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par

Michael Goldman

Sur quelques problèmes variationnels avec pénalisation d'interfaces

Pattern formation in variational problems with interfacial terms

Soutenue le 17/12/2018 devant le jury composé de :

Xavier Blanc	Coordinateur du mémoire
Guy Bouchitté	Examinateur
Yann Brenier	Examinateur
Guy David	Examinateur
Robert V. Kohn	Rapporteur
Etienne Sandier	Rapporteur
Sylvia Serfaty	Examinatrice

Préambule

Le but de ce mémoire est de présenter les résultats que j'ai obtenus depuis la fin de ma thèse sur la question de l'influence de la présence d'énergies d'interface dans des problèmes variationnels issus de la physique.
Les modèles présentés ici ont comme point commun de mettre en jeu une compétition entre un terme de surface (ou de longueur en dimension deux) avec un terme volumique (la seule exception étant l'énergie considérée au Chapitre 6 où ce terme volumique est remplacé par une énergie ponctuelle). Cette interaction complexe et souvent non-locale engendre une variété de comportements et fait donc naître de nombreuses questions mathématiques. Comme nous le verrons, un premier phénomène important est l'absence dans certaines situations de minimiseurs (globaux ou locaux) pour ces problèmes variationnels. En règle générale, le terme d'interface joue un rôle régularisant favorisant l'existence tandis que le terme volumique est de nature répulsive, jouant en sens inverse. Toutefois, dans certaines situations c'est de fait le contraire qui est observé (voir Chapitre 4).
Lorsque les minimiseurs existent ceux-ci peuvent prendre des formes et des motifs très variés. Grand nombre des modèles présentés ici peuvent être vus comme des variantes ou des perturbations du problème isopérimétrique. Celui-ci consiste à minimiser le périmètre parmi tous les ensembles de volume prescrit. Il est connu depuis des centaines d'annes que la solution est donnée par la boule. Nous verrons que ceci reste vrai pour certains des problèmes étudiés dans ce mémoire. Un rôle important est joué ici par la stabilité de la boule pour le problème isopérimétrique (elle même très liée aux inégalités isopérimétriques quantitatives qui ont été le sujet de nombreux travaux de recherche ces dernières années). À nouveau, cette règle générale a ses exceptions tel le cas des condensats de Bose-Einstein à deux composantes étudié au Chapitre 7 où la minimalité de la boule est obtenue grâce au terme volumique.
Toutefois, dans de nombreux cas les configurations minimisantes ont des formes beaucoup plus complexes et peuvent s'arranger selon différents motifs (bandes, motifs branchés etc...). Bien que cela soit rigoureusement démontré que dans très peu de cas, on s'attend à ce que ces motifs soient typiquement périodiques. La plupart des problèmes variationnels étudiés ici peuvent être

vus comme des énergies non-convexes régularisées par un terme d'interface qui fixe l'échelle des oscillations. Ces modèles présentent souvent des échelles multiples. Afin de mieux comprendre celles-ci, la dérivation de modèles réduits dans les régions extrêmes du diagramme de phase est une question naturelle. Nous verrons que ceci fait émerger naturellement des énergies de type Steiner ou de transport branché.

Si un effort de contextualisation a été fait, ce mémoire a pour but principal de présenter mes travaux de recherche et il ne contient donc pas de présentation exhaustive de la littérature existente sur les sujets traités. Je m'excuse donc par avance pour les oublis et les résultats non mentionnés.
L'ordre des chapitres est dicté par des proximités thématiques. Les deux premières parties se concentrent sur des modèles similaires au modèle de Gamow (aussi connu sous le nom de modèle d'Ohta-Kawasaki avec interface nette). La troisième a en commun avec la première de s'intéresser à la question de l'existence/non-existence de solutions à support compact. Les deux parties suivantes s'intéressent à la fonctionnelle de Ginzburg-Landau dans deux régimes opposés. Pour des raisons diamétralement opposées, ces deux problèmes font apparaître de façon asymptotique des énergies de type transport branché. L'avant dernière partie se concentre sur l'étude des phénomènes de ségrégation pour les condensats de Bose-Einstein à deux espèces. Celle-ci a en commun avec la dernière partie (consacrée à la connexité des ensembles isopérimétriques soumis à un potentiel convexe) d'être les seules parties présentant des modèles purement locaux.

Remerciements

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J'aimerais profiter de cette occasion pour remercier à nouveau chaleureusement ceux qui ont joué pour moi le rôle de mentors. Si la liste est longue (à commençer par les enseignants de primaire!), je pense qu'il est juste de débuter par Antonin Chambolle. En encadrant mon stage de License puis ma thèse de Doctorat, il m'a le premier initié aux plaisirs de la recherche mathématique. La thèse a été pour moi l'occasion d'entrer en contact avec Matteo Novaga qui est rapidement passé du statut de co-directeur non officiel à collaborateur et ami. Sa capacité à faire fondre les difficultés mathématiques comme neige au soleil ne cesse de m'impressionner. Je remercie Irene Fonseca et Giovanni Leoni pour leur accueil à Pittsburgh juste après la fin de ma thèse. Ce séjour m'a permis d'entamer un virage vers des mathématiques un peu plus appliquées. Les mots me manquent pour exprimer ma gratitude envers Felix Otto qui a dirigé mon Postdoc à l'Institut Max Planck de Leipzig. Son gout mathématique couplé à une vision large, une technicité très pointue et une immense générosité a eu un impact très fort sur mon travail.

La plupart des résultats présentés dans ce mémoire sont le fruit de collaborations. Sans ces échanges cette thèse serait beaucoup plus courte et assez pauvre. Je remercie donc du fond du coeur Peter Bella, Giovanni Bellettini, Antonin Chambolle, Sergio Conti, Martin Huesmann, Lucia De Luca, Guido De Philippis, Marc Josien, Vincent Millot, Matteo Novaga, Felix Otto, Matthias Röger, Jimena Royo-Letellier, Berardo Ruffini, Eris Runa, Sylvia Serfaty, Marta Strani et Barbara Zwicknagl pour ces heures de travail et tout ce qui va avec. Une mention très spéciale va à Benoît Merlet en tant que collaborateur très spécial.

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Mes derniers remerciements vont à tous ceux, amis et famille, qui n'ont pas été déjà cités mais sans qui tout ceci n'aurait que peu de sens.

Contents

Publication list

In preparation:

1. **How to recognize functions depending only on one set of variables: a non-local and non-convex approach** with B. Merlet.

Preprints:

1. **A two-point function approach to connectedness of drops in convex potentials** with G. De Philippis.
2. **A large-scale regularity theory for the Monge-Ampère equation with rough data and application to the optimal matching problem**, with M. Huesmann and F. Otto.
3. **A Ginzburg-Landau model with topologically induced free discontinuities**, with B. Merlet and V. Millot.
4. **A variational proof of partial regularity for optimal transportation maps**, with F. Otto.
5. **On the optimality of stripes in a variational model with non-local interactions**, with E. Runa.

Published Journal Papers:

1. **A gradient flow approach to relaxation rates for the multi-dimensional Cahn-Hilliard equation**, with L. De Luca et M. Strani, accepted in Math. Annalen.
2. **Quantitative estimates for bending energies and applications to non-local variational problems**, with M. Novaga and M. Röger, accepted in Proc. Roy. Soc. Edimburgh.
3. **Self-similar minimizers of a branched transport functional**, accepted in Indiana U. Math. J..
4. **A branched transport limit of the Ginzburg-Landau functional**, with S. Conti, F. Otto and S. Serfaty, J. École Polytechnique, 5:317–375 (2018).

5. **On minimizers of an isoperimetric problem with long-range interactions and convexity constraint**, with M. Novaga and B. Ruffini, Analysis and PDEs, 11(5):1113–1142 (2018).

6. **Phase segregation for binary mixtures of Bose-Einstein Condensates**, with B. Merlet, SIAM J. Math. Anal. 49 (2017), no. 3, 1947–1981.

7. **New bounds for the inhomogenous Burgers and the Kuramoto-Sivashinsky equations**, with M. Josien and F. Otto, Comm. Partial Differential Equations 40 (2015), no. 12, 2237–2265.

8. **Study of island formation in epitaxially strained films on unbounded domains**, with P. Bella and B. Zwicknagl, ARMA, 218, (2015), no. 1, 163–217.

9. **Sharp interface limit for two components Bose-Einstein condensates**, with J. Royo-Letelier, ESAIM COCV (2015), no.3 603-624.

10. **Nucleation barriers at corners for a cubic-to-tetragonal phase transformation**, with P. Bella, Proc. Roy. Soc. Edimburgh, 145 A (2015), 715-724.

11. **Existence and stability for a non-local isoperimetric model of charged liquid drops**, with M. Novaga and B. Ruffini, ARMA, 217 (2015), no. 1, 1–36.

12. **Fine properties of the subdifferential for a class of one-homogeneous functionals**, with A. Chambolle and M. Novaga, Adv. Calc. Var, vol. 8 n. 1 (2015), 31-42.

13. **Scaling law and reduced models for epitaxially strained films**, with B. Zwicknagl, SIAM J. Math. Analysis, 46 (2014), no. 1, 1–24.

14. **The Γ-limit for singularly perturbed functionals of Perona-Malik type in arbitrary dimension**, with G. Bellettini and A. Chambolle, M3AS, vol. 24, Issue 6 (2014).

15. **Plane-like minimizers and differentiability of the stable norm**, with A. Chambolle and M. Novaga, J. Geometric Analysis, vol. 24, Issue 3 (2014).

16. **Representation, relaxation and convexity for variational problems in Wiener spaces**, with A. Chambolle and M. Novaga, J. Math. Pures Appl, vol. 99 (2013), 419-435.

17. **A geometric approach for convexity in some variational problem in the Gauss space**, Rend. Sem. Mat. Padova, vol. 129 (2013).

18. **Approximation and relaxation of perimeter in the Wiener space**, with M. Novaga, Annales IHP - Analyse Non linéaire, vol. 29, (2012), 525-544.

19. **Volume-constrained minimizers for the prescribed curvature problem in periodic media**, with M. Novaga, Calc. Var. and PDE, vol. 44, Issue 3 (2012), 297-318.

20. **Continuous Primal-Dual Methods for Image Processing**, SIAM Journal of Imaging Science vol. 4, no. 1, (2011).

Conference Proceedings and Review Papers:

1. **Equilibrium shapes of charged droplets and related problems: (mostly) a review**, with B. Ruffini, accepted in Geometric flows.

2. **Existence and qualitative properties of isoperimetric sets in periodic media**, with A. Chambolle and M. Novaga, "Geometric Partial Differential Equations", Edizioni della Normale, CRM Series, vol. 15, (2013).

Chapter 1

Introduction

This thesis presents the results I obtained since the end of my PhD on the influence of interfacial terms on pattern formation in variational problems. In all of these models, there is a competition between a surface (or line in dimension two) energy and a volume term (with the exception of Chapter 6, where the volume term is replaced by point interactions). This complex competition leads to a rich variety of phenomena and thus to many interesting mathematical questions.
The simplest and maybe most iconic example of functionals in this class is the so-called Gamow liquid drop model (also known as the sharp interface version of the Ohta-Kawasaki model), see [CMT17]:

$$\min_{|E|=1} P(E) + Q^2 \int_{E\times E} \frac{1}{|x-y|^{d-\alpha}} dxdy, \tag{1.1}$$

where for $E \subset \mathbb{R}^d$, $P(E)$ stands for the De Giorgi perimeter [AFP00], $Q > 0$ is a parameter representing the charge and $\alpha \in (0, d)$. In this model, which is a non-local, non-convex functional regularized by the perimeter, it is known that (at least in some range of exponents α) for small Q, the minimizer is the ball while for large Q minimizers do not exist [KM14]. Considering now sets constrained to live in a finite box, it is expected that in many regimes minimizers form periodic patterns made of shapes which may be lamellar, spherical or more complex according to the parameters (see [ACO09, RW03, MS14, KMN16, Cri15, AFM13]). One of the aim of this thesis is to investigate similar questions for other variational models motivated by a variety of applications in physics and biology.

1.1 What is contained in this thesis

We now give a very brief overview of the content of each chapter.

Chapter 2 contains results obtained in [GNR15, GNR18b, GR17, GNR18a] in collaboration with M. Novaga, M. Röger and B. Ruffini on the equilibrium shapes of charged liquid drops. We first prove that the natural model is always ill-posed and thus needs to take into account some further regularization mechanism. We then prove that for small mass, the ball is a local minimizer both in the class of convex sets (in dimension two) and under $C^{1,1}$ perturbations. Finally, we investigate the effect of a higher order regularization by a Willmore type energy on a simplified model.

Chapter 3 describes the main achievements of [GR16, GM18] obtained with B. Merlet and E. Runa on the formation of stripe patterns for a variant of (1.1) where the non-local part of the energy is given by an integrable kernel. The main result is a Γ-convergence result (see [DM93]) to a model allowing only one-dimensional competitors. The main ingredient of the proof is a rigidity lemma for a non-local and non-convex energy which characterizes functions depending only on a set of variables. We investigate the fine structure of functions with bounded energy in the borderline case and prove a quantitative version of the rigidity lemma.

Chapter 4 is dedicated to results obtained with P. Bella and B. Zwicknagl in [GZ14, BGZ15] on a variational model for epitaxial growth. We first considered the case in which the film is imposed to live in a bounded region of the space. We proved the optimal scaling law of the energy and studied the asymptotic shape of minimizers, proving that in the regime of strong mismatch, islands must form. We then extended these results to the non-compact case and proved a dichotomy: for small masses minimizers do not exist, while for larger masses they are connected compact and smooth.

Chapter 5 focus on results obtained in collaboration with S. Conti, F. Otto and S. Serfaty in [CGOS18, Gol18] on branching patterns for type-I superconductors. Starting from the full three-dimensional Ginzburg-Landau model we derive in the vanishing magnetic field limit a branched transportation type functional. Then, we study the minimizers of this limit problem and exactly compute them for some parameter regime in the simplified two-dimensional setting.

Chapter 6 contains the main results of [GMM17] obtained in collaboration with B. Merlet and V. Millot on the pattern of defects for the ripple phase in lipid bi-layers. We consider a functional which combines aspect of the two-dimensional Ginzburg-Landau energy with aspects of the Mumford-Shah functional. We first prove that asymptotically the problem is equivalent to an energy which is the sum of (a variant of) the Steiner problem and the renormalized energy of the endpoints. Using a combination of techniques coming from the study of the Ginzburg-Landau and the Mumford-Shah functional we prove that minimizers of the original problem have themselves point singularities connected by Steiner-type graphs.

Chapter 7 reports on results obtained with B. Merlet and J. Royo-Letelier in [GM17, GRL15] on the strong segregation limit for two-component Bose-Einstein condensates. We prove that in the symmetric case the geometry of the minimizers is dictated by a weighted isoperimetric problem while in the non-symmetric case, it is directly given by the Thomas-Fermi limit. In the small asymmetry regime, we prove that there is a symmetry breaking in the sense that for small (but positive) asymmetry minimizers are radial while for stronger asymmetry they are not. Our study confirms some predictions previously made in the physics literature [Van08].

Chapter 8 is dedicated to recent progress made with G. De Philippis in [PG17] to understand the shape of drops subject to a convex external potential. Using a two-point function argument inspired by ideas of B. Andrews [And12], we prove that unconditionally stable critical points must be convex. From this we obtain that minimizers must be connected since otherwise they would have at least one unconditionally stable connected component.

1.2 What is not contained in this thesis

Besides [Gol11, Gol13, GN12a, GN12b, CGN13b] which were part of my PhD thesis, this habilitation thesis does not contain some results I obtained since then. Let me now quickly present them.

Even though they cover questions which are closely connected to the topic of this thesis, I decided not to include here [BCG14, CGN13a, CGN15, CGN14]. Indeed, they could almost be considered as part of my doctoral

dissertation since they were completed shortly afterwards and in collaboration with my PhD advisor(s). The paper [BCG14] contains a derivation of a Mumford-Shah type functional starting from the Perona-Malik energy while [CGN13a, CGN15, CGN14] are concerned with the structure of plane-like minimizers for oscillating interfacial energies.

The paper [BG15] written with P. Bella proves a scaling law for a model describing nucleation barriers for cubic-to-tetragonal phase transitions. This shows that in agreement with the experimental observation, nucleation is energetically favored at corners. This result would have fit perfectly here since the model is roughly speaking a H^{-1} norm regularized by the perimeter (it is a variant of the Kohn-Müller model [KM94]). Nevertheless, I have decided not to include it here because that would require quite a lot of introductory material for a result which is a little isolated from the rest of the text.

In [GJO15], we improved with M. Josien and F. Otto known bounds for the Kuramoto-Sivashinsky equation. The main result is an optimal estimate for the inhomogeneous Burgers equation following the work of [GP13]. This work is about scalar conservation laws and is poorly related to the topic of this thesis.

In [LGS18] we extended with L. De Luca and M. Strani the analysis of [OW14] on the stability of one-dimensional kink states to higher dimension. Since these kink states are strongly related to the perimeter by the result of Modica-Mortola [Mod87] (and since the Cahn-Hilliard equation is related to the Mullins-Sekerka flow), this question is again tightly connected to the topics covered here. However, since we did not consider evolution problems I decided to leave this work aside.

Probably the most important recent paper not included here is the partial regularity result for optimal transport maps obtained in [GO17] together with F. Otto. Using ideas from [ACO09], we set up a Campanato scheme and prove that at every scale, optimal transport maps are close to the gradient of harmonic maps. From this we deduced an $\varepsilon-$regularity theorem. We then extended this result in [GHO18] together with M. Huesmann in order to study the optimal matching problem. Even though I invested a large proportion of my recent time in these projects and even though they grew out (somewhat surprisingly) from my work on branching patterns for type-I superconductors, they are not included here since I felt that they would not

fit nicely.

1.3 General notation and conventions

Since we collect here results from many different papers, notation might not always be consistent. We tried to be as faithful as possible to the original articles while keeping a reasonably unified notation. For instance, we tried to keep d for the dimension of the ambient space but in Chapter 4 (respectively in Chapter 6), we used it for the volume (respectively the degree) since both chapters are about a two-dimensional situation.

We globally adopted the following notation. The symbols $\sim$, $\gtrsim$, $\lesssim$ indicate estimates that hold up to a global constant C, which typically only depends on the dimension d and on the fixed parameters of the problem. For instance, $f \lesssim g$ means that there exists such a constant with $f \leq Cg$, $f \sim g$ means $f \lesssim g$ and $g \lesssim f$. An assumption of the form $f \ll 1$ means that there exists $\varepsilon > 0$, typically only depending on dimension, such that if $f \leq \varepsilon$, then the conclusion holds. For a set E, ν^E will always denote the external normal to E. When clear from the context we will drop the explicit dependence on the set. We write $|E|$ for the Lebesgue measure of a set E and χ_E for the indicator function of E. When no confusion is possible, we will drop the integration measures in the integrals. For $R > 0$ and $x_0 \in \mathbb{R}^d$, $B_R(x_0)$ denotes the ball of radius R centered in x_0. When $x_0 = 0$, we will simply write B_R for $B_R(0)$. The symbol $\mathcal{H}^k$ denotes the $k-$dimensional Hausdorff measure.
We use the notation $P(E)$ for the De Giorgi perimeter of E. We refer to [AFP00] for more on the theory of BV functions and sets of finite perimeter. Let us just point out that for smooth sets, $P(E) = \mathcal{H}^{d-1}(\partial E)$. Since we decided to choose clarity over technical precision, we will not dwell on the regularity issues when it is not crucial for the argument. In particular, the reader should keep in mind that most often we actually work in the class of sets of finite perimeter. These are not smooth and by ∂E we denote the measure theoretic boundary.

Let us conclude by saying that we often present slightly simplified or weaker statements than in the quoted papers in order to improve readability. We tried to include as often as possible proofs or sketch of proofs but considering the length of this thesis it was impossible to include full de-

tails. Therefore, most often when reading the word proof, the reader should understand instead sketch of proof and be indulgent with some imprecisions.

Chapter 2

Equilibrium shapes of charged liquid drops

This Chapter focuses on research done in collaboration with M. Novaga, M. Röger and B. Ruffini in [GNR15, GNR18b, GR17, GNR18a].

2.1 The model

We study (variants of) the following energy:

$$\mathcal{E}_{\alpha,Q}(E) = P(E) + Q^2\mathcal{I}_\alpha(E), \tag{2.1}$$

where $Q > 0$ is a constant representing the charge and $\mathcal{I}_\alpha$ is defined for $\alpha \in (0, d)$ as

$$\mathcal{I}_\alpha(E) = \min_{\mu(E)=1} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{d-\alpha}}, \tag{2.2}$$

and by a slight abuse of notation we denote by $\mathcal{I}_d$ the logarithmic interaction potential. Our aim is to understand existence/non-existence issues and characterization of the minimizers (when they exist) of the following variational problem (where the volume is normalized to the one of the unit ball by a simple scaling):

$$\min_{|E|=|B_1|} \mathcal{E}_{\alpha,Q}(E). \tag{2.3}$$

The mathematical interest of (2.3) lies in the fact that there is a competition between the perimeter which is a local term minimized by the ball

and the non-local electrostatic energy $\mathcal{I}_\alpha$ which is maximized by the ball (at least for $\alpha \leq 2$ [Bet04]). For $\alpha = 2$, $\mathcal{I}_\alpha(E)$ corresponds to the Coulombic interaction energy and (2.3) can be thought as modeling the equilibrium shape of a charged droplet for which surface tension and electric forces compete. Such charged droplets have received considerable attention since the seminal work of Lord Rayleigh [Ray82] and are by now widely used in applications such as electrospray ionization, fuel injection and ink jet printing. Starting with the pioneering experiments of Zeleny [Zel17], the following scenario emerged. For small charge, a spheric drop remains stable but when the charge overcomes a critical threshold Q_c, which depends on the volume of the drop and on the characteristic constants of the liquid (surface tension and dielectric constant), a symmetry breaking occurs. Typically, the drop deforms and quickly develops conical shaped singularities, ejecting a very thin liquid jet [Tay64, FdlM07, FF04]. This jet carries very little mass but a large portion of the charge. This type of behavior has been since then observed in more details and in various experimental setups. We emphasize in particular on [DMV64, AL67], where the disintegration of an evaporating drop is observed, since a model very similar to (2.1) has been proposed in [RK83] to explain these experiments. We should stress the fact that the study of the unstable regime, which is still very poorly understood both experimentally and mathematically (see for instance [Mik81, FF04, FJ15]), is far outside the scope of this thesis. We focus instead on the rather simple variational model (2.3) which hopefully captures, at least for small charges, most of the characteristics of the system. We refer to [MN16] for more physical background and literature.

2.2 Nonlinear instability

The first result of [GNR15], which is also the most surprising one is that the problem (2.3) is actually always ill-posed when $\alpha \in (1, d)$. Roughly speaking this is due to the fact that the perimeter term sees objects of dimension $d-1$ while $\mathcal{I}_\alpha$ naturally lives on object of dimension $d - \alpha$.

Theorem 2.2.1. *For every $d \geq 2$, $\alpha \in (1, d)$ and $Q > 0$,*

$$\inf_{|E|=|B_1|} \mathcal{E}_{\alpha,Q}(E) = \mathcal{H}^{d-1}(\partial B_1).$$

By the isoperimetric inequality, this means that (2.3) *is not attained.*

Proof. For $n \in \mathbb{N}$ and $\beta \in ((d-1)^{-1}, (d-\alpha)^{-1})$, let $r_n = n^{-\beta}$. Consider the competitor E_n made of n balls of radius r_n each carrying a charge n^{-1}

and infinitely far apart together with a ball of radius $R_n \sim 1$ which is free of charge. We can then compute the energy

$$\mathcal{E}_{\alpha,Q}(E_n) = P(B_{R_n}) + nr_n^{d-1}P(B_1) + \frac{Q^2}{n}r_n^{-(d-\alpha)}\mathcal{I}_\alpha(B_1).$$

By the choice of β,

$$\lim_{n\to+\infty} nr_n^{d-1} + n^{-1}r_n^{-(d-\alpha)} = \lim_{n\to+\infty} n^{-(\beta(d-1)-1)} + n^{-(1-\beta(d-\alpha))} = 0,$$

which concludes the proof. □

Performing a more careful analysis it can be shown that actually even local minimizers do not exist for the Hausdorff topology .

Theorem 2.2.2. *Let Ω be a compact subset of $\mathbb{R}^d$ with smooth boundary, and let $0 < m < |\Omega|$. Let E_0 be a solution of the constrained isoperimetric problem*

$$\min\{P(E) : E \subset \Omega,\ |E| = m\}. \tag{2.4}$$

Then, for $\alpha \in (0, d-1)$ and $Q > 0$ we have

$$\inf_{|E|=m,\, E\subset\Omega} \mathcal{E}_{\alpha,Q}(E) = P(E_0) + Q^2\mathcal{I}_\alpha(\Omega). \tag{2.5}$$

If the construction leading to the non-linear instability of the ball described here is made of many disconnected components, it has been proven in [MN16, Th. 2] that (at least in the physical case $d = 3$, $\alpha = 2$) the ball is actually unstable even in the class of smooth graphs over the ball.

Theorem 2.2.3. *Let $d = 3$ and $\alpha = 2$. Then, for every $\delta > 0$, there exists a smooth function $\phi_\delta : \partial B_1 \to (-\delta, \delta)$ such that letting*

$$E_\delta = \left\{x \ :\ |x| \le 1 + \phi_\delta\left(\frac{x}{|x|}\right)\right\}$$

we have $|E_\delta| = |B_1|$ and

$$\mathcal{E}_{2,Q}(E_\delta) < \mathcal{E}_{2,Q}(B_1).$$

In the case $\alpha \in (0, 1]$ one can expect a stronger interaction between both terms in (2.3) which might restore well-posedness. This has been recently investigated in [MNR18], where the case $d = 2$, $\alpha = 1$ corresponding to three-dimensional drops trapped between two very close isolating plates has been completely solved.

2.3 Stability of the ball under "regular" variations

Turning back to the case $\alpha > 1$ where (2.3) is ill-posed, it is natural to wonder if restricting the admissible set could restore well-posedness. This is very much related to the question of the stability of the ball under regular enough variations for small charge (which is expected from the numerical and experimental evidence). A first possibility, explored in [GNR15] is to add a strong constraint on the curvature. For $\delta > 0$, we say that a set E satisfies the $\delta-$ball condition if for every $x \in \partial E$ there are two balls of radius δ touching at x, one of which is contained in E and the other one which is contained in E^c. Notice that this implies in particular that ∂E is $C^{1,1}$ with all the curvatures bounded by δ^{-1}. We set

$$\mathcal{A}_\delta = \Big\{ E \subset \mathbb{R}^d \,:\, |E| = |B_1|,\ E \text{ satisfies the } \delta-\text{ball condition} \Big\}.$$

Under this regularity assumption, it was proven that minimizers exist for small enough charges while non-existence for large charges holds in the case $\alpha > d-1$.

Proposition 2.3.1. *For every $d \geq 2$ and $\alpha \in (0,d)$, there exists $Q_0(d,\alpha) > 0$ such that for every δ small enough and every $Q < Q_0\delta^d$ a minimizer of*

$$\min_{E \in \mathcal{A}_\delta} \mathcal{E}_{\alpha,Q}(E) \tag{2.6}$$

exists. Moreover, if $\alpha > d-1$, there exists $Q_1(d,\alpha) > 0$ such that for every δ small enough and every $Q > Q_1\delta^{-((d-\alpha)(d-1)+1)/2}$, no minimizer of (2.6) *exists.*

Proof. For the existence part the main point is to prove that every minimizing sequence E_n must be connected for $Q < Q_0\delta^d$. By (almost) minimality, we have

$$P(E_n) - P(B_1) \leq Q^2(\mathcal{I}_\alpha(B_1) - \mathcal{I}_\alpha(E_n)). \tag{2.7}$$

The quantitative isoperimetric inequality [FMP08] then implies that we have $|E_n \Delta B_1| \lesssim Q$. Thanks to the $\delta-$ball condition, this yields that E_n is indeed connected for $Q \ll \delta^d$.
The non-existence part is obtained by constructing a competitor made of δ^{-d} balls of radius δ. □

In the Coulombic case $\alpha = 2$, it was shown in [GNR15, Th. 5.6] that for small enough charges, the ball is the unique minimizer of (2.6).

Theorem 2.3.2. *Let $d \geq 2$ and $\alpha = 2$. Then there exists $Q_3(d, \delta)$ such that for $Q \leq Q_3$, B_1 is the only minimizer (up to translation) of problem* (2.6).

The proof of this result is quite long and involved but the basic idea is to argue as in [CL12, KM14, FFM+15, GM17] for instance and show that for small charges minimizers are nearly spherical sets, that is small Lipschitz graphs over ∂B_1. This allows the use of a Taylor expansion of the perimeter for this type of sets given by Fuglede [Fug89]. The main technical lemma is the following.

Lemma 2.3.3. *For $d \geq 2$ and $\alpha = 2$, if E is a nearly spherical set and if the optimal measure μ is bounded in $L^\infty(\partial E)$, then there exists a constant C depending on this L^∞ bound such that*

$$\mathcal{I}_2(B_1) - \mathcal{I}_2(E) \leq C(P(E) - P(B_1)). \tag{2.8}$$

Thanks to the $\delta-$ball condition, it can be proven that for Q small enough, minimizers of (2.6) satisfy the hypothesis of Lemma 2.3.3. The proof of Theorem 2.3.2 is concluded by combining (2.8) together with (2.7).

Remark 2.3.4. *One consequence of Theorem 2.3.2 is the stability of the ball under small $C^{1,1}$ perturbations. This extends a previous result of [FF04] where stability with respect to $C^{2,\alpha}$ perturbations was proven. Let us however point out that in [FF04], the asymptotic stability of the ball is also studied.*

An alternative way to restore well-posedness for (2.3) is to reduce the admissible class to convex sets. If this geometric restriction is not directly comparable with the $\delta-$ball condition, it allows for less regular competitor (Lipschitz). As shown in [GNR18b, Th. 2.3], under the convexity constraint, there is always a minimizer for (2.3).

Theorem 2.3.5. *For every $d \geq 2$, $\alpha \in (0, d]$ and $Q > 0$, there exists a minimizer of*

$$\min \{\mathcal{E}_{\alpha,Q}(E) \,:\, |E| = |B_1|,\ E \text{ convex}\}. \tag{2.9}$$

Having Theorem 2.3.2 in mind, it is natural to wonder if, at least in the Coulombic case, (2.9) is still minimized by the unit ball for small Q. In the bi-dimensional logarithmic case, it has been proven to hold in [GNR18b, Th. 5.1].

Theorem 2.3.6. *Let $d = \alpha = 2$ then for Q small enough, the only minimizer of* (2.9) *is the unit ball.*

The idea is to show that for small charges, minimizers of (2.9) satisfy the hypothesis of Lemma 2.3.3. This is a consequence of the following regularity result.

Theorem 2.3.7. *For $d = \alpha = 2$ and every $Q > 0$, every minimizer of* (2.9) *is $C^{1,1}$, with uniform $C^{1,1}$ bounds for small Q.*

This result proves that in dimension two and when restricted to the class of convex sets, conical singularities never appear. The proof of Theorem 2.3.7 is quite long and technical but the main idea is to show that if E is not regular enough, then we can lower the energy by replacing part of the boundary by a straight line. The major difficulty is to precisely estimate the variation of the non-local term. A crucial technical point is that for convex sets, the optimal charge distribution is in $L^p(\partial E)$ for some $p > 2$ (see [GNR18b, Th. 3.1]). In higher dimensions, it seems difficult to obtain regularity by such simple cutting-by-planes argument but it would be interesting to investigate further this question.

2.4 Regularization by a Willmore type energy

A natural way to restore well-posedness for (2.3) and hopefully obtain the stability of the ball for small charges would be to add an L^2 penalization of the mean curvature (known as the Willmore energy) to the energy. This is however a very difficult problem since the physical case $d = 3$ is critical for the Willmore energy in the sense that a bound on this energy only implies $C^{1,\alpha}$ regularity of ∂E for every $\alpha \in (0, 1)$ but not $C^{1,1}$ regularity which is needed to obtain an $L^\infty(\partial E)$ bound on the charge distribution μ.

In dimension $d = 2, 3$, together with M. Novaga and M. Röger, we decided to investigate in [GNR18a] a simpler problem where the charge μ is assumed to be uniformly distributed on E i.e. we replaced $\mathcal{I}_\alpha$ by

$$V_\alpha(E) = \int_{E\times E} \frac{1}{|x-y|^{d-\alpha}} dxdy$$

and considered for $\lambda, Q \geq 0$, the functional

$$\mathcal{F}_{\lambda,Q}(E) = \lambda^2 P(E) + W(E) + Q^2 V_\alpha(E), \tag{2.10}$$

where

$$W(E) = \begin{cases} \displaystyle\int_{\partial E} H^2 \, d\mathcal{H}^1 & \text{for } d = 2, \\ \displaystyle\frac{1}{4}\int_{\partial E} H^2 \, d\mathcal{H}^2 & \text{for } d = 3. \end{cases}$$

Here H denotes the mean curvature of ∂E i.e. the curvature in dimension two and the sum of the principal curvatures in dimension three.
Notice that if the Willmore term is dropped and $\lambda > 0$, this reduces to the celebrated Gamow liquid drop model for atomic nuclei also known as the sharp interface limit of the Ohta-Kawasaki model for diblock copolymers (recall (1.1)).

For $d = 2, 3$, we define the following classes of admissible sets

$$\begin{aligned} \mathcal{M} &= \{E \subset \mathbb{R}^d \text{ bounded with } W^{2,2}\text{-regular boundary}\}, \\ \mathcal{M}_{sc} &= \{E \in \mathcal{M} \,:\, E \text{ simply connected}\}, \\ \mathcal{M}(|B_1|) &= \{E \in \mathcal{M} \,:\, |E| = |B_1|\}, \\ \mathcal{M}_{sc}(|B_1|) &= \{E \in \mathcal{M}_{sc} \,:\, |E| = |B_1|\}, \end{aligned}$$

and consider the variational problems

$$\min_{\mathcal{M}_{sc}(|B_1|)} \mathcal{F}_{\lambda,Q}(E) \tag{2.11}$$

and

$$\min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda,Q}(E). \tag{2.12}$$

2.4.1 The planar case

We start by considering the planar problem $d = 2$ and first focus on the uncharged case $Q = 0$. For $\lambda = 0$ no global minimizer exists in $\mathcal{M}(|B_1|)$, but it has been recently shown in [BH16, FKN16] that balls minimize the elastic energy under volume constraint in the class of simply connected sets. Our first result is a quantitative version of this fact in the spirit of the quantitative isoperimetric inequality [FMP08, CL12].

Theorem 2.4.1. *For every set $E \in \mathcal{M}_{sc}(|B_1|)$,*

$$W(E) - W(B_1) \gtrsim \min_{x \in \mathbb{R}^2} |E \Delta B_1(x)|^2.$$

Furthermore, there exist $\delta_0 > 0$ and $c_1 > 0$ such that if $W(E) \leq W(B_1) + \delta_0$, then

$$W(E) - W(B_1) \geq c_1(P(E) - P(B_1)).$$

As for Theorem 2.3.2, the proof is based on the idea of [CL12] for the proof of the quantitative isoperimetric inequality to reduce by a contradiction argument to the case of nearly spherical sets and then compute a Taylor expansion along the lines of [Fug89]. As opposed to [CL12] which is based on an improved convergence theorem, we obtain the strong convergence to the ball directly from the energy and a delicate refinement of [BH16].

Still in the case $Q = 0$, we then remove the constraint on the sets to be simply connected but consider the minimization problem (2.12) for $\lambda > 0$.

Theorem 2.4.2. *Let $Q = 0$ and $d = 2$. There exists $\bar{\lambda} > 0$ such that for $\lambda < \bar{\lambda}$, minimizers of* (2.12) *are annuli while for $\lambda > \bar{\lambda}$ they are balls.*

The proof is based on the fact that using [BH16, FKN16], we can first reduce the class of competitors to sets E whose boundary is a union of circles. In this relatively small class, the problem can then be directly solved. Next, we turn to the stability estimates analogous to Theorem 2.4.1.

Theorem 2.4.3. *Let $d = 2$ and $\bar{\lambda}$ be given by Theorem 2.4.2. Then, for any $E \in \mathcal{M}(|B_1|)$ and $\lambda > \bar{\lambda}$*

$$\mathcal{F}_{\lambda,0}(E) - \mathcal{F}_{\lambda,0}(B_1) \gtrsim (\lambda^2 - \bar{\lambda}^2) \min_x |E \Delta B_1(x)|^2,$$

while for any $\lambda_* > 0$ there exists a constant $c(\lambda_*) > 0$ such that for any $\lambda \in [\lambda_*, \bar{\lambda}]$

$$\mathcal{F}_{\lambda,0}(E) - \min_{\mathcal{M}(|B_1|)} \mathcal{F}_{\lambda,0} \geq c(\lambda_*) \min_{\Omega} |E \Delta \Omega|^2,$$

where the minimum is taken among all sets Ω minimizing $\mathcal{F}_{\lambda,0}$ in $\mathcal{M}(|B_1|)$ (which are either balls or annuli depending on λ).

The proof follows the same strategy as for Theorem 2.4.1 but extra care need to be taken due to the non-trivial topology of annuli.

Then, we turn to the study of (2.11) and (2.12) for $Q > 0$. Regarding (2.11) we prove the following.

Theorem 2.4.4. *Let $d = 2$. There exists $Q_0 > 0$ such that for $Q < Q_0$ and all $\lambda \geq 0$, balls are the only minimizers of* (2.11).

The proof is a combination of Theorem 2.4.1 and [KM14]. As for (2.12), we obtain a good understanding of part of the phase diagram (see Figure 2.1).

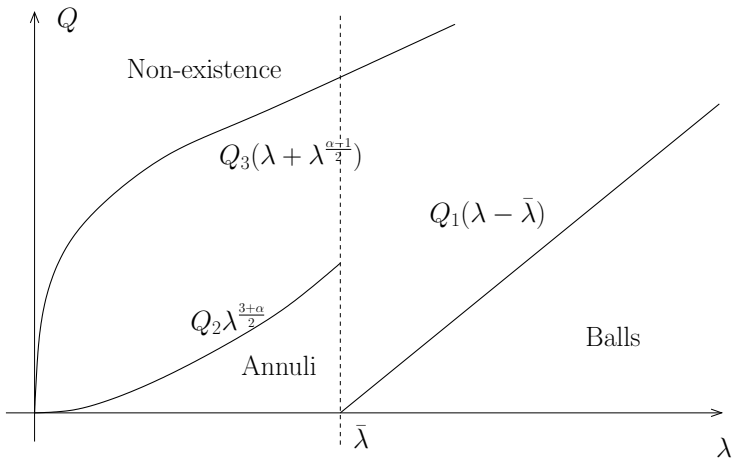

Figure 2.1: The phase diagram.

Theorem 2.4.5. *Let $d = 2$.*

- *There exists $Q_1 > 0$ such that for every $\lambda > \bar{\lambda}$ and every $Q \leq Q_1(\lambda - \bar{\lambda})$, balls are the only minimizers of* (2.12).
- *There exists $Q_2 > 0$ such that for every $\lambda \leq \bar{\lambda}$ and every $Q \leq Q_2\lambda^{\frac{3+\alpha}{2}}$, centered annuli are the only minimizers of* (2.12).
- *For every $\alpha \in (1, 2)$ there exists $Q_3(\alpha)$ such that for every $\lambda \geq 0$ and every $Q \geq Q_3(\alpha)(\lambda + \lambda^{\frac{\alpha-1}{2}})$, no minimizer exists for* (2.12).

The first part of the theorem is a direct consequence of the minimality of the ball for $\mathcal{F}_{\bar{\lambda},0}$ and for $P + Q^2V_\alpha$ for Q small enough. The second point regarding the minimality of centered annuli is the most delicate part of the theorem. It requires first to argue that sets of small energy are almost annuli and then to use the stability of annuli. The last part of the theorem regarding non-existence is obtained by noticing that if a minimizer exists then it must be connected and therefore we can obtain a lower bound on the energy which is not compatible for large Q with an upper bound obtained by constructing a competitor made of a union of annuli.

2.4.2 The three-dimensional case

In the three-dimensional case, a characterization of the energy landscape is even more difficult than in dimension two. As already pointed out, this

dimension is critical for the Willmore energy which is then invariant under rescaling and is globally minimized by balls [Wil65, Sim83]. We can thus focus on the case $\lambda = 0$ where we have competition between the Willmore energy and the Riesz interaction energy. Stability estimates for the Willmore energy have been obtained by De Lellis and Müller [DLM05]. Building on these, on the control of the isoperimetric deficit by the Willmore deficit obtained in [RS12] and a bound on the perimeter, we obtained in [GNR18a] that balls are minimizers of (2.12) for small Q.

Theorem 2.4.6. *For $d = 3$ and $\lambda = 0$, there exists $Q_4 > 0$ such that for every $Q \leq Q_4$, the only minimizers of* (2.12) *are balls.*

Of course, since balls are also minimizers of the isoperimetric problem, a direct consequence of Theorem 2.4.6 is the minimality of the balls for (2.12) for every $\lambda \geq 0$ and every $Q \leq Q_4$. Unfortunately, we are not able to prove or disprove a non-existence regime in the parameter space. This seems to be a difficult problem since it can be shown (see [GNR18a, Prop.4.1]) that

$$\inf_{\mathcal{M}(|B_1|)} \mathcal{F}_{0,Q}(E) \leq 8\pi.$$

Still, we show that if a minimizer exists for every Q then its isoperimetric quotient must degenerate as $Q \to \infty$. This is somewhat reminiscent of earlier results obtained by Schygulla [Sch12].

2.5 Non-existence of perfectly conducting drops in a uniform external field

The problem of finding the equilibrium shape of charged droplets is closely related to the problem of finding the equilibrium shape of a conducting drop submitted to an external electric field. If the understanding of meteorological phenomena has first motivated the study of this question [WT25], the wide spectrum of modern applications ranging from the breakdown of dielectrics due to the presence of water droplets to ink-jet printers might explain the large amount of literature on the subject (see for instance [Mik81, DM07, KDT14]). In the simplest and most considered case of a constant external field $\mathbb{E}$, we have for $E \subset \mathbb{R}^d$ with $d \geq 3$,

$$\mathcal{F}(E) = \min_{\mu(E)=0} \mathcal{I}_2(\mu) - \int_E \mathbb{E} \cdot x d\mu,$$

where the minimum is attained by [GR17, Th. 1.10], and we look for a minimizer of

$$\min_{|E|=|B_1|} P(E) + Q^2\mathcal{F}(E). \tag{2.13}$$

It is quite easy to see that this problem is ill-posed (see [GR17, Th. 3.1]. Moreover, we proved in [GR17, Th. 3.2] that a bit surprisingly and as opposed to the case of the charged liquid drop model (2.1), ill-posedness still holds in the class of convex sets.

Theorem 2.5.1. *For every $d \geq 3$ and every $\mathbb{E} \in \mathbb{R}^d$,*

$$\inf_{\substack{|E|=|B_1| \\ E \text{ convex}}} P(E) + Q^2\mathcal{F}(E) = -\infty.$$

Proof. We may assume that $\mathbb{E} = e_1$. For $n \in \mathbb{N}$, consider the set

$$E_n = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d \,:\, |x_1| \leq \frac{n}{2} \;\; |x_i| \leq \frac{\varepsilon_n}{2},\; i = 2, \dots, d \right\},$$

where

$$\varepsilon_n = \left(\frac{|B_1|}{n} \right)^{\frac{1}{d-1}}.$$

Notice that ε_n is chosen so that $|E_n| = |B_1|$. By definition of E_n we have

$$P(E_n) \lesssim n\varepsilon^{d-2} \lesssim n^{\frac{1}{d-1}}.$$

Moreover, by letting

$$E_n^- = E_n \cap \left\{ x \,:\, x_1 \in \left[-\frac{n}{2}, -\frac{n}{2} + \varepsilon_n \right] \right\}, \qquad E_n^+ = E_n \cap \left\{ x \,:\, x_1 \in \left[\frac{n}{2} - \varepsilon_n, \frac{n}{2} \right] \right\}$$

and then

$$\mu_n = \frac{\chi_{E_n^+}}{|E_n^+|} - \frac{\chi_{E_n^-}}{|E_n^-|},$$

we have that μ_n is admissible for $\mathcal{F}(E_n)$ and thus $\mathcal{F}(E_n) \leq \mathcal{F}(\mu_n)$. Since on the one hand,

$$\int_{E_n} x_1 \, d\mu_n \gtrsim n$$

and on the other hand

$$\mathcal{I}_2(\mu_n) \lesssim \frac{1}{|E_n^+|^2} \int_{E_n^+ \times E_n^+} \frac{dxdy}{|x-y|^{d-2}} \lesssim \varepsilon_n^{-(d-2)} \lesssim n^{\frac{d-2}{d-1}},$$

we find that

$$P(E_n) + \mathcal{F}(E_n) \lesssim n^{\frac{1}{d-1}} + Q^2(n^{\frac{d-2}{d-1}} - n)$$

which diverges to $-\infty$ as $n \to +\infty$. □

Chapter 3

Stripe patterns for a an isoperimetric problem with non-local interactions

This chapter describes results obtained in collaboration with B. Merlet and E. Runa in [GR16, GM18].

3.1 The model and the main result

3.1.1 The model

A well-known conjecture is that minimizers of the Ohta-Kawasaki model under fixed volume fraction* are periodic or at least nearly-periodic. However, besides the one-dimensional situation [Mül93, RW03] (see also [MS14, SCN15] for an almost one-dimensional case) and the low volume fraction limit [CP10, GMS14, KMN16] not much is known. The only result available on periodicity of minimizers for intermediate volume fractions is the uniform local energy distribution [ACO09] as well as minimality in the perimeter dominant regime [ST11, AFM13, Cri15]. In particular, a major open question is the optimality of stripes for volume fraction equal to $\frac{1}{2}$.

*by this we mean minimizers in large periodic cubes $Q_L = [0, L)^d$ under the constraint $\frac{1}{L^d}|E| = m$ for some $m \in (0, 1)$.

Motivated by this question and by recent works [GLL11, GLS14, GS16] on striped patterns in Ising models with competing interactions, we considered with E. Runa for $d \geq 2$, $J, L > 0$ and $p \geq d+2$ the functional

$$\widetilde{\mathcal{F}}_{J,L}(E) = \frac{1}{L^d}\left(J\int_{\partial E\cap Q_L} |\nu^E|_1 d\mathcal{H}^{d-1} - \int_{Q_L\times\mathbb{R}^d} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^p+1}dxdy\right), \tag{3.1}$$

where E is a $[0,L)^d-$periodic set, ν^E is its external normal and $|\cdot|_1$ denotes the $1-$norm. We often denote

$$\mathrm{Per}_1(E,Q_L) = \int_{\partial E\cap Q_L} |\nu^E|_1 d\mathcal{H}^{d-1}.$$

Before proceeding further, let us explicit the connection between (3.1) and Gamow's liquid drop model (1.1). This will give us the occasion to discuss some of the choices we made in (3.1). We begin by noting that if for the normal we replaced the $1-$norm by the Euclidean norm, the first term would simply be the usual relative perimeter of E inside Q_L. We choose to work with the anisotropic $1-$perimeter since our proof is based on a slicing technique which is easier to implement for this perimeter. Second, if we call $K_1(\zeta) = \frac{1}{|\zeta|^p+1}$, then for $p > d$, K_1 is integrable on $\mathbb{R}^d$ (which is the reason why we could not just consider the kernel $K_0(\zeta) = \frac{1}{|\zeta|^p}$) and

$$\begin{aligned}
-\int_{Q_L\times\mathbb{R}^d} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^p+1}dxdy &= -2\int_{(E\cap Q_L)\times E^c} K_1(x-y)dxdy \\
&= 2\int_{(E\cap Q_L)\times E} K_1(x-y)dxdy \\
&\quad - 2\int_{(E\cap Q_L)\times\mathbb{R}^d} K_1(x-y)dxdy \\
&= 2\int_{(E\cap Q_L)\times E} K_1(x-y)dxdy \\
&\quad - 2\left(\int_{\mathbb{R}^d} K_1(\zeta)d\zeta\right)|E\cap Q_L|
\end{aligned}$$

and thus if we fixed the volume fraction $|E\cap Q_L|$, minimizing (3.1) would be equivalent to minimizing

$$\frac{1}{L^d}\left(J\,\mathrm{Per}_1(E,Q_L) + 2\int_{(E\cap Q_L)\times E} \frac{1}{|x-y|^p+1}dxdy\right),$$

which is indeed very reminiscent of (1.1). The restriction on the exponent $p \geq d+2$ may be explained by the fact that in this case, 'angles' are very

expensive which will allow us to remove them in the regime we consider. This can be seen for instance from the fact that a periodic checkerboard has a larger energy scaling (in J) than periodic stripes (see [GLL11]).

It can be shown that if $J \geq J_c = \int_{\mathbb{R}^d} |\zeta_1| K_1(\zeta)$, then the energy is always positive and thus the global minimizer (without any volume constraint) of (3.1) is trivial while for $J < J_c$ there exists non-trivial minimizers. We are interested here in the behavior of these minimizers as $J \uparrow J_c$. Building on the computations made in [GLL11], it is expected that for $\tau = J_c - J$ small enough, minimizers are periodic striped patterns. A simple computation shows that the energy of periodic stripes of width h is of the order of

$$\widetilde{\mathcal{F}}_{J,L}(E_h) \simeq -\frac{\tau}{h} + h^{-(p-d)}.$$

Optimizing in h, we find that the optimal stripes have a width of order $\tau^{-1/(p-d-1)}$ and energy of order $-\tau^{(p-d)/(p-d-1)}$. Letting $\beta = p-d-1$, this motivates the rescaling

$$x = \tau^{-1/\beta}\widehat{x}, \quad L = \tau^{-1/\beta}\widehat{L} \quad \text{and} \quad \widetilde{\mathcal{F}}_{J,L}(E) = \tau^{(p-d)/\beta}\mathcal{F}_{\tau,\widehat{L}}(\widehat{E}), \tag{3.2}$$

which yields stripes of width and energy of order one as τ goes to zero.

After this rescaling and dropping the hats, we are led to study the functional

$$\begin{aligned}\mathcal{F}_{\tau,L}(E) = \frac{1}{L^d}\Bigg(&- \mathrm{Per}_1(E, Q_L) \\ &+ \int_{\mathbb{R}^d} K_\tau(\zeta) \left[\int_{\partial E \cap Q_L} \sum_{i=1}^d |\nu_i^E||\zeta_i| d\mathcal{H}^{d-1} - \int_{Q_L} |D\chi_E(x,\zeta)| dx \right] d\zeta \Bigg),\end{aligned}$$

where for $\tau > 0$, $x \in Q_L$, $\zeta \in \mathbb{R}^d$ and a function u, we introduced the notation

$$K_\tau(\zeta) = \frac{1}{|\zeta|^p + \tau^{p/(p-d-1)}} \qquad \text{and} \qquad Du(x,\zeta) = u(x+\zeta) - u(x).$$

3.1.2 The main result

Our main result is a $\Gamma-$convergence result (see [DM93]) for $\mathcal{F}_{\tau,L}$ (see [GR16])

Theorem 3.1.1. *For $p > d+2$ and $L > 1$, the functionals $\mathcal{F}_{\tau,L}$ $\Gamma-$converge as τ goes to zero with respect to the L^1-convergence to the functional defined*

for sets $E = \widehat{E} \times \mathbb{R}^{d-1}$ *where* $\widehat{E}$ *is* $L-$*periodic with* $\sharp\{\partial\widehat{E} \cap [0, L)\} < +\infty$, *by*

$$\mathcal{F}_{0,L}(E) = \frac{1}{L}\Bigg(- \sharp\{\partial\widehat{E} \cap [0, L)\} + \int_{\mathbb{R}^d} \frac{1}{|\zeta|^p} \left[\sum_{x \in \partial\widehat{E} \cap [0,L)} |\zeta_1| - \int_0^L |D\chi_{\widehat{E}}(x, \zeta_1)| dx \right] d\zeta \Bigg), \tag{3.3}$$

and $\mathcal{F}_{0,L}(E) = +\infty$ *otherwise. Moreover, if* E^τ *is such that* $\sup_\tau \mathcal{F}_{\tau,L}(E^\tau) < +\infty$, *then up to a relabeling of the coordinate axes, there is a subsequence which converges in* L^1 *to some set* E *with* $E = \widehat{E} \times \mathbb{R}^{d-1}$ *and* $\sharp\{\partial\widehat{E} \cap [0, L)\} < +\infty$.

Before giving the main steps of the proof, let us make a few comments. First, in [GR16] the theorem is stated in the more limited range $p > 2d$. This difference comes from the rigidity estimate (see Lemma 3.1.5 below) which in [GR16] was proved under this restricted assumption. The sharp condition was understood only later on (see [GM18, DR17] and Section 3.1.3). Second, we point out that the two terms in the bracket in (3.3) are separately both equal to infinity and it was thus crucial to combine them in the correct way to have a well-defined functional. Last, building on this result, it was later proven by Daneri and Runa in [DR17] that actually minimizers of $\mathcal{F}_{\tau,L}$ are periodic stripes for τ small enough but uniformly in L.

We now discuss the proof of Theorem 3.1.1. The construction of a recovery sequence is straightforward so that we just need to concentrate on the compactness and the liminf inequality which are a bit not standard due to the minus sign in front of $\mathrm{Per}_1(E, Q_L)$. A main step in the proof is to show that the energy actually both controls the perimeter and penalizes curvature. As already alluded to, this is obtained through slicing. For this purpose, one needs first to better decompose the energy. For $i \in [1, d]$ and $\zeta \in \mathbb{R}^d$, we write

$$\zeta = \zeta_i + \zeta_i^\perp,$$

where $\zeta_i \in \mathbb{R}e_i$. The crucial observation is the following elementary lemma.

Lemma 3.1.2. *For every Q_L-periodic set E and every $\tau > 0$, there holds*

$$\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)|D\chi_E(x,\zeta)| \le \int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d |D\chi_E(x,\zeta_i)| \\ - \frac{2}{d}\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d |D\chi_E(x,\zeta_i)||D\chi_E(x,\zeta_i^\perp)|. \quad (3.4)$$

Proof. The proof is obtained by integrating the identity

$$|\chi_E(x)-\chi_E(x+\zeta)| = |\chi_E(x)-\chi_E(x+\zeta_i)| + |\chi_E(x+\zeta_i)-\chi_E(x+\zeta)| \\ - 2|\chi_E(x)-\chi_E(x+\zeta_i)||\chi_E(x+\zeta_i)-\chi_E(x+\zeta)|$$

together with the triangle inequality. □

In particular if we let for $i \in [1,d]$,

$$\mathcal{G}^i_{\tau,L}(E) = \frac{1}{L^d}\int_{\mathbb{R}} \widehat{K}_\tau(\zeta_i)\left[\int_{\partial E\cap Q_L} |\nu_i^E||\zeta_i| - \int_{Q_L} |D\chi_E(x,\zeta_i)|\right],$$

and

$$I_{\tau,L}(E) = \frac{2}{dL^d}\int_{Q_L\times\mathbb{R}^d} K_\tau(\zeta)\sum_{i=1}^d |D\chi_E(x,\zeta_i)||D\chi_E(x,\zeta_i^\perp)|,$$

we have

$$\mathcal{F}_{\tau,L}(E) \ge -\frac{1}{L^d}\operatorname{Per}_1(E,Q_L) + \sum_{i=1}^d \mathcal{G}^i_{\tau,L}(E) + I_{\tau,L}(E).$$

In this form, both $\operatorname{Per}_1(E,Q_L)$ and $\mathcal{G}^i_{\tau,L}(E)$ are playing well with slicing. In particular, for a $L-$periodic set $E \subset \mathbb{R}$, it is useful to introduce the one-dimensional analog of $\mathcal{G}^i_{\tau,L}$

$$\mathcal{G}^{1d}_{\tau,L}(E) = \int_{\mathbb{R}} \widehat{K}_\tau(z)\left(\operatorname{Per}(E,[0,L))|z| - \int_0^L |D\chi_E(x,z)|\right),$$

where

$$\widehat{K}_\tau(z) = \int_{\mathbb{R}^{d-1}} K_\tau(z,\zeta).$$

The main one-dimensional estimate is the following

Lemma 3.1.3. *Let $E \subset \mathbb{R}$ be a $L-$periodic set of finite perimeter. Then, for every $\tau \ge 0$,*

$$L^{-1}\operatorname{Per}(E,[0,L)) \lesssim L^{-1}(1+\mathcal{G}^{1d}_{\tau,L}(E)). \quad (3.5)$$

By slicing, we obtain as a direct consequence of this estimate that the energy actually gives a control on the perimeter

Lemma 3.1.4. *For every $L \gtrsim 1$ and $\tau \lesssim 1$, and every Q_L-periodic set E,*

$$\mathrm{Per}_1(E, Q_L) \lesssim L^d \max(1, \mathcal{F}_{\tau,L}(E)). \tag{3.6}$$

This gives the compactness in L^1 of sequences E^τ of bounded energy. The liminf estimate is then obtained by using a refined version of (3.5) to prove that there is actually convergence of the perimeter. The only point which is left to prove is the fact that every limit set is made of stripes.

3.1.3 The rigidity lemma

Since it can be shown that if E^τ converges in L^1 to some set E,

$$\liminf_{\tau\to 0} I_{\tau,L}(E^\tau) \geq I_{0,L}(E)$$

and using (3.6), the proof is concluded once proven that

Lemma 3.1.5. *Let $p \geq d+2$ and E be a Q_L-periodic set such that for some $i \in [1, d]$,*

$$\int_{Q_L\times\mathbb{R}^d} \frac{|D\chi_E(x,\zeta_i)||D\chi_E(x,\zeta_i^\perp)|}{|\zeta|^p} < \infty, \tag{3.7}$$

then E is either a set invariant along $\mathbb{R}e_i$ or E along $(\mathbb{R}e_i)^\perp$.

In [GR16], this lemma was proven under the stronger hypothesis that $p \geq 2d$, E is a set of finite perimeter and $\sum_{i=1}^d \mathcal{G}_{0,L}(E) < \infty$. The above version of the lemma was later obtained together with B. Merlet in [GM18] as a corollary of a more general statement (see also [DR17]). We assume from now on that without loss of generality that $i = 1$, $L = 1$ and we identify Q_1 with the unit torus.

For a standard convolution kernel ρ_ε, a Q_1-periodic function u and positive numbers θ_1 and θ_2 with $\theta = \theta_1 + \theta_2 \leq 1$, we let

$$\mathcal{E}_0(u) = \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^d} \rho_\varepsilon(\zeta) \int_{Q_1} \frac{|Du(x,\zeta_1)|^{\theta_1}|Du(x,\zeta_1^\perp)|^{\theta_2}}{|\zeta|^2}.$$

The energy $\mathcal{E}_0(u)$ is a non-convex variant of an energy which has attracted a lot of attention recently for the purpose of a non-local characterization of

Sobolev spaces (see in particular [Bre02, DMMS08]).

Let us notice that if $p \geq d+2$, then $|\zeta|^{p-2}$ is not integrable around 0 and thus if (3.7) holds then $\mathcal{E}_0(u) = 0$ for a well chosen kernel ρ_ε. Lemma 3.1.5 thus follows from the more general result:

Theorem 3.1.6. *Let u be such that $\mathcal{E}_0(u) = 0$, then u depends only on x_1 or only on $x_1^\perp$.*

Proof. Up to replacing u by $\arctan(u)$, we may assume that u is bounded. The main insight in the proof of Theorem 3.1.6 is that $\mathcal{E}_0(u)$ controls

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^d} \rho_\varepsilon(\zeta)$$
$$\int_{Q_1} \frac{(|Du(x+\zeta_1^\perp,\zeta_1)| + |Du(x,\zeta_1)|)^{\theta_1}(|Du(x+\zeta_1,\zeta_1^\perp)| + |Du(x,\zeta_1^\perp)|)^{\theta_2}}{|\zeta|^2} dx d\zeta.$$

By triangle inequality and the fact that

$$Du(x+\zeta_1^\perp,\zeta_1)+Du(x,\zeta_1) = Du(x+\zeta_1,\zeta_1^\perp)+Du(x,\zeta_1^\perp) = D[Du(\cdot,\zeta_1^\perp)](x,\zeta_1),$$

this leads to

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2} \rho_\varepsilon(\zeta) \int_{Q_1} \frac{|D[Du(\cdot,\zeta_1^\perp)](x,\zeta_1)|}{|\zeta|^2} = 0.$$

From this it can be inferred that

$$\partial_{e_1}\partial_{e_1^\perp} u = 0$$

so that $u(x) = u_1(x_1) + u_2(x_1^\perp)$. Plugging this back into $\mathcal{E}_0(u) = 0$ this yields that $u_1 = 0$ or $u_2 = 0$. □

Theorem 3.1.6 is sharp in the sense that already in the two-dimensional case, the square $u = \chi_{(\frac{1}{2},\frac{1}{4})^2}$ satisfies $\mathcal{E}_0(u) < \infty$ but is not one-dimensional. However, one can improve it into a quantitative control of the deviation of u from being one-dimensional. To simplify the estimates we will assume from now on that u is bounded with $\|u\|_\infty \leq 1$. We obtain a strong result in dimension two.

Proposition 3.1.7. *Assume $d = 2$. Let u be such that $\|u\|_\infty \leq 1$ and $\mathcal{E}_0(u) < \infty$. Then, there exists a function $\bar{u}$ depending only on x_1 or only on x_2 and such that*

$$\|u - \bar{u}\|_{BV\cap L^\infty} \lesssim \mathcal{E}_0(u) + \mathcal{E}_0(u)^{\frac{1}{2}(1-\theta)} \left(\mathcal{E}_0(u)^{\theta_1} + \mathcal{E}_0(u)^{\theta_2}\right). \tag{3.8}$$

Moreover, if $u = \chi_E$, then there exists $c_0 > 0$ such that if $\mathcal{E}_0(\chi_E) \leq c_0$ then E is made of either vertical or horizontal stripes.

Let us point out that in the case $\theta < 1$, we can actually show that $u - \bar{u} \in SBV(Q_1)$ and that $D(u - \bar{u})$ is purely made of a jump part. If $d \geq 3$, we can only obtain a weaker version of (3.8) where the $BV \cap L^\infty$ norm is replaced by a L^p norm for some p depending on the dimension.

A preliminary but important result in order to prove (3.8) is the fact that if $\mathcal{E}_0(u) < \infty$, then (for any $d \geq 2$)

$$\mu = \partial_{e_1} \partial_{e_1^\perp} u$$

is a Radon measure with $|\mu|(Q_1) \lesssim \mathcal{E}_0(u)$. This is obtained arguing along the lines of the proof of Theorem 3.1.6. One can actually say much more about the structure of this measure μ. Let us first consider the two-dimensional situation.

Proposition 3.1.8. *Let $d = 2$ and u be such that $\|u\|_\infty \leq 1$ and $\mathcal{E}_0(u) < \infty$ then*

- *if $\theta < 1$, then $\mu = \sum_i a_i \delta_{x_i}$ for some $x_i \in Q_1$ and $a_i \in \mathbb{R}$ with*
$$\sum_i |a_i|^\theta \leq \mathcal{E}_0(u). \tag{3.9}$$
- *if $\theta = 1$, then there exists a Borel subset $A \subset Q_1$ such that $\mathcal{H}^1 \llcorner A$ is σ-finite and $|\mu|(Q_1 \setminus A) = 0$.*

Proof. The proof is quite involved, in particular in the case $\theta = 1$ but greatly simplifies in the case $u = \chi_E$. Indeed, in this case the information that $\partial_1 \partial_2 \chi_E$ is a measure is very rigid. Indeed, if we define $w(x) = \mu((0, x_1] \times (0, x_2])$, we have $\partial_1 \partial_2 (\chi_E - w) = 0$ and therefore, there exist u_1 and u_2 such that

$$\chi_E(x) = u_1(x_1) + u_2(x_2) + w(x). \tag{3.10}$$

We now claim that for every rectangle $Q \subset Q_1$,

$$\mu(Q) \in \pm\{0, 1, 2\}.$$

Indeed, letting $\{a, b, c, d\}$ be the vertices of Q (with a the upper left vertex and using then clockwise enumeration), from (3.10),

$$\mu(Q) = \chi_E(b) + \chi_E(d) - \chi_E(a) - \chi_E(c) \in \pm\{0, 1, 2\}.$$

From this we readily obtain that μ must be atomic. $\square$

Again, Proposition 3.1.8 is optimal. Indeed, we already saw the example $u = \chi_{(\frac{1}{2},\frac{1}{4})^2}$ for which μ is atomic, in the case $\theta = 1$, one can consider the "hat" function $u(x_1, x_2) = \min(x_1, x_2, 1 - x_1, 1 - x_2)$ for which μ is concentrated on the diagonals of the cube.
In higher dimension, we focus on the case $\theta > 1$ since for $\theta = 1$ the situation was already very complicated in dimension two. Using the two-dimensional result, slicing and White's rectifiability criterion [Whi99], we can prove the following rectifiability result.

Theorem 3.1.9. *Let u be such that $\|u\|_\infty \leq 1$ and $\mathcal{E}_0(u) < \infty$. Then, $\mu = \partial_{e_1}\partial_{e_1^\perp} u$ is a rectifiable measure in the sense that*

$$\mu = \sigma\, e_1 \otimes \nu^\Sigma\, \mathcal{H}^{d-2} \llcorner \Sigma \tag{3.11}$$

for some $(d-2)-$rectifiable set Σ with normals e_1 and $\nu^\Sigma \in (\mathbb{R}e_1)^\perp$ and some weight function $\sigma : \Sigma \to \mathbb{R}$. Moreover,

$$\int_\Sigma |\sigma|^\theta \, d\mathcal{H}^{d-2} \lesssim \mathcal{E}_0(u). \tag{3.12}$$

Let us point out that since $\theta < 1$, estimate (3.12), which follows by slicing and (3.9), implies that μ should be quite concentrated. This type of energies, which also appear in the context of branched transportation, are often call $\theta-$mass and have been the subject of intensive research in the last couple of years (see [CFM18, CRMS17] for instance). Extending the decomposition of integral currents in indecomposable components (see [Fed69]) to currents with finite $\theta-$mass we proved that for Σ as in (3.11),

$$\Sigma = \cup_i \{x_1^i\} \times \Sigma_i$$

for some $x_1^i \in \mathbb{R}e_1$ and $(d-2)-$rectifiable sets $\Sigma_i \subset (\mathbb{R}e_1)^\perp$. This is quite natural since the normals to Σ are a.e. given by e_1 and a vector $\nu^\Sigma \in (\mathbb{R}e_1)^\perp$.

3.2 Reflection positivity and minimizers of the limit problem

The advantage of the limit functional $\mathcal{F}_{0,L}$ is that its minimizers can be easily computed using the so-called reflection positivity method. This method has been introduced in statistical mechanics [FILS78] and has been extensively used (mainly in the context of one-dimensional Ising type models) to prove periodicity of minimizers (see for instance [GM12, GLS14, GS16, GLL09, GM12]).

Theorem 3.2.1. *There exist an explicit constant $h^\star > 0$ and a constant $C > 0$ such that for every $L > 1$, minimizers of $\mathcal{F}_{0,L}$ are periodic stripes of width h with*

$$|h - h^\star| \leq CL^{-1}.$$

Proof. Recalling the definition of $\mathcal{F}_{0,L}$ in (3.3), and computing the integration on $(\mathbb{R}e_1)^\perp$, we have letting $q = p - d + 1$ for some explicit constant C_q,

$$\mathcal{F}_{0,L}(E) = \frac{1}{L}\Bigg(- \mathrm{Per}(E, [0, L)) + C_q \int_{\mathbb{R}} \frac{1}{|z|^q} \left[\mathrm{Per}(E, [0, L))|z| - \int_0^L |D\chi_E(x, z)| \right] \Bigg), \quad (3.13)$$

Since E is of finite perimeter, we can write it as $E = \cup_{i\in\mathbb{Z}}(s_i, t_i)$ and we may assume that $E \cap [0, L) = \cup_{i=1}^N (s_i, t_i)$ for some $N \in \mathbb{N}$, $s_1 > 0$ and $t_N < L$. We then let for $x \in \partial E$ the width and gaps of E be defined as

$$h(x) = t_i - s_i \quad \text{if } x = s_i \text{ or } x = t_i \qquad \text{and} \qquad g(x) = \begin{cases} t_{i-1} - s_i & \text{if } x = s_i \\ s_{i+1} - t_i & \text{if } x = t_i. \end{cases}$$

For $h > 0$, let $E_h = \cup_{k\in\mathbb{Z}}[(2k)h, (2k + 1)h]$. Then, we define

$$e_\infty(h) = \mathcal{F}_{0,2h}(E_h) = \lim_{L\to+\infty} \mathcal{F}_{0,L}(E_h).$$

The main estimate is the so-called chessboard estimate:

$$\mathcal{F}_{0,L}(E) \geq \frac{1}{2L} \sum_{x\in\partial E\cap[0,L)} \left(h(x)e_\infty(h(x)) + g(x)e_\infty(g(x))\right). \quad (3.14)$$

Indeed, if (3.14) holds then defining $h^\star$ as the minimizer of $e_\infty(h)$,

$$\begin{aligned} \mathcal{F}_{0,L}(E) &\geq \frac{1}{2L} \sum_{x\in\partial E\cap[0,L)} (h(x)e_\infty(h(x)) + g(x)e_\infty(g(x))) \\ &\geq \frac{e_\infty(h^\star)}{2L} \sum_{x\in\partial E\cap[0,L)} (h(x) + g(x)) = e_\infty(h^\star), \quad (3.15) \end{aligned}$$

from which the claim follows quite easily.
Let us give an idea of the proof of (3.14). Since the contribution of the perimeter to both sides of the inequality (3.14) is the same, forgetting about the non-integrability issues, it is enough to prove the estimate for the non-local part. Using Laplace transform i.e. the fact that for $s > 0$, $s^{-q} =$

$\frac{1}{\Gamma(q)}\int_0^{+\infty}\alpha^{q-1}e^{-\alpha s}$, it is enough to prove that the same chessboard estimate holds for every $\alpha > 0$ for the functional

$$-\int_{[0,L]\times\mathbb{R}} |\chi_E(x) - \chi_E(y)|e^{-\alpha|x-y|}dxdy.$$

The method is called reflexion positivity since the estimate is proven through repeated reflexions. To be more specific let us introduce some more notation. For $L_1, L_2 > 0$, and two sets $E_1 \subset [0, L_1)$, $E_2 \subset (L_1, L_1 + L_2)$, we let $L = L_1 + L_2$, $(E_1, E_2) = E_1 \cup E_2$ and

$$\mathcal{J}(E_1, E_2) = -\int_{[0,L]\times[0,L]} |\chi_{(E_1,E_2)}(x) - \chi_{(E_1,E_2)}(y)|e^{-|x-y|}.$$

We then define the set $(E_1, \theta E_1)$ in $[0, 2L_1]$ by

$$\chi_{(E_1,\theta E_1)}(x) = \begin{cases} \chi_{E_1}(x) & \text{for } x \in [0, L_1] \\ 1 - \chi_{E_1}(2L_1 - x) & \text{for } x \in (L_1, 2L_1]. \end{cases}$$

Letting $L_2 = L - L_1$, we similarly define, $(\theta E_2, E_2)$ as a subset of $[L_1 - L_2, L]$ by

$$\chi_{(\theta E_2,E_2)}(x) = \begin{cases} \chi_{E_2}(x) & \text{for } x \in [L_1, L] \\ 1 - \chi_{E_2}(2L_1 - x) & \text{for } x \in (L_1 - L_2, L_1]. \end{cases}$$

The key estimate is

$$\mathcal{J}(E_1, E_2) \geq \frac{1}{2}\left(\mathcal{J}(E_1, \theta E_1) + \mathcal{J}(\theta E_2, E_2)\right).$$

The chessboard estimate (3.14) is then obtained by iterating this inequality. $\square$

Chapter 4

Island formation in heteroepitaxial growth

This chapter contains results obtained together with P. Bella and B. Zwicknagl in [GZ14, BGZ15].

4.1 Background

We are interested in the epitaxial deposition of a thin crystalline film on a relatively thick rigid substrate with a misfit between the lattice parameters of the film and those of the substrate. Experimental and numerical observations suggest that the shape of the film changes with increasing volume (see [BC02, ST10, SS12]). At small volumes, one typically observes a very thin flat layer ("wetting"), while at larger volumes, compact islands form. This transition is often explained as the result of a competition between two opposing types of energies, namely, the stored strain energy due to the crystallographic misfit, and the surface energy of the film's free surface. Heuristically, at small volumes, the surface energy dominates, and complex structures are avoided, while at larger volumes, the film forms patterns to release elastic energy at the price of an additional surface energy.

We study a two-dimensional variational model introduced in [Spe99] (see also [BC02, FFLM07, FM12]), which has received a lot of attention in the past few years (see for instance [Leo16]). In particular, building on ideas

from [CL03] the regularity of minimizers outside of a finite number of cusps and cuts has been established in [FFLM07]. Subsequently it is proven in [FM12] that for small values of the amplitude of the mismatch, the flat configuration is always minimizing no matter how thick is the film. Further, they proved that for larger (but fixed) values of the mismatch, there are increasing thresholds for the volumes such that the flat configuration changes from being an absolute minimizer to being only a local minimizer, before becoming actually unstable. Further work on the dynamical version of this problem has been done in [FFLM15, FFLM12]. See also [DP17] for a recent study of the contact angle between the film and the substrate.

The focus here is on the formation and the understanding of the macroscopic shape of the islands.

4.2 The compact case

In [GZ14], we started by considering the case in which the film is constrained to live in a compact set. Precisely, we consider

$$F_{d,e_0}(u,h) = \int_{\Omega_h} |\nabla u|^2 \, dx \, dy + \int_0^1 \sqrt{1+|h'|^2} \, dx, \tag{4.1}$$

where the Lipschitz function $h : [0,1] \to [0,\infty)$ describes the profile of the deposited film, $\Omega_h = \{(x,y) : 0 \le x \le 1 \, , \, 0 \le y \le h(x)\}$, and $u \in W^{1,2}(\Omega_h;\mathbb{R})$ is the planar displacement. Note that we restrict ourselves to two-dimensional morphologies which correspond to three-dimensional configurations with planar symmetry. We consider film profiles of fixed volume $d > 0$, i.e., $\int_0^1 h(x) = d$, and impose the boundary conditions $h(0) = h(1) = 0$. The parameter $e_0 \in \mathbb{R}$ stands for the amplitude of the crystallographic misfit between the film and the rigid substrate, and is introduced in this model via the boundary condition $u(x,0) = e_0 x$. This condition forces the film to be strained.

Let us point out that in the literature, most of the mathematical analysis of the model (4.1) has been devoted to the geometrically linear small strain approximation (see [FFLM07, FM12, BC02]), corresponding to small deformations, in which $|\nabla u|^2$ is replaced by a quadratic energy depending on the symmetrized gradient $E(u) = \nabla u + \nabla^T u$. Since we are in most part interested in the case of large mismatch e_0, the small strain approximation seems less justified here. However, most of our results extend fairly easily

to the geometrically linear setting. We should also notice that in [GZ14] we considered more general (in particular non-linear) elastic energies but we focus here on the case of the Dirichlet energy for simplicity.

Our first result is a scaling law for the energy.

Theorem 4.2.1. *For every d, $e_0 > 0$,*

$$\max\left\{1,\ d,\ e_0^{2/3}d^{2/3}\right\} \lesssim \inf F_{d,e_0} \lesssim \max\left\{1,\ d,\ e_0^{2/3}d^{2/3}\right\}.$$

Proof. The proof of the upper bound is obtained by a construction. Up to a small regularization, we take $h = \frac{d}{\ell}\chi_{[0,\ell]}$ for some $\ell > 0$ to be chosen. Since

$$\min_{u(x,0)=e_0x} \int_{[0,\ell]\times[0,+\infty)} |\nabla u|^2 = Ce_0^2\ell^2 \tag{4.2}$$

for some $C > 0$, it can be seen that

$$\min_u F_{d,e_0}(u,h) \lesssim e_0^2\ell^2 + 1 + \frac{d}{\ell}$$

so that choosing $\ell_{opt} = \min\left(1, \left(\frac{d}{e_0^2}\right)^{1/3}\right)$, we conclude the proof of the upper bound.

The lower bound is obtained by proving that if h contains only very small islands (with support smaller than ℓ_{opt}, then the interfacial part of the energy is very large. Otherwise, using (4.2) it can be shown that already the elastic energy of the largest connected component is large. □

This results proves that for $d \gtrsim e_0^2$ we expect to see the flat configuration while for $d \lesssim e_0^2$, islands of width of order $\left(\frac{d}{e_0^2}\right)^{1/3}$ and height $e_0^{2/3}d^{2/3}$ should be minimizing. In order to understand this better, we consider the regime $e_0 \to +\infty$ and try to obtain asymptotic models. We set $\widetilde{h} = h/d$, $\Omega_{\tilde{h}} = \{(x,y)\ :\ (x,d\,y) \in \Omega_h\}$, and $\widetilde{u}\,(x,y) = u\,(x,d\,y)$. Dropping the tildes, the energy now reads

$$F_{d,e_0}\,(u,h) = d\left[\int_{\Omega_h}\left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{d^2}\left|\frac{\partial u}{\partial y}\right|^2 + \int_0^1\sqrt{\frac{1}{d^2} + |h'|^2}\right]$$

for (u,h) such that $\int_0^1 h\,dx = 1$, and $u\,(x,0) = e_0x$.

The case when $\min F_{d,e_0} \sim 1$ is not very interesting and therefore we focus on the other regimes. If $d \to +\infty$ and $e_0^2 = o(d)$, then the perimeter dominates and $F_{d,e_0} \simeq d$. Hence, we rescale the energy by d, and consider

$$F_d(u,h) = \int_{\Omega_h} \left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{d^2}\left|\frac{\partial u}{\partial y}\right|^2 + \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2}\,.$$

We then prove

Proposition 4.2.2. *If $d \to +\infty$ and $e_0^2 = o(d)$, then F_d $\Gamma-$converges for the Hausdorff topology on Ω_h to the functional*

$$\bar{F}(h) = \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts}),$$

where Γ_{cuts} denotes the (internal) vertical parts of Ω_h. The minimizer of $\bar{F}$ under the constraint $\int_0^1 h = 1$ is the flat configuration $h = 1$.

If now $e_0 \to +\infty$, $\frac{e_0^2}{d} \to +\infty$ and $e_0^2 d^2 \to +\infty$, the main contribution to the energy comes from the elastic part. Note that this regime also allows for $d \to 0$. Define $\eta = \left(\frac{d}{e_0^2}\right)^{1/3} \to 0$, so that $F_{d,e_0} \simeq \frac{d}{\eta}$. We thus rescale the energy $F_{d,e_0}(u,h)$ by $\frac{d}{\eta}$, and consider

$$F_\eta(h) = \min_{u(x,0)=e_0 x} \eta \left[\int_{\Omega_h} \left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{d^2}\left|\frac{\partial u}{\partial y}\right|^2 dx\,dy + \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2}\,dx\right].$$

In this case, we can show the asymptotic formation of islands.

Proposition 4.2.3. *The functional F_η $\Gamma-$converges for the weak* convergence of measures to the functional defined for atomic measures $\mu = \sum_i d_i \delta_{x_i}$ as*

$$F_0(\mu) = 3C^{1/3} \sum_i d_i^{2/3},$$

where C is the constant from (4.2).

Notice that since the limit energy is concave in the volume, it is minimized by a single island.

4.3 The non-compact case

In [BGZ15], we considered the analog of (4.1) in the case of an unbounded domain. The motivation was to understand how this influences the formation of islands. Since the interfacial term

$$\int_{\mathbb{R}} \sqrt{1 + |h'|^2}$$

is always unbounded, some renormalization needs to be done. In [BGZ15], we considered both the small-slope approximation $\sqrt{1+|h'|^2} \sim 1 + \frac{1}{2}|h'|^2$ and the large-slope approximation $\sqrt{1+|h'|^2} \sim 1 + |h'|$ and obtained comparable results for both models. However, since the results are neater in the small-slope approximation, we focus here on that case. Since we work on an unbounded domain, by rescaling we can get rid of one of the two parameters d or e_0 and consider for $V > 0$

$$F_s(V) = \inf \left\{ \int_{\Omega_h} |\nabla u|^2 + \int_{\mathbb{R}} |h'|^2 \ : \ h \in H^1(\mathbb{R}),\, h \geq 0,\, \int_{\mathbb{R}} h = V,\, u(x,0) = x \right\}.$$

Arguing somewhat similarly as for Theorem 4.2.1, we could prove

Theorem 4.3.1. *There exists $c > 0$ such that*

$$c \min(V, V^{4/5}) \leq F_s(V) \leq \min\left(V, \frac{V^{4/5}}{c}\right). \tag{4.3}$$

Notice that the exponent 4/5 instead of 2/3 reflects the small slope approximation. Our main result is the following:

Theorem 4.3.2. *There exists $\bar{V} > 0$ such that :*

i) if $V < \bar{V}$ then no minimizer of $F_s(V)$ exists and $F_s(V) = V$;

ii) if $V \geq \bar{V}$, then there exists minimizers for $F_s(V)$ and $F_s(V) < V$. Moreover, every such minimizer has compact and connected support, is smooth and has zero contact angle with the substrate $\{y = 0\}$.

Proof. By (4.3), we always have $F_s(V) \leq V$. By suitable rescaling, it can be proven that F_s is a concave function which is strictly concave in $\{F_s(V) < V\}$ so that the region $\{F_s(V) = V\}$ is connected. Still by a scaling argument and concavity, it is easily seen that if $F_s(V) = V$ for some $V > 0$ then no minimizer can exist for $V' < V$. The first part of the statement thus reduces to the proof that if V is small enough then $F_s(V) = V$. As already noted, the upper bound $F_s(V) \leq V$ always holds so that we need to show the reverse inequality.
Considering a given admissible profile h, we let u be the corresponding optimal displacement which then solves the Laplace equation. Testing it with $u - x$ and performing a few integrations by parts, it can be shown that

$$\int_{\Omega_h} |\nabla u|^2 = V - \int_{\Omega_h} \frac{\partial u}{\partial y}(x,y) h'(x) dx dy$$

and thus in order to prove that

$$\int_{\Omega_h} |\nabla u|^2 + \int_{\mathbb{R}} |h'|^2 \geq V,$$

it is enough to show the estimate

$$\int_{\Omega_h} \frac{\partial u}{\partial y}(x,y)h'(x)dxdy \leq \int_{\mathbb{R}} |h'|^2. \tag{4.4}$$

By Cauchy-Schwarz we have

$$\begin{aligned}\int_{\Omega_h} \frac{\partial u}{\partial y}(x,y)h'(x)dxdy &\leq \left(\int_{\Omega_h} \left|\frac{\partial u}{\partial y}\right|^2\right)^{1/2} \left(\int_{\Omega_h} |h'|^2\right)^{1/2} \\ &\leq (\sup h)^{1/2} \left(\int_{\Omega_h} \left|\frac{\partial u}{\partial y}\right|^2\right)^{1/2} \left(\int_{\mathbb{R}} |h'|^2\right)^{1/2}.\end{aligned}$$

Now since by the interpolation inequality

$$\|h\|_{L^\infty(\mathbb{R})} \leq \left(\frac{9}{16}\right)^{1/3} \left(\int_{\mathbb{R}} |h(x)|\right)^{1/3} \left(\int_{\mathbb{R}} h'(x)^2\right)^{1/3},$$

we have that for V small enough $\sup h$ is small and since by another scaling argument,

$$\int_{\Omega_h} \left|\frac{\partial u}{\partial y}\right|^2 = \frac{3}{4}\int_{\mathbb{R}} h'^2,$$

the proof of (4.4) follows.

We now turn to *ii*). In order to prove the existence of minimizers for $F_s(V) < V$, we use a concentration-compactness type argument: on the one hand minimizing sequences cannot vanish since otherwise the elastic energy would be of order V and on the other hand dichotomy cannot occur since F_s is strictly concave for $V > \bar{V}$. This also shows that minimizers are compact and connected. The regularity and angle condition are obtained arguing along the lines of [CL03, FFLM07]. The existence of a minimizer for the borderline case $V = \bar{V}$ is then obtained by proving that the size of the support of the minimizers of $F_s(V)$ remains bounded as $V \to \bar{V}$. □

Let us point out that as opposed to [KM14] or to the situation of Chapter 2 where the perimeter term was favoring existence of minimizers, here it is the elastic term which plays this role.

In the spirit of Proposition 4.2.3, we study the asymptotic behavior of the minimizers of $F_s(V)$ as $V \to \infty$. For this we make the anisotropic rescaling $\widetilde{h}(x) = V^{-3/5}h(V^{2/5}x)$ (which heuristically gives an island of width and height of order one) and consider the rescaled energy

$$G_V(\widetilde{h}) = \int_{\mathbb{R}} \widetilde{h}'^2 + V^{-4/5} \int_{\Omega_h} |\nabla u|^2.$$

Observe that the rescaled $\widetilde{h}$ satisfies $\int_{\mathbb{R}} \widetilde{h} = 1$.

Theorem 4.3.3. *For every sequence $V_n \to +\infty$ and every minimizer h_{V_n} of $F_s(V)$, the corresponding $\widetilde{h}_{V_n}$ (possibly translated) converges, up to a subsequence, in $L^\infty(\mathbb{R})$ to the function*

$$\overline{h}(x) = \begin{cases} \frac{3}{2}\ell^{-3}\left(\ell^2 - x^2\right) & \text{if } x \in [-\ell, \ell] \\ 0 & \text{if } x \notin [-\ell, \ell], \end{cases}$$

where $\ell = \left(\frac{9}{16C}\right)^{1/5}$ with C the constant from (4.2). Up to translation, the function $\bar{h}$ is the only minimizer of the functional

$$G(h) = \left(\inf_{u(x,0)=x} \int_{\{h>0\}\times[0,+\infty)} |\nabla u|^2 \right) + \int_{\mathbb{R}} h'^2 \tag{4.5}$$

under the constraint $\int_{\mathbb{R}} h = 1$.

Finally, using the stability of the minimizer of the limit problem, we can prove that minimizers of G_V are exponentially close to $\bar{h}$. To state our result, we will need the following notation. Let $V > \overline{V}$ and $\widetilde{h}_V$, a minimizer of G_V, be fixed. Then for $s > 0$, we let $\widetilde{I}_s$ be the largest connected component of $\{\widetilde{h}_V > s\}$ and $\bar{h}_s$ be the minimizer of $\int_{\widetilde{I}_s} h'^2$ with the constraint $\int_{\widetilde{I}_s} h = \int_{\widetilde{I}_s} \widetilde{h}_V$ and $h = \widetilde{h}_V$ on the boundary of $\widetilde{I}_s$.

Proposition 4.3.4. *Let $\varepsilon > 0$. Then there exist constants $C_0 = C_0(\varepsilon)$ and $C_1 = C_1(\varepsilon)$ such that for every $V > \overline{V}$ and for every minimizer $\tilde{h}_V$ of G_V,*

$$\|\tilde{h}_V - \bar{h}_s\|_{L^2(\widetilde{I}_s)} \leq C_0 \exp(-C_1 V^{1/5}) \quad \forall s \geq \varepsilon.$$

Chapter 5

Study of branching patterns in type-I superconductors

This chapter contains results obtained in collaboration with S. Conti, F. Otto and S. Serfaty in [CGOS18, Gol18].

5.1 The Ginzburg-Landau energy

In 1911, K. Onnes discovered the phenomenon of superconductivity, manifested in the complete loss of resistivity of certain metals and alloys at very low temperature. W. Meissner discovered in 1933 that this was coupled with the expulsion of the magnetic field from the superconductor at the critical temperature. This is now called the Meissner effect. In the phenomenological model proposed by V. Ginzburg and L. Landau (see (5.1) below), the state of the material is represented by the order parameter $u : \Omega \to \mathbb{C}$, where Ω is the material sample. The density of superconducting electrons is then given by $\rho = |u|^2$. One of the main achievements of the Ginzburg-Landau theory is the prediction and the understanding of the mixed (or intermediate) state below the critical temperature. This is a state in which, for moderate external magnetic fields, normal and superconducting regions coexist. The behavior of the material in the Ginzburg-Landau theory is characterized by two physical parameters. The first is the coherence length ξ which measures the typical length on which u varies, the second is the penetration length λ which gives the typical length on which the magnetic field

penetrates the superconducting regions. The Ginzburg-Landau parameter is then defined as $\kappa = \frac{\lambda}{\xi}$. The Ginzburg-Landau functional is given by

$$\int_\Omega |\nabla_A u|^2 + \frac{\kappa^2}{2}(1-|u|^2)^2 dx + \int_{\mathbb{R}^3} |\nabla \times A - B_{\text{ext}}|^2 dx \tag{5.1}$$

where $A : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic potential (so that $B = \nabla \times A$ is the magnetic field), $\nabla_A u = \nabla u - iAu$ is the covariant derivative of u and B_{ext} is the external magnetic field. In these units, the penetration length λ is normalized to 1. We also introduce the density of superconducting electrons $\rho = |u|^2$. As first observed by A. Abrikosov this theory predicts two types of superconductors. On the one hand, when $\kappa < 1/\sqrt{2}$, there is a positive surface tension which leads to the formation of normal and superconducting regions corresponding to $\rho \simeq 0$ and $\rho \simeq 1$ respectively, separated by interfaces. These are the so-called type-I superconductors. On the other hand, when $\kappa > 1/\sqrt{2}$, this surface tension is negative and one expects to see the magnetic field penetrating the domain through lines of vortices. These are the so-called type-II superconductors. In this chapter we are interested in the former type but we refer the interested reader to [SS07] for more information about the latter type (see also Chapter 6).

In type-I superconductors, it is observed experimentally [PGPP05] that complex patterns appear at the surface of the sample. It is believed that these patterns are a manifestation of branching patterns inside the sample. Although the observed states are highly history-dependent, it is argued in [CKO04, PGPP05] that the hysteresis is governed by low-energy configurations at vanishing external magnetic field. We will thus focus here on the asymptotic behavior of the minimizers in this regime and consider the simplest geometric settingwhere the sample Ω is the box $Q_{L_0,T} = (-\frac{L_0}{2}, \frac{L_0}{2})^2 \times (-T, T)$ for some $T, L_0 > 0$ with periodic lateral boundary conditions. The external magnetic field is taken to be perpendicular to the sample, that is $B_{\text{ext}} = b_{\text{ext}} e_3$ for some $b_{\text{ext}} > 0$ and where we recall that e_3 is the third vector of the canonical basis of $\mathbb{R}^3$. After making an isotropic rescaling, subtracting the bulk part of the energy and dropping lower order terms, minimizing (5.1) can be seen as equivalent to minimizing

$$E_T(u, A) = \frac{1}{L^2} \int_{Q_{L,1}} |\nabla_{TA} u|^2 + (B_3 - \alpha(1-\rho))^2 + |B'|^2 dx \\ + \|B_3 - \alpha\beta\|^2_{H^{-1/2}(\{x_3=\pm 1\})}, \tag{5.2}$$

where we have let $B = (B', B_3) = \nabla \times A$,

$$\kappa T = \sqrt{2}\alpha, \qquad b_{\text{ext}} = \frac{\beta\kappa}{\sqrt{2}} \qquad \text{and} \qquad L = L_0/T.$$

If $u = \rho^{1/2}\exp(i\theta)$, since $|\nabla_{TA}u|^2 = |\nabla\rho^{1/2}|^2 + \rho|\nabla\theta - TA|^2$, in the limit $T \to +\infty$ we obtain, at least formally, that A is a gradient field in the region where $\rho > 0$ and therefore the Meissner condition $\rho B = 0$ holds. Moreover, in the regime $\alpha \gg 1$, from (5.2) we see that $B_3 \simeq \alpha(1-\rho)$ and thus ρ takes almost only values in $\{0, 1\}$ with the first two terms of (5.2) acting as a Modica-Mortola type energy penalizing interfaces between normal and superconducting regions. Hence $\operatorname{div} B = 0$ can be rewritten as

$$\partial_3\chi + \frac{1}{\alpha}\operatorname{div}'\chi B' = 0,$$

where $\chi = (1-\rho)$ and div' denotes the divergence with respect to the first two variables. Therefore, from the Benamou-Brenier formulation of optimal transportation [Vil03] and since from the Meissner condition, $B' \simeq \frac{1}{\chi}B'$, the term

$$\int_{Q_{L,1}} |B'|^2 dx \simeq \int_{Q_{L,1}} \frac{1}{\chi}|B'|^2 dx$$

in the energy (5.2) can be seen as a transportation cost. Because of the last term $\|B_3 - \alpha\beta\|^2_{H^{-1/2}(\{x_3=\pm 1\})}$ in the energy (5.2), one expects $B \simeq \alpha\beta e_3$ outside the sample. This implies that close to the boundary the normal domains have to refine. The interaction between the surface energy, the transportation cost and the penalization of an $H^{-1/2}$ norm leads to the formation of complex patterns (see Figure 5.1).

In this regime, the Ginzburg-Landau functional can thus be seen as a non-convex, non-local (in u) functional favoring oscillations, regularized by a surface term which selects the lengthscales of the microstructures. The appearance of branched structures for this type of problem is shared by many other functionals appearing in material sciences such as shape memory alloys [KM92, KM94, Con00, BG15, CDZ17], uniaxial ferromagnets [OV10, KM11] and blistered thin films [JS01, BCDM02].

It has been proven in [COS16, CCKO08, CKO04] that in the regime $T \gg 1$, $\alpha \gg 1$ and $\beta \ll 1$,

$$\min E_T(u, A) \sim \min\{\alpha^{4/3}\beta^{2/3}, \alpha^{10/7}\beta\}\,. \tag{5.3}$$

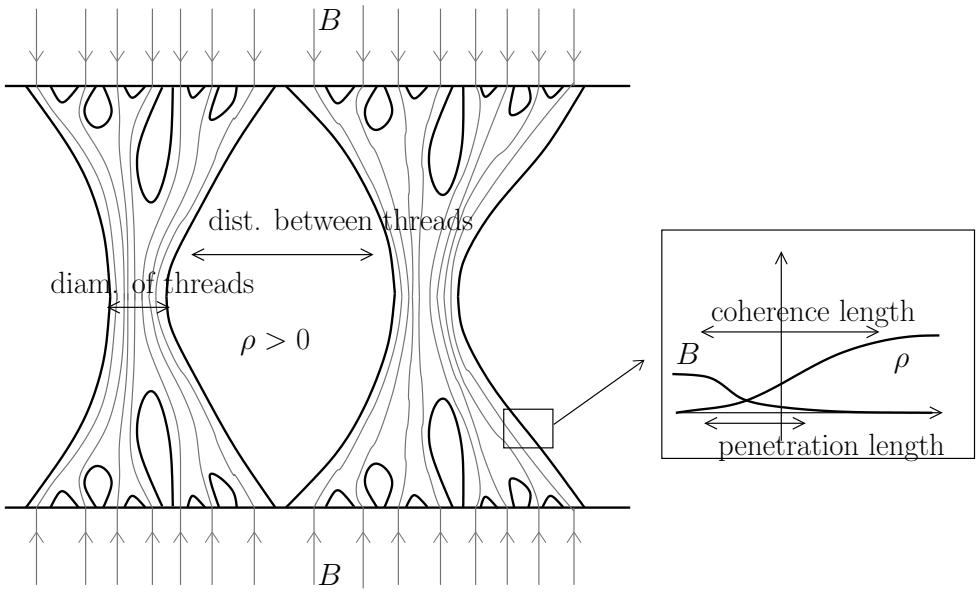

Figure 5.1: The various lengthscales

The scaling $\min E_T(u,A) \sim \alpha^{4/3}\beta^{2/3}$ (relevant for $\alpha^{-2/7} \ll \beta$) corresponds to uniform branching patterns whereas the scaling $\min E_T(u,A) \sim \alpha^{10/7}\beta$ corresponds to non-uniform branching ones. We focus here for definiteness on the regime $\min E_T(u,A) \sim \alpha^{4/3}\beta^{2/3}$, although we believe that our proof can be extended to the other one. Based on the construction giving the upper bounds in (5.3), we expect that in the first regime there are multiple scales appearing (see Figure 5.1):

$$\begin{matrix}\text{penetration}\\ \text{length}\end{matrix} \ll \begin{matrix}\text{coherence}\\ \text{length}\end{matrix} \ll \begin{matrix}\text{diameter of the}\\ \text{threads in the bulk}\end{matrix} \ll \begin{matrix}\text{distance between the}\\ \text{threads in the bulk}\end{matrix}, \tag{5.4}$$

which amounts in our parameters to

$$T^{-1} \ll \alpha^{-1} \ll \alpha^{-1/3}\beta^{1/3} \ll \alpha^{-1/3}\beta^{-1/6}.$$

In order to better describe the minimizers we focus on the extreme region of the phase diagram $T, T\alpha^{-1}, \beta^{-1}, \alpha\beta^{7/2} \to +\infty$, with $L = \alpha^{-1/3}\beta^{-1/6}$. In this regime, we have in particular $\alpha^{-1} \ll \alpha^{-1/3}\beta^{1/3}$ so that the separation of scales (5.4) holds. After making an anisotropic rescaling for which both the mass carried by the trees and the distance between them is of order 1, this leads to the functional

$$\widetilde{E}(u,A) = \int_{Q_{1,1}} \alpha^{-2/3}\beta^{-1/3}\left|\nabla'_{\alpha^{1/3}\beta^{-1/3}TA}u\right|^2 + \alpha^{-4/3}\beta^{-2/3}\left|(\nabla_{\alpha^{1/3}\beta^{-1/3}TA}u)_3\right|^2 \tag{5.5}$$
$$+ \alpha^{2/3}\beta^{-2/3}\left(B_3 - (1-|u|^2)\right)^2 + \beta^{-1}|B'|^2 dx$$
$$+ \alpha^{1/3}\beta^{7/6}\|\beta^{-1}B_3 - 1\|^2_{H^{-1/2}(x_3=\pm 1)}.$$

Notice that in these units, the scale separation (5.4) reads

$$T^{-1}\alpha^{1/3}\beta^{1/6} \ll \alpha^{-2/3}\beta^{1/6} \ll \beta^{1/2} \ll 1. \tag{5.6}$$

5.2 The Γ−convergence result

The main result of [CGOS18] is a Γ−convergence result of the functional $\widetilde{E}$ towards a functional defined on measures μ living on one-dimensional trees. These trees correspond to the normal regions in which $\rho \simeq 0$ and where the magnetic field B penetrates the sample. Roughly speaking, if for a.e. $x_3 \in (-1,1)$ the slice of $\mu = \mu_{x_3} \otimes dx_3$ has the form $\mu_{x_3} = \sum_i \phi_i \delta_{X_i(x_3)}$ where the sum is at most countable, then we let

$$I(\mu) = \int_{-1}^{1} K_* \sum_i \sqrt{\phi_i} + \phi_i |\dot{X}_i|^2 dx_3, \tag{5.7}$$

where $K_* = 8\sqrt{\pi}/3$ and $\dot{X}_i$ denotes the derivative (with respect to x_3) of $X_i(x_3)$. The X_i's represent the graphs of each branch of the tree (parametrized by height) and the ϕ_i's represent the flux carried by the branch. We can now state our main result

Theorem 5.2.1. *Let $T_n, \alpha_n, \beta_n^{-1} \to +\infty$ with $T_n\alpha_n^{-1}, \alpha_n\beta_n^{7/2} \to +\infty$, then:*

1. *For every sequence (u_n, A_n) with $\sup_n \widetilde{E}(u_n, A_n) < +\infty$, up to subsequence , $\beta_n^{-1}(1-|u_n|^2)$ weakly converges to a measure μ of the form $\mu = \mu_{x_3} \otimes dx_3$ with $\mu_{x_3} = \sum_i \phi_i \delta_{X_i}$ for a.e. $x_3 \in (-1,1)$, $\mu_{x_3} \rightharpoonup dx'$ (where dx' denotes the two dimensional Lebesgue measure on Q_1) when $x_3 \to \pm 1$ and such that*

$$\liminf_{n\to+\infty} \widetilde{E}(u_n, A_n) \geq I(\mu). \tag{5.8}$$

2. *If in addition $L_n^2\alpha_n\beta_n T_n \in 2\pi\mathbb{Z}$, where $L_n = \alpha_n^{-1/3}\beta_n^{-1/6}$, then for every measure μ such that $I(\mu) < +\infty$ and $\mu_{x_3} \rightharpoonup dx'$ as $x_3 \to \pm 1$, there exists (u_n, A_n) such that $\beta_n^{-1}(1-|u_n|^2) \rightharpoonup \mu$ and*

$$\limsup_{n\to+\infty} \widetilde{E}(u_n, A_n) \leq I(\mu). \tag{5.9}$$

Within our periodic setting, the quantization condition $L_n^2\alpha_n\beta_n T_n \in 2\pi\mathbb{Z}$ for the flux is a consequence of the fact that the phase circulation of the complex-valued function in the original problem is naturally quantized. It is necessary in order to make our construction but we believe that it is also a necessary condition for having sequences of bounded energy. We remark that scaling back to the original variables this condition is the physically natural one $L_0^2 b_{\text{ext}} \in 2\pi\mathbb{Z}$.

Let us point out that the limiting functional $I(\mu)$ has many similarities with irrigation (or branched transportation) models that have recently attracted a lot of attention (see [BCM09, CDRM18] or the recent paper [BW17] where the connection is also made to some urban planning models).

5.2.1 The lower bound

We discuss here the proof of (5.8). The difficulty is to make the informal discussion below (5.2) on the various contribution of the energy, rigorous and more quantitative.
A first step is to prove that the energy controls the deviation from satisfying the Meissner effect in a weak norm. This is the content of the following lemma.

Lemma 5.2.2. *For every Q_1-periodic test function $\psi \in H^1_{\text{per}}(Q_{1,1})$, if $\|\rho\|_\infty \leq 1$ then*

$$\left|\int_{Q_{1,1}} \rho B_3 \psi dx\right| \lesssim \frac{\alpha^{1/3}\beta^{2/3}}{T}\widetilde{E}(u,A)\|\psi\|_{L^\infty} + \frac{\beta^{1/2}}{T}\widetilde{E}(u,A)^{1/2}\|\nabla'\psi\|_{L^2}, \tag{5.10}$$

One of the main consequences of this lemma is that the energy controls a standard double-well potential. This is similar to (7.10). For $\varepsilon > 0$ fixed, we define the following regularization of the singular double well potential $\chi_{\rho>0}(1-\rho)^2$:

$$W_\varepsilon(\rho) = \eta_\varepsilon(\rho)(1-\rho)^2 \text{ with } \eta_\varepsilon(\rho) = \min\{\rho/\varepsilon, 1\}\,, \tag{5.11}$$

see Figure 5.2.

Lemma 5.2.3. *For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for every (u,A) with $\|\rho\|_\infty \leq 1$ there holds,*

$$\int_{Q_{1,1}} \alpha^{2/3}\beta^{-2/3}W_\varepsilon(\rho)dx \leq \int_{Q_{1,1}} \alpha^{2/3}\beta^{-2/3}\left(B_3 - (1-\rho)\right)^2 dx + C_\varepsilon\widetilde{E}(u,A)\frac{\alpha}{T}. \tag{5.12}$$

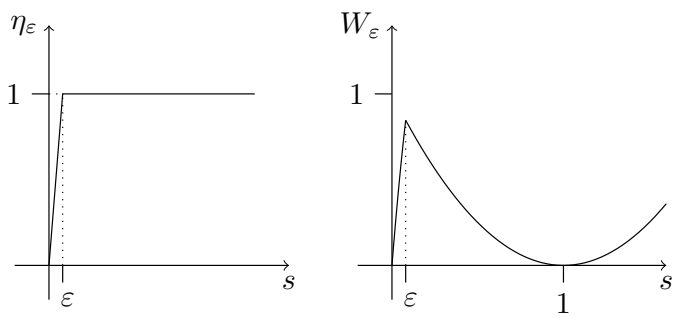

Figure 5.2: The cutoff function η_ε and the two-well potential W_ε used in Lemma 5.2.3.

Proof. To lighten notation, we let $E = \widetilde{E}(u, A)$. Writing $(1-\rho) = B_3 - (B_3 - (1-\rho))$, we obtain by Young's inequality

$$\begin{aligned}(1-\rho)^2 &= B_3(1-\rho) - (B_3 - (1-\rho))(1-\rho)\\ &\le B_3(1-\rho) + \frac{1}{2}(1-\rho)^2 + \frac{1}{2}(B_3 - (1-\rho))^2.\end{aligned}$$

Multiplying by $2\eta_\varepsilon(\rho)$ and using that $0 \le \eta_\varepsilon \le 1$ we obtain for $W_\varepsilon(\rho)$ the estimate

$$W_\varepsilon(\rho) = \eta_\varepsilon(\rho)(1-\rho)^2 \le (B_3 - (1-\rho))^2 + 2\eta_\varepsilon(\rho)(1-\rho)B_3. \tag{5.13}$$

Let $\psi_\varepsilon(s) = 2\frac{\eta_\varepsilon(s)}{s}(1-s) = 2\min\{\frac{1}{\varepsilon}, \frac{1}{s}\}(1-s)$ then ψ_ε is bounded by $1/\varepsilon$ and is Lipschitz continuous in $s^{1/2}$ with constant of order $\varepsilon^{-3/2}$ i.e. $\sup_t |(\psi_\varepsilon(t^2))'| \lesssim \varepsilon^{-3/2}$. Since $2\eta_\varepsilon(\rho)(1-\rho)B_3 = \rho B_3 \psi_\varepsilon(\rho)$, using (5.10) with $\psi = \psi_\varepsilon(\rho)$, we get

$$\begin{aligned}\left|\int_{Q_{1,1}} 2\eta_\varepsilon(\rho)(1-\rho)B_3 dx\right| &\lesssim \frac{\alpha^{1/3}\beta^{2/3}}{T}\left(\varepsilon^{-1}E + \alpha^{-1/3}\beta^{-1/6}E^{1/2}\|\nabla'(\psi_\varepsilon(\rho))\|_{L^2}\right)\\ &\lesssim \frac{\alpha^{1/3}\beta^{2/3}}{T}\left(\varepsilon^{-1}E + \varepsilon^{-3/2}\alpha^{-1/3}\beta^{-1/6}E^{1/2}\|\nabla'\rho^{1/2}\|_{L^2}\right)\\ &\lesssim C_\varepsilon \frac{\alpha^{1/3}\beta^{2/3}}{T}\left(E + \int_{Q_{1,1}} \alpha^{-2/3}\beta^{-1/3}|\nabla'\rho^{1/2}|^2 dx\right)\\ &\lesssim C_\varepsilon \frac{\alpha^{1/3}\beta^{2/3}}{T}E,\end{aligned}$$

where we used that $|\nabla'\rho^{1/2}| \le |\nabla'_{\alpha^{1/3}\beta^{1/3}TA}u|$ and thus $\int_{Q_{1,1}} \alpha^{-2/3}\beta^{-1/3}|\nabla'\rho^{1/2}|^2 \le E$. Estimate (5.12) follows from inserting this estimate into (5.13). □

Neglecting small error terms, this means that $\widetilde{E}(u, A)$ controls

$$\int_{Q_{1,1}} \alpha^{-2/3}\beta^{-1/3}\left|\nabla'\sqrt{\rho}\right|^2 + \alpha^{2/3}\beta^{-2/3}W_\varepsilon(\rho) + \beta^{-1}|B'|^2 dx \\ + \alpha^{1/3}\beta^{7/6}\|\beta^{-1}B_3 - 1\|^2_{H^{-1/2}(x_3=\pm 1)}. \quad (5.14)$$

By the usual Modica-Mortola trick, using the co-area formula, we can find $s \in (0,1)$ such that letting $\chi(x', x_3) = \beta^{-1}(1 - \chi_{\{\rho(\cdot,x_3)>s\}}(x')$,

$$\int_{Q_{1,1}} \alpha^{-2/3}\beta^{-1/3}\left|\nabla'\sqrt{\rho}\right|^2 + \alpha^{2/3}\beta^{-2/3}W_\varepsilon(\rho) \geq C_\varepsilon \int_{Q_{1,1}} \beta^{1/2}|D'\chi|, \quad (5.15)$$

for some constant $C_\varepsilon \overset{\varepsilon\to 0}{\to} \frac{4}{3} = 2\int_0^1(1-t^2)dt$. Now since for a.e. x_3,

$$\int_{Q_1} \chi(\cdot, x_3) \simeq \int_{Q_1} \beta^{-1}B_3(\cdot, x_3) = 2,$$

the fact that $\chi \in \{0, \beta^{-1}\}$ with $\beta \ll 1$ together with isoperimetric effects implies that for a.e. x_3, $\{\chi(\cdot, x_3) = \beta^{-1}\}$ must concentrate into points.

Lemma 5.2.4. *Let $\chi_n \in BV(Q_1, \{0, \beta_n^{-1}\})$ be such that $\lim_{n\to+\infty}\int_{Q_1}\chi_n dx' = 1$ and*

$$\sup_n \int_{Q_1} \beta_n^{1/2}|D'\chi_n| < +\infty.$$

Then, up to a subsequence, $\chi_n \rightharpoonup \sum_i \phi_i\delta_{X_i}$ for some at most countable family of $\phi_i > 0$ and $X_i \in Q_1$, and

$$\liminf_{n\to+\infty}\int_{Q_1} \beta_n^{1/2}|D'\chi_n| \geq 2\sqrt{\pi}\sum_i \sqrt{\phi_i}. \quad (5.16)$$

Combining this with (5.14) and (5.15) (and quite some extra work related to the lack of regularity in x_3) this allows to conclude the proof of (5.8).

5.2.2 The upper bound construction

We now turn to (5.9). Due to the multi-scale aspect of the problem and to the necessary complexity of the patterns close the boundary of the sample (since infinite branching must occur), the construction of a recovery sequence is actually here the most difficult part of the proof. Even though it is a delicate point, we will forget in this discussion for simplicity the quantization issue.

As in many $\Gamma-$convergence result, an important first step is to prove that 'regular' patterns are dense in energy.

Lemma 5.2.5. *For every measure μ with $I(\mu) < +\infty$ and $\mu_{\pm 1} = dx'$, there exists a sequence of measures μ_N with $\mu_N \rightharpoonup \mu$, $\limsup_{N\to+\infty} I(\mu_N) \leq I(\mu)$, $\mu_N(Q_{1,1}) = \mu(Q_{1,1})$ and such that*

(i) The measure μ is finite polygonal.

(ii) All branching points are triple points. This means that any $x \in Q_{1,1}$ belongs to the closures of no more than three segments.

(iii) The traces obey $\mu_1 = \mu_{-1} = N^{-2}\sum_j \delta_{X_j}$, where the X_j are N^2 points on a square grid, spaced by $1/N$.

Proof. The idea is to rescale (in x_3) the measure μ so that $\mu_{1-\varepsilon} = \mu_{-1+\varepsilon} = dx'$ for some $\varepsilon \ll 1$. By symmetry me may focus on $x_3 > 0$. Regarding the boundary, we can make a construction $\tilde{\mu}$ connecting the Lebesgue measure at $x_3 = 1-\varepsilon$ to the Lebesgue measure at $x_3 = 1$ with small energy and such that $\tilde{\mu}_{1-\frac{\varepsilon}{2}} = N^{-2}\sum_j \delta_{X_j}$, where the X_j are N^2 points on a square grid, spaced by $1/N$. We then let

$$\hat{\mu}_{x_3} = \begin{cases} \mu_{x_3} & \text{if } x_3 \leq 1-\varepsilon \\ \tilde{\mu}_{x_3} & \text{if } 1-\varepsilon \leq x_3 \leq 1-\frac{\varepsilon}{2} \\ N^{-2}\sum_j \delta_{X_j} & \text{if } 1-\frac{\varepsilon}{2} \leq x_3 \leq 1. \end{cases}$$

Regarding the bulk part of the domain, we discretize $\hat{\mu}$ in x_3, remove the 'small' branches in order to have a finite sum of Dirac masses at each considered slice and then minimize the energy $I(\mu)$ under the constraint of coinciding with this pruned version of $\hat{\mu}$ on the slices. By definition this new measure has smaller energy and it can be shown that this procedure produces a finite polygonal measure (see Proposition 5.3.1 below). Of course, because of the pruning, this measure has slightly less mass than μ but this can be easily fixed. By performing a small perturbation, we may also make sure that branching points are triple. □

With this density result at hand, we can start describing the construction (which of course is only performed for measures given by Lemma 5.2.5). At the boundary, thanks to (*iii*) of Lemma 5.2.5 our measure is equal to a sum of Dirac masses on a regular grid and we can thus directly use the construction given by [COS16, Lem. 4.7].
In the bulk the construction is made differently away from the triple junctions and close to the triple junctions. Away from the branching points, the construction is quite simple (however it must not only have the correct

energy scaling but also the correct prefactor): if locally $\mu = \phi\delta_{X(x_3)}$, then we take

$$\rho(x) = v\left(\frac{|x' - X(x_3)| - \sqrt{\beta\phi/\pi}}{\alpha^{-2/3}\beta^{1/6}}\right),$$

where v is the optimal transition profile associated to the double-well potential $\chi_{\rho\neq 0}(1-\rho)^2$. Notice that for every x_3, the level-sets of ρ are disks and that in the current units, by (5.6), $\alpha^{-2/3}\beta^{1/6}$ is the coherence length. At the branching points, we thus need to connect one disk of area $\beta\phi$ to two disks of area $\beta\phi_1$ and $\beta\phi_2$ with $\phi = \phi_1 + \phi_2$. Since there are finitely many branching points and since the construction is local, we must make a construction with the correct energy scaling but we are allowed to lose in the prefactor. Therefore, we first transform the disk into a square, then split the square into two rectangles and then retransform each rectangle into a disk.

5.3 Minimizers of the limit energy

5.3.1 The three-dimensional case

One of the advantages of the limit functional $I(\mu)$ is that it is much simpler than the original model (5.1) and thus more can be said about the structure of its minimizers. In particular, we can prove that they contain no-loops (which is a common feature in branched transportation problem, see [BCM09]). As a consequence, we can prove that minimizers are regular in the following sense:

Proposition 5.3.1. *A minimizer of $I(\mu)$ with boundary conditions $\bar{\mu}_+ = \sum_{i=1}^N \phi_i^+\delta_{X_i^+}$ and $\bar{\mu}_- = \sum_{i=1}^N \phi_i^-\delta_{X_i^-}$ (some ϕ_i may be zero) satisfies*

1. *$\mu = \sum_{i=1}^M \frac{\varphi_i}{\sqrt{1+|\dot{X}_i|^2}}\mathcal{H}^1 \llcorner \Gamma_i$ for some $M \in \mathbb{N}$, where $\Gamma_i = \{(X_i(x_3), x_3) : x_3 \in [a_i, b_i]\}$ are disjoint up to the endpoints, and the X_i are absolutely continuous.*

2. *Each X_i is affine.*

3. *If $\bar{\mu}_- = \bar{\mu}_+$ then there exists a symmetric minimizer with respect to the $x_3 = 0$ plane.*

We can then also show that for any boundary, μ is made of locally finitely many branches.

Proposition 5.3.2. *Fix $\bar{\mu}$ a positive measure and let μ be a symmetric minimizer of I subject to $\mu_{\pm 1} = \bar{\mu}$. Then for any $\delta > 0$ sufficiently small, the number of Diracs in each slice $x_3 \in [-1+\delta, 1-\delta]$ is bounded from above by a multiple of $\lesssim \delta^{-4}$.*

The proof is based on the fact that the interfacial part of the energy is concave in the flux and thus favors concentration. Therefore, if there are too many branches, one can construct a better competitor by bundling together small branches.

If these two results are interesting, one would hope to be able to say more about the structure of the minimizers when the boundary conditions are given by the Lebesgue measure (which are the ones coming from Theorem 5.2.1). In particular, it is expected that minimizers are self-similarly refining close to the boundary. One possibility would be to reproduce the strategy of [Con00] and obtain first local bounds on the energy (see also [Vie09, ACO09]). However, this has proven to be a very difficult problem. Indeed, if the relaxed problem in [Con00, Vie09, ACO09] is given by the Poisson equation, in our case it would be the Monge-Ampère equation, which is a nonlinear and degenerate elliptic equation. As a side note, it is precisely the study of this question which led us together with F. Otto to the variational proof of partial regularity for optimal transport maps in [GO17].

5.3.2 The two-dimensional case

Since the precise structure of minimizers is hard to understand in dimension three, I decided to consider in [Gol18] a simplified two-dimensional model. For a measure μ on $\mathbb{R} \times (a, b)$ such that $\mu = \mu_t \otimes dt$ with $\mu_t = \sum_i \phi_i \delta_{X_i}$ for a.e. $t \in (a, b)$ for some (pairwise distinct) $X_i \in \mathbb{R}$, we functional reads as

$$I(\mu) = \int_a^b \sharp\{\phi_i \neq 0\} + \sum_i \phi_i |\dot{X}_i|^2 dt. \tag{5.17}$$

Notice that compared with (5.7), the term $\sharp\{\phi_i \neq 0\}$ replaces here $\sum_i \phi_i^{1/2}$. This is in line with the interpretation of the first term in (5.17) as an interfacial term penalizing the creation of many flux tubes. That is, if we are in $(1+d)-$dimensions, it is proportional to the perimeter of a union of $d-$dimensional balls of volume ϕ_i (which is $2\sqrt{\pi}\sum_i \phi_i^{1/2}$ if $d = 2$ and $2\sharp\{\phi_i \neq 0\}$ if $d = 1$).

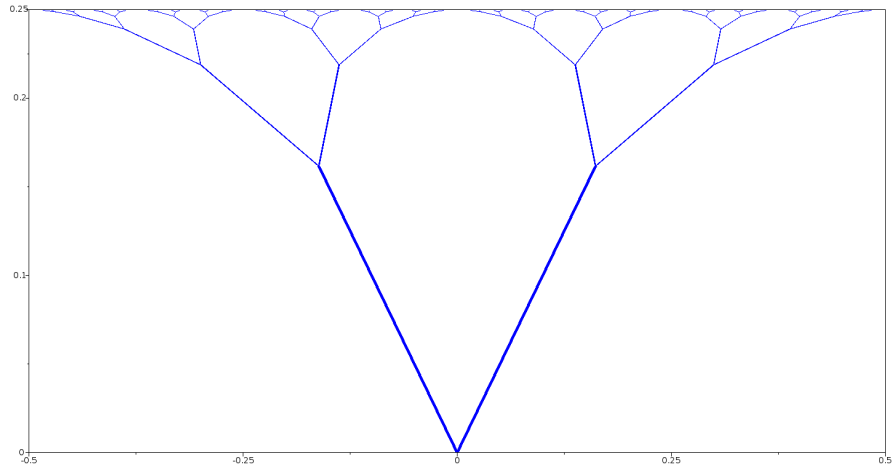

Figure 5.3: The optimal configuration μ^*

Since a no-loop property can also be proven for minimizers of this energy, if μ is a minimizer of $I(\mu)$ under some boundary conditions, then it is made of a finite union of connected components each independent from the other ones. In particular, in the case we are interested in i.e. $a = -T$, $b = T$ and $\mu_{\pm T} = dx$, if in the mid-plane $t = 0$ the minimizer μ is given by $\mu_0 = \sum_i^N \phi_i \delta_{X_i}$, then μ can be easily reconstructed provided we can solve the N irrigation problems of connecting $\phi_i \delta_{X_i}$ to their corresponding Lebesgue measures. Up to rescaling and shearing we are led to study the variational problem

$$\mathcal{I}(T) = \min \{ I(\mu) \ : \ \mu_0 = \delta_0 \text{ and } \mu_T = dx \llcorner [-1/2, 1-2] \} . \tag{5.18}$$

When T is large enough, we were able to compute explicitly the minimizer of (5.18). As explained below, our proof works as long as T is larger than the first branching time which turns out to be equal to $T = 1/4$.

Theorem 5.3.3. *For $T = 1/4$, the dyadically branching measure μ^* given in Figure 5.3.2, is the unique minimizer of* (5.18)*. Moreover, if $T \geq 1/4$, the unique minimizer of* (5.18) *is given by $\mu_t = \delta_0$ for $t \in [0, T - 1/4]$ and $\mu_t = \mu^*_{t-(T-1/4)}$ for $t \in (T - 1/4, T)$, with*

$$I(\mu) = \frac{1}{2 - \sqrt{2}} + T.$$

Proof. Let us sketch the proof of this theorem. It is based on the tree structure of the minimizers of (5.18) which together with invariance by scaling

and shearing leads to a recursive characterization of the minimizers. Indeed, we first show that

$$\mathcal{I}(T) = \min_{\sum_{i=1}^N \phi_i = 1} \sum_{i=1}^N \phi_i^{3/2} \mathcal{I}(T\phi_i^{-3/2}) + \frac{1}{12T}\left(1 - \sum_{i=1}^N \phi_i^3\right). \tag{5.19}$$

This formula reflects the fact that if at level T, the minimizer branches into N pieces of respective masses ϕ_i then up to rescaling and shearing, each of the subtrees solves the exact same problem as the original one (connecting a Dirac mass to the Lebesgue measure). In particular, if we define T_* to be the first branching time, meaning that if $T > T_*$ then the minimizer of $E(T)$ cannot branch for a time $T - T_*$, we may use that $T_*\phi_i^{-3/2} \geq T_*$ to obtain

$$\mathcal{I}(T_*\phi_i^{-3/2}) = \mathcal{I}(T_*) + T_*(\phi_i^{-3/2} - 1)$$

and then rewrite (5.19) in the purely analytical form

$$\frac{\mathcal{I}(T_*) - T_*}{T_*} = \min_{\sum_{i=1}^N \phi_i = 1} \frac{(N-1) + \frac{1}{12T_*^2}\left(1 - \sum_{i=1}^N \phi_i^3\right)}{1 - \sum_{i=1}^N \phi_i^{3/2}}. \tag{5.20}$$

We then want to use (5.20) to prove that $T_* = 1/4$ and that the corresponding minimizer has exactly two branches of mass $1/2$ at time zero. Once this is proven, the conclusion is readily reached thanks to the recursive nature of the problem.

In order to prove that $T_* = 1/4$ and $N = 2$, we introduce for fixed $N \geq 2$ the quantity

$$\alpha_N = \inf_{\phi_i \geq 0} \left\{ \frac{1 - \sum_{i=1}^N \phi_i^3}{1 - \sum_{i=1}^N \phi_i^{3/2}} \ : \ \sum_{i=1}^N \phi_i = 1 \right\}.$$

By (5.20), if at time T_* the minimizer has N branches then since for $\sum_{i=1}^N \phi_i = 1$ there holds $1 - \sum_{i=1}^N \phi_i^{3/2} \leq 1 - N^{-1/2}$, we have the lower bound

$$\frac{\mathcal{I}(T_*) - T_*}{T_*} \geq \frac{N-1}{1 - N^{-1/2}} + \frac{\alpha_N}{12T_*^2}.$$

Using this together with an upper bound on $\mathcal{I}(T_*)$ given by a dyadically branching construction we obtain both that $T_* \leq 1/4$ and that a lower bound on α_N gives a corresponding upper bound on N. We then obtain these lower bounds on α_N using a computer assisted proof. This excludes that $N \geq 3$. The case $N = 2$ is finally studied by hand to prove that $T_* = 1/4$ and that the mass splits in half.

□

As an application we derive a full characterization of symmetric (with respect to $t = 0$) minimizers in the case $a = -b = -T$, $\mu_\pm = dx \llcorner [-1/2, 1/2]$ and $T \geq 1/4$:

Theorem 5.3.4. *For $T \in [\frac{1}{4}, \frac{1}{4(2\sqrt{2}-2)})$, the unique symmetric minimizer of $I(\mu)$ with $\mu_{\pm T} = dx$ is equal in $[0, T]$ to the union of two rescaled copies of the optimal tree given by Theorem 5.3.3. For $T > 4(2\sqrt{2} - 2)$, it is given in $[0, T]$ by the unique minimizer of $\mathcal{I}(T)$ given by Theorem 5.3.3.*

We should however point out that this result is not totally satisfactory since we are essentially able to study only the situation of an isolated microstructure (due to the constraint $T \geq 1/4$) whereas one is typically interested in the case $T \ll 1$ where many microstructures are present and where the lateral boundary conditions have limited effect i.e. one tries to capture an extensive behavior of the system.

Chapter 6

A Ginzburg-Landau model with topologically induced free discontinuities

This chapter contains results obtained in collaboration with B. Merlet and V. Millot in [GMM17].

6.1 The model

We are interested in the study of the asymptotic behavior of a family of functionals combining aspects of both Ginzburg-Landau [BBH94, SS07] and Mumford-Shah [AFP00, Lem16, Dav05] functionals in dimension two. Those extend the standard Ginzburg-Landau energy, and give rise to the formation of vortex points connected by line defects in the small energy regime. Interestingly, vortices and line defects are coupled through topological constraints.

To be more specific, let us introduce the mathematical context. We consider for $m \in \mathbb{N}$, $m \geq 2$, the group of m-th roots of unity $\mathbf{G}_m = \left\{1, \mathbf{a}, \mathbf{a}^2, \ldots, \mathbf{a}^{m-1}\right\}$ with $\mathbf{a} = e^{2i\pi/m}$. We are interested in maps taking values in the quotient space $\mathbb{C}/\mathbf{G}_m$. We identify $\mathbb{C}/\mathbf{G}_m$ with the round cone

$$\mathcal{N} = \left\{(z,t) \in \mathbb{C} \times \mathbb{R} : t = |z|\sqrt{m^2-1}\right\} \subset \mathbb{R}^3$$

by means of the map $\mathrm{P} : \mathbb{C} \to \mathcal{N}$ defined as

$$\mathrm{P}(z) = \frac{1}{m}\Big(\mathrm{p}(z), |z|\sqrt{m^2-1}\Big) \quad \text{with } \mathrm{p}(z) = \frac{z^m}{|z|^{m-1}}\,.$$

The map P induces an isometry between $\mathbb{C}/\mathbf{G}_m$ and $\mathcal{N}$, and restricted to $\mathbb{C}\setminus\{0\}$ it defines a covering map of $\mathcal{N}\setminus\{0\}$ of degree m. For a given open set Ω and $p \geq 1$ we can thus say that $u \in W^{1,p}(\Omega, \mathbb{C}/\mathbf{G}_m)$ if $\mathrm{P}(u) \in W^{1,p}(\Omega, \mathcal{N})$ (where we say that a map $v \in W^{1,p}(\Omega, \mathcal{N})$ if v takes values in $\mathcal{N}$ and $v \in W^{1,p}(\Omega, \mathbb{R}^3)$).

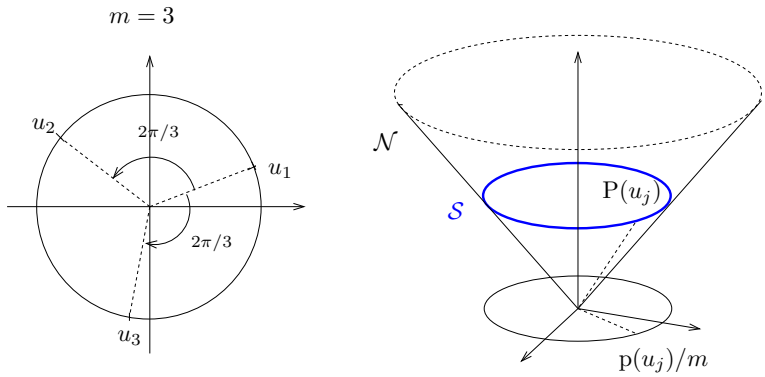

Figure 6.1: The cone $\mathcal{N}$ and the projection P. $\mathrm{P}(u_1) = \mathrm{P}(u_2) = \mathrm{P}(u_3)$.

For a simply connected smooth bounded domain $\Omega \subset \mathbb{R}^2$ and a "small" parameter $\varepsilon > 0$, the standard Ginzburg-Landau energy over Ω of a vector valued $W^{1,2}$-map reads

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\,dx\,.$$

Notice that for a domain $\Omega \times \mathbb{R}$ of $\mathbb{R}^3$ and a function u invariant along e_3, letting $\kappa = \varepsilon^{-1}$ and assuming that there is no magnetic field, i.e. that $A = 0$, (5.1) indeed reduces to E_ε. Compared to Chapter 5, the regime we are considering here would thus correspond to type-II superconductors.

The main functional under investigation is defined for $u \in SBV^2(\Omega)$ satisfying the constraint $\mathrm{P}(u) \in W^{1,2}(\Omega; \mathcal{N})$ by

$$F_\varepsilon(u) = E_\varepsilon\big(\mathrm{P}(u)\big) + \mathcal{H}^1(J_u)\,, \tag{6.1}$$

where J_u denotes the jump set of u. We stress that F_ε extends E_ε, that is $F_\varepsilon(u) = E_\varepsilon(u)$ whenever $u \in W^{1,2}(\Omega)$, which comes from the isometric character of P. In the same way F_ε appears as a Mumford-Shah type functional

since

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\,dx + \mathcal{H}^1(J_u)\,,$$

where ∇u denotes the absolutely continuous part of the measure Du. The constraint $\mathrm{P}(u) \in W^{1,2}(\Omega;\mathcal{N})$ rephrases the fact that the functional is restricted to the class $\big\{u \in SBV^2(\Omega) : u^+/u^- \in \mathbf{G}_m \text{ on } J_u\big\}$. In particular, only specific discontinuities in the orientation are allowed. The case $m = 2$, which consists in identifying u and $-u$, is of special interest as it appears in many physical models, see Section 6.3 below.

We also considered in [GMM17] an Ambrosio-Tortorelli regularization of (6.1) where the jump set J_u is (formally) replaced by the zero set $\{\psi \sim 0\}$ of some scalar phase field function ψ, and the length $\mathcal{H}^1(J_u)$ by a suitable energy of ψ. We introduce a second small parameter η and consider for $u \in L^2(\Omega)$ and $\psi \in W^{1,2}(\Omega;[0,1])$ satisfying $\mathrm{P}(u) \in W^{1,2}(\Omega;\mathcal{N})$ and $u\psi \in W^{1,2}(\Omega)$, the functional

$$F_\varepsilon^\eta(u,\psi) = E_\varepsilon\big(\mathrm{P}(u)\big) + \frac{1}{2}\int_\Omega \eta|\nabla\psi|^2 + \frac{1}{\eta}(1-\psi)^2\,dx\,. \tag{6.2}$$

Compared to the original Ambrosio-Tortorelli functional [AT92], u and ψ are only coupled through the constraint $u\psi \in W^{1,2}(\Omega)$, and not in the functional itself. As for F_ε, the functional F_ε^η extends E_ε in the sense that $F_\varepsilon^\eta(u,1) = E_\varepsilon(u)$ whenever $u \in W^{1,2}(\Omega)$. To keep the exposition simpler, we will focus from now on only on the sharp interface model F_ε.

We aim to study low energy states (in particular minimizers) of the functional F_ε under Dirichlet boundary conditions of the form $u = g$ on $\partial\Omega$ for a prescribed smooth $g \in C^\infty(\partial\Omega;\mathbb{S}^1)$. Concerning F_ε, we work in the class $\mathcal{G}_g(\Omega)$ of maps satisfying $\mathrm{P}(u) = \mathrm{P}(g)$ on $\partial\Omega$. Then, we penalize possible deviations from g on $\partial\Omega$ by considering the modified energy

$$F_{\varepsilon,g}(u) = F_\varepsilon(u) + \mathcal{H}^1\big(\{u \neq g\} \cap \partial\Omega\big)\,.$$

Notice that such a penalization is necessary in order to have lower semicontinuity of the functional (see for instance [GMS79]). In this setting, the functional $F_{\varepsilon,g}$ still extends E_ε restricted to $W_g^{1,2}(\Omega)$, so that

$$\min_{\mathcal{G}_g(\Omega)} F_{\varepsilon,g} \le \min_{W_g^{1,2}(\Omega)} E_\varepsilon. \tag{6.3}$$

As in the classical Ginzburg-Landau theory [BBH94], we assume that the winding number (or degree) is strictly positive, i.e.,

$$d = \deg(g,\partial\Omega) > 0\,.$$

In this way, g does not admit a continuous $\mathbb{S}^1$-valued extension to Ω. This topological obstruction is responsible for the formation of *v*ortices (point singularities) in any configuration of small energy E_ε as $\varepsilon \to 0$, and the minimum value of E_ε over $W_g^{1,2}$ is given by $\pi d|\log\varepsilon|$ at first order. In view of (6.3), creating discontinuities in the orientation may lead to configurations of smaller energy. Indeed, direct constructions of competitors show that the minimum value of $F_{\varepsilon,g}$ is less than $\frac{\pi d}{m}|\log\varepsilon|$ at first order, and thus (almost) minimizers must have line singularities, at least for ε small enough.

6.2 Heuristics

The starting point is the identity

$$E_\varepsilon\big(\mathrm{P}(u)\big) = \frac{1}{m^2}E_\varepsilon\big(\mathrm{p}(u)\big) + \frac{m^2-1}{m^2}E_\varepsilon\big(|\mathrm{p}(u)|\big)\,.$$

Following the standard theory of the Ginzburg-Landau functional [BBH94, SS07], one may expect that for configurations u of small energy, the leading term is $\frac{1}{m^2}E_\varepsilon\big(\mathrm{p}(u)\big)$, and that $\mathrm{p}(u)$ has (classical) Ginzburg-Landau energy E_ε close to the one of the minimizers under the boundary condition $\mathrm{p}(u) = \mathrm{p}(g)$ on $\partial\Omega$. Since $\mathrm{p}(g) = g^m$, its topological degree equals md, and $\mathrm{p}(u)$ should have md distinct vortices of degree $+1$, i.e., md distinct points x_k in Ω such that $\mathrm{p}(u)(x_k) = 0$ and

$$\mathrm{p}(u)(x) \sim \alpha_k\frac{x-x_k}{|x-x_k|} \qquad \text{for } \varepsilon \ll |x-x_k| \ll 1 \text{ and some constant } \alpha_k \in \mathbb{S}^1\,.$$

In terms of E_ε, the energetic cost of each vortex is $\pi|\log\varepsilon|$ at leading order, and therefore $E_\varepsilon\big(\mathrm{P}(u)\big)$ should be less than $\frac{\pi d}{m}|\log\varepsilon|$, again at leading order. This discussion led us to consider the energy regimes

$$F_{\varepsilon,g}(u) \le \frac{\pi d}{m}|\log\varepsilon| + O(1) \tag{6.4}$$

for $u \in \mathcal{G}_g(\Omega)$. Once again, it corresponds to the energy regime of md vortices of degree $+1$ in the variable $\mathrm{p}(u)$. By an elementary topological argument, one can see that any pre-image by p of $\frac{x-x_k}{|x-x_k|}$ must have at least one discontinuity line departing from x_k, and has a (formal) winding number around x_k equal to $1/m$ (in other words, the phase has a jump of $2\pi/m$ around x_k). For this reason, any configuration u satisfying (6.4) must be discontinuous. We actually expect that each connected component of the jump set J_u connects mk vortices for some $k \in \{1,\cdots,d\}$, since

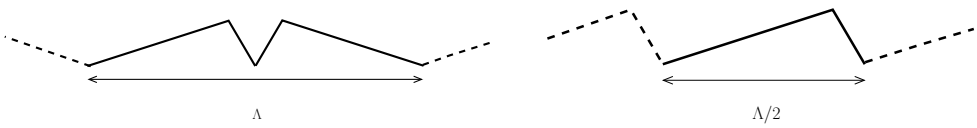

Figure 6.2: Top: profile of the $\Lambda-$phase. Bottom: profile of the $\Lambda/2-$phase.

the winding number around any such connected component must be an integer. The energy associated with discontinuities is their length, and there should be a competition between this term which favors clustered vortices and the so-called renormalized energy from Ginzburg-Landau theory which is a repulsive (logarithmic) point interaction.

6.3 Motivation

Our original motivation for studying the functionals (6.2) and (6.1) stemmed from the analysis of the defect patterns observed in the so-called ripple or $P_{\beta'}$ phase in biological membranes such as lipid bilayers [Sac95, BFL91, RS83]. In this phase, which is intermediate between the gel and the liquid phase, periodic corrugations are observed at the surface of the membranes (see [RS83] for instance). Two different kinds of periodic sawtooth profiles are observed. A symmetric one and an asymmetric one respectively called Λ and $\Lambda/2-$phases (see Figure 6.2 for a schematic representation of a cross-section). In the asymmetric phase, only defects of integer degree are allowed while in the symmetric phase half integer degree vortices are also permitted. Since two vortices of degree 1/2 have an energetic cost of order $\frac{\pi}{2}|\log\varepsilon|$ (where ε is the lengthscale of the vortex) while a vortex of degree 1 has a cost of order $\pi|\log\varepsilon|$, it is expected that even in the regime where the $\Lambda/2-$phase is favored (which happens for nearly flat membranes), a phase transition occurs around the defects with the nucleation of a small island of $\Lambda-$phase leading to the formation of two vortices of degree 1/2 (see Figure 6.3). In the model proposed by [BFL91], the order parameter is given by $f(\varphi)$, where f is a fixed profile (corresponding to the one on the right part of Figure 6.2) and φ is the phase modulation. Their functional corresponds to F_ε^η, for $\varepsilon = \eta$, $m = 2$ and $u = \nabla\varphi$ (so that u represents the local speed at which the profile f is modulated). In [BFL91], the authors further argue that the constraint of u being a gradient can be relaxed so that we recover completely our model.

We also point out that (6.1) and (6.2) have connections with many

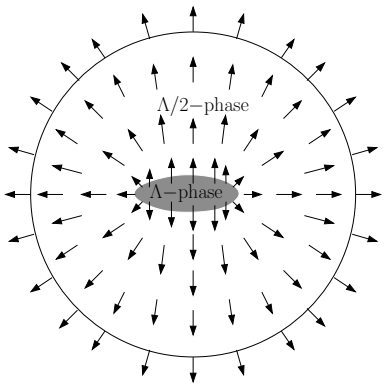

Figure 6.3: Creation of two vortices of degree 1/2.

other models appearing in the literature. As an example, we can mention the issue of orientability of Sobolev vector fields into $\mathbb{R}\mathbf{P}^1$, see [BZ11]. More generally, our functionals resemble the ones suggested recently to model liquid crystals where both points and lines singularities appear, see [Bed16]. Similarly to [BZ11], a central issue here is to find square roots (and more generally $m-$th roots) of $W^{1,p}$-functions into $\mathbb{S}^1$ which is intimately related to the question of lifting of Sobolev functions into $\mathbb{S}^1$, see [BBM00, Dem90].

As explained in [BCDLP18], where a similar Γ-convergence analysis is performed for a discrete model, other motivations come from applications to micromagnetics, and crystal plasticity.

6.4 The $\Gamma-$convergence result

Our first main theorem is a Γ-convergence result in the energy regime (6.4). To describe the limiting functional, we need to introduce the following objects. First, set $\mathcal{A}_d$ to be the family of all atomic measures of the form $\mu = 2\pi \sum_{k=1}^{md} \delta_{x_k}$, for some md distinct points $x_k \in \Omega$. To $\mu \in \mathcal{A}_d$, we associate the so-called canonical harmonic map v_μ defined by

$$v_\mu(x) = e^{i\varphi_\mu(x)} \prod_{k=1}^{md} \frac{x - x_k}{|x - x_k|} \quad \text{with} \quad \begin{cases} \Delta\varphi_\mu = 0 & \text{in } \Omega\,, \\ v_\mu = g^m & \text{on } \partial\Omega\,. \end{cases}$$

In turn, the renormalized energy $\mathbb{W}(\mu)$ can be defined as the finite part of the energy of v_μ, i.e.,

$$\mathbb{W}(\mu) = \lim_{r\downarrow 0} \left\{ \frac{1}{2} \int_{\Omega\setminus B_r(\mu)} |\nabla v_\mu|^2 \, dx - \pi md |\log r| \right\},$$

where for a measure μ, we denote by $B_r(\mu)$ the tubular neighborhood of $\operatorname{supp}\mu$ of radius r. Up to terms coming from the boundary condition g, it is a repulsive logarithmic potential i.e of the form $\sum_{j\neq k} -\log|x_j - x_k|$.

As for the classical Ginzburg-Landau model, the pre-Jacobian is an important quantity which allows to detect the presence of point singularities. It is defined as

$$j(u) = u \wedge \nabla u.$$

Its importance was first recognized by Jerrard and Soner in [JS02] and is due to the fact that the Jacobian determinant $\nabla \times j(u)$ satisfies for instance

$$\nabla \times j(v_\mu) = \mu.$$

Theorem 6.4.1. *The functionals* $\{F_{\varepsilon,g} - \frac{\pi d}{m}|\log\varepsilon|\}$ *restricted to* $\mathcal{G}_g(\Omega)$ *and* Γ*-converge in the strong* L^1*-topology as* $\varepsilon \to 0$ *to the functional*

$$F_{0,g}(u) = \frac{1}{2m^2}\int_\Omega |\nabla\varphi|^2\,dx + \frac{1}{m^2}\mathbb{W}(\mu) + md\gamma_m + \mathcal{H}^1\big(J_u\big) + \mathcal{H}^1(\{u \neq g\}\cap\partial\Omega)$$

defined for $u \in SBV(\Omega;\mathbb{S}^1)$ *such that* $u^m = e^{i\varphi}v_\mu$ *for some* $\mu \in \mathcal{A}_d$ *and* $\varphi \in W^{1,2}(\Omega)$ *satisfying* $e^{i\varphi} = 1$ *on* $\partial\Omega$. *The constant* γ_m, *referred to as core energy only depends on* m.

Proof. As can be expected, the proof of Theorem 6.4.1 combines ideas coming from the study of the Ginzburg-Landau functional [BBH94, SS07, CJ99, AP14, LX99, JS02], together with ideas from free discontinuities problems [AFP00, Bra98, BCS07, AT92].

The Γ-lim inf inequality is a relatively standard (though delicate) combination of techniques developed in [CJ99, AP14, BCS07], while the construction of recovery sequences is a much more delicate issue. The main difficulty (which is actually only present when considering the diffuse interface functional F_ε^η) comes from the constraint $u^m = e^{i\varphi}v_\mu$, which prevents us to apply directly the existing approximation results by functions with a smooth jump set, see e.g. [CT99, BCG14]). Our approach uses a (new) regularization technique which is somehow reminiscent of [AT92]. Roughly speaking, the idea is to approximate u by the solution $\bar{u}$ of

$$\min_{\tilde{u}}\left\{\mathcal{H}^1(J_{\tilde{u}}) + \lambda\int_\Omega |\tilde{u} - u|^2 \;:\; \tilde{u}^m = u^m,\ \mathcal{H}^1(J_{\tilde{u}}\backslash J_u) = 0\right\},$$

for λ large enough. The advantage is that $\bar{u}$ satisfies the density lower bound, $\mathcal{H}^1(J_{\bar{u}} \cap B_r(x)) \geq r/2$ for $\mathcal{H}^1$ a.e. $x \in J_{\bar{u}}$, which is crucial for the

construction. Another difficulty comes from the optimal profile problem defining the core energy γ_m. The underlying minimization problem involves the Ginzburg-Landau energy of $\mathcal{N}$-valued maps, and one has to find almost minimizers which can be lifted into $\mathbb{C}$-valued maps in SBV^2. □

We point out that there is of course a compactness result companion to Theorem 6.4.1. Namely, if a sequence $\{u_\varepsilon\}$ satisfies the energy bound (6.4), and is uniformly bounded in $L^\infty(\Omega)$, then $\{u_\varepsilon\}$ converges up to a subsequence in $L^1(\Omega)$, and $\{\mathrm{p}(u_\varepsilon)\}$ converges (again up to a subsequence) in the weak $W^{1,p}$-topology for every $p < 2$. This compactness result is based on the following lemma applied to $v_\varepsilon = \mathrm{p}(u_\varepsilon)$.

Lemma 6.4.2. *Let $\{v_\varepsilon\} \subset W^{1,2}_{g^m}(\Omega, \mathbb{C})$ be such that $\{v_\varepsilon\}$ is bounded in $L^\infty(\Omega)$,*

$$E_\varepsilon(v_\varepsilon) \leq \pi m d |\log \varepsilon| + O(1)$$

and for which $\mu_\varepsilon = j(v_\varepsilon)$ weakly converges in $(C^{0,1}_0(\Omega))^*$ to some measure $\mu \in \mathcal{A}_d$ as $\varepsilon \to 0$. Then $\{v_\varepsilon\}$ is bounded in $W^{1,p}(\Omega)$ for every $1 \leq p < 2$.*

Although this $W^{1,p}$ bound, which is proven using the ball construction (see [SS07]), is certainly known to the Ginzburg-Landau community (see for instance [CJ99, LX99]), it has never been used before in the context of Γ-convergence. In particular, compared to [AP14, BCDLP18] it allows to obtain compactness for u_ε and not only for the pre-Jacobian.

6.5 Minimizers of $F_{0,g}$ and the Steiner problem

The $\Gamma-$convergence result Theorem 6.4.1 proves that minimizers of $F_{\varepsilon,g}$ converge as $\varepsilon \to 0$ to minimizers of $F_{0,g}$. In this section we characterize precisely the structure of the minimizers of this limit problem.
It is based on the following observations. First, from the explicit form of $F_{0,g}$, it follows that $\varphi = 0$ in the representation $u^m = e^{i\varphi} v_\mu$. In particular, u can be characterized as a solution of the minimization problem

$$\min \left\{ \frac{1}{m^2} \mathbb{W}(\mu) + \mathcal{H}^1\big(J_u\big) + \mathcal{H}^1(\{u \neq g\} \cap \partial\Omega\big) : u \in SBV(\Omega; \mathbb{S}^1)\,, \right.$$
$$\left. u^m = v_\mu \text{ for some } \mu \in \mathcal{A}_d \right\}.$$

In turn, this later can be equivalently rewritten as

$$\min_{\mu\in\mathcal{A}_d}\min\left\{\frac{1}{m^2}\mathbb{W}(\mu)+\mathcal{H}^1\big(J_u\big)+\mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big):u\in SBV(\Omega;\mathbb{S}^1)\,,\ u^m=v_\mu\right\}.$$

As a consequence, fixing $\mu\in\mathcal{A}_d$ and solving

$$L(\mu)=\min\left\{\mathcal{H}^1\big(J_u\big)+\mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big):u\in SBV(\Omega;\mathbb{S}^1)\,,\ u^m=v_\mu\right\},$$

we are left with a finite dimensional problem to recover the minimizers of $F_{0,g}$.

Given $\mu\in\mathcal{A}_d$, we compare in Theorem 6.5.1 below the minimization problem $L(\mu)$ with the following variant of the Steiner problem (see e.g. [GP68]):

$$\begin{aligned}\Lambda(\mu)=\min\Big\{&\mathcal{H}^1(\Gamma):\Gamma\subset\overline{\Omega}\text{ compact with }\operatorname{supp}\mu\subset\Gamma\\&\text{and every connected component }\Sigma\text{ satisfies }\sharp(\Sigma\cap\operatorname{supp}\mu)\in m\mathbb{N}\Big\}.\end{aligned}$$

We shall see that any minimizer Γ of $\Lambda(\mu)$ is made of at most d disjoint Steiner trees, i.e., connected trees made of a finite union of segments meeting either at points of supp μ, or at triple junction making a $120°$ angle. From now on, when talking about triple junctions we always implicitly include this condition on the angles.

Our second main result is the following theorem, in which we assume Ω to be convex (to avoid issues at the boundary).

Theorem 6.5.1. *Assume that Ω is convex. For every $\mu\in\mathcal{A}_d$, $L(\mu)=\Lambda(\mu)$. Moreover, if u is a minimizer for $L(\mu)$, then its jump set J_u is a minimizer for $\Lambda(\mu)$, $u\in C^\infty(\overline{\Omega}\setminus J_u)$, and $u=g$ on $\partial\Omega$. Vice-versa, if Γ is a minimizer for $\Lambda(\mu)$, then there exists a minimizer u for $L(\mu)$ such that $J_u=\Gamma$.*

Proof. If Γ is a minimizer of $\Lambda(\mu)$ then it is not too difficult to construct a function u which is admissible for $L(\mu)$ and such that $J_u=\Gamma$. This shows that $L(\mu)\leq\Lambda(\mu)$.
To prove the converse inequality, we first need to show that the jump set of any minimizer u of $L(\mu)$ is polygonal away from $\operatorname{supp}\mu$. Let us fix an arbitrary ball $B_{2r}(y)\subset\Omega\setminus\operatorname{supp}\mu$. Since v_μ is smooth in $B_{2r}(y)$, we can find a smooth function φ on $\overline{B}_r(y)$ such that $v_\mu=e^{i\varphi}$ in $\overline{B}_r(y)$. The map

$u_\star = e^{i\varphi/m}$ is then smooth on $\overline{B}_r(y)$, and satisfies $u_\star^m = v_\mu$ in $\overline{B}_r(y)$. Since every competitor u_{comp} satisfies $u_{comp}^m = u_\star^m$, it can be written as

$$u_{comp} = \left(\sum_{k=0}^{m} \mathbf{a}^k \chi_{F_k}\right) u_\star \quad \text{in } B_r(y)\,,$$

for some Caccioppoli partition $\{F_k\}_{k=0}^{m-1}$ of $B_r(y)$, and

$$J_{u_{comp}} \cap B_r(y) = \bigcup_{k=0}^{m-1} \partial F_k \cap B_r(y) \quad \text{up to an } \mathcal{H}^1\text{-null set}\,.$$

In addition,

$$\mathcal{H}^1\big(J_{u_{comp}} \cap B_r(y)\big) = \frac{1}{2}\sum_{k=0}^{m-1} \mathcal{H}^1\big(\partial F_k \cap B_r(y)\big)\,. \tag{6.5}$$

As a consequence, the minimizer u of $L(\mu)$ that we consider can be written as $u = \big(\sum_k \mathbf{a}^k \chi_{E_k}\big) u_\star$, for some Caccioppoli partition $\{E_k\}_{k=0}^{m-1}$ of $B_r(y)$ minimizing the right-hand side of (6.5). By classical results on minimal planar clusters, $\cup_k \partial E_k \cap B_r(y)$ is locally a finite union of segments meeting at triple junctions. Since $u_\star$ is smooth in $B_r(y)$, it implies that $J_u \cap B_{r/2}(y) = \cup_k \partial E_k \cap B_{r/2}(y)$.
To complete the proof, one needs to prove that any connected component of J_u contains a multiple of m vortices (possibly equal to zero). Indeed, this would lead to $L(\mu) = \mathcal{H}^1(J_u) \geq \Lambda(\mu)$.

Let us consider Σ a connected component of J_u, and $A \subset \Omega$ a connected smooth open neighborhood of Σ such that $(J_u \setminus \Sigma) \cap \overline{A} = \emptyset$. We may write $A = A_0 \setminus \cup_{n=1}^N \overline{A}_n$ where the A_n are connected and simply connected smooth open sets satisfying $\overline{A}_n \subset A_0$ for $n = 1, \ldots, N$, and $\overline{A}_n$ are pairwise disjoint. Since v_μ and u are smooth on ∂A_n for $n = 0, \ldots, N$, and $u^m = v_\mu$,

$$\mathrm{d}eg(v_\mu, \partial A_n) = m\mathrm{d}eg(u, \partial A_n) \in m\mathbb{N}$$

and thus

$$\sharp(\Sigma \cap \text{supp } \mu) = \mathrm{d}eg(v_\mu, \partial A_0) - \sum_{n=1}^{N} \mathrm{d}eg(v_\mu, \partial A_n) \in m\mathbb{N},$$

concluding the proof. □

The geometry of minimizers for $\Lambda(\mu)$ strongly depends on m, d, and the location of supp μ. In the case $m = 2$, a minimizer for $\Lambda(\mu)$ is always given

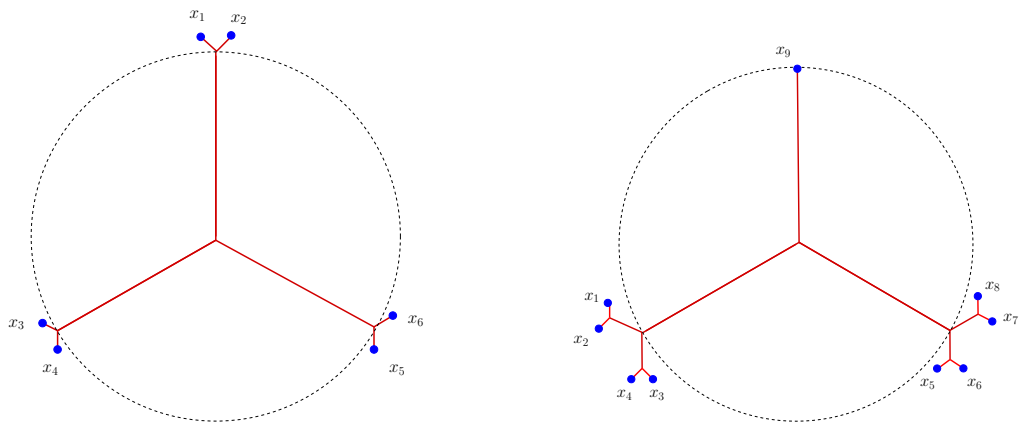

Figure 6.4: Example of connected minimizers with respectively six and nine vortices of degree 1/3.

by a disjoint union of d segments connecting the points of supp μ. However, for $m \geq 3$ and $d \geq 2$, we have explicit examples (see for instance Figure 6.4) for which minimizers are not given by a disjoint union of d Steiner trees containing exactly m vortices.

6.6 Geometry of the minimizers of $F_{\varepsilon,g}$ for small ε

We may now use the characterization of the minimizers of $F_{0,g}$ provided by Theorem 6.5.1 to show that for $\varepsilon > 0$ small enough, minimizers of $F_{\varepsilon,g}$ have essentially the same structure away from the limiting vortices. Let us point out that in [GMM17], we only proved this result for minimizers of the sharp interface model but not for the diffuse interface one F_ε^η.

Theorem 6.6.1. *Assume that Ω is convex. Let $\varepsilon \to 0$, and let u_ε be a minimizer of $F_{\varepsilon,g}$ over $\mathcal{G}_g(\Omega)$. Assume that $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0$ for some minimizer u of $F_{0,g}$. Setting $\mu = \nabla \times j(u^m)$, for every $\sigma > 0$ small enough and every ε small enough, the following holds:*

(i) $J_{u_\varepsilon} \setminus B_\sigma(\mu)$ is a compact subset of $\Omega \setminus B_\sigma(\mu)$ made of finitely many segments, meeting by three at an angle of $120°$ (i.e., triple junctions).

(ii) $u_\varepsilon \in C^\infty\big(\overline{\Omega} \setminus (B_\sigma(\mu) \cup J_{u_\varepsilon})\big)$ and $u_\varepsilon = g$ on $\partial\Omega$.

In addition,

(iv) J_{u_ε} converges in the Hausdorff distance to J_u.

(v) $u_\varepsilon \to u$ *in* $C^k_{loc}(\Omega \setminus J_u) \cap C^{1,\alpha}_{loc}(\overline{\Omega} \setminus J_u)$ *for every* $k \in \mathbb{N}$ *and* $\alpha \in (0,1)$.

We actually prove a stronger result than Theorem 6.6.1 that we now briefly describe. In each (sufficiently small) ball $B_r(x) \subset \Omega \setminus B_\sigma(\mu)$ and ε small enough, u_ε is bounded away from zero, and it can be decomposed as $u_\varepsilon = \phi_\varepsilon w_\varepsilon$ where $\phi_\varepsilon \in SBV^2(B_r(x))$ and w_ε is minimizing the classical Ginzburg-Landau energy $E_\varepsilon(\cdot, B_r(x))$ with respect to its own boundary condition (and as a consequence, w_ε is smooth). The proof of this decomposition relies on the energy splitting discovered by Lassoued & Mironescu [LM99]. Combined with the classical Wente estimate [Wen69], it leads to a lower expansion of the energy of the form

$$F_\varepsilon(u_\varepsilon, B_r(x)) \geq E_\varepsilon(w_\varepsilon, B_r(x)) + \frac{1}{\alpha}\left(\int_{B_r(x)} |\nabla \phi_\varepsilon|^2\, dx + \alpha \mathcal{H}^1\big(J_{\phi_\varepsilon} \cap B_r(x)\big)\right),$$

for some constant $\alpha > 0$. Using suitable competitors, we deduce that ϕ_ε is a Dirichlet minimizer the Mumford-Shah functional in $B_r(x)$. Applying the calibration results of [ABDM03], we infer that ϕ_ε takes values into the finite set $\mathbf{G}_m$, reducing the problem to a minimal partition problem in $B_r(x)$. The classical regularity results on two dimensional minimal clusters then yield the announced geometry of the jump set.

It would be interesting to study the behavior of the minimizers u_ε close to the vortices, i.e., in $B_\sigma(\mu)$. It seems to be a difficult question since it combines both issues related to the presence of an expected singularity in the jump set in the spirit of the so called crack tip (see for instance [Dav05]) for the Mumford-Shah functional, with the fact that $\mathrm{P}(u_\varepsilon)$ should have the same regularity as minimizing harmonic maps with values into the singular cone $\mathcal{N}$. Such harmonic maps satisfy non-standard elliptic equations, and are usually more singular than minimizing harmonic maps with values into a smooth target [Lin89, HL93, AHL17]*.

*quoting [Lin89]: "Unfortunately, the equations satisfied by s and u are so bad that no existing result can be applied".

Chapter 7

Phase segregation for binary mixtures of Bose-Einstein Condensates

This chapter reports on joint works with B. Merlet and J. Royo-Letelier [GM17, GRL15].

7.1 The Gross-Pitaievskii energy and the Thomas-Fermi profile

In this chapter, we investigate the asymptotic behavior as ε goes to zero of minimizers of the Gross-Pitaievskii functional

$$F_\varepsilon(\eta) \;=\; \varepsilon \int_{\mathbb{R}^2} |\nabla\eta_1|^2 + |\nabla\eta_2|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \frac{1}{2}\eta_1^4 + \frac{g}{2}\eta_2^4 + K\eta_1^2\eta_2^2 + (\eta_1^2+\eta_2^2)V, \quad (7.1)$$

under the mass constraint

$$\int_{\mathbb{R}^2} \eta_1^2 = \alpha_1 \qquad \text{and} \int_{\mathbb{R}^2} \eta_2^2 = \alpha_2. \quad (7.2)$$

This functional arises in the study of two component Bose-Einstein condensates. It has been widely studied, both in the physical and mathematical literature (see [GRL15] and the references therein or the book [Aft06]). The potential V is a trapping potential. For simplicity, we only consider here

the harmonic potential $V = |x|^2$. The constant g, measures the asymmetry between the intracomponent repulsive strengths of each component and K represents the intercomponent repulsive strength. Without loss of generality, we will take here $g \geq 1$. The case $K < \sqrt{g}$, where mixing of the two condensates occurs has recently been well understood in [ANS15]. Instead, we focus here on the case $K > \sqrt{g}$, where it is expected both experimentally [MCJ+11, PPW08], numerically [MA11] and theoretically [AC98, Tim98, Bar02, Van08] that segregation occurs.

As we will see, at first order, the behavior of the minimizers η_ε of (7.1) under the mass constraint (7.2) are dictated by the behavior of the minimizers of the Thomas-Fermi energy

$$E(\rho) = \int_{\mathbb{R}^2} \left[\frac{1}{2}\rho_1^2 + \frac{g}{2}\rho_2^2 + K\rho_1\rho_2 + (\rho_1 + \rho_2)V \right], \tag{7.3}$$

under the volume constraint

$$\int_{\mathbb{R}^2} \rho_i = \alpha_i. \tag{7.4}$$

It can be easily seen that in the case $K > 1$, minimizers of (7.3) under (7.4) must be segregated in the sense that $\rho_1\rho_2 = 0$. However, as we will see this problem is much less rigid in the symmetric case $g = 1$ than in the asymmetric case $g \neq 1$.

Let us recall that the ground state for single component condensates is defined as the (unique) positive minimiser $\bar{\eta}_\varepsilon$ of the one-component Gross-Pitaevskii functional

$$G_\varepsilon(\eta) := \frac{\varepsilon}{2} \int_{\mathbb{R}^2} |\nabla\eta|^2 + \frac{1}{\varepsilon}V|\eta|^2 + \frac{1}{2\varepsilon}|\eta|^4 \, dx \,, \tag{7.5}$$

under volume constraint

$$\int_{\mathbb{R}^2} \eta^2 = 1.$$

The behavior as ε goes to zero of η_ε is by now very well understood (see [Aft06, IM06, KS14]). In particular, it is well known that $\bar{\eta}_\varepsilon^2$ converges to the Thomas-Fermi profile $\bar{\rho}$, given by

$$\bar{\rho}(x) = (R^2 - V(x))_+ \tag{7.6}$$

where R is such that $\int_{\mathbb{R}^2} \bar{\rho} = 1$

7.2 The symmetric case $g = 1$

In the symmetric case, for segregated states, the Thomas-Fermi energy (7.3) may be written in terms of $\tilde{\rho} = \rho_1 + \rho_2$ as

$$\int_{\mathbb{R}^2} \left[\frac{1}{2}\tilde{\rho}^2 + \tilde{\rho} V \right],$$

whose minimizer under the volume constraint $\int_{\mathbb{R}^2} \tilde{\rho} = 1$ is exactly $\bar{\rho}$, the Thomas-Fermi profile for the single component condensate. Therefore, any function of the form $\rho = (\chi_E, \chi_{E^c})\bar{\rho}$ with E such that $\int_E \bar{\rho} = \alpha_1$ is a minimizer of (7.3) under the constraint (7.4). Therefore, besides segregation, the Thomas-Fermi limit does not give much information about the shape of the minimizers η_ε of (7.1). This is reminiscent of the situation studied by Modica and Mortola [Mod87] and it suggests that at the next order the behavior of η_ε will be dictated by the solution of a (weighted) isoperimetric problem. In order to see this, it is convenient to use the Lassoued-Mironescu trick [LM99] (recall also Chapter 6) and to make the change of variables,

$$\eta = u\, \bar{\eta}_\varepsilon. \tag{7.7}$$

Using the Euler-Lagrange equation satisfied by $\bar{\eta}_\varepsilon$, we obtain that the energy decomposes as

$$F_\varepsilon(\eta) = G_\varepsilon(\bar{\eta}) + \tilde{F}_\varepsilon(u),$$

where

$$\tilde{F}_\varepsilon(u) = \varepsilon \int_{\mathbb{R}^2} \bar{\eta}_\varepsilon^2 |\nabla u_1|^2 + \bar{\eta}_\varepsilon^2 |\nabla u_2|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \frac{\bar{\eta}_\varepsilon^4}{2} (u_1^2 + u_2^2 - 1)^2 + (K-1)\bar{\eta}_\varepsilon^4 u_1^2 u_2^2, \tag{7.8}$$

and if η satisfies (7.2) then u satisfies

$$\int_{\mathbb{R}^2} \bar{\eta}_\varepsilon^2 u_i^2 = \alpha_i. \tag{7.9}$$

Minimizing (7.1) under the volume constraint (3.1) is thus equivalent to minimizing (7.8) under the volume constraint (7.9). The following result was obtained in [ARL15] under the assumption of very strong segregation $\lim_{\varepsilon \to 0} K = \infty$ and then extended to the case $0 < K < \infty$ in [GRL15]

Theorem 7.2.1. *For $K > 1$, the functional $\tilde{F}_\varepsilon$ $\Gamma-$ converges as ε goes to zero for the strong L^1 topology to*

$$\mathcal{G}_0(u_1, u_2) = \begin{cases} \sigma_{1,K} \displaystyle\int_{\partial E} \bar{\rho}^{3/2} & \text{if } u_1 = \bar{\rho}\chi_E,\ u_2 = \bar{\rho}\chi_{E^c} \text{ and } \displaystyle\int_E \bar{\rho} = \alpha_1, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\sigma_{1,K} > 0$ is defined by the one dimensional optimal transition problem

$$\sigma_{1,K} = \inf \left\{ \int_{\mathbb{R}} |\eta_1'|^2 + |\eta_2'|^2 + \frac{1}{2}\left(\eta_1^2 + \eta_2^2 - 1\right)^2 + (K-1)\eta_1^2\eta_2^2 : \right.$$
$$\left. \lim_{-\infty} \eta_1 = 0,\ \lim_{+\infty} \eta_1 = 1 \right\}.$$

Proof. The main difficulty is to obtain the strong compactness. In [ARL15, GRL15], this was obtained by using another change of variable to obtain a functional which is very reminiscent of the Ambrosio-Tortorelli functional [AT92]. We later understood with B. Merlet in [GM17] that similarly to what we did in [CGOS18] (recall (5.12)), thanks to the segregation effect, the functional directly controls a standard Modica-Mortola type energy. Indeed, letting

$$W_K(s,t) = \frac{1}{2}\left(s^2 + t^2 - 1\right)^2 + (K-1)s^2t^2$$

and then

$$w_K(s) \;=\; \inf_{t\in\mathbb{R}} W_K(s,t) \;=\; \begin{cases} \frac{1}{2}(1-s^2)^2 - \frac{1}{2}(1-Ks^2)^2 & \text{if } 0 \le s < K^{-1/2}, \\ \frac{1}{2}(1-s^2)^2 & \text{if } s \ge K^{-1/2}, \end{cases} \tag{7.10}$$

we see that w_K is a standard double-well potential. In particular, since $w_K(s) = w_K(t) \le W_K(s,t)$, we have for every u,

$$\tilde{F}_\varepsilon(u) \gtrsim \varepsilon \int_{\mathbb{R}^2} \bar{\eta}_\varepsilon^2 |\nabla u_1|^2 + \bar{\eta}_\varepsilon^2 |\nabla u_2|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} w_K(u_1) + w_K(u_2),$$

so that compactness follows by the standard Modica-Mortola argument. The lower bound is obtained by the slicing technique [Bra98] while the construction for the upper bound is done with the help of the optimal profile for $\sigma_{1,K}$ (with some technical difficulties related to the volume constraint). □

It was then proven in [ARL15] (see also [GRL15]) that for intermediate volumes α_1 there is symmetry breaking in the sense that minimizers of $\mathcal{G}_0$ are not radially symmetric

Proposition 7.2.2. *There exists $\alpha_0 \in (0, 1/2)$ such that for every $\alpha_1 \in (\alpha_0, 1-\alpha_0)$, the minimizers of $\mathcal{G}_0$ are not radially symmetric.*

7.3 The asymmetric case $g \neq 1$

In the asymmetric case, the minimizer of the Thomas-Fermi energy (7.3) is unique and radial as proven in [GM17] (and was not totally expected, see [Van08, MA11] or the discussion in [ARL15, Sec. 1.3.4])

Theorem 7.3.1. *For every $\alpha_1, \alpha_2, g, K > 0$, with $K \geq \sqrt{g} > 1$, there exists a unique minimizer $\rho^0 = (\rho_1^0, \rho_2^0)$ of* (7.3) *under the volume constraints* (7.4)*. This minimizer ρ^0 is radially symmetric (and may be explicitly computed). Moreover, we have the following stability result: letting $E_0 = E(\rho^0)$, there exists $C > 0$ (which depends only on α_1, α_2 and g) such that if ρ satisfies the constraints* (7.4) *then*

$$\|\rho - \rho^0\|_1 \leq C\sqrt{E(\rho) - E_0}. \tag{7.11}$$

Proof. In order to prove the first part of the statement, the idea of the proof is to compute the minimizer ρ^0 of (7.3) under the constraint (7.4) in the class of radially symmetric functions and then prove that $E(\rho) \geq E(\rho^0)$ for any other function ρ satisfying (7.4). For this we may assume without loss of generality that $K = \sqrt{g}$, which is the worst case. Then, we write,

$$\rho = \rho^0 + \delta\rho, \qquad \text{with } \delta\rho = (\delta\rho_1, \delta\rho_2).$$

Since the energy is quadratic, it expands as

$$E(\rho) \;=\; E(\rho^0) + L_1(\delta\rho_1) + L_2(\delta\rho_2) + Q(\delta\rho),$$

with

$$L_1(\delta\rho_1) \;=\; \int_{\mathbb{R}^2} (\rho_1^0 + \sqrt{g}\rho_2^0 + |x|^2)\delta\rho_1, \qquad L_2(\delta\rho_2) \;=\; \int_{\mathbb{R}^2} (g\rho_2^0 + \sqrt{g}\rho_1^0 + |x|^2)\delta\rho_2,$$

$$Q(\delta\rho) \;=\; \frac{1}{2}\int_{\mathbb{R}^2} (\delta\rho_1 + \sqrt{g}\delta\rho_2)^2.$$

Since the last term is positive, it is enough to prove that $L_1(\delta\rho_1)$ and $L_2(\delta\rho_2)$ are also positive. This is then obtained by using the explicit form of ρ^0.
A more careful inspection of this proof lead to the quantitative estimate (7.11). □

One crucial point which explains the difference between the asymmetric case and the symmetric one is that here, there is a gap between the two Thomas-Fermi profiles ρ_1^0 and ρ_2^0 in the sense that there exists $r_0 > 0$ such that $\operatorname{supp} \rho_0^1 \subset \overline{B(0, r_0)}$ and $\operatorname{supp} \rho_0^2 \subset \mathbb{R}^2 \setminus B(0, r_0)$ with

$$\inf_{r<r_0} \rho_1^0(r) \;>\; \sup_{r>r_0} \rho_2^0(r).$$

As a consequence of this stability result, we prove that minimizers of F_ε converge to $(\sqrt{\rho_1^0}, \sqrt{\rho_2^0})$.

Theorem 7.3.2. *There exists $C > 0$ such that for $\varepsilon \in (0, 1]$, any minimizer η^ε of F_ε under the constraints* (7.2) *satisfies*

$$\left\| \eta^\varepsilon - \left(\sqrt{\rho_1^0}, \sqrt{\rho_2^0} \right) \right\|_2 \leq C\varepsilon^{1/4}. \tag{7.12}$$

This theorem establishes that in the non-symmetric case, the Thomas-Fermi limit already provides full information on the limiting behavior of the minimizers of (7.1). It is quite surprising that even without using isoperimetric effects, we were able to obtain strong convergence of the minimizers in the form of (7.12).

7.4 The cross-over case $g = 1 + \varepsilon\xi$

We now study the crossover case where $g = 1 + \varepsilon\xi$ for some $\xi > 0$. Since in that case F_ε is a continuous (for the L^1 topology) perturbation of the case $g = 1$, a direct consequence of Theorem 7.2.1 is the following

Theorem 7.4.1. *For $K > 1$, the functional $(F_\varepsilon - G_\varepsilon(\overline{\eta}_\varepsilon))$ $\Gamma-$ converges as ε goes to zero for the strong L^1 topology to*

$$\mathcal{G}_\xi(u_1, u_2) = \begin{cases} \sigma_{1,K} \displaystyle\int_{\partial E} \bar{\rho}^{3/2} + \xi \int_{E^c} \bar{\rho}^2 & \text{if } u_1 = \bar{\rho}\chi_E,\ u_2 = \bar{\rho}\chi_{E^c} \text{ and } \displaystyle\int_E \bar{\rho} = \alpha_1, \\ +\infty & \text{otherwise.} \end{cases}$$

In [GM17], we studied the minimizers of the limiting functional $\mathcal{G}_\xi$:

Theorem 7.4.2. *The following holds:*

- *there exists $\alpha_0 \in (0, 1/2]$ such that for every $\alpha_1 \in (\alpha_0, 1 - \alpha_0)$ there exists $\xi^1_{\alpha_1}$ such that the minimizer of $\mathcal{G}_\xi$ is not radially symmetric for $\xi \leq \xi^1_{\alpha_1}$,*
- *for every $\alpha_1 \in (0, 1)$, there exists $\xi^2_{\alpha_1}$ such that the minimizer of $\mathcal{G}_\xi$ is the centered ball for $\xi \geq \xi^2_{\alpha_1}$.*

The regime $g = 1 + \varepsilon\xi$ corresponds to the numerical simulations of [MA11]. In that paper, the observed numerical results (droplets) fit with the first point of Theorem 7.4.2 (see in particular [MA11, Fig 1.c, Fig. 3.a]). The first part of Theorem 7.4.2 is a consequence of the symmetry breaking result Proposition 7.2.2. The second part follows from a combination of two results. The first is a stability result for the functional $\int_{E^c} \bar{\rho}^2$:

Proposition 7.4.3. *For every $\alpha \in (0,1)$, there exists $C = C(\alpha) > 0$ such that for every measurable set $E \subset \mathbb{R}^2$ with $\int_E \bar{\rho} = \alpha$, we have,*

$$\int_{E^c} \bar{\rho}^2 - \int_{B_r^c} \bar{\rho}^2 \;\geq\; C\left(\int_{E\Delta B_r} \bar{\rho}\right)^2$$

where r is such that $\int_{B_r} \bar{\rho} = \alpha$.

The second is an estimate on the potential instability of the ball for the weighted isoperimetric problem.

Proposition 7.4.4. *For every $\alpha \in (0,1)$, there exists $c = c(\alpha) > 0$ such that for every set $E \subset \mathbb{R}^2$ with locally finite perimeter and with $\int_E \bar{\rho} = \alpha$, there holds*

$$\int_{\partial E} \bar{\rho}^{3/2} - \int_{\partial B_r} \bar{\rho}^{3/2} \;\geq\; -c\left(\int_{E\Delta B_r} \bar{\rho}\right)^2, \tag{7.13}$$

where r is such that $\int_{B_r} \bar{\rho} = \alpha$.

The peculiar aspect of the rigidity result given by Theorem 7.4.2 is that here rigidity does not come from the isoperimetric term but rather from the volume term. Nevertheless, the proof of (7.13) bounding the *instability* of the ball follows as for Theorem 2.3.2 or Theorem 2.4.1 the strategy of [CL12] to reduce ourselves to the case of nearly spherical sets where the estimate can be obtained following ideas of [Fug89]. To the best of our knowledge, it is the first time that this strategy has been implemented to control the *instability* of the ball. Let us also point out that one of the ingredients in our proof is the following isoperimetric inequality:

Lemma 7.4.5. *There exists a constant $c > 0$ such that for every measurable set $E \subset \mathbb{R}^2$ satisfying $\int_E \bar{\rho} \leq \alpha/2$, there holds*

$$\int_{\partial E} \bar{\rho}^{3/2} \geq c\left(\int_E \bar{\rho}\right)^{5/6}.$$

In recent years, there has been an increasing interest in studying isoperimetric problems with densities (see [MP13, FM13, DPFP17] for instance). However, most of these authors consider either problems where the same density is used for weighting the volume and the perimeter or weights which are increasing at infinity.

7.5 The case $g \neq 1$ with stiff potential

In this last section, we come back to the situation $g > 1$ but consider an infinitely stiff trapping potential. That is, we assume that V is equal to zero inside some given open set Ω and is infinite outside. This is somehow the setting which is considered in [Van08, VI15]. In this case, it is easier to work with slightly different parameters. After a new rescaling in the spirit of the Lassoued-Mironescu trick, and some simple algebraic manipulations, the problem can be seen to be equivalent to minimizing

$$J_\varepsilon(\eta) = \varepsilon \int_\Omega |\nabla \eta_1|^2 + \lambda^2 |\nabla \eta_2|^2 + \frac{1}{\varepsilon} \int_\Omega \frac{1}{2}(\eta_1^2 + \eta_2^2 - 1)^2 + (K-1)\eta_1^2\eta_2^2 \quad (7.14)$$

with the volume constraint

$$\int_\Omega \eta_1^2 = \alpha_1 \qquad \text{and} \qquad \int_\Omega \eta_2^2 = \alpha_2,$$

for some $\lambda \leq 1$, $K > 1$ and $\alpha_1, \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = |\Omega|$. Arguing as for Theorem 7.2.1, we obtain (notice that in the case $\lambda \neq 1$, the nonlinear sigma model representation used in [ARL15, GRL15] does not seem to work):

Theorem 7.5.1. *When $\varepsilon \to 0$, J_ε $\Gamma-$converges for the strong L^1 topology to*

$$\mathcal{J}(\eta_1, \eta_2) = \begin{cases} \sigma_{\lambda,K} P(E, \Omega) & \text{if } \eta_1 = \chi_E = 1 - \eta_2 \text{ and } |E| = \alpha_1 \\ +\infty & \text{otherwise,} \end{cases}$$

where $\sigma_{\lambda,K} > 0$ is defined by the one dimensional optimal transition problem

$$\sigma_{\lambda,K} = \inf \left\{ \int_{\mathbb{R}} |\eta_1'|^2 + \lambda^2 |\eta_2'|^2 + \frac{1}{2}\left(\eta_1^2 + \eta_2^2 - 1\right)^2 + (K-1)\eta_1^2\eta_2^2 : \right.$$
$$\left. \lim_{-\infty} \eta_1 = 0, \ \lim_{+\infty} \eta_1 = 1 \right\}.$$

Motivated by predictions from the physics literature [Bar02, Van08, AC98, Tim98] we studied in [GRL15] in the case $\lambda = 1$ (see also the paper [AS17]) and in [GM17] the general case, the asymptotic behavior of $\sigma_{\lambda,K}$ as K goes to one (mixing) or K goes to infinity (strong segregation). In particular, we proved that

$$\lim_{K \to 1} \frac{\sigma_{\lambda,K}}{\sqrt{K-1}} = \frac{2}{3} \frac{1-\lambda^3}{1-\lambda^2}, \qquad \sigma_{\lambda,K} - \sigma_{\lambda,\infty} \overset{\frac{\lambda^2}{K} \downarrow 0}{\sim} -\left(\frac{1}{K^{1/2}} + \frac{\lambda^{1/2}}{K^{1/4}} \right),$$

The proof uses once again the lower bound on the energy given by (7.10).

Chapter 8

A two-point function approach to connectedness of drops in convex potentials

This last chapter contains a result obtained in collaboration with G. De Philippis in [PG17].

8.1 Background

Crystals and drops subject to the action of an external potential are usually described by the following free energy (see [HC72])

$$\mathcal{F}(E) = \int_{\partial E} \Phi(\nu^{\partial E}) + \int_E g, \tag{8.1}$$

The potential $g : \mathbb{R}^d \to \mathbb{R}$ accounts for the external forces and $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a *one-homogeneous* and *convex* function which describes the (typically anisotropic) surface tension. Note that when the surface tension is isotropic, i.e. $\Phi(\nu) = |\nu|$, the surface term reduces to the classical perimeter, $P(E)$.

It is commonly assumed that minimisers of (8.1) under a volume constraint give a good description of the equilibrium shapes of drops. We thus consider the following variational problem:

$$\min_{|E|=V} \mathcal{F}(E) \tag{P_V}$$

and its unconstrained counterpart:

$$\min_E \mathcal{F}(E). \tag{P}$$

Let us also note that the above variational problems naturally appear in various other contexts such as:

- In the limiting case in which g takes only the values 0 and $+\infty$, (P_V) reduces to the isoperimetric problem in the domain $\Omega = \{g < +\infty\}$ (see [ACC05]);

- The minimisation problem (P) appears as one step of the Almgren, Taylor and Wang approximation of the mean curvature flow (cf. [ATW93]). There, $g = \mathrm{s}dist(x, E_{k-1})$ is the signed distance to the $(k-1)$-th step of the scheme.

A natural question is to understand how properties of the surface tension Φ and of the potential g influence the shape of E. In this paper we investigate the following question which is attributed to Almgren (see [McC98]):

Question: *Let E be a minimiser of (P_V) and let us assume that g is convex. Is it true that E is convex?*

Note that, if the answer to this question is positive, it can only result from a delicate interaction between the surface and the volume terms in (8.1). Indeed, it is known that the answer is positive both for very small and very large volumes but for totally different reasons. On the one hand, in the regime $V \ll 1$ and assuming that Φ and g are sufficiently smooth, Figalli and Maggi showed in [FM11] that E is a smooth perturbation of the Wulff shape associated to Φ (the ball if $\Phi(\nu) = |\nu|$) and is thus convex. On the other hand, in the regime $V \gg 1$, the set E should resemble the level set of g having volume V and therefore must be convex as well, see [CC06].

In the regime of intermediate mass $V \sim 1$, no term in the functional is predominant and very little is known about the minimisers. To the best of our knowledge, the only available result in this general setting is due to McCann (see [McC98]). Building on an unpublished paper of Okikiolu, he proved that when $d = 2$, every connected component of E must be convex. Moreover, using ideas from optimal transport theory he proved that every such connected component is uniquely minimising (P_V) for its own volume in the class of convex sets.

8.2 The main results

The main result of this chapter asserts that minimisers of (P_V) are always connected. More precisely we have the following result which is new even in the case of isotropic surface tensions, $\Phi(\nu) = |\nu|$.

Theorem 8.2.1. *Let E be a minimiser of (P_V) and assume that $g \in C^{1,\alpha}(\mathbb{R}^d)$ is a convex and coercive function and that $\Phi \in C^{3,\alpha}(\mathbb{R}^d \setminus \{0\})$ is uniformly elliptic, i.e.*

$$\langle D^2\Phi(\nu)\xi, \xi\rangle \geq |\xi - \langle \nu, \xi\rangle \nu|^2 \qquad \forall\ |\xi| = |\nu| = 1. \tag{8.2}$$

Assume moreover then g is strictly convex, then, ∂E is connected, in particular E is indecomponible (and thus connected). In case $d = 2$, the strict convexity assumption is not needed.*

From the result of McCann (see [McC98]) this immediately positively answers Almgren's question in dimension 2.

Corollary 8.2.2. *Assume $d = 2$ and let $E \subset \mathbb{R}^2$ be a minimiser of (P_V) with $g \in C^{1,\alpha}(\mathbb{R}^2)$ a convex and coercive function and $\Phi \in C^{3,\alpha}(\mathbb{R}^2 \setminus \{0\})$ uniformly elliptic. Then E is convex and unique.*

For the unconstrained problem (P) we are actually able to show convexity of minimisers.

Theorem 8.2.3. *Let E be a minimiser of (P) and assume that $g \in C^{1,\alpha}(\mathbb{R}^d)$ is a convex and coercive function and that $\Phi \in C^{3,\alpha}(\mathbb{R}^d \setminus \{0\})$ satisfies (8.2), then E is convex.*

Remark 8.2.4. *In Theorem 8.2.1 we must require strict convexity of g in the case $d > 2$ since in that case we are not able to guarantee the existence of a stable connected component of ∂E intersecting† $\partial co(E)$.*

8.3 Idea of the proof

We first recall the following classical result.

*Recall that a set of finite perimeter E is said to be indecomponible if for every partition $E = E_1 \cup E_2$ with $|E_1 \cap E_2| = 0$ and $P(E) = P(E_1) + P(E_2)$ then either $|E_1| = 0$ or $|E_2| = 0$.

†Here and in the sequel $\mathrm{co}(F)$ denotes the *closed convex envelope* of a set F.

Theorem 8.3.1. *Let us assume that $g \in C^{1,\alpha}(\mathbb{R}^d)$ satisfies*

$$\lim_{|x|\to+\infty} g(x) = +\infty$$

and that $\Phi \in C^{3,\alpha}(\mathbb{R}^d \setminus \{0\})$ is uniformly elliptic (recall (8.2)). Then, for every $V \in (0,+\infty)$ there exists a minimiser of (P_V) (resp. (P)). Moreover any minimiser E of (P_V) (resp. (P)) satisfies:

(i) Let Σ be the singular set of ∂E, i.e.

$$\Sigma = \big\{x \in \partial E \ : \ \partial E \text{ does not have a tangent plane at } x\big\}.$$

Then, Σ is closed, $\mathcal{H}^{d-3}(\Sigma) = 0$ and $\partial E \setminus \Sigma$ is a (relatively open) C^3 manifold oriented by ν^E.

(ii) There exists a constant μ such that letting the anisotropic mean curvature be defined as $H^\Phi = \operatorname{div}_{\partial E}(D\Phi(\nu^E)) = tr((D^2\Phi(\nu^E)A)$ (where A is the second fundamental form of ∂E),

$$H^\Phi + g = \mu \qquad \text{for all } x \in \partial E \setminus \Sigma \qquad (\text{resp } H^\Phi + g = 0), \tag{8.3}$$

$$\int_{\partial E\setminus\Sigma} \langle D^2\Phi(\nu^E)\nabla\varphi, \nabla\varphi\rangle - tr(D^2\Phi(\nu^E)A^2)\varphi^2 + \frac{\partial g}{\partial \nu}\,\varphi^2 \geq 0$$
$$\text{for all } \varphi \in C^1_c(\partial E \setminus \Sigma) \text{ such that } \int_{\partial E} \varphi = 0 \text{ (resp. for all } \varphi \in C^1_c(\partial E \setminus \Sigma)). \tag{8.4}$$

If E satisfies (8.4) for every $\varphi \in C^1_c(\partial E \setminus \Sigma)$, we say that E is unconditionally stable.

Inspired by the two-point function technique introduced by Andrews in [And12] to prove preservation of an interior ball condition along the mean curvature flow (see also [Bre13] or the review paper [And15]), we consider for E a minimizer of (P_V) or (P), the function defined on $\partial E\setminus\Sigma$ by

$$S(x) = \max_{y\in\partial E} \nu^E(x)\cdot(y-x).$$

Notice that $S \geq 0$ and that convexity of E is characterized by the fact that $S = 0$. The idea is to prove that if E is an unconditionally stable critical point then S gives a negative variation to the energy and therefore must vanish. This is enough to conclude in the case of (P). For (P_V), we can

argue that if ∂E is not connected then it must have at least on unconditionally stable connected component. From this we conclude from the previous argument that E was actually convex which contradicts the fact that ∂E is disconnected.

Let us now explain why S produces a negative variation. We first define the Jacobi operator

$$L_\Phi \varphi = \operatorname{div}_{\partial E}(D^2\Phi(\nu^E)\nabla\varphi) + \operatorname{tr}(D^2\Phi(\nu^E)A^2)\varphi. \tag{8.5}$$

Notice that by integration by parts,

$$\begin{aligned}\int_{\partial E\setminus\Sigma} &(-L_\Phi\varphi)\varphi + \frac{\partial g}{\partial \nu}\,\varphi^2 \\ &= \int_{\partial E\setminus\Sigma} \langle D^2\Phi(\nu^E)\nabla\varphi, \nabla\varphi\rangle - \operatorname{tr}(D^2\Phi(\nu^E)A^2)\varphi^2 + \frac{\partial g}{\partial \nu}\,\varphi^2. \end{aligned} \tag{8.6}$$

Lemma 8.3.2. *Let $\bar{x} \in \partial E\setminus\Sigma$ and assume that $\bar{y}$ is such that*

$$S(\bar{x}) = \nu^E(x)\cdot(\bar{y}-\bar{x}).$$

Then,

$$L_\Phi S(\bar{x}) \geq H^\Phi(\bar{x}) - H^\Phi(\bar{y}) + \langle \nabla H^\Phi(\bar{x}), \bar{y}-\bar{x}\rangle. \tag{8.7}$$

This inequality is proven in the viscosity sense i.e. considering a function φ such that

$$\varphi(x) - S_{\partial E}(x) \geq \varphi(\bar{x}) - S_{\partial E}(\bar{x}) = 0,$$

and using the first and second order optimality conditions of the function $G(x,y) = \varphi(x) - S(x,y)$ which achieves its minimum at $(\bar{x},\bar{y})$. In particular, we use the fact that

$$0 \leq (\nabla_i^x + \nabla_i^y)(\nabla_i^x + \nabla_i^y)\, G(\bar{x},\bar{y}).$$

We may now show that

$$L_\Phi S - \frac{\partial g}{\partial \nu} S \geq 0 \tag{8.8}$$

Multiplying this by $(-S)$, integrating and using (8.3.2) with $\varphi = S$ we would conclude the proof.

Differentiating (8.3) to get $\nabla H^\Phi = -\nabla g$ and subtracting to both side of inequality (8.7)

$$\frac{\partial g}{\partial \nu}(\bar{x})S(\bar{x}) = \frac{\partial g}{\partial \nu}(\bar{x})\nu^E(\bar{x})\cdot(\bar{y}-\bar{x}),$$

we obtain that indeed

$$\begin{aligned} L_\Phi S(\bar{x}) - \frac{\partial g}{\partial \nu}(\bar{x}) S(\bar{x}) & \\ &\geq H^\Phi(\bar{x}) - H^\Phi(\bar{y}) - \nabla g(\bar{x}) \cdot (\bar{y} - \bar{x}) - \frac{\partial g}{\partial \nu}(\bar{x}) \nu^E(\bar{x}) \cdot (\bar{y} - \bar{x}) \\ &\overset{(8.3)}{=} g(\bar{y}) - g(\bar{x}) - Dg(\bar{x}) \cdot (\bar{y} - \bar{x}) \geq 0 \end{aligned}$$

where the last inequality follows by convexity of g.

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