DENSITY ESTIMATES AND FURTHER PROPERTIES OF BLAKE & ZISSERMAN FUNCTIONAL

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Abstract - *We prove some properties of strong minimizers for functionals depending on free discontinuities, free gradient discontinuities and second derivatives, which are related to image segmentation.*

Key words: Calculus of Variations, special functions with bounded variation, free discontinuity problems, image segmentation.

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1. Introduction

In previous papers ([CLT3],[CLT4]) we proved the existence of minimizers for the following functional

$$
\begin{aligned}
F(K_0, K_1, u) \; :=& \\
& \int_{\Omega\setminus(K_0\cup K_1)} \left(|D^2u|^2 + \mu|u-g|^q\right)\, dy + \alpha\mathcal{H}^1(K_0\cap\Omega) + \beta\mathcal{H}^1((K_1\setminus K_0)\cap\Omega)\;,
\end{aligned}
\tag{1.1}
$$

where $\Omega \subset \mathbf{R}^2$ is an open set, $\mathcal{H}^1$ denotes the length (in the sense of the one dimensional Hausdorff measure), and $\alpha, \beta, \mu, q \in \mathbf{R}$, with

$$
q \ge 1\;,\;\mu > 0\;,\; 0 < \beta \le \alpha \le 2\beta\;,\; g \in L^{2q}_{loc}(\Omega) \cap L^q(\Omega)\;,
\tag{1.2}
$$

are given; while $K_0,\ K_1 \subset \mathbf{R}^2$ are Borel sets (a priori unknown) with $K_0 \cup K_1$ closed, $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ and it is approximately continuous on $\Omega \setminus K_0$.
Functional (1.1) (called *thin plate surface under tension* in computer vision modelling) was proposed by Blake & Zisserman as an energy to be minimized in order to achieve a segmentation of a monochromatic picture ([BZ]). In this context g describes the light intensity level of the image Ω, μ is a scale parameter, α is a contrast parameter and a measure of immunity to noise, β is a gradient-contrast parameter.

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The elements of a minimizing triplet (K_0, K_1, u) play respectively the role of edges, creases and smoothly varying intensity in the region $\Omega \setminus (K_0 \cup K_1)$ for the segmented image. This second-order model (1.1) was introduced to overcome the over-segmentation of steep gradients (ramp effect) and other inconvenients which occur in lower order models as in case of Mumford & Shah functional ([MS],[DGCL]).

The energy (1.1) is a functional depending on bulk energy and a surfacic discontinuity energy; their coupling is rather intriguing, since there is dependence on second derivatives, while there is no bound on the first derivatives. Moreover the discontinuities take place on the sets $K_0,\ K_1$ which are "a priori" unknown, hence the associated minimization problem turns out to be essentially nonconvex, and non uniqueness of minimizers may develop for suitable data g.

The existence of minimizers of (1.1) was proved by a suitable definition of weak solutions ([CLT3]), and hence by showing regularity properties for them ([CLT4]).

Our technique was based on the application of a new Poincaré type inequality ([CLT4]) which entails a suitable decay outside the singular set of weak solutions.

In this paper we prove additional properties of the solutions: *upper and lower energy density* of the *essential minimizing triplet* (see Definition in Section 3); an *elimination property* and a precise estimate of the *Minkowski content* for the segmentation set $K_0 \cup K_1$ (see Theorems in Section 4).

In particular the elimination property states that, when an optimal segmentation has length, in a small ball, less than an absolute constant times the radius of the ball, then such segmentation does not intersect the ball with half the radius. This is also a useful information for numerical analysis of the problem, in the sense that a suitable algorithm can eliminate such essential isolated parts of $K_0 \cup K_1$ because they are "needless energy" for the segmentation.

The result about the Minkowski content expresses the agreement between the Hausdorff one dimensional measure and the Minkowski content of the segmentation $K_0 \cup K_1$. Roughly speaking, the theorem says that a uniform fattening of an optimal segmentation is a reasonable approximation of the segmentation itself.

The assumption $g \in L^{2q}_{loc}(\Omega)$ is sharp in the sense that for every $s < 2q$ there exist $g \in L^s(\Omega)$ such that functional (1.1) has no minimizing triplet. This is proved here by a counterexample in section 5.

The outline of the paper is the following.

1. Introduction.
2. Notation and preliminary results.
3. Local weak minimizers and essential minimizing triplets.
4. Density estimates for essential minimizing triplets.
5. A counterexample.

2. Notation and preliminary results

From now on we denote by Ω an open set in $\mathbf{R}^2$. Given two vectors a, b, we set $a \cdot b = \sum_i a_i b_i$.

For a given set $U \subset \mathbf{R}^2$ we denote by ∂U its topological boundary, by $\mathcal{H}^1(U)$ its one–dimensional Hausdorff measure and by $|U|$ its Lebesgue outer measure; χ_U is the characteristic function of U. We indicate by $B_\rho(x)$ the open ball $\{y \in \mathbf{R}^2; |y-x| < \rho\}$. If Ω, Ω' are open subsets in $\mathbf{R}^2$, by $\Omega \subset\subset \Omega'$ we mean that $\overline{\Omega}$ is compact and $\overline{\Omega} \subset \Omega'$.

We say that a subset E of $\mathbf{R}^2$ is *countably* $(\mathcal{H}^1, 1)$ *rectifiable* if it is $\mathcal{H}^1$ measurable and E (up to a set of vanishing $\mathcal{H}^1$ measure) is the countable union of C^1 images of bounded subsets of $\mathbf{R}$; if in addition $\mathcal{H}^1(E) < +\infty$ then we say that E is $(\mathcal{H}^1, 1)$ *rectifiable.*

For any Borel function $v : \Omega \to \mathbf{R}$ the approximate upper and lower limits of v are the Borel functions $v^+, v^- : \Omega \to \overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$ defined for any $x \in \Omega$ by

$$v^+(x) = \inf\{t \in \overline{\mathbf{R}} : \lim_{\rho\to 0} \rho^{-2} |\{v > t\} \cap B_\rho(x)| = 0\},$$
$$v^-(x) = \sup\{t \in \overline{\mathbf{R}} : \lim_{\rho\to 0} \rho^{-2} |\{v < t\} \cap B_\rho(x)| = 0\}.$$

The set

$$S_v = \{x \in \Omega;\ v^-(x) < v^+(x)\ \}$$

is a Borel set, of negligible Lebesgue measure (see e.g. [F], 2.9.13); we say that v is approximately continuous on $\Omega \setminus S_v$ and we denote by $\tilde{v} : \Omega \setminus S_v \to \bar{\mathbf{R}}$ the function

$$\tilde{v}(x) = \operatorname{ap}\lim_{y\to x} v(y) = v^+(x) = v^-(x).$$

Let $x \in \Omega \setminus S_v$ be such that $\tilde{v}(x) \in \mathbf{R}$; we say that v is approximately differentiable at x if there exists a vector $\nabla v(x) \in \mathbf{R}^2$ (the approximate gradient of v at x) such that

$$\operatorname{ap}\lim_{y\to x} \frac{|v(y) - \tilde{v}(x) - \nabla v(x) \cdot (y-x)|}{|y-x|} = 0.$$

If v is a smooth function then ∇v is the classical gradient. In the following with the notation $|\nabla v|$ we mean the euclidean norm of ∇v and we set $\nabla_i v = (\mathbf{e}_i \cdot \nabla) v$, $\{\mathbf{e}_i\}$ denoting the canonical base of $\mathbf{R}^2$. We recall the definition of the space of real valued functions with bounded variation in Ω :

$$BV(\Omega) = \{v \in L^1(\Omega); Dv \in \mathcal{M}(\Omega)\}$$

where $Dv = (D_1 v, D_2 v)$ denotes the distributional gradient of v and $\mathcal{M}(\Omega)$ denotes the space of vector–valued Radon measure with finite total variation. We denote by $\int_\Omega |Dv|$ the total variation of the measure Dv in Ω.

For every $v \in BV(\Omega)$ the following properties hold ([F]):

1) $v^+(x), v^-(x) \in \mathbf{R}$ for $\mathcal{H}^1$–almost all $x \in \Omega$;
2) S_v is countably $(\mathcal{H}^1, 1)$ rectifiable;

3) ∇v exists a.e. in Ω and coincides with the Radon–Nikodym derivative of Dv with respect to the Lebesgue measure;
4) for $\mathcal{H}^1$ almost all $x \in S_v$ there exists a unique $\nu = \nu_v(x) \in \partial B_1(0)$ such that, setting $B_\rho^+ = \{y \in B_\rho(x) : (y-x)\cdot\nu > 0\}$ and $B_\rho^- = \{y \in B_\rho(x) : (y-x)\cdot\nu < 0\}$, then

$$\lim_{\rho\to 0}\left(\fint_{B_\rho^+} |v(y) - v^+(x)|^2 dy + \fint_{B_\rho^-} |v(y) - v^-(x)|^2 dy \right) = 0.$$

Moreover $\nu_v(x)$ is an approximate normal vector to S_v at x, and also

$$\int_\Omega |Dv| \geq \int_\Omega |\nabla v|\, dy + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^1.$$

We recall the definitions of some function spaces related to first derivatives which are De Giorgi special measures, and we refer to [DGA] and [A] for their properties.

Definition 2.1 - *$SBV(\Omega)$ denotes the class of functions $v \in BV(\Omega)$ such that*

$$\int_\Omega |Dv| = \int_\Omega |\nabla v|\, dy + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^1.$$

Moreover we set

$$\begin{aligned} SBV_{loc}(\Omega) \; &:= \; \{v \in SBV(\Omega') : \; \forall \Omega' \subset\subset \Omega\}\,, \\ GSBV(\Omega) \; &:= \; \big\{v : \Omega \to \mathbf{R} \text{ Borel function}; -k \vee v \wedge k \in SBV_{loc}(\Omega)\ \forall k \in \mathbf{N}\big\}. \end{aligned}$$

Notice that, if $v \in GSBV(\Omega)$, then S_v is countably $(\mathcal{H}^1, 1)$ rectifiable and ∇v exists a.e. in Ω.

We recall also a function space related to second derivatives, that allows the definition of a finite energy set of competing functions (see [CLT1,2,3,4]).

Definition 2.2 - *We set*

$$GSBV^2(\Omega) \; := \; \big\{v \in GSBV(\Omega),\ (\nabla_1 v, \nabla_2 v) \in [GSBV(\Omega)]^2\big\}.$$

Notice that $Dv \neq \nabla v$ in $GSBV^2(\Omega)$; moreover we set

$$S_{\nabla v} = S_{\nabla_1 v} \cup S_{\nabla_2 v}.$$

Eventually we define the strong and weak energy functionals.

Definition 2.3 - (Strong formulation of Blake & Zisserman functional)
For $\Omega \subset \mathbf{R}^2$ open set, $A \subset \Omega$ Borel set , $K_0, K_1 \subset \mathbf{R}^2$ Borel sets with $K_0 \cup K_1$ closed, $v \in C^2(\Omega \setminus (K_0 \cup K_1))$ and approximately continuous on $\Omega \setminus K_0$, under the assumptions (1.2), we set

$$F(K_0, K_1, v, A) \; := \tag{2.1}$$
$$\int_{A \setminus (K_0 \cup K_1)} \left(|D^2 v|^2 + \mu |v - g|^q \right) \, dy + \alpha \mathcal{H}^1(K_0 \cap A) + \beta \mathcal{H}^1((K_1 \setminus K_0) \cap A).$$

We shortly write $F(K_0, K_1, v)$ when $A = \Omega$.

The space $X(\Omega) := GSBV^2(\Omega) \cap L^q(\Omega)$ is the natural space for a weak formulation of functional (2.1).

Definition 2.4 - (Weak formulation of Blake & Zisserman functional)
For $\Omega \subset \mathbf{R}^2$ open set, under the assumptions (1.2), we define $\mathcal{F} : X(\Omega) \to [0, +\infty]$ by

$$\mathcal{F}(v) \; := \; \int_\Omega (|\nabla^2 v|^2 + \mu |v - g|^q) \, dy + \alpha \mathcal{H}^1(S_v) + \beta \mathcal{H}^1(S_{\nabla v} \setminus S_v) \, . \tag{2.2}$$

We proved the following results in [CLT3],[CLT4].

Theorem 2.5 - (Existence of weak solutions)
Let $\Omega \subset \mathbf{R}^2$ be an open set and assume (1.2). Then there is $v_0 \in X(\Omega)$ such that

$$\mathcal{F}(v_0) \leq \mathcal{F}(v) \qquad \forall v \in X(\Omega).$$

We recall that assumption $\beta \leq \alpha \leq 2\beta$ is necessary for lower semicontinuity of $\mathcal{F}$.

Theorem 2.6 - (Existence of strong solutions)
Let $\Omega \subset \mathbf{R}^2$ be an open set. Assume (1.2). Then there is at least one triplet among $K_0, K_1 \subset \mathbf{R}^2$ Borel sets with $K_0 \cup K_1$ closed and $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ approximately continuous on $\Omega \setminus K_0$ minimizing the functional (1.1) with finite energy. Moreover the sets $K_0 \cap \Omega$ and $K_1 \cap \Omega$ are $(\mathcal{H}^1, 1)$ rectifiable.

Theorem 2.7 - *Let $\Omega \subset \mathbf{R}^2$ be an open set. Assume (1.2) and $\alpha = \beta$. Then there is at least one pair among $K \subset \mathbf{R}^2$ closed set and $u \in C^2(\Omega \setminus K)$ minimizing the functional*

$$\int_{\Omega \setminus K} \left(|D^2 u|^2 + \mu |u - g|^q \right) \, dy + \alpha \mathcal{H}^1(K \cap \Omega)$$

with finite energy. Moreover the set $K \cap \Omega$ is $(\mathcal{H}^1, 1)$ rectifiable.

3. Local weak minimizers and essential minimizing triplets.

We refine the definition of solution by the concept of essential minimizing triplet, in order to eliminate unimportant points in the segmentation set.
We start with a localization of the functional $\mathcal{F}$ and of the notion of weak minimizers.

Definition 3.1 - *For every Borel subset $A \subset \Omega$ and $v \in X(\Omega)$ we define*

$$\mathcal{F}(v, A) := \int_A (|\nabla^2 v|^2 + \mu|v - g|^q)\, dy + \alpha\mathcal{H}^1(S_v \cap A) + \beta\mathcal{H}^1((S_{\nabla v} \setminus S_v) \cap A).$$

Definition 3.2 - (**Local weak minimizers**)
A function $v \in X(\Omega)$ is a local weak minimizer of $\mathcal{F}(\cdot, \Omega)$ if, for every compact set $H \subset \Omega$,

$$\mathcal{F}(v, H) = \min_{w \in X(\Omega)} \{\mathcal{F}(w, H) : \; w = v \text{ a.e. in } \Omega \setminus H\} < +\infty.$$

From [CLT4] we get a decay property for local weak minimizers.

Theorem 3.3 - (**Decay Theorem**)
Let $\Omega \subset \mathbf{R}^2$ be an open set and assume (1.2). Then there is an absolute positive constant $c_{2,q}$ (depending only on the dimension and the exponent) such that for every $k > 2$, $\eta, \sigma \in (0,1)$ with $\eta^\sigma < \frac{1}{c_{2,q}}$, there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and $\overline{B}_\rho(x) \subset \Omega$, if $u \in GSBV^2(\Omega)$ is a local weak minimizer of $\mathcal{F}(\cdot, \Omega)$ with

$$\rho \le \varepsilon^k, \qquad \int_{B_\rho(x)} |g|^{2q} dy \le \varepsilon^k$$

and

$$\alpha\mathcal{H}^1(S_u \cap B_\rho(x)) + \beta\mathcal{H}^1\big((S_{\nabla u} \setminus S_u) \cap B_\rho(x)\big) \le \varepsilon\rho,$$

then

$$\mathcal{F}(u, B_{\eta\rho}(x)) \le \eta^{2-\sigma}\mathcal{F}(u, B_\rho(x)).$$

Definition 3.4 - (**Strong minimizing triplet**)
A triplet (T_0, T_1, v) such that, $T_0, T_1 \subset \mathbf{R}^2$ are Borel sets, $T_0 \cup T_1$ is a closed set, $v \in C^2\big(\Omega \setminus (T_0 \cup T_1)\big)$ and approximately continuous in $\Omega \setminus T_0$, is a strong minimizing triplet of the functional (1.1) if

$$(T_0, T_1, v) \;\in\; \operatorname{argmin} F \;.$$

Remark 3.5 - Notice that, if (T_0, T_1, v) is a strong minimizing triplet then v is a weak minimizer and $F(T_0, T_1, v) = \mathcal{F}(v) = \min \mathcal{F}$ (see [CLT4], Lemma 3.2 and section 6).

Definition 3.6 - (**Essential minimizing triplet**)
Given a strong minimizing triplet (T_0, T_1, v) of the functional (1.1), there is another triplet (K_0, K_1, u) , called essential minimizing triplet, uniquely defined by

$$\begin{aligned} K_0 &= \overline{T_0 \cap K} \setminus (S_{\nabla v} \setminus S_v) \\ K_1 &= \overline{T_1 \cap K} \setminus S_v \\ u &= \tilde{v} \end{aligned}$$

where K is the smallest closed subset of $T_0 \cup T_1$ such that $\tilde{v} \in C^2(\Omega \setminus K)$.

Remark 3.7 - For every $v \in X(\Omega)$ that is a weak minimizer of $\mathcal{F}$, we set

$$\Omega_0 = \left\{ x \in \Omega : \lim_{\varrho \to 0_+} \varrho^{-1} \mathcal{F}(v, B_\varrho(x)) = 0 \right\}.$$

By the argument of section 6 in [CLT4], we get that Ω_0 is an open set. Moreover

$$\Omega_0 \cap (\overline{S_v \cup S_{\nabla v}}) = \emptyset,$$

$$\mathcal{H}^1 \left(\Omega \cap \left((\overline{S_v \cup S_{\nabla v}}) \setminus (S_v \cup S_{\nabla v}) \right) \right) = 0.$$

4. Density estimates for essential minimizing triplets.

In this section we state and prove the main results.
We enphasize that in all the statements of this section it is assumed that the open set Ω is contained in $\mathbf{R}^2$ and the assumptions (1.2) are always understood.

Theorem 4.1 - (**Density upper bound for the functional F**)
Let (K_0, K_1, u) be a strong minimizing triplet for the functional (1.1) with $g \in L^{2q}_{loc}(\Omega)$. Then for every $0 < \rho \leq 1$ and for every $x \in \Omega$ such that $\overline{B}_\rho(x) \subset \Omega$ we have

$$F(K_0, K_1, u, \overline{B}_\rho(x)) \leq c_0 \rho$$

where $c_0 = \pi^{\frac{1}{2}} \mu \|g\|^q_{L^{2q}(B_\rho(x))} + 2\pi\alpha$.

If $q = 2$ and $g \in L^\infty(\Omega)$, then $c_0 = \pi\mu\|g\|^2_{L^\infty(\Omega)} + 2\pi\alpha$.

Proof - By minimality of (K_0, K_1, u) for F we get

$$F(K_0, K_1, u) \leq F(Q_0, Q_1, w)\,,$$

where

$$w = u\chi_{\Omega\setminus B_\rho(x)}, \quad Q_0 = \big(K_0 \setminus B_\rho(x)\big) \cup \partial B_\rho(x), \; Q_1 = K_1.$$

Taking into account $\beta \le \alpha$, by subtraction we obtain

$$\begin{aligned}
\int_{B_\rho(x)\setminus(K_0\cup K_1)} & \big(|D^2u|^2 + \mu|u-g|^q\big)\,dy \\
&\qquad + \alpha\mathcal{H}^1(K_0 \cap \overline{B}_\rho(x)) + \beta\mathcal{H}^1\big((K_1\setminus K_0)\cap \overline{B}_\rho(x)\big) \\
&\le \mu \int_{B_\rho(x)} |g|^q\,dy + \alpha\mathcal{H}^1(\partial B_\rho(x)) \\
&\le \mu\|g\|^q_{L^{2q}(B_\rho(x))}(\pi\rho^2)^{\frac{1}{2}} + 2\pi\alpha\rho,
\end{aligned}$$

hence we achieve the proof. ■

Theorem 4.2 - (**Density lower bound for the functional** F)
Let (K_0, K_1, u) be an essential minimizing triplet for the functional (1.1) with $g \in L^{2q}(\Omega)$. Then there exist $\varepsilon_0 > 0$, $\varrho_0 > 0$ such that

$$F(K_0, K_1, u, B_\varrho(x)) \;\; \ge \;\; \varepsilon_0\varrho \qquad \forall x \,\in\, K_0 \cup K_1, \quad \forall \varrho \le \varrho_0.$$

Proof - By Remark 3.5 u is a weak minimizer and $F(K_0, K_1, u, B_\rho(x)) = \mathcal{F}(u, B_\rho(x))$ for every $B_\rho(x) \subset \Omega$. Let k, η, σ and ε_0 be as in the Decay Theorem 3.3. Let $\rho_0 > 0$ such that $\rho_0 \le \varepsilon_0^k$, $\int_{B_{\rho_0}(x)} |g|^{2q}dy \le \varepsilon_0^k$ for every $x \in \Omega$. Assume, by contradiction, that the thesis is false. In such case, there exist $x \in K_0 \cup K_1$ and $0 < \rho \le \rho_0$ such that

$$F(K_0, K_1, u, B_\rho(x)) < \varepsilon_0\rho.$$

Then, by Theorem 3.3, for every $h \in \mathbf{N}$,

$$F(K_0, K_1, u, B_{\eta^h\rho}(x)) < \eta^{h(2-\sigma)}\varepsilon_0\rho$$

so that

$$\lim_{\varrho\to 0_+} \varrho^{-1}\, F\,(K_0, K_1, u, B_\varrho(x)) \,=\, 0,$$

hence $x \in \Omega_0$. Since Ω_0 is open, then u is a C^2 function in a neighbourhood of x and this contradicts the assumption that the minimizing triplet is an essential one. ■

Theorem 4.3 - (**Density lower bound for the segmentation length**)
Let (K_0, K_1, u) be an essential minimizing triplet for the functional (1.1) with $g \in L^{2q}(\Omega)$. Then there exist $\varepsilon_1 > 0, \varrho_1 > 0$ such that

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(x)\right) \quad \geq \quad \varepsilon_1 \varrho \qquad \forall x \in K_0 \cup K_1, \quad \forall \varrho \leq \varrho_1.$$

Proof - Let k, η, σ and ε_0, ρ_0 be as in Theorem 4.2. We can fix $h_0 \in \mathbf{N}$ such that $\eta^{h_0(1-\sigma)} c_0 < \varepsilon_0$, where c_0 is given in Theorem 4.1. Define $\varepsilon_1 = \frac{\varepsilon_0}{\alpha} \eta^{h_0 - 1}$ and $\rho_1 = \min\{\rho_0, 1\}$. If we assume, by contradiction, that there exist $x \in K_0 \cup K_1$ and $\rho \leq \rho_1$ such that

$$\mathcal{H}^1((K_0 \cup K_1) \cap B_\rho(x)) < \varepsilon_1 \rho,$$

then we can use Theorem 3.3 h_0 times until we get

$$F(K_0, K_1, u, B_{\eta^{h_0}\rho}(x)) \leq \eta^{h_0(2-\sigma)} F(K_0, K_1, u, B_\rho(x)) \leq \eta^{h_0(1-\sigma)} c_0 \eta^{h_0} \rho < \varepsilon_0(\eta^{h_0}\rho)$$

which contradicts Theorem 4.2. ∎

Theorem 4.4 - (**Elimination Property**)
Let (K_0, K_1, u) be an essential minimizing triple for the functional (1.1) with $g \in L^{2q}(\Omega)$ and let $\varepsilon_1 > 0, \varrho_1 > 0$ as in Theorem 4.3 and $\rho \leq \rho_1$. If $x \in \Omega$ and

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(x)\right) \quad < \quad \frac{\varepsilon_1}{2} \rho$$

then

$$(K_0 \cup K_1) \cap B_{\rho/2}(x) = \emptyset.$$

Proof - Assume, by contradiction, that there exists $y \in (K_0 \cup K_1) \cap B_{\rho/2}(x)$. Then $B_{\rho/2}(y) \subset B_\rho(x)$, hence

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_{\varrho/2}(y)\right) \quad \leq \quad \mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(x)\right) \quad < \quad \varepsilon_1 \left(\frac{\rho}{2}\right),$$

therefore $y \notin K_0 \cup K_1$ by Theorem 4.3. ∎

Theorem 4.5 - (**Minkowski content of the segmentation**)
Let (K_0, K_1, u) be an essential minimizing triplet for the functional (1.1) with $g \in L^{2q}(\Omega)$. Then
(i) $K_0 \cup K_1$ is $(\mathcal{H}^1, 1)$ rectifiable;
(ii) for every $\Omega' \subset\subset \Omega$ the following equality holds

$$\lim_{\rho \to 0} \frac{|\{x \in \Omega\,;\ dist(x, (K_0 \cup K_1) \cap \Omega') < \rho\ \}|}{2\rho} = \mathcal{H}^1\left((K_0 \cup K_1) \cap \Omega'\right)\ .$$

Proof - By Remark 3.5 the function u is a weak minimizer of $\mathcal{F}$, so that $S_u \cup S_{\nabla u}$ is $(\mathcal{H}^1, 1)$ rectifiable and $\mathcal{H}^1\big((K_0 \cup K_1) \cap \Omega \setminus (S_u \cup S_{\nabla u})\big) = 0$. Hence (i) follows. The result (ii) is achieved with the same argument as in [AT] and in [CL], by using the previous Theorem 4.3. ■

We enphasize that the various constants $c_0, \varepsilon_0, \varepsilon_1, \rho_0, \rho_1$ depend on the data α, β, μ, g.

5. A counterexample

In this section we show that the functional (1.1) does not achieve the infimum when g has not enough summability. This fact holds true in any dimension: in this section Ω denotes an open set in $\mathbf{R}^n$, for any integer $n \geq 2$.
We use in the n dimensional case the notation introduced in section 2 and we set $\omega_n = |B_1(0)|$.
We show that for any $s < nq$ there is a function $g \in L^s(\Omega) \cap L^q(\Omega)$ such that there are no minimizing triplets of the following functional

$$
\begin{aligned}
G(K_0, K_1, u) = & \int_{\Omega \setminus (K_0 \cup K_1)} \left(|D^2 u|^p + \mu |u - g|^q\right) \, dy \\
& + \alpha \mathcal{H}^{n-1}(K_0 \cap \Omega) + \beta \mathcal{H}^{n-1}((K_1 \setminus K_0) \cap \Omega)
\end{aligned}
\tag{5.1}
$$

where $\Omega \subset \mathbf{R}^n$ is an open set, $n \in \mathbf{N}$, $n \geq 2$, $\mathcal{H}^{n-1}$ denotes the $(n-1)$ dimensional Hausdorff measure and $\alpha, \beta, \mu, p, q \in \mathbf{R}$, with

$$
p > 1, \; q \geq 1 \;, \; \mu > 0 \;, \; 0 < \beta \leq \alpha \leq 2\beta \;, \tag{5.2}
$$

are given; while $K_0, \; K_1 \subset \mathbf{R}^n$ are Borel sets (a priori unknown) with $K_0 \cup K_1$ closed, $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ and it is approximately continuous on $\Omega \setminus K_0$.
Notice that (5.1) reduces to (1.1) if $n = p = 2$.

Let $b > \left(\dfrac{n\alpha}{\mu}\right)^{\frac{1}{q}}$, $a > 1$, $r_h = a^{-h}$ for every $h \in \mathbf{N}$, and $\{x_n\}_{n \in \mathbf{N}}$ a dense sequence in Ω. Setting $A_h = B_{r_h}(x_h)$, we define

$$
g = b \sum_{h=1}^{\infty} {r_h}^{-\frac{1}{q}} \, \chi_{A_h}.
$$

Then for every $s < nq$ we have $g \in L^s(\Omega)$, but $g \notin L^{nq}(\Omega)$. In fact, if $s < nq$

$$
\sum_{h=1}^{\infty} \|{r_h}^{-\frac{1}{q}} \chi_{A_h}\|_{L^s(\Omega)} = \sum_{h=1}^{\infty} {r_h}^{-\frac{1}{q}} |A_h|^{\frac{1}{s}} = {\omega_n}^{\frac{1}{s}} \sum_{h=1}^{\infty} {r_h}^{\frac{n}{s} - \frac{1}{q}} < +\infty,
$$

while

$$\int_\Omega \left| \sum_{h=1}^{\infty} {r_h}^{-\frac{1}{q}} \chi_{A_h} \right|^{nq} dy \;\geq\; \int_\Omega \sum_{h=1}^{\infty} \left| {r_h}^{-\frac{1}{q}} \chi_{A_h} \right|^{nq} dy \;=\; \omega_n \sum_{h=1}^{\infty} 1 \;=\; +\infty\ .$$

Assume by contradiction that (K_0, K_1, u) is a minimizing triplet for G.
Let $\bar{B} \subset \Omega \setminus (K_0 \cup K_1)$ a closed ball. Taking into account that $u \in C^2(\bar{B})$ and $r_h \to 0$, we choose $k_0 \in \mathbf{N}$ such that $A_{k_0} \subset \bar{B}$ and

$$b{r_{k_0}}^{-\frac{1}{q}} \;\geq\; \max_{\bar{B}} u.$$

Let $k_1 > k_0$ be an index such that $A_{k_1} \subset A_{k_0}$ and set

$$Q_0 \;=\; (K_0 \setminus A_{k_1}) \cup \partial A_{k_1} \qquad Q_1 \;=\; K_1$$
$$v \;=\; \max\left\{ u\,,\; b\,\left({r_{k_0}}^{-\frac{1}{q}} + {r_{k_1}}^{-\frac{1}{q}} \right) \chi_{A_{k_1}} \right\}.$$

Now (Q_0, Q_1, v) is an admissible triplet for the functional G. Since $g \geq v \geq u + b\,(r_{k_1})^{-\frac{1}{q}}$ in A_{k_1} we get $0 \leq g - v \leq g - u - b\,(r_{k_1})^{-\frac{1}{q}}$ in A_{k_1} and

$$\begin{aligned}
\int_{\Omega\setminus(Q_0\cup Q_1)} |v-g|^q\,dy \;&=\; \int_{\Omega\setminus(K_0\cup K_1\cup A_{k_1})} |u-g|^q\,dy \;+\; \int_{A_{k_1}} |v-g|^q\,dy \\
&\leq\; \int_{\Omega\setminus(K_0\cup K_1)} |u-g|^q\,dy \;-\; \int_{A_{k_1}} b^q\;(r_{k_1})^{-1}\,dy \\
&<\; \int_{\Omega\setminus(K_0\cup K_1)} |u-g|^q\,dy \;-\; \frac{\alpha}{\mu}\, n\omega_n\,(r_{k_1})^{n-1}.
\end{aligned}$$

Moreover we get easily

$$\int_{\Omega\setminus(Q_0\cup Q_1)} |D^2 v|^p\,dy \;\leq\; \int_{\Omega\setminus(K_0\cup K_1)} |D^2 u|^p\,dy,$$

$$\mathcal{H}^{n-1}(Q_0\cap\Omega) \;\leq\; \mathcal{H}^{n-1}(K_0\cap\Omega) + n\,\omega_n\,(r_{k_1})^{n-1},$$

$$\mathcal{H}^{n-1}((Q_1\setminus Q_0)\cap\Omega) \;\leq\; \mathcal{H}^{n-1}((K_1\setminus K_0)\cap\Omega).$$

Hence $G(Q_0, Q_1, v) \;<\; G(K_0, K_1, u)$, which contradicts the minimality of (K_0, K_1, u).

We remark that also in the lower order model (Mumford & Shah functional) the infimum is not achieved for a poorly summable datum g (see [L]).

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