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Introduction

In this thesis we study the convergence of sequences of optimal control problems of different kinds. The main tools we use are the Γ-convergence technique and the theory of viscosity solutions for Hamilton-Jacobi-Bellman (HJB) equation. More precisely, we start by studying sequences of optimal control problems of the form

$$(\mathcal{P}_n) \qquad \min\left\{\int_0^T f_n(t,y,u)dt \; : \; (y,u)\in\mathcal{A}_n\right\}$$

where

$$\mathcal{A}_n = \{(y,u)\in Y\times\mathcal{U} : y'(t) = a_n(t,y) + B_n(t,u), \quad y(0) = y_n^0\}.$$

In some previous papers Buttazzo, Cavazzuti, Dal Maso and Freddi (see [BDM82], [BC89], [BF93], [BF95] and [Fre98]), have developed the theory of Γ-limits for this and more general optimal control problems. Here we recall that the Γ-limit problem $(\mathcal{P})$ of the sequence $(\mathcal{P}_n)$ is, roughly speaking, a problem such that:
a) the optimal controls u_n of $(\mathcal{P}_n)$ converge to the optimal control u of $(\mathcal{P})$;
b) the optimal trajectories y_n of $(\mathcal{P}_n)$ converge to the optimal trajectory y of $(\mathcal{P})$.
In Chapter 1 we will see that if the functions B_n converge continuously then the limit problem $(\mathcal{P})$ has the same form of problem $(\mathcal{P}_n)$ while if the functions B_n are only weakly convergent, in the limit problem the state equation may disappear and may be replaced by an integral contribution in the cost functional.
Since this behaviour is somewhat unexpected our first approach was to understand the behaviour of the (HJB) equations related to those problems. Thus, in the second chapter we have developed the theory of viscosity solutions for the (HJB) equation

$$\begin{cases} -\frac{\partial}{\partial t}v(t,x) + H(t,x,Dv(t,x)) = 0 \text{ in } (0,T)\times\mathbb{R}^N \\ \qquad\qquad\qquad\qquad v(T,x) = 0 \text{ in } \mathbb{R}^N \end{cases}$$

where

$$H(t,x,p) = \sup_{u\in\mathbb{R}^M}\{-p\cdot B(t,u) - f(t,x,u)\} - p\cdot a(t,x).$$

Our standard assumptions are that the Hamiltonian is measurable in time and that the control set is unbounded. This is why the existence and the uniqueness result for the viscosity solution of the (HJB) equation are not standard. However, the way to obtain those results is not very different from the standard one. In Section 2.2 we prove a Dynamic Programming

Principle and the continuity of the value function and, using this, in Section 2.3 we prove the existence and uniqueness result. We define the viscosity solution as in [Is85] but in the uniqueness proof we use also one of the equivalent definitions given in [PLP87]. The uniqueness is a consequence of a comparison result proved in Section 2.3.2.
The last section of Chapter 2 is devoted to the case of a linear-quadratic problem, that is $B_n(t,u) = b_n(t)u(t)$, and $f_n(t,y,u) = g(y) + \frac{1}{2}|u|^2$. Note that for this problem the Hamiltonian can be explicitly computed and we obtain

$$H(t,x,p) = -p \cdot a(t,x) - g(x) + \frac{1}{2}\, p^2\, b_n(t)^2.$$

In that section we discuss the convergence hypotheses $b_n(t) \to b(t)$ a.e. on (0,T) and $b_n(t) \to b(t)$ weakly$^* L^\infty(0,T)$. However, since our control space is $L^2(0,T)$ the natural assumption on the convergence of the coefficients $b_n(t)$ is $b_n(t) \to b(t)$ weakly $L^2(0,T)$. In this case measures naturally appear in the Γ-limit and one may suppose that $b_n(t)^2 \to \mu$ weakly$^*\mathcal{M}^+(0,T)$ (this behaviour is explained in Section 1.3.2). Thus the Hamiltonian of the limit problem should be the following (for all details see Chapter 3).

$$H(t,x,p) = -p \cdot a(t,x) - g(x) + \frac{1}{2}p^2\mu.$$

Since the presence of a measure in the Hamiltonian is something new we have tried to better understand this behaviour by studying an example in all details (see Section 3.2). Starting from this, in Chapter 3 we give a definition of viscosity solution for this problem by using the idea of "reparametrization" introduced for ordinary differential equation in some previous papers (see, among others [Su78], [Br87], [Mi89]).
With this technique we obtain an optimal control problem that we will call "reparametrized optimal control problem"; for this new problem we write an (HJB) equation using the result obtained in Chapter 2 and we characterize the value function as its unique viscosity solution.
We remark that from the point of view of viscosity solutions the assumptions that the Hamiltonian is merely measurable with respect to t, and that the set of controls is unbounded may seem useless and artificial generalizations. However, in the framework of Γ-convergence t-measurability is a natural assumption since in this kind of problems one deals with weak convergences which generally preserve measurability but not continuity. Moreover, the case of measures is ruled out if we suppose the control set bounded.
The relationship between the model with measures and the one with unbounded control set has been also studied in [BJM93], where the authors study a (HJB) equation for optimal control problem with measures as controls, which is slightly different from our case.
Once we have given this definition we consider a more general fully non linear problem, i.e.

$$\min\left\{\int_0^T \left(l_0(t,y) + \sum_{i=1}^{M} l_i(t,y)u_i(t) + \sum_{i=1}^{M} \tilde{l}_i(t,y)u_i(t)^2\right) dt \;:\; (y,u) \in \mathcal{A}_n\right\}$$

where

$$\mathcal{A}_n = \left\{(y,u) \in Y \times U : y' = a^n(t,y) + \sum_{i=1}^{M} g_i^n(t,y) b_i^n(t) u_i(t)\;;\; y(0) = y_n^0\right\}.$$

In a work in progress that is briefly described in Section 3.3, we see that if b_n are uniformly bounded we can expect the same as above (the Γ-limit result is given in [BC89]) while if $b_n(t) \in L^2(0,T)$ and converge only weakly in $L^2(0,T)$ a new phenomenon may occur. More precisely, we have obtained an example that shows how the conditions

$$b_n(t) \to b(t) \text{ weakly } L^2(0,T), \quad b_n(t)^2 \to \mu \text{ weakly}^* \mathcal{M}^+(0,T)$$

do not characterize the Γ-limit. By using the reparametrization technique introduced for the previous case, we are able to build an optimal control problem which characterizes the Γ-limit.
We end Chapter 3 by proving a chain rule in dimension one for distributional derivatives of functions with bounded variation which are the composition of an absolutely continuous function with a monotone increasing function. The standard measure theory that we use in this proof and in Chapter 3 is briefly described in Appendix B.
The results of Chapters 2 and 3 are collected in [Bri(I)] and [Bri(II)].
In 1995 Buttazzo and Freddi (see [BF95]) developed the theory of Γ-limit for more general systems of the form

$$(\mathcal{P}_n) \qquad \min\left\{\int_0^T f_n(t,y,u)dt \;:\; (y,u) \in \mathcal{A}_n\right\}$$

where $\mathcal{A}_n = \{(y,u) \in Y \times \mathcal{U} : A_n(y) = B_n(u)\}$ and $A_n : Y \to V$, $B_n : U \to V$ are general operators.
In Chapter 4 we apply this theory to the study of optimal control problems governed by parabolic partial differential state equations and different kinds of boundary conditions. The control is always acting on the boundary.
More precisely in the first part we proved that we can approximate a Dirichlet boundary condition with a more regular boundary condition. Note that this kind of approximations are widely used in numerical computations, so a stability result of this type is useful.
In the application it is also natural to think about boundary conditions of mixed type, i.e. Dirichlet conditions on a part of the boundary and Neumann conditions on the rest. Section 4.2 is devoted the study of this kind of problems. The Riccati equation and the optimal control in a feedback form are derived using the representation formula obtained by Acquistapace, Flandoli and Terreni in [AFT91]. To get these results we use the semigroup theory, whose basic notions are briefly recalled in Appendix C.
This thesis is organized as follows.
Chapter 1 is devoted to the description of two basic methods for the analysis of general optimal control problems, i.e. the Dynamic Programming approach and the variational approach.
In Chapter 2 we develop the theory of viscosity solutions for (HJB) equation with measurable time dependent Hamiltonians. An existence and uniqueness result is obtained in Section 2.3.
In Chapter 3 we give a definition of viscosity solution of (HJB) equation with a measure in the Hamiltonian and some hints for the fully non linear case.
In Chapter 4 we describe some applications of the Γ-convergence theory for infinite dimensional optimal control problems. In particular we consider different kinds of boundary control

problems.
Finally we remark that the definitions of all functional spaces involved are collected in Appendix A, the basic measure theory we used is described in Appendix B and in Appendix C we briefly recall some results on semigroup theory and fractional powers of operators.

Chapter 1

Two basic theories for optimal control problems

1.1 A class of optimal control problems

In this thesis we consider a class of *finite horizon* deterministic optimal control problem. Fixed a finite time T we want to minimize the following *cost functional*

$$J = \int_0^T f(t, y, u)\, dt \tag{1.1.1}$$

where $f : (0, T) \times \mathbb{R}^N \times \mathbb{R}^M \to [0, \infty]$ is a Borel function and y is the solution of the *state equation* with *control* u. We will consider two kinds of state equations, in Chapters 2 and 3 y will be the solution of an ordinary differential equation, i.e.

$$\begin{cases} y'(s) = g(s, y, u) & \text{in } (t, T] \\ y(t) = x \end{cases} \tag{1.1.2}$$

where $g : (0, T) \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is a Borel function, while in Chapter 4 we will discuss the case of parabolic partial differential equation, i.e.

$$\begin{cases} \frac{\partial y}{\partial t} - Ay = f(t, x) & \text{in } \Omega \times (0, T) \\ \\ y(t, x) = u(t, x) & \text{on } \partial\Omega \times (0, T) \\ \\ y(0, x) = y_0(x) & \text{in } \Omega \end{cases} \tag{1.1.3}$$

where Ω is an open bounded connected subset of $\mathbb{R}^N$ with regular boundary and A is a uniform elliptic operator on Ω.
Two of the most important approaches to the study of the optimal control systems are the Dynamic Programming approach and the variational approach. The ideas behind these theories and their goals are quite different. In the following two sections we will try to describe them and to outline the main results. We will not give details or proofs but we will always refer to the literature.

1.2 Dynamic Programming approach

1.2.1 The general setting

Since our aim is only to give the main ideas of this approach, we do not give a detailed list of the precise assumptions but the complete description of the finite horizon problem will be given in Chapter 2. For a complete and general reference we quote, among others, [FR75], [FS92] and [BCD97] while for a first and historical reference we recall [Be57].
The first step in the standard Dynamic Programming approach is to introduce the *value function*, defined by

$$v(t,x) = \inf_{u \in \mathcal{U}} \{J(t,x,u)\} \tag{1.2.1}$$

where $J(t,x,u)$ is the cost functional

$$J(t,x,u) = \int_t^T f(s, y^{t,x}(s), u(s))\, ds,$$

$\mathcal{U}$ is the space of controls u and (t,x) are the initial data of the state equation (1.1.2).
The fundamental idea of Dynamic Programming is that v satisfies a functional equation, often called the *Dynamic Programming Principle* (DPP), and when v is smooth enough, its differential version, the *Hamilton-Jacobi-Bellman* (HJB) equation.
This equation is important for two reasons. First, it contains all the relevant information needed to design, trough the so-called synthesis procedure, an optimal feedback map for the problem. (We will see an example in the next section). Second, this equation can be numerically solved in order to compute approximate optimal controls in feedback form and, as consequence, approximate optimal trajectories.
This is why the goal is to characterize v as the unique solution of an (HJB) equation. Moreover in the application it is important to have a stability property of the value function with respect to suitable perturbations of the data f and g.
The (DPP) for the finite horizon problem has the following form

$$v(t,x) = \inf_{u \in \mathcal{U}} \left\{ \int_t^\tau f(s, y^{t,x}(s;u), u(s))\, ds + v(\tau, y^{t,x}(\tau;u)) \right\} \tag{1.2.2}$$

where $\tau \in [t,T]$ and we denote by $y^{t,x}$ the solution of (1.1.2) with initial data (t,x).
The idea is that one can split the computation of the value function at the point (t,x) taking the infimum over the controls u of the cost functional in the subinterval (t,τ) plus the value function computed at time τ and starting with the value of the solution $y^{t,x}$ at the same time τ.
If the function v is differentiable it is easy to see by (1.2.2) that v is a solution of the following (HJB) equation (divide both sides of (1.2.2) by $\tau - t$ and let $\tau \to t$...)

$$\begin{cases} -\frac{\partial}{\partial t} v(t,x) + H(t,x,Dv(t,x)) = 0 \text{ in } (0,T) \times \mathbb{R}^N \\ v(T,x) = 0 \text{ in } \mathbb{R}^N \end{cases} \tag{1.2.3}$$

where $Dv(t,x)$ denotes the gradient with respect to the space variables of $v(t,x)$, and the Hamiltonian is given by

$$H(t,x,p) = \sup_{u \in \mathbb{R}^M} \{-p \cdot g(t,x,u) - f(t,x,u)\} \tag{1.2.4}$$

being $\cdot$ the scalar product in $\mathbb{R}^N$.
It is well known, however, that everywhere differentiability is a too restrictive assumption on v, so the derivation of (HJB) should be performed in some weak sense. First, it was given the definition of the so-called *generalized solution*, i.e. locally Lipschitz continuous functions satisfying the equation almost everywhere. The point was that this definition does not have uniqueness and stability properties. A classical example is the following.

Example 1.2.1 Let us consider the following (HJB) equation

$$\begin{cases} |v'| = 1 & \text{in } (0,1) \\ v(0) = v(1) = 0. \end{cases}$$

It is easy to show that this equation does not have any $C^1(0,1)$ solution. The first generalized solution we choose is

$$v^+(x) = \frac{1}{2} - |\frac{1}{2} - x|$$

while as second one we take $v^-(x) = -v^+(x)$. Actually, in order to build infinitely many solutions of this equations it is enough to alternate segments with slope 1 to segments with slope -1. More precisely one can consider a sequence such that

$$v_n(0) = 0$$

$$v_n'(x) = \begin{cases} 1 & x \in]\frac{2k}{2^n}, \frac{2k+1}{2^n}[\\ \\ -1 & x \in]\frac{2k+1}{2^n}, \frac{2k+2}{2^n}[\end{cases}$$

for $k = 0, 1, ..., 2^n - 1$.
Note that this is a sequence of generalized solutions that converges uniformly to $v \equiv 0$ which is not a generalized solution. So with this definition of solution there is no hope to get uniqueness and stability properties for the (HJB) equation (1.2.3).
We remark that the "good solutions" for the optimal control point of view are v^+ and v^-. In fact

$$v^+(x) = d(x, \partial(]0,1[)) := \inf_{y \in \partial(]0,1[)} (|y - x|)$$

$$v^-(x) = -d(x, \partial(]0,1[)) := \sup_{y \in \partial(]0,1[)} (-|y - x|).$$

$\square$

This is why in the early '80s the notion of *viscosity solution* was introduced by M.G. Crandall and P.L. Lions, [CL81, CL83], for general Hamiltonians. (The link between this notion and the previous one is described in [PLi82]). The (HJB) equation are in fact nonlinear elliptic or parabolic partial differential equation. Although in this thesis we consider only the first order parabolic case, just to give the idea behind this definition we consider the following elliptic second order equation,

$$H(x, u, Du, D^2u) = 0 \text{ in } \Omega, \tag{1.2.5}$$

where Ω is an open subset in $\mathbb{R}^N$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N \to \mathbb{R}$ (S^N is the vector space of symmetric matrices $N \times N$ and D^2u is the Hessian matrix of u). If we suppose that the elliptic condition is fulfilled, i.e.

$$M_1 \leq M_2 \implies H(x, u, p, M_1) \leq H(x, u, p, M_2),$$

then the following Maximum Principle holds for classical solutions. (For a proof see [Ba94], Theorem 2.1, page 10).

Theorem 1.2.2 *(Maximum Principle).*
A function $u \in C^2(0, T)$ is a classical solution of (1.2.5) if and only if

(i) *for every $\phi \in C^2(0, T)$, if x_0 is a local maximum for $u - \phi$ then*

$$H(x_0, u(x_0), D\phi(x_0), D^2(\phi(x_0))) \leq 0;$$

(ii) *for every $\phi \in C^2(0, T)$, if x_0 is a local minimum for $u - \phi$ then*

$$H(x_0, u(x_0), D\phi(x_0), D^2(\phi(x_0))) \geq 0.$$

This theorem gives an equivalent notion of classical solution, the important remark is that in conditions (i) and (ii) there isn't any request on the regularity of u. This is one of the reasons why this formulation has been used to define the viscosity solutions.
For the sake of simplicity we give this definition directly for the parabolic (HJB) equation (1.2.3), in the case of continuous Hamiltonians.

Definition 1.2.3 *Let v be a continuous function. We say that:*
v is a viscosity subsolution *of (1.2.3) if, for every $\phi \in C^1((0, T) \times \mathbb{R}^N)$ it holds*

$$-\frac{\partial}{\partial t}\phi(t_0, x_0) + H(t_0, x_0, D\phi(t_0, x_0)) \leq 0$$

at any (t_0, x_0) local maximum point for $v - \phi$;
v is a viscosity supersolution *of (1.2.3) if, for every $\phi \in C^1((0, T) \times \mathbb{R}^N)$ it holds*

$$-\frac{\partial}{\partial t}\phi(t_0, x_0) + H(t_0, x_0, D\phi(t_0, x_0)) \geq 0$$

at any (t_0, x_0) local minimum point for $v - \phi$;
v is a viscosity solution *of (1.2.3) if it is both viscosity subsolution and viscosity supersolution of (1.2.3).*

Remark 1.2.4 Looking again at Example 1.2.1 it is easy to prove that v^+ is the viscosity solution of

$$\begin{cases} |v'| - 1 = 0 & \text{in } (0,1) \\ v(0) = v(1) = 0 \end{cases}$$

while v^- is the viscosity solution of

$$\begin{cases} 1 - |v'| = 0 & \text{in } (0,1) \\ v(0) = v(1) = 0. \end{cases}$$

This underlines the fact that in some sense the viscosity solution is, among all the generalized solutions, the "good one" from the optimal control point of view. Notice that the different sign changes the solution. □

Crandall, Evans and Lions gave rather simple proofs of good comparison and stability properties of viscosity solutions [CEL84]. We do not discuss them here because in Chapter 2 we will prove these results for measurable time dependent Hamiltonians and compare them with the known results.
The definition of viscosity solution has been generalized in many directions (semicontinuous value functions, second order (HJB) equations, stochastic optimal control problems, etc...) and a considerable amount of work has been subsequently devoted to applications of viscosity solutions to optimal controls and differential games.
A comprehensive account of the theory of viscosity solutions can be found in [CIL92], for a less general but more systematic approach see [Ba94]. For the applications to optimal control we refer to [BCD97] and the references therein.

1.2.2 The infinite dimensional linear quadratic problem

In this section we want to describe briefly the classical infinite dimensional linear quadratic problem. We consider the following optimal control problem.
(P) *To find a control $u \in L^2(0,T;U) := \mathcal{U}$ that minimizes the cost functional*

$$J(t,x,u) = \int_t^T \left(\| Cx(s) \|_Y^2 + \| u(s) \|_U^2 \right) ds$$

under the condition

$$\begin{cases} x'(s) = Ax(s) + Bu(s) & s \geq t \\ x(t) = x \end{cases}$$

where $x \in H$, $A : D(A) \subset H \to H$, $B : U \to H$, $C : H \to Y$ are linear operators and H is a Hilbert space with scalar product $< \cdot, \cdot >$.
In order to apply the Dynamic Programming approach to this problem we first define the value function

$$v(t,x) = \inf_{u \in \mathcal{U}} \{ J(t,x,u) \} .$$

The Dynamic Programming Principle can be proved and one can derive from this the following (HJB) equation, (we always suppose that $v(t,x)$ is differentiable),

$$\begin{cases} -\frac{\partial}{\partial t}v(t,x) + \sup_{u\in U}\{- < Ax+Bu, Dv(t,x) > - \| Cx \|_Y^2 - \| u \|_U^2\} = 0 \\ \\ v(T,x) = 0. \end{cases} \tag{1.2.6}$$

(See, for example, [LY95], Chapter 6). The supremum in this equation can be easily calculated by "completing the square"; it is attained for

$$u = -\frac{1}{2}\, B^*\, Dv(t,x) \tag{1.2.7}$$

(where B^* is the adjoint operator of B), thus the Hamiltonian $H : (0,T)\times H\times H \to \mathbb{R}$ is

$$\begin{aligned} H(t,x,p) &:= \sup_{u\in U}\Big\{- < Ax+Bu, p > - \| Cx \|_Y^2 - \| u \|_U^2\Big\} = \\ &= - < Ax, p > - \| Cx \|_Y^2 + \frac{1}{4} \| B^* p \|_U^2 \,. \end{aligned}$$

Let us see now that the unique optimal control of problem $(\mathcal{P})$ can be obtained in the following feedback form

$$\hat{u}(t) = -\frac{1}{2}\, B^*\, Dv(t,\hat{x}(t)) \tag{1.2.8}$$

where $\hat{x}(\cdot)$ is the solution of

$$\begin{cases} x'(s) = Ax(s) - \frac{1}{2}\, B\, B^*\, Dv(s,x(s)) \qquad s \geq t \\ x(t) = x. \end{cases} \tag{1.2.9}$$

Let $u(\cdot) \in L^2(t,T;U)$ and $x(\cdot)$ be the solution of

$$x'(t) = Ax(t) + Bu(t) \quad , \quad x(t) = x \in H \tag{1.2.10}$$

then, if $G(s) = v(s,x(s))$ we have (using also the (HJB) equation)

$$\begin{aligned} G'(s) &= \frac{\partial}{\partial t}v(s,x(s)) + < Ax(s)+Bu(s), Dv(s,x(s)) >= \\ &= \frac{1}{4} \| B^* Dv(s,x(s)) \|_U^2 - \| Cx(s) \|_Y^2 + < Bu(s), Dv(s,x(s)) >= \\ &= \Big\| \frac{1}{2} B^* Dv(s,x(s)) + u(s) \Big\|_U^2 - \| u(s) \|_U^2 - \| Cx(s) \|_Y^2 \,. \end{aligned}$$

Now integrating on $[t,T]$ we get

$$v(T,x(T)) - v(t,x) =$$

$$= \int_t^T \Big\| \frac{1}{2} B^* Dv(s,x(s)) + u(s) \Big\|_U^2 \; ds - \int_t^T \Big(\| u(s) \|_U^2 + \| Cx(s) \|_Y^2\Big)\, ds$$

and using the (HJB) equation

$$v(t,x) + \int_t^T \| \frac{1}{2} B^* Dv(s,x(s)) + u(s) \|_U^2 \ ds =$$

$$= \int_t^T \left(\| u(s) \|_U^2 + \| Cx(s) \|_Y^2 \right) ds. \tag{1.2.11}$$

It is then clear, recalling the definition of cost functional, that the control $\hat{u}(t) = -\frac{1}{2} B^* Dv(t, \hat{x}(t))$ is optimal on $[t,T]$, in fact

$$v(t,x) = J(t,x,\hat{u}) = \int_t^T \left(\| \hat{u}(s) \|_U^2 + \| C\hat{x}(s) \|_Y^2 \right).$$

In order to show that $\hat{u}$ is unique let us suppose that $\overline{u} \in L^2(0,T;U)$ is a different optimal control and $\overline{x}$ the optimal state corresponding to $\overline{u}$. Identity (1.2.11) implies that

$$v(0,x) + \int_0^T \left(\| \frac{1}{2} B^* Dv(s,\overline{x}(s)) + \overline{u}(s) \|_U^2 \right) ds =$$

$$= \int_0^T \left(\| \overline{u}(s) \|_U^2 + \| C\overline{x}(s) \|_Y^2 \right) ds = J(0,x,\overline{u}),$$

and so

$$\int_0^T \| \frac{1}{2} B^* Dv(s,\overline{x}(s)) + \overline{u}(s) \|_U^2 \ ds = 0$$

which gives

$$\overline{u}(s) = -\frac{1}{2} B^* \ Dv(s,\overline{x}(s)) \ \forall s \in [0,T].$$

The uniqueness follows then from the uniqueness of the solution of the state equation (1.2.9). When we deal with a linear quadratic problem the (HJB) equation can be simplified and transformed in the so called Riccati equation. In fact, if we look for a solution of the (HJB) equation of the form

$$v(t,x) = < P(t)x, x >$$

where $P : H \to H$ is a selfadjoint and non negative definite operator, the (HJB) equation

$$-\frac{\partial}{\partial t} v(t,x) - < Ax, Dv(t,x) > - \| Cx \|_Y^2 + \frac{1}{4} \| B^* Dv(t,x) \|_U^2 = 0$$

becomes

$$- < \dot{P}(t)x, x > - < Ax, 2P(t)x > - < Cx, Cx > + \frac{1}{4} \| B^* 2P(t)x \|_U^2 = 0.$$

In order to derive an equation for $P(t)$ we write it as

$$- < \dot{P}(t)x, x > - < P(t)Ax, x > - < x, A^* P(t)x > +$$

$$- < C^*Cx, x > + < PBB^*Px, x >= 0$$

thus, $P(t)$ has to be the solution of the following Riccati equation

$$\begin{cases} -\dot{P}(t) - P(t)A - A^*P(t) - C^*C + P(t)BB^*P(t) = 0 \\ P(T) = 0. \end{cases}$$

The technique can be resumed as follows. First one solves the Riccati equation and finds the function $P(t)$, then the value function is $v(t,x) =< P(t)x, x >$ and the optimal control is given in the feedback form

$$\hat{u}(t) = -\, B^*\, P(t)\, \hat{x}(t)$$

where $\hat{x}(t)$ is the solution of

$$\begin{cases} x'(s) = Ax(s) - B\,B^*\,P(s)x(s) \qquad s \geq t \\ x(t) = x. \end{cases}$$

Note that in order to describe the method we have supposed $v(t,x)$ differentiable which is a too restrictive assumption. A lot of work has been devoted to obtain the same kind of characterization of the value function and of the optimal control under weaker assumptions. For a general approach to the theory of control see [BDDM92], while the theory of viscosity solution for infinite dimensional problems is developed in a series of seven papers of M. Crandall and P.L. Lions published on the Journal of Functional Analysis (1985-1994).

Remark 1.2.5 If one thinks about the linear quadratic problem in finite dimension (just take $U = \mathbb{R}^M$, $H = Y = \mathbb{R}^N$) note that this procedure transforms the (HJB) equation, which is a partial differential equation in the Riccati equation, which is an ordinary differential equation. This fact will be used in Chapter 3. □

1.3 Variational approach

1.3.1 The general setting

In order to explain the variational approach we consider a more general framework. Let us denote by Y the topological space of *states*, by $\mathcal{U}$ the topological space of *controls* and by $\mathcal{A} \subseteq Y \times \mathcal{U}$ the set of *admissible pairs* (i.e. the subset of pairs $(y,u) \in Y \times \mathcal{U}$ such that y is linked to u through the state equation). If the cost functional is $J : Y \times \mathcal{U} \to \overline{\mathbb{R}}$ the optimal control problem can be written as

$$(\mathcal{P}) \qquad \min \{J(y,u) \;\; : \;\; (y,u) \in \mathcal{A}\}$$

or equivalently, if we denote by χ_E the indicator function of the set E defined by

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E, \end{cases}$$

as

$$(\mathcal{P}) \qquad \min\{(J+\chi_{\mathcal{A}})(y,u) \ : \ (y,u) \in Y \times \mathcal{U}\}.$$

The basic idea of the variational approach is then to study the minimum of the optimal control problem looking at the minimum of the functional $F = J + \chi_{\mathcal{A}}$. The main tool in this direction is the *Direct Method of Calculus of Variation.* To state this theorem we need the following definition.

Definition 1.3.1 *Let (X, τ) be a topological space and let $F : X \to \overline{\mathbb{R}}$ be a functional.*

a) *We say that F is* τ-lower semicontinuous (shortly τ-l.s.c.) *if for every $t \in \mathbb{R}$ the set $\{x \in F \ : \ F(x) \leq t\}$ is τ-closed in X.*
   We say that F is sequentially τ-l.s.c. *if for every $x \in X$*

$$F(x) \leq \liminf_{n\to\infty} F(x_n)$$

   for every sequence (x_n) converging to x in (X, τ).

b) *We say that F is* τ-coercive *if for every $t \in \mathbb{R}$ there exists a τ-compact (and τ-closed) subset K_t of X such that $\{x \in F \ : \ F(x) \leq t\} \subset K_t$.*
   We say that F is sequentially τ-coercive *if for every $t \in \mathbb{R}$, every sequence in $\{x \in F \ : \ F(x) \leq t\}$ has a subsequence that τ-converges to a point of $\{x \in F \ : \ F(x) \leq t\}$.*

Theorem 1.3.2 *Assume that*

1) *F is τ-l.s.c., (respectively sequentially τ-l.s.c.);*

2) *F is τ-coercive, (respectively sequentially τ-coercive).*

Then F admits a minimum point on X.

(For the proof see, for example, [Bu89], Proposition 1.2.2, page 10). If the minimum does not exist it is interesting to find a functional that describes the behaviour of the minimizing sequences of F (i.e. the sequences (x_n) such that $\lim_{n\to\infty} F(x_n) = \inf_{x\in X} F(x)$). This is the so-called *relaxed* functional.

Definition 1.3.3 *Let (X, τ) be a topological space and let $F : X \to \overline{\mathbb{R}}$ be a functional. The* relaxed functional $\overline{F} : X \to \overline{\mathbb{R}}$ *is defined by*

$$\overline{F}(x) := \sup\Big\{G(x) : X \to \overline{\mathbb{R}} \text{ such that } G \text{ is } \tau\text{-l.s.c. and } G \leq F\Big\}.$$

Note that the supremum above is actually attained, so that $\overline{F}$ turns out to be the greatest τ-l.s.c. functional less than or equal to F. The properties of the relaxed functional are listed in the following Proposition. (The proof is given in [Bu89], Prop. 1.3.1, page 16).

Proposition 1.3.4 *The following properties hold:*

(i) $\overline{F}$ *is a* τ*-l.s.c. functional;*

(ii) *if* F *is* τ*-coercive then* $\overline{F}$ *is* τ*-coercive;*

(iii) $\inf_{x\in X} F = \min_{x\in X} \overline{F}$*;*

(iv) *for every* τ*-continuous functional* $G : X \to \overline{\mathbb{R}}$ *it is* $\overline{F+G} = \overline{F} + G$*;*

(v) *If* (x_n) *is a minimizing sequence for* F *which* τ*-converges to some point* $x \in X$*, then* x *is a minimum point for* $\overline{F}$*.*

So if F is a τ-coercive functional, there exists a minimum of $\overline{F}$ on X (apply the Direct Method to $\overline{F}$ using (i) and (ii)) and all minimizing sequences of F converge to this minimum. This is why the main point in this approach is the characterization of the relaxed functional.
The following example shows how a relaxed functional for an optimal control problem with an ordinary differential state equation can be constructed. As a general reference for this section we recall the book of Buttazzo [Bu89].

Definition 1.3.5 *Let* X *be a Banach space and let* X^* *be its topological dual space.*
If $F : X \to]-\infty,\infty]$ *is a proper functional (i.e.* $F \not\equiv \infty$*) we define the* dual functional $F^* : X^* \to]-\infty,\infty]$ *as*

$$F^* := \sup_{x\in X} \{< x^*, x > -F(x)\}$$

where $<\cdot,\cdot>$ *denotes the duality between* X *ad* X^**, and* $F^{**} : X \to]-\infty,\infty]$ *as*

$$F^{**} := \sup_{x^*\in X^*} \{< x^*, x > -F^*(x^*)\}.$$

Note that F^* is always, by definition, a convex and l.s.c. functional on X^* and that $F^{**} = F$ for every convex and l.s.c. F. (Theorem I.10 of [Bre92]).

Example 1.3.6 We consider the following finite horizon optimal control problem

$$(\mathcal{P}) \qquad \min\left\{\int_0^T f(t,y,u)\,dt + \chi_{\mathcal{A}}(y,u) \;:\; (y,u) \in Y \times \mathcal{U}\right\}$$

where $\mathcal{A} = \{(y,u) \,:\, y'(t) = a(t,y) + B(t,u) \;\; y(0) = y_0\}$. We suppose that f and a are only measurable with respect to the t variable and Lipschitz continuous with respect to the y variable. We may then take $Y = W^{1,1}(0,T;\mathbb{R}^N)$ and $\mathcal{U} = L^p(0,T;\mathbb{R}^M)$ $(1 < p \le \infty)$ endowed with their weak topologies.
Assume also that f is coercive on the variable u and l.s.c. in the pair (y,u). Due to the nonlinearity on u of the state equation this problem in general does not have existence of minima. (For all details and proofs see [Bu89], Section 5.3).
In order to derive the relaxed problem we perform 3 steps.
Step 1). We introduce an auxiliary variable $v = B(t,u)$. The new control space is $\mathcal{U} \times V$ where V is $L^q(0,T;\mathbb{R}^N)$ provided that

$$|B(t,u)| \le \beta_0(t) + C|u|^{p/q} \quad \text{with } q > 1 \text{ and } \beta_0 \in L^q(0,T).$$

Problem $(\mathcal{P})$ is then equivalent to a problem with linear state equation in the control but modified cost functional, i.e.

$$(\mathcal{P}) \qquad \min_{Y\times\mathcal{U}\times V} \left\{ \int_0^T \Big(f(t,y,u) + \chi_{\{v=B(t,u)\}}(t,u,v) + \chi_{\{y'(t)-a(t,y)=v\}}(t,y,v) \Big) \, dt \right\}.$$

Step 2). The term $\int_0^T \chi_{\{y'(t)-a(t,y)=v\}}(t,y,v)\, dt$ gives now a continuous contribution so, by point (iv) of Proposition 1.3.4, in order to relax the problem we can apply the general theory of relaxation for functional defined on L^p spaces to

$$\int_0^T f(t,y,u) + \chi_{\{v=B(t,u)\}}(t,y,u)\, dt.$$

Since our goal is just to give an example we refer to Chapter 2 of [Bu89] for all the precise results. Here, applying in particular Theorem 2.6.4 and Remark 2.6.5 page 74-75, we obtain the relaxed problem

$$(\overline{\mathcal{P}}) \qquad \min_{Y\times\mathcal{U}\times V} \left\{ \int_0^T \Big(f(t,y,u) + \chi_{\{v=B(t,u)\}}(t,y,u) \Big)^{**} \, dt + \int_0^T \chi_{\{y'(t)-a(t,y)=v\}}(t,y,u)\, dt \right\}.$$

Step 3). To write the final expression we have then to eliminate the v variable. We get

$$(\overline{\mathcal{P}}) \qquad \min \left\{ \int_0^T \psi(t,y,u,y'-a(t,y))\, dt \ : (y,u) \in Y\times\mathcal{U} \right\}$$

where $\psi(t,y,u,v) = \Big(f(t,y,u) + \chi_{v=B(t,u)} \Big)^{**}$. So, when this function can be explicitly calculated, problem $(\overline{\mathcal{P}})$ is completely known. □

Is this technique stable under perturbations? More precisely, if we consider a sequences of optimal control problems as follows

$$(\mathcal{P}_n) \qquad \min \{ J_n(y,u) \ : (y,u) \in \mathcal{A}_n \}$$

can we give a meaning to the sentence “pass to the limit”? Moreover, can we identify a “limit problem”? The answer to these questions is given by the theory of Γ-limits, that we will describe in the next section.

1.3.2 Γ-convergence

Setting $F_n(y,u) = (J_n + \chi_{\mathcal{A}_n})(y,u)$ the sequence of optimal control problems we consider is

$$(\mathcal{P}_n) \qquad \min \{ F_n(y,u) \ : (y,u) \in Y\times\mathcal{U} \}.$$

We will say that (y_n,u_n) is an *optimal pair* for the problem $(\mathcal{P}_n)$ if $F_n(y_n,u_n) = \min_{Y\times\mathcal{U}} F_n$. A good definition of “pass to the limit” will be a definition that guarantees the following property: “*If (y_n,u_n) is an optimal pair of $(\mathcal{P}_n)$, or simply a minimizing sequence, and if (y_n,u_n) tends to (y,u) in $Y\times\mathcal{U}$ then (y,u) is an optimal pair for the limit problem $(\mathcal{P})$.* ” The next theorem gives a definition of limit problem such that this property holds (for the proof see [BDM82], Proposition 2.1, page 388).

Theorem 1.3.7 *Let Y and $\mathcal{U}$ be two topological spaces and let $F_n : \mathcal{U} \times Y \to \overline{\mathbb{R}}$ be a sequence of functions. Let (y_n, u_n) be a minimum point for F_n, or simply a pair such that*

$$\lim_{n\to\infty} F_n(y_n, u_n) = \lim_{n\to\infty}[\inf_{Y\times\mathcal{U}} F_n].$$

Assume that (y_n, u_n) converges to (y, u) in $Y \times \mathcal{U}$ and there exists

$$F(y,u) = \Gamma(\mathbb{N}, \mathcal{U}^-, Y^-) \lim_{n\to\infty} F_n(y_n, u_n). \tag{1.3.1}$$

Then we have
(i) *(y, u) is a minimum point for F on $Y \times \mathcal{U}$;*
(ii) $\lim_{n\to\infty}[\inf_{Y\times\mathcal{U}} F_n] = \min_{Y\times\mathcal{U}} F$.

This is why we call *limit problem* :

$$(\mathcal{P}) \qquad \min\{F(y,u) \ : \ (y,u) \in Y \times \mathcal{U}\}$$

where F is defined by the multiple Γ-limit (1.3.1). A multiple Γ-limit is in general defined as follows.

Definition 1.3.8 *Let X and W be two topological spaces and let $F_n : X \times W \to \overline{\mathbb{R}}$ be a sequence of functions; by $Z(+)$ we shall denote the "sup" operator and by $Z(-)$ the "inf" one. For every $x \in X$ and $w \in W$ we define*

$$\Gamma(\mathbb{N}^\alpha, X^\beta, W^\gamma) \lim_{n\to\infty} F_n(w,x) = \underset{(x_n)\in S(x)}{\mathrm{Z}(\beta)} \ \underset{(w_n)\in S(w)}{\mathrm{Z}(\gamma)} \ \underset{k\in\mathbb{N}}{\mathrm{Z}(-\alpha)} \ \underset{n\geq k}{\mathrm{Z}(\alpha)} F_n(w_n, x_n)$$

where α, β, γ are signs $+$ or $-$, and $S(x)$ and $S(w)$ respectively denote the set of all sequences $x_n \to x$ in X and $w_n \to w$ in W. Note that when the Γ-limit does not depends on the signs $+$ or $-$, this sign will be omitted.

It is clear that now the limit problem is well defined and our next goal is to give some tools to calculate it explicitly (see [BDM82], [BC89], [BF93], [BF95] and [Fre98]). We will describe now some particular cases that we will use in the following chapters. (For a complete theory of Γ-limits we refer to [DM93]).
We consider sets of admissible pairs of the following form $\mathcal{A}_n = \{(y,u) \ : \ A_n(y) = B_n(u)\}$ where for fixed $n \in \mathbb{N}$ the operator A_n and B_n map respectively Y and $\mathcal{U}$ in a third topological space V. The sequence of optimal control problems is then

$$(\mathcal{P}_n) \qquad \min\{(J_n + \chi_{\mathcal{A}_n})(y,u) \ : (y,u) \in Y \times \mathcal{U}\}.$$

The main trick of this calculation is, as in Example 1.3.6, the introduction of an auxiliary variable $v = B_n(u)$. The functional $F_n = J_n + \chi_{\mathcal{A}_n}$ can then be split as follows

$$J_n(y,u) + \chi_{\{v=B_n(u)\}}(u,v) + \chi_{\{A_n(y)=v\}}(y,v) := G_n(y,u,v) + \chi_{A_n(y)=v}(y,v) \tag{1.3.2}$$

and the identification of (1.3.1) is given by the following theorem (for the proof see [BF95], Proposition 3.3).

Theorem 1.3.9 *Assume that*

(HAn) $$A_n \ G\text{-converges to } A.$$

(HBn) $$u_n \to u \text{ in } \mathcal{U} \implies B_n(u_n) \text{ is relatively compact in } V.$$

If there exists the Γ*-limit*

(HGn) $$G(u,v,y) := \Gamma(\mathbb{N}, \mathcal{U} \times V^-, Y^-) \lim_{n\to\infty} G_n(y,u,v)$$

then

$$F(y,u) = G(y,u,A(y)).$$

This reduces the calculation of the limit (1.3.1) to the calculation of (HGn) and to the identification of the G-limit of the operators (A_n). We recall the definition of G-convergence of operators.

Definition 1.3.10 *We say that the sequence* (A_n) G-converges *to* A *if for every* $v \in V$ *and* $y \in Y$

$$\Gamma(\mathbb{N}, V, Y^-) \lim_{n\to\infty} \chi_{\{A_n(y)=v\}}(v,y) = \chi_{\{A(y)=v\}}(v,y)$$

that is, if the following conditions are satisfied

i) *if* $y_n \to y$ *in* Y*,* $v_n \to v$ *in* V *and* $A_n(y_n) = v_n$ *for infinitely many* $n \in \mathbb{N}$*, then* $A(y) = v$*.*

ii) *if* $y \in Y$ *and* $v \in V$ *are such that* $A(y) = v$ *and* $v_n \to v$ *in* V*, then there exists* $y_n \to y$ *in* Y *such that* $A_n(y_n) = v_n$ *for every* $n \in \mathbb{N}$ *large enough.*

A lot of literature has been developed about the characterization of those conditions (we quote among others [Sp76]).
We can resume the identification of the limit problem in 3 steps:
Step 1). We introduce the auxiliary variable $v \in V$.
Step 2). We calculate the operator A given by (HAn) and the Γ-limit G in (HGn).
Step 3). We eliminate the v variable in order to write the final expression of the limit problem setting $F(y,u) = G(y,u,A(y))$.
From now on we will always suppose that A_n G-converge to an operator A and we will focus on the behaviour of the input operators B_n.
The simplest case is when the operators B_n converge continuously to an operator B, that is

$$u_n \to u \text{ in } \mathcal{U} \implies B_n(u_n) \to B(u) \text{ in } V. \tag{1.3.3}$$

(this implies hypothesis (HBn)). In order to characterize condition (HGn) we require some kind of uniform continuity on the cost functionals. More precisely we assume that

(HJn) there exist a function $\Psi : \mathcal{U} \to \mathbb{R}$ bounded on the $\mathcal{U}$-bounded sets, a function $\omega : Y \times Y \to \mathbb{R}$ with $\lim_{z \to y} \omega(y,z) = 0$ for every $y \in Y$, such that for every $u \in \mathcal{U}, y, z \in Y$ and $n \in \mathbb{N}$,

$$J_n(y,u) \leq J_n(z,u) + \Psi(u)\omega(y,z).$$

In this case the limit problem has the same form of the problems $(\mathcal{P}_n)$ and it is simply (see [BF95], Section 3),

$$(\mathcal{P}) \qquad \min\left\{J(y,u) \ : \ A(y) = B(u)\ ,\ (y,u) \in Y \times \mathcal{U}\right\}$$

where $J(y,u) := \Gamma(\mathbb{N}, \mathcal{U}^-) \lim_{n\to\infty} J_n(y,u)$.
The point is that we can ask really less than hypothesis (1.3.3) to get (HBn). To describe the limit problems under more general assumptions on the operators B_n we consider a particular optimal control system (for the general case see [BF95], Section 4).
Fixed two positive integers N, M our state space Y is the Sobolev space $W^{1,1}(0,T;\mathbb{R}^N)$ endowed with the $L^\infty(0,T;\mathbb{R}^N)$ topology and our control space $\mathcal{U}$ is $L^p(0,T;\mathbb{R}^M)$ endowed with its weak topology. We suppose $1 < p < \infty$. We want to minimize the cost functional

$$J_n(y,u) = \int_0^T f_n(t,y,u)\,dt \tag{1.3.4}$$

where $f_n : (0,T) \times \mathbb{R}^N \times \mathbb{R}^M \to [0,+\infty]$ are Borel functions and the state equations are

$$\begin{cases} y'(t) = a_n(t,y) + B_n(t,u) \\ y(0) = y_n^0 \end{cases} \tag{1.3.5}$$

where $a_n : (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ and $B_n : (0,T) \times \mathbb{R}^M \to \mathbb{R}^N$ are Borel functions. Therefore the optimal control problems we are concerned with are

$$(\mathcal{P}_n) \qquad \min\left\{\int_0^T f_n(t,y,u)\,dt \ : \ (y,u) \in \mathcal{A}_n\right\}$$

where

$$\mathcal{A}_n = \{(y,u) \in Y \times \mathcal{U} : y'(t) = a_n(t,y) + B_n(t,u), \ \ y(0) = y_n^0\}. \tag{1.3.6}$$

We will detail the list of assumption on a_n , B_n and f_n at the beginning of Chapter 2, here we recall only that we assume a_n and f_n to be Lipschitz continuous with respect to y and only measurable with respect to t and f_n coercive with respect to u.
We want to perform now the program presented above.
Step 1). We consider $v \in L^q(0,T;\mathbb{R}^N)$ where $1 < q < \infty$ is such that $B_n(t,u) \in L^q(0,T;\mathbb{R}^N)$ and we set

$$g_n(t,y,u,v) := f_n(t,y,u) + \chi_{\{B_n(t,u)=v\}}(t,u,v)$$

so that

$$G_n(y,u,v) = \int_0^T g_n(t,y,u,v)\,dt.$$

Step 2). If

$$a_n(\cdot,y) \longrightarrow a(\cdot,y) \text{ weakly } L^1(0,T;\mathbb{R}^N) \ \ \forall y \in \mathbb{R}^N, \tag{1.3.7}$$

$$y_n^0 \to y_0 \ \text{ in } \ \mathbb{R}^N, \tag{1.3.8}$$

then assumption (HAn) is fulfilled with $A(y) := y' - a(t,y)$.
We consider the dual function of g_n with respect to the pair (u,v), i.e.

$$g_n^*(t,y,u^*,v^*) = \sup_{u\in\mathbb{R}^M\ v\in\mathbb{R}^N} [u\cdot u^* + v\cdot v^* - g_n(t,y,u,v)] \tag{1.3.9}$$

and we suppose that

$$g_n^*(\cdot,y,u^*,v^*) \to \varphi(\cdot,y,u^*,v^*) \text{ weakly } L^q(0,T;\mathbb{R}^N) \tag{1.3.10}$$

for every $(y,u^*,v^*) \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N$. Then Proposition 3.2 page 158 of [BC89] let us conclude that

$$G(y,u,v) = \int_0^T \varphi^*(t,y,u,v)\,dt \tag{1.3.11}$$

where φ^* is the dual of φ with respect to the pair (u^*,v^*). By Theorem 1.3.9 the expression of the limit problem is

$$(\mathcal{P}) \qquad \min_{Y\times\mathcal{U}\times V} \left\{ \int_0^T \varphi^*(t,y,u,v)\ \ dt\ :\ y' = a(t,y)+v\ ,\ y(0)=y_0 \right\}.$$

Step 3). We eliminate the auxiliary variable to get as final expression of the limit problem

$$(\mathcal{P}) \qquad \min_{Y\times\mathcal{U}} \left\{ \int_0^T \varphi^*(t,y,u,y'-a(t,y))\ \ dt\ :\ (y,u)\in Y\times\mathcal{U}\ ,\ y(0)=y_0 \right\}.$$

Remark 1.3.11 We observe that if we have only a single problem instead of a sequence, this procedure coincides with the calculation of the relaxed problem in the Example 1.3.6. The Γ-limits are, in fact, the generalization of the relaxation procedure to a sequence of optimal control problems. □

If we consider the linear - quadratic case, i.e. $B_n(t,u) = b_n(t)u(t)$, and $f_n(t,y,u) = \frac{|u|}{2}^2 + |y-\phi(t)|^2$ (suppose $N = M = 1$ for simplicity, and $p = q = 2$), we can calculate explicitly the expression (1.3.9) and we obtain

$$g_n^*(t,y,u^*,v^*) = \frac{|u^*(t)+b_n(t)v^*(t)|^2}{2} + |y-\phi(t)|^2. \tag{1.3.12}$$

It is then clear that the limit problem does not depend only on the limit of b_n but also on the limit of b_n^2, more precisely if

$$b_n(t) \to b(t) \text{ a. e. on } (0,T) \tag{1.3.13}$$

then the (1.3.3) is fulfilled so that the limit problem has the same form of the problem $(\mathcal{P}_n)$ that is

$$(\mathcal{P}) \qquad \min\left\{\int_0^T \left(\frac{|u(t)|^2}{2} + |y-\phi(t)|^2\right)\,dt \ :\ (y,u)\in\mathcal{A}\right\}$$

where

$$\mathcal{A} = \{(y,u)\in Y\times\mathcal{U} : y'(t) = a(t,y) + b(t)u(t),\ \ y(0)=y_0\}.$$

On the contrary, if we suppose that $b_n \in L^\infty(0,T)$ and

$$\begin{array}{ll} b_n(t) \to b(t) & \text{weakly}^*\ L^\infty(0,T) \\ b_n(t)^2 \to \beta(t) & \text{weakly}^*\ L^\infty(0,T) \end{array} \tag{1.3.14}$$

the functions g_n converge weakly* $L^\infty(0,T)$ to

$$\varphi(t,y,u^*,v^*) = \frac{1}{2}\left((u^*)^2 + 2b(t)u^*v^* + \beta(t)(v^*)^2\right) + |y-\phi(t)|^2.$$

Finally, calculating the dual of this function we obtain the following limit problem

$$(\mathcal{P}) \qquad \min_{Y\times\mathcal{U}}\left\{\int_0^T \frac{|u|^2}{2} + |y-\phi(t)|^2 + \frac{|y'(t)-a(t,y)-b(t)u(t)|^2}{2(\beta-b^2)}\,dt,\ y(0)=y_0\right\}.$$

Note that in this case in the limit problem the state equation has disappeared and it is replaced by an integral contribution in the cost functional. This is the case, for instance, when the coefficients are highly oscillating. For example if we take $B_n(t,u) = sin(nt)u(t)$ this yields

$$(\mathcal{P}) \qquad \min_{Y\times\mathcal{U}}\left\{\int_0^T \frac{|u|^2}{2} + |y-\phi(t)|^2 + |y'-a(t,y)|^2\,dt,\ y(0)=y_0\right\}.$$

In this second case we have supposed $b_n \in L^\infty(0,T)$ so that we can choose $V = L^2(0,T)$ to fulfill hypothesis (HBn), but a more natural assumption would be $b_n \in L^2(0,T)$ and b_n^2 converging only weakly $L^2(0,T)$. In this case $b_n(t)u(t) \in L^1(0,T)$ so in order to have still compactness, i.e. to verify (HBn), we have to choose V as the space of Borel measures. Note that if we assume $u\in K$ (K compact subset of $\mathbb{R}$) the sequence (b_nu_n) is bounded so we do not need to consider the space of measures to fulfill (HBn).
The characterization of the limit problem in this case has been studied by Buttazzo and Freddi in [BF93] for the linear case and by Freddi in [Fre98] for the nonlinear case. We recall the result only in the linear case because we will use it in Chapter 3. If

$$\begin{array}{ll} b_n(t) \to b(t) & \text{weakly } L^2(0,T) \\ b_n(t)^2 \to \mu & \text{weakly}^*\ \mathcal{M}^+(0,T) \end{array} \tag{1.3.15}$$

then

$$g_n^*(t,y,u^*,v^*) \to \frac{1}{2}\left((u^*)^2 + 2b(t)u^*v^* + \mu(v^*)^2\right) + |y-\phi(t)|^2$$

and applying Proposition 4.2 of [BF93] we obtain the following limit problem

$$(\mathcal{P}) \qquad \min_{Y\times\mathcal{U}}\{\int_0^T \left(\frac{|u|^2}{2} + |y-\phi|^2 + \frac{|y'-a(t,y)-bu|^2}{2(\mu^a-b^2)}\right)\,dt + \int_{[0,T)} \frac{1}{2}\left|\frac{d(y')^s}{d\mu^s}\right|^2\,d\mu^s +$$

$$+\frac{|y^+(0)-y_0|^2}{\mu(0)} \ : \ y' << dt+\mu^s\},$$

where now $Y = BV(0,T)$. For all notations about measure theory and BV functions see Appendix B.

Chapter 2

Measurable time dependent Hamiltonians: stability and uniqueness

In the first chapter we have seen that when one wants to study limits of optimal control problems one of the most important questions is whether or not the theory is in some sense "stable".
In the variational approach this has been discussed describing the theory of Γ-limits while in the Dynamic Programming approach these results are of the following form : "if v_n are the viscosity solutions of a sequence of (HJB) equations having as Hamiltonians H_n and if (v_n) converges to a function v and (H_n) converges to H then v is the viscosity solution of the (HJB) equation having H as Hamiltonian".
In this chapter we want to study the relationship between these two approaches in order to better understand the behaviour of the limit problems.
More precisely we deal with sequences of optimal control problems of the form

$$(\mathcal{P}_n) \qquad \min\left\{\int_0^T f_n(t,y,u)dt \;:\; (y,u)\in\mathcal{A}_n\right\}$$

where

$$\mathcal{A}_n = \{(y,u)\in Y\times\mathcal{U} : y'(t) = a_n(t,y) + B_n(t,u), \;\; y(0) = y_n^0\}.$$

In Chapter 1 we have seen that if the functions B_n converge continuously then the limit problem $(\mathcal{P})$ has the same form of problem $(\mathcal{P}_n)$ while if the functions B_n are only weakly convergent, in the limit problem the state equation may disappear and may be replaced by an integral contribution in the cost functional.
Since this behaviour is somewhat unexpected, we look for a justification of the limit problem via (HJB) equation's theory. In fact we show that the Hamiltonians of the problem $(\mathcal{P}_n)$ converge in a natural way to the Hamiltonian of the problem $(\mathcal{P})$ without any Γ-convergence tool.
To obtain this result we start by associating to every problem $(\mathcal{P}_n)$ its value function v_n

and its (HJB) equation characterized by a Hamiltonian H_n. We define in the same way the value function v of the limit problem $(\mathcal{P})$ and its Hamiltonian H. We observe then that by the Γ-convergence theory we can conclude that the value functions v_n converge pointwise to v and we prove that the Hamiltonians H_n converge to H. In order to conclude we need to show that for each fixed n the value function v_n is the solution of the corresponding (HJB) equation and that v is the solution of the (HJB) equation having H as Hamiltonian.
We can resume this construction in the following scheme

$$\begin{array}{ccccc} (\mathcal{P}_n) & \Longleftrightarrow & (v_n) & \Longleftrightarrow & (H_n) \\ \downarrow & & \downarrow & & \Downarrow \\ (\mathcal{P}) & \Longleftrightarrow & v & \Longleftrightarrow & H \end{array}$$

where we represent by a single line the results following by Γ-convergence and by a double line our work.
We remark that, since we are considering an unbounded control set, the proof of the continuity of the value function is not a standard result and will be given in Section 2.2. Moreover, since the Hamiltonian is only measurable in time, we cannot apply the results available in the literature to show that v_n and v are viscosity solutions of the respective (HJB) equations, so we prove it directly in Section 2.3.1. For the same reasons we need to prove a uniqueness result in Section 2.3.2.
We will see in the next section that for these problems the link between the two approaches is clear and is given by an explicit formula that links the Hamiltonians H_n to the functions g_n^* that appear in the calculation of the Γ-limit (see Remark 2.1.3).
This chapter is organized as follows. In Section 2.1 we describe the control problems in all details and we prove the convergence results for the Hamiltonians and for the value functions. In Section 2.2 we prove the Dynamic Programming Principle and the continuity of the value function. In Section 2.3 we recall the definitions of viscosity solution in this case, we write the (HJB) equation and we prove an existence (Subsection 2.3.1) and uniqueness (Subsection 2.3.2) result. The last section is devoted to the details of the interesting case of a linear-quadratic problem.

2.1 The framework

We consider the time interval $(0,T)$ with $T<\infty$. Fixed two positive integers N, M our state space Y is the Sobolev space $W^{1,1}(0,T;\mathbb{R}^N)$ endowed with the $L^\infty(0,T;\mathbb{R}^N)$ topology and our control space $\mathcal{U}$ is $L^p(0,T;\mathbb{R}^M)$ endowed with its weak topology. We suppose $1<p<\infty$.
We want to minimize the cost functional

$$J_n(y,u)=\int_0^T f_n(t,y,u)\,dt$$

where $f_n:(0,T)\times\mathbb{R}^N\times\mathbb{R}^M\to[0,\infty]$ are Borel functions and the state equations are

$$\begin{cases} y'(t)=a_n(t,y)+B_n(t,u) \\ y(0)=y_n^0 \end{cases} \tag{2.1.1}$$

where $a_n : (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ and $B_n : (0,T) \times \mathbb{R}^M \to \mathbb{R}^N$ are Borel functions. Therefore the optimal control problems we are concerned with are

$$(\mathcal{P}_n) \qquad \min \left\{ \int_0^T f_n(t,y,u)\,dt \;:\; (y,u) \in \mathcal{A}_n \right\}$$

where

$$\mathcal{A}_n = \{(y,u) \in Y \times \mathcal{U} : y'(t) = a_n(t,y) + B_n(t,u), \;\; y(0) = y_n^0\}.$$

We denote by $\|\cdot\|_r$ the $L^r(0,T;\mathbb{R}^N)$ norm for $r \in [1,\infty)$. We make the following assumptions on a_n , B_n and f_n.

(Ha1) There exists a sequence of functions M_n from $(0,T)$ into $\mathbb{R}$ with $\| M_n \|_1 \leq \tilde{M} < \infty$ such that, for every $t \in (0,T)$

$$|a_n(t,0)| \leq M_n(t) \quad \forall n \in \mathbb{N};$$

(Ha2) there exists a sequence of functions α_n from $(0,T)$ into $\mathbb{R}$ with $\| \alpha_n \|_1 \leq \tilde{\alpha} < \infty$ such that, for every $t \in (0,T)$ and for every $y_1, y_2 \in \mathbb{R}^N$

$$|a_n(t,y_1) - a_n(t,y_2)| \leq \alpha_n(t)|y_1 - y_2| \quad \forall n \in \mathbb{N};$$

(HB) there exist $q > 1$, a sequence of functions $G_n \in L^q(0,T)$ with $\| G_n \|_q \leq \tilde{G}$, and a constant $D \geq 0$ such that, for every $(t,u) \in (0,T) \times \mathbb{R}^M$

$$|B_n(t,u)| \leq G_n(t) + D|u|^{p/q} \quad \forall n \in \mathbb{N};$$

(Hf1) for a.e. $t \in (0,T)$ and for every $y \in \mathbb{R}^N$, $f_n(t,y,\cdot)$ is l.s.c. on $\mathbb{R}^M$, for every $n \in \mathbb{N}$;

(Hf2) there exists a function $C(t,v,w)$, from $(0,T) \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, integrable with respect to t, increasing and continuous in v and w such that, for every $u \in \mathbb{R}^M$ and for every $y, z \in \mathbb{R}^N$

$$|f_n(t,y,u) - f_n(t,z,u)| \leq C(t,|y|,|z|)|y - z| \quad \forall n \in \mathbb{N};$$

(Hf3) there exist two constants $a > 0$ and $b \geq 0$ such that, for every $(t,y,u) \in (0,T) \times \mathbb{R}^N \times \mathbb{R}^M$

$$f_n(t,y,u) \geq a|u|^p - b \quad \forall n \in \mathbb{N};$$

(Hf4) there exists a control function $u_0 \in \mathcal{U}$ such that, for every $y \in Y$ the sequence of functions $f_n(\cdot,y(\cdot),u_0(\cdot))$ is bounded in $L^1(0,T;\mathbb{R})$;

(HBf) there exists a sequence of control functions $u_n \in L^\infty(0,T;\mathbb{R}^M)$ such that, for every $y \in Y$

$$B_n(\cdot, u_n(\cdot)) \in L^\infty(0,T;\mathbb{R}^N) \text{ and } \int_0^T f_n(t, y(t), u_n(t))dt < \infty.$$

Let us recall that by definition $g_n(t,y,u,v) := f_n(t,y,u) + \chi_{\{B_n(t,u)=v\}}(t,u,v)$ and

$$g_n^*(t,y,u^*,v^*) = \sup_{u\in\mathbb{R}^M\ v\in\mathbb{R}^N} [u\cdot u^* + v\cdot v^* - g_n(t,y,u,v)].$$

In the first chapter we have seen (Section 1.3.2) that if

$$a_n(\cdot,y) \longrightarrow a(\cdot,y) \text{ weakly } \ L^1(0,T;\mathbb{R}^N) \ \ \forall y \in \mathbb{R}^N, \tag{2.1.2}$$

$$y_n^0 \to y_0 \ \text{ in } \ \mathbb{R}^N, \tag{2.1.3}$$

$$g_n^*(\cdot,y,u^*,w^*) \to \varphi(\cdot,y,u^*,w^*) \text{ weakly } L^q(0,T;\mathbb{R}^N) \tag{2.1.4}$$

for every $(y,u^*,w^*) \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N$, then the limit problem is given by

$$(\mathcal{P}) \qquad \min\left\{\int_0^T \varphi^*(t,y,u,y'-a(t,y))\ dt\ :\ (y,u)\in Y\times\mathcal{U}\ ,\ y(0)=y_0\right\}.$$

Our purpose is now to associate to the sequence of problems $(\mathcal{P}_n)$ a sequence of Hamiltonians H_n and to show that it converges to the Hamiltonian H associated to the limit problem $(\mathcal{P})$. By the standard theory of (HJB) equations (see [BCD97] or [FS92]) it is known that the Hamiltonians associated to problems $(\mathcal{P}_n)$ are

$$H_n(t,x,p) = \sup_{u\in\mathbb{R}^M} \{-p\cdot B_n(t,u) - f_n(t,x,u)\} - p\cdot a_n(t,x) \tag{2.1.5}$$

while the value functions are

$$v_n(t_0,x_0) = \inf_{u\in\mathcal{U}} \left\{\int_{t_0}^T f_n(t, y^0(t;u), u(t))\ dt\right\} \tag{2.1.6}$$

where $y^0(t;u)$ is the solution of (2.1.1) with initial condition $y(t_0) = x_0$. If we look at the limit problem $(\mathcal{P})$ we see that the state equation disappears and the derivative $y'(t)$ appears now in the cost. In order to write the value function and the Hamiltonian of this problem we introduce a new variable as follows. We take $v \in L^q(0,T;\mathbb{R}^N)$ (where q is given by (HB)) and we consider as control the vector $U = (u,v)$; the space of controls is then $\mathcal{U} = L^p(0,T;\mathbb{R}^M) \times L^q(0,T;\mathbb{R}^N)$ and the state equation is given by

$$\begin{cases} y'(t) = a(t,y) + C(U(t)) & t\in(0,T) \\ y(t_0) = x_0 \end{cases} \tag{2.1.7}$$

where $C : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^N$ is the linear mapping $C(z_1, z_2) = z_2$. It follows that the value function is

$$v(t_0, x_0) = \inf_{U \in \mathcal{U}} \left\{ \int_{t_0}^{T} \alpha(t, y^0(t; U), U(t))\, dt \right\} \tag{2.1.8}$$

where $y^0(t; U)$ is the solution of (2.1.7) and $\alpha(t, y, U) := \varphi^*(t, y, u, v)$ while the Hamiltonian is

$$H(t, x, p) = \sup_{u \in \mathbb{R}^N\ v \in \mathbb{R}^N} \{-p \cdot v - \varphi^*(t, y, u, v)\} - p \cdot a(t, x). \tag{2.1.9}$$

Remark 2.1.1 This procedure coincides in fact with considering as limit problem the Γ-limit problem before eliminating the auxiliary variable v (as described in Section 1.3.2) which in this case is

$$(\mathcal{P}) \qquad \min_{Y \times \mathcal{U} \times V} \left\{ \int_0^T \varphi^*(t, y, u, v)\ dt\ :\ y' = a(t, y) + v\ ,\ \ y(0) = y_0 \right\}.$$

□

Our convergence result for the Hamiltonians is the following.

Theorem 2.1.2 *Let H_n be as in (2.1.5) and let H be as in (2.1.9), under the assumptions (2.1.2) and (2.1.4)*

$$H_n(\cdot, x, p) \to H(\cdot, x, p) \quad \textit{weakly} \quad L^1(0, T)$$

for every $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$.

Proof. By definition of g_n and g_n^* we can write

$$\begin{aligned} H_n(t, x, p) &= \sup_{u \in \mathbb{R}^M\ v \in \mathbb{R}^N} \{-p \cdot v - g_n(t, x, u, v)\} - p \cdot a_n(t, x) = \\ &= g_n^*(t, x, 0, -p) - p \cdot a_n(t, x) \end{aligned}$$

so that, by hypotheses (2.1.2) and (2.1.4),

$$H_n(\cdot, x, p) \to \varphi(\cdot, x, 0, -p) - p \cdot a(\cdot, x) \quad \text{weakly} \quad L^1(0, T)$$

for every $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$. Since by (2.1.9),

$$H(\cdot, x, p) = \varphi^{**}(\cdot, x, 0, -p) - p \cdot a(\cdot, x) = \varphi(\cdot, x, 0, -p) - p \cdot a(\cdot, x)$$

the proof is achieved. □

Remark 2.1.3 The relationship between the two approaches is clear in the first equality of this proof, i.e.

$$H_n(t, x, p) = g_n^*(t, x, 0, -p) - p \cdot a_n(t, x),$$

this makes evident that the hypotheses on the convergence of g_n^* and a_n imply the convergences of the Hamiltonians. We will see in Section 3.3 that in a more general case this equality does not hold but Theorem 2.1.2 remains true. □

The convergence result for the value functions is :

Proposition 2.1.4 *If v_n and v are defined as in (2.1.6) and (2.1.8), and we assume* (Ha1), (Ha2), (HB), (Hf1), (Hf2), (Hf3), (Hf4), (HBF), *(2.1.2), (2.1.3) and (2.1.4), then*

$$\lim_{n\to\infty} v_n(t,x) = v(t,x) \quad \text{for all } (t,x) \in (0,T)\times \mathbb{R}^N.$$

Proof. This is nothing but a different way of writing the Γ-convergence result of (P_n) to (P) (see Chapter 1, Section 1.3.2, Theorem 1.3.7 (ii)). □

In order to link the previous convergence results (Theorem 2.1.2 and Proposition 2.1.4) we have to show that v_n is the viscosity solution of the (HJB) equation having H_n as Hamiltonian and that v is the viscosity solution of the (HJB) equation having H as Hamiltonian.

Theorem 2.1.5 *Assume* (Ha1), (Ha2), (HB), (Hf2), (Hf3), (Hf4), *(2.1.2), (2.1.3), (2.1.4) and that*

$$M_n \to \overline{M} \quad \text{weakly } L^1(0,T), \tag{2.1.10}$$

$$\alpha_n(t) \to \overline{\alpha} \quad \text{weakly } L^1(0,T), \tag{2.1.11}$$

$$f_n(\cdot, y(\cdot), u_0) \text{ is bounded in } L^1(0,T) \quad \forall y \in Y, \text{ where } u_0 \text{ is given by (Hf4)}. \tag{2.1.12}$$

Then the functions $a(t,y)$ and $\alpha(t,y,U)$ in (2.1.7) and (2.1.8) verify (Ha1), (Ha2), (HB), (Hf2), (Hf3) *and* (Hf4).

Proof. Let us take $\varphi \in L^\infty(0,T;\mathbb{R}^N)$ and consider

$$\begin{aligned}\int_0^T a(t,0)\cdot\varphi(t)dt &= \int_0^T (a(t,0)-a_n(t,0))\cdot\varphi(t)dt + \int_0^T a_n(t,0)\cdot\varphi(t)dt \le \\ &\le \int_0^T (a(t,0)-a_n(t,0))\cdot\varphi(t)dt + \int_0^T M_n(t)|\varphi(t)|dt\end{aligned}$$

by (Ha1). Passing to the limit by (2.1.2) and (2.1.10) we obtain

$$\int_0^T a(t,0)\cdot\varphi(t)dt \le \int_0^T \overline{M}(t)|\varphi(t)|dt$$

and choosing $\varphi(t) = \frac{a(t,0)}{|a(t,0)|} 1_{[t_0-\varepsilon;t_0+\varepsilon]}(t)$,

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} |a(t,0)|dt \le \int_{t_0-\varepsilon}^{t_0+\varepsilon} \overline{M}(t)dt \le \tilde{M}\, 2\varepsilon \tag{2.1.13}$$

(where we recall that by definition the indicator function is $1_A(x) = \left\{\begin{array}{ll} 1 & x\in A \\ 0 & x\notin A\end{array}\right.$).

Dividing by 2ε in (2.1.13) and letting $\varepsilon \to 0$ we finally get

$$|a(t,0)| \le \tilde{M}$$

which is hypothesis (Ha1) for the function $a(t,x)$. The (Ha2) can be proved by the same technique using (Ha2) for the sequence (a_n) and (2.1.11). Hypothesis (HB) is trivially satisfied with $D = 0$ and $G_n \equiv v$ while (Hf4) follows form (2.1.12) and the definition of Γ-limit. By definition of dual function and by (Hf2) for the sequence (f_n) we have

$$|g_n(t,y,u^*,v^*) - g_n(t,z,u^*,v^*)| \le C(t,|y|,|z|)|y-z|.$$

By the same technique used to prove (Ha1) and by (2.1.4) we get

$$|\varphi(t,y,u^*,v^*) - \varphi(t,z,u^*,v^*)| \le C(t,|y|,|z|)|y-z|,$$

that implies, always by definition of dual function,

$$|\varphi^*(t,y,u,v) - \varphi^*(t,z,u,v)| \le C(t,|y|,|z|)|y-z|,$$

and this gives us the (Hf2) for $\alpha(t,y,U)$ (recall that $\alpha(t,y,U) := \varphi^*(t,y,u,v)$).
Note that by definition $F \ge G \implies F^* \le G^*$. Using this, hypothesis (Hf3) on (f_n) and (2.1.4), one can easily get (Hf3) for $\alpha(t,y,U)$ and verify that $\alpha(t,y,U)$ takes values in $[0,\infty]$.
□

This result tells us that the study of the two (HJB) equations (at n fixed and the limit one) can be done, in fact, by studying a general optimal control problem with cost

$$J(t,x,u) = \int_t^T f(s, y^{x,t}(s;u), u(s))\, dt$$

and state equation

$$\begin{cases} y'(t) = a(t,y) + B(t,u) & t \in (0,T) \\ y(t) = x \end{cases} \tag{2.1.14}$$

that verifies (Ha1), (Ha2), (HB), (Hf2), (Hf3) and (Hf4). (Set $f := f_n$, $a := a_n$, $B := B_n$, $M := M_n$, $\alpha := \alpha_n$, $G := G_n$). To this study will be devoted the rest of the chapter.

2.2 Dynamic Programming Principle and regularity of the value function

In order to study our general optimal control problem, in this section we begin by proving the Dynamic Programming Principle and the regularity of the value function. We recall the definition of value function:

$$v(t,x) = \inf_{u \in \mathcal{U}} \{J(t,x,u)\} = \inf_{u \in \mathcal{U}} \left\{ \int_t^T f(s, y^{x,t}(s;u), u(s))\, dt \right\} \tag{2.2.1}$$

where we denote by $y^{x,t}(s;u)$ (or $y^{x,t}(s)$) the solution of (2.1.14) with initial condition $y(t) = x$ and control u at any time $s \in [0,T]$.

Theorem 2.2.1 *(Dynamic Programming Principle). If* (Ha1), (Ha2), (HB) *are fulfilled and the space of controls is* $\mathcal{U} = L^p(0,T;\mathbb{R}^N)$ $(p \in [1,\infty])$ *then for every* $\tau \in [t,T]$

$$v(t,x) = \inf_{u \in \mathcal{U}} \left\{ \int_t^\tau f(s, y^{x,t}(s;u), u(s))\, ds + v(\tau, y^{x,t}(\tau)) \right\}. \tag{2.2.2}$$

Proof. Let us denote by $w(t,x)$ the right hand side of (2.2.2). We first observe that by definition (2.2.1) $v(T, y^{x,t}(T)) = 0$ so we have

$$w(t,x) = \inf_{u \in \mathcal{U}} \left\{ \int_t^T f(s, y^{x,t}(s;u), u(s))\, ds \right\}$$

which is the definition of the value function (2.2.1). Therefore (2.2.2) is proved for $\tau = T$. Let us consider now the case $\tau < T$. Fix $u \in \mathcal{U}$ and write

$$J(t,x,u) = \int_t^\tau f(s, y^{x,t}(s;u), u(s))\, ds + \int_\tau^T f(s, y^{x,t}(s;u), u(s))\, ds.$$

By (Ha1), (Ha2) and (HB) we have uniqueness for the solution of (2.1.14). Hence, setting $\mu = y^{x,t}(\tau;u)$ we have $y^{x,t}(s;u) = y^{\mu,\tau}(s;u)$, for every $s \geq \tau$. Using this we can write

$$\begin{aligned} J(t,x,u) &= \int_t^\tau f(s, y^{x,t}(s;u), u(s))\, ds + \int_\tau^T f(s, y^{\mu,\tau}(s;u), u(s))\, ds = \\ &= \int_t^\tau f(s, y^{x,t}(s;u), u(s))\, ds + J(\tau, \mu, u) \geq \\ &\geq \int_t^\tau f(s, y^{x,t}(s;u), u(s))\, ds + v(\tau, y^{x,t}(\tau)) \end{aligned}$$

by definition (2.2.1). Taking the infimum over $u \in \mathcal{U}$ we have the inequality $v(t,x) \geq w(t,x)$. In order to prove the opposite inequality, fix $\varepsilon > 0$ and let $u_\varepsilon \in \mathcal{U}$ be such that

$$v(\tau, \mu) + \varepsilon > J(\tau, \mu, u_\varepsilon) \tag{2.2.3}$$

where $\mu = y^{x,t}(\tau;u)$ and u is fixed. We set

$$\overline{u}(s) = \begin{cases} u(s) & t \leq s \leq \tau \\ u_\varepsilon(s) & \tau < s \leq T, \end{cases}$$

by this definition and by uniqueness of the solution of (2.1.14) we have

$$y^{x,t}(s;\overline{u}) = \begin{cases} y^{x,t}(s;u) & t \leq s \leq \tau \\ y^{\mu,\tau}(s;u_\varepsilon) & \tau < s \leq T. \end{cases}$$

So by definition (2.2.1)

$$\begin{aligned} v(t,x) &\leq J(t,x,\overline{u}) = \\ &= \int_t^\tau f(s, y^{x,t}(s;\overline{u}), \overline{u}(s))\, ds + \int_\tau^T f(s, y^{x,t}(s;\overline{u}), \overline{u}(s))\, ds = \\ &= \int_t^\tau f(s, y^{x,t}(s;u), u(s))\, ds + \int_\tau^T f(s, y^{\mu,\tau}(s;u_\varepsilon), u_\varepsilon(s))\, ds = \\ &= \int_t^\tau f(s, y^{x,t}(s;u), u(s))\, ds + J(\tau, \mu, u_\varepsilon) < \\ &< \int_t^\tau f(s, y^{x,t}(s;u), u(s))\, ds + v(\tau, \mu) + \varepsilon \end{aligned}$$

where we used (2.2.3). Taking the infimum over $u \in \mathcal{U}$ and let ε going to 0 we get the inequality $v(t,x) \leq w(t,x)$ which completes the proof. □

To prove the regularity of the value function we will need some preliminary results that we state below.

Lemma 2.2.2 *Under hypotheses* (Ha1), (Ha2) *and* (HB), *the solution $y^0(t;u)$ of the state equation (2.1.14) with initial condition $y(t_0) = x_0$ satisfies the inequality*

$$|y^0(t;u)| \leq \vartheta(|x_0|, u) \tag{2.2.4}$$

where $\vartheta(|x_0|,u) = A\{|x_0| + \tilde{M} + \parallel B(\cdot, u(\cdot)) \parallel_1\}$ and $A = 1 + \parallel \alpha \parallel_1 e^{\|\alpha\|_1}$. Moreover, if we denote by $y^1(t;u)$ the solution of (2.1.14) with initial condition $y(t_1) = x_1$ we have

$$|y^0(t;u) - y^1(t;u)| \leq A\left\{|x_0 - x_1| + \int_{t_0}^{t_1} \gamma(s, |x_0|, u)\, ds\right\} \text{ if } t_0 < t_1$$

$$|y^0(t;u) - y^1(t;u)| \leq A\left\{|x_0 - x_1| + \int_{t_1}^{t_0} \gamma(s, |x_1|, u)\, ds\right\} \text{ if } t_1 < t_0$$

where $\gamma(s, |x|, u) = \alpha(s)\vartheta(|x|, u) + M(s) + |B(s, u(s))|$.

Proof. The proof is a standard application of Gronwall's Lemma so it is omitted. □

Remark 2.2.3 It is easy to see that $\vartheta(|x|, u)$ and $\gamma(s, |x|, u)$ are both increasing functions on the variable $|x|$, and $\gamma(\cdot, x, u) \in L^1(0,T)$ for every fixed $(x,u) \in \mathbb{R}^N \times \mathbb{R}^M$. □

From now on we will denote by $\omega(\delta, x)$ a function such that $\lim_{\delta\to 0} \omega(\delta, x) = 0$ and that is monotone increasing on $|x|$ and by $\omega(\delta)$ a function such that $\lim_{\delta \to 0} \omega(\delta) = 0$. In the same way we will denote by $K(|x|)$ a function monotone increasing on $|x|$ and by K an arbitrary constant. We remark that ω and K may possibly vary from line to line.

Lemma 2.2.4 *Under the hypotheses* (Ha1), (Ha2), (HB), (Hf2), (Hf3), (Hf4), *for every $t \in [0,T]$, for every $x \in \mathbb{R}^N$, for every $\varepsilon > 0$ if the control u_ε is such that*

$$v(t,x) + \varepsilon > \int_t^T f(s, y^{x,t}(s, u_\varepsilon), u_\varepsilon(s))\, ds \tag{2.2.5}$$

we have the following estimate

$$\parallel B(\cdot, u_\varepsilon(\cdot)) \parallel_{L^q(t,T)} \leq K(|x|) + K\varepsilon \tag{2.2.6}$$

where the constant K does not depend on u_ε and t. Moreover

$$\parallel B(\cdot, u_\varepsilon(\cdot)) \parallel_{L^1(t,T)} \leq K(|x|) + K\varepsilon. \tag{2.2.7}$$

Proof. We first verify that $v(t,x)$ is locally bounded in $(0,T)\times\mathbb{R}^N$. Let u_0 be given by hypothesis (Hf4), by (2.2.1) and (Hf2) we can write

$$v(t,x) \le \int_t^T |f(s,y^{x,t}(s;u_0),u_0(s)) - f(s,0,u_0(s))|\,ds + \int_t^T f(s,0,u_0(s))\,ds \le$$
$$\le \int_t^T C(s,|y^{x,t}(s;u_0)|,0)|y^{x,t}(s;u_0)|\,ds + \int_t^T f(s,0,u_0(s))\,ds.$$

Applying (2.2.4) in Lemma 2.2.2 we have that

$$v(t,x) \le \int_t^T C(s,\vartheta(|x|,u_0),0)\vartheta(|x|,u_0)\,ds + \int_t^T f(s,0,u_0(s))\,ds$$

which, together with Remark 2.2.3, implies that $v(t,x)$ is locally bounded in $(0,T)\times\mathbb{R}^N$, i.e.

$$v(t,x) \le K(|x|). \tag{2.2.8}$$

Let u_ε satisfy (2.2.5); by (HB) and (Hf3)

$$\int_t^T |B(s,u_\varepsilon(s))|^q\,ds \le 2^q \int_t^T (G(s)^q + D^q|u_\varepsilon(s)|^p)\,ds \le$$
$$\le 2^q\left\{\tilde G^q + \frac{D^q}{a}\left(\int_t^T f(s,y^{x,t}(s,u_\varepsilon),u_\varepsilon(s))\,ds + bT\right)\right\}$$

then, by hypothesis (2.2.5) and by (2.2.8), we can write

$$\int_t^T |B(s,u_\varepsilon(s))|^q\,ds < 2^q\left\{\tilde G^q + \frac{D^q}{a}(v(t,x)+\varepsilon+bT)\right\} \le$$
$$\le 2^q\left(\tilde G^q + \frac{D^q}{a}(K(|x|)+\varepsilon+b)\right) = K(|x|) + K\varepsilon$$

which gives us the estimate (2.2.6). Finally the (2.2.7) is a consequence of the Hölder's inequality. □

We are now in a position to prove that the value function is continuous in both variables.

Theorem 2.2.5 *If we suppose* (Ha1), (Ha2), (HB), (Hf2), (Hf3), (Hf4)*, then the value function $v(t,x)$ is a continuous function on $[0,T]\times\mathbb{R}^N$.*

Proof. We split the proof in 3 steps.
1) Continuity in x.
Fix $\varepsilon > 0$ and let u_ε be a control such that

$$v(t,w) + \varepsilon > \int_t^T f(s,y_\varepsilon^{w,t}(s),u_\varepsilon(s))\,ds$$

where $y_\varepsilon^{w,t}(s)$ is the solution of (2.1.14) with control u_ε and initial condition w at time t. By definition we have

$$v(t,x) \le \int_t^T f(s,y_\varepsilon^{x,t}(s),u_\varepsilon(s))\,ds$$

where similarly $y_\varepsilon^{x,t}(s)$ is the solution of (2.1.14) with control u_ε but initial condition x at time t. Then by (Hf2) and using Lemma 2.2.2,

$$\begin{aligned} v(t,x) - v(t,w) &\le \int_t^T f(s, y_\varepsilon^{x,t}(s), u_\varepsilon(s))\, ds - \int_t^T f(s, y_\varepsilon^{w,t}(s), u_\varepsilon(s))\, ds + \varepsilon \le \\ &\le A|w-x| \int_t^T C(s, \vartheta(|w|, u_\varepsilon), \vartheta(|x|, u_\varepsilon))\, ds + \varepsilon. \end{aligned}$$

By definition

$$\vartheta(|w|, u_\varepsilon) = A\{|w| + \tilde{M} + \| B(\cdot, u_\varepsilon(\cdot)) \|_{L^1(t,T)}\} \le A\{|w| + \tilde{M} + K(|x|) + K\varepsilon\}$$

by Lemma 2.2.4. Since w will tend to x we can suppose that there exists a constant K such that $|x| + |x-w| \le K$, then, using this in the previous estimate we get

$$\vartheta(|w|, u_\varepsilon) \le K(|x|, |x-w|) + K\varepsilon.$$

With a similar argument we also have

$$\vartheta(|x|, u_\varepsilon) \le A\{|x| + \tilde{M} + K(|x|) + K\varepsilon\} = K(|x|) + K\varepsilon.$$

Our estimate becomes

$$\begin{aligned} v(t,x) - v(t,w) &\le A|w-x| \int_t^T C(s, K(|x|, |x-w|) + K\varepsilon, K(|x|) + K\varepsilon) ds + \varepsilon \le \\ &\le A|w-x| \int_0^T C(s, K(|x|, |x-w|) + K\varepsilon, K(|x|) + K\varepsilon) ds + \varepsilon. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ we obtain, using also hypothesis (Hf2),

$$\begin{aligned} v(t,x) - v(t,w) &\le A|w-x| \int_0^T C(s, K(|x|, |x-w|), K(|x|)) ds = \\ &= K(|x|, |x-w|)|w-x|. \end{aligned}$$

By a similar argument we can easily get the reverse inequality, to obtain the following estimate

$$|v(t,x) - v(t,w)| \le K(|x|, |w-x|)|w-x| \tag{2.2.9}$$

which gives us the continuity in x of the value function.
2) Continuity in t.
We begin by proving the following inequality:

$$v(t,x) - v(t+\delta, x) \le K(|x|)\omega(\delta, x). \tag{2.2.10}$$

We consider first the case $\delta > 0$. By Dynamic Programming Principle we have

$$v(t,x) \le \int_t^{t+\delta} f(s, y^{x,t}(s; u_0), u_0)\, ds + v(t+\delta, y^{x,t}(t+\delta; u_0))$$

where the control u_0 satisfies hypothesis (Hf4). Then

$$v(t,x) - v(t+\delta,x) =$$

$$= v(t,x) - v(t+\delta, y^{x,t}(t+\delta;u_0)) + v(t+\delta, y^{x,t}(t+\delta;u_o)) - v(t+\delta,x) \leq$$

$$\leq \int_t^{t+\delta} f(s, y^{x,t}(s;u_0), u_0)\, ds + v(t+\delta, y^{x,t}(t+\delta;u_0)) - v(t+\delta,x). \tag{2.2.11}$$

Using hypothesis (Hf2) we can estimate

$$\int_t^{t+\delta} f(s, y^{x,t}(s;u_0), u_0)\, ds \leq$$

$$\leq \int_t^{t+\delta} \Big(f(s, y^{x,t}(s;u_0), u_0) - f(s,0,u_0)\Big)\, ds + \int_t^{t+\delta} f(s,0,u_0)\, ds \leq$$

$$\leq \int_t^{t+\delta} C(s, |y^{x,t}(s;u_0)|, 0)|y^{x,t}(s;u_0)|\, ds + \int_t^{t+\delta} f(s,0,u_0)\, ds \leq$$

$$\leq \vartheta(|x|,u_0) \int_t^{t+\delta} C(s, \vartheta(|x|,u_0), 0)\, ds + \int_t^{t+\delta} f(s,0,u_0)\, ds$$

where we used (2.2.4). Thus, by (Hf2) and (Hf4) we obtain:

$$\int_t^{t+\delta} f(s, y^{x,t}(s;u_0), u_0)\, ds \leq K(|x|)\omega(\delta,x). \tag{2.2.12}$$

In order to estimate the second part of (2.2.11) we need the following inequalities which can be easily deduced from Gronwall's Lemma, Hölder's inequality and hypotheses (Ha1), (Ha2):

$$|y^{x,t}(t+\delta;u_0) - x| \leq \omega(\delta,x) \quad , \quad |y^{x,t}(t+\delta;0) - x| \leq K(|x|). \tag{2.2.13}$$

Using the continuity in x of the value function, in particular (2.2.9), we can write

$$v(t+\delta, y^{x,t}(t+\delta;u_0)) - v(t+\delta,x) \leq |y^{x,t}(t+\delta;u_0) - x| K(|y^{x,t}(t+\delta;u_0) - x|, |x|)$$

that, by (2.2.13), becomes

$$v(t+\delta, y^{x,t}(t+\delta;u_0)) - v(t+\delta,x) \leq \omega(\delta,x)K(|x|). \tag{2.2.14}$$

Finally we can write the estimate for $\delta > 0$ inserting (2.2.12) and (2.2.14) in (2.2.11)

$$v(t,x) - v(t+\delta,x) \leq K(|x|)\omega(\delta,x). \tag{2.2.15}$$

We consider now the case $\delta < 0$. Fix $\varepsilon > 0$ and let u_ε^δ be the control such that

$$v(t+\delta,x) + \varepsilon > \int_{t+\delta}^T f(s, y^{x,t+\delta}(s;u_\varepsilon^\delta), u_\varepsilon^\delta)\, ds.$$

By definition of value function we get

$$v(t,x) \leq \int_t^T f(s, y^{x,t}(s; u_\varepsilon^\delta), u_\varepsilon^\delta)\; ds.$$

Then

$$v(t,x) - v(t+\delta, x) \leq$$

$$\leq \int_t^T f(s, y^{x,t}(s; u_\varepsilon^\delta), u_\varepsilon^\delta)\; ds - \int_{t+\delta}^T f(s, y^{x,t+\delta}(s; u_\varepsilon^\delta), u_\varepsilon^\delta)\; ds + \varepsilon \leq$$

$$\leq \int_t^T \Big(C(s, |y^{x,t}(s; u_\varepsilon^\delta)|, |y^{x,t+\delta}(s; u_\varepsilon^\delta)|) |y^{x,t}(s; u_\varepsilon^\delta) - y^{x,t+\delta}(s; u_\varepsilon^\delta)| \Big)\; ds + \varepsilon \tag{2.2.16}$$

where we used (Hf2) and that f is a positive function. By Lemma 2.2.2 and (2.2.7) we estimate

$$|y^{x,t}(s; u_\varepsilon^\delta)| \leq A\left(|x| + \tilde{M} + \| \; B(\cdot, u_\varepsilon^\delta(\cdot)) \; \|_{L^1(t,T)} \right) \leq$$

$$\leq A\left(|x| + \tilde{M} + \| \; B(\cdot, u_\varepsilon^\delta(\cdot)) \; \|_{L^1(t+\delta,T)} \right) \leq$$

$$\leq A\Big(|x| + \tilde{M} + (K(|x|) + K\varepsilon) \Big) = K(|x|) + K\varepsilon \tag{2.2.17}$$

and

$$|y^{x,t+\delta}(s; u_\varepsilon^\delta)| \leq A\Big(|x| + \tilde{M} + (K(|x|) + K\varepsilon) \Big) = K(|x|) + K\varepsilon. \tag{2.2.18}$$

By Lemma 2.2.2 we can estimate

$$|y^{x,t}(s; u_\varepsilon^\delta) - y^{x,t+\delta}(s; u_\varepsilon^\delta)| \leq$$

$$\leq A\left\{ \int_{t+\delta}^t (\alpha(s)(K(|x|) + K\varepsilon) + M(s))\; ds + (-\delta)^{1/q'}(K(|x|) + K\varepsilon) \right\} \tag{2.2.19}$$

were we used (2.2.18) and Lemma 2.2.4. Plugging (2.2.17), (2.2.18), and (2.2.19) in (2.2.16) we have

$$v(t,x) - v(t+\delta, x) \leq$$

$$\leq \int_t^T (C(s, K(|x|) + K\varepsilon, K(|x|) + K\varepsilon))\; ds \;\cdot$$

$$\cdot \left(A\left\{ \int_{t+\delta}^t (\alpha(s)(K(|x|) + K\varepsilon) + M(s))\; ds + (-\delta)^{1/q'}(K(|x|) + K\varepsilon) \right\} \right) + \varepsilon$$

and letting ε go to zero we obtain

$$v(t,x) - v(t+\delta,x) \leq$$

$$\leq \int_t^T C(s,K(|x|),K(|x|))\, ds \left(A \left\{ \int_{t+\delta}^t (\alpha(s)K(|x|) + M(s))\, ds + (-\delta)^{1/q'} K(|x|) \right\} \right).$$

By hypotheses (Ha1), (Ha2) and (Hf2) we can conclude

$$v(t,x) - v(t+\delta,x) \leq K(|x|)(\omega(\delta,x) + \omega(\delta)) = K(|x|)\omega(\delta,x)$$

which, with (2.2.15) gives us the required inequality. With the same arguments we obtain the reverse inequality, i.e.

$$v(t+\delta,x) - v(t,x) \leq K(|x|)\omega(\delta,x). \tag{2.2.20}$$

3) Final estimate.
Let $t \in [0,T]$, $\delta > 0$ and $x, w \in \mathbb{R}^N$, by (2.2.9), (2.2.10) and (2.2.20) we obtain

$$|v(t,x) - v(t+\delta,w)| \leq K(|x|)\omega(\delta,x) + K(|x|,|x-w|)|x-w|.$$

This implies

$$\lim_{\delta\to 0, w\to x} |v(t,x) - v(t+\delta,w)| = 0$$

which completes the proof. □

2.3 Hamilton-Jacobi-Bellman equation

We consider the (HJB) equation

$$\begin{cases} -\frac{\partial}{\partial t} v(t_0,x_0) + H(t_0,x_0,Dv(t_0,x_0)) = 0 \text{ in } (0,T)\times\mathbb{R}^N \\ v(T,x_0) = 0 \text{ in } \mathbb{R}^N \end{cases} \tag{2.3.1}$$

where $Dv(t,x)$ denotes the gradient with respect to the space variables of $v(t,x)$, and the Hamiltonian is given by

$$H(t,x,p) = \sup_{u\in\mathbb{R}^M} \{-p\cdot B(t,u) - f(t,x,u)\} - p\cdot a(t,x). \tag{2.3.2}$$

In the first chapter we have introduced the definition of viscosity solutions for continuous Hamiltonians. This definition has been adapted to the case of measurable time dependent Hamiltonians by Ishii in [Is85] in the following manner.

Definition 2.3.1 *Let $v(t,x)$ be a continuous function on $(0,T)\times \mathbb{R}^N$. We say that:*
$v(t,x)$ is a viscosity subsolution *of (2.3.1) in $(0,T)\times\mathbb{R}^N$ if the following condition is satisfied. If $b \in L^1(0,T)$, $F(t,x,p) \in C([0,T]\times\mathbb{R}^N\times\mathbb{R}^N)$, and F satisfies for some $\delta > 0$, $(t_0,x_0) \in (0,T)\times\mathbb{R}^N$, $\phi \in C^1([0,T]\times\mathbb{R}^N)$,*

$$-b(t) + F(t,x,p) \le H(t,x,p) \tag{2.3.3}$$

for $|x-x_0| < \delta$, $|p - D\phi(t_0,x_0)| < \delta$, a.e. $|t-t_0| < \delta$ and if

$$v(t,x) + \int_0^t b(s)ds - \phi(t,x)$$

has a local maximum at (t_0,x_0) then

$$-\frac{\partial}{\partial t}\phi(t_0,x_0) + F(t_0,x_0,D\phi(t_0,x_0)) \le 0.$$

$v(t,x)$ is a viscosity supersolution *of (2.3.1) in $(0,T)\times\mathbb{R}^N$ if the following condition is satisfied. If $b \in L^1(0,T)$, $F(t,x,p) \in C([0,T]\times\mathbb{R}^N\times\mathbb{R}^N)$, and F satisfies for some $\delta > 0$, $(t_0,x_0) \in (0,T)\times\mathbb{R}^N$, $\phi \in C^1([0,T]\times\mathbb{R}^N)$,*

$$-b(t) + F(t,x,p) \ge H(t,x,p) \tag{2.3.4}$$

for $|x-x_0| < \delta$, $|p - D\phi(t_0,x_0)| < \delta$, a.e. $|t-t_0| < \delta$ and if

$$v(t,x) + \int_0^t b(s)ds - \phi(t,x)$$

has a local minimum at (t_0,x_0) then

$$-\frac{\partial}{\partial t}\phi(t_0,x_0) + F(t_0,x_0,D\phi(t_0,x_0)) \ge 0.$$

$v(t,x)$ is a viscosity solution *of (2.3.1) in $(0,T)\times\mathbb{R}^N$ if it is both viscosity subsolution and viscosity supersolution of (2.3.1) in $(0,T)\times\mathbb{R}^N$ and if*

$$v(T,x) = 0 \;\forall\; x \in \mathbb{R}^N.$$

Starting from this work a lot of definitions have been given, depending on the regularity of the Hamiltonian and of the value function. For our case (continuous value function and measurable time-dependent Hamiltonian) we recall the work of Lions and Perthame [PLP87]. In this work they show that their definition and Ishii's one are in fact equivalent and they obtain various equivalent forms of this notion. In the proof of uniqueness we will use the following one.

Definition 2.3.2 *Let $v(t,x)$ be a continuous function on $(0,T)\times\mathbb{R}^N$. We say that:*
$v(t,x)$ is a viscosity subsolution *of (2.3.1) in $(0,T)\times\mathbb{R}^N$ if for every $\phi \in C^1(\mathbb{R}^N)$ and $b \in L^1(0,T)$, if (t_0,x_0) is a local maximum for*

$$v(t,x) + \int_0^t b(s)ds - \phi(x)$$

then

$$\lim_{\delta\downarrow 0^+} \operatorname*{ess\,inf}_{|t-t_0|<\delta} \inf\{H(t,x,p)+b(t) \ : \ |x-x_0|\le\delta \ , \ |p-D\phi(x_0)|\le\delta\} \le 0.$$

$v(t,x)$ *is a* viscosity supersolution *of (2.3.1) in* $(0,T)\times\mathbb{R}^N$ *if for every* $\phi\in C^1(\mathbb{R}^N)$ *and* $b\in L^1(0,T)$, *if* (t_0,x_0) *is a local minimum for*

$$v(t,x)+\int_0^t b(s)ds-\phi(x)$$

then

$$\lim_{\delta\downarrow 0^+} \operatorname*{ess\,inf}_{|t-t_0|<\delta} \inf\{H(t,x,p)+b(t) \ : \ |x-x_0|\le\delta \ , \ |p-D\phi(x_0)|\le\delta\} \ge 0.$$

$v(t,x)$ *is a* viscosity solution *of (2.3.1) in* $(0,T)\times\mathbb{R}^N$ *if it is both viscosity subsolution and viscosity supersolution of (2.3.1) in* $(0,T)\times\mathbb{R}^N$ *and if*

$$v(T,x)=0 \ \forall \ x\in\mathbb{R}^N.$$

In order to get the equivalence of these two notions we have to apply Theorem 1 in [PLP87]. Hypothesis (6) and (7) (page 615 of [PLP87]) can be deduced from the estimates on our Hamiltonian proved in Lemma 2.3.3 below. We need the following spaces.

$$\Gamma(a)=\{\gamma\in C([0,a]) \ : \ \gamma \text{ is nondecreasing in } [0,\infty) \ \forall s\ge 0\} \text{ for } a>0;$$

$$\begin{aligned}\Sigma=\{&\sigma\in C([0,\infty)\times[0,\infty)) \text{ such that}\\ &\sigma(\cdot,r)\in\Gamma(a) \text{ for every } r\ge 0 \quad \text{for every } a>0,\\ &r\to\sigma(s,r) \text{ is nondecreasing in } [0,\infty) \text{ for every } s\ge 0\};\end{aligned}$$

$$\begin{aligned}\Sigma_T=\{&\sigma:[0,\infty)\times[0,\infty)\times[0,\infty)\to[0,\infty) \text{ such that}\\ &\sigma(t,\cdot,\cdot)\in\Sigma \quad \text{for every } 0<t<T, \ \ \sigma(\cdot,s,r)\in L^1(0,T) \text{ for every } s,r\ge 0\};\end{aligned}$$

$$\begin{aligned}\Gamma_\gamma(a)=\{&\gamma:(0,T)\times[0,a]\to[0,\infty) \text{ such that}\\ &\gamma(t,\cdot)\in\Gamma(a) \quad \text{for every } t\in(0,T), \ \ \gamma(\cdot,s)\in L^1(0,T) \text{ for every } s\in[0,a]\}.\end{aligned}$$

Here and in the sequel we denote by $B(0,R)$ the ball of center 0 and radius R in $\mathbb{R}^N$.

Lemma 2.3.3 *If* $H(t,x,p)$ *is given by (2.3.2) and* (Ha1), (Ha2), (HB), (Hf2) *and* (Hf3) *are fulfilled then the following estimates holds.*

(HH1) *For some* $\sigma\in\Sigma_T$ *the inequality*

$$|H(t,x,p)-H(t,x,q)|\le\sigma(t,|p-q|,R)$$

holds for all $p,q\in B(0,R)$, $R>0$, $x\in B(0,\overline{C})$, $\overline{C}>0$ *and a.e.* $t\in(0,T)$.

(HH2) *For some* $a > 0$ *and* $\gamma \in \Gamma_\gamma(a)$ *the inequality*

$$|H(t,x,p) - H(t,y,p)| \leq \gamma(t, |x-y|(1+|p|))$$

holds for all $p \in \mathbb{R}^N$, $x, y \in B(0, \overline{C})$, $\overline{C} > 0$ *with* $|x-y|(1+|p|) \leq a$ *and a.e.* $t \in (0,T)$.

(HH3) $t \to H(t,x,p)$ *is integrable in* $(0,T)$ *for any* $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$.

Proof. First we remark that given $R > 0$ and a compact set $K \subset B(0,R) \times B(0,R) \subset \mathbb{R}^N \times \mathbb{R}^N$, there exists and increasing function $K_R(t) \in L^{p/q}(0,T)$ such that

$$H(t,x,p) = H_k(t,x,p) := \sup_{|u| \leq K_R(t)} \left\{ -p \cdot (a(t,x) + B(t,u)) - f(t,x,u) \right\}. \tag{2.3.5}$$

In fact, by (Ha1), (Ha2) and (HB), for every $(x,p) \in K$ we have

$$-p \cdot (a(t,x) + B(t,u)) - f(t,x,u) \leq M_R(t) + b + RD|u|^{p/q} - a|u|^p \tag{2.3.6}$$

where $M_R(t) = R|a(t,0)| + R^2\alpha(t) + RG(t) \in L^1(0,T)$. Since $q > 1$ this implies, for every fixed $t \in [0,T]$

$$\lim_{|u| \to \infty} \left\{ \sup_{(x,p) \in K} -p \cdot (a(t,x) + B(t,u)) - f(t,x,u) \right\} = -\infty.$$

This means in particular that

$$\forall C_R(t) > 0 \, \exists K_R(t) : \ |u| > K_R(t) \Rightarrow -p \cdot (a(t,x) + B(t,u)) - f(t,x,u) \leq C_R(t). \tag{2.3.7}$$

Since, by (Hf2), (Ha1) and (Ha2),

$$-p \cdot (a(t,x) + B(t,0)) - f(t,x,0) \geq -M_R(t) - |f(t,0,0)| - C(t,R,0)R$$

choosing $C_R(t) = M_R(t) + |f(t,0,0)| + C(t,R,0)R \in L^1(0,T)$ in (2.3.7) we can conclude that the supremum in the definition of H is attained for $|u| \leq K_R(t)$. We remark that by construction $C_R(t)$ is an increasing function, which implies that $K_R(t)$ is also increasing. Moreover the estimate (2.3.6) allows us to choose $K_R(t) \in L^{p/q}(0,T)$.
Let us now verify (HH1). By definition of H for every $\varepsilon > 0$ there exists a control u_ε such that

$$H(t,x,p) - \varepsilon < -p \cdot (a(t,x) + B(t,u_\varepsilon)) - f(t,x,u_\varepsilon)$$

and

$$H(t,x,q) \geq -p \cdot (a(t,x) + B(t,u_\varepsilon)) - f(t,x,u_\varepsilon).$$

Then we can estimate, using also (Ha1), (Ha2) and (HB)

$$\begin{aligned} H(t,x,p) - H(t,x,q) &\leq |p-q||a(t,x)| + |B(t,u_\varepsilon)||p-q| + \varepsilon \leq \\ &\leq |p-q|(\alpha(t)|x| + |a(t,0)| + G(t) + D|u|^{p/q}) + \varepsilon. \end{aligned}$$

By (2.3.5) we get, letting $\varepsilon \to 0$,

$$H(t,x,p) - H(t,x,q) \leq |p-q|(\alpha(t)\overline{C} + |a(t,0)| + G(t) + DK_R(t)^{p/q}).$$

With the same calculation we get the reverse inequality and we can conclude that

$$|H(t,x,p) - H(t,x,q)| \leq |p-q|(\alpha(t)\overline{C} + |a(t,0)| + G(t) + DK_R(t)^{p/q}).$$

Setting $\sigma(t,|p-q|,R) = |p-q|(\alpha(t)\overline{C} + |a(t,0)| + G(t) + DK_R(t)^{p/q})$ thanks to (Ha1), (Ha2) and (HB) it is easy to verify that $\sigma \in \Sigma_T$ so that (HH1) is fulfilled.
To prove (HH2) we fix again $\varepsilon > 0$ and take a u_ε such that

$$H(t,x,p) - \varepsilon < -p \cdot (a(t,x) + B(t,u_\varepsilon)) - f(t,x,u_\varepsilon).$$

Again by definition of H we get

$$H(t,x,p) - H(t,y,p) \leq$$

$$\leq -p \cdot (a(t,x) + B(t,u_\varepsilon)) - f(t,x,u_\varepsilon) + p \cdot (a(t,x) + B(t,u_\varepsilon)) + f(t,x,u_\varepsilon) + \varepsilon \leq$$

$$\leq |p|\alpha(t)|x-y| + C(t,|x|,|y|)|x-y| + \varepsilon$$

where we used also (Ha2) and (Hf2). Letting $\varepsilon \to 0$ we get

$$H(t,x,p) - H(t,y,p) \leq |p|\alpha(t)|x-y| + C(t,\overline{C},\overline{C})|x-y|.$$

The reverse inequality can be obtained in the same way and we can conclude

$$|H(t,x,p) - H(t,y,p)| \leq |x-y|(1+|p|)max(C(t,\overline{C},\overline{C}),\alpha(t))$$

which gives us (HH2) with $\gamma(t,|x-y|(1+|p|) = |x-y|(1+|p|)max(C(t,\overline{C},\overline{C}),\alpha(t))$. The (HH3) is an easy consequence of the definition of H and hypotheses (Ha1), (Ha2), (HB), (Hf2) and (Hf4). □

2.3.1 An existence result

We prove now that the value function (2.2.1) is a viscosity solution of the (HJB) equation (2.3.1).

Theorem 2.3.4 *If we suppose* (Ha1), (Ha2), (HB), (Hf2), (Hf3) *and if there exists a control function* $u_0 \in \mathbb{R}^M$ *such that, for every* $y \in Y$ *and for every* $n \in \mathbb{N}$

$$f(\cdot, y(\cdot), u_0) \in L^1(0,T;\mathbb{R}^N), \tag{2.3.8}$$

then, $v(t,x)$ *defined in (2.2.1) is a continuous viscosity solution of (2.3.1) in* $(0,T) \times \mathbb{R}^N$.

Proof. We observe that by definition of v we have

$$v(T, x_0) = 0 \quad \forall x_0 \in \mathbb{R}^N.$$

In order to show that $v(t,x)$ is a viscosity solution of (2.3.1) in $(0,T) \times \mathbb{R}^N$ we have to prove that it is a viscosity subsolution and a viscosity supersolution of (2.3.1) in $(0,T) \times \mathbb{R}^N$.
We start proving that $v(t,x)$ is a viscosity subsolution. Following the definition we take a test function $\phi \in C^1([0,T] \times \mathbb{R}^N)$, a $b \in L^1(0,T)$ and a function $F(t,x,p) \in C([0,T] \times \mathbb{R}^N \times \mathbb{R})$ such that (2.3.3) is fulfilled. Let (t_0, x_0) be a local maximum for $v(t,x) + \int_0^t b(s)\, ds - \phi(t,x)$, so there exists a $\delta > 0$ such that

$$|t - t_0| \leq \delta, |x - x_0| \leq \delta \Rightarrow \int_{t_0}^t b(s)\, ds + \phi(t_0, x_0) - \phi(t,x) \leq v(t_0, x_0) - v(t,x). \quad (2.3.9)$$

Let $u_0 \in \mathbb{R}^M$ be fixed satisfying hypothesis (2.3.8), and we denote by $y^0(t)$ the solution of our state equation (2.1.14) with initial condition $y(t_0) = x_0$ and control u_0. The Dynamic Programming Principle gives for every $\tau \in [t_0, T]$

$$v(t_0, x_0) \leq \int_{t_0}^{\tau} f(t, y^0(t), u_0)\, dt + v(\tau, y^0(\tau)). \quad (2.3.10)$$

By Gronwall's Lemma we can easily deduce that there exists a $\tau \in]t_0, t_0 + \delta]$ such that $|y^0(t) - x_0| \leq \delta$ for all $t \in]t_0, \tau]$. It follows that in (2.3.9) we can choose $x = y^0(\tau)$ and $t = \tau$ to get (using also (2.3.10))

$$\int_{t_0}^{\tau} b(s)\, ds + \phi(t_0, x_0) - \phi(\tau, y^0(\tau)) \leq \int_{t_0}^{\tau} f(t, y^0(t), u_0)\, dt. \quad (2.3.11)$$

Since

$$\phi(t_0, x_0) - \phi(\tau, y^0(\tau)) = \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} -\frac{d}{dt}\phi(\tau, y^0(t))\, dt =$$

$$= \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} -D\phi(\tau, y^0(t)) \cdot (a(t, y^0(t)) + B(t, u_0))\, dt$$

(2.3.11) becomes

$$\int_{t_0}^{\tau} b(s)\, ds + \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} -D\phi(\tau, y^0(t)) \cdot (a(t, y^0(t)) + B(t, u_0))\, dt \leq$$

$$\leq \int_{t_0}^{\tau} f(t, y^0(t), u_0)\, dt.$$

Dividing by $\tau - t_0$ and passing to the limit as $\tau \to t_0$ we have for a.e. t_0

$$b(t_0) - \frac{\partial}{\partial t}\phi(t_0, x_0) + \{-D\phi(t_0, x_0) \cdot (a(t_0, x_0) + B(t_0, u_0)) - f(t_0, x_0, u_0)\} \leq 0.$$

Taking the supremum over $u \in \mathbb{R}^M$ and recalling the definition of the Hamiltonian (2.3.2) we can write

$$b(t_0) - \frac{\partial}{\partial t}\phi(t_0, x_0) + H(t_0, x_0, D\phi(t_0, x_0)) \leq 0 \quad \text{a.e. with respect to } \ t_0$$

which implies

$$-\frac{\partial}{\partial t}\phi(t_0, x_0) + F(t_0, x_0, D\phi(t_0, x_0)) \leq 0$$

and this completes the subsolution proof.
To prove that v is a viscosity supersolution of (2.3.1) in $(0, T) \times \mathbb{R}^N$ we take $\phi \in C^1([0, T] \times \mathbb{R}^N)$, $b \in L^1(0, T)$ and a function $F(t, x, p) \in C([0, T] \times \mathbb{R}^N \times \mathbb{R})$ such that (2.3.4) is fulfilled. Let (t_0, x_0) be a local minimum for $v(t, x) + \int_0^t b(s)\ ds - \phi(t, x)$, so there exists a $\delta > 0$ such that

$$|t - t_0| \leq \delta, |x - x_0| \leq \delta \Rightarrow \int_{t_0}^{t} b(s)\ ds + \phi(t_0, x_0) - \phi(t, x) \geq v(t_0, x_0) - v(t, x). \tag{2.3.12}$$

Applying the Dynamic Programming Principle, for every $\varepsilon > 0$ there exists a u_ε such that, denoting by $y_\varepsilon^0(t)$ the solution of (2.1.14) with initial condition $y(t_0) = x_0$ and control u_ε,

$$v(t_0, x_0) + \varepsilon > \int_{t_0}^{\tau} f(t, y_\varepsilon^0(t), u_\varepsilon(t))\ dt + v(\tau, y_\varepsilon^0(\tau)) \tag{2.3.13}$$

for every $\tau \in [t_0, T]$. Then in (2.3.12) we have for $|x - x_0| \leq \delta$ and $|t - t_0| \leq \delta$

$$\int_{t_0}^{t} b(s)\ ds + \phi(t_0, x_0) - \phi(t, x) \geq$$

$$\geq \int_{t_0}^{\tau} f(t, y_\varepsilon^0(t), u_\varepsilon(t))\ ds - \varepsilon + v(\tau, y_\varepsilon^0(\tau)) - v(t, x). \tag{2.3.14}$$

For any $\tau \in [t_0, T]$, by the Hölder's inequality we have

$$\int_{t_0}^{\tau} |B(t, u_\varepsilon(t))|\ dt \leq \left\{ \int_{t_0}^{\tau} |B(t, u_\varepsilon(t))|^q\ dt \right\}^{1/q} (\tau - t_0)^{1/q'}$$

(where $1/q + 1/q' = 1$). From hypotheses (HB) and (Hf3) we deduce then

$$\int_{t_0}^{\tau} |B(t, u_\varepsilon(t))|^q\ dt \leq 2^q \tilde{G}^q + \frac{b}{a} 2^q (\tau - t_0) + \frac{2^q\ D^q}{a} \left(\int_{t_0}^{\tau} f(t, y_\varepsilon^0(t), u_\varepsilon(t))\ dt \right) \leq$$

$$\leq 2^q \tilde{G}^q + \frac{b}{a} 2^q (\tau - t_0) + \frac{2^q\ D^q}{a} (v(t_0, x_0) + \varepsilon)$$

where we used the (2.3.13). Moreover, since $v(t, x)$ is locally bounded, i.e. there exists a function $V(|x_0|)$ monotone increasing on $|x_0|$ such that $v(t_0, x_0) \leq V(|x_0|)$,

$$\int_{t_0}^{\tau} |B(t, u_\varepsilon(t))|\ dt \leq (\overline{B} + \frac{2^q\ D^q}{a} \varepsilon)^{1/q}\ (\tau - t_0)^{1/q'} \tag{2.3.15}$$

where $\overline{B} = 2^q(\tilde{G}^q + \frac{D^q}{a}\ V(|x_0|) + \frac{b}{a}T)$. Applying Gronwall's Lemma, from (Ha1), (Ha2), (HB) and (2.3.15) it follows that

$$|y^0_\varepsilon(\tau) - x_0| \leq A \int_{t_0}^{\tau} (\alpha(s)|x_0| + M(s))\, ds + (\overline{B} + \frac{2^q\ D^q}{a}\varepsilon)^{1/q}\ (\tau - t_0)^{1/q'}.$$

Since it is not restrictive to suppose $\varepsilon \leq 1$, we can fix a $\tau \in]t_0, t_0+\delta]$ such that $|y^0_\varepsilon(t) - x_0| \leq \delta$ for all $t \in]t_0, \tau]$. So, in the inequality (2.3.14) we can take $x = y^0_\varepsilon(\tau)$ and $t = \tau$ to obtain

$$\int_{t_0}^{\tau} b(s)\ ds + \phi(t_0, x_0) - \phi(\tau, y^0_\varepsilon(\tau)) \geq \int_{t_0}^{\tau} f(t, y^0_\varepsilon(t), u_\varepsilon(t))\ dt - \varepsilon. \tag{2.3.16}$$

Since

$$\phi(t_0, x_0) - \phi(\tau, y^0_\varepsilon(\tau)) =$$

$$= \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} -D\phi(\tau, y^0_\varepsilon(t)) \cdot (a(t, y^0_\varepsilon(t)) + B(t, u_\varepsilon(t)))\ dt$$

(2.3.16) becomes

$$\int_{t_0}^{\tau} b(s)\ ds + \int_{t_0}^{\tau} \Big[-D\phi(\tau, y^0_\varepsilon(t)) \cdot (a(t, y^0_\varepsilon(t)) + B(t, u_\varepsilon(t))) - f(t, y^0_\varepsilon(t), u_\varepsilon(t))\Big]\ dt +$$

$$+\phi(t_0, x_0) - \phi(\tau, x_0) \geq -\varepsilon. \tag{2.3.17}$$

Before passing to the limit as $\varepsilon \to 0$ we observe that in this case y^0_ε depends on ε, so we first estimate its distance from x_0 in all the terms. By hypothesis (Hf2), inequality (2.3.17) becomes

$$\int_{t_0}^{\tau} b(s)\ ds + \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} (-f(t, x_0, u_\varepsilon(t)) + C(t, |x_0| + \delta, |x_0|)\ \delta)\ dt +$$

$$+ \int_{t_0}^{\tau} \Phi|a(t, x_0) - a(t, y^0_\varepsilon(t))|\ dt + \int_{t_0}^{\tau} \omega(\delta)|a(t, x_0) + B(t, u_\varepsilon(t))|\ dt +$$

$$+ \int_{t_0}^{\tau} -D\phi(\tau, x_0) \cdot (a(t, x_0) + B(t, u_\varepsilon(t)))\ dt \geq -\varepsilon$$

where Φ is such that $|D\phi(y^0_\varepsilon(t), \tau)| \leq \Phi$. Using the estimate (2.3.15) and recalling the definition of the Hamiltonian (2.3.2) we obtain

$$\int_{t_0}^{\tau} b(s)\ ds + \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} H(t, x_0, D\phi(\tau, x_0))\ dt + \varepsilon + \delta \int_{t_0}^{\tau} \eta(t)\ dt +$$

$$+ \int_{t_0}^{\tau} \omega(\delta)|a(t, x_0)|\ dt + \omega(\delta)(\overline{B} + \frac{2^q\ D^q}{a}\varepsilon)^{1/q}(\tau - t_0)^{1/q'} \geq 0$$

where $\eta(t) = C(t, |x_0| + \delta, |x_0|) + \Phi\alpha(t) \in L^1(0,T)$ and $|a(t,x_0)| \in L^1(0,T)$.
It easy to check that $H(t, x_0, D\phi(\tau, x_0)) \in L^1(0,T)$, then passing to the limit as $\varepsilon \to 0$ we have

$$\int_{t_0}^{\tau} b(s)\, ds + \phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} H(t, x_0, D\phi(\tau, x_0))\, dt +$$

$$+\delta \int_{t_0}^{\tau} \eta(t)\, dt + \int_{t_0}^{\tau} \omega(\delta)|a(t,x_0)|\, dt + \omega(\delta)\overline{B}^{1/q}(\tau - t_0)^{1/q'} \geq 0. \tag{2.3.18}$$

Finally, dividing by $\tau - t_0$ and passing to the limit as $\delta \to 0$ (this implies $\tau \to t_0$), inequality (2.3.18) becomes

$$b(t_0) - \frac{\partial}{\partial t}\phi(t_0, x_0) + H(t_0, x_0, D\phi(t_0, x_0)) \geq 0 \ \text{ a.e. with respect to } \ t_0.$$

This implies

$$-\frac{\partial}{\partial t}\phi(t_0, x_0) + F(t_0, x_0, D\phi(t_0, x_0)) \geq 0$$

which proves that $v(t,x)$ is a viscosity supersolution of (2.3.1) and completes the proof. □

2.3.2 A uniqueness result

We begin this section by stating a general comparison result for the (HJB) equation (2.3.1). The proof is based on the local result (in the sense of initial data) of Ishii (Theorem 8.1 page 54 in [Is85]) and is generalized to a global case with the technique used in [Is97] where a similar result is proved in the case of continuous Hamiltonians.
This proof it is not standard for two reasons, the time measurability of the Hamiltonian and the unboundness of the control space.
In the case of continuous H but unbounded control set we recall also [BDL97] and [RS98] while for time measurable H we recall [BJ87], [Nu90] and [Nu92].

Theorem 2.3.5 *Assume that $H(t,x,p)$ satisfies* (HH1), (HH2), (HH3) *and*

(HH4) *the following inequality holds*

$$H(t, x, \lambda p) \leq \lambda H(t, x, p)$$

for every $0 < \lambda < 1$, $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in (0,T)$.

(HH5) *for every $0 < \lambda < 1$ and $\varepsilon > 0$ there exist $C > 0$ and $M_{\varepsilon,\lambda} > 0$ such that for all $x \in \mathbb{R}^N$*

$$H(t, x, \lambda p - \varepsilon\xi(x)x) \leq \lambda H(t, x, p) + \varepsilon C + M_{\varepsilon,\lambda} + (1 - \lambda)b$$

where $\xi(x) = \frac{1}{|x|^2+1}$ and $M_{\varepsilon,\lambda}$ is such that $\lim_{\varepsilon \to 0} M_{\varepsilon,\lambda} = 0$.

If $u, v \in C((0,T)\times \mathbb{R}^N)$ are respectively a viscosity subsolution and a viscosity supersolution of (2.3.1) such that

$$\inf_{(0,T)\times\mathbb{R}^N} v(t,x) > -\infty \quad \textit{and} \quad u(x,0) \le v(x,0)\ \forall x \in \mathbb{R}^N \tag{2.3.19}$$

then

$$u(x,t) \le v(x,t) \quad \textit{in } (0,T)\times\mathbb{R}^N.$$

Proof. Fix $0<\lambda<1$ and $\varepsilon>0$ we define

$$u^{\varepsilon,\lambda}(t,x) = \lambda u(t,x) - \varepsilon\frac{1}{2}\log(|x|^2+1) + t(\varepsilon C + M_{\varepsilon,\lambda} + (1-\lambda)b) \tag{2.3.20}$$

where $u(t,x)$ is given by hypothesis and $M_{\varepsilon,\lambda}$ and C are given in (HH5).
Assume that we have

(a) for every $\varepsilon>0$ and $0<\lambda<1$ the function $u^{\varepsilon,\lambda}(t,x)$ is a subsolution of (2.3.1);

(b) there exists $R>0$ such that

$$u^{\varepsilon,\lambda}(t,x) \le v(t,x)\ \forall\ (t,x) \in ((0,T)\times\partial B(0,R)) \cup (0\times\overline{B(0,R)}).$$

Then for a fixed $R>0$ given in (b) and thanks to (HH1), (HH2), (HH3) and (2.3.19) we can apply the local comparison Theorem 8.1 at page 54 in [Is85] to obtain

$$u^{\varepsilon,\lambda}(t,x) \le v(t,x) \text{ in } (0,T)\times B(0,R).$$

This implies

$$u^{\varepsilon,\lambda}(t,x) \le v(t,x) \text{ in } (0,T)\times\mathbb{R}^N. \tag{2.3.21}$$

Since by construction $\lim_{\lambda\to1}(\lim_{\varepsilon\to0} u^{\varepsilon,\lambda}(t,x)) = u(t,x)$, passing to the limit as $\varepsilon\to0$ first and then as $\lambda\to1$ in (2.3.21) we can conclude that

$$u(t,x) \le v(t,x) \text{ in } (0,T)\times\mathbb{R}^N.$$

So to end the proof we have to show (a) and (b).
Proof of (a). Our aim is to prove that $u^{\varepsilon,\lambda}(t,x)$ is a subsolution of (2.3.1) following the Definition 2.3.2. Let $\phi\in C^1(\mathbb{R}^N)$, $b\in L^1(0,T)$ and (t_0,x_0) a local maximum of

$$u^{\varepsilon,\lambda}(t,x) + \int_0^t b(s)ds - \phi(x) \tag{2.3.22}$$

the thesis is

$$\lim_{\delta\downarrow0^+}\ \operatorname*{ess\,inf}_{|t-t_0|<\delta}\ \inf\{H(t,x,p)+b(t)\ :\ |x-x_0|<\delta\ ,\ |p-D\phi(x_0)|\le\delta\} \le 0. \tag{2.3.23}$$

By definition of $u^{\varepsilon,\lambda}(t,x)$ (see (2.3.20)), (t_0,x_0) is a local maximum of

$$u(t,x)+\int_0^t \frac{1}{\lambda}(b(s)+N_{\varepsilon,\lambda})ds-\left(\frac{\varepsilon}{2\lambda}\log(|x|^2+1)+\frac{\phi(x)}{\lambda}\right)$$

where $N_{\varepsilon,\lambda}=\varepsilon C+M_{\varepsilon,\lambda}+(1-\lambda)b$. Since u is a subsolution of (2.3.1), choosing as test functions $\tilde{b}(t)=\frac{1}{\lambda}(b(t)+N_{\varepsilon,\lambda})$ and $\tilde{\phi}(x)=\frac{\varepsilon}{2\lambda}\log(|x|^2+1)+\frac{\phi(x)}{\lambda}$ and applying Definition 2.3.2 we obtain

$$\lim_{\delta\downarrow 0^+}\ \operatorname*{ess\,inf}_{|t-t_0|<\delta}\ \inf\left\{H(t,x,\tilde{p})+\tilde{b}(t)\ :\ |x-x_0|<\delta\ \ ,|\tilde{p}-D\tilde{\phi}(x_0)|\le\delta\right\}\le 0. \tag{2.3.24}$$

Using hypothesis (HH5) we have

$$\begin{aligned}H(t,x,p)+b(t)&=H(t,x,\lambda\frac{p}{\lambda})+\lambda\frac{b(t)}{\lambda}\le\\&\le H(t,x,\lambda\left(\frac{p}{\lambda}+\frac{\varepsilon\xi(x)x}{\lambda}\right)-\varepsilon\xi(x)x)+\lambda\frac{b(t)}{\lambda}\le\\&\le \lambda H(t,x,\left(\frac{p}{\lambda}+\frac{\varepsilon\xi(x)x}{\lambda}\right))+N_{\varepsilon,\lambda}+\lambda\frac{b(t)}{\lambda}.\end{aligned}$$

This implies

$$H(t,x,p)+b(t)\le\lambda\left[H(t,x,\left(\frac{p}{\lambda}+\frac{\varepsilon\xi(x)x}{\lambda}\right))+\frac{N_{\varepsilon,\lambda}}{\lambda}+\frac{b(t)}{\lambda}\right]. \tag{2.3.25}$$

We set $\eta=\lambda\delta<\delta$ then

$$\inf\{H(t,x,p)+b(t)\ :\ |x-x_0|<\delta\ \ ,|p-D\phi(x_0)|\le\delta\}\le$$

$$\le\inf\{H(t,x,p)+b(t)\ :\ |x-x_0|<\eta\ \ ,|p-D\phi(x_0)|\le\eta\}\le$$

$$\le\inf\left\{H(t,x,p)+b(t)\ :\ |x-x_0|<\eta\ \ ,\ \left|\frac{p}{\lambda}-\frac{D\phi(x_0)}{\lambda}\right|\le\delta\right\}\le$$

$$\le\lambda\inf\left\{H(t,x,\left(\frac{p}{\lambda}+\frac{\varepsilon\xi(x)x}{\lambda}\right))+\tilde{b}(t)\ :\ |x-x_0|<\eta\ \ ,\ \ \left|\frac{p}{\lambda}-\frac{D\phi(x_0)}{\lambda}\right|\le\delta\right\} \tag{2.3.26}$$

where we used (2.3.25). Setting $\tilde{p}=\frac{p}{\lambda}+\frac{\varepsilon\xi(x)x}{\lambda}$ we get

$$\left|\tilde{p}-D\tilde{\phi}(x_0)\right|\le\frac{\varepsilon}{\lambda}\left|\xi(x)x-\xi(x_0)x_0\right|+\left|\frac{p}{\lambda}-\frac{D\phi(x_0)}{\lambda}\right|$$

then (2.3.26) becomes

$$\inf\{H(t,x,p)+b(t)\ :\ |x-x_0|<\delta\ \ ,\ \ |p-D\phi(x_0)|\le\delta\}\le$$

$$\leq \lambda \inf \left\{ H(t,x,\tilde{p}) + \tilde{b}(t) \ : \ |x-x_0| \leq \eta \ \ , \ \ |\tilde{p} - D\tilde{\phi}(x_0)| \leq \left(\frac{\varepsilon}{\lambda}+1\right)\delta \right\}.$$

This implies

$$\operatorname*{ess\,inf}_{|t-t_0|\leq\delta} \ \inf \left\{ H(t,x,p) + b(t) \mid |x-x_0| \leq \delta \ \ , \ \ |p - D\phi(x_0)| \leq \delta \right\} \leq$$

$$\leq \operatorname*{ess\,inf}_{|t-t_0|\leq\eta} \ \inf \left\{ H(t,x,\tilde{p}) + \tilde{b}(t) \ : \ |x-x_0| \leq \eta \ \ , \ \ |\tilde{p} - D\tilde{\phi}(x_0)| \leq \left(\frac{\varepsilon}{\lambda}+1\right)\delta \right\}.$$

Then, letting $\delta \to 0$,(hence $\eta \to 0$) by using (2.3.24),

$$\lim_{\delta\downarrow 0^+} \ \operatorname*{ess\,inf}_{|t-t_0|<\delta} \ \inf \left\{ H(t,x,p) + b(t) \ : \ |x-x_0| < \delta \ \ , \ \ |p - D\phi(x_0)| \leq \delta \right\} \leq$$

$$\leq \lambda \lim_{\delta\downarrow 0^+} \ \operatorname*{ess\,inf}_{|t-t_0|<\eta} \ \inf \left\{ H(t,x,\tilde{p}) + \tilde{b}(t) \ : |x-x_0| < \eta, |\tilde{p} - D\tilde{\phi}(x_0)| \leq \left(\frac{\varepsilon}{\lambda}+1\right)\delta \right\} \leq 0.$$

This is (2.3.23) and completes the proof of (a).

Proof of (b). We first observe that by hypothesis (2.3.19) adding to u and v a positive constant, we may assume that $\inf_{(0,T)\times\mathbb{R}^N} v(t,x) \geq 0$. Moreover, since the maximum of two subsolutions is again a subsolution, we may assume, by replacing u by $\max\{u,0\}$, that $u \geq 0$ on $(0,T)\times\mathbb{R}^N$. (Since $H(t,x,0) \leq 0$ it is easy to see that $u(t,x) \equiv 0$ is a subsolution of (2.3.1)).
Next, fix any $K \geq 0$ and define $g \in C(\mathbb{R})$ by $g(t) = \min\{t, K\}$, if we prove that

$$u \text{ is a subsolution } \implies g(u) \text{ is a subsolution} \tag{2.3.27}$$

we may assume that u is bounded from above. Thus by definition of $u^{\varepsilon,\lambda}(t,x)$ if follows that there exists $R > 0$ such that

$$u^{\varepsilon,\lambda}(t,x) \leq v(t,x) \quad \forall (t,x) \in ((0,T)\times\partial B(0,R)) \cup (0\times\overline{B(0,R)}).$$

To end the proof we show then implication (2.3.27). Let g_n be a sequence such that

$$\lim_{n\to\infty} g_n(t) = g(t); \tag{2.3.28}$$

$$g_n \text{ is linear for } t \geq Kn; \tag{2.3.29}$$

$$0 < g_n'(t) < 1. \tag{2.3.30}$$

For example, for $n > 1$ we can choose,

$$g_n(t) = \begin{cases} t(1-\frac{1}{n}) & 0 \leq t \leq K - \frac{1}{n} \\ (t-\frac{1}{n})\frac{k-\left(K-\frac{1}{n}\right)\left(1-\frac{1}{n}\right)}{Kn-\left(K-\frac{1}{n}\right)} & K - \frac{1}{n} < t < K \\ \frac{t}{n} & t \geq Kn. \end{cases}$$

We want to show that if u is a subsolution then $g_n(u)$ is a subsolution. Let $\phi(x) \in C^1(\mathbb{R}^N)$, $b \in L^1(0,T)$ and (t_0, x_0) a local maximum of

$$g_n(u) + \int_0^t b(s)ds - \phi(x)$$

we have to prove that

$$\lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-t_0|<\delta} \inf \{H(t,x,p) + b(t) \ : \ |x - x_0| < \delta \ \ , \ \ |p - D\phi(x_0)| \leq \delta\} \leq 0. \tag{2.3.31}$$

Since it is not restrictive to suppose that $g_n(u)(t_0,x_0) + \int_0^{t_0} b(s)ds - \phi(x_0) = 0$ and $\phi(x) \geq Kn$ near x_0, (t_0,x_0) is also a local maximum of

$$u(t,x) - g_n^{-1}(\int_0^t -b(s)ds + \phi(x)) = u(t,x) + g_n^{-1}(\int_0^t b(s)ds) - g_n^{-1}(\phi(x))$$

by (2.3.29), thus by Definition 2.3.2

$$\lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-t_0|<\delta} \inf \left\{ H(t,x,\tilde{p}) + \frac{b(t)}{\lambda} \ : \ |x - x_0| \leq \delta \ \ , \ \ \left|\tilde{p} - \frac{D\phi(x_0)}{\lambda}\right| \leq \delta \right\} \leq 0 \tag{2.3.32}$$

where (2.3.29) allows us to choose $\lambda = \frac{1}{(g_n^{-1})'}$. Setting $\eta = \lambda\delta < \delta$ by (2.3.30), we have

$$\inf \{H(t,x,p) + b(t) \ : \ |x - x_0| \leq \delta \ \ , \ \ |p - D\phi(x_0)| \leq \delta\} \leq$$

$$\leq \inf \{H(t,x,p) + b(t) \ : \ |x - x_0| \leq \eta \ \ , \ \ |p - D\phi(x_0)| \leq \eta\} \leq$$

$$\leq \inf \left\{ H(t,x,p) + b(t) \ : \ |x - x_0| \leq \eta \ \ , \ \ \left|\frac{p}{\lambda} - \frac{D\phi(x_0)}{\lambda}\right| \leq \delta \right\}. \tag{2.3.33}$$

By hypothesis (HH4) we have

$$H(t,x,p) + b(t) = H(t,x,\lambda\frac{p}{\lambda}) + \lambda\frac{b(t)}{\lambda} \leq \lambda \left[H(t,x,\frac{p}{\lambda}) + \frac{b(t)}{\lambda} \right].$$

So (2.3.33) becomes

$$\inf \{H(t,x,p) + b(t) \ : \ |x - x_0| \leq \delta \ \ , \ \ |p - D\phi(x_0)| \leq \delta\} \leq$$

$$\leq \lambda \inf \left\{ H(t,x,\frac{p}{\lambda}) + \frac{b(t)}{\lambda} \ : \ |x - x_0| < \eta \ \ , \ \ \left|\frac{p}{\lambda} - \frac{D\phi(x_0)}{\lambda}\right| \leq \delta \right\}.$$

Choosing $\tilde{p} = \frac{p}{\lambda}$ and letting $\delta \to 0$ (hence $\eta \to 0$) we have

$$\lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-t_0|<\delta} \inf \{H(t,x,p) + b(t) \ : \ |x - x_0| \leq \delta \ \ , \ \ |p - D\phi(x_0)| \leq \delta\} \leq$$

$$\leq \lambda \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-t_0|<\eta} \inf \left\{ H(t,x,\tilde{p}) + \frac{b(t)}{\lambda} \ : \ |x - x_0| < \eta \ \ , \ \ \left|\tilde{p} - \frac{D\phi(x_0)}{\lambda}\right| \leq \delta \right\} \leq 0$$

by (2.3.32). This gives us (2.3.31) and implies that g_n is a subsolution. By (2.3.28) and applying the stability result of Ishii, Proposition 7.1 page 52 in [Is85] we can conclude that if u is a subsolution of (2.3.1) then $g(u)$ is a subsolution of (2.3.1) and this completes the proof. $\square$

We are finally ready to prove that the value function (2.2.1) is the unique viscosity solution of the (HJB) equation (2.3.1).

Theorem 2.3.6 *Let v be the value function defined in (2.2.1) and suppose* (Ha1), (Ha2) *with $\alpha(t) \in L^\infty(0,T)$,* (HB) *with $G(t) \in L^\infty(0,T)$,* (Hf2), (Hf3) *and (2.3.8) then v is the unique viscosity solution of the* (HJB) *equation (2.3.1) where the Hamiltonian is given in (2.3.2).*

Proof. Theorem 2.3.4 implies that $v(t,x)$ is a viscosity solution of (2.3.1), thus we can choose v as subsolution and supersolution in Theorem 2.3.5. Since $v(t,x) \geq 0$ by definition, hypothesis (2.3.19) is fulfilled. Moreover Lemma 2.3.3 gives us (HH1), (HH2) and (HH3). In order to verify (HH4) we have to fix $u \in \mathbb{R}^M$ and compute for $0 < \lambda < 1$

$$-\lambda p \cdot (a(t,x) + B(t,u)) - f(t,x,u) =$$

$$= \lambda \left[p \cdot (a(t,x) + B(t,u)) - f(t,x,u)\right] + (\lambda - 1) f(t,x,u) \leq$$

$$\leq \lambda H(t,x,p).$$

Taking the supremum over the controls u we have by definition $H(t,x,\lambda p) \leq \lambda H(t,x,p)$ so that (HH4) is fulfilled.
In order to prove (HH5) we first observe that by (Ha2)

$$(a(t,x) + B(t,u)) \cdot x \leq (a(t,0) + B(t,u)) \cdot x + \alpha(t) x^2$$

for all $x \in \mathbb{R}^N$. Moreover, since $\varepsilon\xi(x) \geq 0$, $|\xi(x)x| \leq 1$ and $\xi(x)x^2 \leq 1$,

$$\begin{aligned} -(a(t,x) + B(t,u)) \cdot (-\varepsilon\xi(x)x) &\leq -(|a(t,0)| + |B(t,u)|)|\xi(x)x|\varepsilon + \varepsilon\alpha(t)\xi(x)x^2 \leq \\ &\leq \varepsilon(2\alpha(t) + G(t) + D|u|^{p/q}) \end{aligned}$$

where we also used (HB) and (Ha1). Thus setting $C = 2 \parallel \alpha \parallel_\infty + \parallel G \parallel_\infty$ we obtain

$$-(a(t,x) + B(t,u)) \cdot (-\varepsilon\xi(x)x) \leq \varepsilon C + \varepsilon D |u|^{p/q}. \tag{2.3.34}$$

Fixed $0 < \lambda < 1$ we compute now, using (2.3.34) and (Hf3)

$$-\lambda p \cdot (a(t,x) + B(t,u)) + [-(a(t,x) + B(t,u)] \cdot (-\varepsilon\xi(x)x) - f(t,x,u) \leq$$

$$\leq \lambda \left[-p \cdot (a(t,x) + B(t,u)) - f(t,x,u)\right] + (\lambda - 1) f(t,x,u) + \varepsilon C + \varepsilon D|u|^{p/q} \leq$$

$$\leq \lambda H(t,x,p) + \varepsilon C + (1-\lambda) b + \varepsilon D |u|^{p/q} - (1-\lambda) a |u|^p$$

By a calculation we get that the function $\psi(u) = \varepsilon D|u|^{p/q} - (1-\lambda)a|u|^p$ has a maximum at $\overline{u} = (\frac{\varepsilon D}{a(1-\lambda)q})^{\frac{q}{p(q-1)}}$ so setting

$$M_{\varepsilon,\lambda} = \psi(\overline{u}) = \varepsilon D \left[\frac{\varepsilon D}{a(1-\lambda)q}\right]^{1/(q-1)} - (1-\lambda)a \left[\frac{\varepsilon D}{a(1-\lambda)q}\right]^{\frac{q}{q-1}}$$

we get

$$-\lambda p \cdot (a(t,x) + B(t,u)) + [-(a(t,x) + B(t,u)] \cdot (-\varepsilon\xi(x)x) - f(t,x,u) \leq$$

$$\leq \lambda H(t,x,p) + \varepsilon C + (1-\lambda)b + M_{\varepsilon,\lambda}.$$

Taking the supremum over $u \in \mathbb{R}^M$ we finally have

$$H(t,x,\lambda p - \varepsilon\xi(x)x) \leq \lambda H(t,x,p) + \varepsilon C + M_{\varepsilon,\lambda} + (1-\lambda)b.$$

Since by construction $\lim_{\varepsilon\to 0} M_{\varepsilon,\lambda} = 0$ we have verified (HH5). Thus Theorem 2.3.5 applies and we can conclude that $v(t,x)$ is the unique viscosity solution of (2.3.1). □

2.4 The linear quadratic case

We consider now the case of a linear state equation in the control variable u and a quadratic cost functional on the control variable u. The Hamiltonians can be explicitly written in this case, and the convergence result becomes more clear. For the sake of simplicity we will take $N = 1$ and $M = 1$. We want to minimize the cost functional

$$J_n(t,x,u) = \int_t^T (\frac{|u(s)|^2}{2} + g(y))\, ds \tag{2.4.1}$$

where y is the solution of

$$\begin{cases} y'(s) = a_n(s,y) + b_n(s)u(s) & s \in (t,T] \\ y(t) = x. \end{cases}$$

Our control space is $\mathcal{U} = L^2(0,T)$, we choose b_n in $L^\infty(0,T)$, so that (HB) is fulfilled with $p = q = 2$. We suppose that (a_n) verifies (Ha1), (Ha2), and that $g : \mathbb{R} \to [0,\infty]$ is a Borel function such that

(Hg) there exists a function $C(v,w)$, from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, increasing in v and w, such that for every $y, z \in \mathbb{R}$

$$|g(y) - g(z)| \leq C(|y|,|z|)|y-z|.$$

It is easy to see that (Hf1), (Hf2), (Hf3), (Hf4), and (HBf) are fulfilled. In Chapter 1, Section 1.3.2, we have seen how to characterize the Γ-limit problem under the following set of assumptions,

$$\begin{cases} a_n(\cdot, y) \to a(\cdot, y) \text{ weakly } L^1(0,T)\ \forall y \in \mathbb{R} \\ b_n(\cdot) \to b(\cdot) \qquad\qquad \text{a .e. } t \in (0,T) \end{cases} \tag{2.4.2}$$

$$\begin{cases} a_n(\cdot, y) \to a(\cdot, y) \text{ weakly } L^1(0,T) \quad \forall y \in \mathbb{R} \\ b_n(\cdot) \to b(\cdot) \qquad\quad \text{weakly}^* L^\infty(0,T) \\ b_n(\cdot)^2 \to \beta(\cdot) \qquad \text{weakly}^* L^\infty(0,T) \end{cases} \tag{2.4.3}$$

so we will not recall it here. This example is interesting because we can explicitly calculate the Hamiltonians in all cases. Indeed, at n fixed we have

$$H_n(t,x,p) = -pa_n(t,x) - g(x) + \frac{1}{2} b_n(t)^2 p^2. \tag{2.4.4}$$

From this expression it is trivial to pass to the limit and to obtain in the first case (hypothesis (2.4.2))

$$H(t,x,p) = -pa(t,x) - g(x) + \frac{1}{2} b(t)^2 p^2 \tag{2.4.5}$$

and in the second (hypothesis (2.4.3))

$$H(t,x,p) = -pa(t,x) - g(x) + \frac{1}{2} \beta(t)^2 p^2. \tag{2.4.6}$$

We observe that a direct calculation of the Hamiltonians of the Γ-limit problems obtained in Section 1.3.2 gives us the same expressions.
Note that if we assume $\alpha(t) \in L^\infty(0,T)$ in (Ha2), Theorem 2.3.6 applies to all these cases to ensure that the value functions are the unique viscosity solutions of the (HJB) equations (2.3.1) with Hamiltonians respectively given by (2.4.4), (2.4.5) and (2.4.6).

Remark 2.4.1 We observe that using Theorem 2.3.6, Proposition 2.1.4 and Theorem 2.1.2 we have proved the following result.

Corollary 2.4.2 *Assume* (Ha1), (Ha2) *with* $\alpha(t) \in L^\infty(0,T)$, $b_n(t) \in L^\infty(0,T)$, (Hf2), (Hf3), *(2.3.8) and (2.4.3). If v_n is the viscosity solution of (2.3.1) with Hamiltonian H_n given in (2.4.4) and v is the viscosity solution of (2.3.1) with Hamiltonian H given in (2.4.6) then*

$$\lim_{n\to\infty} v_n(t,x) = v(t,x) \textit{ for all } (t,x) \in [0,T] \times \mathbb{R}.$$

This is in fact a stability result for (HJB) equations related to optimal control problems. We observe that it is only a different way to get Proposition 7.1 of [Is85]. In this framework we recall also the convergence results proved in [Ch87] and [Bar93]. □

Remark 2.4.3 All this section can be done in the same way in the case $N > 1$ and $M > 1$. We just want to remark that in this case the control space is $\mathcal{U} = L^2(0, T; \mathbb{R}^M)$ and the coefficients $b_n(t)$ are $N \times M$ matrices, so that in hypothesis (2.4.3) one has to give the convergence of the matrix $b_n(t)b_n(t)^T$ instead of the convergence of $b_n(t)^2$. (We denote by $b_n(t)^T$ the transpose of the matrix $b_n(t)$). □

We end this chapter noting that in the linear quadratic case the existence of the viscosity solution of the (HJB) equation can be obtained also if we suppose $b_n \in L^2(0, T)$ and $u \in L^2(0, T)$ (this implies $q = 1$ which is not allowed in (HB)). The result is the following.

Theorem 2.4.4 *We assume* (Ha1), (Ha2), (Hg) *and* $b_n \in L^2(0, T)$. *If* J_n *is given in (2.4.1) and the control space is* $\mathcal{U} = L^2(0, T)$ *then the value function*

$$v_n(t, x) = \inf_{u \in \mathcal{U}} \{J_n(t, x, u)\}$$

is a viscosity solution of the (HJB) *equation*

$$\begin{cases} -\frac{\partial}{\partial t} v_n(t, x) - \frac{\partial}{\partial x} v_n(t, x) a_n(t, x) - g(x) + \frac{1}{2} b_n(t)^2 \left(\frac{\partial}{\partial x} v_n(t, x)\right)^2 = 0 \text{ in } (0, T) \times \mathbb{R} \\ v(T, x) = 0 \text{ in } \mathbb{R}. \end{cases}$$

Proof. This proof is a slight modification of the proof in the case $q > 1$. (For simplicity we set $b := b_n$, $v := v_n$, $a := a_n$, $J := J_n$ and $\alpha := \alpha_n$).
In the proof of the Dynamic Programming Principle (Theorem 2.2.1) and in Lemma 2.2.2 nothing changes. In Lemma 2.2.4 estimates (2.2.7) has to be obtained in the following way. Let u_ε satisfy (2.2.5), then

$$\begin{aligned} \int_t^T |b(s) u_\varepsilon(s)| ds &\leq \frac{1}{2} \parallel b \parallel_2^2 + \int_t^T \frac{|u_\varepsilon(s)|^2}{2} ds \leq \frac{1}{2} \parallel b \parallel_2^2 + \int_t^T (\frac{u_\varepsilon(s)^2}{2} + g(y)) ds \leq \\ &\leq \frac{1}{2} \parallel b \parallel_2^2 + v(t, x) + \varepsilon \leq \frac{1}{2} \parallel b \parallel_2^2 + K(|x|) + \varepsilon = \\ &= K(|x|) + K\varepsilon. \end{aligned}$$

The regularity of the value function can be proved similarly to what done in Theorem 2.2.5. More precisely, the continuity in x and the first estimate in the time variable (case $\delta > 0$) can be obtained exactly as for $q > 1$, while for the case $\delta < 0$ we can follow the proof until estimate (2.2.18). Then by Lemma 2.2.2 we have

$$\begin{aligned} & |y^{x,t}(s; u_\varepsilon^\delta) - y^{x,t+\delta}(s; u_\varepsilon^\delta)| \leq \\ & \leq A \left\{ \int_{t+\delta}^t \left[\alpha(s)(K(|x|) + K\varepsilon) + M(s) + |B(s) u_\varepsilon^\delta(s)| \right] ds \right\}. \end{aligned} \tag{2.4.7}$$

In order to estimate the last integral we calculate

$$\int_{t+\delta}^t |u_\varepsilon^\delta(s)|^2 \, ds \leq 2 \int_{t+\delta}^t \left(\frac{1}{2} |u_\varepsilon^\delta(s)|^2 + g(y) \right) \, ds \leq 2 \left(v(t + \delta, x) + \varepsilon \right) \leq K(|x|) + \varepsilon$$

where we used (2.2.8) and the property of u_ε^δ. Using this we obtain

$$\int_{t+\delta}^{t} |B(s)u_\varepsilon^\delta(s)|ds \leq \omega(\delta)\,[K(|x|)+\varepsilon]^{1/2}$$

that plugged in (2.4.7) gives

$$|y^{x,t}(s;u_\varepsilon^\delta) - y^{x,t+\delta}(s;u_\varepsilon^\delta)| \leq$$

$$\leq A\left\{\int_{t+\delta}^{t} \alpha(s)(K(|x|)+K\varepsilon) + M(s)ds + \omega(\delta)\,[K(|x|)+\varepsilon]^{1/2}\right\}. \tag{2.4.8}$$

Inserting (2.2.17), (2.2.18), and (2.4.8) in (2.2.16) we have

$$v(t,x) - v(t+\delta,x) \leq \int_t^T C(K(|x|)+K\varepsilon, K(|x|)+K\varepsilon)ds \,\cdot$$

$$\cdot\left(A\left\{\int_{t+\delta}^{t} (\alpha(s)(K(|x|)+K\varepsilon) + M(s))ds + \omega(\delta)\,[K(|x|)+\varepsilon]^{1/2}\right\}\right) + \varepsilon$$

and letting ε go to zero we obtain

$$v(t,x) - v(t+\delta,x) \leq$$

$$\leq \int_t^T C(K(|x|),K(|x|))ds\left(A\left\{\int_{t+\delta}^{t}(\alpha(s)K(|x|)+M(s))ds + \omega(\delta)K(|x|)\right\}\right).$$

This, hypotheses (Ha1), (Ha2) and (Hg) give

$$v(t,x) - v(t+\delta) \leq K(|x|)\omega(\delta,x).$$

Thus we have obtained the estimate (2.2.20) and we can conclude that $v \in C((0,T)\times\mathbb{R}^N)$ as in Theorem 2.2.5.
In order to show that $v(t,x)$ is a viscosity solution of the (HJB) equation the only difference with the proof of Theorem 2.3.4 is estimate (2.3.15) that can be obtained as follows

$$\begin{aligned}\int_{t_0}^{\tau} |b(s)u_\varepsilon(s)|ds &\leq \left(\int_{t_0}^{\tau}|b(s)|^2 ds\right)\left(\int_{t_0}^{\tau}|u_\varepsilon(s)|^2 ds\right) \leq \\ &\leq \left(\int_{t_0}^{\tau}|b(s)|^2 ds\right) 2\left(\int_{t_0}^{\tau}\left(\frac{|u_\varepsilon(s)|^2}{2} + g(y)\right)ds\right) \leq \\ &\leq \omega(\tau-t_0)(v(t_0,x_0)+\varepsilon) \leq \omega(\tau-t_0)(K(|x_0|)+\varepsilon).\end{aligned}$$

□

Chapter 3

A (HJB) equation with measures arising in Γ-convergence of optimal control problems

In Chapter 2 we have developed the theory of viscosity solution for the (HJB) equation related to Γ-limit of optimal control problems when measures do not appear. At the end of Chapter 1 we have described the Γ-limit problem for a linear quadratic optimal control problem under the hypotheses

$$\begin{cases} a_n(\cdot, y) \to a(\cdot, y) & \text{weakly } L^1(0,T)\ \forall y \in \mathbb{R} \\ b_n(t) \to b(t) & \text{weakly } L^2(0,T) \\ b_n(t)^2 \to \mu & \text{weakly}^*\ \mathcal{M}^+(0,T). \end{cases} \tag{3.0.1}$$

In this chapter we want to discuss the (HJB) equation of this limit problem and to give a definition of viscosity solution in this case. Since our aim is to justify the study of this Hamiltonian and to show the ideas behind our definition, we will describe the problem in the framework of Γ-limits of optimal control problems.
For every $n \in \mathbb{N}$ consider the state equation

$$\begin{cases} \dot{x}_n(t) = a_n(t, x_n) + b_n(t)u(t)\ t \in (\tau, T] \\ x_n(\tau) = z \end{cases}$$

where $b_n(t) \in L^2(0,T)$ and $u \in \mathcal{U} \equiv L^2(0,T)$ with $T < \infty$. The cost functionals are defined by

$$J_n(\tau, z, u) = \int_\tau^T \left(g(x_n(t)) + \frac{1}{2}|u(t)|^2 \right) dt$$

so that the value functions, $v_n(\tau, z) = \inf_{u \in \mathcal{U}} J_n(\tau, z, u)$, are viscosity solutions of the (HJB) equations

$$\begin{cases} -\frac{\partial v_n}{\partial t}(\tau, z) + H_n(\tau, z, \frac{\partial}{\partial x} v_n(\tau, z)) = 0 \text{ in } [0,T] \times \mathbb{R} \\ \qquad\qquad v_n(T, z) = 0 \text{ in } \mathbb{R} \end{cases}$$

having as Hamiltonian (this fact has been proved in Theorem 2.4.4 at the end of Section 2.4)

$$H_n(\tau, z, p) = -g(z) - p\, a_n(\tau, z) + \frac{1}{2}p^2 b_n(\tau)^2.$$

At the end of Section 1.3.2 we have described the Γ-limit problem under the convergence assumptions (3.0.1). We can write this as an optimal control problem having as state equation the following identity between measures

$$\begin{cases} \dot{x}(t) = (a(t,x) + v(t)) \cdot dt + \xi(t) \cdot \mu^s & \text{in } (\tau, T] \\ x(\tau^-) = z \end{cases}$$

where $v \in L^2(0,T)$ and $\xi \in L^2_{\mu^s}(0,T)$; and as cost functional

$$J(\tau, z, U) = \int_\tau^T f(t, x(t), U(t))dt + \int_{[\tau,T)} \frac{1}{2}\xi^2(t)d\mu^s$$

where $U(t) = (u(t), v(t), \xi(t)) \in \mathcal{U} = L^2(0,T) \times L^2(0,T) \times L^2_{\mu^s}(0,T)$ and

$$f(t, y, U) = \frac{|u|^2}{2} + g(y) + \frac{|v - bu|^2}{2(\mu^a - b^2)}.$$

In Remark 2.4.1 (Section 2.4) we proved that if we assume, instead of (3.0.1), that $b_n(t) \to b(t)$ weakly* $L^\infty(0,T)$ and $b_n(t)^2 \to \beta(t)$ weakly* $L^\infty(0,T)$ then the Hamiltonians H_n converge to the Hamiltonian of the Γ-limit problem. In order to extend this result to the measure case, we expect as Hamiltonian of the limit problem the limit of $H_n(\tau, z, p)$ under the convergence assumptions (3.0.1). This limit is given by

$$H(\tau, z, p) = -g(z) - p\; a(\tau, z) + \frac{1}{2}p^2 \cdot \mu. \tag{3.0.2}$$

Note that if we formally apply the definition of the Hamiltonian, i.e.

$$H(t, x, p) = \sup_{U \in \mathbb{R}^3} \left\{ -p(a(t,x) + v + \xi \cdot \mu^s) - f(t, x, U) - \frac{1}{2}\xi^2 \cdot (\mu^s)^2 \right\},$$

we also obtain the same expression (3.0.2). The problem is now to understand in which sense the value function, $v(\tau, z) = \inf_{U \in \mathcal{U}} J(\tau, z, U)$, is a solution of the (HJB) equation

$$\begin{cases} -\frac{\partial v}{\partial t}(\tau, z) + H(\tau, z, \frac{\partial v}{\partial x}(\tau, z)) = 0 \text{ in } [0,T] \times \mathbb{R} \\ v(T, z) = 0 \text{ in } \mathbb{R} \end{cases} \tag{3.0.3}$$

if μ^s is concentrated on a jump of Dv. In order to explain this problem in a precise way we consider the very special case $a_n = 0$, $g(z) = |z|^2$, $b_n(t) = 1_{]1,1+1/n[}(t)\sqrt{n}$ and $T = 2$, that will be detailed in Section 3.2. The limit (HJB) equation is formally given by

$$\begin{cases} -\frac{\partial v}{\partial t}(\tau, z) - |z|^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}(\tau, z)\right)^2 \cdot \delta_1 = 0 & \text{in } [0,2] \times \mathbb{R} \\ v(2, z) = 0 & \text{in } \mathbb{R} \end{cases} \tag{3.0.4}$$

and the limit value function can be explicitly computed:

$$v(\tau, z) = \begin{cases} |z|^2(\frac{4}{3} - \tau) & 0 \leq \tau \leq 1 \\ |z|^2(2 - \tau) & 1 < \tau \leq 2. \end{cases}$$

In this case it is clear that a difficulty arises at point $\tau = 1$ where $\frac{\partial v}{\partial x}$ has a jump and δ_1 is concentrated. Since $\frac{\partial v}{\partial t}(\tau, z) = -|z|^2 + \frac{2}{3}|z|^2 \cdot \delta_1$ the question is to understand the meaning of the term $\frac{1}{2}\left(\frac{\partial v}{\partial x}(\tau, z)\right)^2 \cdot \delta_1$ in order to obtain $\frac{2}{3}|z|^2 \cdot \delta_1$ from it.
Since this example is a linear quadratic problem we can suppose $v(\tau, z) = p(\tau)|z|^2$, the Riccati equation for p is then (the Riccati equation for an optimal control problem in finite dimension is clearly a particular case of the one described in Section 1.2.2)

$$\begin{cases} \dot{p}(\tau) = -1 + 2p(\tau) \cdot \delta_1 & \text{in } [0, 2) \\ p(2) = 0. \end{cases} \tag{3.0.5}$$

It is clear that we still have a problem for $\tau = 1$ but now we deal with an ordinary differential equation. This kind of problem has been widely studied in the literature. We can refer, for example to [Ri65], [Su78], [Br87], [BR88], [Mi89] or [DMR91]. In this work we refer in particular to [DMR91]. The basic idea is to consider a graph's reparametrization of the primitive of the measure appearing in the equation and to construct a more regular equivalent equation. Without going into details we only observe that computing the system $(E)_s$, page 741 of [DMR91], for the (3.0.5) we obtain

$$\begin{cases} \dot{p}(\tau) = -1 + \frac{2}{3} \cdot \delta_1 & \text{in } [0, 2) \\ p(2) = 0, \end{cases}$$

which is solved by $p(\tau)$ in the sense of equality between measures.
Since (3.0.3) is in general a partial differential equation we apply here the same method but starting from the state equations: hence we obtain one reparametrized optimal control problem at every fixed n and one for the limit problem. The advantage is that for this limit problem we can define the Hamiltonian and give a meaning to the (HJB) equation applying the result obtained in Chapter 2.
For the sake of simplicity we choose $N = 1$. The case $N > 1$ can be treated in a similar way, more precisely the functions $w_n(t)$ and $w(t)$ introduced in the following section have to contain all the terms b_i^2 converging to μ_i. Moreover this is a subcase of the nonlinear case described in Section 3.3.
This approach to optimal control problems has been already introduced in the literature, see for example [MR96]. But all the papers in this direction deal with problems where the measure appears in the state equation as control while in our case the measures are coefficients in the state equation. This is why our problem is different from the ones previously treated.
This chapter is organized as follows. In Section 3.1 we construct the reparametrized optimal control problems and we justify our definition. Section 3.2 is devoted to the description of an example in all details. In Section 3.3 we discuss in which direction it is possible to generalize this definition and what happens if we consider a fully nonlinear problem.
In Section 3.4 we prove the fundamental result about the linking between the optimal control

problem and the reparametrized one. To get this we prove the chain rule in dimension one for distributional derivatives of functions with bounded variation which are the composition of an absolutely continuous function with a monotone increasing function.

3.1 Viscosity solutions

We start recalling the main notations we will use in this chapter.

Definition 3.1.1 *A function $f:[0,T]\to\mathbb{R}$ is said to have bounded variation on the subinterval $[a,b]\subseteq[0,T]$ if there exists a constant $C\geq 0$ such that, for each finite set of points $t_0,t_1,...,t_p$ satisfying*

$$a=t_0<t_1<...<t_p=b$$

the inequality

$$\sum_{k=1}^{p}|f(t_k)-f(t_{k-1})|\leq C$$

holds. The least C which satisfies the above condition is called the variation of f on $[a,b]$, and is denoted by $V_a^b(f)$. The number $V_0^T(f)$ will be called the total variation of f.

The symbol $BV^-([0,T])$ will indicate the set of all left continuous functions with bounded total variation which are right continuous at 0. For every $u\in BV^-([0,T])$, the distributional derivatives $\dot{u}$ is a signed measure on $]0,T[$ characterized by

$$\dot{u}([t_1,t_2[)=u(t_2)-u(t_1)$$

for every subinterval $[t_1,t_2[\subseteq]0,T[$. In particular, for every $t\in]0,T[$,

$$\dot{u}(t)=[u](t)=u(t^+)-u(t^-)$$

where $u(t^+)=\lim_{s\to t^+}u(s)$ and $u(t^-)=\lim_{s\to t^-}u(s)$.
As we said in the Introduction our aim is to justify our definition in the framework of $\Gamma-$limits of optimal control problems. This is why instead of giving the definition directly we begin with the reparametrization of the optimal control problem at every fixed n. We recall that our state is the solution of the following equation

$$\begin{cases}\dot{x}_n(t)=a_n(t,x_n)+b_n(t)u(t)\ t\in(\tau,T]\\ x_n(\tau)=z\end{cases}\tag{3.1.1}$$

where $\tau\in[0,T]$, $b_n\in L^2(0,T)$ and $u\in L^2(0,T)$. Our control space is $\mathcal{U}=L^2(0,T)$. We will denote by $\dot{f}$ the t-derivative of a function f that can be in general a measure. The cost functional is

$$J_n(\tau,z,u)=\int_\tau^T\left(g(x_n(t))+\frac{1}{2}|u(t)|^2\right)dt\tag{3.1.2}$$

so that the value function is defined by

$$v_n(\tau,z)=\inf_{u\in\mathcal{U}}J_n(\tau,z,u).\tag{3.1.3}$$

We recall that on the Borel functions $a_n:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to[0,+\infty]$ we assume:

(Ha1) there exists a sequence of functions M_n from $[0,T]$ into $\mathbb{R}$ with $\| M_n \|_1 \leq M < \infty$ such that, for every $t \in [0,T]$

$$|a_n(t,0)| \leq M_n(t) \ \forall n \in \mathbb{N}; \tag{3.1.4}$$

(Ha2) there exists a sequence of functions α_n from $[0,T]$ into $\mathbb{R}$ with $\| \alpha_n \|_1 \leq \alpha < \infty$ such that, for every $t \in [0,T]$ and for every $x_1, x_2 \in \mathbb{R}$

$$|a_n(t,x_1) - a_n(t,x_2)| \leq \alpha_n(t)|x_1 - x_2| \ \forall n \in \mathbb{N}; \tag{3.1.5}$$

(Hg) there exists a function $C(v,w)$ from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, increasing in v and w such that, for every $y, z \in \mathbb{R}$

$$|g(y) - g(z)| \leq C(|y|,|z|)|y - z|. \tag{3.1.6}$$

We recall also that by Theorem 2.4.4, the value function v_n is a viscosity solution of the (HJB) equation

$$\begin{cases} -\frac{\partial v_n}{\partial t}(\tau,z) + H_n(\tau,z,\frac{\partial}{\partial x}v_n(\tau,z)) = 0 \text{ in } [0,T] \times \mathbb{R} \\ v_n(T,z) = 0 \text{ in } \mathbb{R} \end{cases} \tag{3.1.7}$$

where the Hamiltonian $H_n : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$H_n(\tau,z,p) = -g(z) - pa_n(\tau,z) + \frac{1}{2}p^2 b_n(\tau)^2. \tag{3.1.8}$$

Since the idea is to reparametrize the graph of the primitive of the measure μ that appears in the limit problem, by hypothesis (3.0.1), (more precisely $b_n(t)^2 \to \mu$ weakly* $\mathcal{M}^+(0,T)$), it is clear that at every fixed n we have to reparametrize the graph of a function $B_n(t)$ such that $\dot{B}_n(t) = b_n(t)^2$. Let us define $w_n(t) : [0,T] \to [0,1]$ by

$$w_n(t) = \frac{t + V_0^t(B_n)}{T + V_0^T(B_n)}$$

and

$$\varphi_n(s) = (\varphi_n^0(s), \varphi_n^1(s)) = (t, B_n(t)) \text{ if } \ s = w_n(t), \quad \forall s \in [0,1]. \tag{3.1.9}$$

If we change variable in the state equation (3.1.1) as follows

$$y_n(s) := x_n(\varphi_n^0(s)) \tag{3.1.10}$$

we obtain our reparametrized state equations,

$$\begin{cases} \frac{dy_n}{ds}(s) = \{a_n(\varphi_n^0(s), y_n(s)) + b_n(\varphi_n^0(s))u(\varphi_n^0(s))\} \frac{d\varphi_n^0}{ds}(s) \ s \in (s_1, 1] \\ y_n(s_1) = z \end{cases} \tag{3.1.11}$$

where s_1 is such that $\varphi_n^0(s_1) = \tau$.

Remark 3.1.2 It is easy to verify that by construction $x_n(t) = y_n(w_n(t))$ for all $t \in (\tau, T]$. □

We want to apply the same change of variable in the cost functional, so if $t = \varphi_n^0(s)$ formula (3.1.2) becomes

$$J_n(\tau, z, u) = \int_{s_1}^{1} \left(g(y_n(s)) + \frac{1}{2}|\tilde{u}(s)|^2 \right) \frac{d\varphi_n^0}{ds}(s)\ ds$$

by definition of $y_n(s)$ and where $\tilde{u}(s) = u(\varphi_n^0(s))$. Now, if we define

$$\overline{J}_n(s_1, z, \tilde{u}) = \int_{s_1}^{1} \left(g(y_n(s)) + \frac{1}{2}|\tilde{u}(s)|^2 \right) \frac{d\varphi_n^0}{ds}(s)\ ds \tag{3.1.12}$$

by construction we get

$$J_n(\tau, z, u) = \overline{J}_n(s_1, z, \tilde{u}),$$

and, since $\mathcal{U} \equiv L^2(0, T) = L^2_{\nu_n}(0, 1)$, (where $\nu_n = \frac{d\varphi_n^0}{ds}(s)ds$), we can conclude that

$$v_n(\tau, z) = \overline{v}_n(s_1, z) \tag{3.1.13}$$

where

$$\overline{v}_n(s_1, z) = \inf_{\tilde{u} \in \mathcal{U}} \overline{J}_n(s_1, z, \tilde{u}). \tag{3.1.14}$$

Remark 3.1.3 We underline that, since $\tau = \varphi_n^0(s_1)$ if and only if $s_1 = w_n(\tau)$ then (3.1.13) can be written as the relationship between the state equations (see Remark 3.1.2), i.e.

$$v_n(\tau, z) = \overline{v}_n(w_n(\tau), z). \tag{3.1.15}$$

□

Concerning the (HJB) equation of the problem (3.1.1)- (3.1.2) we can prove the following.

Theorem 3.1.4 *Assume (3.1.4), (3.1.5) with $\alpha(t) \in L^\infty(0, T)$ and (3.1.6). Then the value function $\overline{v}_n$ is the viscosity solution of the following (HJB) equation*

$$\begin{cases} -\frac{\partial}{\partial s}\overline{v}_n(s_1, z) + \overline{H}_n(s_1, z, \frac{\partial}{\partial y}\overline{v}_n(s_1, z)) = 0 \ in\ (0, 1) \times \mathbb{R} \\ \overline{v}_n(1, z) = 0 \ in\ \mathbb{R} \end{cases} \tag{3.1.16}$$

where the Hamiltonian $\overline{H}_n : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$\overline{H}_n(s_1, z, p) = \left(-p\ a_n(\varphi_n^0(s_1), z) - g(z) \right) \frac{d\varphi_n^0}{ds}(s_1) + \frac{1}{2}p^2 b_n(\varphi_n^0(s_1))^2 \frac{d\varphi_n^0}{ds}(s_1).$$

Proof. In order to apply Theorem 2.3.6 we set $a(s,y) = a_n(\varphi_n^0(s), y_n^1(s))\frac{d\varphi_n^0}{ds}(s)$. Since

$$\int_0^1 a_n(\varphi_n^0(s), y)\frac{d\varphi_n^0}{ds}(s)\ ds = \int_0^T a_n(t,y)\ dt$$

for every $y \in \mathbb{R}$, hypotheses (3.1.4) and (3.1.5) imply assumptions (Ha1) and (Ha2) with $\alpha(t) \in L^\infty(0,T)$.
If $B(t,u) = b(t)u(t)$ and $b(s) = b_n(\varphi_n^0(s))\left(\frac{d\varphi_n^0}{ds}(s)\right)^{1/2}$, $u(s) = u(\varphi_n^0(s))\left(\frac{d\varphi_n^0}{ds}(s)\right)^{1/2}$ setting $t = \varphi_n^0(s)$ we can easily deduce that $U \in L^2(0,1)$. Moreover, by construction

$$b(s) = b_n(\varphi_n^0(s))\left(\frac{d\varphi_n^0}{ds}(s)\right)^{1/2} = \frac{b_n(\varphi_n^0(s))}{\sqrt{(1+(b_n(\varphi_n^0(s)))^2)}}\sqrt{T + \int_0^T (b_n(\varphi_n^0(s)))^2)ds},$$

thus $b \in L^\infty(0,T)$ and (HB) is fulfilled with $q=2$, $D = \frac{1}{2}$ and $G(t) = \frac{b(t)^2}{2}$.
In order to verify assumptions (Hf2), (Hf3) and (2.3.8) we set

$$f(t,y,U) = \left(g(y_n(s)) + \frac{1}{2}|u(\varphi_n^0(s))|^2\right)\frac{d\varphi_n^0}{ds}(s)$$

then, (Hf2) follows from assumption (3.1.6), (2.3.8) is fulfilled with $u_0 \equiv 0$ and by

$$f(t,y,U) > \frac{1}{2}|u(\varphi_n^0(s))|^2\frac{d\varphi_n^0}{ds}(s) = \frac{1}{2}|U|^2$$

we obtain (Hf3). □

Furthermore, since by definition $b_n(\varphi_n^0(\cdot))^2 = \dot{B}_n(\varphi_n^0(\cdot))$ using (3.1.9) we get

$$b_n(\varphi_n^0(s))^2(\frac{d\varphi_n^0}{ds}(s)) = \frac{d\varphi_n^1}{ds}(s) \quad \forall s \in [0,1], \tag{3.1.17}$$

so that the Hamiltonian can be written in the simpler form

$$\overline{H}_n(s_1,z,p) = \left(-p\ a_n(\varphi_n^0(s_1),z) - g(z)\right)\frac{d\varphi_n^0}{ds}(s_1) + \frac{1}{2}\ p^2\ \frac{d\varphi_n^1}{ds}(s_1). \tag{3.1.18}$$

Remark 3.1.5 If we observe that

$$\overline{H}_n(s_1,z,p) = H_n(\tau,z,p)\frac{d\varphi_n^0}{ds}(s_1)$$

$$-\frac{\partial\overline{v}_n}{\partial s}(s_1,z) = -\frac{\partial v_n}{\partial t}(\tau,z)\frac{d\varphi_n^0}{ds}(s_1)$$

by an easy calculation we can conclude that v_n is a viscosity solution of (3.1.7) if and only if $\overline{v}_n$ is a viscosity solution of (3.1.16). For every fixed n the two problems are then equivalent. □

We consider now the limit problem. The state is given by the solution of the following system

$$\begin{cases} \dot{x}(t) = (a(t,x) + v(t)) \cdot dt + \xi(t) \cdot \mu^s \text{ in } (\tau, T] \\ x(\tau^-) = z. \end{cases} \tag{3.1.19}$$

For the definition of solution we refer to [DMR91], Definition 2.2 page 744; here we recall only that the solution $x(t) \in BV^-([0,T])$ is implicitly given by

$$x(t) = \int_\tau^t (a(s,x(s)) + v(s)) \; ds + \int_{[\tau,t)} \xi(s) \; d\mu^s \quad \forall t > \tau$$

$$x(t) = z \quad \forall t \leq \tau.$$

The cost functional and the value function are

$$J(\tau, z, U) = \int_\tau^T f(t, x(t), U(t))dt + \int_{[\tau,T)} \frac{1}{2}\xi^2(t)d\mu^s \tag{3.1.20}$$

$$v(\tau, z) = \inf_{U \in \mathcal{U}} \{J(\tau, z, U)\} \tag{3.1.21}$$

where $U(t) = (u(t), v(t), \xi(t))$, so that the control space is now $\mathcal{U} \equiv L^2(0,T) \times L^2(0,T) \times L^2_{\mu^s}(0,T)$. We recall that

$$f(t, y, U) = \frac{|u|^2}{2} + g(y) + \frac{|v - bu|^2}{2(\mu^a - b^2)}.$$

In order to reparametrize the limit problem we consider $B(t)$ such that $\dot{B}(t) = \mu(t)$ and we construct a *graph completion* of B following [DMR91]. Let $\mathcal{T}$ be a countable subset of $[0,T)$ which contains 0 and all the discontinuity points of B. Furthermore, let $(\psi_t)_{t\in\mathcal{T}}$ be a family of Lipschitz continuous maps from $[0,1]$ into $\mathbb{R}$ such that

$$\sum_{t\in\mathcal{T}} V_0^1(\psi_t) < \infty \qquad \text{and} \qquad \psi_t(0) = B(t^-) \qquad \psi_t(1) = B(t^+) \quad \forall t \in \mathcal{T} \tag{3.1.22}$$

(if $t = 0$ we require only $\psi_0(1) = B(0^+)$). Since $\mathcal{T}$ is countable we can write $\mathcal{T} = \{t_i : i \in \mathbb{N}\}$. For every $i \in \mathbb{N}$ let us set

$$a_i := V_0^1(\psi_{t_i}) \quad a := \sum_{i=1}^\infty a_i$$

and define $\mathcal{W} : [0,T] \to [0,1]$ as follows

$$\mathcal{W}(t) := \frac{1}{1+a}(w(t) + \sum_{t_i<t} a_i) \tag{3.1.23}$$

where

$$w(t) = \frac{t + V_0^t(B)}{T + V_0^T(B)}.$$

Let us define the graph completion of B corresponding to the family $(\psi_t)_{t\in\mathcal{T}}$ as follows

$$\varphi(s)=(\varphi^0(s),\varphi^1(s))=\begin{cases}(t,\ B(t)) & \text{if}\quad s=\mathcal{W}(t) \qquad t\in[0,T]\setminus\mathcal{T}\\ \\ (t_i,\ \psi_{t_i}\left(\frac{s-\mathcal{W}(t_i)}{[\mathcal{W}](t_i)}\right)) & \text{if}\quad s\in[\mathcal{W}(t_i),\mathcal{W}(t_i^+)]\ t_i\in\mathcal{T}.\end{cases} \tag{3.1.24}$$

Remark 3.1.6 We remark that if B is a continuous function then the graph completion is natural and coincides in fact with the change of variables defined in (3.1.10), so the reparametrizations of the problems at every fixed n and of the limit problem are exactly the same. □

We consider now the following system on the variable $y(s):[0,1]\to\mathbb{R}$

$$\begin{cases}\frac{dy}{ds}(s)=\{a(\varphi^0(s),y(s))+v(\varphi^0(s))\}\,\frac{d\varphi^0}{ds}(s)+\xi(\varphi^0(s))\tilde{\mu}(s) \qquad s\in(s_1,1]\\ y(s_1)=z\end{cases} \tag{3.1.25}$$

where $\tilde{\mu}(s)=\left(\frac{d\varphi^1}{ds}(s)-\mu^a(\varphi^0(s))\frac{d\varphi^0}{ds}(s)\right)$ and $s_1=\mathcal{W}(\tau)$ (this implies $\varphi^0(s_1)=\tau$ for all $\tau\in[0,T]$). The following theorem, that will be proved is Section 3.4, shows us that, using this reparametrization we can get the same relationship as in Remark 3.1.2 between the solution of (3.1.19) and of (3.1.25).

Theorem 3.1.7 *Let $B\in BV^-([0,T])$ and let $(\psi_t)_{t\in\mathcal{T}}$ be a family satisfying (3.1.22). If $y(s)$ is the solution of (3.1.25), then the map*

$$x(t):=y(\mathcal{W}(t)) \tag{3.1.26}$$

is a solution of (3.1.19).

With the same idea we consider the following cost functional

$$\overline{J}(s_1,z,U)=\int_{s_1}^1\left(f(\varphi^0(s),y(s),U(\varphi^0(s)))\frac{d\varphi^0}{ds}(s)+\frac{1}{2}\xi(\varphi^0(s))^2\tilde{\mu}(s)\right)ds$$

so that the value function will be

$$\overline{v}(s_1,z)=\inf_{U\in\mathcal{U}}\overline{J}(s_1,z,U). \tag{3.1.27}$$

We observe now that we can prove the same relationship as in (3.1.15) between the value functions:

Theorem 3.1.8 *If $\overline{v}$ and v are respectively defined in (3.1.27) and (3.1.21) then*

$$v(\tau,z)=\overline{v}(\mathcal{W}(\tau),z) \qquad \forall\tau\in[0,T].$$

(The proof will be given later in this Section). The main point is that now we can show that the value function $\overline{v}$ is the viscosity solution of the (HJB) equation related to the reparametrized problem.

Theorem 3.1.9 *Assume that the functions M_n and α_n in (3.1.4), (3.1.5) satisfy*

$$M_n \to \overline{M} \quad \text{weakly } L^1(0,T) \tag{3.1.28}$$

$$\alpha_n(t) \to \overline{\alpha} \quad \text{weakly } L^1(0,T) \tag{3.1.29}$$

then $\overline{v}$ defined in (3.1.27) is the viscosity solution of

$$\begin{cases} -\frac{\partial}{\partial s}\overline{v}(s_1,z)) + \overline{H}(s_1,z,\frac{\partial}{\partial y}\overline{v}(s_1,z)) = 0 \text{ in } (0,1)\times\mathbb{R} \\ \overline{v}(1,z) = 0 \text{ in } \mathbb{R} \end{cases} \tag{3.1.30}$$

with Hamiltonian

$$\overline{H}(s_1,z,p) = \Big(-p\, a(\varphi^0(s_1),z) - g(z)\Big)\frac{d\varphi^0}{ds}(s_1) + \frac{1}{2}\,p^2\,\frac{d\varphi^1}{ds}(s_1). \tag{3.1.31}$$

Proof. In this case we set $a(s,y) = a(\varphi^0(s),y^1(s))\frac{d\varphi^0}{ds}(s)$; then (3.1.28) and (3.1.29) imply (Ha1) and (Ha2) with $\alpha(t)\in L^\infty(0,T)$ (using the change of variable $t=\varphi^0(s)$ and hypotheses (3.1.4), (3.1.5)). We consider

$$B(s) = \left(0\ ,\ \left(\frac{d\varphi^0}{ds}(s)\right)^{1/2}\ ,\ (\tilde{\mu}(s))^{1/2}\right)$$

and

$$U(s) = \left(u(\varphi^0(s))\left(\frac{d\varphi^0}{ds}(s)\right)^{1/2}\ ,\ v(\varphi^0(s))\left(\frac{d\varphi^0}{ds}(s)\right)^{1/2}\ ,\ \xi(\varphi^0(s))\,(\tilde{\mu}(s))^{1/2}\right)$$

so that we can prove that $B\in L^\infty(0,1;\mathbb{R}^3)$ and $U\in L^2(0,1;\mathbb{R}^3)$ using Lemma 3.1.12 below. Hypothesis (HB) is then fulfilled with $q=2$, $D=\frac{1}{2}$ and $G(t)=\frac{|B(t)|}{2}^2$. With an abuse of notation we set

$$f(s,y,U) = f(y^0(s),y(s),U(y^0(s)))\frac{d\varphi^0}{ds}(s) + \frac{1}{2}\xi(\varphi^0(s))^2\tilde{\mu}(s).$$

Then if $u_0\equiv 0$ we obtain (2.3.8), the (Hf2) follows from (3.1.6) and from

$$f(s,y,U) > (a(|u|^2+|v|^2)-b)\frac{d\varphi^0}{ds}(s) + \frac{1}{2}|\xi|^2\tilde{\mu} \ge \min(a,1/2)|U|^2 - b\frac{d\varphi^0}{ds}(s)$$

we get the (Hf3). Theorem 2.3.6 applies then to this case too. □

Theorem 3.1.8 and Theorem 3.1.9 motivate the following definition.

Definition 3.1.10 *We say that a function $v(\tau,z):[0,T]\times\mathbb{R}\to\mathbb{R}$ is the viscosity solution of (3.0.3) with Hamiltonian (3.0.2) if there exist a family $(\psi_t)_{t\in\mathcal{T}}$ satisfying (3.1.22) and a function $\overline{v}(s,z):[0,1]\times\mathbb{R}\to\mathbb{R}$ such that $\overline{v}(\mathcal{W}(\tau),z)=v(\tau,z)$ and $\overline{v}(s,z)$ is the viscosity solution of (3.1.30) with Hamiltonian (3.1.31) where $\mathcal{W}(r)$ and $\varphi(s)=(\varphi^0(s),\varphi^1(s))$ are respectively given in (3.1.23) and (3.1.24).*

Remark 3.1.11 We observe now the term "containing the measure" in the Hamiltonians. At level n, looking at $\overline{H}_n(s_1, z, p)$ and $H_n(\tau, z, p)$ we get

$$\frac{1}{2}\, p^2\, \frac{d\varphi_n^1}{ds}(s) = \frac{1}{2}\, p^2\, b_n(\varphi_n^0(s))^2\, \frac{d\varphi_n^0}{ds}(s).$$

The idea is that our construction gives, in some sense, the same relationship between $\frac{1}{2}p^2\frac{d\varphi^1}{ds}(s)$ and $\frac{1}{2}p^2 \cdot \mu\frac{d\varphi^0}{ds}(s)$. This sense is given in the following Lemma.

Lemma 3.1.12 *For any function $g \in L^2_\mu(0,T)$,*

$$\int_\tau^T g(t)d\mu = \int_{s_1}^1 g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)ds.$$

$\square$

Proof of Lemma 3.1.12. For simplicity we give the proof for $\tau = 0$. First we isolate the contribution of the jumps of B in the first integral to obtain

$$\int_0^T g(t)d\mu(t) = \int_{[0,T]\setminus\mathcal{T}} g(t)d\mu + \sum_{t_i\in\mathcal{T}} g(t_i)\left(B(t_i^+) - B(t_i)\right). \tag{3.1.32}$$

So, by construction for every $t_i \in \mathcal{T}$ we get

$$\begin{aligned}\int_{\mathcal{W}(t_i)}^{\mathcal{W}(t_i^+)} g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)\, ds &= \int_{\mathcal{W}(t_i)}^{\mathcal{W}(t_i^+)} g(t_i)\frac{d\psi_{t_i}}{ds}(\frac{s-\mathcal{W}(t_i)}{[\mathcal{W}](t_i)})\,\frac{1}{[\mathcal{W}](t_i)}\, ds = \\ &= g(t_i)\left(B(t_i^+) - B(t_i)\right)\end{aligned}$$

then

$$\sum_{t_i\in\mathcal{T}} g(t_i)\left(B(t_i^+) - B(t_i)\right) = \sum_{t_i\in\mathcal{T}}\int_{\mathcal{W}(t_i)}^{\mathcal{W}(t_i^+)} g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)\, ds. \tag{3.1.33}$$

Now, if $t \in [0,T]\setminus\mathcal{T}$ by the general chain rule for distributional derivatives of function with bounded variation (see Theorem 3.1 page 746 in [DMR91]) we can deduce

$$B(t) = \varphi^1(\mathcal{W}(t)) \Longrightarrow \dot{B}(t) = \frac{d\varphi^1}{ds}(\mathcal{W}(t))\dot{\mathcal{W}}(t)$$

$$t = \varphi^0(\mathcal{W}(t)) \Longrightarrow dt = \frac{d\varphi^0}{ds}(\mathcal{W}(t))\dot{\mathcal{W}}(t)$$

(where the second identities are between measures) that imply

$$\int_{[0,T]\setminus\mathcal{T}} g(t)d\mu = \int_{[0,T]\setminus\mathcal{T}} g(t)\dot{B}(t) =$$

$$= \int_{[0,T]\setminus\mathcal{T}} g(t)\frac{d\varphi^1}{ds}(\mathcal{W}(t))\left(\frac{d\varphi^0}{ds}(\mathcal{W}(t))\right)^{-1} dt = \int_E g(\varphi^0(s))\frac{d\varphi^1}{ds}(s)\, ds, \tag{3.1.34}$$

where we have set $t = \varphi^0(s)$ and $E = [0,1] \setminus \cup_{t_i \in \mathcal{T}}[\mathcal{W}(t_i), \mathcal{W}(t_i^+)]$. Since

$$\sum_{t_i \in \mathcal{T}} \int_{\mathcal{W}(t_i)}^{\mathcal{W}(t_i^+)} g(\varphi^0(s)) \frac{d\varphi^1}{ds}(s)\, ds + \int_E g(\varphi^0(s)) \frac{d\varphi^1}{ds}(s)\, ds = \int_0^1 g(\varphi^0(s)) \frac{d\varphi^1}{ds}(s)\, ds$$

recalling (3.1.32), (3.1.33) and (3.1.34) we conclude the proof. □

Proof of Theorem 3.1.8. Fix $u \in \mathcal{U}$, by definition

$$\overline{J}(\mathcal{W}(\tau), z, U) = \int_{\mathcal{W}(\tau)}^1 \left(f(\varphi^0(s), y(s), U(\varphi^0(s))) \frac{d\varphi^0}{ds}(s) + \frac{1}{2}\xi(\varphi^0(s))^2 \tilde{\mu}(s) \right) ds =$$

$$= \int_\tau^T f(t, x(t), U(t)) dt + \int_{[\tau,T)} \frac{1}{2}\xi(t)^2 d\mu - \int_\tau^T \frac{1}{2}\xi(t)^2 \mu^a(t) dt$$

where we set $t = \varphi^0(s)$ and we used Lemma 3.1.12. Thus by definition (3.1.20) we obtain

$$\overline{J}(\mathcal{W}(\tau), z, U) = J(\tau, z, U). \tag{3.1.35}$$

Again by Lemma 3.1.12 and setting $t = \varphi^0(s)$ we can deduce that

$$\mathcal{U} \equiv L^2(0,T) \times L^2(0,T) \times L^2_{\mu^s}(0,T) = L^2_\nu(0,1) \times L^2_\nu(0,1) \times L^2_{\tilde{\mu}}(0,1)$$

where $\nu = \frac{d\varphi^0}{ds}(s)\, ds$, so that taking the infimum over $U \in \mathcal{U}$ in (3.1.35) we obtain the desired result. □

As we said in the Introduction, if we suppose (3.0.1), we have the following convergence for the Hamiltonians:

$$H_n(\cdot, z, p) \to H(\cdot, z, p) \text{ weakly}^* \mathcal{M}^+(0,T)$$

for every $(z,p) \in \mathbb{R} \times \mathbb{R}$. This implies, for the reparametrized problems the following convergence result.

Proposition 3.1.13 *For every $\psi \in C(0,T)$ and for every $(z,p) \in \mathbb{R} \times \mathbb{R}$,*

$$\lim_{n \to \infty} \int_0^1 \overline{H}_n(s, z, p) \psi(\varphi_n^0(s))\, ds = \int_0^1 \overline{H}(s, z, p) \psi(\varphi^0(s))\, ds.$$

Proof. Using the natural changes of variable, $t = \varphi_n^0(s)$ and $t = \varphi^0(s)$, we first obtain

$$\int_0^1 \left(-p\, a_n(\varphi_n^0(s), z) - g(z)\right) \psi(\varphi_n^0(s)) \frac{d\varphi_n^0}{ds}(s)\, ds = \int_0^T (-p\, a_n(t,z) - g(z))\, \psi(t)\, dt$$

$$\int_0^1 \left(-p\, a(\varphi^0(s), z) - g(z)\right) \psi(\varphi^0(s)) \frac{d\varphi^0}{ds}(s)\, ds = \int_0^T (-p\, a(t,z) - g(z))\, \psi(t)\, dt,$$

which imply, by assumption (3.0.1),

$$\lim_{n\to\infty}\int_0^1 \Big(-p\ a_n(\varphi_n^0(s),z)-g(z)\Big)\,\psi(\varphi_n^0(s))\frac{d\varphi_n^0}{ds}(s)\ ds =$$

$$=\int_0^1 \Big(-p\ a(\varphi^0(s),z)-g(z)\Big)\,\psi(\varphi^0(s))\frac{d\varphi^0}{ds}(s)\ ds.$$

Then, by hypothesis (3.0.1) we get

$$\lim_{n\to\infty}\int_0^T \psi(t)b_n(t)^2dt=\int_0^T \psi(t)d\mu(t)$$

so that to end the proof we have only to show that

$$\int_0^T \psi(t)b_n(t)^2dt=\int_0^1 \psi(\varphi_n^0(s))\frac{d\varphi_n^1}{ds}(s)\ ds \qquad \forall n\in\mathbb{N} \tag{3.1.36}$$

$$\int_0^T \psi(t)d\mu(t)=\int_0^1 \psi(\varphi^0(s))\frac{d\varphi^1}{ds}(s)\ ds. \tag{3.1.37}$$

The first one can be easily deduced setting $t=\varphi_n^0(s)$ and recalling (3.1.17), while the second one has been proved in Lemma 3.1.12. □

3.2 An example

We describe now an example where all details can be calculated. Our state equation for every fixed n is

$$\begin{cases} \dot x_n(t)=b_n(t)\ u(t)\ t\in(\tau,2] \\ x_n(\tau)=z \end{cases} \tag{3.2.1}$$

where $\tau\in[0,2]$, $b_n(t)=1_{]1,1+1/n[}(t)\sqrt{n}$ and $u\in L^2(0,2)=\mathcal{U}$. As usual we denote by $1_E(t)$ the function

$$1_E(t)=\begin{cases} 1 & t\in E \\ 0 & t\notin E. \end{cases}$$

As cost functional we take

$$J_n(\tau,z,u)=\int_\tau^2 \left(|x_n(t)|^2+\frac{1}{2}|u(t)|^2\right)dt$$

so that the value function will be a solution of the (HJB) equation (3.1.7) with Hamiltonian

$$H_n(\tau,z,p)=-|z|^2+\frac{1}{2}p^2b_n(\tau)^2. \tag{3.2.2}$$

The Γ-convergence theory gives us the following limit problem (see [BF93]). The state equation is the following equality between measures

$$\begin{cases} \dot{x}(t) = c \cdot \delta_1(t) \text{ in } (\tau, 2] \\ x(\tau^-) = z \end{cases}$$

where $c \in \mathbb{R}$ and δ_1 is the Dirac measure for $t = 1$. The cost functional is

$$J(\tau, z, c) = \int_\tau^2 |x(t)|^2 dt + \int_{[\tau,2)} \frac{c^2}{2} d\delta_1(t) = \begin{cases} |z|^2(1-\tau) + |z+c|^2 + \frac{1}{2}c^2 & 0 \le \tau \le 1 \\ |z|^2(2-\tau) & 1 < \tau \le 2 \end{cases}$$

which is discontinuous for $\tau = 1$ if $c \neq 0$. Moreover taking the infimum over $c \in \mathbb{R}$ we get

$$v(\tau, z) = \begin{cases} |z|^2(\frac{4}{3} - \tau) & 0 \le \tau \le 1 \\ |z|^2(2-\tau) & 1 < \tau \le 2 \end{cases}$$

which is discontinuous for $\tau = 1$!
We want to see if $v(\tau, z)$ is a viscosity solution of the (HJB) equation (3.0.3) with Hamiltonian

$$H(\tau, z, p) = -|z|^2 + \frac{1}{2}p^2 \cdot \delta_1$$

following the Definition 3.1.10. So we look for a function $\overline{v}(s, z)$ that will be the value function of the reparametrized limit problem. We consider then

$$B(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 1 & 1 < t \le 2 \end{cases}$$

we have $\dot{B}(t) = \delta_1(t)$ and the graph completion is

$$w(t) = \frac{t + V_0^t(B)}{T + V_0^T(B)} = \begin{cases} \frac{t}{3} & 0 \le t \le 1 \\ \frac{1}{3}(t+1) & 1 < t \le 2 \end{cases}$$

which is discontinuous in $t = 1$. So for the calculation of φ we have to use the following definition

$$\varphi(s) = \begin{cases} (t, B(t)) & \text{if} \quad s = w(t) \\ (t, B(t) + \frac{s - w(t)}{[w](t)}[B](t)) & \text{if} \quad s \in (w(t); w(t^+)). \end{cases}$$

We get

$$\varphi(s) = \begin{cases} (3s, 0) & 0 \le s \le \frac{1}{3} \\ (1, 3s-1) & \frac{1}{3} < s < \frac{2}{3} \\ (3s-1, 1) & \frac{2}{3} \le s \le 1. \end{cases}$$

We consider now the following system on the variable $y(s)$:

$$\begin{cases} \frac{dy}{ds}(s) = c \, \frac{d\varphi^1}{ds}(s) & s \in (s_1, 1] \\ y(s_1) = z. \end{cases} \tag{3.2.3}$$

As a cost functional we take

$$\overline{J}(s_1, z, c) = \int_{s_1}^{1} \left(|y(s)|^2 \frac{d\varphi^0}{ds}(s) + \frac{1}{2} c^2 \frac{d\varphi^1}{ds}(s) \right) ds$$

so that the value function will be $\overline{v}(s_1, z) = \inf_{c \in \mathbb{R}} \overline{J}(s_1, z, c)$, which is a continuous function. In fact

$$\overline{J}(s_1, z, c) = \begin{cases} 3|z|^2(\frac{1}{3} - s_1) + \frac{1}{2}c^2 + |z + c|^2 & 0 \le s_1 \le \frac{1}{3} \\ \frac{3}{2}c^2(\frac{2}{3} - s_1) + |2c + z - 3cs_1|^2 & \frac{1}{3} < s_1 < \frac{2}{3} \\ 3|z|^2(1 - s_1) & \frac{2}{3} \le s_1 \le 1 \end{cases}$$

and taking the infimum over $c \in \mathbb{R}$ we get

$$\overline{v}(s_1, z) = \begin{cases} |z|^2(\frac{4}{3} - 3s_1) & 0 \le s_1 \le \frac{1}{3} \\ \frac{|z|^2}{(5 - 6s_1)} & \frac{1}{3} < s_1 < \frac{2}{3} \\ 3|z|^2(1 - s_1) & \frac{2}{3} \le s_1 \le 1. \end{cases}$$

By simple calculations one can verify $\overline{v}(w(\tau), z) = v(\tau, z)$ and that $\overline{v}$ is a solution of the (HJB) equation

$$\begin{cases} -\frac{\partial}{\partial s}\overline{v}(s_1, z)) + \overline{H}(s_1, z, \frac{\partial}{\partial y}\overline{v}(s_1, z)) = 0 \text{ in } (0, 1) \times \mathbb{R} \\ \overline{v}(1, z) = 0 \text{ in } \mathbb{R} \end{cases} \tag{3.2.4}$$

with Hamiltonian

$$\overline{H}(s_1, z, p) = -|z|^2 \frac{d\varphi^0}{ds}(s_1) + \frac{1}{2} p^2 \frac{d\varphi^1}{ds}(s_1).$$

In fact, since

$$\frac{\partial \overline{v}}{\partial s}(s_1, z) = \begin{cases} -3|z|^2 & 0 \le s_1 \le \frac{1}{3} \\ \frac{6|z|^2}{(5 - 6s_1)^2} & \frac{1}{3} < s_1 < \frac{2}{3} \\ -3|z|^2 & \frac{2}{3} \le s_1 \le 1 \end{cases}$$

$$\frac{\partial \overline{v}}{\partial y}(s_1, z) = \begin{cases} 2z(\frac{4}{3} - 3s_1) & 0 \le s_1 \le \frac{1}{3} \\ \frac{2z}{(5 - 6s_1)^2} & \frac{1}{3} < s_1 < \frac{2}{3} \\ 6z(1 - s_1) & \frac{2}{3} \le s_1 \le 1 \end{cases}$$

setting $p = \frac{\partial \overline{v}}{\partial y}$, we obtain that (3.2.4) is fulfilled. Thus Definition 3.1.10 is verified.
In order to complete the example we calculate also the reparametrization of the problem at level n. Let $B_n(t)$ such that $\dot{B}_n(t) = b_n(t)^2$, i.e.

$$B_n(t) = \begin{cases} 0 & 0 \le t \le 1 \\ n(t - 1) & 1 < t < 1 + \frac{1}{n} \\ 1 & 1 + \frac{1}{n} \le t \le 2 \end{cases}$$

so the function w_n is given by

$$w_n(t) = \begin{cases} \frac{t}{3} & 0 \le t \le 1 \\ \frac{t}{3}(1+n) - \frac{n}{3} & 1 < t < 1 + \frac{1}{n} \\ \frac{1}{3}(t+1) & 1 + \frac{1}{n} \le t \le 2. \end{cases}$$

Then the $\varphi_n(s)$, which by definition is

$$\varphi_n(s) = (\varphi_n^0(s), \varphi_n^1(s)) = (t, B_n(t)) \quad \text{if} \quad s = w_n(t)$$

becomes

$$\varphi_n(s) = \begin{cases} (3s, 0) & 0 \le s \le \frac{1}{3} \\ (\frac{3s+n}{1+n}, \frac{n}{1+n}(3s-1)) & \frac{1}{3} < s < \frac{2n+1}{3n} \\ (3s-1, 1) & \frac{2n+1}{3n} \le s \le 1. \end{cases}$$

The reparametrized state equation is

$$\begin{cases} \frac{dy_n}{ds}(s) = b_n(\varphi_n^0(s)) u(\varphi_n^0(s)) \frac{d\varphi_n^0}{ds}(s) & s \in (s_1, 1] \\ y_n(s_1) = z \end{cases}$$

where s_1 is such that $\varphi_n^0(s_1) = \tau$. The reparametrized cost functional is by definition

$$\overline{J}_n(s_1, z, \tilde{u}) = \int_{s_1}^1 \left(|y_n(s)|^2 + \frac{1}{2} |\tilde{u}(s)|^2 \right) \frac{d\varphi_n^0}{ds}(s)\, ds$$

while the value function is

$$\overline{v}_n(s_1, z) = \inf_{\tilde{u} \in U} \overline{J}_n(s_1, z, \tilde{u}).$$

The (HJB) equation is

$$\begin{cases} -\frac{\partial}{\partial s} \overline{v}_n(s_1, z) + \overline{H}_n(s_1, z, \frac{\partial}{\partial y} \overline{v}_n(s_1, z)) = 0 \text{ in } [0,1] \times \mathbb{R} \\ \overline{v}_n(1, z) = 0 \text{ in } \mathbb{R} \end{cases} \tag{3.2.5}$$

with Hamiltonian:

$$\overline{H}_n(s_1, z, p) = -|z|^2 \frac{d\varphi_n^0}{ds}(s_1) + \frac{1}{2}\, p^2 \frac{d\varphi_n^1}{ds}(s_1).$$

The convergence of the Hamiltonians is in this case stronger than the one of the general framework (see Proposition 3.1.13). In fact, it is clear that the pointwise convergence of the Hamiltonians is a consequence of the following limits

$$\lim_{n \to \infty} \frac{d\varphi_n^0}{ds}(s) = \frac{d\varphi^0}{ds}(s)$$

$$\lim_{n \to \infty} \frac{d\varphi_n^1}{ds}(s) = \frac{d\varphi^1}{ds}(s)$$

which can be easily calculated. Thus we have the following result

$$\lim_{n \to \infty} \overline{H}_n(s, z, p) = \overline{H}(s, z, p) \quad \forall s \in [0, 1]$$

for every fixed $(z, p) \in \mathbb{R} \times \mathbb{R}$.

Remark 3.2.1 We remark that in this example one can verify also the convergence of the optimal controls and of the optimal trajectories. This is in fact a consequence of the Γ-convergence of the optimal control problems. □

Remark 3.2.2 As we observed in the limit problem the value function v it is not continuous in the t-variable, but it is left continuous and only right lower semi-continuous. This regularity can be verified also in the general case with arguments similar to the ones used in Theorem 2.2.5. Thus the reparametrized limit problem has also the advantage of having a continuous value function. □

3.3 A possible generalization to the nonlinear case

All this section is a joint work in progress with Prof. Rampazzo, so we will give just the main ideas. For the complete description of the results we refer to a forthcoming paper. We consider a problem with a nonlinear state equation, i.e.

$$\begin{cases} \dot{x}_n(t) = a^n(t,x) + \sum_{i=1}^{M} g_i^n(t,x) b_i^n(t) u_i(t) & t \in (\tau, T] \\ x_n(\tau) = z \end{cases} \tag{3.3.1}$$

and a quadratic cost functional

$$J_n(\tau, z, u) = \int_\tau^T \left(l_0(t,x) + \sum_{i=1}^{M} l_i(t,x) u_i(t) + \sum_{i=1}^{M} \tilde{l}_i(t,x) u_i(t)^2 \right) dt. \tag{3.3.2}$$

We will refer to this problem as problem $(\mathcal{P}_n)$.
We suppose $g_i^n : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$, $b_i^n(t) \in L^2(0,T)$, $u_i(t) \in L^2(0,T) := \mathcal{U}$ $(i = 1, ..., M)$, and for $i = 1, ..., M$ we assume $l_0(t,x) : [0,T] \times \mathbb{R}^N \to [0,\infty]$, $l_i(t,x) : [0,T] \times \mathbb{R}^N \to [0,\infty]$, $\tilde{l}_i(t,x) : [0,T] \times \mathbb{R}^N \to [0,\infty]$ and hypotheses (Ha1)-(Ha2) for $a^n(t,x)$ and $g_i^n(t,x)$, (Hf2) for $l_0(t,x)$, $l_i(t,x)$, $\tilde{l}_i(t,x)$ and (Hf3) for $\tilde{l}_i(t,x)$.
The value function is defined as usual as

$$v_n(t,x) = \inf_{u \in \mathcal{U}} \{ J_n(t,x,u) \} \tag{3.3.3}$$

and we can see that it is a viscosity solution of the (HJB) equation

$$\begin{cases} -\frac{\partial}{\partial t} v_n(t,x) + H_n(t,x,Dv_n(t,x)) = 0 \text{ in } (0,T) \times \mathbb{R}^N \\ v_n(T,x) = 0 \text{ in } \mathbb{R}^N \end{cases}$$

where the Hamiltonian is

$$H_n(t,x,p) = -p \cdot a^n - l_0 + \frac{1}{4} l^t L^{-1} l + \frac{1}{2} p^t G^n B^n L^{-1} l + \frac{1}{4} p G^n B^n L^{-1} B^n (G^n)^t p^t. \tag{3.3.4}$$

For simplicity, we have set

$$L = \begin{pmatrix} \tilde{l}_1 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \tilde{l}_M \end{pmatrix}$$

$$B^n = \begin{pmatrix} b_1^n & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & b_M^n \end{pmatrix}$$

$$U = \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_M \end{pmatrix}$$

$$l = \begin{pmatrix} l_1 \\ \cdot \\ \cdot \\ \cdot \\ l_M \end{pmatrix}$$

$$G^n = \begin{pmatrix} (g_1^1)^n & \cdots & (g_1^M)^n \\ (g_1^2)^n & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ (g_1^N)^n & \cdots & (g_N^M)^n \end{pmatrix}$$

so that (3.3.1) and (3.3.2) respectively become

$$\begin{cases} \dot{x}_n(t) = a^n(t,x) + G^n(t,x)\, B^n(t)\, U(t) & t \in (\tau, T] \\ x_n(\tau) = z \end{cases}$$

$$J_n(\tau, z, u) = \int_\tau^T \Big(l_0(t,x) + l(t,x)\, U^t(t) + U^t(t)\, L(t,x)\, U(t) \Big)\, dt.$$

In the particular case $g_i^n(t,x) \equiv 1$, $l_0(t,x) \equiv 0$ and $l_i(t,x) \equiv 0$ for $i = 1, ..., M$ we have seen in Chapter 1 how the Γ-limit problem can be characterized under different sets of hypotheses. We will always assume that

$$\begin{cases} a^n(\cdot, x) \to a(\cdot, x) \text{ weakly } L^1(0,T;\mathbb{R}^N) & \forall y \in \mathbb{R}^N \\ g_i^n(\cdot, y) \to g(\cdot, y) \ \ L^2(0,T;\mathbb{R}^N) & \forall y \in \mathbb{R}^N \end{cases} \tag{3.3.5}$$

and we consider three different cases. First, if we suppose that

$$b_i^n(t) \to b_i(t) \quad \text{a. e. on } [0,T], \tag{3.3.6}$$

Buttazzo and Dal Maso have proved in [BDM82] that the limit problem has the same form as problem $(\mathcal{P}_n)$. More precisely

$$(\mathcal{P}) \qquad \min\left\{ \int_\tau^T \left(l_0(t,x) + \sum_{i=1}^M l_i(t,x) u_i(t) + \sum_{i=1}^M \tilde{l}_i(t,x) u_i(t)^2 \right) dt \ : \ (x,u) \in \mathcal{A} \right\}$$

where

$$\mathcal{A} = \left\{ (x,u) \in Y \times U : x' = a(t,x) + \sum_{i=1}^{M} g_i(t,x) b_i(t) u_i(t) \ ; \ x(\tau) = z \right\}$$

($Y = W^{1,1}(0,T;\mathbb{R}^N)$). Second, Buttazzo and Cavazzuti in [BC89] have proved that if

$$\begin{cases} b_i^n(t) \in L^\infty(0,T) \\ b_i^n(t) \to b_i(t) & \text{weakly}^* L^\infty(0,T) \\ b_i^n(t)^2 \to \beta_i(t) & \text{weakly}^* L^\infty(0,T) \end{cases} \tag{3.3.7}$$

then the limit problem can be characterized as follows

$$(\mathcal{P}) \qquad \min_{u \in L^2(0,T)\,,\, v \in L^2(0,T)} \left\{ \int_\tau^T \varphi^*(t,x,u,v)dt \ : \ x' = a(t,x) + \sum_{i=1}^{M} g_i(t,x) v_i(t) \right\}$$

where

$$\varphi^*(t,x,u,v) = l_0 + \sum_{i=1}^{M} l_i u_i + \sum_{i=1}^{M} \tilde{l}_i u_i^2 + \sum_{i=1}^{M} \tilde{l}_i \frac{(b_i u_i - v_i)^2}{(\beta_i - b_i^2)}.$$

Note that, if the matrix G is invertible, one can derive v from the state equation and eliminate this auxiliary variable in the final formulation of the limit problem. For this problem we have proved that, although there is not a direct relationship between the Hamiltonian H_n and the function g_n that appears in the calculation of the Γ-limit (see Remark 2.1.3), if we calculate the Hamiltonian of this problem we get

$$H(t,x,p) = -p \cdot a - l_0 + \frac{1}{4} l^t L^{-1} l + \frac{1}{2} p^t GBL^{-1} l + \frac{1}{4} pGBL^{-1}BG^t p^t. \tag{3.3.8}$$

So the convergence of the Hamiltonian it is obtained also in this case: just compare (3.3.4) and (3.3.8). The point is, what happens if we suppose only

$$\begin{cases} b_i^n(t) \to b_i(t) & \text{weakly } L^2(0,T) \\ b_i^n(t)^2 \to \mu_i(t) & \text{weakly}^* \mathcal{M}^+(0,T) \ ? \end{cases} \tag{3.3.9}$$

In this case something new happens. The information (3.3.9) it is not enough! In fact we can construct an example where two sequences of problems $(\mathcal{P}_n)$ with coefficients b_i^n converging to the same b_i and μ_i, Γ-converge to two different optimal control problems.
The idea is that this phenomenon is due to the non commutativity of the vector fields $g_i^n(t,x)$. This means that if we have in particular a problem $(\mathcal{P}_n)$ with a family of vector fields that commute (i.e. all the Lie brackets are null) the knowledge of (3.3.9) is enough to characterize the Γ-limit.
In order to deal with the general noncommutative case our idea has been to construct a sequence of reparametrized optimal control problems as in the previous section. The advantage is that the new coefficients b_n are always in L^∞ (see the proof of Theorem 3.1.4), and we can apply the result of Buttazzo and Cavazzuti to characterize the Γ-limit.

3.4 The chain rule for distributional derivatives for BV functions

This Section is devoted to the proof of Theorem 3.1.7. This proof is a modification of the proof of Theorem 5.2 page 750 in [DMR91]. The main difference is that since we are considering an unbounded control set our state y it is not a Lipschitz function, so the general chain rule for distributional derivatives of function with bounded variation (see [DMR91], Theorem 3.1 page 746) does not apply in this case. Anyway in the sequel we prove this result in the one dimensional case also when y is only absolutely continuous.
For all notations in this section we refer to Appendix B.

Theorem 3.4.1 *Let $\psi : [0,1] \to \mathbb{R}$ be an absolutely continuous function and let $z : [0,T] \to [0,1]$ be an increasing function. If the map α is defined by*

$$\alpha(t) := \psi(z(t)) \qquad \forall t \in [0,T]$$

then

i) $\alpha \in BV([0,T])$;

ii) *the identity of measures*

$$\dot{\alpha} = \widehat{\psi_*(z)}\dot{z} \tag{3.4.1}$$

holds, where ψ_* denotes any Borel function coinciding with the derivatives $\frac{d\psi}{ds}$ almost everywhere with respect to the Lebesgue measure and we define (Volpert's Average Superposition)

$$\widehat{\psi_*(z)}(t) = \int_0^1 \psi_*(z(t^-) + \sigma(z(t^+) - z(t^-)))\, d\sigma.$$

A fundamental tool in the sequel is the following Lemma.

Lemma 3.4.2 *Let $u \in BV(\Omega)$ (Ω a open subset of $\mathbb{R}$) and let S_u be the set of all the discontinuity points of u. If $N \subset \mathbb{R}$ is a set of null Lebesgue measure, then*

$$|\dot{z}|(z^{-1}(N) \setminus S_u) = 0$$

where $|\dot{z}|$ is the total variation of the measure $\dot{z}$.

Proof of Theorem 3.4.1. First, without loss of generality, we may suppose that ψ is an increasing function. Thus, since α is the composition of two increasing functions, i) is proved.
Before proving ii) we underline that (3.4.1) makes sense in all [0,T]. We first consider the

countable set S_z of all the discontinuity points of z. Since ψ is continuous this set coincides with the set of the discontinuity points of α, so for a fixed $t \in S_z$ we have

$$\dot{\alpha}(t) = \Big(\alpha(t^+) - \alpha(t^-)\Big)\,\delta_t.$$

Always by the continuity of ψ we can say that $\alpha(t^+) = \psi(z(t^+))$ and $\alpha(t^-) = \psi(z(t^-))$, to get

$$\dot{\alpha}(t) = \Big(\psi(z(t^+)) - \psi(z(t^-))\Big)\,\delta_t = \frac{\psi(z(t^+)) - \psi(z(t^-))}{z(t^+) - z(t^-)}\Big(z(t^+) - z(t^-)\Big)\,\delta_t =$$

$$= \left(\int_0^1 \frac{d\psi}{ds}(z(t^-) + \sigma(z(t^+) - z(t^-)))d\sigma\right)\Big(z(t^+) - z(t^-)\Big)\,\delta_t = \widehat{\frac{d\psi}{ds}}(z)(t)\dot{z}(t)$$

by an easy calculation and recalling the definition of Volpert's Average Superposition. This give us the result for all points $t \in S_z$.
Let $E = \left\{s \in [0,1] : \nexists\ \frac{d\psi}{ds}\right\}$, by Lemma 3.4.2, we have

$$|\dot{z}|(z^{-1}(E) \setminus S_z) = 0 \quad \text{and} \quad |\dot{\alpha}|(\alpha^{-1}(M) \setminus S_z) = 0$$

where $M = \psi(E)$. We approximate the function ψ as follows. We set

$$g_n = \frac{d\psi}{ds} \wedge n \quad \text{and} \quad \psi_n(s) = \int_0^s g_n(s)ds + \psi(0),$$

hence

$$\psi_n \text{ is a Lipschitz function on } [0,1], \tag{3.4.2}$$

$$\lim_{n\to\infty} \psi_n(s) = \psi(s) \text{ uniformly in } [0,1], \tag{3.4.3}$$

$$\frac{d\psi_n}{ds} = g_n \text{ a.e.} \quad \text{and} \quad \frac{d\psi}{ds} = \sup_{n\in\mathbb{N}} g_n. \tag{3.4.4}$$

Now setting $\alpha_n(t) = \psi_n(z(t))$ for every $n \in \mathbb{N}$, by (3.4.2) the general chain rule (we always refer to Theorem 3.1 of [DMR91]) can be applied to obtain

$$\dot{\alpha}_n = \widehat{\psi_{n,*}(z)}\dot{z}$$

where $\psi_{n,*}$ denotes any Borel function coinciding with the derivatives $\frac{d\psi_n}{ds}$ almost everywhere with respect to the Lebesgue measure. (We remark that we can give a sense to this equality in all [0,T] as we have done for the (3.4.1)).
Thus, to obtain the result we have to show

$$\dot{\alpha}_n \to \dot{\alpha} \text{ weak}^* \ \mathcal{M}(0,1) \tag{3.4.5}$$

$$\widehat{\psi_{*,n}(z)}\dot{z} \to \widehat{\psi_*(z)}\dot{z} \quad \text{weak}^* \ \mathcal{M}(0,1). \tag{3.4.6}$$

Indeed, uniqueness of the weak limit gives us (3.4.1) which completes the proof.
To prove (3.4.5) we fix $\phi \in C_c^\infty(0,1)$; by definition of distributional derivatives

$$\int_{(0,1)} \phi(t) d\dot{\alpha}_n = -\int_{(0,1)} \phi'(t)\psi_n(z(t))dt,$$

(3.4.3) implies then

$$\lim_{n\to\infty} \int_{(0,1)} \phi'(t)\psi_n(z(t))dt = \int_{(0,1)} \phi'(t)\psi(z(t))dt$$

and, again by definition of distributional derivatives (using also i)),

$$-\int_{(0,1)} \phi'(t)\psi(z(t))dt = \int_{(0,1)} \phi(t) d\dot{\alpha}.$$

So, for every $\phi \in C_c^\infty(0,1)$

$$\lim_{n\to\infty} \int_A \phi(t) d\dot{\alpha}_n = \int_A \phi(t) d\dot{\alpha},$$

and, since by the uniform convergence of the ψ_n the $|\dot{\alpha}_n|$ are equibounded, this convergence is equivalent to the weak$^*\mathcal{M}(0,1)$ (see Proposition B.1.8 in Appendix B).
In order to prove (3.4.6) we fix $\phi \in C_0(0,1)$ (which is the closure of $C_c(0,1)$ in the sup norm); by definition

$$\int_{(0,1)} \phi(t)\widehat{\psi_{n,*}(z)}(t)\dot{z} = \int_{(0,1)\setminus S_z} \phi(t)\psi_{n,*}(z(t))\dot{z} + \sum_{t\in S_z} \Big(\psi_n(z(t^+)) - \psi_n(z(t^-))\Big)\,\delta_t. \tag{3.4.7}$$

Let

$$F_n^1 = \left\{ s \in [0,1] : \nexists \ \frac{d\psi_n}{ds} \right\} \quad F_n^2 = \left\{ s\in[0,1] : \frac{d\psi_n}{ds} \neq g_n \right\}$$

$$F_n^3 = \left\{ s \in [0,1] : \frac{d\psi_n}{ds} \neq \psi_{n,*} \right\} \quad F^1 = \left\{ s\in[0,1] : \frac{d\psi}{ds}(s) \neq \psi_* \right\}$$

$$F^2 = \left\{ s \in [0,1] : \lim_{n\to\infty} g_n(s) \neq \frac{d\psi}{ds}(s) \right\}$$

$$\text{and} \quad K = F^1 \cup E \cup F^2 \cup \Big(\cup_{n\in\mathbb{N}} (F_n^1 \cup F_n^2 \cup F_n^3)\Big).$$

Since the Lebesgue measure of K is zero, applying Lemma 3.4.2 we get

$$\int_{(0,1)\setminus S_z} \phi(t)\psi_{n,*}(z(t))\dot{z} = \int_{(0,1)\setminus(S_z\cup z^{-1}(K))} \phi(t)\frac{d\psi_n}{ds}(z(t))\dot{z}.$$

Then by the (3.4.4) we have

$$\lim_{n\to\infty}\int_{(0,1)\setminus(S_z\cup z^{-1}(K))}\phi(t)\frac{d\psi_n}{ds}(z(t))\dot z=\int_{(0,1)\setminus(S_z\cup z^{-1}(K))}\phi(t)\frac{d\psi}{ds}(z(t))\dot z$$

and applying the Lemma 3.4.2 again we get

$$\int_{(0,1)\setminus(S_z\cup z^{-1}(K))}\phi(t)\frac{d\psi}{ds}(z(t))\dot z=\int_{(0,1)\setminus S_z}\phi(t)\psi_*(z(t))\dot z.$$

Thus, letting $n\to\infty$ in (3.4.7) and recalling also (3.4.3), we obtain

$$\lim_{n\to\infty}\int_{(0,1)}\phi(t)\widehat{\psi_{n,*}(z)}(t)\dot z=\int_{(0,1)\setminus S_z}\phi(t)\psi_*(z(t))\dot z+\sum_{t\in S_z}\left[\psi(z(t^+))-\psi(z(t^-))\right]\delta_t=$$

$$=\int_{(0,1)}\phi(t)\widehat{\psi_*(z)}(t)\dot z$$

for every fixed $\phi\in C_0(0,1)$, which is the (3.4.6) and completes the proof. □

Proof of Lemma 3.4.2. If $u\in BV$, the co-area formula (see [Am97], Theorem 1.3.8 or Appendix B, Theorem B.2.5) gives us

$$|\dot u|(u^{-1}(N)\setminus S_u)=\int_{-\infty}^{+\infty}P(\{z>t\},u^{-1}(N)\setminus S_u)\,dt \tag{3.4.8}$$

where $P(\{z>t\},\Omega)$ is the perimeter of the set of $x\in\Omega$ such that $z(x)>t$. If we show that

$$\int_{-\infty}^{+\infty}P(\{z>t\},u^{-1}(N)\setminus S_u)\,dt\le\int_{-\infty}^{+\infty}\mathcal{H}^0(\{u=t\}\cap(u^{-1}(N)\setminus S_u))\,dt \tag{3.4.9}$$

where $\mathcal{H}^0$ is the null-dimensional Hausdorff measure, to conclude we have only to show that

$$\mathcal{H}^0(\{u=t\}\cap(u^{-1}(N)\setminus S_u))=0 \text{ for a.e. } t\in[0,T].$$

This is equivalent to show that for almost every t and for all $x\in u^{-1}(N)\setminus S_u$ we have $u(x)\neq t$ which is true because the Lebesgue measure of E is zero (we recall that $E=\left\{s\in[0,1]:\nexists\ \frac{d\psi}{ds}\right\}$).
Our aim is then to prove (3.4.9). If t is such that $A=\{u>t\}$ has finite perimeter (which is true for almost every t), we have following inequality between measures

$$|D\chi_A|\le\mathcal{H}^0\llcorner\partial A$$

where ∂A is the topological boundary of A. So, for every Borel set B,

$$P(A;B\setminus S_u)\le\mathcal{H}^0(\partial A\cap(B\setminus S_u)).$$

Now it is easy to see that

$$\partial\{u>t\}\setminus S_u=\{u=t\}\setminus S_u$$

so we can conclude that

$$P(\{u > t\}, u^{-1}(N) \setminus S_u) \leq \mathcal{H}^0(\{u = t\} \cap (u^{-1}(N) \setminus S_u))$$

for almost every t. This implies inequality (3.4.9) which completes the proof. $\square$

Proof of Theorem 3.1.7. Let y be the solution of (3.1.25); if we set $x(\cdot) := y(\mathcal{W}(\cdot))$ we want to show that it is a solution of (3.1.19). By definition y is an absolutely continuous function from $[0,1]$ into $\mathbb{R}$ and $\mathcal{W}(t)$ is an increasing function from $[0,T]$ into $[0,1]$, thus we can apply Theorem 3.4.1 to get the following identity between measures

$$\dot{x} = \widehat{\frac{dy}{ds}}(\mathcal{W})\dot{\mathcal{W}}. \tag{3.4.10}$$

Fix $A \subset [0,T] \setminus \mathcal{T}$ then (3.1.25) implies

$$\begin{aligned}
\int_A \dot{x} &= \int_A \frac{dy}{ds}(\mathcal{W})(t)\dot{\mathcal{W}}(t) = \\
&= \int_A \left(a(\varphi^0(\mathcal{W}(t)), y(\mathcal{W}(t))) + v(\varphi^0(\mathcal{W}(t)))\right) \frac{d\varphi^0}{ds}(\mathcal{W}(t))\dot{\mathcal{W}} + \\
&\quad + \int_A \xi(\varphi^0(\mathcal{W}(t)))\tilde{\mu}(\mathcal{W}(t))\dot{\mathcal{W}}.
\end{aligned}$$

Since from the general chain rule we can deduce for $t \in A$

$$B(t) = \varphi^1(\mathcal{W}(t)) \Longrightarrow \dot{B}(t) = \frac{d\varphi^1}{ds}(\mathcal{W}(t))\dot{\mathcal{W}}(t)$$

$$t = \varphi^0(\mathcal{W}(t)) \Longrightarrow dt = \frac{d\varphi^0}{ds}(\mathcal{W}(t))\dot{\mathcal{W}}(t)$$

we get (also using the definition of x)

$$\int_A \dot{x} = \int_A (a(t, x(t)) + v(t))\, dt + \int_A \xi(t) d\mu^s(t). \tag{3.4.11}$$

Let $t_j \in]0, T[\cap \mathcal{T}$, again by (3.4.10) we have

$$\dot{x}(t_j) = \widehat{\frac{dy}{ds}}(\mathcal{W})(t_j)\dot{\mathcal{W}}(t_j). \tag{3.4.12}$$

By definition

$$\widehat{\frac{dy}{ds}}(\mathcal{W})(t_j) = \int_0^1 \frac{dy}{ds}(\mathcal{W}(t_j) + \sigma\dot{\mathcal{W}}(t_j))\, d\sigma.$$

Now, if $\sigma \in [0,1]$ then $s = \mathcal{W}(t_j) + \sigma\dot{\mathcal{W}}(t_j) \in [\mathcal{W}(t_j), \mathcal{W}(t_j^+)]$ and since y is the solution of (3.1.25), we have

$$\widehat{\frac{dy}{ds}}(\mathcal{W})(t_j) = \int_0^1 \frac{dy}{ds}(\mathcal{W}(t_j) + \sigma\dot{\mathcal{W}}(t_j))\, d\sigma = \int_0^1 \xi(t_j)\frac{d\varphi^1}{ds}\left(\mathcal{W}(t_j) + \sigma\dot{\mathcal{W}}(t_j)\right) =$$

$$= \int_0^1 \xi(t_j) \frac{d\psi_{t_j}}{ds}(\sigma) \frac{1}{\dot{\mathcal{W}}(t_j)} \; d\sigma = \frac{\xi(t_j)}{\dot{\mathcal{W}}(t_j)} \left(B(t_j^+) - B(t_j^-) \right)$$

where we used the definition of φ^1. So, from (3.4.12) we obtain for every $t_j \in \mathcal{T}$,

$$\dot{x}(t_j) = \xi(t_j) \left(B(t_j^+) - B(t_j^-) \right).$$

This identity and the (3.4.11) tell us that x is the solution of (3.1.19). Observing that if $y(s_1) = z$ then $x(\tau^-) = y(\mathcal{W}(\tau^-)) = y(\mathcal{W}(\tau)) = y(s_1) = z$ the initial condition is also fulfilled. $\square$

Chapter 4

Γ-convergence for infinite dimensional optimal control problems

In this chapter we will describe some applications of the general Γ-convergence result to infinite dimensional optimal control problems.
Our first curiosity has been the following. Fix $\varepsilon > 0$, we consider this state equation

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) - \sum_{i,j=1}^n D_i(a_{ij}(x)D_j y(t,x)) = 0 & \text{in } [0,T]\times\overline{\Omega} \\ \varepsilon\left(\sum_{i,j=1}^n a_{ij}(x)\cdot D_j y(t,x)\nu_i(x)\right) + y(t,x) = u(t,x) & \text{in } [0,T]\times\partial\Omega \\ y(0,x) = y_0(x) & \text{in } \overline{\Omega} \end{cases} \tag{4.0.1}$$

and the following cost functional

$$J(y,u) = \int_0^T \left(\| u(s,\cdot) \|^2_{L^2(\partial\Omega;\mathbb{C}^N)} + \| y_\varepsilon(s,\cdot) \|^2_{L^2(\Omega;\mathbb{C}^N)}\right) ds.$$

This problem can be studied in all details and the optimal control can be characterized in feedback form. What happens if $\varepsilon \to 0$?
It is clear that one expects to have Γ-convergence to an optimal control problem having as state equation a parabolic partial differential equation with Dirichlet boundary condition and the same cost functional. In fact in Section 4.1 we prove this applying the general Γ-convergence Theorem 1.3.9 in its stronger form.
Note that often in the programs that calculate numerically the solution of partial differential equations of this type the Neumann and Dirichlet conditions are treated by considering a unique boundary condition of type

$$a\left(\sum_{i,j=1}^n a_{ij}(x)\cdot D_j y(t,x)\nu_i(x)\right) + b\, y(t,x) = u(t,x)$$

and plugging the biggest constant near the condition one wants to consider (an example is the program "free F.E.M., a language for the finite element method", of Pironneau and Prud'homme). Hence the stability result we have proved can be useful from this point of view.

Moreover, in the applications, it is natural to think about a Neumann control in a part of the boundary and a Dirichlet control elsewhere. So, we tried to look at what happens if we consider the state equation (4.0.1) with the boundary condition

$$\vartheta_\varepsilon(x)\left(\sum_{i,j=1}^{n} a_{ij}(x)\cdot D_j y(t,x)\nu_i(x)\right) + (1-\vartheta_\varepsilon(x))\, y(t,x) = u(t,x)$$

where $\vartheta_\varepsilon(x)$ is such that

$$\lim_{\varepsilon\to 0}\vartheta_\varepsilon(x) = \begin{cases} 0 & x \text{ in } \Gamma_0 \\ 1 & x \text{ in } \Gamma_1 \end{cases}$$

($\partial\Omega = \Gamma_0 \cup \Gamma_1$); the cost functional is the same as before. Clearly one would like that this problem Γ-converges to an optimal control problem having the following state equation

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) - \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j y(t,x)) = 0 & \text{in } [0,T]\times\overline{\Omega} \\ y(t,x) = u_0(t,x) & \text{in } [0,T]\times\Gamma_0 \\ \left(\sum_{i,j=1}^{n} a_{ij}(x)\cdot D_j y(t,x)\nu_i(x)\right) = u_1(t,x) & \text{in } [0,T]\times\Gamma_1 \\ y(0,x) = y_0(x) & \text{in } \overline{\Omega} \end{cases}$$

and the same cost functional.
We started with the study of this problem, characterizing the optimal control in feedback form by the technique used in [AFT91]; all details are described in Section 4.2. The main point here is that the appropriate control space turns out to be $\mathcal{U} = L^2(0,T;H^{1/2}(\Gamma_0;\mathbb{C}^N)\times H^{-1/2}(\Gamma_1;\mathbb{C}^N))$; so, in order to have coerciveness we need the presence in the cost functional of the norm $U = H^{1/2}(\Gamma_0;\mathbb{C}^N)\times H^{-1/2}(\Gamma_1;\mathbb{C}^N)$, whereas in the approximating problems only the norm of $L^2(\partial\Omega;\mathbb{C}^N)$ was required.
For this reason we can not apply the Γ-convergence theory unlike we did in Section 4.1 and, at present, we are not able to show the desired approximation result.

4.1 Approximation of an optimal control problem with Dirichlet boundary conditions

We begin this section giving a description of the optimal control problem we want to approximate.
Fix $T>0$, let $\mathcal{U} := L^2(0,T;U)$ and $Y := L^2(0,T;H)$ with $U := L^2(\partial\Omega;\mathbb{C}^N)$ and $H := L^2(\Omega;\mathbb{C}^N)$, we look for

$$(\mathcal{P}) \qquad \min_{Y\times U}\{J(y,u)\}$$

where

$$J(y,u) = \int_0^T \left(\| u(s,\cdot)\|_U^2 + \| y(s,\cdot)\|_H^2\right) ds$$

and, for each control $u \in \mathcal{U}$, y is the solution of

$$(S)\begin{cases} \frac{\partial y}{\partial t}(t,x) - \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j y(t,x)) = 0 & \text{in } [0,T]\times\overline{\Omega} \\ y(t,x) = u(t,x) & \text{in } [0,T]\times\partial\overline{\Omega} \\ y(0,x) = y_0(x) & \text{in } \overline{\Omega}. \end{cases}$$

Our basic assumptions are:

(H0) Ω is a bounded connected open set of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^2.

(H1) $y_0(x) \in H^2(\Omega;\mathbb{C}^N)$.

(H2) $a_{ij}(x) \in C^1(\overline{\Omega};\mathbb{C}^N)$ and

$$Re \sum_{i,j=1}^{n} (a_{ij}(x)\cdot \eta_j|\eta_i) \geq \nu \sum_{i=1}^{n} |\eta_i|^2$$

for every $\eta_1, ..., \eta_n \in \mathbb{C}^N$, for every $(t,x) \in [0,T]\times\overline{\Omega}$, $\nu > 0$.

Note that in all this chapter $(\cdot|\cdot)$ will denote the scalar product in $\mathbb{C}^N$, $\cdot$ will denote the usual matrix multiplication and $< \cdot,\cdot >_{XX'}$ will denote the duality product between the space X and its dual space X'. We will denote by C a constant possibly vary from line to line and by M the $\sum_{i,j=1}^{n}\{\sup_{\overline{\Omega}}|a_{ij}(x)|\}$.
In order to write the solution of system (S) with the representation formula given in [AFT91] we define the following operators.

$$\begin{cases} D(A_0) := H^2(\Omega;\mathbb{C}^N)\cap H_0^1(\Omega;\mathbb{C}^N), \\ \\ A_0 v := A(\cdot, D)v \end{cases} \tag{4.1.1}$$

where

$$A(x,D) := \sum_{i,j=1}^{n} D_i[a_{ij}(x)D_j v]. \tag{4.1.2}$$

$$u := G_0 h \Leftrightarrow \begin{cases} A(x,D)u = 0 & \text{in } \Omega \\ u = h & \text{on } \partial\Omega. \end{cases} \tag{4.1.3}$$

Note that by Proposition 2.4 in [AFT91], A_0 is the infinitesimal generator of an analytic semigroup on $L^2(\Omega;\mathbb{C}^N)$ and the estimate

$$\| (\lambda - A_0)^{-1} f \|_{L^2(\Omega;\mathbb{C}^N)} \leq \frac{C}{|\lambda|} \| f \|_{L^2(\Omega;\mathbb{C}^N)} \quad \forall f \in L^2(\Omega;\mathbb{C}^N) \tag{4.1.4}$$

holds for every λ in $S_{\vartheta_0} := \{\lambda \in \mathbb{C} \, : \, |arg(\lambda)| < \vartheta_0\}$ with $\vartheta_0 = \pi - \arctan(\frac{2M}{\nu})$. We observe that by standard arguments one can prove that this estimate holds in fact for any $\vartheta_0 \in]\frac{\pi}{2}, \pi[$ with a constant C depending on ϑ_0.
The main properties of analytic semigroups and of fractional powers of operators are described in Appendix C.
Proposition 2.13 of [AFT91] applies and we obtain:

Proposition 4.1.1 *Assume* (H0), (H1), (H2). *Fix* $\alpha \in]0,1[$.
Let $u \in C^{\alpha}([0,T]; H^2(\Omega;\mathbb{C}^N)) \cap C^{\alpha+1}([0,T]; L^2(\Omega;\mathbb{C}^N))$ *and assume the compatibility condition* $y_0(x) = u(x,0)$ *a.e. on* $\partial\Omega$.
Then the solution of problem (S) *is given by*

$$y(t,\cdot) = e^{tA_0} y_0(\cdot) + \int_0^t \left[(-A_0^*)^{1-\vartheta}(e^{(t-s)A_0})^*\right]^* [-A_0]^{\vartheta}\, G_0 u(s,\cdot)\, ds$$

for $t \in [0,T]$ *and* $\vartheta \in]0, \frac{1}{4}[$.

We may rewrite this formula in a shorter, although improper, form, namely

$$y(t,\cdot) = e^{tA_0} y_0(\cdot) - \int_0^t e^{(t-s)A_0} A_0 G_0 u(s,\cdot)\, ds. \tag{4.1.5}$$

For the sake of simplicity we will refer to this formulation.
As we have seen in Section 1.2.2 when we deal with a linear quadratic optimal control problem the optimal control can be characterized in a feedback form through the Riccati equation.
In our framework this result is summarized in Theorem 3.14 of [AFT91] that for this optimal control problem becomes the following.

Theorem 4.1.2 *Let* (H0), (H1), (H2) *be fulfilled. Then:*

(i) *There exists a unique optimal control* $\hat{u} \in L^2(0,T;U)$ *for problem* $(\mathcal{P})$*;*

(ii) *Denoting by* $P_0(\cdot)$ *the solution of the Riccati equation, given in integral form,*

$$P_0(t) = \int_t^T \left[e^{(s-t)A_0}\right]^* [I - P_0(s)\, A_0\, G_0\, G_0^*\, A_0^*\, P_0(s)]\, e^{(s-t)A_0}\, ds$$

and by $\hat{y} \in L^2(0,T;H)$ *the optimal trajectory, we have*

$$J(\hat{y},\hat{u}) = < P_0(0) y_0, y_0 >_H;$$

(iii) *We have the feedback formula for* $\hat{u}$*:*

$$\hat{u}(t,\cdot) = \; G_0^*\, A_0^*\, P_0(t)\, \hat{y}(t,\cdot) \quad t \in [0,T[;$$

(iv) *The optimal trajectory is expressed by* $\hat{y}(t,\cdot) = \Phi_0(t,0) y_0(\cdot)$ *where* $\Phi_0(t,s)$ *is defined by the integral equation*

$$\Phi_0(t,s)x = e^{(t-s)A_0}x - \int_s^t e^{(t-r)A_0}\, A_0\, G_0\, G_0^*\, A_0^*\, P_0(r)\, \Phi_0(r,s)x\, dr \quad t \in [s,T[.$$

In (ii), (iii) *and in* (iv) *we have set*

$$\begin{aligned} G_0^* A_0^* P_0(s) &= -\left[[-A_0]^{\vartheta}\, G_0\right]^* [-A_0^*]^{1-\vartheta}\, P_0(s), \\ e^{sA_0} A_0 G_0 &= -[-A_0]^{1-\vartheta}\, e^{sA_0}\, [-A_0]^{\vartheta}\, G_0 \end{aligned}$$

where $\vartheta \in]0, \frac{1}{4}[$.

4.1.1 The approximating problems

Our aim is to approximate problem $(\mathcal{P})$ with the following problem which is more regular in the sense explained in Remark 4.1.6.

$$(\mathcal{P}_\varepsilon) \qquad \min_{Y\times U}\{J(y_\varepsilon,u)\}$$

where now, for each control $u\in\mathcal{U}$, y_ε is the solution of

$$(S_\varepsilon)\begin{cases} \frac{\partial y}{\partial t}(t,x)-\sum_{i,j=1}^n D_i(a_{ij}(x)D_jy(t,x))=0 & \text{in } [0,T]\times\overline{\Omega}\\ \varepsilon\left(\sum_{i,j=1}^n a_{ij}(x)\cdot D_jy(t,x)\nu_i(x)\right)+y(t,x)=u(t,x) & \text{in } [0,T]\times\partial\overline{\Omega}\\ y(0,x)=y_0(x) & \text{in } \overline{\Omega}\end{cases}$$

with fixed $\varepsilon>0$. We will assume (H0), (H1), (H2) and the compatibility condition

$$\varepsilon\left(\sum_{i,j=1}^n a_{ij}(x)\cdot D_jy(0,x)\nu_i(x)\right)+y(0,x)=u(0,x) \text{ a. e. on } \partial\Omega.$$

In order to find a representation formula for the solution y_ε applying, as before, the result of [AFT91] we define the operators

$$\begin{cases} D(A_\varepsilon):=\left\{v\in H^2(\Omega;\mathbb{C}^N)\ :\ \varepsilon\frac{\partial v}{\partial\nu_a}+v=0\ x\in\partial\Omega\right\}\\ \\ A_\varepsilon v:=A(\cdot,D)v\end{cases} \tag{4.1.6}$$

and

$$u:=G_\varepsilon h\Leftrightarrow\begin{cases} A(x,D)u=0 & \text{in } \Omega\\ u+\varepsilon\frac{\partial u}{\partial\nu_a}=h & \text{on } \partial\Omega,\end{cases} \tag{4.1.7}$$

where we denote

$$\frac{\partial u}{\partial\nu_a}=\sum_{i,j=1}^n a_{ij}(x)\nu_s(x)\cdot D_ju.$$

We start proving that A_ε is the infinitesimal generator of an analytic semigroup.

Lemma 4.1.3 *Let* (H0), (H1), (H2) *be fulfilled. The operator A_ε defined by (4.1.6) is the infinitesimal generator of an analytic semigroup in $L^2(\Omega;\mathbb{C}^N)$. Moreover the estimate*

$$\|(\lambda-A_\varepsilon)^{-1}f\|_{L^2(\Omega;\mathbb{C}^N)}\le\frac{C}{|\lambda|}\|f\|_{L^2(\Omega;\mathbb{C}^N)} \tag{4.1.8}$$

holds for every $\lambda\in S_{\vartheta_0}:=\{\lambda\in\mathbb{C}\ :\ |arg(\lambda)|\le\vartheta_0\}$ for any $\vartheta\in]\frac{\pi}{2},\pi[$, and the constant C does not depend on ε.

Proof. By a slight modification of the classical Lumer Phillips Theorem it can be proved that if E is an Hilbert space and $A : D(A) \subset E \to E$ is a selfadjoint closed operator such that

$$\exists\ \omega \in \mathbb{R}\ \ : < Ax, x > \leq \omega \parallel x \parallel_E^2\ \ \forall x \in D(A),$$

then A is the infinitesimal generator of an analytic semigroup in E and $\sigma(A) \subseteq]-\infty, \omega]$. This result applies to the operator A_ε with $E = L^2(\Omega; \mathbb{C}^N)$ and $\omega = 0$, thus $\sigma(A_\varepsilon) \subseteq]-\infty, 0]$. This allows us to choose $\lambda \in S_{\vartheta_0}$ with any $\vartheta_0 \in]\frac{\pi}{2}, \pi[$ to estimate $u_\varepsilon := (\lambda - A_\varepsilon)^{-1} f$. Multiplying by u_ε in $L^2(\Omega; \mathbb{C}^N)$ and integrating by parts we have

$$\lambda \int_\Omega |u_\varepsilon|^2\, dx + \varepsilon \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_a} \right|^2\, d\sigma + \int_\Omega (\sum_{i,j=1}^n a_{ij}(x) D_j u_\varepsilon | D_i u_\varepsilon)\, dx = \int_\Omega (f|u_\varepsilon)\, dx.$$

Taking the real part we have

$$Re\lambda \int_\Omega |u_\varepsilon|^2\, dx + \varepsilon \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_a} \right|^2\, d\sigma + \nu \int_\Omega |Du_\varepsilon|^2\, dx \leq \parallel f \parallel_{L^2(\Omega;\mathbb{C}^N)} \parallel u_\varepsilon \parallel_{L^2(\Omega;\mathbb{C}^N)}$$

while taking the imaginary part

$$\begin{aligned} |Im\lambda| \int_\Omega |u_\varepsilon|^2\, dx &\leq \parallel f \parallel_{L^2(\Omega;\mathbb{C}^N)} \parallel u_\varepsilon \parallel_{L^2(\Omega;\mathbb{C}^N)} + \left| \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) D_j u_\varepsilon | D_i u_\varepsilon \right) \right| \leq \\ &\leq \parallel f \parallel_{L^2(\Omega;\mathbb{C}^N)} \parallel u_\varepsilon \parallel_{L^2(\Omega;\mathbb{C}^N)} \left[1 + \frac{M}{\nu} \right]. \end{aligned}$$

Now, fixed $\delta > 0$, if $|Im\lambda| \geq \delta |Re\lambda|$ then,

$$\parallel u_\varepsilon \parallel_{L^2(\Omega;\mathbb{C}^N)} \leq \frac{\left(1 + \frac{M}{\nu}\right) \sqrt{\frac{1}{\delta^2} + 1}}{|\lambda|} \parallel f \parallel_{L^2(\Omega;\mathbb{C}^N)}$$

whereas if $Re\lambda > 0$ and $\delta(Re\lambda) > |Im\lambda|$,

$$\parallel u_\varepsilon \parallel_{L^2(\Omega;\mathbb{C}^N)} \leq \frac{\sqrt{1+\delta^2}}{|\lambda|} \parallel f \parallel_{L^2(\Omega;\mathbb{C}^N)}\ .$$

So, for every $\lambda \in S_{\vartheta_0}$ with any $\vartheta_0 \in]\frac{\pi}{2}, \pi[$ we have

$$\parallel u_\varepsilon \parallel_{L^2(\Omega;\mathbb{C}^N)} \leq \frac{C}{|\lambda|} \parallel f \parallel_{L^2(\Omega;\mathbb{C}^N)}$$

with $C = \sqrt{1+\delta^2} \left[\left(1 + \frac{M}{\nu}\right) + \frac{1}{\delta} \right]$ and $\delta = \tan(\pi - \vartheta_0)$. Since C does not depend on ε this completes the proof. □

We need the following.

Proposition 4.1.4 *Let* (H0), (H1), (H2) *be fulfilled. Fix* $\varepsilon > 0$*; if* A_ε *is defined by (4.1.6) then the operator* G_ε *is well defined from* $L^2(\partial\Omega; \mathbb{C}^N)$ *into* $D([-A_\varepsilon]^\vartheta)$ *for each* $\vartheta \in]0, \frac{3}{4}[$.

Proof. We start proving that if $v \in H^1(\Omega;\mathbb{C}^N)$ then

$$\| v \|^2_{L^2(\Omega;\mathbb{C}^N)} \leq C \left[\| v \|^2_{L^2(\partial\Omega;\mathbb{C}^N)} + \| Dv \|^2_{L^2(\Omega;\mathbb{C}^N)} \right]. \tag{4.1.9}$$

Suppose, by contradiction, that $\forall n \in \mathbb{N}$, $\exists v_n \in H^1(\Omega;\mathbb{C}^N)$ such that

$$\| v_n \|^2_{L^2(\Omega;\mathbb{C}^N)} > n \left[\| v_n \|^2_{L^2(\partial\Omega;\mathbb{C}^N)} + \| Dv_n \|^2_{L^2(\Omega;\mathbb{C}^N)} \right].$$

Setting $w_n = \frac{v_n}{\|v_n\|_{L^2(\Omega;\mathbb{C}^N)}}$ we have

$$\| w_n \|_{L^2(\Omega;\mathbb{C}^N)} = 1 \quad \text{and} \quad \| w_n \|^2_{L^2(\partial\Omega;\mathbb{C}^N)} + \| Dw_n \|^2_{L^2(\Omega;\mathbb{C}^N)} \leq \frac{1}{n}.$$

This implies (passing possibly to a subsequence)

$$Dw_n \longrightarrow 0 \text{ in } L^2(\Omega;\mathbb{C}^N), \tag{4.1.10}$$

$$w_n \longrightarrow 0 \text{ in } L^2(\partial\Omega;\mathbb{C}^N), \tag{4.1.11}$$

$$w_n \longrightarrow z \text{ weakly } L^2(\Omega;\mathbb{C}^N). \tag{4.1.12}$$

Therefore, since D is a closed operator, $Dz = 0$ in $L^2(\Omega;\mathbb{C}^N)$, thus z is constant. Moreover, by (4.1.10) and (4.1.12) $w_n \to z$ weakly $H^1(\Omega;\mathbb{C}^N)$; by Rellich's Theorem, $w_n \to z$ in $L^2(\Omega;\mathbb{C}^N)$ so we can conclude that $w_n \to z$ in $H^1(\Omega;\mathbb{C}^N)$.
In particular $w_n \to z$ in $L^2(\partial\Omega;\mathbb{C}^N)$; thus by (4.1.11) we can conclude that $z = 0$ which contradicts $\| w_n \|_{L^2(\Omega;\mathbb{C}^N)} = 1$. So (4.1.9) is proved.
Now, fix $\varepsilon > 0$, let u_ε be the solution of

$$\begin{cases} A(x,D)u_\varepsilon = 0 & \text{in } \Omega \\ u_\varepsilon + \varepsilon \frac{\partial u_\varepsilon}{\partial \nu_a} = h & \text{on } \partial\Omega. \end{cases} \tag{4.1.13}$$

By a classical result of [ADN64], if $h \in H^{1/2}(\partial\Omega;\mathbb{C}^N)$ then $u_\varepsilon \in H^2(\Omega;\mathbb{C}^N)$ and

$$\| u_\varepsilon \|_{H^2(\Omega;\mathbb{C}^N)} \leq C \| h \|_{H^{1/2}(\partial\Omega;\mathbb{C}^N)}. \tag{4.1.14}$$

We want now estimate u_ε in a lower norm. We multiply by $\varepsilon u_\varepsilon$ in $L^2(\Omega;\mathbb{C}^N)$ in (4.1.13) and we integrate by parts, thus

$$\begin{aligned} 0 &= \int_\Omega \varepsilon \left(A(x,D)u_\varepsilon | u_\varepsilon \right) \, dx = \\ &= \int_{\partial\Omega} (h|u_\varepsilon) \, d\sigma - \int_{\partial\Omega} |u_\varepsilon|^2 \, d\sigma - \int_\Omega \varepsilon \left(\sum_{i,j=1}^n a_{ij}(x) \cdot D_j u_\varepsilon | D_i u_\varepsilon \right) dx. \end{aligned}$$

Hence, by (H2),

$$\begin{aligned} \int_{\partial\Omega} |u_\varepsilon|^2 \, d\sigma + \int_\Omega \varepsilon\nu |Du_\varepsilon|^2 \, dx \leq \int_{\partial\Omega} (h|u_\varepsilon) \, d\sigma &= \\ &= | < h, u_\varepsilon >_{H^{1/2}(\partial\Omega;\mathbb{C}^N) H^{-1/2}(\partial\Omega;\mathbb{C}^N)} | \leq \\ &\leq \| h \|_{H^{-1/2}(\partial\Omega;\mathbb{C}^N)} \| u_\varepsilon \|_{H^{1/2}(\partial\Omega;\mathbb{C}^N)} \leq \\ &\leq C \| h \|_{H^{-1/2}(\partial\Omega;\mathbb{C}^N)} \| u_\varepsilon \|_{H^1(\Omega;\mathbb{C}^N)}. \end{aligned}$$

Summarizing

$$\| u_\varepsilon \|^2_{L^2(\partial\Omega;\mathbb{C}^N)} + \varepsilon\nu \ \| Du_\varepsilon \|^2_{L^2(\Omega;\mathbb{C}^N)} \leq C \| h \|_{H^{-1/2}(\partial\Omega;\mathbb{C}^N)} \| u_\varepsilon \|_{H^1(\Omega;\mathbb{C}^N)} . \tag{4.1.15}$$

Since $u_\varepsilon \in H^1(\Omega;\mathbb{C}^N)$ applying (4.1.9) we have

$$\| u_\varepsilon \|^2_{L^2(\Omega;\mathbb{C}^N)} \leq C \left(\| u_\varepsilon \|^2_{L^2(\partial\Omega;\mathbb{C}^N)} + \| Du_\varepsilon \|^2_{L^2(\Omega;\mathbb{C}^N)} \right),$$

thus, by (4.1.15),

$$\| u_\varepsilon \|^2_{L^2(\Omega;\mathbb{C}^N)} \leq C_\varepsilon \| h \|_{H^{-1/2}(\partial\Omega;\mathbb{C}^N)} \| u_\varepsilon \|_{H^1(\Omega;\mathbb{C}^N)} .$$

Therefore

$$\| u_\varepsilon \|_{H^1(\Omega;\mathbb{C}^N)} \leq C_\varepsilon \| h \|_{H^{-1/2}(\partial\Omega;\mathbb{C}^N)} . \tag{4.1.16}$$

We now interpolate between (4.1.14) and (4.1.16), using Theorems 7.7 and 9.4 of [LM68]. The result of interpolation is the estimate

$$\| u_\varepsilon \|^2_{H^{3/2}(\Omega;\mathbb{C}^N)} \leq C_\varepsilon \| h \|_{L^2(\Omega;\mathbb{C}^N)} \quad , \ \forall h \in H^{1/2}(\partial\Omega;\mathbb{C}^N) \tag{4.1.17}$$

which shows that the linear operator $h \to u_\varepsilon$ can be boundedly extended to an operator $G_\varepsilon : L^2(\partial\Omega;\mathbb{C}^N) \to H^{3/2}(\Omega;\mathbb{C}^N)$.
Moreover, Theorem 2 page 83 of [Fu67] tells us that

$$D([-A_\varepsilon]^\vartheta) = H^{2\vartheta}(\Omega;\mathbb{C}^N) \quad 0 \leq \vartheta < \frac{3}{4} \tag{4.1.18}$$

and this completes the proof. □

This result lets us apply Proposition 2.13 of [AFT91] to get

$$y_\varepsilon(t,\cdot) = e^{tA_\varepsilon} y_0(\cdot) + \int_0^t \left[(-A_\varepsilon^*)^{1-\vartheta} (e^{(t-s)A_\varepsilon})^* \right]^* [-A_\varepsilon]^\vartheta \, G_\varepsilon u(s,\cdot) \, ds \tag{4.1.19}$$

for $t \in [0,T]$ and $\vartheta \in]0, \frac{3}{4}[$.
Also in this case, for simplicity, we will refer to the shorter, although improper, following formulation

$$y_\varepsilon(t,\cdot) = e^{tA_\varepsilon} y_0(\cdot) - \int_0^t e^{(t-s)A_\varepsilon} A_\varepsilon G_\varepsilon u(s,\cdot) \, ds. \tag{4.1.20}$$

In this case also we can summarize the characterization of the optimal pair in the following theorem (and we refer to Theorem 3.14 in [AFT91] for the proof).

Theorem 4.1.5 *Let* (H0), (H1), (H2) *be fulfilled. Then:*

(i) *There exists a unique optimal control $\hat{u}_\varepsilon \in L^2(0,T;U)$ for problem $(\mathcal{P}_\varepsilon)$;*

(ii) *Denoting by $P_\varepsilon(\cdot)$ the solution of the Riccati equation, given in integral form,*

$$P_\varepsilon(t) = \int_t^T \left[e^{(s-t)A_\varepsilon}\right]^* [I - P_\varepsilon(s)\, A_\varepsilon\, G_\varepsilon\, G_\varepsilon^*\, A_\varepsilon^*\, P_\varepsilon(s)]\, e^{(s-t)A_\varepsilon}\, ds$$

and by $\hat{y}_\varepsilon \in L^2(0,T;H)$ the optimal trajectory, we have

$$J(\hat{y}_\varepsilon, \hat{u}) = < P_\varepsilon(0) y_O, y_O >_H;$$

(iii) *We have the feedback formula for $\hat{u}_\varepsilon$:*

$$\hat{u}_\varepsilon(t,\cdot) = \ G_\varepsilon^*\, A_\varepsilon^*\, P_\varepsilon(t)\, \hat{y}_\varepsilon(t,\cdot) \quad t \in [0,T[;$$

(iv) *The optimal trajectory is expressed by $\hat{y}_\varepsilon(t,\cdot) = \Phi_\varepsilon(t,0) y_0(\cdot)$ where $\Phi_\varepsilon(t,s)$ is defined by the integral equation*

$$\Phi_\varepsilon(t,s)x = e^{(t-s)A_\varepsilon}x - \int_s^t e^{(t-r)A_\varepsilon}\, A_\varepsilon\, G_\varepsilon\, G_\varepsilon^*\, A_\varepsilon^*\, P_\varepsilon(r)\, \Phi_\varepsilon(r,s)x\, dr \quad t \in [s,T[.$$

In (ii), (iii) *and in* (iv) *we have set*

$$G_\varepsilon^* A_\varepsilon^* P_\varepsilon(s) = -\left[[-A_\varepsilon]^\vartheta\, G_\varepsilon\right]^* [-A_\varepsilon^*]^{1-\vartheta}\, P_\varepsilon(s),$$
$$e^{sA_\varepsilon} A_\varepsilon G_0 = -\,[-A_\varepsilon]^{1-\vartheta}\, e^{sA_\varepsilon}\, [-A_\varepsilon]^\vartheta\, G_\varepsilon$$

where $\vartheta \in]0, \frac{3}{4}[$.

Remark 4.1.6 As Proposition 4.1.4 shows the operator G_ε has a better behaviour than G_0 and in fact the state y_ε turns out to belong to $C(0,T;H)$. Moreover, by the feedback formula in Theorem 4.1.5 (iii), the control $\hat{u}_\varepsilon$ has a weaker singularity at $t = T$. In this sense problem $(\mathcal{P}_\varepsilon)$ is more regular than problem $(\mathcal{P})$. □

4.1.2 Γ-convergence

We want to prove now that, for $\varepsilon \to 0$, problems $(\mathcal{P}_\varepsilon)$ Γ-converge to problem $(\mathcal{P})$.
As we said in the first chapter this tells us that the optimal controls $\hat{u}_\varepsilon$ and the optimal trajectories $\hat{y}_\varepsilon$ of $(\mathcal{P}_\varepsilon)$ converge respectively to the optimal control u and to the optimal trajectory y of $(\mathcal{P})$.
Our aim is to apply Theorem 1.3.9, thus we define

$$H_\varepsilon(y) := y - e^{tA_\varepsilon} y_0 \ :\ L^2(0,T;H) \to L^2(0,T;H) \tag{4.1.21}$$

$$H(y) := y - e^{tA_0} y_0 \ :\ L^2(0,T;H) \to L^2(0,T;H) \tag{4.1.22}$$

$$B_\varepsilon(u) := \int_0^t e^{(t-s)A_\varepsilon} A_\varepsilon G_\varepsilon(s) u(s,\cdot)\, ds \ :\ L^2(0,T;U) \to L^2(0,T;U) \tag{4.1.23}$$

$$B(u) := \int_0^t e^{(t-s)A_0} A_0 G_0(s) u(s,\cdot)\, ds \ :\ L^2(0,T;U) \to L^2(0,T;U). \tag{4.1.24}$$

If we prove

(a) $H_\varepsilon(y)$ G-converges to H.

(b) $u_\varepsilon \to u$ in $L^2(0,T;U) \Longrightarrow B_\varepsilon(u_\varepsilon) \to B(u)$ in $L^2(0,T;H)$.

(c) $J(y,u)$ fulfills hypothesis (HJn) in Chapter 1.

Recalling that (4.1.20) implies $H_\varepsilon(y) = B_\varepsilon(u)$ and (4.1.5) implies $H(y) = B(u)$, Theorem 1.3.9 applies and we have obtained that the Γ-limit of problems $(\mathcal{P}_\varepsilon)$ is problem $(\mathcal{P})$. (For the details of this particular case of Γ-convergence one can see [BF95], Section 3).
Since it is immediate to verify that $J(u,y)$ fulfills (HJn) the rest of this section is devoted to the proof of (a) and (b).

Theorem 4.1.7 *Let* (H0), (H1), (H2) *be fulfilled. If* H_ε *is defined by (4.1.21) and* H *is defined by (4.1.22) then*

$H_\varepsilon(y)$ G-converges to H.

Proof. Suppose first to have proved that, for $\varepsilon \to 0$

$$e^{tA_\varepsilon}y_0 \to e^{tA_0}y_0 \text{ in } L^2(0,T;H). \tag{4.1.25}$$

This implies the following properties.
(i) If $y_\varepsilon \to y$ in $L^2(0,T;H)$ and $v_\varepsilon \to v$ in $L^2(0,T;H)$ are such that $v_\varepsilon = H(y_\varepsilon) := y_\varepsilon - e^{tA_\varepsilon}y_\varepsilon$ for infinitely many ε, then, letting $\varepsilon \to 0$, we have $v = y - e^{tA_0}$, i.e. $H(y) = v$.
(ii)Let $v = H(y) := y - e^{tA_0}y$ and v_ε be a sequence such that $v_\varepsilon \to v$ in $L^2(0,T;H)$, setting $y_\varepsilon := v_\varepsilon + e^{tA_\varepsilon}y_0$ we have $H_\varepsilon(y_\varepsilon) = v_\varepsilon$ for infinitely many ε and $y_\varepsilon \to y$ in $L^2(0,T;H)$.
Recalling Definition 1.3.10, properties (i) and (ii) tell us that H_ε G-converges to H, so the proof is achieved if we prove (4.1.25).
We begin with some estimate on the resolvent of the operators A_ε and A_0.
Note that by Lemma 4.1.3 and Proposition 2.14 in [AFT91] (see also (4.1.4)) $\sigma(A_\varepsilon)$ and $\sigma(A_0)$ are contained in $]-\infty,0]$. Thus there exists a common sector of $\mathbb{C}$ contained in the resolvent of A_ε and A_0. We can then fix λ in this sector to estimate $u_\varepsilon = R(\lambda, A_\varepsilon)f$, i.e.

$$\begin{cases} (\lambda - A(x,D))u_\varepsilon = f & \text{in } \Omega \\ u_\varepsilon + \varepsilon\frac{\partial u_\varepsilon}{\partial \nu_a} = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying by $\varepsilon u_\varepsilon$ in $L^2(\Omega;\mathbb{C}^N)$ and integrating by parts we have

$$\begin{aligned} 0 &= \int_\Omega ((\lambda - A)u_\varepsilon - f|\varepsilon u_\varepsilon)\,dx = \\ &= \int_\Omega \varepsilon\lambda|u_\varepsilon|^2\,dx - \int_{\partial\Omega}\varepsilon\left(\frac{\partial u_\varepsilon}{\partial\nu_a}|u_\varepsilon\right)d\sigma + \int_\Omega \varepsilon\left(\sum_{i,j=1}^n a_{ij}(x)D_iu_\varepsilon|D_ju_\varepsilon\right)dx + \\ &\quad -\varepsilon\int_\Omega (f|u_\varepsilon)\,dx, \end{aligned}$$

hence, by the coercitivity assumption (H2),

$$\varepsilon Re\lambda\int_\Omega |u_\varepsilon|^2\,dx + \int_{\partial\Omega}\varepsilon^2\left|\frac{\partial u_\varepsilon}{\partial\nu_a}\right|^2 d\sigma + \varepsilon\nu\int_\Omega |Du_\varepsilon|^2\,dx \le \int_\Omega \varepsilon\,|(f|u_\varepsilon)|\,dx,$$

that, using also the resolvent estimate (4.1.8) becomes

$$\int_{\partial\Omega} \varepsilon^2 \left|\frac{\partial u_\varepsilon}{\partial \nu_a}\right|^2 d\sigma \leq \int_\Omega \varepsilon \left|(f|u_\varepsilon)\right| \, dx \leq C\frac{\varepsilon}{|\lambda|} \parallel f \parallel^2_{L^2(\Omega;\mathbb{C}^N)} . \tag{4.1.26}$$

Now, if $u = R(\lambda, A_0)f$, the function $u - u_\varepsilon$ solves

$$\begin{cases} (\lambda - A(x,D))(u-u_\varepsilon) = 0 & \text{in } \Omega \\ u - u_\varepsilon = -\varepsilon\frac{\partial u_\varepsilon}{\partial \nu_a} & \text{on } \partial\Omega, \end{cases}$$

hence we can apply Proposition 2.12 of [AFT91], precisely estimate (2.59), to obtain

$$\parallel u - u_\varepsilon \parallel_{H^{1/2}(\Omega;\mathbb{C}^N)} \leq C \left\| \varepsilon\frac{\partial u_\varepsilon}{\partial \nu_a} \right\|_{L^2(\partial\Omega;\mathbb{C}^N)} . \tag{4.1.27}$$

Estimates (4.1.26) and (4.1.27) let us conclude that

$$\parallel R(\lambda, A_\varepsilon)f - R(\lambda, A_0)f \parallel_{L^2(\Omega;\mathbb{C}^N)} \leq C\sqrt{\frac{\varepsilon}{|\lambda|}} \parallel f \parallel_{L^2(\Omega;\mathbb{C}^N)} . \tag{4.1.28}$$

Applying the classical properties of the analytic semigroups we have

$$\parallel e^{tA_\varepsilon}x - e^{tA_0}x \parallel_{L^2(\Omega;\mathbb{C}^N)} = \left\| \frac{1}{2\pi i}\int_\gamma e^{t\lambda}\left[R(\lambda, A_\varepsilon)x - R(\lambda, A_0)x\right] \, d\lambda \right\|_{L^2(\Omega;\mathbb{C}^N)} =$$

$$= \left\| \frac{1}{2\pi i}\int_\gamma e^{\sigma}\left[R\left(\frac{\sigma}{t}, A_\varepsilon\right)x - R\left(\frac{\sigma}{t}, A_0\right)x\right] \, \frac{d\sigma}{t} \right\|_{L^2(\Omega;\mathbb{C}^N)} \tag{4.1.29}$$

where $\gamma = \{\lambda \in \mathbb{C} \, : |arg(\lambda)| = \eta, |\lambda| \geq \delta\} \cup \{\lambda \in \mathbb{C} \, : |arg(\lambda)| \leq \eta, |\lambda| = \delta\}$ with $\frac{\pi}{2} < \eta < \pi$ and $\delta > 0$ and we set $\sigma = \lambda t$.
If $0 < \vartheta < 1$, by the resolvent estimates (4.1.4), (4.1.8) and (4.1.28),

$$\left\| R\left(\frac{\sigma}{t}, A_\varepsilon\right)x - R\left(\frac{\sigma}{t}, A_0\right)x \right\|^{\vartheta+1-\vartheta}_{L^2(\Omega;\mathbb{C}^N)} \leq$$

$$\leq C\frac{\varepsilon^{\vartheta/2}}{\sigma^{\vartheta/2}}t^{\vartheta/2} \parallel x \parallel^{\vartheta/2}_{L^2(\Omega;\mathbb{C}^N)} \left(C\frac{t^{1-\vartheta}}{|\sigma|^{1-\vartheta}} \parallel x \parallel^{1-\vartheta}_{L^2(\Omega;\mathbb{C}^N)} \right) \leq$$

$$\leq C \parallel x \parallel^{1-(\vartheta/2)}_{L^2(\Omega;\mathbb{C}^N)} \frac{t^{1-(\vartheta/2)}}{|\sigma|^{1-(\vartheta/2)}}\varepsilon^{\vartheta/2},$$

hence, using this in (4.1.29),

$$\parallel e^{tA_\varepsilon}x - e^{tA_0}x \parallel_{L^2(\Omega;\mathbb{C}^N)} \leq C \parallel x \parallel^{1-(\vartheta/2)}_{L^2(\Omega;\mathbb{C}^N)} t^{-\vartheta/2}\varepsilon^{\vartheta/2}.$$

This estimate allows us to conclude by the following calculation

$$\begin{aligned}\| e^{tA_\varepsilon}x - e^{tA_0}x \|_{L^2(0,T;H)} &= \int_0^T \| e^{tA_\varepsilon}x - e^{tA_0}x \|^2_{L^2(\Omega;\mathbb{C}^N)} \ dt \le \\ &\le \int_0^T C \| x \|^{2-\vartheta}_{L^2(\Omega;\mathbb{C}^N)} t^{-\vartheta}\varepsilon^\vartheta \, dt = \\ &= C\varepsilon^\vartheta \| x \|^{2-\vartheta}_{L^2(\Omega;\mathbb{C}^N)} \frac{T^{1-\vartheta}}{1-\vartheta}\end{aligned}$$

which clearly implies (4.1.25) setting $x = y_0$. □

This theorem proves (a), to prove (b) we need two preliminary results.

Lemma 4.1.8 *Assume* (H0), (H1), (H2). *Let G_0 and G_ε be respectively defined by (4.1.3) and (4.1.7). If $u_\varepsilon \to u$ in $L^2(0,T;U)$ then*

$$G_\varepsilon u_\varepsilon \to G_0 u \text{ in } L^2(0,T;H^{1/2}(\Omega;\mathbb{C}^N)).$$

Proof. Fix $s \in [0,T]$, we set $v_\varepsilon(s,x) = G_\varepsilon u_\varepsilon(s,x)$ and $v(s,x) = G_0 u(s,x)$. The function $v - v_\varepsilon$ solves then

$$\begin{cases} A(x,D)(v - v_\varepsilon) = 0 & \text{in } \Omega \\ v - v_\varepsilon = u_\varepsilon - u - \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a} & \text{on } \partial\Omega. \end{cases}$$

Applying estimate (2.59) in Proposition 2.12 of [AFT91] we have

$$\| v_\varepsilon - v \|_{H^{1/2}(\Omega;\mathbb{C}^N)} \le \| u_\varepsilon - u \|_{L^2(\partial\Omega;\mathbb{C}^N)} + \left\| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a} \right\|_{L^2(\partial\Omega;\mathbb{C}^N)}.$$

Integration on $[0,T]$ gives

$$\int_0^T \| v_\varepsilon(t,\cdot) - v(t,\cdot) \|^2_{H^{1/2}(\Omega;\mathbb{C}^N)} \ dt \le$$

$$\le 2 \left(\int_0^T \| u_\varepsilon(t,\cdot) - u(t,\cdot) \|^2_{L^2(\partial\Omega;\mathbb{C}^N)} \ dt + \int_0^T \left\| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a} \right\|^2_{L^2(\partial\Omega;\mathbb{C}^N)} dt \right)$$

so to end the proof we need

$$\lim_{\varepsilon \to 0} \left(\int_0^T \left\| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a} \right\|^2_{L^2(\partial\Omega;\mathbb{C}^N)} dt \right) = 0. \tag{4.1.30}$$

Multiplying by $\varepsilon v_\varepsilon$ in $L^2(\Omega;\mathbb{C}^N)$ the equation $A(x,D)v_\varepsilon = 0$, integrating by parts, using the coercitivity assumption (H2) and the boundary condition we get the estimate

$$\frac{1}{2} \| v_\varepsilon \|^2_{L^2(\partial\Omega;\mathbb{C}^N)} + \varepsilon\nu \| Dv_\varepsilon \|^2_{L^2(\Omega;\mathbb{C}^N)} \le \frac{1}{2} \| u_\varepsilon \|^2_{L^2(\partial\Omega;\mathbb{C}^N)} \tag{4.1.31}$$

that integrated on $[0,T]$ is

$$\frac{1}{2}\int_0^T \parallel v_\varepsilon(t,\cdot) \parallel^2_{L^2(\partial\Omega;\mathbb{C}^N)}\ dt + \nu\varepsilon\int_0^T \parallel Dv_\varepsilon(t,\cdot) \parallel^2_{L^2(\Omega;\mathbb{C}^N)}\ dt \le$$

$$\le \frac{1}{2}\int_0^T \parallel u_\varepsilon(t,\cdot) \parallel^2_{L^2(\partial\Omega;\mathbb{C}^N)}\ dt.$$

Since by hypothesis $u_\varepsilon \to u$ in $L^2(0,T;U)$ this implies, up to subsequences,

$$v_\varepsilon \to w \text{ weakly } L^2(0,T;U), \tag{4.1.32}$$

$$\sqrt{\varepsilon}Dv_\varepsilon \to z \text{ weakly } L^2(0,T;H). \tag{4.1.33}$$

We can write $v_\varepsilon = G_\varepsilon u_\varepsilon$ as

$$\begin{cases} A(x,D)(\varepsilon v_\varepsilon) = 0 & \text{in } \Omega \\ \frac{\partial}{\partial \nu_a}(\varepsilon v_\varepsilon) = u_\varepsilon - v_\varepsilon & \text{on } \partial\Omega \end{cases}$$

thus, using estimate (2.64) in Proposition 2.12 of [AFT91], we have

$$\parallel \varepsilon v_\varepsilon \parallel_{H^{3/2}(\Omega;\mathbb{C}^N)} \le C \parallel u_\varepsilon - v_\varepsilon \parallel_{L^2(\partial\Omega;\mathbb{C}^N)}$$

that integrated on $[0,T]$ becomes

$$\int_0^T \parallel \varepsilon v_\varepsilon(t,\cdot) \parallel^2_{H^{3/2}(\Omega;\mathbb{C}^N)}\ dt \le$$

$$\le C\left(\int_0^T \parallel u_\varepsilon(t,\cdot) \parallel^2_{L^2(\partial\Omega;\mathbb{C}^N)}\ dt + \int_0^T \parallel v_\varepsilon(t,\cdot) \parallel^2_{L^2(\partial\Omega;\mathbb{C}^N)}\ dt\right).$$

By the convergence hypothesis on u_ε and by (4.1.31) this implies, up to subsequences,

$$\varepsilon v_\varepsilon \to y \text{ weakly } L^2(0,T;H^{3/2}(\Omega;\mathbb{C}^N)). \tag{4.1.34}$$

Recalling that the inclusion $H^{3/2}(\Omega;\mathbb{C}^N) \subset H^1(\Omega;\mathbb{C}^N)$ is compact, by (4.1.34) we have

$$\varepsilon v_\varepsilon \to y \text{ in } L^2(0,T;H^1(\Omega;\mathbb{C}^N)). \tag{4.1.35}$$

This and (4.1.33) imply $Dy = 0$, so y is a constant. Moreover, since the trace operator is continuous, (4.1.32) implies that $y|_{\partial\Omega} = 0$, hence $y = 0$.
We need now a different estimate, multiplying by $\varepsilon v_\varepsilon$ in $L^2(\Omega;\mathbb{C}^N)$ the equation $A(x,D)v_\varepsilon = 0$ and integrating by parts we have

$$0 = \int_\Omega (\varepsilon v_\varepsilon | A(x,D)v_\varepsilon)\ dt =$$

$$= \varepsilon\int_{\partial\Omega}\left(\frac{\partial v_\varepsilon}{\partial \nu_a}|v_\varepsilon\right)\ d\sigma - \varepsilon\int_\Omega\left(\sum_{i,j=1}^n a_{ij}(x)D_i v_\varepsilon|D_j v_\varepsilon\right)\ dx$$

which easily implies, using the boundary condition $v_\varepsilon = u_\varepsilon - \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a}$,

$$\varepsilon^2 \int_{\partial\Omega} \left| \frac{\partial v_\varepsilon}{\partial \nu_a} \right|^2 d\sigma + \varepsilon \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) D_i v_\varepsilon | D_j v_\varepsilon \right) dx = \varepsilon \int_{\partial\Omega} \left(\frac{\partial v_\varepsilon}{\partial \nu_a} | u_\varepsilon \right) d\sigma. \tag{4.1.36}$$

Using assumption (H2) this becomes

$$\frac{\varepsilon^2}{2} \int_{\partial\Omega} \left| \frac{\partial v_\varepsilon}{\partial \nu_a} \right|^2 d\sigma + \varepsilon\nu \int_\Omega |Dv_\varepsilon|^2 \, dx \leq \frac{1}{2} \int_{\partial\Omega} |u_\varepsilon|^2 \, d\sigma.$$

Integration on [0,T] gives

$$\frac{1}{2} \int_0^T \left\| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a}(t,\cdot) \right\|^2_{L^2(\partial\Omega;\mathbb{C}^N)} dt + \varepsilon\nu \int_0^T \| Dv_\varepsilon(t,\cdot) \|^2_{L^2(\Omega;\mathbb{C}^N)} \, dt \leq$$

$$\leq \frac{1}{2} \int_0^T \| u_\varepsilon(t,\cdot) \|^2_{L^2(\partial\Omega;\mathbb{C}^N)} \, dt$$

that by the convergence assumption on u_ε implies, up to subsequences,

$$\varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a} \to \lambda \text{ weakly } L^2(0,T;U). \tag{4.1.37}$$

Let us see now that $\lambda = 0$. The operator $\frac{\partial}{\partial \nu_a}$ is well defined from $H^{3/2}(\Omega;\mathbb{C}^N)$ in $L^2(\partial\Omega;\mathbb{C}^N)$ so the adjoint operator is $\left(\frac{\partial}{\partial \nu_a}\right)^* : L^2(\partial\Omega;\mathbb{C}^N) \to (H^{3/2}(\Omega;\mathbb{C}^N))'$. Fix $\phi \in L^2(0,T;U)$: then

$$< \phi, \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a} >_{L^2(0,T;U)L^2(0,T;U)} = \int_0^T < \left(\frac{\partial}{\partial \nu_a}\right)^* \phi, \varepsilon v_\varepsilon >_{(H^{3/2}(\Omega;\mathbb{C}^N))' H^{3/2}(\Omega;\mathbb{C}^N)} \, dt,$$

and since $y = 0$ in (4.1.34)

$$\lim_{\varepsilon\to 0} \int_0^T < \left(\frac{\partial}{\partial \nu_a}\right)^* \phi, \varepsilon v_\varepsilon >_{(H^{3/2}(\Omega;\mathbb{C}^N))' H^{3/2}(\Omega;\mathbb{C}^N)} \, dt = 0$$

thus $\lambda = 0$ in (4.1.37). Now, integrating on $[0,T]$ the (4.1.36) we have

$$\int_0^T \left\| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a}(t,\cdot) \right\|^2_{L^2(\partial\Omega;\mathbb{C}^N)} dt + \varepsilon \int_0^T \int_\Omega \sum_{i,j=1}^n (a_{ij}(x) D_j v_\varepsilon | D_i v_\varepsilon) \, dx \, dt \leq$$

$$\leq \left| \int_0^T < u_\varepsilon, \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a} >_{L^2(\partial\Omega;\mathbb{C}^N) L^2(\partial\Omega;\mathbb{C}^N)} (t,\cdot) \, dt \right|$$

but $u_\varepsilon \to u$ in $L^2(0,T;L^2(\partial\Omega;\mathbb{C}^N))$ and $\varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a} \to 0$ weakly $L^2(0,T;L^2(\partial\Omega;\mathbb{C}^N))$ so

$$\lim_{\varepsilon\to 0} \left| \int_0^T < u_\varepsilon, \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_a} >_{L^2(\partial\Omega;\mathbb{C}^N) L^2(\partial\Omega;\mathbb{C}^N)} (t,\cdot) \, dt \right| = 0$$

this, thanks to the previous estimate, implies (4.1.30), and completes the proof. □

Lemma 4.1.9 *Let* (H0), (H1), (H2) *be fulfilled and* A_ε *and* G_ε *be respectively defined by (4.1.6) and (4.1.7). If* $0 < \vartheta < \frac{1}{4}$, *then*

$$\| (-A_\varepsilon)^\vartheta \, G_\varepsilon u_\varepsilon(s,\cdot)\|_{L^2(\Omega;\mathbb{C}^N)} \le C(\vartheta)\|G_\varepsilon u_\varepsilon(s,\cdot)\|_{H^{1/2}(\Omega;\mathbb{C}^N)}$$

with C *independent of* ε.

Proof. By Theorem 2 in [Fu67] we have that if $f \in H^{2\vartheta}(\Omega;\mathbb{C}^N)$ for $0 < \vartheta < \frac{1}{4}$ then $f \in D([-A_\varepsilon]^\vartheta)$ and

$$\|f\|_{D([-A_\varepsilon]^\vartheta)} \le C(\vartheta)\|f\|_{H^{2\vartheta}(\Omega;\mathbb{C}^N)} \tag{4.1.38}$$

with C independent of ε. Moreover, we prove now that if $0 < \vartheta' < \vartheta < \frac{1}{4}$ and $f \in D([-A_\varepsilon]^\vartheta)$ then

$$\| (-A_\varepsilon)^{\vartheta'} \, f \, \|_{L^2(\Omega;\mathbb{C}^N)} \le C(\vartheta,\vartheta') \, \| \, f \, \|_{D([-A_\varepsilon]^\vartheta)} \tag{4.1.39}$$

with C independent of ε. Suppose first that $f \in D(A_\varepsilon)$, then

$$\begin{aligned} \| (-A_\varepsilon)^{\vartheta'} \, f\|_{L^2(\Omega;\mathbb{C}^N)} &= \| (-A_\varepsilon)^{\vartheta'-1} \, (A_\varepsilon f) \, \|_{L^2(\Omega;\mathbb{C}^N)} = \\ &= \left\| \frac{1}{2\pi i} \int_\gamma (-\lambda)^{\vartheta'-1} \, R(\lambda, A_\varepsilon) \, (A_\varepsilon f) \; d\lambda \right\|_{L^2(\Omega;\mathbb{C}^N)} \le \\ &\le C \left(\int_\gamma |\lambda|^{\vartheta'-\vartheta-1} |d\lambda| \right) \left\| \lambda^\vartheta A_\varepsilon R(\lambda, A_\varepsilon) f \right\|_{L^2(\Omega;\mathbb{C}^N)}. \end{aligned}$$

This implies (4.1.39) for $f \in D(A_\varepsilon)$. Now, if $f \in D([-A_\varepsilon]^\vartheta)$ then $f \in D([-A_\varepsilon]^{\vartheta'})$ and setting $f_n = nR(n, A_\varepsilon)$, we have $f_n \in D(A_\varepsilon)$ and

$$f_n \longrightarrow f \text{ in } D([-A_\varepsilon]^{\vartheta'}), \tag{4.1.40}$$

$$\|f_n\|_{D([-A_\varepsilon]^\vartheta)} \le C\|f\|_{D([-A_\varepsilon]^\vartheta)}. \tag{4.1.41}$$

Hence

$$\begin{aligned} \| (-A_\varepsilon)^{\vartheta'} \, f\|_{L^2(\Omega;\mathbb{C}^N)} &\le \| (-A_\varepsilon)^{\vartheta'} \, (f_n - f)\|_{L^2(\Omega;\mathbb{C}^N)} + \| (-A_\varepsilon)^{\vartheta'} \, f_n\|_{L^2(\Omega;\mathbb{C}^N)} \le \\ &\le \| (-A_\varepsilon)^{\vartheta'} \, (f_n - f)\|_{L^2(\Omega;\mathbb{C}^N)} + C(\vartheta,\vartheta') \, \| \, f_n \, \|_{D([-A_\varepsilon]^\vartheta)} \le \\ &\le \| (-A_\varepsilon)^{\vartheta'} \, (f_n - f)\|_{L^2(\Omega;\mathbb{C}^N)} + C(\vartheta,\vartheta') \, \| \, f \, \|_{D([-A_\varepsilon]^\vartheta)} \end{aligned}$$

by (4.1.39) and (4.1.41). Letting $n \to \infty$ and using (4.1.40) we finally get (4.1.39) for all $f \in D([-A_\varepsilon]^\vartheta)$.
Now, fixed $\vartheta \in]0, \frac{1}{4}[$ by Proposition 4.1.4, $G_\varepsilon u_\varepsilon \in H^{2\vartheta}(\Omega;\mathbb{C}^N)$ and $G_\varepsilon u_\varepsilon \in D([-A_\varepsilon]^\vartheta)$. Hence using (4.1.39) and (4.1.38) we can conclude with the estimate

$$\begin{aligned} \| (-A_\varepsilon)^{\vartheta'} \, G_\varepsilon u_\varepsilon\|_{L^2(\Omega;\mathbb{C}^N)} &\le C(\vartheta,\vartheta')\|G_\varepsilon u_\varepsilon\|_{D([-A_\varepsilon]^\vartheta)} \le C(\vartheta,\vartheta')\|G_\varepsilon u_\varepsilon\|_{H^{2\vartheta}(\Omega;\mathbb{C}^N)} \le \\ &\le C(\vartheta,\vartheta')\|G_\varepsilon u_\varepsilon\|_{H^{1/2}(\Omega;\mathbb{C}^N)} \end{aligned}$$

which implies, for $\vartheta \in]0, \frac{1}{4}[$,

$$\| (-A_\varepsilon)^\vartheta \, G_\varepsilon u_\varepsilon(s,\cdot)\|_{L^2(\Omega;\mathbb{C}^N)} \leq C(\vartheta) \|G_\varepsilon u_\varepsilon(s,\cdot)\|_{H^{1/2}(\Omega;\mathbb{C}^N)}$$

and completes the proof. □

We are finally ready to prove (b).

Theorem 4.1.10 *Assume* (H0), (H1), (H2). *Let B_ε be defined by (4.1.23) and B be defined by (4.1.24).*
If $u_\varepsilon \to u$ in $L^2(0,T;U)$ then $B_\varepsilon(u_\varepsilon) \to B(u)$ in $L^2(0,T;H)$.

Proof. In all this proof we will set, referring to the simplest formulations (4.1.5) and (4.1.20), $f_\varepsilon(s) := A_\varepsilon e^{(t-s)A_\varepsilon} G_\varepsilon u_\varepsilon(s,\cdot)$ and $f_0(s) := A_0 e^{(t-s)A_0} G_0 u(s,\cdot)$.
Fix $0 < s < t$ we have

$$\begin{aligned} f_\varepsilon(s) - f_0(s) &= \frac{1}{2\pi i}\int_\gamma \lambda e^{(t-s)\lambda}\left[R(\lambda,A_\varepsilon)G_\varepsilon u_\varepsilon(s,\cdot) - R(\lambda,A_0)G_0 u(s,\cdot)\right]\, d\lambda = \\ &= \frac{1}{2\pi i}\int_\gamma \lambda e^{(t-s)\lambda} R(\lambda,A_\varepsilon)\left[G_\varepsilon u_\varepsilon(s,\cdot) - G_0 u(s,\cdot)\right]\, d\lambda + \\ &\quad + \frac{1}{2\pi i}\int_\gamma \lambda e^{(t-s)\lambda}\left[R(\lambda,A_\varepsilon) - R(\lambda,A_0)\right](G_0 u(s,\cdot))\, d\lambda. \end{aligned}$$

where $\gamma := \{\lambda \in \mathbb{C} \,:\, |arg(\lambda)| = \eta \,,\, |\lambda| \geq \delta\} \cup \{\lambda \in \mathbb{C} \,:\, |arg(\lambda)| \leq \eta \,,\, |\lambda| = \delta\}$ is contained in the common sector of $\mathbb{C}$ contained in $\rho(A_\varepsilon)$ and $\rho(A_0)$.
Setting $\sigma = (t-s)\lambda$, it becomes

$$\begin{aligned} &\|f_\varepsilon(s) - f_0(s)\|_{L^2(\Omega;\mathbb{C}^N)} \leq \\ &\leq \left\| \frac{1}{2\pi i}\int_\gamma \frac{\sigma}{t-s} e^\sigma R(\frac{\sigma}{t-s}, A_\varepsilon)\left[G_\varepsilon u_\varepsilon(s,\cdot) - G_0 u(s,\cdot)\right]\, \frac{d\sigma}{t-s}\right\|_{L^2(\Omega;\mathbb{C}^N)} + \\ &+ \left\| \frac{1}{2\pi i}\int_\gamma \frac{\sigma}{t-s} e^\sigma \left[R(\frac{\sigma}{t-s} A_\varepsilon) - R(\frac{\sigma}{t-s}, A_0)\right] G_0 u(s,\cdot)\, d\sigma\right\|_{L^2(\Omega;\mathbb{C}^N)} \leq \\ &\leq \frac{C}{t-s} \parallel G_\varepsilon u_\varepsilon(s,\cdot) - G_0 u(s,\cdot) \parallel_{L^2(\Omega;\mathbb{C}^N)} + \frac{C\sqrt{\varepsilon}}{(t-s)^{3/2}} \parallel G_0 u(s,\cdot) \parallel_{L^2(\Omega;\mathbb{C}^N)} \end{aligned}$$

where we used (4.1.4), (4.1.8) and (4.1.28). Hence, recalling also Lemma 4.1.8, we can conclude that, for each fixed $s \in [0,T]$,

$$\lim_{\varepsilon\to 0} \|f_\varepsilon(s) - f_0(s)\|_{L^2(\Omega;\mathbb{C}^N)} = 0. \tag{4.1.42}$$

Moreover, for $0 < \vartheta < \frac{1}{4}$, applying Lemma 2.7 of [AFT91],

$$\begin{aligned}
\| f_\varepsilon(s) \|_{L^2(\Omega;\mathbb{C}^N)} = \| (-A_\varepsilon)^{1-\vartheta} e^{(t-s)A_\varepsilon}(-A_\varepsilon)^\vartheta G_\varepsilon u_\varepsilon(s,\cdot) \|_{L^2(\Omega;\mathbb{C}^N)} &\leq \\
\leq \frac{C}{(t-s)^{1-\vartheta}} \| (-A_\varepsilon)^\vartheta G_\varepsilon u_\varepsilon(s,\cdot) \|_{L^2(\Omega;\mathbb{C}^N)} &\leq \\
\leq \frac{CC(\vartheta)}{(t-s)^{1-\vartheta}} \| G_\varepsilon u_\varepsilon(s,\cdot) \|_{L^2(\Omega;\mathbb{C}^N)} &\leq \\
\leq \frac{C(\vartheta)h(s)}{(t-s)^{1-\vartheta}}
\end{aligned}$$

where the last two estimates are respectively consequences of Lemma 4.1.9 and Lemma 4.1.8 and h is a function in $L^2(0,T;\mathbb{C}^N)$.
Integrating first on $(0,t)$ and then on $(0,T)$ we have

$$\begin{aligned}
\int_0^T \left(\int_0^t \| f_\varepsilon(s) \|_{L^2(\Omega;\mathbb{C}^N)} \, ds \right)^2 dt &\leq C \int_0^T \left(\int_0^t \frac{h(s)}{(t-s)^{1-\vartheta}} \, ds \right)^2 dt \leq \\
&\leq C \int_0^T \left(\int_0^t \frac{1}{(t-s)^{\frac{1-\vartheta}{2}}} \frac{h(s)}{(t-s)^{\frac{1-\vartheta}{2}}} \, ds \right)^2 dt \leq \\
&\leq C \int_0^T \frac{t^\vartheta}{\vartheta} \left(\int_0^t \frac{|h(s)|^2}{(t-s)^{1-\vartheta}} \, ds \right) dt = \\
&= \frac{C}{\vartheta} \int_0^T \left(\int_s^T \frac{t^\vartheta}{(t-s)^{1-\vartheta}} \, dt \right) |h(s)|^2 \, ds \leq \\
&\leq C \frac{T}{\vartheta} \int_0^T (T-s)^\vartheta |h(s)|^2 \, ds \leq \\
&\leq C \frac{T^{2\vartheta}}{\vartheta} \int_0^T |h(s)|^2 \, ds.
\end{aligned}$$

Summarizing

$$\int_0^T \left(\int_0^t \| f_\varepsilon(s) \|_{L^2(\Omega;\mathbb{C}^N)} \, ds \right)^2 dt \leq C \frac{T^{2\vartheta}}{\vartheta} \int_0^T |h(s)|^2 \, ds. \tag{4.1.43}$$

By definition we get

$$\int_0^T \| B_\varepsilon(u_\varepsilon)(t,\cdot) - B(u)(t,\cdot) \|^2_{L^2(\Omega;\mathbb{C}^N)} \, dt \leq \int_0^T \left(\int_0^t \| f_\varepsilon(s) - f_0(s) \|_{L^2(\Omega;\mathbb{C}^N)} \, ds \right)^2 dt$$

so by (4.1.42) and (4.1.43) we can apply the Lebesgue Theorem and conclude. □

4.2 Optimal control problems with mixed boundary conditions

Let us describe now the optimal control problem having as cost functional

$$J(y,u) = \int_0^T \left(\| u(s,\cdot) \|^2_U + \| y_\varepsilon(s,\cdot) \|^2_H \right) ds \tag{4.2.1}$$

where now the control is $u = (u_0, u_1) \in \mathcal{U} := L^2(0,T;U)$ with $U := H^{1/2}(\Gamma_0;\mathbb{C}^N) \times H^{-1/2}(\Gamma_1;\mathbb{C}^N)$ and for each control u the function y is the solution of

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) - \sum_{i,j=1}^n D_i(a_{ij}(x)D_j y(t,x)) = 0 & \text{in } [0,T]\times\overline{\Omega} \\ y(t,x) = u_0(t,x) & \text{in } [0,T]\times\Gamma_0 \\ \frac{\partial}{\partial\nu_a} y(t,x) = u_1(t,x) & \text{in } [0,T]\times\Gamma_1 \\ y(0,x) = y_0(x) & \text{in } \overline{\Omega}. \end{cases} \tag{4.2.2}$$

We assume (H0), (H1), (H2), and

(H3) Γ_0 and Γ_1 are of class C^2 and such that $\Gamma_0 \subset \partial\Omega$, $\Gamma_1 = \partial\Omega\backslash\overline{\Gamma_0}$ and $\overline{\Gamma_0}\cap\overline{\Gamma_1} \neq \emptyset$.

From now on we will denote by $\Sigma := \partial\Omega\times[0,T]$, $\Sigma_0 := \Gamma_0\times[0,T]$ and $\Sigma_1 := \Gamma_1\times[0,T]$. For the solution of system (4.2.2) we can prove the following existence and uniqueness result.

Theorem 4.2.1 *We assume* (H0), (H1), (H2), (H3).
If $u(t,x) \in \mathcal{U} = L^2(0,T;U)$ with $U = H^{1/2}(\Gamma_0;\mathbb{C}^N) \times H^{-1/2}(\Gamma_1;\mathbb{C}^N)$, then there exists a unique solution $y \in L^\infty(0,T;H^1(\Omega;\mathbb{C}^N))$ of (4.2.2).

Proof. Suppose first that $u_0 \in C^\infty(\Sigma_0;\mathbb{C}^N)$ and $u_1 \in C^\infty(\Sigma_1;\mathbb{C}^N)$ and assume the compatibility condition

$$y_0(x) = u_0(0,x) \text{ on } \Gamma_0 \text{ and } \frac{\partial}{\partial\nu_a}y_0(x) = u_1(0,x) \text{ on } \Gamma_1. \tag{4.2.3}$$

By Theorem 4.1 in [Sa97(II)] there exists a unique y solution of

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) - A(x,D)y(t,x) = 0 & \text{in } [0,T]\times\overline{\Omega} \\ y(t,x) = u_0(t,x) & \text{in } [0,T]\times\Gamma_0 \\ \frac{\partial}{\partial\nu_a} y(t,x) = u_1(t,x) & \text{in } [0,T]\times\Gamma_1 \\ y(0,x) = y_0(x) & \text{in } \overline{\Omega}. \end{cases} \tag{4.2.4}$$

Moreover $y \in C(0,T;H^1(\Omega;\mathbb{C}^N))$ and $\frac{\partial y}{\partial t} \in L^2(0,T;L^2(\Omega;\mathbb{C}^N))$. Thus we can multiply by y in $L^2(\Omega;\mathbb{C}^N)$ and integrate on $]\tau,s[$.

$$\begin{aligned} 0 &= \int_\tau^s \int_\Omega (\frac{\partial y}{\partial t}(t,x) - A(x,D)y(t,x)|y(t,x))\,dx\,dt = \\ &= \int_\tau^s \int_\Omega \left[\frac{1}{2}\frac{d}{dt}|y(t,x)|^2 - (A(x,D)y(t,x)|y(t,x))\right]\,dx\,dt = \\ &= \frac{1}{2}\left[\int_\Omega |y(s,x)|^2\,dx - \int_\Omega |y(\tau,x)|^2\,dx\right] - \int_\tau^s \int_{\partial\Omega} (\frac{\partial y}{\partial\nu_a}(t,x)|y(t,x))\,d\sigma\,dt + \\ &\quad + \int_\tau^s \int_\Omega (\sum_{i,j=1}^n a_{ij}(x)D_i y(t,x)|D_j y(t,x))\,dx\,dt. \end{aligned}$$

Applying the boundary condition in (4.2.4) and hypothesis (H2), this yields

$$\frac{1}{2}\left[\int_\Omega |y(s,x)|^2\,dx - \int_\Omega |y(\tau,x)|^2\,dx\right] + \nu\int_\tau^s \int_\Omega |Dy(t,x)|^2\,dx\,dt \leq$$

$$\leq \left| \int_\tau^s \int_{\Gamma_0} (\frac{\partial y}{\partial \nu_a}(t,x)|u_0(t,x))\, d\sigma\, dt \right| + \left| \int_\tau^s \int_{\Gamma_1} (y(t,x)|u_1(t,x))\, d\sigma\, dt \right| \leq$$

$$\leq \varepsilon \int_\tau^s \left\| \frac{\partial y}{\partial \nu_a}(t,\cdot) \right\|^2_{H^{-1/2}(\partial\Omega;\mathbb{C}^N)} dt + C(\varepsilon) \int_\tau^s \|u_0(t,\cdot)\|^2_{H^{1/2}(\Gamma_0;\mathbb{C}^N)}\, dt +$$

$$+\varepsilon \int_\tau^s \|y(t,\cdot)\|^2_{H^{1/2}(\partial\Omega;\mathbb{C}^N)}\, dt + C(\varepsilon) \int_\tau^s \|u_1(t,\cdot)\|^2_{H^{-1/2}(\Gamma_1;\mathbb{C}^N)}\, dt \leq$$

$$\leq C\, \varepsilon \int_\tau^s \|y(t,\cdot)\|^2_{H^1(\Omega;\mathbb{C}^N)}\, dt + C(\varepsilon) \int_\tau^s \|u_0(t,\cdot)\|^2_{H^{1/2}(\Gamma_0;\mathbb{C}^N)}\, dt +$$

$$+C(\varepsilon) \int_\tau^s \|u_1(t,\cdot)\|^2_{H^{-1/2}(\Gamma_1;\mathbb{C}^N)}\, dt. \tag{4.2.5}$$

Consequently, choosing $\tau = 0$, we have for each $s \in [0,T]$

$$\frac{1}{2} \int_\Omega |y(s,x)|^2\, dx + \| \, Dy \, \|_{L^2(0,s;L^2(\Omega;\mathbb{C}^N))} \leq$$

$$\leq \frac{1}{2} \| \, y_0 \, \|^2_{L^2(\Omega;\mathbb{C}^N)} + C\, \varepsilon \, \| \, y \, \|^2_{L^2(0,T;H^1(\Omega;\mathbb{C}^N))} +$$

$$+C(\varepsilon) \int_0^T \| \, u_0(t,\cdot) \, \|^2_{H^{1/2}(\Gamma_0;\mathbb{C}^N)}\, dt + C(\varepsilon) \int_0^T \| \, u_1(t,\cdot) \, \|^2_{H^{-1/2}(\Gamma_1;\mathbb{C}^N)}\, dt.$$

Hence, choosing a sufficient small ε,

$$\sup_{s\in[0,T]} \| \, y(s,\cdot) \, \|^2_{L^2(\Omega;\mathbb{C}^N)} + \| \, Dy \, \|^2_{L^2(0,T;L^2(\Omega;\mathbb{C}^N))} \leq$$

$$\leq C \, \| \, y_0 \, \|^2_{L^2(\Omega;\mathbb{C}^N)} + C \, \| \, u_0 \, \|_{L^2(0,T;H^{1/2}(\Gamma_0;\mathbb{C}^N))} + C \, \| \, u_1 \, \|_{L^2(0,T;H^{-1/2}(\Gamma_1;\mathbb{C}^N))} . \tag{4.2.6}$$

Now if we suppose only $u_0 \in L^2(0,T;H^{1/2}(\Gamma_0;\mathbb{C}^N))$ and $u_1 \in L^2(0,T;H^{-1/2}(\Gamma_1;\mathbb{C}^N))$ estimate (4.2.6) lets us conclude by density arguments. □

This result allows us to choose $H := H^1(\Omega;\mathbb{C}^N)$.

Remark 4.2.2 Note that in the proof of Theorem 4.2.1, leaving r and s arbitrary in estimate (4.2.5) we have

$$\| \, y(s,\cdot) \, \|^2_{L^2(\Omega;\mathbb{C}^N)} - \| \, y(\tau,\cdot) \, \|^2_{L^2(\Omega;\mathbb{C}^N)} \leq$$

$$\leq C \int_\tau^s \|u_0(t,\cdot)\|^2_{H^{1/2}(\Gamma_0;\mathbb{C}^N)}\, dt + C \int_\tau^s \|u_1(t,\cdot)\|^2_{H^{-1/2}(\Gamma_1;\mathbb{C}^N)}\, dt.$$

Therefore, applying a density argument, we can conclude that under the hypotheses of Theorem 4.2.1 the solution y is in $C(0,T;H^1(\Omega;\mathbb{C}^N))$. □

In order to obtain a representation for the solution like (4.1.5) or (4.1.19) we define the following operators.

$$\begin{cases} D(A) := \left\{ v \in H^1(\Omega;\mathbb{C}^N) \ : \ A(x,D)v \in L^2(\Omega;\mathbb{C}^N)\ ,\ v|_{\Gamma_0} = 0\ ,\ \frac{\partial v}{\partial \nu_a}|_{\Gamma_1} = 0 \right\} \\ \\ Av := A(\cdot, D)v \end{cases} \tag{4.2.7}$$

where $A(x,D)$ is defined in (4.1.2).

$$v := G(u_0,u_1) \Leftrightarrow \begin{cases} A(x,D)v = 0 & \text{in } \Omega \\ v = u_0 & \text{on } \Gamma_0 \\ \frac{\partial v}{\partial \nu_a} = u_1 & \text{on } \Gamma_1. \end{cases} \tag{4.2.8}$$

First we need to prove that A is the generator of an analytic semigroup in $L^2(\Omega;\mathbb{C}^N)$.

Proposition 4.2.3 *Under the assumption* (H0), (H2), (H3) *we have:*

(i) *A is the infinitesimal generator of an analytic semigroup.*

(ii) *$D(A)$ is dense in $L^2(\Omega;\mathbb{C}^N)$.*

Proof. First we observe that by the result of [Sa97(I)] (page 827) the resolvent set of the operator A, $\rho(A)$, contains the positive real axis.
To prove (i) we need then the following estimate

$$\| R(\lambda, A) f \|_{L^2(\Omega;\mathbb{C}^N)} \leq \frac{C}{|\lambda|} \| f \|_{L^2(\Omega;\mathbb{C}^N)} \qquad \forall \lambda \in S_{\vartheta_0} \tag{4.2.9}$$

in a suitable sector $S_{\vartheta_0} = \{ z \in \mathbb{C} : |arg(z)| \leq \vartheta_0 \}$ with $\vartheta_0 \in]\frac{\pi}{2}, \pi[$.
Fix $\lambda > 0$ and let $v = R(\lambda, A) f$. Then

$$\begin{cases} \lambda v - A(x,D)v = f & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_0 \\ \frac{\partial v}{\partial \nu_a} = 0 & \text{on } \Gamma_1. \end{cases}$$

Multiplying by v in $L^2(\Omega;\mathbb{C}^N)$ and integrating by parts we get

$$\int_\Omega \lambda (v|v)\, dx + \int_\Omega \sum_{i,j=1}^n (a_{ij}(x) D_j v | D_i v)\, dx = \int_\Omega (f|v)\, dx. \tag{4.2.10}$$

By taking the real part and recalling hypothesis (H2) we obtain

$$Re\lambda \int_\Omega |v|^2\, dx + \nu \int_\Omega |Dv|^2\, dx \leq \| f \|_{L^2(\Omega;\mathbb{C}^N)} \| v \|_{L^2(\Omega;\mathbb{C}^N)}$$

while taking the imaginary part,

$$|Im\lambda| \int_\Omega |v|^2\, dx \leq \| f \|_{L^2(\Omega;\mathbb{C}^N)} \| v \|_{L^2(\Omega;\mathbb{C}^N)} + M \int_\Omega |Dv|^2\, dx.$$

Now if $|Re\lambda| \leq \delta |Im\lambda|$, then

$$|\lambda| \int_\Omega |v|^2\, dx \leq \left(\sqrt{1+\delta^2}\right)\ |Im\lambda| \int_\Omega |v|^2\, dx \leq$$
$$\leq \sqrt{1+\delta^2}\left(1+\frac{M}{\nu}\right) \| f \|_{L^2(\Omega;\mathbb{C}^N)} \| v \|_{L^2(\Omega;\mathbb{C}^N)}$$

whereas if $Re\lambda$ and $\delta|Im\lambda| < Re\lambda$ we have

$$|\lambda| \int_\Omega |v|^2\, dx \leq \left(\sqrt{1+\frac{1}{\delta^2}}\right) Re\lambda \int_\Omega |v|^2\, dx \leq$$
$$\leq \sqrt{1+\frac{1}{\delta^2}} \| f \|_{L^2(\Omega;\mathbb{C}^N)} \| v \|_{L^2(\Omega;\mathbb{C}^N)} .$$

These two estimates show that for any $\delta > 0$ we have

$$|\lambda| \| v \|_{L^2(\Omega;\mathbb{C}^N)} \leq \sqrt{1+\delta^2}\left[\left(1+\frac{M}{\nu}\right)+\frac{1}{\delta}\right] \| f \|_{L^2(\Omega;\mathbb{C}^N)} . \tag{4.2.11}$$

Note that the bigger is δ the better is our estimate and the best δ we can find is $\delta = 1/\left(1+\frac{M}{\nu}\right)^{1/3}$.
Summarizing, by the result in [Sa97(I)] we can choose $\lambda > 0$; estimate (4.2.11), by standard arguments, shows that we can choose λ in the half plane $\{\lambda \in \mathbb{C} \ :\ Re\lambda > 0\}$ so that Proposition 2.1.11 in [Lu95] implies that A is sectorial. (See also Appendix C). Therefore A is the infinitesimal generator of an analytic semigroup and setting $C = \sqrt{1+\delta^2}\left[\left(1+\frac{M}{\nu}\right)+\frac{1}{\delta}\right]$ with $\delta = 1/\left(1+\frac{M}{\nu}\right)^{1/3}$ in estimate (4.2.11) we get

$$\| v \|_{L^2(\Omega;\mathbb{C}^N)} \leq \frac{C}{|\lambda|} \| f \|_{L^2(\Omega;\mathbb{C}^N)} \quad \forall \lambda \in S_{\vartheta_0}$$

where $\vartheta_0 = \pi - \arctan\delta$. This gives us estimate (4.2.9) and completes the proof of (i).
The proof of part (ii) is obvious. $\square$

The adjoint operator of A is the following

$$\begin{cases} D(A^*) := \Big\{ v \in H^1(\Omega;\mathbb{C}^N)\ :\ \overline{A(x,D)}v \in L^2(\Omega;\mathbb{C}^N)\,,\ v|_{\Gamma_0} = 0\,, \\ \qquad \left(\sum_{i,j=1}^n \overline{a_{ij}(x)}^t \nu_i \cdot D_j v\right)|_{\Gamma_1} = 0 \Big\} \\ A^* v := \overline{A(\cdot,D)}v \end{cases} \tag{4.2.12}$$

where $\overline{a_{ij}(x)}^t$ is the matrix whose elements are the conjugates of the elements of the transposed $a_{ij}(x)^t$ of $a_{ij}(x)$ and $\overline{A(x,D)}v := \sum_{i,j=1}^n D_i\left[\overline{a_{ij}(x)}^t \cdot D_j v\right]$. Consequently it is clear that the following result holds.

Proposition 4.2.4 *Under the assumption* (H0), (H2), (H3) *we have:*

(i) A^* *is the infinitesimal generator of an analytic semigroup.*

(ii) *$D(A^*)$ is dense in $L^2(\Omega;\mathbb{C}^N)$.*

On the operator G we prove the following.

Proposition 4.2.5 *Let* (H0), (H1), (H2), (H3) *be fulfilled. If A is defined in (4.2.7) then the operator G is well defined from $H^{1/2}(\Gamma_0;\mathbb{C}^N)\times H^{-1/2}(\Gamma_1;\mathbb{C}^N)$ into $D([-A]^\vartheta)$ for $0\le\vartheta<\frac{1}{4}$.*

Proof. Theorems 2 and 3 at page 876 in [Sa97(I)] tell us that for a fixed $(u_0,u_1)\in H^{1/2}(\Gamma_0;\mathbb{C}^N)\times H^{-1/2}(\Gamma_1;\mathbb{C}^N)$ there exists a unique $v=G(u_0,u_1)$ such that $v\in H^1(\Omega;\mathbb{C}^N)$. On the other hand Theorem 3.1 at page 84 of [Ban94] ensure us that $D([-A]^\vartheta)=H^{2\vartheta}(\Omega;\mathbb{C}^N)$ for $0\le\vartheta<\frac{1}{4}$.
Recalling that $H^1(\Omega;\mathbb{C}^N)\subset H^{2\vartheta}(\Omega;\mathbb{C}^N)$ for every $0\le\vartheta<\frac{1}{4}$ the proof is achieved. □

We are finally ready to prove the representation formula for the solution.

Theorem 4.2.6 *Assume* (H0), (H1), (H2), (H3).
If $u=(u_0,u_1)\in H^{1/2}(\Gamma_0;\mathbb{C}^N)\times H^{-1/2}(\Gamma_1;\mathbb{C}^N)$ then the solution of (4.2.2) is given by

$$y(t,\cdot)=e^{tA}y_0(\cdot)+\int_0^t\left[(-A^*)^{1-\vartheta}(e^{(t-s)A})^*\right]^*\,(-A)^\vartheta\,G\,u(s,\cdot)\,ds \tag{4.2.13}$$

for $t\in[0,T]$ and $\vartheta\in]0,\frac{1}{4}[$.

Proof. By Theorem 4.2.1 we know that system (4.2.2) have a solution

$$y\in L^\infty(0,T;H^1(\Omega;\mathbb{C}^N)). \tag{4.2.14}$$

Consider the function $y(t,\cdot)-Gu(t,\cdot)$, we verify that $y(t,\cdot)-Gu(t.\cdot)\in D(A)$.
Indeed, by (4.2.14) $y(t,\cdot)\in H^1(\Omega;\mathbb{C}^N)$, Theorem 2 and 3 of [Sa97(I)] imply $Gu(t,\cdot)\in H^1(\Omega;\mathbb{C}^N)$ and by definition $(y(t,\cdot)-Gu(t,\cdot))\,|_{\Gamma_0}=0$ and $\frac{\partial}{\partial\nu_a}(y(t,\cdot)-Gu(t,\cdot))\,|_{\Gamma_1}=0$.
Moreover,

$$A\,[y(t,\cdot)-Gu(t,\cdot)]=A(\cdot,D)y(t,\cdot). \tag{4.2.15}$$

Now, fix $t\in[0,T]$ let $z\in D(A^*)$ and define

$$h(s):=<y(s,\cdot),\left(e^{(t-s)A}\right)^*z>_{L^2(\Omega;\mathbb{C}^N)}. \tag{4.2.16}$$

For simplicity let us write $<\cdot,\cdot>$ for $<\cdot,\cdot>_{L^2(\Omega;\mathbb{C}^N)}$. By the standard results about analytic semigroup, that we can apply thanks to Proposition 4.2.3 (i), we may compute

$$\begin{aligned}
h'(s)&=<\frac{d}{ds}y(s,\cdot),\left(e^{(t-s)A}\right)^*z>+<y(s,\cdot),\frac{d}{ds}\left(e^{(t-s)A}\right)^*z>=\\
&=<\frac{d}{ds}y(s,\cdot),\left(e^{(t-s)A}\right)^*z>+<y(s,\cdot),-A^*\left(e^{(t-s)A}\right)^*z>=\\
&=<\frac{d}{ds}y(s,\cdot),\left(e^{(t-s)A}\right)^*z>-<A\,[y(t,\cdot)-Gu(t,\cdot)]\,,\left(e^{(t-s)A}\right)^*z>+\\
&\quad-<Gu(s,\cdot),A^*\left(e^{(t-s)A}\right)^*z>=\\
&=<A(\cdot,D)y(s,\cdot),\left(e^{(t-s)A}\right)^*z>-<A(\cdot,D)y(s,\cdot),\left(e^{(t-s)A}\right)^*z>+\\
&\quad-<Gu(s,\cdot),A^*\left(e^{(t-s)A}\right)^*z>=\\
&=-<Gu(s,\cdot),A^*\left(e^{(t-s)A}\right)^*z>
\end{aligned}$$

where we used (4.2.15) and the fact that y solves (4.2.2). Consequently, by Proposition 4.2.5, we have for $0 \leq \vartheta < \frac{1}{4}$,

$$h'(s) =< (-A)^{\vartheta} (Gu(s,\cdot)), \left((-A)^{1-\vartheta}\right)^* \left(e^{(t-s)A}\right)^* z > . \tag{4.2.17}$$

Now, Proposition 4.2.5 tells us that $(-A)^{\vartheta} Gu(s,\cdot) \in L^2(\Omega;\mathbb{C}^N)$ and corollary 2.10 of [AFT91] gives

$$\left\| \left[(-A^*)^{\gamma} \left(e^{(t-s)A}\right)^*\right]^* \right\|_{\mathcal{L}\left(L^2(\Omega;\mathbb{C}^N)\right)} \leq \frac{M_\gamma}{(t-s)^\gamma} \quad 0 \leq \gamma \leq 1\,,\;\; 0 \leq s < t. \tag{4.2.18}$$

So we can write (4.2.17) as

$$h'(s) =< \left[\left((-A)^{1-\vartheta}\right)^* \left(e^{(t-s)A}\right)^*\right]^* (-A)^{\vartheta} Gu(s,\cdot), z > \tag{4.2.19}$$

and $h \in L^1(0,t)$. Hence integrating on $]0,t[$ we get, recalling (4.2.16),

$$< y(t,\cdot), z > - < y(0,\cdot), \left(e^{(t-s)A}\right)^* z >=$$

$$= \int_0^t < \left[\left((-A)^{1-\vartheta}\right)^* \left(e^{(t-s)A}\right)^*\right]^* (-A)^{\vartheta} (Gu(s,\cdot)), z > \; ds. \tag{4.2.20}$$

Since, by Proposition 4.2.4, $D(A^*)$ is dense in $L^2(\Omega;\mathbb{C}^N)$ we deduce (4.2.13) from (4.2.20) by a density argument. □

With this formulation we can finally apply Theorem 3.14 of [AFT91] to characterize the optimal control through the Riccati equation. The result is summarized in the following theorem.

Theorem 4.2.7 *Let* (H0), (H1), (H2), (H3) *be fulfilled. Then:*

(i) *There exists a unique optimal control* $\hat{u} \in L^2(0,T;U)$*;*

(ii) *Denoting by* $P(\cdot)$ *the solution of the Riccati equation, given in integral form,*

$$P(t) = \int_t^T \left[e^{(s-t)A}\right]^* [I - P(s)\, A\, G\, G^*\, A^*\, P(s)]\, e^{(s-t)A}\, ds$$

and by $\hat{y} \in L^2(0,T;H)$ *the optimal trajectory, we have*

$$J(\hat{y},\hat{u}) =< P(0)y_0, y_0 >_H;$$

(iii) *We have the feedback formula for* $\hat{u}$*:*

$$\hat{u}(t,\cdot) = \; G^*\, A^*\, P(t)\, \hat{y}(t,\cdot) \quad t \in [0,T[;$$

(iv) *The optimal trajectory is expressed by* $\hat{y}(t,\cdot) = \Phi(t,0)y_0(\cdot)$ *where* $\Phi(t,s)$ *is defined by the integral equation*

$$\Phi(t,s)x = e^{(t-s)A}x - \int_s^t e^{(t-r)A}\, A\, G\, G^*\, A^*\, P(r)\, \Phi(r,s)x\, dr \quad t \in [s,T[.$$

In (ii), (iii) *and in* (iv) *we have set*

$$\begin{aligned} G^* A^* P(s) &= -\left[[-A]^{\vartheta}\, G\right]^* [-A^*]^{1-\vartheta}\, P(s), \\ e^{sA} A G &= -\,[-A]^{1-\vartheta}\, e^{sA}\, [-A]^{\vartheta}\, G \end{aligned}$$

where $\vartheta \in]0, \frac{1}{4}[$.

Appendix A

Some functional spaces

(I) Let $\Omega \subset \mathbb{R}^m$ with $m \in \mathbb{N}$, X be a Banach space, $k \in \mathbb{N}$ and $\vartheta \in]0,1[$.

$$C^k(\Omega; X) := \{f : \Omega \to X \ : \ f \text{ is } k \text{ times continuously differentiable}\}$$

$$C^{k+\vartheta}(\Omega; X) := \left\{f : \Omega \to X \ : \ f \in C^k(\Omega; X),\ f^{(k)} \text{ is } \vartheta - \text{Hölder continuous}\right\}$$

$$C_c^k(\Omega; X) := \{f : \Omega \to X \ : \ f \text{ is } k \text{ times continuously differentiable with compact support on } \Omega\}.$$

The set $C_0(\Omega; X)$ is the closure of $C_c^k(\Omega; X)$ in the supremum norm.
If $k = 0$ we write $C(\Omega; X)$ for $C^0(\Omega; X)$.
If $X = \mathbb{R}$ we set $C^k(\Omega)$ for $C^k(\Omega; \mathbb{R})$.

(II) Let X be a Banach space with norm $\|\cdot\|_X$.

$$\mathcal{L}(X) := \{T : X \to X \text{ bounded linear operator}\}$$

$$\|T\|_{\mathcal{L}(X)} := \sup_{0 \neq x \in X} \frac{\| Tx \|_X}{\| x \|_X}.$$

(III) Let X be a Banach space with norm $\|\cdot\|_X$ and $1 \leq p < \infty$ an integer.

$$L^p(0,T; X) := \left\{f : (0,T) \to X \ : \ f \text{ is measurable, } \int_0^T \| f(t) \|_X^p \ dt < \infty\right\}$$

$$\|f\|_{L^p(0,T;X)} := \left(\int_0^T \| f(t) \|_X^p \ dt\right)^{1/p}$$

$$W^{1,1}(0,T; X) := \left\{f : (0,T) \to X \ : \ f \in L^1(0,T; X) \ , \ f' \in L^1(0,T; X)\right\}$$

$$L^\infty(0,T;X) := \{f : (0,T) \to X \ : \ f \text{ is measurable}, \exists\, C > 0 \ : \| f(t) \|_X \le C \text{ a.e. on } (0,T)\}$$

$$\|f\|_{L^\infty(0,T;X)} := \inf\{C \ : \| f(t) \|_X \le C \text{ a.e. on } (0,T)\}.$$

If $X = \mathbb{R}$ we write $L^p(0,T)$ for $L^p(0,T;\mathbb{R})$ for $1 \le p \le \infty$.

(IV) Let μ be a Borel measure on $(0,T)$. Let $1 \le p < \infty$ be an integer.

$$L^p_\mu(0,T) := \left\{ f \ : \ (0,T) \to \mathbb{R} \ : \ f \text{ is } \mu - \text{measurable}, \ \int_{[0,T]} |f(t)|^p \, d\mu < \infty \right\}$$

$$\| f \|_{L^p_\mu(0,T)} := \left(\int_{[0,T]} |f(t)|^p \, d\mu \right)^{1/p} .$$

(V) Fix $T > 0$.

$$\mathcal{M}(0,T) := \{\mu \text{ Borel measure on } (0,T)\}$$

$$\mathcal{M}^+(0,T) := \{\mu \text{ non negative Borel measure on } (0,T)\} .$$

If $(a,b) \subseteq [0,T]$.

$$BV(a,b) := \{f : (0,T) \to \mathbb{R} \ : \ \exists\, C \ge 0 \ : \ \forall \text{ finite set of points } a = t_0 < t_1 < ... < t_p = b \text{ the inequality } \sum_{k=1}^{p} |f(t_k) - f(t_{k-1})| \le C \text{ holds. } \}$$

In particular we will denote by $BV^-([0,T])$ the subset of left continuous BV functions.
Let $\Omega \subset \mathbb{R}^p$ be an open subset.

$$BV(\Omega) := \left\{ u \in L^1(\Omega) \ : \ |Du|(\Omega) := \sup \left\{ \int_\Omega u \, div \, g \ dx \ \ : \ g \in C^1_0(\Omega;\mathbb{R}^p) \ \ |g| \le 1 \right\} < \infty \right\} .$$

(VI) Let Ω be an open subset of $\mathbb{R}^n$.

$$L^2(\Omega;\mathbb{C}^N) := \left\{ f : \Omega \to \mathbb{C}^N \ : \ f \text{ is measurable}, \ \int_\Omega |f(x)|^2 \, dx < \infty \right\}$$

$$\|f\|_{L^2(\Omega;\mathbb{C}^N)} := \left(\int_\Omega |f(x)|^2 \, dx \right)^{1/2} .$$

Let $m \ge 1$ be an integer.

$$H^m(\Omega;\mathbb{C}^N) := \left\{ f : \Omega \to \mathbb{C}^N \ : \ D^\alpha f \in L^2(\Omega;\mathbb{C}^N) \ \forall \alpha \ : \ |\alpha| \le m \right\}$$

where $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_m\}$, $D^\alpha = \frac{\partial^{\alpha_1+\alpha_2+...+\alpha_m}}{\partial x_1^{\alpha_1}...\partial x_m^{\alpha_m}}$ and $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_m$.

$$\| f \|_{H^m(\Omega;\mathbb{C}^N)} := \left(\sum_{|\alpha|\leq m} \| D^\alpha f \|^2_{L^2(\Omega;\mathbb{C}^N)} \right)^{1/2}.$$

The set $H_0^m(\Omega;\mathbb{C}^N)$ is the closure of $C_c^\infty(\Omega;\mathbb{C}^N)$ in the norm of $H^m(\Omega;\mathbb{C}^N)$.

(VII) If $\vartheta \geq 0$ is a real number we set $\vartheta = m + \alpha$ with $m \in \mathbb{N}$ and $0 < \alpha < 1$. Let $\Omega \subseteq \mathbb{R}^n$.

$$H^\vartheta(\Omega;\mathbb{C}^N) := \left\{ f : \Omega \to \mathbb{C}^N \ : \ f \in H^m(\Omega;\mathbb{C}^N)\ ,\ |D^\beta f|_{\alpha,2,\Omega} < \infty \text{ for } |\beta| = m \right\}$$

where

$$|f|^2_{\alpha,2,\Omega} := \int_\Omega \left(\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+2\vartheta}}\, dy \right) dx.$$

The set $H_0^\vartheta(\Omega;\mathbb{C}^N)$ is the closure of $C_c^\infty(\Omega;\mathbb{C}^N)$ in the norm of $H^\vartheta(\Omega;\mathbb{C}^N)$.

$$H^{-1/2}(\Omega;\mathbb{C}^N) := \left[H_0^{1/2}(\Omega;\mathbb{C}^N) \right]'.$$

Appendix B

Basic notation on measure theory and functions of bounded variation

This Appendix is devoted only to the main definitions and results we used in this thesis. For all proofs and details we refer to [Am97] or any classical book on measure theory.

B.1 Measure theory

Let Ω be a separable and locally compact space. We will denote by $\mathcal{B}(\Omega)$ the σ-algebra of Borel subsets of Ω. We start by recalling the definition of measure.

Definition B.1.1 *A* Borel measure *μ is an application $\mu : \Omega \to \mathbb{R}^p$ $(p \geq 1)$ such that:*

(1) $\mu(\emptyset) = 0$.

(2) *For each family $\{B_h\}_{h\in\mathbb{N}}$ such that $B_h \in \mathcal{B}(\Omega)$, $\cup_{h\in\mathbb{N}} B_h = B$ and $B_h \cap B_k = \emptyset$ for each $h \neq k$ we have*
$$\mu(B) = \sum_{h\in\mathbb{N}} \mu(B_h).$$

A measure is said to be non negative if the application μ takes values in $[0, \infty]$.

In this thesis we deal with the following space of measures.

$$\mathcal{M}^+(0,T) := \{\mu \text{ non negative Borel measure on } (0,T)\}.$$

We need the following definitions.

Definition B.1.2 *If $\mu : \Omega \to \mathbb{R}^p$ is a Borel measure the* total variation*, $|\mu|$, of μ is*

$$|\mu|(B) := \sup\left\{\sum_{i=1}^{\infty} |\mu(B_i)| \ : \ B_i \cap B_J = \emptyset \ \forall i, j \ \ \cup_{i=1}^{\infty} B_i = B\right\}.$$

Definition B.1.3 *If $\mu : \Omega \to \mathbb{R}^p$ is a Borel measure the* support *of μ, $supp(\mu)$, is the smallest closed subset C of Ω such that $|\mu|(\Omega \setminus C) = 0$.*

Definition B.1.4 *Given two Borel measures $\mu : \Omega \to [0, \infty]$ and $\lambda : \Omega \to \mathbb{R}^p$, we say that:*

(i) λ *is* absolutely continuous *with respect to* μ, $\lambda << \mu$, *if* $\mu(B) = 0 \Rightarrow |\lambda|(\mu) = 0$.

(ii) λ *is* concentrated *on* E *if* $B \in \mathcal{B}(\Omega)$, $B \cap E = \emptyset \Rightarrow |\lambda|(B) = 0$.

(iii) λ *and* μ *are* singular, $\lambda \perp \mu$, *if there exist* $A, B \in \mathcal{B}(\Omega)$ *such that* $A \cap B = \emptyset$, λ *is concentrated on* A *and* μ *is concentrated on* B.

We state now one of the main results in measure theory.

Theorem B.1.5 Radon-Nikodym.
Let $\mu : \Omega \to [0, \infty]$ and $\lambda : \Omega \to \mathbb{R}^p$ be two Borel measures. Then there exist two measures $\lambda^a : \Omega \to \mathbb{R}^p$ and $\lambda^s : \Omega \to \mathbb{R}^p$ such that $\lambda^a << \mu$ and $\lambda^s \perp \mu$.
Moreover, there exists a function $f \in L^1_\mu(\Omega; \mathbb{R}^p)$ such that

$$\lambda = f \cdot \mu + \lambda^s.$$

The function f is usually denoted by $\frac{d\lambda}{d\mu}$ and called the derivative of λ with respect to μ. We define

$$L^1_\mu(\Omega; \mathbb{R}^p) := \left\{ f : \Omega \to \mathbb{R}^p \ : \ f \text{ is } \mu - \text{measurable} \ \ , \ \int_\Omega |f(x)| \, d\mu < \infty \right\}.$$

The space of measures on Ω can be described with a more functional approach.
Let $C_c(\Omega; \mathbb{R}^p)$ be the space of continuous functions $f : \Omega \to \mathbb{R}^p$ with compact support on Ω and $C_0(\Omega; \mathbb{R}^p)$ its closure on the supremum norm. One can prove that

$$\phi \in C_0(\Omega; \mathbb{R}^p) \iff \forall \varepsilon > 0 \quad \exists K \subset \Omega \text{ compact } : \ |\phi| < \varepsilon \text{ on } \Omega \setminus K$$

and that $C_0(\Omega; \mathbb{R}^p)$ is a separable Banach space. The space of measures turns out to coincide with the dual of this space. More precisely one can prove.

Theorem B.1.6 Riesz.
For each functional L in the dual of $C_0(\Omega; \mathbb{R}^p)$ there exists a unique measure $\mu : \Omega \to \mathbb{R}^p$ such that

$$L(\phi) = \sum_{i=1}^{p} \int_\Omega \phi_i(x) d\mu_i(x) \quad \forall \phi \in C_0(\Omega; \mathbb{R}^p)$$

and conversely. Moreover $\| L \|_{\mathcal{L}(C_0(\Omega;\mathbb{R}^p);\mathbb{R}^p)} = |\mu|(\Omega)$.

From this characterization follows naturally the definition of a weak convergence for a sequence of measures.

Definition B.1.7 *A sequence of measures $(\mu_h)_{h \in \mathbb{N}}$ weakly* converges to a measure μ if the sequence of functional $(L_{\mu_h})_{h \in \mathbb{N}}$ weakly* converges to the functional L_μ. More precisely if*

$$\lim_{h \to \infty} \sum_{i=1}^{p} \int_\Omega \phi_i(x) d\mu_{h,i}(x) = \sum_{i=1}^{p} \int_\Omega \phi_i(x) d\mu_i(x) \quad \forall \phi \in C_0(\Omega; \mathbb{R}^p).$$

This convergence can be characterized as follows.

Proposition B.1.8 *Let $\mu : \Omega \to \mathbb{R}^p$ and $\mu_h : \Omega \to \mathbb{R}^p$ be Borel measures. The following statements are equivalent.*

(i) *The sequence $(\mu_h)_{h\in\mathbb{N}}$ weakly* converges to μ.*

(ii) $\sup_{h\in\mathbb{N}} |\mu_h|(\Omega) < \infty$ *and*

$$\lim_{h\to\infty} \sum_{i=1}^{p} \int_\Omega \phi_i(x) d\mu_{h,i}(x) = \sum_{i=1}^{p} \int_\Omega \phi_i(x) d\mu_i(x) \quad \forall \phi \in C_0^\infty(\Omega; \mathbb{R}^p).$$

We state now the main properties of the weak* convergence.

Proposition B.1.9 *Let $(\mu_h)_{h\in\mathbb{N}}$ be a sequence of Borel measures from Ω in $\mathbb{R}^p$ such that the sequence $(|\mu_h|(\Omega))_{h\in\mathbb{N}}$ is bounded. Then*

(i) *there exists a subsequence $(\mu_h)_{h\in\mathbb{N}}$ weakly* converging on Ω.*

(ii) *If $p = 1$ and $\mu_h : \Omega \to [0, \infty]$ then $\mu_h \to \mu$ weakly* implies:*

$$\liminf_{h\to\infty} \mu_h(A) \geq \mu(A) \text{ for each open } A \subset \Omega,$$

$$\limsup_{h\to\infty} \mu(K) \leq \mu(K) \text{ for each compact } K \subset \Omega.$$

(iii) *If $\mu_h \to \mu$ weakly* and $|\mu_h| \leq \sigma$ then $|\mu| \leq \sigma$ and*

$$\lim_{h\to\infty} \mu_h(E) = \mu(E)$$

for each Borel $E \subset\subset \Omega$ such that $\sigma(\partial E) = 0$.

We denote by ∂E the topological boundary of E.
We end this section by giving the definition of Hausdorff measure.

Definition B.1.10 *Let $K \geq 0$ and $B \subset \mathbb{R}^p$. The k-dimensional Hausdorff measure of B is*

$$\mathcal{H}^k(B) := \sup_{\delta>0} \mathcal{H}^k_\delta(B)$$

where for $0 < \delta \leq \infty$, $\mathcal{H}^k_\delta$ is defined by

$$\mathcal{H}^k_\delta := \frac{\omega_k}{2^k} \inf \left\{ \sum_{i\in I} (diam(B_i))^k \ : \ diam(B_i) < \delta \ \ B \subset \cup_{i\in I} B_I \right\}$$

and the constant ω_k is $\frac{\pi^{k/2}}{\Gamma(1+k/2)}$, Γ being the Euler function $\Gamma(t) := \int_0^\infty s^{t-1} e^{-s}\, ds$.

B.2 Functions of bounded variation

We start by considering the functions defined on $\mathbb{R}$.

Definition B.2.1 *A function $f : [0,T] \to \mathbb{R}$ is said to have bounded variation on the subinterval $[a,b] \subseteq [0,T]$ if there exists a constant $C \geq 0$ such that, for each finite set of points $t_0, t_1, ..., t_p$ satisfying*

$$a = t_0 < t_1 < ... < t_p = b$$

the inequality

$$\sum_{k=1}^{p} |f(t_k) - f(t_{k-1})| \leq C$$

holds. The least C which satisfies the above condition is called the variation of f on $[a,b]$, and is denoted by $V_a^b(f)$. The number $V_0^T(f)$ will be called the total variation of f.

The symbol $BV^-([0,T])$ will indicate the set of all left continuous functions with bounded total variation which are right continuous at 0. For every $u \in BV^-([0,T])$, the distributional derivatives $\dot{u}$ is a signed measure on $]0,T[$ characterized by

$$\dot{u}([t_1,t_2[) = u(t_2) - u(t_1)$$

for every subinterval $[t_1,t_2[\subseteq]0,T[$. In particular, for every $t \in]0,T[$,

$$\dot{u}(t) = [u](t) = u(t^+) - u(t^-)$$

where $u(t^+) = \lim_{s\to t^+} u(s)$ and $u(t^-) = \lim_{s \to t^-} u(s)$.
We consider now functions defined on $\mathbb{R}^p$. The definition of function with bounded variation can be given in the following way.

Definition B.2.2 *Let $\Omega \subset \mathbb{R}^p$ be an open subset and let $u \in L^1(\Omega)$. We say that u is a function of* bounded variation *in Ω if*

$$|Du|(\Omega) := \sup \left\{ \int_\Omega u \; div \; g \;\, dx \quad : \; g \in C_0^1(\Omega;\mathbb{R}^p) \;\; |g| \leq 1 \right\} < \infty.$$

We will denote the space of this functions by $BV(\Omega)$.

This space can be characterized also in a distributional way as follows.

Proposition B.2.3 *Let $u \in L^1(\Omega)$. The following statements are equivalent.*

(i) $u \in BV(\Omega)$.

(ii) *There exists a measure $Du = (Du_1, ..., Du_n)$ on Ω such that $|Du|(\Omega) \leq M < \infty$ and*

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega \phi(x) dD_i u \quad \forall \phi \in C_c^1(\Omega),$$

i.e. Du is the distributional derivatives of u.

Moreover the smallest M in (ii) *coincides with the supremum in Definition B.2.2.*

We end this appendix by giving the definition of set of finite perimeter and recalling the formula that links this notion with the BV functions.

Definition B.2.4 *Let E be a measurable subset of $\mathbb{R}^p$. We say that E is a set of* finite perimeter *in an open subset Ω of $\mathbb{R}^p$ if*

$$P(E,\Omega) := \sup\left\{\int_E div\ g\ dx \quad : \ g \in C_0^1(\Omega;\mathbb{R}^p)\ \ |g| \le 1\right\} < \infty.$$

Theorem B.2.5 Coarea formula of Fleming-Rischel.
For each $u \in L^1(\Omega)$ we have

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} P(\{u > t\},\Omega)\, dt.$$

In particular if $u \in BV(\Omega)$ then the set $\{u > t\}$ has finite perimeter in Ω for almost every $t \in \mathbb{R}$.

Appendix C

Basic theory on analytic semigroups and fractional powers of operators

C.1 Analytic semigroups

In this appendix we will recall only the main properties of analytic semigroups we used in Chapter 4. For the general theory one can see, among others, [Pa83] and [Go85].
In all this Appendix X will denote a complex Banach space with norm $\|\cdot\|_X$.
First we recall the definition of semigroup.

Definition C.1.1 *A family of linear operators* $\{G(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ *is said to be a* semigroup *if*

$$\begin{cases} G(t)G(s) = G(t+s) & t, s \geq 0 \\ \\ G(0) = I. \end{cases}$$

A semigroup $G(t)$ is said to be *analytic* if the function $t \to G(t)$ is analytic in $]0, \infty[$ with values in $\mathcal{L}(X)$. It is said to be *strongly continuous* if for each $x \in X$ the function $t \to G(t)x$ is continuous on $[0, \infty[$.
Let $G(t)$ be a semigroup and let us set

$$\Delta_h := \frac{G(h) - I}{h} \quad h > 0.$$

The *infinitesimal generator* of $G(t)$ is the following operator.

$$\begin{cases} D(A) := \{x \in X \ : \ \exists \lim_{h\to 0} \Delta_h x \ \text{ in } \ \|\cdot\|_X\} \\ \\ Ax := \lim_{h\to 0} \Delta_h x. \end{cases}$$

The following theorem gives a sufficient condition for an operator to be the infinitesimal generator of a strongly continuous semigroup in an Hilbert space E.

Theorem C.1.2 Lumer Phillips.
Let $A : D(A) \subset E \to E$ be a linear closed operator with $\overline{D(A)} = X$. If there exist $\omega \in \mathbb{R}$ such that

$$Re < Ax, x >_E \leq \omega \parallel x \parallel_E^2 \ \forall x \in D(A) \text{ and } Re < x, A^* x >_E \leq \omega \parallel x \parallel_E^2 \ \forall x \in D(A^*)$$

then A is the infinitesimal generator of a strongly continuous semigroup $G(t)$ and

$$\parallel G(t) \parallel_{\mathcal{L}(E)} \leq e^{\omega t}.$$

Let $A : D(A) \subset X \to X$ be a linear operator with domain $D(A)$. We will denote by $R(\lambda, A)$ the operator $(\lambda - A)^{-1}$, by $\rho(A) := \{\lambda \in \mathbb{C} : R(\lambda, A) \in \mathcal{L}(X)\}$ the *resolvent* set of A and by $\sigma(A) := \mathbb{C} \setminus \rho(A)$ the *spectrum* of A. We define.

Definition C.1.3 *An operator $A : D(A) \subset X \to X$ is said to be* sectorial *if there are constants $\omega \in \mathbb{R}$, $\vartheta \in]\frac{\pi}{2}, \pi[$ and $M > 0$ such that*

$$\begin{cases} (i) \ \ \rho(A) \supset S_{\vartheta,\omega} := \{\lambda \in \mathbb{C} \ : \ \lambda \neq \omega \ , \ |arg(\lambda - \omega)| < \vartheta\} \, , \\ \\ (ii) \ \|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|} \ \forall \lambda \in S_{\vartheta,\omega}. \end{cases}$$

We give now a sufficient condition to be a sectorial operator. (For the proof see Proposition 2.1.11 in [Lu95]).

Proposition C.1.4 *Let $A : D(A) \subset X \to X$ be a linear operator such that $\rho(A)$ contains a half plane $\{\lambda \in \mathbb{C} \ : \ Re\lambda \geq \omega\}$, and*

$$\parallel R(\lambda, A) \parallel_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|} \qquad Re\lambda \geq \omega,$$

with $\omega \in \mathbb{R}$, $M > 0$. Then A is sectorial.

If A is a sectorial operator, for every $t > 0$ we can define a linear bounded operator e^{tA} on X, by means of the Dunford integral,

$$\begin{cases} e^{tA} := \frac{1}{2\pi i} \int_{\omega + \gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) \, d\lambda \\ \\ e^{0A} := I \end{cases} \tag{C.1.1}$$

where $r > 0$, $\eta \in]\frac{\pi}{2}, \vartheta[$ and $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathbb{C} \ : \ |arg(\lambda)| \leq \eta \, , \ |\lambda| = r\}$, oriented counterclockwise.
Note that, since the function$\lambda \to e^{t\lambda} R(\lambda, A)$ is holomorphic in $S_{\vartheta,\omega}$ the definition of e^{tA} is independent of the choice of r and η.
In the following proposition we state the main properties of e^{tA}. (For the proof see, for example, [Lu95], Section 2.1).

Proposition C.1.5 *If $A : D(A) \subset X \to X$ is a sectorial operator then:*

(i) $e^{tA} \in \mathcal{L}(X)$ *and* $e^{tA} e^{sA} = e^{(t+s)A}$ *for every* $t, s \geq 0$.

(ii) $e^{tA}x \in D(A^k)$ *for every* $t > 0$, $x \in X$, $k \in \mathbb{N}$. *If* $x \in D(A^k)$ *then*

$$A^k e^{tA} x = e^{tA} A^k x \quad , \quad \forall t \geq 0.$$

(iii) *There are constants* $M_0, M_1, M_2, \ldots$, *such that*

$$\begin{cases} \text{(a)} \parallel e^{tA} \parallel_{\mathcal{L}(X)} \leq M_0 e^{\omega t} & t > 0 \\ \text{(b)} \parallel t^k (A - \omega I)^k e^{tA} \parallel_{\mathcal{L}(X)} \leq M_k e^{\omega t} & t > 0 \end{cases}$$

where ω *is given by definition C.1.3.*
In particular from (b) *it follows that for every* $\varepsilon > 0$ *and* $k \in \mathbb{N}$ *there exists* $C_{k,\varepsilon} > 0$ *such that*

$$\parallel t^k A^k e^{tA} \parallel_{\mathcal{L}(X)} \leq C_{k,\varepsilon}\, e^{(\omega+\varepsilon)t} \quad t > 0.$$

(iv) *The function* $t \to e^{tA}$ *belongs to* $C^\infty(]0,\infty[;\mathcal{L}(X))$ *and*

$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA} \quad t > 0,$$

moreover it has an analytic extension in the sector $S := \{\lambda \in \mathbb{C} \,:\, \lambda \neq 0\,, |arg(\lambda)| < \vartheta - \frac{\pi}{2}\}$.

(v) $x \in \overline{D(A)} \Leftrightarrow \lim_{t\to 0} \parallel e^{tA}x - x \parallel_X = 0.$

(vi) *There exists* $x \in X$ *such that*

$$\lim_{t \to 0} \left\| \frac{e^{tA}x - x}{t} - y \right\|_X = 0 \Leftrightarrow x \in D(A) \quad , \; Ax = y \in \overline{D(A)}.$$

This list of properties shows that the sectorial operator A is the infinitesimal generator of the analytic semigroup e^{tA} defined in C.1.1.
It may be useful in several circumstances to approximate e^{tA} by a sequence of semigroups generated by bounded operators. The result is the following. (For the proof see Proposition 2.1.2 in [Lu95]).

Proposition C.1.6 *For every integer* $n > \omega$ *set*

$$A_n : X \to X \quad , \quad A_n := nAR(n, A).$$

Then $\rho(A) \subset \rho(A_n)$ *and* $R(\lambda, A_n) \to R(\lambda, A)$ *in* $\mathcal{L}(X)$ *as* $n \to \infty$, *for every* $\lambda \in \rho(A)$. *Moreover* $e^{tA_n} \to e^{tA}$ *in* $\mathcal{L}(X)$ *as* $n \to \infty$, *for every* $t > 0$.

C.2 Fractional powers of operators

From now on we assume that $A : D(A) \subset X \to X$ is a sectorial operator such that for $t > 0$

$$\| e^{tA} \|_{\mathcal{L}(X)} \leq M_0 e^{-\omega t} \quad , \quad \| tAe^{tA} \|_{\mathcal{L}(X)} \leq M_1 e^{-\omega t}$$

with $M_0, M_1, \omega > 0$. For every $\alpha > 0$ we define

$$(-A)^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{tA} \, dt$$

where the Euler Γ function is defined by

$$\Gamma(\vartheta) = \int_0^{+\infty} e^{-\vartheta t} t^{\vartheta-1} \, dt.$$

By definition one can verify for $\alpha = n \in \mathbb{N}$, $n \geq 1$ the operator $(-A)^{-n}$ defined above coincides with the inverse of $(-A)^n$. Moreover if $\alpha, \beta > 0$ then $(-A)^{-\alpha}(-A)^{-\beta} = (-A)^{-(\alpha+\beta)}$ thus $(-A)^{-\alpha}$ is one to one. (This facts are proved, for example, in [Lu95], Section 2.2.2). We define now the operator $(-A)^\alpha$ for $\alpha > 0$.

Definition C.2.1 *For $\alpha > 0$, we set $D(-A)^\alpha = Range\,((-A)^{-\alpha})$, and*

$$(A)^\alpha = \left((-A)^{-\alpha}\right)^{-1}.$$

Moreover, we set $(-A)^0 = I$.
$D(-A)^\alpha$ is endowed with the norm $\| x \|_{D(-A)^\alpha} = \| (-A)^\alpha x \|$. Since $(-A)^\alpha$ is one to one, such norm is equivalent to the graph norm.

The operators $(-A)^\alpha$, with α not integer, are usually called *fractional powers* of $-A$. We list now some properties that follows easily from the definition.

Lemma C.2.2 (i) *If $\alpha > \beta > 0$ then $D((-A)^\alpha) \subset D((-A)^\beta)$,*

(ii) *for every $\alpha, \beta \in \mathbb{R}$ we have $(-A)^\alpha(-A)^\beta = (-A)^{\alpha+\beta}$ on $D((-A)^\gamma)$ with $\gamma = \max\{\alpha, \beta, \alpha+\beta\}$,*

(iii) *For $t > 0$*

$$\| e^{tA} \|_{\mathcal{L}(D((-A)^\alpha))} \leq M_0 e^{-\omega t}.$$

One of the most useful properties is that for $0 < \alpha < 1$ the domains of the operators $(-A)^\alpha$ are intermediate spaces between X and $D(A)$. For a more precise description of this property and for a general theory on fractional powers of operators we refer, for example, to [Ta79] or [Lu95].

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