A RELAXATION RESULT FOR NON-CONVEX AND NON-COERCIVE SIMPLE INTEGRALS

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Abstract. We consider the following classical autonomous variational problem

$$\text{minimize}\left\{F(u)=\int_a^b f(u(x),u'(x))\,\mathrm{d}x\,:\,u\in AC([a,b]),u(a)=\alpha,u(b)=\beta,\,u([a,b])\subseteq I\right\}$$

where I is a real interval, $\alpha,\beta\in I$, and $f:I\times\mathbb{R}\to[0,+\infty)$ is possibly neither continuous, nor coercive, nor convex; in particular $f(s,\cdot)$ may be not convex at 0. Assuming the solvability of the relaxed problem, we prove under mild assumptions that the above variational problem has a solution, too.

1. Introduction

A classical problem of the Calculus of Variations is

$$\text{minimize}\left\{F(u):=\int_a^b f(u(x),u'(x))\,\mathrm{d}x\,:\,u\in\Omega\right\}, \tag{P}$$

where

$$\Omega:=\{u\in AC([a,b])\,:\,u(a)=\alpha,\,u(b)=\beta,\,u([a,b])\subseteq I\},$$

I is a real interval, $\alpha,\beta\in I$, and $f:I\times\mathbb{R}\to[0,+\infty)$ is a Borel measurable function.

If f is not convex or coercive w.r.t. the second variable, then the direct method of the Calculus of Variations cannot be applied and the variational problem can have no solution. The solvability of (P) has been object of investigation in many papers and many results have been proved under different sets of hypotheses. Besides the intrinsic interest of this problem, a further motivation for the study of autonomous, one-dimensional problems is the minimization of a functional along the different parametrizations of a given curve in $\mathbb{R}^n$.
A recurrent hypothesis in the literature is that $f(s,\cdot)$ is *convex at* 0, i.e.,

$$f(s,0)=f^{**}(s,0)\qquad\text{for every } s\in I, \tag{1.1}$$

where f^{**} denotes the convex envelope of f with respect to the second variable. This assumption is usually coupled with a coercivity condition, see e.g. Marcellini [18] and Fusco et al. [15]. In particular, [15] deals with the sum case $f(s,z)=g(s)+h(z)$, and the authors prove that weak regularity properties on g and h suffice to get the existence of minimizers. Further generalizations have been obtained by Ornelas, see [20] and [21]. In case of a lack of coercivity, but with f convex, we address to the papers by Cellina-Ferriero [9] and Clarke [11], see also [2], [3], [4] and [16] for non-autonomous functionals.

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Recently, existence and relaxation results have been obtained by Cupini et al. in [13] and [14], see also [12], for quite general possibly non-convex, non-coercive and discontinuous Lagrangians, and, again, assuming the convexity assumption at 0. Here and in the sequel, by relaxation result we mean a result stating that, under suitable assumptions, the solvability of (P^{**}) implies the solvability of (P); as usual, (P^{**}) stands for the variational problem (P) with f replaced by f^{**}. Differently from the papers listed above, in the quite recent article by Celada-Perrotta [6] a relaxation result is proved without assuming (1.1). If f is real valued, as in our setting, the authors assume the continuity both of f and f^{**}, a growth condition compatible with either superlinearity or certain cases of linear growth at infinity, and some more delicate additional assumptions; indeed, if (1.1) is removed then alternative assumptions must be considered, otherwise the solvability of (P) cannot be expected even in presence of coercive and regular energies, as the well known Bolza's example shows (see Section 7).
We also mention the result by Cellina [7], where it is proved that $\inf(P) = \inf(P^{**})$, with a Lipschitz continuous minimizer of (P) when the infimum is attained, assuming that $f(s,\cdot)$ is affinely minorized and satisfies a slightly weaker growth condition than that one in [6]. Of course this list of papers is far to be exhaustive and we refer to [5] and [6] for more details and references on this subject.

In this paper we prove a relaxation result without any coercivity condition, in absence of (1.1) and under mild regularity assumptions on f. The main advantages with respect to [6] consist in the weakening of the regularity assumptions on f and f^{**} and mainly in the removal of the growth condition on $f(s,\cdot)$, which is essentially replaced by the hypothesis that the convex envelope of $\{f^{**}(s,\cdot) = f(s,\cdot)\}$ coincides with $\mathbb{R}$. We refer to Remark 6.4 for a more detailed comparison between our Theorem 6.1 and [6, Theorem 2.2]. As it is now standard, to prove a relaxation result one can reson as follows: if u minimizes (P^{**}), and the equality $f^{**}(u,u') = f(u,u')$ holds a.e., then u minimizes (P), too; otherwise it has to be modified, so to get a new minimizer v of (P^{**}) satisfying the equality $f^{**}(v,v') = f(v,v')$. The basic instrument at our disposal, which is also our starting point, is the *a priori* monotonicity properties of minimizers, recently proved in [14] and a refinement of it, see the results in Section 3 below. Notice that a study of the monotonicity properties of minimizers was also carried out in [15] and [20] for integrands having the sum structure $f(s,z) = g(s) + h(z)$ with h coercive but non-convex; subsequently, this investigation was extended in [22] for coercive integrands having general structure. Thanks to this monotonicity property of u we will not use the procedure of modification used in [6], but two different approaches, the first one established in [17] and the other one based on the resolution of two suitable ordinary differential equations (see Sections 4 and 5, respectively).
Our main relaxation result is Theorem 6.1 in Section 6, see also Theorems 6.2 and 6.3 for existence results in special cases. These results are stated assuming that $\alpha \leq \beta$, since analogous results hold for $\alpha > \beta$, with straightforward changes.
Section 7 conlcudes the paper; there we provide some applications of our main relaxation result and we analyse the sharpness of our assumptions.

2. Notations

As mentioned in Introduction, we consider the autonomous variational problem (P), where $f : I \times \mathbb{R} \to [0, +\infty)$ is a Borel-measurable function and

$$\Omega := \{u \in AC([a,b]) \,:\, u(a) = \alpha,\ u(b) = \beta,\ u([a,b]) \subseteq I\},$$

with I interval in $\mathbb{R}$, $\alpha, \beta \in I$.

From now on, we assume $\alpha \leq \beta$. The case $\alpha > \beta$ can be treated with straightforward changes.

In the sequel (P^{**}) is the variational problem analogous to (P), with the functional F, now denoted F^{**}, having the integrand function f^{**} in place of f. Notice that (P^{**}) is well defined if $s \mapsto f^{**}(s,0)$ is a Borel-measurable function, as the following result states.

Lemma 2.1 (Lemma 2.1 in [14])**.** *Let $f : I \times \mathbb{R} \to [0, +\infty)$ be a Borel-measurable function. If $s \mapsto f^{**}(s,0)$ is a Borel-measurable function, then the function $s \mapsto f^{**}(s,z)$ is Lebesgue-measurable for every $z \in \mathbb{R}$ and the function $x \mapsto f^{**}(v(x), v'(x))$ is Lebesgue-measurable for every $v \in W^{1,1}_{\mathrm{loc}}(a,b)$.*

Remark 2.2. In [14, Lemma 2.1] it was assumed $f(s,0) = f^{**}(s,0)$ for all $s \in I$ instead of the Borel measurability of $f^{**}(\cdot, 0)$. However, the proof works also in the present setting.

Our approach to study the optimality of problem (P) is based on the *a priori* monotonicity properties of minimizers. In fact, as we will show in Section 3, we can reduce the competition set to its following subset:

$$\Omega^* := \{u \in \Omega \,:\, u \,\text{satisfies property (2.2) below}\} \tag{2.1}$$

$$\begin{aligned} &\text{there exist } \tau_1, \tau_2 \in [a,b],\ \tau_1 \leq \tau_2, \text{ such that } u \text{ is constant in } [\tau_1, \tau_2], \\ &\text{strictly monotone in } [a, \tau_1] \text{ and in } [\tau_2, b] \text{ and } u' \neq 0 \text{ a.e. in } [a,\tau_1] \cup [\tau_2, b]. \end{aligned} \tag{2.2}$$

Obviously, $\Omega^* = \Omega_+ \cup \Omega^+_+ \cup \Omega^+_- \cup \Omega^-_+$, where

$$\Omega_+ := \{u \in \Omega^* \,:\, u'(x) > 0 \text{ a.e. in } [a,b]\},$$

$$\begin{aligned} \Omega^+_+ := \{u \in \Omega^* \quad : \quad & a \leq \tau_1 < \tau_2 \leq b, u'(x) > 0 \text{ a.e. in } [a, \tau_1], \\ & u'(x) = 0 \text{ a.e. in } [\tau_1, \tau_2], u'(x) > 0 \text{ a.e. in } [\tau_2, b]\}, \end{aligned}$$

$$\begin{aligned} \Omega^+_- := \{u \in \Omega^* \quad : \quad & a < \tau_1 \leq \tau_2 < b, u'(x) < 0 \text{ a.e. in } [a, \tau_1], \\ & u'(x) = 0 \text{ a.e. in } [\tau_1, \tau_2], u'(x) > 0 \text{ a.e. in } [\tau_2, b]\}, \end{aligned}$$

$$\begin{aligned} \Omega^-_+ := \{u \in \Omega^* \quad : \quad & a < \tau_1 \leq \tau_2 < b, u'(x) > 0 \text{ a.e. in } [a, \tau_1], \\ & u'(x) = 0 \text{ a.e. in } [\tau_1, \tau_2], u'(x) < 0 \text{ a.e. in } [\tau_2, b]\}. \end{aligned}$$

If $u \in \Omega^*$ then s_u is the value of u in $[\tau_1, \tau_2]$. To allow a unified presentation, in the sequel if $u \in \Omega_+$ then $\tau_1 = \tau_2 = b$.

We will use also the following sets

$$
\begin{aligned}
Q_+^+ &:= \{s \in [\alpha,\beta] \,:\, f^{**}(s,0) = \min_{\sigma\in[\alpha,\beta]} f^{**}(\sigma,0)\},\\
Q_-^+ &:= \{s \in I\cap(-\infty,\alpha) \,:\, f^{**}(s,0) < f^{**}(\sigma,0) \ \text{ for all}\, \sigma\in(s,\beta]\},\\
Q_+^- &:= \{s \in I\cap(\beta,+\infty) \,:\, f^{**}(s,0) < f^{**}(\sigma,0) \ \text{ for all}\, \sigma\in[\alpha,s)\}.
\end{aligned}
\tag{2.3}
$$

Since we aim at proving the existence of minimizers of (P) when (P^{**}) is solvable, then it will be advantageous to compare the function f with respect to f^{**}. Thus, for every $s \in I$, we define

$$\mathcal{C}_s^+ := \{z > 0 \,:\, f(s,z) = f^{**}(s,z)\}, \quad \mathcal{C}_s^- := \{z < 0 \,:\, f(s,z) = f^{**}(s,z)\}, \tag{2.4}$$

$$\mathcal{A}_s^+ = \{z \geq 0 \,:\, f^{**}(s,\cdot) \text{ is affine in}\, [0,z]\}, \quad \mathcal{A}_s^- = \{z \leq 0 \,:\, f^{**}(s,\cdot) \text{ is affine in}\, [z,0]\}. \tag{2.5}$$

Morever, we define the set

$$\mathcal{D}^0 := \{s \in I \,:\, f(s,0) > f^{**}(s,0)\}. \tag{2.6}$$

We conclude saying that co A stands for the convex envelope of a set A and recalling the notion of subdifferential in the sense of Convex Analysis, i.e.,

$$\partial f(s,z) := \{\xi \in \mathbb{R} \,:\, f(s,w) - f(s,z) \geq \xi(w-z) \text{ for every } w \in \mathbb{R}\}.$$

3. Monotonicity properties of minimizers

As in [14, Theorem 3.1], under suitable assumptions, including (1.1), fixed $u \in \Omega$ it is possible to find $w \in \Omega^*$ such that $F(w) \leq F(u)$ and $F^{**}(w) \leq F^{**}(u)$. Now, we give a similar but more precise and complete statement than that one presented in [14]. First of all since we will use this result only for F^{**} we write the assumptions to get the inequality in this context, then we add the assumptions needed to get the corresponding inequality for F. Moreover, we classify the value s_w of w in the interval (possibly reduced to a point) $[\tau_1,\tau_2]$, where it is constant: if $w \notin \Omega_+$, we have that s_w has to belong to Q_+^+, Q_-^+ or Q_+^-.

Theorem 3.1. *Suppose that $f : I \times \mathbb{R} \to [0,+\infty)$ is a Borel-measurable function, $f^{**}(\cdot,0)$ is lower semicontinuous and*

$$\textit{there exists a Lebesgue-measurable selection } g(\cdot) \in \partial f^{**}(\cdot,0) \textit{ with } g \in L^\infty_{\mathrm{loc}}(I). \tag{3.1}$$

We have that

(a) $\inf_{\Omega} F^{**} = \inf_{\Omega^*} F^{**}$,

(b) *if (P^{**}) has a solution then there exists a solution $w \in \Omega^*$ such that one of the following properties holds:*

  (b1) $w \in \Omega_+$,

  (b2) $w \in \Omega_+^+$ *and s_w is any element in Q_+^+,*

  (b3) $w \in \Omega_-^+$ *and s_w is a suitable element in Q_-^+,*

  (b4) $w \in \Omega_+^-$ *and s_w is a suitable element in Q_+^-.*

Moreover, if $f(s,0) = f^{**}(s,0)$ for every $s \in I$ and g in (3.1) *is Borel-measurable then (a) and (b) hold also for F and (P).*

Proof. We begin noticing that (a) follows if we prove that for every $u \in \Omega$ there exists $w \in \Omega^*$ such that $F^{**}(w) \leq F^{**}(u)$. We remark also that such w satisfies $w([a,b]) \subseteq u([a,b])$.

If $u \in \Omega_+$ then $w = u$ and we conclude.

Let $u \in \Omega \setminus \Omega_+$ and define $\tilde{f^{**}}(s,z) = f^{**}(s,z) - g(s)z$. It is easy to verify that $\tilde{f^{**}}(s,0) \leq \tilde{f^{**}}(s,z)$ for every $s \in I$ and every $z \in \mathbb{R}$. Moreover, $s \mapsto \tilde{f^{**}}(s,0)$ is lower semicontinuous and there exists $k \in \mathbb{R}$ such that for every $v \in \Omega$ we get

$$F^{**}(v) = \tilde{F^{**}}(v) + k,$$

where $\tilde{F^{**}}(v)$ stands for $\int_a^b \tilde{f^{**}}(v(x), v'(x))\, \mathrm{d}x$, see [14, Lemma 2.2]. Notice that by Lemma 2.1 F^{**} and $\tilde{F^{**}}$ are well defined.
Thus, without loss of generality we can assume that $f^{**}(s,0) \leq f^{**}(s,z)$ for every $s \in I$ and every $z \in \mathbb{R}$.

Suppose that there exists $\bar{s} \in [\alpha, \beta]$ such that

$$f^{**}(\bar{s}, 0) = \min_{s \in u([a,b])} f^{**}(s,0). \tag{3.2}$$

In particular, since $[\alpha,\beta] \subseteq u([a,b])$ then $\bar{s} \in Q_+^+$. By the proof of [14, Theorem 3.1] there exists $w \in \Omega_+^+$ such that $F^{**}(w) \leq F^{**}(u)$ and $w(x) = \bar{s}$ in $[\tau_1, \tau_2]$. Defining $s_w = \bar{s}$, we have that s_w satisfies the property stated in (b2).

If $\bar{s} \in [\alpha, \beta]$ satisfying (3.2) does not exist, then, by the lower semicontinuity of $f^{**}(\cdot, 0)$, $u([a,b]) \setminus [\alpha,\beta]$ must contain at least one element, denoted again by $\bar{s}$, such that (3.2) holds and

$$f^{**}(\bar{s},0) < f^{**}(s,0) \quad \text{for every}\, s \in [\alpha,\beta]. \tag{3.3}$$

Suppose that $\bar{s} < \alpha$. We define

$$\tilde{s} := \max\{s \in [\bar{s}, \alpha] \;:\; f^{**}(s,0) = f^{**}(\bar{s},0)\} < \alpha,$$

where the inequality follows by the lower semicontinuity of $f^{**}(\cdot,0)$, (3.2) and (3.3). Obviously $\tilde{s} \in Q_-^+$. Modifying u as explained in the proof of [14, Theorem 3.1], with $\tilde{s}$ as above, we get $w \in \Omega_-^+$, strictly decreasing in $[a, \tau_1]$, $w \equiv \tilde{s}$ in $[\tau_1, \tau_2]$, strictly increasing in $[\tau_2, b]$ with $a < \tau_1 \leq \tau_2 < b$, and satisfying $F^{**}(w) \leq F^{**}(u)$. In particular, $w([a,b]) \subseteq u([a,b]) \cap (-\infty, \beta]$. Defining $s_w = \tilde{s}$, we have that s_w satisfies (b3).

If $\bar{s} > \beta$ then we proceed in an analogous way, defining

$$\tilde{s} = \min\{s \in [\beta, \bar{s}] \;:\; f^{**}(s,0) = f^{**}(\bar{s},0)\} > \beta,$$

obtaining in this case that $w \in \Omega_+^-$, $s_w = \tilde{s} \in Q_+^-$ and $w([a,b]) \subseteq u([a,b]) \cap [\alpha + \infty)$.

We now turn to the proof of the last part of the claim. Suppose that $f(s,0) = f^{**}(s,0)$ for every $s \in I$ and g is Borel-measurable. If we define $\tilde{f}(s,z) = f(s,z) - g(s)z$ and $\tilde{F}(v) = \int_a^b \tilde{f}(v(x), v'(x))\, \mathrm{d}x$, then $\tilde{f}$ is Borel-measurable and $\tilde{F}$ is well defined in Ω, so we can repeat the same proof of [14, Theorem 3.1]. Also in this case, a function w will be defined, which coincides with w above, because the procedure used to modify $u \in \Omega$ into w depends only on u and $f^{**}(\cdot,0)$, by assumption equal to $f(\cdot,0)$. $\square$

4. Modification of the minimizer in $[a, \tau_1]$ and in $[\tau_2, b]$

From the results in the previous section we have that if (P^{**}) has a minimizer, then there exists a minimizer belonging to Ω^*, see (2.1).

We claim that under suitable assumptions a minimizer $u \in \Omega^*$ of (P^{**}) can be modified so to obtain another minimizer $\tilde{u} \in \Omega^*$ of (P^{**}) satisfying

$$f(\tilde{u}(x), \tilde{u}'(x)) = f^{**}(\tilde{u}(x), \tilde{u}'(x)) \tag{4.1}$$

for almost every $x \in [a, b]$.

We postpone to the next section the modification of u in the interval $[\tau_1, \tau_2]$, where u is constant. In this section we discuss the modification of the restrictions of u to intervals $[a, \tau_1]$ and $[\tau_2, b]$, where u is strictly monotone. We recall that if $u \in \Omega_+$ then we set $\tau_1 = \tau_2 = b$.

Proposition 4.1. *Let $f : I \times \mathbb{R} \to [0, +\infty)$ be a Borel-measurable function, with $f^{**}(\cdot, 0)$ lower semicontinuous. Assume that $u \in \Omega^*$; i.e., u is strictly monotone in $[a, \tau_1]$ and in $[\tau_2, b]$ with $u'(x) \neq 0$ a.e., and $u \equiv s_u$ in $[\tau_1, \tau_2]$.*

Suppose that

$$\mathcal{C}^+_{u(x)} \text{ is a not empty, closed set in } (0, +\infty) \quad \text{and} \quad u'(x) \in \operatorname{co} \mathcal{C}^+_{u(x)} \tag{4.2}$$

for a.e. $x \in [a, b]$ s.t. $u'(x) > 0$.

Analogously, suppose that

$$\mathcal{C}^-_{u(x)} \text{ is a not empty, closed set in } (-\infty, 0) \quad \text{and} \quad u'(x) \in \operatorname{co} \mathcal{C}^-_{u(x)} \tag{4.3}$$

for a.e. $x \in [a, b]$ s.t. $u'(x) < 0$.

Then there exists $\tilde{u} \in \Omega^*$ such that $F^{**}(\tilde{u}) = F^{**}(u)$, $\tilde{u}$ has the same strict monotonicity properties of u in $[a, \tau_1]$ and $[\tau_2, b]$,

$$f(\tilde{u}(x), \tilde{u}'(x)) = f^{**}(\tilde{u}(x), \tilde{u}'(x)) \quad \text{for a.e. } x \in [a, \tau_1] \cup [\tau_2, b] \tag{4.4}$$

and

$$\tilde{u}(x) = u(x) = s_u \quad \text{for every } x \in [\tau_1, \tau_2]. \tag{4.5}$$

Proof. Let u, τ_1, τ_2 be as in the statement. Let us deal with the interval $[a, \tau_1]$ and suppose that u is strictly increasing on this set; the proof in the decreasing case goes in a similar way.

We reason as in the proof of [14, Theorem 4.2]. For the sake of completeness, we provide here the proof, apart from the measurability of the functions ξ_i, ψ_i $(i = 1, 2)$ below, for whom we refer to [14].

By (4.2), the functions

$$\xi_1(x) := \sup\{z \leq u'(x) : f(u(x), z) = f^{**}(u(x), z)\},$$
$$\xi_2(x) := \inf\{z \geq u'(x) : f(u(x), z) = f^{**}(u(x), z)\}$$

are well-defined and positive. Moreover, they are measurable functions.

By (4.2), since u is a strictly monotone and absolutely continuous function, then $\mathcal{C}^+_s$ is a not empty and closed set in $(0, +\infty)$ for a.e. $s \in [\alpha, s_u]$. Therefore,

$$f(u(x), \xi_i(x)) = f^{**}(u(x), \xi_i(x)) \quad \text{for a.e. } x \in (a, \tau_1), \tag{4.6}$$

and $f^{**}(u(x), \cdot)$ is affine in $[\xi_1(x), \xi_2(x)]$ (notice that this interval can be degenerate).

Define the measurable functions

$$\psi_i(s) := \frac{1}{\xi_i(u^{-1}(s))}, \qquad \text{for } s \in (\alpha, s_u),\ i = 1, 2. \tag{4.7}$$

Since $\xi_1(x) \le u'(x) \le \xi_2(x)$, we have $\psi_2(s) \le \frac{1}{u'(u^{-1}(s))} \le \psi_1(s)$, hence there exists a measurable weight function $\lambda : (\alpha, s_u) \to [0,1]$ such that

$$\frac{1}{u'(u^{-1}(s))} = \lambda(s)\psi_1(s) + (1-\lambda(s))\psi_2(s). \tag{4.8}$$

In particular, the map $s \mapsto \lambda(s)\psi_1(s) + (1-\lambda(s))\psi_2(s)$ is summable in (α, s_u).

Let us define $\tilde{f}, \tilde{f^{**}} : I \times (0,+\infty) \to \mathbb{R}$ as

$$\tilde{f}(s,z) := f(s, \frac{1}{z})z, \quad \tilde{f^{**}}(s,z) := f^{**}(s, \frac{1}{z})z.$$

In [17, Lemma 5] it was proved that $\tilde{f^{**}}(s,\cdot)$ is affine in (c,d), $c>0$, if and only if $f^{**}(s,\cdot)$ is affine in $(\frac{1}{d}, \frac{1}{c})$. Therefore, by (4.6) we deduce that

$$\tilde{f}(s,\psi_i(s)) = \tilde{f^{**}}(s,\psi_i(s)) \qquad \text{a.e. in } (\alpha, s_u),\ i=1,2 \tag{4.9}$$

and $\tilde{f^{**}}(s,\cdot)$ is affine in $[\psi_2(s), \psi_1(s)]$. Hence, by (4.8)

$$\tilde{f^{**}}(s, \frac{1}{u'(u^{-1}(s))}) = \lambda(s)\tilde{f^{**}}(s,\psi_1(s)) + (1-\lambda(s))\tilde{f^{**}}(s,\psi_2(s)) \tag{4.10}$$

so, in particular, also the map $s \mapsto \lambda(s)\tilde{f^{**}}(s,\psi_1(s)) + (1-\lambda(s))\tilde{f^{**}}(s,\psi_2(s))$ is summable in (α, s_u), since

$$\int_\alpha^{s_u} \tilde{f^{**}}(s, \frac{1}{u'(u^{-1}(s))})\ \mathrm{d}s = \int_a^{\tau_1} f^{**}(u(x), u'(x))\ \mathrm{d}x \le F^{**}(u).$$

Thus, we can apply an extension of Liapunov's Theorem on the range of vector measures, see [10, Chap.16], [8, page 103] or [19, Theorem 2], to the functions $g_i(s) := \Big(\psi_i(s), \tilde{f^{**}}(s,\psi_i(s))\Big)$, $i=1,2$, deducing the existence of a decomposition of (α, s_u) into disjoint measurable subsets F_1, F_2 such that, if we define $\gamma(s) := \psi_1(s)\chi_{F_1}(s) + \psi_2(s)\chi_{F_2}(s)$, then, by (4.8),

$$\int_\alpha^{s_u} \gamma(s)\ \mathrm{d}s = \int_\alpha^{s_u} \frac{1}{u'(u^{-1}(s))}\ \mathrm{d}s \tag{4.11}$$

and, by (4.9) and (4.10),

$$\begin{aligned} &\textstyle\int_\alpha^{s_u} \tilde{f}(s,\gamma(s))\ \mathrm{d}s = \int_{F_1} \tilde{f^{**}}(s,\psi_1(s))\ \mathrm{d}s + \int_{F_2} \tilde{f^{**}}(s,\psi_2(s))\ \mathrm{d}s \\ &= \textstyle\int_\alpha^{s_u} [\lambda(s)\tilde{f^{**}}(s,\psi_1(s)) + (1-\lambda(s))\tilde{f^{**}}(s,\psi_2(s))]\ \mathrm{d}s = \int_\alpha^{s_u} \tilde{f^{**}}(s, \frac{1}{u'(u^{-1}(s))})\ \mathrm{d}s. \end{aligned} \tag{4.12}$$

Consider the absolutely continuous function

$$U(s) := a + \int_\alpha^s \gamma(\sigma)\ \mathrm{d}\sigma.$$

By (4.11) and a change of variables (see e.g. [1, Corollary 5.4.4])

$$U(s_u) = a + \int_\alpha^{s_u} \gamma(s)\ \mathrm{d}s = a + \int_\alpha^{s_u} \frac{1}{u'(u^{-1}(s))}\ \mathrm{d}s = a + \int_a^{\tau_1} \frac{1}{u'(x)} u'(x)\ \mathrm{d}x = \tau_1.$$

Finally, since $U'(s) > 0$ a.e. in $[\alpha, s_u]$, then also the inverse function $\tilde{u}(x) := U^{-1}(x)$, $x \in [a, \tau_1]$, is absolutely continuous (see e.g. [1, page 389]). Moreover, $\tilde{u}(a) = \alpha$, $\tilde{u}(\tau_1) = s_u$ and, by definition of γ, (4.7) and (4.9), the equality (4.4) follows for a.e. $x \in [a, \tau_1]$.

Finally by a change of variable and (4.12)

$$\int_a^{\tau_1} f^{**}(\tilde{u}(x),\tilde{u}'(x))\ \mathrm{d}x = \int_a^{\tau_1} f(\tilde{u}(x),\tilde{u}'(x))\ \mathrm{d}x = \int_\alpha^{s_u} f(s,\tilde{u}'(U(s)))U'(s)\ \mathrm{d}s$$
$$= \int_\alpha^{s_u} \tilde{f}(s,\gamma(s))\ \mathrm{d}s = \int_\alpha^{s_u} \tilde{f^{**}}(s,\frac{1}{u'(u^{-1}(s))})\ \mathrm{d}s = \int_a^{\tau_1} f^{**}(u(x),u'(x))\ \mathrm{d}x.$$

A similar construction can be done in the interval $[\tau_2,b]$ so that the modified trajectory $\tilde{u}$ (kept equal to s_u in $[\tau_1,\tau_2]$) satisfies $F^{**}(\tilde{u}) = F^{**}(u)$. □

5. Modification of the minimizer in $[\tau_1,\tau_2]$

In this section, we show how to modify a given $u\in\Omega^*$ in the interval $[\tau_1,\tau_2]$ where it is constant, in such a way to obtain $v\in\Omega$ satisfying

(a) $v = u$ in $[a,\tau_1]\cup[\tau_2,b]$,
(b) $f(v(x),v'(x)) = f^{**}(v(x),v'(x))$ for a.e. $x\in[\tau_1,\tau_2]$.
(c) $F^{**}(v)\le F^{**}(u)$.

Of course, if the interval $[\tau_1,\tau_2]$ is degenerate, in particular this happens if $u\in\Omega_+$ or if $f(s_u,0) = f^{**}(s_u,0)$, then there is nothing to prove. Therefore in this section we consider the case $u\in\Omega^*\setminus\Omega_+$ and $f(s_u,0) > f^{**}(s_u,0)$.

To face our problem, we introduce two properties, (5.1*) and (5.1**), both depending on some $\bar{s}\in I$ that will be specified each time. We point out that *decreasing* and *increasing* have to be intended as *non-increasing* and *non-decreasing*, respectively, and that the sets $\mathcal{C}_s^\pm$, $\mathcal{A}_s^\pm$ are defined in (2.4) and (2.5).
The first property is the following:

$$\exists\,\delta>0\ :\ f^{**}(\cdot,0)\text{ is increasing in }[\bar{s}-\delta,\bar{s}]\subseteq I\text{ and }(*)\text{ holds,}\tag{5.1*}$$

where $(*)$ means that there exist two measurable functions

$$g^+:[\bar{s}-\delta,\bar{s}]\to(0,+\infty)\quad\text{and}\quad g^-:[\bar{s}-\delta,\bar{s}]\to(-\infty,0)$$

such that

(a^*) $g^+(s)\in\mathcal{C}_s^+\cap\mathcal{A}_s^+$ and $g^-(s)\in\mathcal{C}_s^-\cap\mathcal{A}_s^-$ for a.e. $s\in(\bar{s}-\delta,\bar{s})$,
(b^*) $\dfrac{1}{g^+},\dfrac{1}{g^-}\in L^1(\bar{s}-\delta,\bar{s})$.

Analogously, the second property is the following:

$$\exists\,\delta>0\ :\ f^{**}(\cdot,0)\text{ is decreasing in }[\bar{s},\bar{s}+\delta]\subseteq I\text{ and }(**)\text{ holds,}\tag{5.1**}$$

where $(**)$ means that there exist two measurable functions

$$g^+:[\bar{s},\bar{s}+\delta]\to(0,+\infty)\quad\text{and}\quad g^-:[\bar{s},\bar{s}+\delta]\to(-\infty,0)$$

such that

(a^{**}) $g^+(s)\in\mathcal{C}_s^+\cap\mathcal{A}_s^+$ and $g^-(s)\in\mathcal{C}_s^-\cap\mathcal{A}_s^-$ for a.e. $s\in(\bar{s},\bar{s}+\delta)$,
(b^{**}) $\dfrac{1}{g^+},\dfrac{1}{g^-}\in L^1(\bar{s},\bar{s}+\delta)$.

Remark 5.1. We claim that if f and f^{**} are continuous with superlinear growth (or satisfy the weaker growth condition (H3) in [6], that includes the superlinearity growth and some linear growth as well) then $(*)$ and $(**)$ hold for any $\bar{s}$ such that $f(\bar{s},0) > f^{**}(\bar{s},0)$. In fact, by continuity there exist $\epsilon, \delta > 0$ such that: $\inf C_s^+ = \min C_s^+ > \epsilon$ and $\sup C_s^- = \max C_s^- < -\epsilon$ for every $s \in [\bar{s}-\delta, \bar{s}+\delta]$. Moreover, both $s \mapsto \inf C_s^+$ and $s \mapsto \sup C_s^-$ are bounded, with $s \mapsto \inf C_s^+$ lower semicontinuous and $s \mapsto \sup C_s^-$ upper semicontinuous (see [6, Proposition 3.1]). Thus, defining $g^+(s) = \inf C_s^+$ and $g^-(s) = \sup C_s^-$, $(*)$ and $(**)$ hold true.

Before stating the result we notice that the regularity assumption on $f^{**}(\cdot,0)$ is not the lower semicontinuity as in Proposition 4.1, but only the Borel-measurability.

Proposition 5.2. *Let $f : I \times \mathbb{R} \to [0,+\infty)$ be a Borel-measurable function, with $f^{**}(\cdot,0)$ Borel-measurable. Let $u \in \Omega^* \setminus \Omega_+$ satisfying (b2), (b3) or (b4) in Theorem 3.1. In particular, $u(x) = s_u$ for every $x \in [\tau_1,\tau_2]$, with $\tau_1 < \tau_2$ as in* (2.2).
Suppose that

$$f(s_u,0) > f^{**}(s_u,0) \quad \textit{and} \quad (5.1^*) \textit{ or } (5.1^{**}) \textit{ holds with } \bar{s} = s_u. \tag{5.2}$$

Then there exists $v \in \Omega$ such that $F^{**}(v) \le F^{**}(v)$,

$$f(v(x),v'(x)) = f^{**}(v(x),v'(x)) \quad \textit{for a.e. } x \in [\tau_1,\tau_2] \tag{5.3}$$

and

$$v(x) = u(x) \quad \textit{for every } x \in [a,\tau_1] \cup [\tau_2,b]. \tag{5.4}$$

Moreover, if $g^{\pm}$ are in $L^\infty(s_u-\delta,s_u)$ (if (5.1^*) *holds) or in $L^\infty(s_u,s_u+\delta)$ (if* (5.1^{**}) *holds), then $v \in W^{1,\infty}(\tau_1,\tau_2)$.*

Proof. Let us suppose (5.1^{**}). Then $f^{**}(\cdot,0)$ is decreasing in $[s_u,s_u+\delta] \subseteq I$ and $(**)$ holds. Define

$$G^{\pm}(s) := \int_{s_u}^{s} \frac{1}{g^{\pm}(t)}\,\mathrm{d}t \qquad s \in [s_u,s_u+\delta].$$

It is readily seen that G^+ is a strictly increasing, nonnegative, absolutely continuous function onto $[0,G^+(s_u+\delta)]$; analogously, G^- is a strictly decreasing, nonpositive, absolutely continuous function onto $[G^-(s_u+\delta),0]$. Moreover, both G^+ and G^- have an absolutely continuous inverse, since $(G^{\pm})'(s) \neq 0$ a.e. (see e.g. [1]). Let now consider

$$u^+ : [0,G^+(s_u+\delta)] \to [s_u,s_u+\delta], \quad u^+(x) := (G^+)^{-1}(x)$$

and

$$u^- : [G^-(s_u+\delta),0] \to [s_u,s_u+\delta], \quad u^-(x) := (G^-)^{-1}(x).$$

They are absolutely continuous functions satisfying

$$(u^+)'(x) = g^+(u^+(x)) \text{ for a.e. } x \text{ in } [0,G^+(s_u+\delta)], \quad u^+(0) = s_u \tag{5.5}$$

and

$$(u^-)'(x) = g^-(u^-(x)) \text{ for a.e. } x \text{ in } [G^-(s_u+\delta),0], \quad u^-(0) = s_u. \tag{5.6}$$

By (a^*)

$$f(u^+(x),(u^+)'(x)) = f^{**}(u^+(x),(u^+)'(x)) \qquad f(u^-(x),(u^-)'(x)) = f^{**}(u^-(x),(u^-)'(x))$$

for a.e. x in $[0,G^+(s_u+\delta)]$ and in $[G^-(s_u+\delta),0]$, respectively.

For every $t \in [0, \delta]$ define the periodic and absolutely continuous function $u_t : \mathbb{R} \to [s_u, s_u + \delta]$ with period $G^+(s_u + t) - G^-(s_u + t)$,

$$u_t(x) := \begin{cases} u^-(x) & \text{if } x \in [G^-(s_u + t), 0], \\ u^+(x) & \text{if } x \in [0, G^+(s_u + t)]. \end{cases}$$

Notice that $u_t(G^-(s_u + t)) = u_t(G^+(s_u + t)) = s_u + t$.

We claim that there exists $\bar{t} \in (0, \delta]$ such that $G^+(s_u + \bar{t}) - G^-(s_u + \bar{t}) = \frac{\tau_2 - \tau_1}{n}$ for some $n \in \mathbb{N}$. In fact, $G^+(s_u) - G^-(s_u) = 0$ and $G^+(s_u + \delta) - G^-(s_u + \delta) > 0$; by continuity the claim follows. Then

$$u_{\bar{t}}(\tau_2 - \tau_1) = u_{\bar{t}}(n(G^+(s_u + \bar{t}) - G^-(s_u + \bar{t})) = u_{\bar{t}}(0) = s_u.$$

Now, defining

$$v(x) := \begin{cases} u(x) & \text{if } x \in [a, b] \setminus [\tau_1, \tau_2], \\ u_{\bar{t}}(x - \tau_1) & \text{if } x \in [\tau_1, \tau_2], \end{cases}$$

we get (5.3) and (5.4).

We claim that $F^{**}(v) \le F^{**}(u)$. In fact, for all $s \in (s_u - \delta, s_u + \delta)$ there exists a measurable function $m(s)$ $(= [f^{**}(s, g^+(s)) - f^{**}(s, 0)]/g^+(s))$ such that

$$f^{**}(s, z) = f^{**}(s, 0) + m(s)z \text{ for all } z \in (\inf \mathcal{A}_s^-, \sup \mathcal{A}_s^+).$$

Hence, by the periodicity of $u_{\bar{t}}$ and a change of variables,

$$\begin{aligned} \int_{\tau_1}^{\tau_2} f^{**}(v(x), v'(x))\, \mathrm{d}x =& n \int_{G^-(s_u+\bar{t})}^{G^+(s_u+\bar{t})} f^{**}(u_{\bar{t}}(x), u_{\bar{t}}'(x))\, \mathrm{d}x \\ =& n \int_{G^-(s_u+\bar{t})}^{G^+(s_u+\bar{t})} \{f^{**}(u_{\bar{t}}(x), 0) + m(u_{\bar{t}}(x))u_{\bar{t}}'(x)\}\, \mathrm{d}x. \end{aligned}$$

Another change of variables implies,

$$\int_{G^-(s_u+\bar{t})}^{G^+(s_u+\bar{t})} m(u_{\bar{t}}(x))u_{\bar{t}}'(x)\, \mathrm{d}x = \int_{G^-(s_u+\bar{t})}^{0} m(u_{\bar{t}}(x))u_{\bar{t}}'(x)\, \mathrm{d}x + \int_{0}^{G^+(s_u+\bar{t})} m(u_{\bar{t}}(x))u_{\bar{t}}'(x)\, \mathrm{d}x = 0,$$

therefore, by the monotonicity assumption on $f^{**}(\cdot, 0)$,

$$\begin{aligned} \int_{G^-(s_u+\bar{t})}^{G^+(s_u+\bar{t})} f^{**}(u_{\bar{t}}(x), u_{\bar{t}}'(x))\, \mathrm{d}x = \int_{G^-(s_u+\bar{t})}^{G^+(s_u+\bar{t})} f^{**}(u_{\bar{t}}(x), 0)\mathrm{d}x \le& \int_{G^-(s_u+\bar{t})}^{G^+(s_u+\bar{t})} f^{**}(s_u, 0)\, \mathrm{d}x \\ =& \frac{1}{n} \int_{\tau_1}^{\tau_2} f^{**}(u(x), u'(x))\, \mathrm{d}x \end{aligned}$$

and the claim follows. The Lipschitz regularity result follows by (5.5) and (5.6).

If (5.1*) holds, the proof goes in a similar way with straightforward changes. □

6. The relaxation results

In this section we state three results. Theorem 6.1 is our main result; there we prove that under suitable assumptions the solvability of (P) follows from that one of (P^{**}). Of course, combining the assumptions of this theorem with those of some existence results for (P^{**}), e.g. f^{**} has superlinear growth with respect to the second variable, so to apply the direct method of the Calculus of Variations, or the assumptions of the recent [13, Theorem 7.1], then we obtain existence results for (P), without assuming *a priori* the solvability of (P^{**}). In the other results, Theorems 6.2 and 6.3, we consider two special cases. Precisely, in the first one it is assumed that $\alpha < \beta$ and that (P^{**}) has a minimizer in Ω_+; in the second one we assume that $\alpha = \beta$ and that the constant function $u \equiv \alpha$ is a solution to (P^{**}).

As far as Theorem 6.1 is concerned, we need the following assumption, also used in [14]:

$$\text{meas}\,\{s \in [\alpha,\beta] \,:\, f^{**}(s,0) = \min_{\sigma\in[\alpha,\beta]} f^{**}(\sigma,0) \quad \text{and} \quad \inf \mathcal{C}_s^+ > 0\} = 0, \tag{6.1}$$

with the position $\inf \mathcal{C}_s^+ = 0$ if $\mathcal{C}_s^+ = \emptyset$. We point out also that a main point in the proof of Theorem 6.1 is the classification of the values of a minimizer in Ω^* of (P^{**}) in the interval $[\tau_1, \tau_2]$, where it is constant, see Theorem 3.1 (b).

Theorem 6.1. *Let $\alpha \leq \beta$. Let $f : I \times \mathbb{R} \to [0,+\infty)$ be a Borel-measurable function, with $f^{**}(\cdot,0)$ lower semicontinuous and satisfying* (3.1) *and* (6.1).

Assume

$$\text{co}\,\{z \in \mathbb{R} \,:\, f(s,z) = f^{**}(s,z)\} = \mathbb{R} \quad \textit{for a.e. } s \in I, \tag{6.2}$$

$$\mathcal{C}_s^+ \textit{is a closed set in } (0,+\infty) \;\; \textit{for a.e. } s \in I \tag{6.3}$$

$$\mathcal{C}_s^- \textit{is a closed set in } (-\infty,0) \;\; \textit{for a.e. } s \in I \tag{6.4}$$

and the following properties:

(i) *there exists $s^* \in Q_+^+$ such that one of the three properties below holds:*
  (1) $f(s^*,0) = f^{**}(s^*,0)$,
  (2) $s^* = \alpha$ *and* (5.1*) *holds with α in place of $\bar{s}$,*
  (3) $s^* = \beta$ *and* (5.1**) *holds with β in place of $\bar{s}$,*
(ii) (5.1*) *holds for every $\bar{s} \in Q_-^+ \cap \mathcal{D}^0$,*
(iii) (5.1**) *holds for every $\bar{s} \in Q_+^- \cap \mathcal{D}^0$.*

If (P^{**}) is solvable, then (P) is solvable.

Theorem 6.2. *Let $\alpha < \beta$. Let $f : I \times \mathbb{R} \to [0,+\infty)$ be a Borel-measurable function, with $f^{**}(\cdot,0)$ lower semicontinuous.*
Suppose that

$$\mathcal{C}_s^+ \textit{ is a not empty, closed set in } (0,+\infty) \textit{ for a.e. } s \in [\alpha,\beta]. \tag{6.5}$$

If $v \in \Omega_+$ is a solution to (P^{**}), such that

$$v'(x) \in \text{co}\, \mathcal{C}_{v(x)}^+ \textit{ for a.e. } x \in [a,b], \tag{6.6}$$

then there exists $u \in \Omega_+$ solution to (P).

Proof. Since $v \in \Omega_+$ then (6.5) and (6.6) obviously imply (4.2). Of course, (4.3) is trivially satisfied. The conclusion follows by Proposition 4.1. □

Differently than in Theorems 6.1 and 6.2, whose proof make use of Proposition 4.1, where the lower semicontinuity of $f^{**}(\cdot,0)$ is requested, the proof of Theorem 6.3 relies on Proposition 5.2 only, where the regularity of assumption on $f^{**}(\cdot,0)$ is weakened to Borel measurability.

Theorem 6.3. *Let $\alpha = \beta$. Let $f : I \times \mathbb{R} \to [0,+\infty)$ be a Borel-measurable function, with $f^{**}(\cdot,0)$ Borel-measurable.*

Suppose that the constant function $u(x) = \alpha$ is a solution to (P^{**}) and if $f(\alpha,0) > f^{**}(\alpha,0)$ suppose that

$$(5.1^*) \textit{ or } (5.1^{**}) \textit{ holds with } \bar{s} = \alpha. \tag{6.7}$$

Then there exists a solution to (P).
Moreover, if $g^{\pm}$ *are in* $L^\infty(\alpha-\delta,\alpha)$ *(if* (5.1*) *holds) or in* $L^\infty(\alpha,\alpha+\delta)$ *(if* (5.1**) *holds), then* (P) *has a Lipschitz solution.*

Proof. If $f(\alpha,0)=f^{**}(\alpha,0)$ then u is a minimizer of (P). If instead (6.7) holds, then the claim is a direct consequence of Proposition 5.2. □

We now give the proof of our main result.

Proof of Theorem 6.1. If (P^{**}) is solvable, then by Theorem 3.1 there exists $u\in\Omega^*$, solution to (P^{**}), satisfying one of the four cases listed in Theorem 3.1 (b). In particular, there exist τ_1,τ_2 in $[a,b]$, $a\le\tau_1\le\tau_2\le b$, such that u is strictly monotone in $[a,\tau_1]$ and $[\tau_2,b]$, $u'(x)\neq 0$ for a.e. $x\in[a,\tau_1]\cup[\tau_2,b]$, and $u(x)=s_u$ for every $[\tau_1,\tau_2]$.

We write a complete proof for the second case (b2); i.e., $u\in\Omega^+_+$ and s_u is any element in Q^+_+. This is the only case where (6.2) is used. By assumption (i), there exists $s^*\in Q^+_+$ such that (1), (2) or (3) holds. If (1) holds, without loss of generality we can assume $s_u=s^*$. If instead (2) or (3) holds, then $s_u=\alpha$ or $s_u=\beta$, respectively, and $f(s_u,0)>f^{**}(s_u,0)$.

Let us consider the intervals $[\alpha,\tau_1]$ and $[\tau_2,\beta]$.
If (4.4) holds, with $\tilde u$ replaced by u, then u does not neeed to be modified in $[a,\tau_1]\cup[\tau_2,b]$. If instead (4.4) does not hold, we modify u in $[a,\tau_1]$ and $[\tau_2,b]$ using Proposition 4.1. Precisely, since u is strictly increasing in $[a,\tau_1]$ and in $[\tau_2,b]$, with $u'(x)>0$ for a.e. x, then $\{x:\,u'(x)<0\}$ is a negligible set and (4.3) holds. Moreover, since the restrictions of u to $[a,\tau_1]$ and to $[\tau_2,b]$ have an absolutely continuous inverse function, see e.g. [1], then (6.2) and (6.3) imply that

$$\mathcal{C}^+_{u(x)} \text{ is a not empty, closed set in } (0,+\infty) \quad \text{for a.e. } x\in[a,b].$$

We want to prove that

$$u'(x)\in \operatorname{co}\mathcal{C}^+_{u(x)} \quad \text{for a.e. } x\in[a,\tau_1]\cup[\tau_2,b].$$

By (6.2), to prove the statements above it suffices to prove that

$$u'(x)\ge \inf\mathcal{C}^+_{u(x)} \quad \text{for a.e. } x\in[a,\tau_1]\cup[\tau_2,b]. \tag{6.8}$$

To do this we use a similar argument to that used in the proof of [14, Lemma 4.5].
Let v denote the restriction of u in (a,τ_1); then v has an absolutely continuous inverse. Recalling that $f^{**}(s_u,0)=\min_{\sigma\in[\alpha,\beta]}f^{**}(\sigma,0)$, by (6.1) we have

$$\begin{aligned}
&\operatorname{meas}\Big\{x\in(a,\tau_1): f^{**}(u(x),0)=f^{**}(s_u,0) \text{ and } \inf\mathcal{C}^+_{u(x)}>u'(x)\Big\}\\
&\le \operatorname{meas}\Big\{x\in(a,\tau_1)\,:\, f^{**}(u(x),0)=f^{**}(s_u,0) \text{ and } \inf\mathcal{C}^+_{u(x)}>0\Big\}\\
&=\operatorname{meas} v^{-1}(\{s\in(\alpha,s_u)\,:\, f^{**}(s,0)=f^{**}(s_u,0) \text{ and } \inf\mathcal{C}^+_s>0\})=0. \qquad (6.9)
\end{aligned}$$

Observe that u is a minimizer of the constrained problem

$$\text{minimize}\left\{\int_a^{\tau_1} f^{**}(w(x),w'(x))\,\mathrm{d}x:\ w\in\Upsilon^+\right\},$$

where

$$\Upsilon^+:=\{w\in AC([a,\tau_1])\,:\,w(a)=\alpha,\,w(\tau_1)=s_u,\,w'(x)\ge 0 \text{ a.e. in } (a,\tau_1)\}$$

Hence, by [17, Theorem 7 and Remark 4] u satisfies the following DuBois-Reymond condition

$$f^{**}(u(x),u'(x))-c\in u'(x)\partial f^{**}(u(x),u'(x)) \qquad \text{a.e. in } (a,\tau_1) \tag{6.10}$$

for some constant $c \le f^{**}(s_u, 0)$. For every s there exists $m(s)$ such that

$$f^{**}(s,z) = f^{**}(s,0) + m(s)z \quad \text{for every } z \in [0, \inf \mathcal{C}_s^+].$$

Thus, by (6.10) we deduce the existence of a constant $c \le f^{**}(s_u, 0)$ such that

$$f^{**}(u(x),0) + m(u(x))u'(x) - c = u'(x)m(u(x)) \quad \text{for a.e. } x \in (a,\tau_1) \text{ with } u'(x) < \inf \mathcal{C}_{u(x)}^+.$$

Hence by (6.9) we get

$$f^{**}(s_u,0) < f^{**}(u(x),0) = c \le f^{**}(s_u,0) \quad \text{for a.e. } x \in (a,\tau_1) \text{ such that } u'(x) < \inf \mathcal{C}_{u(x)}^+;$$

this is an absurd, so it must be

$$u'(x) \ge \inf \mathcal{C}_{u(x)}^+ \quad \text{for a.e. } x \in (a,\tau_1).$$

Reasoning in a similar way it can be proved that $u'(x) \ge \inf \mathcal{C}_{u(x)}^+$ for a.e. $x \in (\tau_2, b)$ and (6.8) follows. We have so proved that (4.2) holds. By Proposition 4.1 we obtain the existence of a minimizer $\tilde{u} \in \Omega^*$ of (P^{**}) satisfying (4.4) and $\tilde{u} = u(x) = s_u$ in $[\tau_1, \tau_2]$. Now, let us consider the interval $[\tau_1, \tau_2]$. If

$$f(s_u,0) = f^{**}(s_u,0) = f^{**}(u(x),u'(x)) \quad \text{for every } x \in (\tau_1,\tau_2),$$

as it happens when (1) holds, then $\tilde{u}$ does not need to be modified in the interval $[\tau_1, \tau_2]$. Otherwise, by Proposition 5.2 there exists $v \in \Omega$ minimizer of (P^{**}) such that

$$f(v(x),v'(x)) = f^{**}(v(x),v'(x)) \quad \text{for a.e. } x \in [\tau_1,\tau_2]$$

and

$$v(x) = \tilde{u}(x) \quad \text{for every } x \in [a,\tau_1] \cup [\tau_2, b].$$

Thus, v is a minimizer of (P).

Let us now sketch the proof assuming the third case, (b3), listed in Theorem 3.1 (b), i.e., $u \in \Omega_-^+$ and s_u is a suitable element in Q_-^+; in particular, $s_u < \alpha$.
If (4.4) does not hold, we can apply Proposition 4.1 to modify u in $[a,\tau_1] \cup [\tau_2, b]$. In fact, u is strictly decreasing in $[a,\tau_1]$ and strictly increasing in $[\tau_2, b]$, with $u'(x) \neq 0$ for a.e. x. By (6.3) and (6.4)

$$\mathcal{C}_{u(x)}^+ \text{ is a closed set in } (0,+\infty) \quad \text{for a.e. } x \in [\tau_2, b]$$

and

$$\mathcal{C}_{u(x)}^- \text{ is a closed set in } (-\infty,0) \quad \text{for a.e. } x \in [a,\tau_1].$$

We claim that

$$\begin{aligned} u'(x) &\ge \inf \mathcal{C}_{u(x)}^+ \quad \text{for a.e. } x \in [\tau_2, b], \\ u'(x) &\le \sup \mathcal{C}_{u(x)}^- \quad \text{for a.e. } x \in [a,\tau_1]. \end{aligned} \tag{6.11}$$

Indeed, by $s_u \in Q_-^+$, and without using (6.1), it follows that

$$f^{**}(u(x),0) > f^{**}(s_u,0) \qquad \text{for all } x \in (a,\tau_1) \cup (\tau_2, b).$$

Reasoning as in the previous case, we can prove that

$$f^{**}(s_u,0) < f^{**}(u(x),0) \le f^{**}(s_u,0) \quad \text{for a.e. } x \in (\tau_2,b) \text{ such that } u'(x) < \inf \mathcal{C}_{u(x)}^+,$$

and

$$f^{**}(s_u,0) < f^{**}(u(x),0) \le f^{**}(s_u,0) \quad \text{for a.e. } x \in (a,\tau_1) \text{ such that } u'(x) > \sup \mathcal{C}_{u(x)}^-.$$

Thus (6.11) holds true and by (6.2) we have that $u'(x) \in \operatorname{co}\mathcal{C}^+_{u(x)}$ for a.e. $x \in [\tau_2, b]$ and $u'(x) \in \operatorname{co}\mathcal{C}^-_{u(x)}$ for a.e. $x \in [a, \tau_1]$. Therefore, (4.2) and (4.3) hold. By Proposition 4.1 we obtain the existence of a function $\tilde{u} \in \Omega^*$, which is a minimizer of (P^{**}) satisfying (4.4) and $\tilde{u}(x) = u(x) = s_u$ in $[\tau_1, \tau_2]$.
Now, by (ii), either $f(s_u, 0) = f^{**}(s_u, 0)$ or $f(s_u, 0) > f^{**}(s_u, 0)$ with s_u satisfying (5.1*). In the first case, define $v = \tilde{u}$ and we have concluded, since v is a minimizer of (P). In the second case, we apply Proposition 5.2 obtaining a function v, minimizer of (P^{**}), satisfying (5.3) and $v = \tilde{u}$ in $[a, \tau_1] \cup [\tau_2, b]$. Thus, v is a solution to (P).

The remaining cases in (b) of Theorem 3.1, that is $u \in \Omega_+$ and $u \in \Omega_+^-$, can be treated in a similar way. □

Remark 6.4. A comparison with the relaxation result proved in [6] is in order.
In [6] f may assume value $+\infty$, but if f is a real valued function, as in our setting, both the continuity of f and f^{**} is assumed, together with a growth condition including the superlinearity growth and some linear growth as well, see [6, (H3)]. These assumptions are weakened in our result; indeed in our case f and f^{**} may be irregular and the growth condition is now replaced by the weaker assumption (6.2), see [6, Proposition 3.1], including cases of no growth.
Let us compare the other assumptions. Of course, the continuity of f and f^{**} imply (6.3) and (6.4) (it would be sufficient to have the lower semicontinuity of $z \mapsto f(s, z)$ for a.e. $s \in I$). Moreover, in [6] there are two technical assumptions, see [6, (2.13) and (2.14)]. Roughly speaking, (2.13) says that if $Ef^{**}(s, \bar{z})$ is the value at 0 of the tangent line to the graph of $z \mapsto f^{**}(s, z)$ in $\bar{z}$, then the map $s \mapsto Ef^{**}(s, \bar{z})$ is piecewise monotone in a neighbour of $\bar{s}$, for every $\bar{s}$ such that $f(\bar{s}, \bar{z}) > f^{**}(\bar{s}, \bar{z})$; moreover, if $\bar{z} = 0$ then the above assumption is strenghtened assuming that $s \mapsto Ef^{**}(s, 0) = f^{**}(s, 0)$ has no strict local minima on $\mathcal{D}^0$.
A consequence of [6, (2.13) and (2.14)] is that if $f(\bar{s}, 0) > f^{**}(\bar{s}, 0)$ then there exists $\delta > 0$ such that either $s \mapsto f^{**}(s, 0)$ is monotone in $[\bar{s} - \delta, \bar{s} + \delta]$, or it is increasing in $[\bar{s} - \delta, \bar{s}]$ and decreasing in $[\bar{s}, \bar{s} + \delta]$ (we remind here that with *increasing* we mean *non-decreasing*). Now, if $\bar{s} \in Q^+_-$ then these cases reduce to assume that there exists $\delta > 0$ such that $s \mapsto f^{**}(s, 0)$ is increasing in $[\bar{s} - \delta, \bar{s} + \delta]$; in fact, $f^{**}(\cdot, 0)$ cannot be decreasing in a right neighbour of $\bar{s}$, otherwise a contradiction arises with the request $\bar{s} \in Q^+_-$. Analogously, if $\bar{s} \in Q^-_+$ then $s \mapsto f^{**}(s, 0)$ has to be decreasing in a neighbour of $\bar{s}$. Therefore, taking also into account what previously observed in Remark 5.1, we conclude that the assumptions in [6] imply (ii) and (iii).

As far as (i) is concerned, suppose that $\alpha = \beta$. If $f(\alpha, 0) = f^{**}(\alpha, 0)$ then (i) holds true. In the opposite case $f(\alpha, 0) > f^{**}(\alpha, 0)$, using [6, (2.13) and (2.14)] and reasoning as above we get that there exists $\delta > 0$ such that $s \mapsto f^{**}(s, 0)$ is either increasing in $[\alpha - \delta, \alpha]$ or decreasing in $[\alpha, \alpha + \delta]$. Thus, taking into account Remark 5.1 once again we get that both the second and the third item in (i) hold. Moreover, $\alpha = \beta$ trivially implies (6.1), so we conclude that if $\alpha = \beta$ then our result includes [6, Theorem 2.2].

A discrepancy among our relaxation result and [6, Theorem 2.2] turns out if $\alpha < \beta$. In this case (i) and (H5) are not a consequence of the assumptions in [6]; however, they have the benefit to avoid any request on $s \mapsto Ef^{**}(s, \bar{z})$ when $\bar{z} \neq 0$.

7. Examples

In this section we present some examples of problem (P), so to better understand the role played by the assumptions in Theorem 6.1 and their sharpness.

- **Assumption** (6.2)

An example which shows that (6.2) cannot be removed is the following.

Example 7.1. *Let* $f(z) = z^2 e^{-z^2}$ *and* $\Omega = \{u \in AC([0,1]) : u(0) = 0, u(1) = \beta\}$ *with* $\beta > 0$. *Then* $f^{**}(z) = 0$ *for all* z *and assumption* (6.2) *is not satified. However, all the other assumptions of Theorem 6.1 holds.*
Of course, $u(x) = \beta x$ *solves* (P^{**}). *Moreover,* $\inf F = 0$. *To prove this, fixed* $n > \beta$ *consider*

$$u_n(x) = \begin{cases} nx & \text{if } 0 \le x \le \frac{1}{2} + \frac{\beta}{2n} \\ \beta - n(x-1) & \text{if } \frac{1}{2} + \frac{\beta}{2n} \le x \le 1. \end{cases}$$

Then $F(u_n) \to 0$. *Now,* (P) *does not have a solution, because no absolutely continuous functions in* Ω *can have derivative equal to* 0 *a.e.*

- **Assumption** (6.3)

Let us consider the following example.

Example 7.2. *Let*

$$f(s,z) = \begin{cases} -|s|z + |s| & \text{if } z < 0 \\ |s| & \text{if } 0 \le z < 1 \\ z - 1 & \text{if } z \ge 1. \end{cases}$$

and consider

$$\Omega := \{u \in AC([0,1]) : u(0) = 0, u(1) = 1\}.$$

Then

$$f^{**}(s,z) = \begin{cases} -|s|z + |s| & \text{if } z \le 1 \\ z - 1 & \text{if } z > 1. \end{cases}$$

In particular, $f(0,z) = f^{**}(0,z)$ *and* $f(s,z) = f^{**}(s,z)$ *for all* $s \neq 0$ *and* $z \in (-\infty, 0] \cup [1, +\infty)$. *All the assumptions of Theorem 6.1 are satisfied and* $u(x) = x$ *is a minimizer both of* (P^{**}) *and* (P).

If in Example 7.2 we change the definition of $f(s,\cdot)$ at $z = 1$ we may lose the solvability of (P), see the proposition below. Notice that in this case (6.3) fails.

Proposition 7.3. *Consider*

$$\text{minimize}\left\{F(u) := \int_a^b f(u(x), u'(x))\,\mathrm{d}x\right\}, \quad u \in \Omega, \tag{P}$$

where

$$\Omega := \{u \in AC([0,1]) : u(0) = 0, u(1) = 1\},$$

and

$$f(s,z) = \begin{cases} -|s|z + |s| & \text{if } z < 0 \\ |s| & \text{if } 0 \le z \le 1 \\ z - 1 & \text{if } z > 1. \end{cases}$$

Then (P^{**}) is solvable, but there are no solutions to (P).
In particular, all the assumptions in Theorem 6.1 are satisfied, with the exception of (6.3) *that fails to hold.*

Proof. It is obvious that

$$f^{**}(s,z) = \begin{cases} -|s|z + |s| & \text{if } z \leq 1 \\ z - 1 & \text{if } z > 1. \end{cases}$$

In particular, $f^{**}(s,1) = 0$, so $u(x) = x$ is a minimizer of (P^{**}). Moreover, $f^{**}(s,0) = |s|$, therefore, $\min_{s\in[0,1]} f^{**}(s,0) = 0$ and (6.1) holds true.

Notice that Q^+_+ in (2.3) is equal to $\{0\}$ and $f^{**}(0,0) = 0 = f(0,0)$ so that (i) in Theorem 6.1 is satisfied. It is obvious that Q^+_- and Q^-_+ are both empty sets, which implies (ii) and (iii) in Theorem 6.1. Assumptions (3.1), (6.2) and (6.4) trivially hold. However, (6.3) fails, since $\mathcal{C}^+_s = (1,+\infty)$ for all $s \neq 0$, and it is not a closed set in $(0,+\infty)$.

To conclude we remark that (P) has not a solution. In fact, if

$$u_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{n} \\ \frac{n}{n-1}x - \frac{1}{n-1} & \text{if } \frac{1}{n} < x \leq 1 \end{cases}$$

then

$$F(u_n) = \int_0^{\frac{1}{n}} f(0,0)\ \mathrm{d}x + \int_{\frac{1}{n}}^1 f\left(\frac{n}{n-1}x - \frac{1}{n-1}, \frac{n}{n-1}\right)\ \mathrm{d}x = \left(\frac{n}{n-1} - 1\right)\left(1 - \frac{1}{n}\right) = \frac{1}{n}.$$

Thus, $\inf F = 0$, but F never attains this value. □

- **Assumption (i) in Theorem 6.1**

In the well known Bolza's example $f(s,z) = s^2 + (z^2-1)^2$, $\alpha = \beta = 0$ and $I = \mathbb{R}$. The corresponding problem (P) has not a minimizer. In particular, the variational problem (P^{**}) has the constant function $u(x) = 0$ as a solution and the Langrangian $f(s,z)$ satisfies neither (6.7) of Theorem 6.3 nor (i) of Theorem 6.1. Thus, this example shows that (i) of Theorem 6.1 cannot be removed. Now, we observe that the alternative (2) in (i) of Theorem 6.1 cannot be weakened (a similar example can be produced for (3)); precisely, (b^{**}) in (5.1^{**}) is sharp, that is if $\dfrac{1}{g^+}$ or $\dfrac{1}{g^-}$ is not in L^1 then (P) may have no solutions (see in particular Claim 4 in the proof).

Proposition 7.4. *Fixed $\gamma \in \mathbb{R}$, consider $f : \mathbb{R}\times\mathbb{R} \to [0,+\infty)$, depending on γ, $f(s,z) = g(s) + h(s,z)$ with*

$$g(s) = \begin{cases} 1 & \text{if } s < 0 \\ 0 & \text{if } s \geq 0, \end{cases} \qquad h(s,z) = \begin{cases} 1 - \dfrac{|z|}{|s|^\gamma} & \text{if } s \neq 0 \text{ and } |z| < |s|^\gamma \\ z - |s|^\gamma & \text{if } s \neq 0 \text{ and } z \geq |s|^\gamma \\ 0 & \text{if } s \neq 0 \text{ and } z \leq -|s|^\gamma \\ \max\{z,0\} & \text{if } s = 0 \text{ and } z \neq 0 \\ 1 & \text{if } s = 0 \text{ and } z = 0. \end{cases}$$

Consider

$$\text{minimize}\left\{F(u) := \int_a^b \big(g(u(x)) + h(u(x),u'(x))\big)\ \mathrm{d}x\right\}, \quad u \in \Omega, \qquad (P)$$

where

$$\Omega := \{u \in AC([0,1]) \,:\, u(0) = 0,\, u(1) = 0\}.$$

There exists a minimizer of (P) *if and only if* $\gamma < 1$.

Proof. Problem (P^{**}) is solvable for any $\gamma \in \mathbb{R}$. In fact, define $f(s,z) = g(s) + h(s,z)$. Notice that $f(s,z) \geq 0$. So, f^{**} is a non-negative function, too. Since $f^{**}(0,0) = 0$ then $u(x) \equiv 0$ is a solution of (P^{**}).

Let us proof the thesis of the statement using Theorem 6.1.

Claim 1. $f^{**}(\cdot, 0)$ is a lower semicontinuous function and (3.1), (6.2)–(6.4) are satisfied.

Proof of Claim 1. Of course, $f(s,0) = 1$ for every $s \in \mathbb{R}$ and for any value of $z \in \mathbb{R}$ we have that

$$f^{**}(s,z) = \begin{cases} g(s) & \text{if } s \neq 0 \text{ and } z < |s|^\gamma \\ g(s) + z - |s|^\gamma & \text{if } s \neq 0 \text{ and } z \geq |s|^\gamma \\ \max\{z, 0\} & \text{if } s = 0 \end{cases}$$

Since $f^{**}(s,0) = g(s)$, then $f^{**}(\cdot,0)$ is a lower semicontinuous function. Moreover, (3.1) holds since $0 \in \partial f^{**}(\cdot, 0)$. For all $s \neq 0$,

$$\mathcal{C}_s^+ = [|s|^\gamma, +\infty), \qquad \mathcal{C}_s^- = (-\infty, -|s|^\gamma]$$

and $\mathcal{C}_0^+ = (0, +\infty)$, $\mathcal{C}_0^- = (-\infty, 0)$; hence assumptions (6.2)–(6.4) are satisfied.

Claim 2. Assumption (i) of Theorem 6.1 is satisfied if and only if $\gamma < 1$.

Proof of Claim 2. Since $Q_+^+ = \{0\}$ and $f(0,0) = 1 > 0 = f^{**}(0,0)$, then (1) does not hold. Assumption (5.1*) with $\bar{s} = 0$ fails as well, because $f^{**}(s,0) = g(s)$ for all s, and it is not increasing in $[-\delta, 0]$ for any positive δ. As far as the validity of (3) is concerned, the first claim of (5.1**) is satisfied since $f^{**}(\cdot,0)$ is constant in $[0,+\infty)$. Let us consider $(**)$. For any $\delta > 0$, the only possible choice for $g^+ : [0,\delta] \to \mathbb{R}$ is $g^+(s) = s^\gamma$, and $\frac{1}{g^+} \in L^1(0,\delta)$ if and only if $\gamma < 1$. For $g^- : [0,\delta] \to \mathbb{R}$ we can choose any function such that $g^-(s) \leq -s^\gamma$ for all s and that satisfies the corresponding assumptions in $(**)$, see (5.1**). Thus, (b^{**}) holds if and only if $\gamma < 1$.

Claim 3. If $\gamma < 1$ there exists a minimizer of (P).

Proof of Claim 3. Since both Q_-^+ and Q_+^- are empty sets, then (ii) and (iii) hold true. Therefore, Theorem 6.1 implies that if $\gamma < 1$ there exists a minimizer of (P).

Claim 4. If $\gamma \geq 1$ then (P) has no solutions.

Proof of Claim 4. Fixed $n \geq 2$, let v_n^+ and v_n^- be the solutions of the Cauchy problems

$$y' = y^\gamma \ \text{ in } \left[\frac{1}{n}, \frac{1}{2}\right], \qquad y\left(\frac{1}{n}\right) = \frac{1}{n}$$

and

$$y' = -y^\gamma \ \text{ in } \left[\frac{1}{2}, 1 - \frac{1}{n}\right], \qquad y\left(1 - \frac{1}{n}\right) = \frac{1}{n},$$

respectively.

Define $u_n : [0,1] \to [0, +\infty)$,

$$u_n(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{n} \\ v_n^+(x) & \text{if } \frac{1}{n} \leq x \leq \frac{1}{2} \\ v_n^-(x) & \text{if } \frac{1}{2} \leq x \leq 1 - \frac{1}{n} \\ 1 - x & \text{if } 1 - \frac{1}{n} \leq x \leq 1. \end{cases}$$

Then,

$$F(u_n) = \int_0^{\frac{1}{n}} f(x,1)\ \mathrm{d}x + \int_{\frac{1}{n}}^{\frac{1}{2}} f(v_n^+(x),(v_n^+)'(x))\ \mathrm{d}x + \int_{\frac{1}{2}}^{1-\frac{1}{n}} f(v_n^-(x),(v_n^-)'(x))\ \mathrm{d}x$$
$$+ \int_{1-\frac{1}{n}}^{1} f(1-x,-1)\ \mathrm{d}x = \int_0^{\frac{1}{n}} (1-|x|^\gamma)\ \mathrm{d}x + \int_{1-\frac{1}{n}}^{1} (1-|1-x|^\gamma)\, dx \le \frac{2}{n},$$

which implies that $\inf_\Omega F = 0$. Thus, taking into account that $f(s,z) \ge 1$ for all $s < 0$ and every z, there exists a minimizer of (P) if and only if there exists $u \in AC([0,1])$ such that $u(0) = u(1) = 0$, $u(x) > 0$ for a.e. $x \in (a,b)$, and it satisfies

$$u'(x) = (u(x))^\gamma \qquad \text{for a.e. } x \text{ s.t. } u'(x) > 0 \tag{7.1}$$
$$u'(x) \le -(u(x))^\gamma \qquad \text{for a.e. } x \text{ s.t. } u'(x) < 0. \tag{7.2}$$

Such a function u does not exist. By contradiction, suppose that u exists. Let $x_0 \in (a,b)$ be such that $u(x_0) > 0$ and let $v \in C^1([0,1])$ be the unique solution of the Cauchy problem

$$v'(x) = (v(x))^\gamma, \quad v(x_0) = u(x_0).$$

Obviously, there exists c, depending on γ, such that $v(x) \ge c > 0$ for every $x \in [0,x_0]$. Let $x_1 = \max\{x \in [0,x_0] : u(x) = 0\}$ and $x_2 = \min\{x \in [x_1,x_0] : u(x) \ge v(x)\}$.

Then, by (7.1) and (7.2)

$$v(x_1) = (u(x_2)-v(x_2))-(u(x_1)-v(x_1)) = \int_{x_1}^{x_2} (u'(x)-v'(x))\ \mathrm{d}x \le \int_{x_1}^{x_2} ((u(x))^\gamma-(v(x))^\gamma)\ \mathrm{d}x < 0,$$

an absurd. □

- **Assumption** (6.1)

The following proposition shows that also assumption (6.1) cannot be removed.

Proposition 7.5. *Consider*

$$\textit{minimize}\left\{F(u) := \int_a^b f(u'(x))\ \mathrm{d}x\right\}, \quad u \in \Omega, \tag{P}$$

where

$$\Omega := \{u \in AC([0,1]) : u(0) = 0,\, u(1) = 1\},$$

and

$$f(z) = \begin{cases} -z & \text{if } z < 0 \\ |z-2| & \text{if } z \ge 0. \end{cases}$$

Then (P^{**}) is solvable, but there are no solutions to (P).
In particular, all the assumptions in Theorem 6.1 are satisfied, with the exception of (6.1) *that fails to hold.*

Proof. Since

$$f^{**}(z) = \begin{cases} -z & \text{if } z < 0 \\ 0 & \text{if } 0 \le z \le 2 \\ z-2 & \text{if } z > 2 \end{cases}$$

then $u(x) = x$ is a solution to (P^{**}).

We claim that assumption (6.1) does not hold. In fact, f^{**} is independent of s, $s \mapsto \inf \mathcal{C}_s^+$ is constant and takes value 2. Now, it is easy to verify that all the other assumptions in Theorem 6.1 hold. To prove that (P) has not a minimizer, for every integer $n \geq 3$ define

$$u_n(x) = \begin{cases} -\frac{2}{n-2}x & \text{if } 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ 2x - 1 & \text{if } \frac{1}{2} - \frac{1}{n} < x \leq 1. \end{cases}$$

Then

$$F(u_n) = \int_0^{\frac{1}{2}-\frac{1}{n}} f\left(-\frac{2}{n-2}\right)\,\mathrm{d}x = \frac{2}{n-2}\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{n};$$

therefore $\inf F = 0$. It is obvious that no functions $u \in \Omega$ exist such that $F(u) = 0$. □

We conclude observing that if $\alpha = \beta$ then the relaxation result proved in [6] is a included in our Theorem 6.1 (see Remark 6.4); thus, if $\alpha = \beta$ then [6, Example 2.5] is an example of a variational problem for whom Theorem 6.1 applies. Moreover, this example can be generalized choosing suitable non-smooth functions $a(\eta)$, $b(\eta)$, $c(\eta)$.

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