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On the validity of the Euler–Lagrange equation

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Abstract

The purpose of the present paper is to establish the validity of the Euler–Lagrange equation for the solution $\hat{x}$ to the classical problem of the calculus of variations.

Keywords: Euler–Lagrange equation

1. Introduction

The purpose of the present paper is to establish (under Carathéodory's conditions) the validity of the Euler–Lagrange equation (E–L) for the solution $\hat{x}$ to the classical problem of the calculus of variations consisting in minimizing the functional

$$J(x) = \int_I L\big(t, x(t), x'(t)\big)\, dt,$$

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where $I = (a, b) \subset R$, on the set of those absolutely continuous functions $x : I \to R^s$ satisfying the boundary conditions $x(a) = A$, $x(b) = B$. Establishing the validity of the Euler–Lagrange equation amounts to proving that

$$\int_I \left[\left\langle \nabla_\xi L\big(t, \hat{x}(t), \hat{x}'(t)\big), \eta'(t)\right\rangle + \left\langle \nabla_x L\big(t, \hat{x}(t), \hat{x}'(t)\big), \eta(t)\right\rangle\right] dt = 0$$

for every variation η in some suitable class. A large number of papers has been devoted to this classical problem, e.g., [4–6,8–10]. The example obtained by Ball and Mizel [1], modifying an earlier example of Maniá [7], provides a variational problem where the integrability of $\nabla_x L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ does not hold and, as a consequence, (E–L) is not true along the solution. Hence, some condition on the term $\nabla_x L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ has to be imposed in order to ensure the validity of (E–L). A result of Clarke [5] implies that the following assumption on the term $\nabla_x L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$:

there exists a function $S(t)$ integrable on I such that, for y in a neighborhood of the solution,

$$\left\|\nabla_x L\big(t, y, \hat{x}'(t)\big)\right\| \leqslant S(t)$$

is sufficient to establish the validity of (E–L). This condition implies that, locally along the solution, $x \to L(t, x, x')$ is Lipschitzian of Lipschitz constant $S(t)$. However, there are simple and meaningful examples of variational problems where this Lipschitzianity condition is not verified.

Consider the Lagrangian defined by $L(x, \xi) = (\xi\sqrt{|x|} - 2/3)^2$, and the problem (P) of minimizing

$$\int_0^1 L\big(x(t), x'(t)\big)\, dt$$

over the absolutely continuous functions x with $x(0) = 0$, $x(1) = 1$.

One can easily verify that $\hat{x}(t) = t^{2/3}$ is a minimizer for (P) (indeed, $L(\hat{x}(t), \hat{x}'(t)) = 0$ on $[0, 1]$, and L is non-negative everywhere). In this case, although L is not differentiable everywhere, $L_x(\hat{x}(t), \hat{x}'(t))$ exists a.e. (it is a.e. zero) and it is integrable. The purpose of the present paper is to provide a result on the validity of (E–L) that is satisfied by Lagrangians that are Lipschitzian in x, but that applies as well to the non-Lipschitzian cases as the example before.

In the proof we first show that the fact that $\hat{x}$ is a solution implies the integrability of $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$. Then, using this result, we establish the validity of (E–L) under Carathéodory's condition.

Note that we do not assume any convexity hypothesis on the Lagrangian. Moreover, no growth condition whatsoever is assumed so that, as far as we know, relaxation theorems cannot be applied.

2. Integrability of $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$

Consider the problem of minimizing the functional

$$J(x) = \int_I L\big(t, x(t), x'(t)\big)\, dt$$

on the set of those absolutely continuous functions $x : I \to R^s$ satisfying the boundary conditions $x(a) = A$, $x(b) = B$. Let $\hat{x}$ be a (weak local) minimizer yielding a *finite* value for the functional J, and set $\mu = \sup_{t\in[a,b]} \|\hat{x}(t)\|$.

Our results will depend on the following assumption.

Assumption A. (i) L is differentiable in x along $\hat{x}$, for a.e. t, and the map $\nabla_x L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ is integrable on I;

(ii) there exists a function $S(t)$ integrable on I such that, for any $y \in B(0, \mu + 1)$,

$$L\big(t, y, \hat{x}'(t)\big) \leqslant L\big(t, \hat{x}(t), \hat{x}'(t)\big) + S(t)\big\|y - \hat{x}(t)\big\|.$$

Consider problem (P) as presented in the Introduction. L and $\hat{x}$ satisfy Assumption A: $L_x(\hat{x}(t), \hat{x}'(t))$ exists a.e. (identically zero, hence integrable), $S(t) = t^{-2/3}$ verifies the inequality

$$L\big(y, \hat{x}'(t)\big) \leqslant S(t)\big|y - \hat{x}(t)\big|,$$

since

$$\left(\frac{2}{3}t^{-1/3}\sqrt{|y|} - \frac{2}{3}\right)^2 = \frac{4(\sqrt{|y|} - t^{1/3})}{9t^{2/3}(\sqrt{|y|} + t^{1/3})}\big(|y| - t^{2/3}\big) \leqslant \frac{4}{9t^{2/3}}|y - t^{2/3}|.$$

This is our first result on the term $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$. In what follows, $\bar{R}$ denotes $R \cup \{+\infty\}$.

Theorem 2.1. *Suppose that $L : I \times R^s \times R^s \to \bar{R}$ is an extended valued function, finite on its effective domain of the form* $\operatorname{dom} L = I \times R^s \times G$, *where $G \subset R^s$ is an open set, and that it satisfies Carathéodory's conditions, i.e., $L(\cdot, x, \xi)$ is measurable for fixed (x, ξ) and $L(t, \cdot, \cdot)$ is continuous for almost every t. Moreover assume that L is differentiable in ξ on* $\operatorname{dom} L$ *and that $\nabla_\xi L$ satisfies Carathéodory's conditions on* $\operatorname{dom} L$. *Suppose that Assumption* A *holds. Then,*

$$\int_I \big\|\nabla_\xi L\big(t, \hat{x}(t), \hat{x}'(t)\big)\big\|\, dt < +\infty.$$

Proof. (1) By assumption, $L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \in L^1(I)$, hence setting $S_0 = \{t \in I\colon \hat{x}'(t) \notin G\}$, we have $m(S_0) = 0$. Given $\epsilon > 0$, we can cover S_0 by an open set O_1 of measure $m(O_1) < \epsilon/2$. We have also that $\nabla_\xi L$ is a Carathéodory's function and that $\hat{x}'$ is measurable in I. Hence, by the theorems of Scorza Dragoni and of Lusin, for the given $\epsilon > 0$ there exists an open set O_2 such that $m(O_2) < \epsilon/2$ and at once $\hat{x}'$ is continuous in $I \setminus O_2$, and $\nabla_\xi L$ is continuous in $(I \setminus O_2) \times R^s \times G$. By taking $K_\epsilon = I \setminus (O_1 \cup O_2)$, we have that K_ϵ

is a closed set such that on it $\hat{x}'$ is continuous with values in G, $\nabla_\xi L$ is continuous on $K_\epsilon \times R^s \times G$ and $m(I \setminus K_\epsilon) < \epsilon$. For $n \geqslant 1$ set $\epsilon_n = (b-a)/2^{n+1}$ and $K_n = K_{\epsilon_n}$; set also $C_n = \bigcup_{j=1}^n K_j$. Then C_n are closed sets, $C_n \subset C_{n+1}$, $\hat{x}'$ is continuous on C_n with values in G, $\nabla_\xi L$ is continuous in $C_n \times R^s \times G$ and $\lim_{n\to+\infty} m(I \setminus C_n) = 0$.

From these properties it follows that there exists $k_n > 0$ such that, for all $t \in C_n$,

$$\left\|\nabla_\xi L\big(t, \hat{x}(t), \hat{x}'(t)\big)\right\| < k_n.$$

There is no loss of generality in assuming $k_n \geqslant k_{n-1}$. Moreover, we have that $m(C_1) \geqslant (b-a)/2$ and $\sum_{n=2}^\infty m(C_n \setminus C_{n-1}) \leqslant (b-a)/2$.

For all $n > 1$, we set $A_n = C_n \setminus C_{n-1}$. Hence we obtain that $C_m = C_1 \bigcup_{n=2}^m A_n$ and that $I = E \cup C_1 \bigcup_{n>1} A_n$, where $m(E) = 0$.

(2) Consider the function

$$\theta(t) = \begin{cases} 0 & \text{if } \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) = 0, \\ \frac{\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))}{\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|} & \text{otherwise,} \end{cases}$$

and

$$v_n = \int_{A_n} \theta(t)\,dt,$$

so that $\|v_n\| \leqslant m(A_n)$. There exists a closed set $B_n \subseteq C_1$ such that $m(B_n) = \|v_n\|$. Set

$$\theta_n'(t) = -\theta(t)\chi_{A_n}(t) + \frac{v_n}{\|v_n\|}\chi_{B_n}(t).$$

We have that

$$\int_I \theta_n'(t)\,dt = -\int_{A_n} \theta(t)\,dt + v_n = 0.$$

Hence, setting $\theta_n(t) = \int_a^t \theta_n'(\tau)\,d\tau$, we see that the functions $\theta_n(t)$ are admissible variations. Moreover we obtain

$$\|\theta_n\|_\infty \leqslant \sup_{t\in I} \int_a^t \left|\theta_n'(\tau)\right| d\tau \leqslant \int_I \left|\theta_n'(\tau)\right| d\tau \leqslant 2m(A_n).$$

(3) For t in A_n, we have $\|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| < k_n$; for t in B_n, $\|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| < k_1 \leqslant k_n$. Recalling that $\bar{A}_n \subset C_n$, we infer that, for all $t \in \bar{A}_n \cup B_n$,

$$\left\|\nabla_\xi L\big(t, \hat{x}(t), \hat{x}'(t)\big)\right\| \leqslant k_n.$$

We wish to obtain an uniform bound for $\|\nabla_\xi L\|$ computed in a suitable neighborhood of the solution $(\hat{x}(\cdot), \hat{x}'(\cdot))$. Consider the set $(\bar{A}_n \cup B_n) \times R^s \times G$ as a metric space M_n with distance $d((t,x,\xi),(t',x',\xi')) = \sup(|t-t'|, |x-x'|, |\xi-\xi'|)$. On M_n, $\nabla_\xi L$ is continuous. Moreover, its subset

$$G_n = \left\{\big(t, \hat{x}(t), \hat{x}'(t)\big)\colon\ t \in \bar{A}_n \cup B_n\right\}$$

is compact and, on G_n, $\|\nabla_\xi L\|$ is bounded by k_n. Hence there exists $\delta_n > 0$ such that, for $(t,x,\xi) \in M_n$ with $d((t,x,\xi),(t,\hat{x}(t),\hat{x}'(t))) < \delta_n$, we have $\|\nabla_\xi L(t,x,\xi)\| < k_n + 1$.

(4) For $|\lambda| < \min\{1/2m(A_n), \delta_n/2m(A_n), \delta_n\}$, consider the integrals

$$\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)+\lambda\theta_n'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt$$
$$= \int_{A_n\cup B_n} \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)+\lambda\theta_n'(t)\big)-L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)\big]\,dt$$
$$+\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt.$$

For every $t \in A_n \cup B_n$ there exists $\zeta_\lambda(t) \in (0,\lambda)$ such that

$$\frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)+\lambda\theta_n'(t)\big)-L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)\big]$$
$$=\big\langle\nabla_\xi L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)+\zeta_\lambda(t)\theta_n'(t)\big),\theta_n'(t)\big\rangle$$
$$\leqslant\big\|\nabla_\xi L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)+\zeta_\lambda(t)\theta_n'(t)\big)\big\|$$

and from the choice of λ, $\|\nabla_\xi L(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)+\zeta_\lambda(t)\theta_n'(t))\| < k_n+1$. Hence, we can apply the dominated convergence theorem to obtain that

$$\lim_{\lambda\to 0}\int_{A_n\cup B_n} \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)+\lambda\theta_n'(t)\big)-L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)\big]\,dt$$
$$=\int_{A_n\cup B_n}\big\langle\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\theta_n'(t)\big\rangle\,dt.$$

(5) Set $f^+(s)=\max\{0,f(s)\}$, $f^-(s)=\max\{0,-f(s)\}$. Since

$$0\leqslant\frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]^+\leqslant S(t)\big\|\theta_n(t)\big\|,$$

by the dominated convergence theorem,

$$\lim_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]^+\,dt$$
$$=\int_I \lim_{\lambda\to 0}\frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]^+\,dt.$$

By the Fatou's lemma,

$$\liminf_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]^-\,dt$$
$$\geqslant\int_I \liminf_{\lambda\to 0}\frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]^-\,dt.$$

We have obtained that

$$\begin{aligned}
&\limsup_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt\\
&\quad\leqslant \lim_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]^+\\
&\qquad -\liminf_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]^-\,dt\\
&\quad\leqslant \int_I \limsup_{\lambda\to 0}\frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt\\
&\quad= \int_I \big\langle\nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big),\theta_n(t)\big\rangle\,dt.
\end{aligned}$$

(6) Since $\hat{x}$ is a minimizer, we have

$$\begin{aligned}
0 &\leqslant \int_{A_n\cup B_n}\big\langle\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\theta_n'(t)\big\rangle\,dt\\
&\quad+\limsup_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\theta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt\\
&\leqslant \int_{A_n\cup B_n}\big\langle\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\theta_n'(t)\big\rangle\,dt+\int_I\big\langle\nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big),\theta_n(t)\big\rangle\,dt. \qquad (*)
\end{aligned}$$

Since $\theta_n'(t)=-\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))/\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|$, for any t in A_n, it follows that $-\langle\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)),\theta_n'(t)\rangle\chi_{A_n}(t)=\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|\chi_{A_n}(t)$. Hence, we obtain that $(*)$ can be written as

$$\begin{aligned}
&\int_{A_n}\big\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big\|\,dt\\
&\quad\leqslant \int_{B_n}\big\langle\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\theta_n'(t)\big\rangle\,dt+\int_I\big\langle\nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big),\theta_n(t)\big\rangle\,dt.
\end{aligned}$$

On B_n, $\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|$ is bounded by k_1; from Hölder's inequality and the estimate on $\|\theta_n\|_\infty$ obtained in (2) we have that there exists a constant C (independent of n) such that

$$\int_{A_n}\big\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big\|\,dt\leqslant Cm(A_n).$$

(7) As $m\to+\infty$, the sequence of functions $(\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|\chi_{\{\bigcup_{n=2}^m A_n\}}(t))_m$ converges monotonically to the function $\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|\chi_{\{\bigcup_{n>1}A_n\}}(t)$. From the estimate above and monotone convergence, we obtain

$$\int_{I\setminus C_1} \left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\| dt = \int_I \left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\| \chi_{\{\bigcup_{n>1} A_n\}}\, dt$$
$$\leqslant Cm\Big(\bigcup_{n>1} A_n\Big).$$

On C_1, $\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\| < k_1$. Hence

$$\int_I \left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\| dt < +\infty. \qquad \square$$

3. Additional regularity and the validity of the Euler–Lagrange equation

Corollary 3.1. *Under the same assumptions as in Theorem* 2.1, *for every variation* η, $\eta(a)=0$, $\eta(b)=0$ *and* $\eta' \in L^\infty(I)$, *we have*

$$\int_I \Big[\big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta'(t)\big\rangle + \big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta(t)\big\rangle\Big]\, dt = 0.$$

Proof. We shall prove that, for every η in $AC(I)$ with bounded derivative, such that $\eta(a) = \eta(b) = 0$, we have

$$\int_I \Big[\big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta'(t)\big\rangle + \big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta(t)\big\rangle\Big]\, dt \geqslant 0.$$

Fix η, let $\|\eta'(t)\| \leqslant K$ for almost every t in I.

(1) Define C_n and k_n as in point (1) of the proof of Theorem 2.1. Set

$$v_n = \int_{I\setminus C_n} \eta'(t)\, dt.$$

We have that $\lim_{n\to+\infty} \|v_n\| = 0$. In particular, for $n \geqslant \nu$, there exists $B_n \subseteq C_1$ such that $m(B_n) = \|v_n\|$. Set

$$(\eta_n)'(t) = \begin{cases} 0 & \text{for } t \in I \setminus C_n, \\ \eta'(t) & \text{for } t \in C_n \setminus B_n, \\ \frac{v_n}{\|v_n\|} + \eta'(t) & \text{for } t \in B_n. \end{cases}$$

We obtain

$$\int_I \eta_n'(t)\, dt = \int_{C_n\setminus B_n} \eta'(t)\, dt + \int_{B_n} \left[\frac{v_n}{\|v_n\|} + \eta'(t)\right] dt = \int_{C_n} \eta'(t)\, dt + v_n$$
$$= \int_I \eta'(t)\, dt = 0.$$

Hence, setting $\eta_n(t) = \int_a^t \eta_n'(\tau)\, d\tau$, we have that the functions $\eta_n(t)$ are variations and that, for almost every t in I, $\|\eta_n'(t)\| \leqslant (1+K)$, so that $\|\eta_n\|_\infty \leqslant (1+K)(b-a)$.

(2) As in point (3) of the proof of Theorem 2.1, there exists $\delta_n > 0$ such that for $(t,x,\xi) \in C_n \times R^s \times G$, with $d((t,x,\xi),(t,\hat{x}(t),\hat{x}'(t))) < \delta_n$, we have $\|\nabla_\xi L(t,x,\xi)\| < k_n + 1$.

(3) For $|\lambda| < \min\{1/(1+K)(b-a), \delta_n/(1+K)(b-a), \delta_n/(1+K)\}$, consider the integrals

$$\begin{aligned}
&\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)+\lambda\eta_n'(t)\big) - L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt \\
&\quad = \int_{C_n} \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)+\lambda\eta_n'(t)\big) - L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)\big)\big]\,dt \\
&\qquad + \int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)\big) - L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt.
\end{aligned}$$

For almost every $t \in C_n$, there exists $\zeta_\lambda(t) \in (0,\lambda)$ such that

$$\begin{aligned}
&\frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)+\lambda\eta_n'(t)\big) - L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)\big)\big] \\
&\quad = \big\langle \nabla_\xi L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)+\zeta_\lambda(t)\eta_n'(t)\big), \eta_n'(t)\big\rangle \\
&\quad \leqslant \big\|\nabla_\xi L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)+\zeta_\lambda(t)\eta_n'(t)\big)\big\|(1+K) < (k_n+1)(1+K).
\end{aligned}$$

Hence, we can apply the dominated convergence theorem to obtain that

$$\begin{aligned}
&\lim_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)+\lambda\eta_n'(t)\big) - L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)\big)\big]\,dt \\
&\quad = \int_{C_n} \big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta_n'(t)\big\rangle\,dt = \int_I \big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta_n'(t)\big\rangle\,dt.
\end{aligned}$$

(4) Following the point (5) of Theorem 2.1, we obtain that

$$\begin{aligned}
&\limsup_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)\big) - L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt \\
&\quad \leqslant \int_I \big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta_n(t)\big\rangle\,dt.
\end{aligned}$$

(5) Since $\hat{x}$ is a minimizer, we have

$$\begin{aligned}
0 &\leqslant \int_I \big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta_n'(t)\big\rangle\,dt \\
&\qquad + \limsup_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)\big) - L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt \\
&\leqslant \int_I \big[\big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta_n'(t)\big\rangle + \big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta_n(t)\big\rangle\big]\,dt.
\end{aligned}$$

(6) Since

$$\lim_{n\to+\infty} \eta_n'(t) = \eta'(t)$$

and

$$\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\| \left\|\eta_n'(t)\right\| \leqslant \left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|(1+K),$$

and

$$\lim_{n\to+\infty} \eta_n(t) = \eta(t)$$

and

$$\left\|\nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\| \left\|\eta_n(t)\right\| \leqslant \left\|\nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|(1+K)(b-a),$$

by the dominated convergence we obtain

$$\lim_{n\to+\infty} \int_I \big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta_n'(t)\big\rangle\, dt = \int_I \big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta'(t)\big\rangle\, dt$$

and

$$\lim_{n\to+\infty} \int_I \big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta_n(t)\big\rangle\, dt = \int_I \big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta(t)\big\rangle\, dt.$$

It follows that

$$\int_I \big[\big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta'(t)\big\rangle + \big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big), \eta(t)\big\rangle\big]\, dt \geqslant 0. \qquad \square$$

Even if the validity of the Euler–Lagrange equations already follows by the previous Corollary 3.1 and the DuBois–Reymond's lemma [3], we give an alternative proof in Corollary 3.3.

In the following theorem we prove an additional regularity result for the Lagrangian evaluated along the minimizer.

Theorem 3.2. *Under the same assumptions as in Theorem* 2.1*, the map* $\nabla_\xi L(\cdot,\hat{x}(\cdot),\hat{x}'(\cdot))$ *is in* $L^\infty(I)$.

Proof. Using an iteration process, we shall prove that for every p in N, $\nabla_\xi L(\cdot,\hat{x}(\cdot),\hat{x}'(\cdot))$ is in $L^p(I)$. Since the L^p are nested, this proves that $\nabla_\xi L(\cdot,\hat{x}(\cdot),\hat{x}'(\cdot)) \in \bigcap_{p\geqslant 1} L^p(I)$. At the same time, we shall prove that there exists a constant $K>0$ such that, for every $1 \leqslant p < +\infty$, $\|\nabla_\xi L\|_p \leqslant K$, thus proving that $\nabla_\xi L(\cdot,\hat{x}(\cdot),\hat{x}'(\cdot))$ is in $L^\infty(I)$.

Suppose that: (1) From Theorem 2.1, we know that $\nabla_\xi L(\cdot,\hat{x}(\cdot),\hat{x}'(\cdot))$ is in $L^1(I)$. Starting the iteration process, fix $p \in N$ and suppose that $\nabla_\xi L(\cdot,\hat{x}(\cdot),\hat{x}'(\cdot)) \in L^p(I)$, to prove that $\nabla_\xi L(\cdot,\hat{x}(\cdot),\hat{x}'(\cdot)) \in L^{p+1}(I)$.

(2) We can assume $\|\nabla_\xi L\|_p \neq 0$. Define C_n, A_n, k_n as in point (1) of the proof of Theorem 2.1. For all $n>1$, set

$$v_n^p = \frac{(b-a)}{2\|\nabla_\xi L\|_p^p} \int_{A_n} \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|^{p-1} dt.$$

Since

$$m(C_1) \geqslant (b-a)/2$$

and

$$\|v_n^p\| \leqslant \frac{(b-a)}{2\|\nabla_\xi L\|_p^p} \int_{A_n} \|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^p \leqslant (b-a)/2,$$

there exists a set $B_n^p \subset C_1$ such that $m(B_n^p) = \|v_n^p\|$, so that

$$\frac{2\|\nabla_\xi L\|_p^p}{(b-a)} \int_{B_n^p} \frac{v_n^p}{\|v_n^p\|}\,dt = \int_{A_n} \nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)) \|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^{p-1}.$$

Set

$$\begin{aligned}(\theta_n^p)'(t) &= -\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)) \|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^{p-1} \chi_{A_n}(t) \\ &\quad + \frac{2\|\nabla_\xi L\|_p^p}{(b-a)} \frac{v_n^p}{\|v_n^p\|} \chi_{B_n^p}(t)\end{aligned}$$

and $\theta_n^p(t) = \int_a^t (\theta_n^p)'(\tau)\,d\tau$. We obtain that $\|\theta_n^p\|_\infty \leqslant 2\int_{A_n} \|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^p\,dt$. The variations θ_n^p have bounded derivatives so we can apply Corollary 3.1 to obtain that

$$\int_I \left[\langle \nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)), (\theta_n^p)'(t)\rangle + \langle \nabla_x L(t,\hat{x}(t),\hat{x}'(t)), \theta_n^p(t)\rangle\right] dt = 0.$$

It follows that

$$\begin{aligned}&\int_{A_n} \|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^{p+1}\,dt \\ &\quad = \frac{2\|\nabla_\xi L\|_p^p}{(b-a)} \int_{B_n^p} \langle \nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)), v_n^p/\|v_n^p\|\rangle\,dt \\ &\qquad + \int_I \langle \nabla_x L(t,\hat{x}(t),\hat{x}'(t)), \theta_n^p(t)\rangle\,dt \\ &\quad \leqslant k_1 \int_{A_n} \|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^p\,dt \\ &\qquad + 2\int_{A_n} \|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^p\,dt \int_I \|\nabla_x L(t,\hat{x}(t),\hat{x}'(t))\|\,dt \\ &\quad \leqslant \tilde{C} \int_{A_n} \|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^p\,dt,\end{aligned}$$

where $\tilde{C}$ is independent of n and p (suppose $\tilde{C} \geqslant 1$). The sequence of maps

$$\left(\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^{p+1} \chi_{\bigcup_{n=2}^m A_n}(t)\right)_m$$

converges monotonically to $\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^{p+1}\chi_{\{\bigcup_{n>1}A_n\}}(t)$, and each integral $\int_I \|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|^{p+1}\chi_{\{\bigcup_{n=2}^m A_n\}}(t)\,dt$ is bounded by the same constant

$$\tilde{C}\sum_{n=2}^{\infty}\int_{A_n}\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|^p dt = \tilde{C}\int_{I\setminus C_1}\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|^p dt.$$

Hence, by the monotone convergence theorem,

$$\int_{I\setminus C_1}\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|^{p+1} dt \leqslant \tilde{C}\int_{I\setminus C_1}\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|^p dt < +\infty.$$

On C_1, $\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\| < k_1$, proving that $\nabla_\xi L(\cdot,\hat{x}(\cdot),\hat{x}'(\cdot)) \in L^{p+1}(I)$. Moreover, we have also obtained that

$$\int_{I\setminus C_1}\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|^{p+1} dt \leqslant \tilde{C}^p\int_{I\setminus C_1}\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\| dt,$$

so that

$$\left(\int_{I\setminus C_1}\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|^{p+1} dt\right)^{1/(p+1)} \leqslant \tilde{C}S,$$

where $S = \max\{1, \int_{I\setminus C_1}\|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))\|\,dt\}$. Setting $T = \max\{1, m(C_1)\}$ we have that, for all $p \in N$,

$$\begin{aligned}
\|\nabla_\xi L\|_{(p+1)} &\leqslant \left(k_1^{(p+1)}m(C_1) + \int_{I\setminus C_1}\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|^{p+1} dt\right)^{1/(p+1)} \\
&\leqslant k_1 m(C_1)^{1/(p+1)} + \left(\int_{I\setminus C_1}\left\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\right\|^{p+1} dt\right)^{1/(p+1)} \\
&\leqslant k_1 T + \tilde{C}S = K. \qquad \square
\end{aligned}$$

Corollary 3.3. *Under the same conditions as in Theorem* 2.1, *for every variation* η, $\eta(a) = 0$, $\eta(b) = 0$ *and* $\eta' \in L^1(I)$, *we have*

$$\int_I \left[\left\langle\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta'(t)\right\rangle + \left\langle\nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta(t)\right\rangle\right] dt = 0.$$

As a consequence, $t \to \nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))$ *is absolutely continuous.*

Proof. We shall prove that, for every η in $AC(I)$, such that $\eta(a) = \eta(b) = 0$, we have

$$\int_I \left[\left\langle\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta'(t)\right\rangle + \left\langle\nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta(t)\right\rangle\right] dt \geqslant 0.$$

(1) Fix η. Through the same steps as in point (1) of the proof of Theorem 2.1, for every $n \in N$ we can define a closed set C_n such that on it η' is continuous, $\hat{x}'$ is continuous with values in G, $\nabla_\xi L$ is continuous in $C_n \times R^s \times G$ and $\lim_{n\to+\infty} m(I \setminus C_n) = 0$. In particular, it follows that there are constants k_n and $c_n > 0$ such that, for all $t \in C_n$,

$$\left\|\nabla_\xi L\big(t, \hat{x}(t), \hat{x}'(t)\big)\right\| < k_n \quad \text{and} \quad \left\|\eta'(t)\right\| < c_n.$$

Define v_n, B_n, η_n' and η_n as in the proof of Corollary 3.1. Since, for all $t \in I$, $\|\eta_n'(t)\| \leqslant 1 + \|\eta'(t)\|$, it follows that $\|\eta_n'\|_1 \leqslant (b-a) + \|\eta'\|_1$. Moreover, $\|\eta_n\|_\infty \leqslant \|\eta_n'\|_1 \leqslant (b-a) + \|\eta'\|_1$.

(2) As in point (3) of the proof of Theorem 2.1, there exists $\delta_n > 0$ such that for $(t, x, \xi) \in C_n \times R^s \times G$, with $d((t, x, \xi), (t, \hat{x}(t), \hat{x}'(t))) < \delta_n$, we have $\|\nabla_\xi L(t, x, \xi)\| < k_n + 1$.

(3) For $|\lambda| < \min\{1/((b-a) + \|\eta'\|_1), \delta_n/((b-a) + \|\eta'\|_1), \delta_n/(c_n + 1)\}$, consider the integrals

$$\begin{aligned}
&\int_I \frac{1}{\lambda}\big[L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta_n'(t)\big) - L\big(t, \hat{x}(t), \hat{x}'(t)\big)\big]\,dt \\
&\quad = \int_I \frac{1}{\lambda}\big[L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta_n'(t)\big) - L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)\big)\big]\,dt \\
&\qquad + \int_I \frac{1}{\lambda}\big[L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)\big) - L\big(t, \hat{x}(t), \hat{x}'(t)\big)\big]\,dt \\
&\quad = \int_{C_n} \frac{1}{\lambda}\big[L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta_n'(t)\big) - L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)\big)\big]\,dt \\
&\qquad + \int_I \frac{1}{\lambda}\big[L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)\big) - L\big(t, \hat{x}(t), \hat{x}'(t)\big)\big]\,dt.
\end{aligned}$$

For every $t \in C_n$, there exists $\zeta_\lambda(t) \in (0, \lambda)$ such that

$$\begin{aligned}
&\frac{1}{\lambda}\big[L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta_n'(t)\big) - L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)\big)\big] \\
&\quad = \big\langle \nabla_\xi L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \zeta_\lambda(t)\eta_n'(t)\big), \eta_n'(t)\big\rangle \\
&\quad \leqslant \left\|\nabla_\xi L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \zeta_\lambda(t)\eta_n'(t)\big)\right\| c_n < (k_n + 1)c_n.
\end{aligned}$$

Hence, we can apply the dominated convergence theorem to obtain that

$$\begin{aligned}
&\lim_{\lambda\to 0} \int_I \frac{1}{\lambda}\big[L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta_n'(t)\big) - L\big(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)\big)\big]\,dt \\
&\quad = \int_{C_n} \big\langle \nabla_\xi L\big(t, \hat{x}(t), \hat{x}'(t)\big), \eta_n'(t)\big\rangle\,dt = \int_I \big\langle \nabla_\xi L\big(t, \hat{x}(t), \hat{x}'(t)\big), \eta_n'(t)\big\rangle\,dt.
\end{aligned}$$

(4) Following the point (5) of Theorem 2.1, we obtain that

$$\limsup_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt$$
$$\leqslant \int_I \big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta_n(t)\big\rangle\,dt$$

and, since $\hat{x}$ is a minimizer, we have

$$0\leqslant \int_I \big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta_n'(t)\big\rangle$$
$$+\limsup_{\lambda\to 0}\int_I \frac{1}{\lambda}\big[L\big(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)\big)-L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big]\,dt$$
$$\leqslant \int_I \big[\big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta_n'(t)\big\rangle+\big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta_n(t)\big\rangle\big]\,dt.$$

(5) Finally we have

$$\lim_{n\to+\infty}\eta_n'(t)=\eta'(t)$$

and

$$\big\|\nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big\|\,\|\eta_n'(t)\|\leqslant \big\|\nabla_\xi L\big(\cdot,\hat{x}(\cdot),\hat{x}'(\cdot)\big)\big\|_\infty\big(1+\|\eta'(t)\|\big),$$

and

$$\lim_{n\to+\infty}\eta_n(t)=\eta(t)$$

and

$$\big\|\nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big\|\,\|\eta_n(t)\|\leqslant \big\|\nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big)\big\|\big((b-a)+\|\eta'\|_1\big),$$

so that, by dominated convergence, we obtain

$$\lim_{n\to+\infty}\int_I \big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta_n'(t)\big\rangle\,dt=\int_I \big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta'(t)\big\rangle\,dt$$

and

$$\lim_{n\to+\infty}\int_I \big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta_n(t)\big\rangle\,dt=\int_I \big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta(t)\big\rangle\,dt.$$

Hence, it follows that

$$\int_I \big[\big\langle \nabla_\xi L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta'(t)\big\rangle+\big\langle \nabla_x L\big(t,\hat{x}(t),\hat{x}'(t)\big),\eta(t)\big\rangle\big]\,dt\geqslant 0.$$

The absolute continuity of $t\to\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))$ is classical (e.g., [2]). □

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