf{u}_k \overset{w^* BD}{\rightharpoonup} \mathbf{u} \in SBD(\Omega)\,, \\
& \mathcal{E}(\mathbf{u}_k) \overset{w\, L^2}{\rightharpoonup} \mathcal{E}(\mathbf{u})\,, \\
& \mathbf{u}_k \;\to\; \mathbf{u} \;\text{ strongly } L^s(\Omega) \quad \forall s \in [\,1, n/(n-1)\,)\,, \\
& \mathbf{u}_k \rightharpoonup \mathbf{u} \;\text{ weakly in }\; L^{n/(n-1)}(\Omega)\,, \\
& \Gamma(\mathbf{u}_k) \;\rightharpoonup\; \Gamma(\mathbf{u}) \;\text{ weak}^* \text{ in } \mathcal{M}(\Omega), \\
& L(\mathbf{u}_k) \to L(\mathbf{u}),
\end{aligned}
$$

$$
\liminf_k \int_\Omega Q(\mathcal{E}(\mathbf{u}_k)\, d\mathbf{x} \;\geq\; \int_\Omega Q(\mathcal{E}(\mathbf{u})\, d\mathbf{x}\;.
$$

Then we get

$$
-\infty \;<\; G^N(\mathbf{u}) \;\leq\; \liminf_k\, G^N(\mathbf{u}_k)
$$

so that $\mathbf{u}$ minimize G^N. □

Remark 2.2 - Any minimizer $\mathbf{v}$ of the functional G^N over the whole space $SBD(\Omega)$, with regular jump set $J_\mathbf{v}$ is a variational (weak) solution of the *Neumann problem for the system of linear elasticity with free cracks*:

$$\begin{cases} -\mu\Delta\mathbf{v} - (\lambda+\mu)D\,(\operatorname{div}\mathbf{v}) \;=\; \mathbf{f} & \Omega\setminus J_{\mathbf{v}} \\ \lambda(\operatorname{div}\mathbf{v})\,\nu \,+\, 2\,\mu\,\mathbf{e}(\mathbf{v})\nu \;=\; \mathbf{0} & \partial\Omega\cup J_{\mathbf{v}}\,, \end{cases}$$

where ν denotes the outward normal on $\partial\Omega$ and $\nu_{\mathbf{v}}$ on $J_{\mathbf{v}}$.

3. Dirichlet problem in linear elasticity with free cracks

Let us consider an elastic body, with reference configuration Ω, which undergoes a prescribed displacement $\mathbf{w}$ at the boundary $\partial\Omega$, with

(3.1) $\Omega \subset\subset B_R(\mathbf{x}) \subset \mathbf{R}^n$, $n = 2, 3$, non empty connected Lipschitz open set.

As usual in problems with linear growth, we prescribe the non homogeneous Dirichlet boundary condition by imposing the coincidence of the admissible displacements (defined in the whole set $A \overset{\text{def}}{=} B_R(\mathbf{x})$) with the datum outside $\overline{\Omega}$. We assume that the Dirichlet datum $\mathbf{w}$ verifies

$$\mathbf{w} \in SBD(A) \tag{3.2}$$

and we restrict the admissible deformations in Ω to the ones having an $SBD(A)$ extension coincident with w in $A \setminus \Omega$.
The body may undergo small deformation and damage: say the deformations may be discontinuous. We assume that the stored energy due to a displacement $\mathbf{v}$ in $SBD(\Omega)$ is given by

$$\begin{aligned} E_{\overline{\Omega}}(\mathbf{v}) \;=\; &\int_\Omega \left(\mu|\mathcal{E}(\mathbf{v})|^2 + \frac{\lambda}{2}|Tr\,\mathcal{E}(\mathbf{v})|^2\right) d\mathbf{x} + \\ &+\, \alpha\mathcal{H}^{n-1}(J_{\mathbf{v}}\cap\overline{\Omega}) \;+\gamma\, \int_{J_{\mathbf{v}}\cap\overline{\Omega}} |[\mathbf{v}]\odot\nu_{\mathbf{v}}|\, d\mathcal{H}^{n-1} \end{aligned} \tag{3.3}$$

where $\mathcal{E}(\mathbf{v}) \;=\; \frac{d}{d\mathcal{L}^n}\mathbf{e}(\mathbf{v})\,,\; \mathbf{e}(\mathbf{v}) = \operatorname{sym} D\mathbf{v},$ and λ, μ are the Lamé constants of the material, with

$$\mu > 0, \qquad 2\mu + n\lambda > 0, \qquad \alpha > 0, \qquad \gamma > 0. \tag{3.4}$$

The body is loaded by a dead force field $\mathbf{f}$ such that

$$\mathbf{f} \in L^p(\Omega, \mathbf{R}^n), \qquad p \geq n, \tag{3.5}$$

then the load energy associated to the displacement $\mathbf{v}$ is

$$L = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}\,. \tag{3.6}$$

The total energy associated to the displacement field $\mathbf{v}\in SBD(\Omega)$ is

$$E_{\overline{\Omega}}(\mathbf{v}) - L(\mathbf{v}). \tag{3.7}$$

By setting, for any $\mathbf{v}\in SBD(A)$:

$$\begin{aligned} E_A(\mathbf{v}) \;=\; & \int_A \left(\mu|\mathcal{E}(\mathbf{v})|^2 + \frac{\lambda}{2}|Tr\,\mathcal{E}(\mathbf{v})|^2\right) d\mathbf{x} + \\ & + \alpha\mathcal{H}^{n-1}(J_{\mathbf{v}}) \;+\; \gamma\int_{J_{\mathbf{v}}} |[\mathbf{v}]\odot\nu_{\mathbf{v}}|\,d\mathcal{H}^{n-1}, \end{aligned} \tag{3.8}$$

$$L_A(\mathbf{v}) = \int_A \widetilde{\mathbf{f}}\cdot\mathbf{v}\,d\mathbf{x}\,, \qquad \text{where } \widetilde{\mathbf{f}} = \mathbf{f} \text{ in } \Omega,\ \widetilde{\mathbf{f}} = 0 \text{ in } A\setminus\Omega, \tag{3.9}$$

$$G^D(\mathbf{v}) \;=\; E_A(\mathbf{v}) - L_A(\mathbf{v}) \qquad \forall \mathbf{v}\in SBD(\Omega), \tag{3.10}$$

we are led to the problem

$$\text{minimize} \quad G^D(\mathbf{v}) \quad \text{among } \mathbf{v}\in SBD(A)\;:\;\mathbf{v} = \mathbf{w} \text{ in } A\setminus\overline{\Omega}. \tag{3.11}$$

Theorem 3.1. *Assume (3.1)-(3.10) and*

$$\|\mathbf{f}\|_{L^p(\Omega)} \;<\; \frac{\gamma}{c_2\,|\Omega|^{\frac{1}{n}-\frac{1}{p}}} \qquad (\textbf{safe load})\,, \tag{3.12}$$

where $c_2 = c_2(\Omega)$ is the constant in the Korn-Poincaré inequality (1.6). Then the functional

$$G^D(\mathbf{v}) = E_A(\mathbf{v}) - L_A(\mathbf{v})$$

achieves a finite minimum over

$$W \;=\; \{\mathbf{v}\in SBD(A):\ \mathrm{spt}(\mathbf{v}-\mathbf{w})\;\subset\;\overline{\Omega}\}.$$

Proof – We set

$$G(\mathbf{v}) = G^D(\mathbf{v}) + \chi_W(\mathbf{v})$$

where $\chi_W(\mathbf{v}) = 0$ if $\mathbf{v} \in W$ and $\chi_W(\mathbf{v}) = +\infty$ if $\mathbf{v} \notin W$. We apply the sequential recession functionals theory (see Definitions 1.2, 1.3, 1.4), and test the assumptions of Theorem 1.7 on the functional G^D.
We show that G_∞ is nonnegative and its kernel is trivial, that is:

$$G_\infty(\mathbf{v}) \geq 0, \qquad \ker(G_\infty) = \{\mathbf{0}\}.$$

Once established these properties, the compatibility (iii) of Theorem 1.7 follows. More precisely, by adding and subtracting $\mathbf{w}$, using (1.6) and Hölder inequality, we get

$$\begin{aligned}
&|L(\mathbf{v})| \leq |L(\mathbf{v}-\mathbf{w})| + |L(\mathbf{w})| \leq \\
&\leq \|\mathbf{f}\|_{L^p(\Omega)} \left(\|\mathbf{v}-\mathbf{w}\|_{L^{p'}(\Omega)} + \|\mathbf{w}\|_{L^{p'}(\Omega)}\right) \leq \\
&\leq \|\mathbf{f}\|_{L^p(\Omega)} |\Omega|^{\frac{1}{p'}-\frac{1}{n'}} c_2 \, |\mathbf{e}(\mathbf{v}-\mathbf{w})|_{T(\overline{\Omega})} + \|\mathbf{f}\|_{L^p(\Omega)} \|\mathbf{w}\|_{L^{p'}(\Omega)} \leq \\
&\leq \|\mathbf{f}\|_{L^p(\Omega)} |\Omega|^{\frac{1}{n}-\frac{1}{p}} c_2 \left(|\mathbf{e}(\mathbf{v})|_{T(\overline{\Omega})} + |\mathbf{e}(\mathbf{w})|_{T(\overline{\Omega})}\right) + \|\mathbf{f}\|_{L^p(\Omega)} \|\mathbf{w}\|_{L^{p'}(\Omega)}.
\end{aligned} \tag{3.13}$$

As like as in (2.11),(2.12) with the notation (2.10) we get

$$\int_A Q(\mathcal{E}(\mathbf{v}))\, d\mathbf{x} \;\geq\; \delta\, q \int_A |\mathcal{E}(\mathbf{v})|\, d\mathbf{x} - \frac{\gamma^2\, |A|}{4\, q}. \tag{3.14}$$

By summarizing, with any choice $\delta \geq \gamma/q$, we obtain

$$\begin{aligned}
&G^D(\mathbf{v}) \geq \\
&\geq \delta\, q \int_A |\mathcal{E}(\mathbf{v})| d\mathbf{x} + \gamma \int_{J_{\mathbf{v}}} |[\mathbf{v}] \odot \nu_{\mathbf{v}}| d\mathcal{H}^{n-1} - L(\mathbf{v}) - \frac{\gamma^2\, |A|}{4\, q} \geq \\
&\geq \left(\gamma - c_2\, |\Omega|^{\frac{1}{n}-\frac{1}{p}} \, \|\mathbf{f}\|_{L^p}\right) |e(\mathbf{v})|_{T(A)} + \\
&- \left(\frac{\gamma^2\, |A|}{4\, q} + \|\mathbf{f}\|_{L^p(\Omega)} \, \|\mathbf{w}\|_{L^{p'}(\Omega)} + c_2 |\Omega|^{\frac{1}{n}-\frac{1}{p}} \|\mathbf{f}\|_{L^p} |e(\mathbf{w})|_{T(A)}\right).
\end{aligned} \tag{3.15}$$

To evaluate exactly the sequential recession functional in the homogeneous case, we make a choice of the norm: it is not restrictive to assume that $\operatorname{spt} w \subset\subset A$, since the minimization takes into account only the behavior of w near $\partial\Omega$. In such case an equivalent norm in $BD(A)$ is given by the seminorm $|\mathbf{e}(\mathbf{v})|_{T(A)}$. This does not affect the constant c_2 in (1.6). With the choice $\|\mathbf{v}\|_{BD(\Omega)} = |\mathbf{e}(\mathbf{v})|_{T(A)}$, by comparison of the recession functionals of the right and left hand-side in the inequality (3.15) and taking into account (3.12), we get

$$G^D{}_\infty(\mathbf{v}) \;\geq\; \gamma - |\Omega|^{\frac{1}{n}-\frac{1}{p}} c_2 \, \|\mathbf{f}\|_{L^p} \;>\; 0 \qquad \forall \mathbf{v} \in W \setminus R.$$

The Dirichlet condition makes W an affine space and W is convex and sequentially $w^*\, BD$ closed. Then, by Lemma 1.5, we obtain $(\chi_W)_\infty = \chi_{W_\infty} = \chi_{W^\infty}$ and

$$W_\infty \cap R \;=\; \{\mathbf{0}\}\,,$$

$$G_\infty(\mathbf{z}) \;=\; +\infty \qquad \forall\, \mathbf{z} \in \mathcal{R} \setminus \{\mathbf{0}\}$$

and

$$G_\infty(\mathbf{v}) \;\geq\; \gamma \;-\; |\Omega|^{\frac{1}{n}-\frac{1}{p}}\, c_2\, \|\mathbf{f}\|_{L^p} \;>\; 0 \qquad \forall \mathbf{v} \in W \setminus \{\mathbf{0}\}$$

hence, by summarizing,

$$\ker(G_\infty) \;=\; \{\mathbf{0}\}, \qquad G_\infty(\mathbf{v}) \geq 0 \quad \forall \mathbf{v} \in W.$$

Then G^D satisfies the compatibility (iii).
Since R is finite dimensional, the compactness (ii) of Theorem 1.7 is fulfilled. By arguing as for G^N in Section 2 and applying Theorem 1.8 with (1.22), G^D is $w^*\, BD(A)$ seq. l.s.c.. Hence , taking into account that W is seq. w^* closed convex subset of $BD(\Omega)$, both functionals χ_W and G^D are seq. l.s.c. with respect to the $w^*\, BD$ topology, say assumption (i) is fulfilled too. □

Remark 3.2 - We underline that the main difference about assumptions in Theorems 2.1 and 3.1 (the absence of a condition similar to (2.7) which was a necessary condition in the Neumann problem), relies on the fact that W_∞ contains no nontrivial rigid body motion.

Remark 3.3 - The minimization (3.11) solves the *non homogeneous Dirichlet problem for the system of linear elasticity with free cracks* in the sense that the boundary condition is either assumed or penalized: if a minimizer $\mathbf{u}$ has a non empty intersection $J_{\mathbf{u}} \cap \partial\Omega$, then the stored energy $E_{\overline{\Omega}}$ will include the following amount for crack appearing at the boundary $\partial\Omega$

$$\alpha \mathcal{H}^{n-1}(J_{\mathbf{u}} \cap \partial\Omega) \;+\; \gamma \int_{J_{\mathbf{u}} \cap \partial\Omega} |[\mathbf{w} - \Gamma(\mathbf{u})] \odot \nu_{\mathbf{u}}|\, d\mathcal{H}^{n-1}.$$

Any minimizer $\mathbf{u}$ verifies:

$$\begin{cases} -\mu \Delta \mathbf{u} - (\lambda + \mu) D(\mathrm{div}\mathbf{u}) = \mathbf{f} & \Omega \setminus \overline{J_{\mathbf{u}}} \\ \mathbf{u} = \mathbf{w} & \partial\Omega \setminus J_{\mathbf{u}}. \end{cases}$$

4. Obstacle problem in linear elasticity with free cracks

Let us consider an elastic body whose reference configuration is Ω s.t.

$$\Omega \subset \mathbf{R}^3, \qquad \text{non empty connected Lipschitz open set.} \tag{4.1}$$

The body is free at the boundary. The admissible deformations are constrained to stay in a given rigid box; the contact with the obstacle (boundary of the box) is assumed frictionless. For simplicity we assume that the reference configuration is contained in the upper half-space:

$$\Omega \subset \{\mathbf{x} \in \mathbf{R}^3 : \ \mathbf{x}_3 \geq 0\}, \tag{4.2}$$

and that the unilateral constraint is expressed by

$$\mathbf{x}_3 + \mathbf{v}_3(\mathbf{x}) \geq 0 \qquad \text{a.e. } \mathbf{x} \in \Omega, \tag{4.3}$$

i.e. the body is simply supported by the rigid plane $\mathbf{x}_3 = 0$.
In (4.2),(4.3) and in the following of this section we set $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ denote the canonical basis of $\mathbf{R}^3$, and the indexes denote the components of a vector (not the label of a sequence).
The body may undergo small deformation and damage (say the deformations may be discontinuous): we assume that the stored energy due to a displacement $\mathbf{v}$ in $SBD(\Omega)$ is given by

$$\begin{aligned} F(\mathbf{v}) \ = \ & \int_\Omega \left(\mu |\mathcal{E}(\mathbf{v})|^2 + \frac{\lambda}{2} |Tr\, \mathcal{E}(\mathbf{v})|^2 \right) d\mathbf{x} + \\ & + \ \alpha \mathcal{H}^{n-1}(J_{\mathbf{v}}) \ + \ \gamma \int_{J_{\mathbf{v}}} |[\mathbf{v}] \odot \nu_{\mathbf{v}}| \, d\mathcal{H}^{n-1} \end{aligned} \tag{4.4}$$

where $\mathcal{E}(\mathbf{v}) = \frac{d}{d\mathcal{L}^n} \mathbf{e}(\mathbf{v})$, $\mathbf{e}(\mathbf{v}) = \mathrm{sym}\, D\mathbf{v}$, and λ, μ are the Lamé constants of the material, with

$$\mu > 0, \qquad 2\mu + 3\lambda > 0, \qquad \alpha > 0, \qquad \gamma > 0. \tag{4.5}$$

The body is loaded by a dead force field with density $\mathbf{f}$ such that

$$\mathbf{f} \in L^p(\Omega, \mathbf{R}^3), \quad p \geq 3, \tag{4.6}$$

then the load energy associated to the displacement $\mathbf{v}$ is expressed by

$$L(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}. \tag{4.7}$$

The total energy G^N associated to the displacement field $\mathbf{v}$ is

$$G^N(\mathbf{v}) \;=\; F(\mathbf{v}) - L(\mathbf{v}) \qquad \forall \mathbf{v} \in SBD(\Omega). \tag{4.8}$$

To take into account the "box constraint" we set

$$K \;=\; \big\{ \mathbf{v} \in SBD(\Omega) : \; \mathbf{x} + \mathbf{v}(\mathbf{x}) \in \{ \mathbf{x} \in \mathbf{R}^3 : \, \mathbf{x}_3 \geq 0 \} \;\; \text{a.e.}\, \mathbf{x} \in \Omega \big\}. \tag{4.9}$$

We look for minimizers of the functional

$$G^S(\mathbf{v}) \;=\; F(\mathbf{v}) - L(\mathbf{v}) + \chi_K(\mathbf{v}) \;=\; G^N(\mathbf{v}) + \chi_K(\mathbf{v}) \quad \forall \mathbf{v} \in SBD(\Omega), \tag{4.10}$$

where $\chi_K(\mathbf{v}) = 0$ if $\mathbf{v} \in K$ and $\chi_K(\mathbf{v}) = +\infty$ if $\mathbf{v} \notin K$.

Theorem 4.1. *Assume (4.1)-(4.10),*

$$\|\mathbf{f}\|_{L^p(\Omega)} \;<\; \frac{\gamma}{c_1 \, |\Omega|^{\frac{1}{3} - \frac{1}{p}}} \qquad (\textbf{safe load}) \tag{4.11}$$

where $c_1 = c_1(\Omega)$ is given by (1.5),

$$\int_\Omega \mathbf{f}_1 \, d\mathbf{x} = \int_\Omega \mathbf{f}_2 \, d\mathbf{x} = 0, \quad \int_\Omega \mathbf{f}_3 \, d\mathbf{x} \,<\, 0, \; (\textbf{resultant compatibility}), \tag{4.12}$$

and there is $\mathbf{P}$ in $co\,\Omega$ such that

$$\int_\Omega \mathbf{f} \times (\mathbf{x} - \mathbf{P}) \, d\mathbf{x} \;=\; 0 \qquad\qquad (\textbf{torque compatibility}). \tag{4.13}$$

Then the functional $G^N(\mathbf{v})$ achieves a finite minimum over $\mathbf{v}$ in K.

Proof – We apply the theory of recession functionals by verifying that $G^S = G^N + \chi_K$ fulfills the assumptions of Theorem 1.7. We endow $SBD(\Omega)$ with the equivalent norm (see (1.4))

$$|\mathbf{e}(\mathbf{v})|_{T(\Omega)} + \|\mathcal{R}(\mathbf{v})\|_{L^{\frac{n}{n-1}}(\Omega)} \,. \tag{4.14}$$

This choice does not affect the constant c_1 in (1.5).

Then we choose $\sigma = w^*$ topology of $BD(\Omega)$.
The functional G^N is seq. w^* l.s.c. by Theorem 1.8 with (1.14),(1.23). The set K is closed convex and seq. w^* closed. Then $\chi_K,\ G^N$ and G^S are seq. w^* l.s.c. and K_∞ is seq. w^* closed convex cone. Lemma 1.5 gives

$$K \ \neq\ K^\infty \ =\ K_\infty \ =\ \{\mathbf{v}:\ \mathbf{v}_3(\mathbf{x}) \geq 0 \text{ a.e. } \Omega\}\,, \tag{4.15}$$

$$\chi_{K_\infty} = \chi_{K^\infty}\ . \tag{4.16}$$

We show that ${G^S}_\infty(\mathbf{v}) \geq 0$ for any $\mathbf{v} \in SBD(\Omega)$. We emphasize that now ${G^S}_\infty$ has a non trivial kernel. By the same computations made in (2.9)-(2.13), we get

$$|L(\mathbf{v})| \ \leq\ c_1|\Omega|^{\frac{1}{3}-\frac{1}{p}}\|\mathbf{f}\|_{L^p(\Omega)}\,|\mathbf{e}(\mathbf{v})|_{T(\Omega)} \qquad \forall \mathbf{v} \in SBD(\Omega) \tag{4.17}$$

$$G^N(\mathbf{v}) \geq \Big(\gamma - c_1|\Omega|^{\frac{1}{3}-\frac{1}{p}}\|\mathbf{f}\|_{L^p(\Omega)}\Big)\,|\mathbf{e}(\mathbf{v})|_{T(\Omega)} - \frac{\gamma^2\,|\Omega|}{4q}\,,\ \ \forall \mathbf{v} \in SBD(\Omega). \tag{4.18}$$

By taking into account the choice (4.14) of the norm, we find

$${G^N}_\infty(\mathbf{v}) \ \geq\ \gamma - c_1|\Omega|^{\frac{1}{n}-\frac{1}{p}}\|\mathbf{f}\|_{L^p(\Omega)} \ >\ 0 \qquad \forall \mathbf{v} \in SBD(\Omega)\setminus R\,. \tag{4.19}$$

Since R is finite dimensional, (4.19) entails the compactness (ii) of Theorem 1.7. Since ${G^S}_\infty \ \geq\ {G^N}_\infty + \chi_{K_\infty}$ we have to test the compatibility (iii) of Theorem 1.7 only in the set $R \cap K_\infty$.
Though $R \cap K_\infty = R \cap \{\mathbf{v}:\mathbf{v}_3 \geq 0\}$, then $R \cap K_\infty$ is a seq. w^* closed convex cone. Then (4.12),(4.13) entail ${G^S}_\infty(\mathbf{v}) \geq 0$ for any $\mathbf{v} \in SBD(\Omega)$ and

$$\ker({G^S}_\infty) \ =\ R \cap \ker L \cap K_\infty \ =\ R\ \cap\ \ker L\ \cap \{\mathbf{v} : \mathbf{v}_3 \geq 0\}. \tag{4.20}$$

So the compatibility (iii) will follow if we show that $\ker\,({G^S}_\infty)$ is a linear space. We know that $\mathbf{v} \in \ker({G^S}_\infty)$ entails $\mathbf{v}_3(\mathbf{x}) \geq 0$, a.e $\mathbf{x} \in \Omega$, and, hence, also $\mathbf{v}_3(\mathbf{x}) \geq 0$, a.e $\mathbf{x} \in \overline{\mathrm{co}\,\Omega}$.
Any $\mathbf{v} \in \ker({G^S}_\infty)$ is a rigid body motion $\mathbf{v}$, so it has the representation $\mathbf{v}(\mathbf{x}) = \mathbf{a} \times (\mathbf{x} - \mathbf{P}) + \mathbf{b}$, for suitable vectors $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3$.
Then (4.12),(4.13) and $\mathbf{v} \in \ker({G^S}_\infty)$ imply $\mathbf{b}_3 = 0$.

Otherwise: $\mathbf{b}_3 < 0$ contradicts the facts: $\mathbf{P} \in \mathrm{co}\,\Omega$ and $\mathbf{v}_3(\mathbf{x}) \geq 0$ for a.e $\mathbf{x}$ in $\overline{\mathrm{co}\,\Omega}$; while $\mathbf{b}_3 > 0$ together with (4.12) contradicts $\mathbf{v} \in \ker L$.

Since $\mathbf{P}$ belongs to the open set $\mathrm{co}\,\Omega$, there is $\varrho > 0$ s.t. $B_\varrho(\mathbf{P}) \subset \mathrm{co}\,\Omega$.
Let $\mathbf{v} \in \ker({G^S}_\infty)$, then $\mathbf{v}(\mathbf{x}) = \mathbf{a} \times (\mathbf{x} - \mathbf{P}) + \mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3$, $\mathbf{v}(B_\varrho(\mathbf{P}))$ is a 2d flat disk of radius $|\mathbf{a}|\,\varrho$, centered at $\mathbf{b}$, orthogonal to $\mathbf{a}$ and

$$\mathbf{v}(B_\varrho(\mathbf{P})) \subset \{\mathbf{x}_3 \geq 0\}. \tag{4.21}$$

All the above requirements on $\mathbf{v}(B_\varrho(\mathbf{P}))$ impose $\mathbf{b}_3 \geq 0$, but $\mathbf{v} \in \ker L$ together with (4.12) exclude $\mathbf{b}_3 > 0$.
Then $\mathbf{b}_3 = 0$, and (4.21) entail $\mathbf{a} = t\mathbf{u}_3$ for some $t \in \mathbf{R}$. So that

$$\mathbf{v}(\mathbf{x}) = (\mathbf{b}_1, \mathbf{b}_2, 0) + t\mathbf{u}_3 \times (\mathbf{x} - \mathbf{P})$$

$$-\mathbf{v} \ \in\ R \cap \ker L \cap \{\mathbf{v} : \mathbf{v}_3 \geq 0\} \ =\ \ker({G^S}_\infty)\,.$$

$\square$

Remark 4.2 - The physical meaning of assumptions (4.12),(4.13) is that the load has a non vanishing resultant pointing downward, the torque with respect to the central axis is null and the central axis crosses the interior of the convex hull of Ω.

Remark 4.3 - In [6] the case of a bounded box was studied with $\gamma = 0$. The Signorini problem in Hencky plasticity was studied in [21],[23]. In [13] second order functional with free gradient discontinuity are studied.

Remark 4.4 - Any minimizer (with regular jump set $J_\mathbf{v}$) of the functional G^D is a weak solution of the *Signorini problem for the system of linear elasticity with free cracks* ([16]).

Remark 4.5 - The method of the proof for Theorem 4.1 can be extended to the case of deformations forced to stay in any convex or non convex box Σ. More precisely: given any $\Sigma \subset \mathbf{R}^3$ s.t. $\overline{\Omega} \subset \Sigma$, impose the constraint $\mathbf{v} \in K = \{\mathbf{v} : \mathbf{x} + \mathbf{v}(\mathbf{x}) \in \Sigma,\ \text{ a.e. } \mathbf{x} \in \Omega, \}$. Then Theorem 4.1 holds true with (4.12) substituted by

$$
(4.22) \qquad \begin{cases} \displaystyle\int_{\Omega} \mathbf{f}\cdot\mathbf{b}\, d\mathbf{x} \;\leq\; 0, \quad \forall \mathbf{b}\in\Sigma_\infty \\[2ex] \big\{\mathbf{b}\in\Sigma_\infty\,,\; L(\mathbf{b})=0\big\} \quad\Rightarrow\quad \mathbf{q}+t\mathbf{b}\in\Sigma \quad \forall\mathbf{q}\in\Sigma,\; \forall t\in\mathbf{R}\,. \end{cases}
$$

where Σ_∞ is the set of sequentially unbounded directions of Σ with respect to the euclidian topology of $\mathbf{R}^3$.

Remark 4.6 - As it is clear from the proofs, the Theorems 2.1, 3.1 and 4.1 hold true if the quadratic form Q (defined in (2.11) and to be evaluated in $\mathcal{E}(\mathbf{v})$) is substituted by any quadratic form which is positive definite on symmetric matrices.

References

[1] L.Ambrosio, A.Coscia, G.Dal Maso: Fine properties of functions with bounded deformation, in: *Arch. Rational Mech. Anal.*, **139** (1997), 201-238.

[2] L.Ambrosio, N.Fusco, D.Pallara: Functions of Bounded Variations and Free Discontinuity Problems, *Oxford Mathematical Monographs*, Clarendon Press, Oxford (2000).

[3] C.Baiocchi, G.Buttazzo, F.Gastaldi, F.Tomarelli: General Existence Theorems for Unilateral Problems in Continuum Mechanics, *Arch. Rational Mech. Anal.*, **100** (1988), 149-189.

[4] G.I.Barenblatt: The formation of equilibrium cracks during brittle fracture, general ideas and hypotheses. Axially symmetric cracks, in: *Appl. Math. Mech. (PMM)*, **23** (1959), 622-636.

[5] Z.P.Bažant: Instability, ductility and size effect in strain-softening concrete, in: *J. Eng. Mech. Div. ASCE,* **102** (1976), 331-344.

[6] G.Bellettini, A.Coscia, G.Dal Maso: Compactness and lower semicontinuity properties in *SBD*(Ω), in: *Math. Zeit.*, **228** (1998), 337-351.

[7] G.Bouchitté, P.Suquet: Equi-coercivty of a sequence of variational problems: the role played by recession functions, in: *Séminaire EDP du Collège de France*, Pitman Res. Notes in Maths, (1994).

[8] A.Braides, G.Dal Maso, A.Garroni: Variational formulation of softening phenomena in fracture mechanics: the one dimensional case, in: *Arch. Rational Mech. Anal.*, **146** (1999), 23-58.

[9] G.Buttazzo, F.Tomarelli: Compatibility conditions for nonlinear Neumann problems, in: *Adv. Math.,* **89** (1991), 127-143.

[10] M.Carriero, A.Leaci, F.Tomarelli: Special Bounded Hessian and Elastic-Plastic Plate, in: *Rend. Acc.Naz.Sci. detta dei XL, Memorie di Matematica* 110^0 **XVI**, fasc.13 (1992), 223-258.

[11] M.Carriero, A.Leaci, F.Tomarelli: Plastic free discontinuities and special bounded hessian, in: *C. R. Acad. Sci. Paris,* **314** *(1992)*, 595-600.

[12] M.Carriero, A.Leaci, F.Tomarelli: Strong solution for an elastic-plastic plate, in: *Calc. Var.*, **2** (1994), 219-240.

[13] M.Carriero, A.Leaci, F.Tomarelli: Second Order Variational Problems with Free Discontinuity and Free Gradient Discontinuity , in: *Calculus of Variations: Topics from the Mathematical Heritage of Ennio De Giorgi*, D. Pallara (Ed.), Quaderni di Matematica, 14, Series edited by Dipartimento di Matematica, Seconda Università di Napoli, (2004).

[14] G.Dal Maso, G.Francfort, R.Toader in: *Quasistatic crack growth inite elasticity*, Preprint (2004).

[15] G.Del Piero, L.Truskinovsky: One dimensional model for localized and distributed failure, in: *Proc. Euromech-Mecamat Conference "Mechanics of materials with intrinsic length scale"*, Magdeburg 1998.

[16] G.Fichera: Boundary Value Problems in Elasticity with Unilateral Constraints, in: *Handbbuch der Physik*, **VIa/2**, ed. S.Flugge, Springer, (1972) 391-424.

[17] A.A.Griffith: The phenomenon of rupture and flow in solids, in: *Phyl. Trans. Royal Soc. London*, **A 221** (1920), 163-198.

[18] D.Percivale, F.Tomarelli: Scaled Korn-Poincaré inequality in *BD* and a model of elastic plastic cantilever, in: *Asympt. Anal.*, **23** (2000), 291-311.

[19] D.Percivale, F.Tomarelli: From *SBD* to SBH: the elastic plastic plate, in: *Interfaces Free Bound.*, **4** (2002), 137-165.

[20] D.Percivale, F. Tomarelli: From special bounded deformation to special bounded hessian: the elastic–plastic beam, in: *Quad. 551/P, Dip. Mat. Politecnico di Milano*, (2003), 1–41.

[21] B.D.Reddy, F.Tomarelli: The obstacle problem for an elastoplastic body, in: *Appl. Math. Optim.*, **21** (1990), 89-110.

[22] R. Temam: *Problèmes Mathematiques en Plasticité, Gauthier–Villars,* (1983) Paris.

[23] F.Tomarelli: Signorini problem in Hencky plasticity, in: *Ann. Univ. Ferrara Sez. VII (N.S.)*, **36** (1991), 73-84.

[24] R.Temam, G.Strang: Functions of Bounded Deformation, in: *Arch. Rational Mech. Anal.,* **75** (1980), 7-21.