Blow-up of Optimal Sets in the Irrigation Problem

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Abstract: We consider the minimization problem for an average distance functional in the plane, among all compact connected sets of prescribed length. For a minimizing set, the blow-up sequence in the neighbourhood of any point is investigated. We show existence of the blow up limits and we characterize them, using the results to get some partial regularity statements.

1 Introduction

The problem we deal with, known as "irrigation problem", was first introduced in [4], in the general framework of transport problems endowed with regions (called Dirichlet regions) in which transport is free of charge.

Although the formulation given in [4] is quite general, in this paper we are concerned with the following instance of the problem:

$$\min F(\Sigma) = \int_\Omega d(x,\Sigma)\,\mu(dx) : \Sigma \subset \overline{\Omega} \text{ compact and connected, } \mathcal{H}^1(\Sigma) \leq l, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open domain, $d(x,\Sigma)$ denotes the distance from the point x to the set Σ, μ is a given probability measure on Ω, and the Hausdorff measure $\mathcal{H}^1(\Sigma)$ cannot exceed a prescribed value $l > 0$.

As a possible interpretation, we may regard Σ as a resource, whose amount is limited by the constraint $\mathcal{H}^1(\Sigma) \leq l$, to be distributed over a region Ω. Since the functional $F(\Sigma)$ is the average distance of a point $x \in \Omega$ to Σ, minimizing $F(\Sigma)$ means letting the resource be as widespread as possible throughout the region Ω. Of course, the measure μ reflects the fact that some subregions of Ω might have a higher or lower priority in being close to the resource Σ. Finally, the imposed connectedness of Σ prevents the infimum of F from being zero, and is a natural constraint in several applications, such as image reconstruction (trying to recover a line Σ from a pixel cloud μ in a picture) or urban planning (when Σ is a subway network in a city Ω with population density μ).

Existence results for the minimum problem (1.1) can be achieved by Gołab's theorem (see, for instance, [3]) and are the starting point of [4].

An interesting feature of this problem is its link with the theory of optimal transportation: indeed, for any Σ, there holds

$$F(\Sigma) = \inf \{ W_1(\mu, \nu) \,|\, spt(\nu) \subset \Sigma \} , \tag{1.2}$$

where W_1 is the Wasserstein distance (see [2]).

In the first paper on the subject [4], after proving the existence of solutions, some questions concerning the minimizers are posed and some partial answers are provided. Stronger results on the structure of a minimizer Σ are obtained in [5]. To recall the results from [4, 5] which are relevant to the present paper, let us introduce some notation which will be used throughout.

Let Σ be a minimizer of (1.1). We denote by $t : \Omega \to \Sigma$ a fixed measurable selection of the projection multimap, i.e. t is a measurable map such that, for every $x \in \Omega$, there holds $|x - t(x)| = d(x, \Sigma)$. The measure μ and the map t induce a measure ψ on Σ defined by $\psi = t_\sharp \mu$, that is

$$\psi(E) = \mu(t^{-1}(E)) \quad \text{for every Borel set } E \subset \mathbb{R}^2. \tag{1.3}$$

Of course, ψ is a probability measure concentrated on Σ. Estimates on the measure ψ will be crucial in the sequel.

The result from [5] which is most relevant to our purpose states that, topologically, Σ is equivalent to a finite graph without cycles, whose vertices have order at most three. More precisely (see [5] for more details):

- Σ is the union of finitely many injective Lipschitz curves with endpoints (called "branches" of Σ).
- Any two branches have at most one endpoint in comnmon.
- Σ has no loops (i.e. $\mathbb{R}^2 \setminus \Sigma$ is connected).
- For every $x \in \Sigma$, exactly one of the following three possibilities occurs:

1. x is in the relative interior of one branch and belongs to no other branch of Σ. In this case we say that x is an *ordinary point* of Σ.
2. x is an endpoint of exactly one branch of Σ. We say that x is an *endpoint* of Σ.
3. x is an endpoint of exactly three branches of Σ. We say that x is a *triple junction.*

- Every endpoint of Σ is an atom for the measure ψ.
- There are at least two endpoints in Σ.

In this paper we focus on the existence and characterization of blow-up limits of Σ. More precisely, we say that Σ has a blow-up limit K at $x_0 \in \Sigma$ if the localized and rescaled sets $(\Sigma \cap \overline{B_r(x_0)} - x_0)/r$ converge, in the Hausdorff distance as $r \to 0$, to some set $K \subset \overline{B_1}$ (here and throughout, $B_r(x_0)$ denotes the ball of radius r centered at x_0, whereas B_r denotes the ball centered at the origin).

In particular we prove (see Section 3) that Σ has a blow-up limit K at every point x_0, and we are able to characterize K according to the nature of the point x_0. More precisely, it turns out that K is a radius of the unit ball if x_0 is an endpoint, is the union of two radii if x_0 is an ordinary points (the twoo radii forming a diameter unless x_0 is an atom for ψ), or is the union of three radii forming angles of 120° if x_0 is a triple junction.

Concerning the regularity of Σ, we can prove a partial result (see Section 4), namely that every branch of Σ is a $C^{1,1}$ curve in a neighbourhood of a point x_0, provided that $diam(t^{-1}(x_0))$ is sufficiently small. Curiously enough, it turns out that this assumption is always satisfied if x_0 is a triple junction, hence the three branches starting at a triple junction are indeed $C^{1,1}$ near the singularity, and their tangents form angles of 120 degrees at x_0.

Finally, we stress the fact that the existence of the blow-up limits in the Hausdorff distance at a point x_0 is linked to differentiability of Σ at x_0. In fact, parametrizing every branch of Σ by arc length curves, it is not difficult to check that our results on blow-ups imply the existence of the derivative as a unit vector in the classical sense, or at least the existence of the derivatives from each side in the unlucky case of the limit being a corner.

This paper involves several different variational techniques to get necessary optimality conditions; most of them are simply based on first-order perturbations of the minimizer, i.e. the key ingredient is stationarity instead of minimality. We cite in connection to this feature of our work the classical paper [1] on regularity of stationary 1-dimensional structures. However, some cases studied in this paper have been solved by more variational techniques, such as by $\Gamma-$convergence.

2 Notation and auxiliary results

For the setting of the problem we refer to the introductory section. From now on Ω will be a convex and bounded open subset of $\mathbb{R}^2$ and Σ a fixed minimizer of Problem 1.1. Since Ω is supposed to be convex we know that $\Sigma \subset \overline{\Omega}$ is a solution also to Problem 2.1:

$$\min F(\Sigma) = \int_\Omega d(x,\Sigma)\,\mu(dx) : \Sigma \subset \mathbb{R}^2 \text{ compact and connected, } \mathcal{H}^1(\Sigma) \leq l. \tag{2.1}$$

This is a consequence of what proven in [6], i.e. that Σ is always contained in the convex hull of the support of μ, and so there is no matter if we enlarge the domain. In the sequel, to get rid of possible boundary difficulties, we will silently use the fact that Σ minimizes also in Problem 2.1.

The measure μ will be considered to be absolutely continuous with respect to the two dimensional Lebesgue measure $\mathcal{L}^2$ and its density to belong to $L^\infty(\Omega)$. We will denote by $t : \Omega \to \Sigma$ a fixed measurable selection of the projection multimap, i.e. t is a measurable map such that, for every x, it holds $|x - t(x)| = d(x,\Sigma)$. By ψ we will denote the measure on Σ given by $t_\sharp\mu$, which does not depend on the choice of the selection t, since t is uniquely defined $\mathcal{L}^2-$almost everywhere.

We will call C any positive, finite constant depending only on Ω, μ and Σ that may be enlarged at will. Every time a new, larger constant C is needed, the value of the former will be considered to be enlarged as necessary, without changing the notation C.

Finally, given two sets $A, B \in \mathbb{R}^2$, we denote by $\mathrm{d_H}(A,B)$ their Hausdorff distance, i.e. the infimum of those positive numbers h such that $A \subset (B)_h$ and $B \subset (A)_h$, where $(A)_h$ denotes a neighboourhood of A of width h.

Remark 1. Let Γ be a Lipschitz curve with endpoints x, y and let $\overline{xy}$ be the segment from x to y. Then the Hausdorff distance $\mathrm{d_H}(\Gamma, \overline{xy})$ coincides with the smallest $h \geq 0$ such that Γ is contained in a h_0-neighbourhood of $\overline{xy}$. Indeed, letting h_0 denote such smallest h, it suffices to observe that $\overline{xy}$ is contained in a h-neighbourhood of Γ: in fact, for every $z \in \overline{xy}$, there is at least a point of Γ on the line through z perpendicular to $\overline{xy}$.

Remark 2. Moreover, we have

$$\mathcal{H}^1(\Gamma) \geq \sqrt{4\,\mathrm{d_H}(\Gamma, \overline{xy})^2 + |x-y|^2}. \tag{2.2}$$

To see this, take a point $p \in \Gamma$ a distance exactly $\mathrm{d_H}(\Gamma, \overline{xy})$ to the segment $\overline{xy}$ (such a point exists due to the previous remark): then clearly

$$\mathcal{H}^1(\Gamma) \geq |x-p| + |p-y|,$$

and minimizing the last expression over all possible p gives the desired estimate (the minimum is achieved at a point p belonging to the axis of $\overline{xy}$).

We use here the fact that endpoints are atoms for ψ to establish a basic estimate which will be useful in the sequel.

Lemma 2.1. *There exists $C > 0$ with the following properties. Let U be any open subset of Σ and let V be a compact set in $\mathbb{R}^2$ such that $(\Sigma \setminus U) \cup V$ is connected. If $\Sigma \setminus U$ contains at least one endpoint of ψ, then*

$$\mathcal{H}^1(U) \leq \mathcal{H}^1(V) + C\psi(U \setminus V) \max_{z \in U} d(z, V) \tag{2.3}$$

and

$$\mathcal{H}^1(U) \leq \mathcal{H}^1(V) + C\psi(U \setminus V)\,\mathrm{d_H}(U, V). \tag{2.4}$$

The proof uses some ideas from [5]. For the sake of completeness, we provide all the details.

Proof. We can assume that $\mathcal{H}^1(V) \leq \mathcal{H}^1(U)$.

Let $a \in \Sigma \setminus U$ be an endpoint of Σ (hence an atom for ψ), and set $A = t^{-1}(a)$. Since A is the intersection of a convex set and Ω, and $\mu(A) = \psi(\{a\}) > 0$, then A has nonempty interior. Let $B(y, \rho)$ be a small closed ball of radius $\rho > 0$, centred at some point $y \in A$ and contained in A, such that $a \notin \overline{B(y, \rho)}$ and $\mu(B(y, \rho)) > 0$. Letting $l = \mathcal{H}^1(U) - \mathcal{H}^1(V)$, we construct a new competitor Σ' as follows:

$$\Sigma' = (\Sigma \setminus U) \cup V \cup s_l,$$

where s_l is the closed segment of length l, that lies on the half-line from a to y and has a as one endpoint. Clearly, $\mathcal{H}^1(\Sigma') = \mathcal{H}^1(\Sigma)$. Moreover, Σ' is compact and connected (the segment s_l touches $\Sigma \setminus U$ at a, and $(\Sigma \setminus U) \cup V$ is connected by assumption), hence the minimality of Σ implies that

$$\int_\Omega d(x,\Sigma)\,\mu(dx) \le \int_\Omega d(x,\Sigma')\,\mu(dx). \tag{2.5}$$

Note that, by construction, $d(x,\Sigma') \le d(x,\Sigma)$ for all $x \in \Omega$ such that $t(x) \notin U \setminus V$, hence in particular, from (2.5) we find

$$\int_{B(y,\rho)} (d(x,\Sigma) - d(x,\Sigma')))\,\mu(dx) \le \int_{t^{-1}(U\setminus V)} (d(x,\Sigma') - d(x,\Sigma))\;\mu(dx). \tag{2.6}$$

For every $x \in B(y,\rho)$, we have $d(x,\Sigma) = |x-a|$, and $d(x,\Sigma') \le d(x,s_l)$. Hence we find

$$\int_{B(y,\rho)} (d(x,\Sigma) - d(x,\Sigma')))\,\mu(dx) \ge \int_{B(y,\rho)} (|x-a| - d(x,s_l))\;\mu(dx) =: I(l). \tag{2.7}$$

Considering for a while l as a parameter, we want to estimate from below the last integral $I(l)$ as a function of l (i.e. the length of the segment s_l), on the interval $[0, \mathcal{H}^1(\Sigma)]$ (having defined $l = \mathcal{H}^1(U) - \mathcal{H}^1(V)$, we are not interested in $I(l)$ when $l > \mathcal{H}^1(\Sigma)$): it is clear that $I(l)$ is non decreasing in l, and one can easily check that when l is small enough (for instance, such that $2l < |y-a| - \rho$), there holds $I(l) \ge \varepsilon l$ for some $\varepsilon > 0$ (which depends only on ρ, μ and on the distance $|a-y|$). Since $I(l)$ is nondecreasing, reducing if necessary the value of ε we obtain that an estimate of the kind $I(l) \ge \varepsilon l$ holds for all $l \in [0, \mathcal{H}^1(\Sigma)]$. Therefore, plugging this estimate in (2.7) and using (2.6), we obtain

$$\varepsilon\left(\mathcal{H}^1(U) - \mathcal{H}^1(V)\right) = \varepsilon l \le I(l) \le \int_{t^{-1}(U\setminus V)} (d(x,\Sigma') - d(x,\Sigma))\;\mu(dx). \tag{2.8}$$

Note that ε can also be made independent of the particular endpoint a that we have chosen in $\Sigma \setminus U$, since Σ has only finitely many endpoints: therefore, we may work with some ε which depends only on Σ, and not on U. Then, observing that

$$\sup_{x \in t^{-1}(U\setminus V)} (d(x,\Sigma') - d(x,\Sigma)) \le \sup_{x \in t^{-1}(U)} (d(x,V) - d(x,U)) \le \sup_{z \in U} d(z,V),$$

from (2.8) one obtains (2.3). Finally, (2.4) is an easy consequence of (2.3). $\square$

We face an important particular case when the set V is a segment.

Lemma 2.2. *There exists C with the following property. Let $\Gamma \subset \Sigma$ be a closed injective arc, with endpoints x, y, such that $\Gamma \setminus \{x, y\}$ contains no triple junctions of Σ and $C\psi(\Gamma \setminus \{x, y\}) < 1/2$. Then*

$$\begin{aligned} \mathcal{H}^1(\Gamma) &\leq |x-y| + C\psi(\Gamma \setminus \{x,y\})\, \mathrm{d_H}(\Gamma, \overline{xy}), && (2.9)\\ \mathrm{d_H}(\Gamma, \overline{xy}) &\leq C\psi(\Gamma \setminus \{x,y\})|x-y|, && (2.10)\\ \mathcal{H}^1(\Gamma) &\leq |x-y|\big(1 + C\psi(\Gamma \setminus \{x,y\})^2\big), && (2.11)\\ \mathcal{H}^1(\Gamma) &\leq 2|x-y|. && (2.12) \end{aligned}$$

Proof. We apply the previous lemma with $U = \Gamma \setminus \{x, y\}$ and V the segment from x to y (note that $(\Sigma \setminus \Gamma) \cup V$ is connected since we have replaced a simple curve by another curve with the same endpoints). Then (2.9) follows from (2.4). Moreover, on squaring (2.2) one obtains

$$4\, \mathrm{d_H}(\Gamma, \overline{xy})^2 \leq \big(\mathcal{H}^1(\Gamma) + |x-y|\big)\big(\mathcal{H}^1(\Gamma) - |x-y|\big) \leq 2\mathcal{H}^1(\Gamma)\big(\mathcal{H}^1(\Gamma) - |x-y|\big). \tag{2.13}$$

If we temporarily set $\Delta = \mathcal{H}^1(\Gamma) - |x-y|$ and we square (2.9), we get

$$\Delta^2 \leq C\psi(\Gamma \setminus \{x,y\})^2\, \mathrm{d_H}(\Gamma, \overline{xy})^2 \leq C\psi(\Gamma \setminus \{x,y\})^2 \mathcal{H}^1(\Gamma)\Delta,$$

where we used also (2.13). Then we get

$$\Delta \leq C\psi(\Gamma \setminus \{x,y\})^2 \mathcal{H}^1(\Gamma). \tag{2.14}$$

To estimate $\mathcal{H}^1(\Gamma)$ we use the estimate on Δ:

$$\mathcal{H}^1(\Gamma) - |x-y| = \Delta \leq C\psi(\Gamma \setminus \{x,y\})^2 \mathcal{H}^1(\Gamma),$$

which, under the assumption $C\psi(\Gamma \setminus \{x, y\}) < 1/2$, provides

$$\frac{1}{2}\mathcal{H}^1(\Gamma) \leq \big(1 - C\psi(\Gamma \setminus \{x,y\})^2\big)\, \mathcal{H}^1(\Gamma) \leq |x-y|, \tag{2.15}$$

which is (2.12). Then, from (2.14) and (2.15), we get also the estimate

$$\Delta \leq C\psi(\Gamma \setminus \{x,y\})|x-y|$$

and, by recalling (2.13) and (2.15), (2.10) is valid as well. Finally, (2.11) follows from (2.9) and (2.10). □

2.1 The function θ and its variation

Take a point $x_0 \in \Sigma$ and consider a branch of Σ starting at x_0: we may regard it as an injective Lipschitz curve $\gamma : [0,T] \to \Sigma$, parameterized by arclength, such that $\gamma(0) = x_0$ and $\gamma(T)$ is either an endpoint or a triple point of Σ. Of course, we suppose that $T > 0$ and that γ contains neither endpoints nor triple junctions in its relative interior.

We will prove that, if $r > 0$ is small enough, then γ touches $\partial B(x_0, r)$ at exactly one point: in this way, we can define for small $r > 0$ the function $\theta(r)$, i.e. (choosing polar coordinates centred at x_0) the angular coordinate θ of the (unique) point where γ touches $\partial B(x_0, r)$. We will also prove some regularity properties of the function $\theta(r)$.

Choose a certain radius $r_0 > 0$, such that the ball $B(x_0, r_0)$ contains no endpoint and no triple junction of Σ, with the only possible exception of x_0. In particular, γ meets $\partial B(x_0, r)$ at least once, for every $r \le r_0$.

We have the following

Theorem 2.3. *Consider $x_0 \in \Sigma$ and $r_0 > 0$ such that $B(x_0, r_0)$ contains no endpoint and triple junction other than, possibly, x_0 itself. For any $r < r_0$, set*

$$t_1 = \min\{t \ge 0 \mid \gamma(t) \in \partial B(x_0, r)\}, \quad t_2 = \max\{t \le T \mid \gamma(t) \in \partial B(x_0, r)\}.$$

If $C\psi(\gamma((0, t_2]) < 1$, then $t_1 = t_2$, i.e. γ touches $\partial B(x_0, r)$ exactly once.

Proof. Let us set for brevity $\gamma_{0,1} = \gamma([0, t_1])$, $\gamma_{0,2} = \gamma([0, t_2])$ and $\gamma_{1,2} = \gamma([t_1, t_2])$. We apply Lemma 2.1, with $U = \gamma_{0,2} \setminus \{\gamma(0), \gamma(t_2)\}$ and V equal to a suitable rotation of $\gamma_{0,1}$ around x_0, of an angle $\Delta\theta$, in such a way that the point $\gamma(t_1)$, after the rotation, overlaps with $\gamma(t_2)$. Observing that $\mathcal{H}^1(U) - \mathcal{H}^1(V) = \mathcal{H}^1(\gamma_{1,2})$, (2.3) implies that

$$\mathcal{H}^1(\gamma_{1,2}) \le C\psi\left(\gamma_{0,2} \setminus \{x_0\}\right) \max_{x \in \gamma_{0,2}} d(x, V). \tag{2.16}$$

To estimate the max in the right hand side, let us split $\gamma_{0,2} = \gamma_{0,1} \cup \gamma_{1,2}$. Since V is a rotation of $\gamma_{0,1}$ which sends the boundary point $\gamma(t_1)$ to $\gamma(t_2)$, there holds

$$\max_{x \in \gamma_{0,1}} d(x, V) \le |\gamma(t_1) - \gamma(t_2)| \le \mathcal{H}^1(\gamma_{1,2}).$$

Moreover, since $\gamma(t_2) \in V$ we find

$$d(x, V) \le |x - \gamma(t_2)| \le \mathrm{diam}(\gamma_{1,2}) \le \mathcal{H}^1(\gamma_{1,2})$$

since $\gamma_{1,2}$ is connected. Plugging these estimates into (2.16), we find

$$\mathcal{H}^1(\gamma_{1,2}) \leq C\mathcal{H}^1(\gamma_{1,2})\psi(\gamma((0,t_2])). \tag{2.17}$$

Under the assumption $C\psi(\gamma((0,t_2])) < 1$ it is clear that we get $\mathcal{H}^1(\gamma_{1,2}) = 0$ and also $t_1 = t_2$, since γ is injective. □

If we want the last result to be useful, it is necessary to establish the following.

Lemma 2.4. *For any $x_0 \in \Sigma$ there exists $r_0 = r_0(x_0) > 0$ sufficiently small such that, for any $r < r_0$, the ball $B(x_0, r)$ contains no triple junction nor endpoint other than, possibly, x_0 itself, and $C\psi(\gamma((0,t_2])) < 1$.*

Proof. It is immediate to satisfy the first constraint (no triple junction nor endpoint in the ball) since such points are finite. To satisfy the second requirement it is sufficient to show that the diameter of $\gamma((0,t_2])$ tends to 0 when $r \to 0$. In fact, proven this, we would have $\psi(\gamma((0,t_2])) \leq \psi(B(x_0, \delta(r) \setminus \{x_0\})$ with $\delta(r) \to 0$. Since the measure of the ball without the centre tends to vanish with the radius, the thesis would be achieved.

To prove that $diam(\gamma((0,t_2]))$ tend to 0 suppose, on the contrary, that there exists $\delta > 0$ and a sequence of radii $r_j \to 0$ with $diam(\gamma((0,t_2^{r_j}]) \geq \delta$ and $|x_0 - \gamma((0,t_2^{r_j}]) = r_j$. In the limit we would get a loop in this branch of Σ, and this is a contradiction. □

To strengthen the result, we can make it quite uniform. This uniformization result will be useful in the sequel too.

Theorem 2.5. *For any $\Sigma_1 \subset \Sigma$ compactly contained in the complement of the atoms of mass at least $(2C)^{-1}$ and of triple junctions and endpoints (which is the complement of a finite set) there exists $r_0 = r_0[\Sigma_1]$ such that, if $r < r_0$ and $x_0 \in \Sigma_1$, then $C\psi(B(x_0, r)) < 1/2$, no triple junction nor endpoint belongs to $B(x_0, r)$, and $C\psi(\gamma((0,t_2]) < 1$ (as in Theorem 2.3).*

Proof. We can consider separately the three requirements, and then choose the smallest radius.

It is easy to deal with the statement on balls: otherwise there would exist a sequence of centres x_0^n and of radii $r_n \to 0$ with the mass of the corresponding balls greater than $(2C)^{-1}$. By passing to a converging subsequence it would be straightforward to get the existence of a point $\overline{x_0}$ which would be an atom of mass at least $(2C)^{-1}$, obtained as a limit of the considered sequence of centres, which is a contradiction.

The requirement on triple junctions and endpoints is easily satisfied thanks to the assumption on Σ_1.

As far as the curves $\gamma((0,t_2])$ are concerned it is a little more difficult. Suppose on the contrary that there exists a sequence of arcs $\gamma_n([0,t_2^n])$ for which the distance $|\gamma_n(0)-\gamma_n(t_2^n)|=r_n$ is arbitrarily small and the measure $\psi(\gamma_n([0,t_2^n]))$ larger than C^{-1}. Up to subsequences we may assume that all this arcs are contained in one of the finitely many parts Σ_i consisting in the support of an injective simple curve γ and that they converge in the Hausdorff distance to a closed subset of Σ_i. The map γ provides an omeomorphisme between Σ_i and the interval $[0,1]$. Because it is well known that Hausdorff convergence on compact sets depends only on topology and not on metric (see, for instance, [7]) we can deduce that we have convergence also of the images of the arcs in $[0,1]$. The images are clearly segments and so the same holds for the limit. The condition that the distance between the extremal points x_n, y_n of the arcs goes to 0 says that, for a certain $x\in\Sigma_1$, we have $x_n\to x$, $y_n\to x$ and this fact is conserved by the omeomorphisme. As a consequence also the extremal points of the segments in $[0,1]$ collapse to the same point in the limit and so the limit must be a single point. Regarding this fact in Σ_i it is easy to deduce that we have a limit consisting in a single point which must be an atom of at least mass C^{-1}, which is a contradiction. □

As a consequence of what we have just proved, there exists well defined and continuous a function $\theta:(0,r_0]\to S^1$ such that $\theta(r)$ is the angle of the unique point of the curve γ which lies on $\partial B(x_0,r)$. The value r_0 has to be small enough and can be chosen quite uniformly according to Theorem 2.5 or depending on x_0 if x_0 is one of the dangerous points (triple junctions, endpoints, atoms of mass at least $(2C)^{-1}$). From now on, r_0 will always denote such a radius chosen in this way.

Theorem 2.6. *The function θ is locally Lipschitz on $(0,r_0]$. Moreover, for almost every $r\in(0,r_0)$ we have*

$$|\theta'(r)|\leq C\frac{\psi\left(\overline{B(x_0,r)}\setminus\{x_0\}\right)}{r}.$$

Proof. Consider two rays $0<r<R\leq r_0$ and the variation $\Delta\theta$ of the angle θ between the two values of the ray.

Let $\Delta r = R - r$, $y = \Sigma \cap \partial B_r$ and $x = \Sigma \cap \partial B_R$. We apply Lemma 2.1 with $U = \gamma \cap B_R \setminus \{x_0\}$, and V given by two parts: a rotation of $\gamma \cap \overline{B_r}$ of an angle $\Delta\theta$ around x_0 (in such a way that the image y' of y under the rotation is collinear with x_0 and x), and the segment $\overline{y'x}$.

Setting $\Gamma := \gamma \cap \overline{B_R} \setminus B_r$, the Hausdorff distance from Γ to the segment $\overline{yx}$ can be bounded by $C|y-x|\psi(\Gamma \setminus \{x\})$ due to (2.10), whereas the distance from $\overline{yx}$ to $\overline{y'x}$ equals $|y - y'|$, hence

$$\max_{z\in\Gamma} d(z,V) \le C|y-x|\psi(\Gamma\setminus\{x\})+|y-y'| \le C(r\Delta\theta+\Delta r)\psi(B_R\setminus\{x_0\})+r\Delta\theta. \tag{2.18}$$

Moreover, for every point in $\gamma \cap \overline{B_r}$ there is a point in V at a distance less than $r\Delta\theta$ (just follow the point along the arc, as it rotates), hence combining this with (2.18) we find

$$\max_{z\in U} d(z,V) \le r\Delta\theta+C(r\Delta\theta+\Delta r)\psi(B_R\setminus\{x_0\}) \le Cr\Delta\theta+C\Delta r\psi(B_R\setminus\{x_0\}). \tag{2.19}$$

Then from (2.4) we find

$$\mathcal{H}^1(U) - \mathcal{H}^1(V) \le C\psi(U\setminus V)\big(r\Delta\theta + \Delta r\psi(B_R\setminus\{x_0\})\big).$$

By our construction, the left hand side equals $\mathcal{H}^1(\Gamma) - \Delta r \ge |y-x| - \Delta r$, hence we find

$$|y-x| - \Delta r \le C\psi(B_R\setminus\{x_0\})\big(r\Delta\theta + \Delta r\psi(B_R\setminus\{x_0\})\big), \tag{2.20}$$

having used $U \setminus V \subset B_R \setminus \{x_0\}$. Now for the left hand side a simple computation yields

$$|y-x| - \Delta r = \sqrt{(\Delta r)^2 + 2r(r+\Delta r)(1-\cos\Delta\theta)} - \Delta r \ge$$
$$= \Delta r\left(\sqrt{1+\frac{r^2}{(\Delta r)^2}(1-\cos\Delta\theta)} - 1\right) \ge C\Delta r\left(\frac{r^2(\Delta\theta)^2}{(\Delta r)^2}\wedge\frac{r\Delta\theta}{\Delta r}\right),$$

having used elementary estimates such as $\sqrt{1+x^2} - 1 \ge C(x^2 \wedge x)$ and $(1-\cos t) \ge Ct^2$. Now, getting back to (2.20) and writing ψ_R for $\psi(B_R\setminus\{x_0\})$, we see that either

$$\Delta r\frac{r^2(\Delta\theta)^2}{(\Delta r)^2} \le C\psi_R r\Delta\theta + C\psi_R^2\Delta r$$

or

$$\Delta r \frac{r\Delta\theta}{\Delta r} \leq C\psi_R r\Delta\theta + C\psi_R^2\Delta r$$

is satisfied.

The first case provides a quadratic estimate like $A^2 \leq CAB + CB^2$, which implies $A \leq (1 \vee 2C)B$, where $A = \Delta\theta/\Delta r$ and $B = \psi_R/r$. The second one, under the assumption that $2C\psi_R < 1$, gives $r\Delta\theta \leq 2C\psi_R^2\Delta r \leq \psi_R\Delta r$ and so the same linear estimate.

So far we have obtained

$$\frac{\Delta\theta}{\Delta r} \leq C\frac{\psi_R}{r},$$

which gives local Lipschitz continuity of θ, as far as ψ_R/r remains bounded, i.e. as far as r stays bounded away from 0. By passing to the limit as $R \to r$ we get also the bound on the derivative required by the statement of the theorem.

□

Remark 3. What we have just proved is useful when one wants to show uniqueness of the possible limits of subsequences of $(\Sigma \cap B_r)/r$: it is often enough to find a limit to $\theta(r)$ as $r \to 0$. To achieve it, it would be enough to have $\theta \in BV(0, r_0)$, since any function with bounded variation near 0 satisfies a Cauchy condition near the same point, and so it is enough

$$\int_0^{r_0} \frac{\psi(B_r)}{r} dr < +\infty.$$

2.2 Blow-up limits, up to subsequences

Lemma 2.7. *Choose a point $x_0 \in \Sigma$ and, for every $r > 0$, let $\Sigma_r = \Sigma \cap B(x_0, r)$. The family of rescaled sets $r^{-1}(\Sigma_r - x_0)$ is compact, in the metric space of all nonempty compact subsets of $\overline{B_1}$ endowed with the Hausdorff distance. If $r_j^{-1}(\Sigma_{r_j} - x_0)$ converge to some set $K \subseteq B_1$ for a suitable subsequence $r_j \to 0$, then:*

1. *If x_0 is an endpoint, then K is a radius of $\overline{B_1}$.*

2. *If x_0 is a simple point, then K is the union of two radii of $\overline{B_1}$, which form an angle of at least 120 degrees.*

3. If x_0 is a triple junction, then K is the union of three radii of $\overline{B_1}$, forming angles of 120 degrees.

Proof. We can assume that x_0 is the origin of the coordinates.

According to the results in [5], there are p branches of Σ going out of x_0, with $p \in \{1, 2, 3\}$ according to the nature of x_0. We can regard these p branches as p injective curves $\gamma_i : [0, 1] \to \Sigma$, with $\gamma_i(0) = x_0$. Moreover, these curves meet only at the starting point x_0 (otherwise Σ would have a loop), and Theorem 2.3 implies that, when r is small enough, each γ_i has a unique intersection with the circle ∂B_r. As a consequence, we can reparameterize each γ_i in such a way that for $r \in [0, r_0]$ we have $\gamma_i \cap \partial B_r = \{\gamma_i(r)\}$, and hence also $\Sigma_r = \bigcup_i \gamma_i([0, r])$.

Thanks to the choice of r_0 (small enough), we can suppose that Σ_r contains no triple junctions, other than (possibly) x_0. Then applying Lemma 2.2 with $\Gamma = \gamma_i([0, r])$ for some i, (2.10) and (2.11) yields

$$\mathrm{d_H}\left(\gamma_i([0, r]), \overline{x_0\gamma_i(r)}\right) \le Cr\psi\left(B_r \setminus \{x_0\}\right), \tag{2.21}$$

Moreover, letting $K_r = \bigcup_i \overline{x_0\gamma_i(r)}$ denote the union of the p radii of B_r from the centre to the points $\gamma_i(r)$, using (2.21) for $i = 1, \ldots, p$ yields

$$\mathrm{d_H}(\Sigma_r, K_r) \le Cr\psi\left(B_r \setminus \{x_0\}\right),$$

Now observe that $\psi\left(B_r \setminus \{x_0\}\right) \to 0$ as $r \to 0$. Then such an estimate is the key tool. Indeed, given a sequence of radii $r_j \to 0$, passing to subsequences (not relabelled) we may suppose that the sets $r_j^{-1}K_{r_j}$ converge, in the Hausdorff distance, to some set K, which clearly is the union of q radii of B_1, with $1 \le q \le p$. When $p = 1$ this proves the lemma.

Now suppose that $p = 2$. To complete the proof, it suffices to prove that $q = 2$ (i.e. that no two radii of K_r/r overlap in the limit).

To this end, we suppose that there exists some $\varepsilon > 0$ and a subsequence of radii (still denoted by r_j) such that the angle formed by, say, the radii $\overline{x_0\gamma_1(r_j)}$ and $\overline{x_0\gamma_2(r_j)}$ is less than $2/3\pi - \varepsilon$ for every j, and we seek a contradiction.

Let us apply Lemma 2.1 to the sets $U = (\gamma_1([0, r_j]) \cup \gamma_2([0, r_j])) \setminus \{x_0, \gamma_1(r_j), \gamma_2(r_j)\}$, and V equal to the Steiner connection of the three points x_0, $\gamma_1(r_j)$ and $\gamma_2(r_j)$. Since clearly $\mathcal{H}^1(U) \ge 2r_j$ and $\mathrm{d_H}(U, V) \le 2r_j$, from (2.4) we find

$$2r_j \le \mathcal{H}^1(V) + C\psi\left(B_{r_j} \setminus \{x_0\}\right)2r_j.$$

Now, due to our assumption on the angle, one can check that there exists δ, depending only on $\varepsilon > 0$, such that $\mathcal{H}^1(V) \le (2-\delta)r_j$. Then we obtain

$$2r_j \le (2-\delta)r_j + C\psi(B_{r_j} \setminus \{x_0\})2r_j \quad \forall j,$$

and we get a contradiction for small enough r_j since $\psi(B_{r_j} \setminus \{x_0\}) \to 0$.

When $p = 2$, this shows that the two radii forming K_r tend to form an angle, in the limit, which is at least $120°$. Hence the limit set K is, in this case, the union of exactly two radii of B_1, whose angle is at least $120°$.

Finally, when $p = 3$ is suffices to repeat the same argument, to every pair of radii in K_r. □

For the case of ordinary points, we give also a stronger result.

Lemma 2.8. *Suppose x_0 is a simple point with $\psi(\{x_0\}) = 0$ and $r_j \to 0$ a sequence of radii such that $r_j^{-1}(\Sigma_{r_j} - x_0)$ converge to a set $K \subset \overline{B_1}$. Then K is a diameter (i.e. the angle between the two radii given by Lemma 2.7 is in fact $180°$).*

Proof. As usual, we will suppose $x_0 = 0$. By Lemma 2.7 we know that K is the union of two radii, forming an angle $\alpha > 120°$. If we set $\Sigma \cap \partial B_{r_j} = \{x_j^1, x_j^2\}$, we may say

$$2 = \mathcal{H}^1(K) \le \liminf_{j\to+\infty} \mathcal{H}^1\left(\frac{\Sigma_{r_j}}{r_j}\right) \le \liminf_{j\to+\infty} \frac{|x_j^1 - x_j^2|}{r_j}\big(1 + C\psi(\Sigma_{r_j})^2\big) = 2\sin\left(\frac{\alpha}{2}\right).$$

Here the first inequality is a consequence of Gołab's Theorem, while the second comes from (2.11). This easily implies $\alpha = 180°$ and the thesis. □

2.3 Γ−Convergence

We want here to give a useful Γ−convergence result finding a Γ−limit to functionals minimized by sets of the form $\Sigma_r := \Sigma \cap B(x_0, r)$. Here we state our theorem by considering only the case when x_0 is an endpoint, but the same result is true, with small modifications, also for any point of Σ which is an atom for ψ. A slightly more sofisticated Γ−convergence result concerning atomic ordinary point will be developed in section 3.

Let us consider an endpoint of Σ which we will call 0 and a small B_r around it. Let x_r be the only point of intersection of Σ and the boundary of the ball B_r. It is clear that Σ_r minimizes, among all compact connected sets Γ such that $x_r \in \Gamma$ and $\mathcal{H}^1(\Gamma) \leq \mathcal{H}^1(\Sigma_r)$, the quantity

$$\int_{A_r} d(x,\Gamma)\,\mu(dx), \tag{2.22}$$

where $A_r = t^{-1}(\Sigma_r)$ is the set of points projected to Σ_r. What we want to investigate now is whether we can find a limit of a proper rescaling of the functional appearing in (2.22), in order to get information on the limits of Σ_r/r. Let us consider the functionals defined on the set X of compact connected sets contained in the closed ball of radius 2 and of length less than 2 (which is a compact metric space if endowed with the Hausdorff topology, as a consequence of Gołab's Theorem), given by:

$$F_r(\Gamma) = \begin{cases} \int_{A_r/r} (d(x,\Gamma) - d(x,0))\, m_{r\sharp}\mu(dx) & \text{if } \frac{x_r}{r} \in \Gamma,\ \mathcal{H}^1(\Gamma) \leq \frac{\mathcal{H}^1(\Sigma_r)}{r}; \\ +\infty & \text{otherwise.} \end{cases} \tag{2.23}$$

Here we denote by m_r the division by r, i.e.: $m_r(x) = x/r$.

Each functional F_r is then minimized by Σ_r/r, as far as such sets have length smaller than 2. This happens for small r, since it holds

$$\lim_{r\to 0} \frac{\mathcal{H}^1(\Sigma_r)}{r} = 1,$$

as a consequence of (2.11).

Let us denote by D the application $D(x) = x/|x|$ that gives the direction of a non-zero vector and by ν the measure on S^1 given by $\nu = D_\sharp(\mu I_{A_0})$, where A_0 is the set of point projected to the endpoint 0.

Lemma 2.9. *Suppose that on a certain subsequence (not relabeled) it holds $x_r/r \to \bar{x}$: then it holds $F_r \xrightarrow{\Gamma} F$, where the $\Gamma-$convergence is intended with respect to the Hausdorff convergence on X and F is given by*

$$F(\Gamma) = \begin{cases} \int_{S^1} -\delta^*(v|\Gamma)\nu(dv) & \textit{if } x \in \Gamma \textit{ and } \mathcal{H}^1(\Gamma) \leq 1; \\ +\infty & \textit{otherwise.} \end{cases} \tag{2.24}$$

As usual, $\delta^*(v|\Gamma) = \sup_{y\in\Gamma} v \cdot y$. As a particular consequence Σ_r/r converges in the Hausdorff distance to a minimizer of F.

Proof. Let us start by the $\Gamma - \liminf$ inequality. We have to prove that, for any $\Gamma \in X$ and $\Gamma_r \to \Gamma$ we have $\liminf F_r(\Gamma_r) \geq F(\Gamma)$. We may suppose that $F_r(\Gamma_r)$ is finite for a subsequence, otherwise the $\liminf$ is $+\infty$, and so we have a sequence of sets Γ_r approaching Γ in such a way that $x_r \in \Gamma_r$ and $\mathcal{H}^1(\Gamma_r) \leq \mathcal{H}^1(\Sigma_r)/r \to 1$. It is clear so far that Γ satisfies the constraints $\bar{x} \in \Gamma$ and $\mathcal{H}^1(\Gamma) \leq 1$, as a consequence of Haudorff convergence's properties.

For every x we denote by $y_x(\Gamma_r)$ (one of) the nearest point(s) to x belonging to Γ_r and by $z_x(\Gamma_r)$ (one of) the point(s) realizing the max in $\sup_{z\in\Gamma_r} x\cdot z$. Notice that $z_x(\Gamma_r)$ depends actually only on $D(x)$. For every point x we have

$$d(x,\Gamma_r) - d(x,0) = d(x, y_x(\Gamma_r)) - d(x,0) \geq -\frac{x}{|x|}\cdot y_x(\Gamma_r) \geq -\frac{x}{|x|}\cdot z_x(\Gamma_r).$$

So we may estimate

$$\begin{aligned} F_r(\Gamma_r) &\geq \int_{A_r/r} -\frac{x}{|x|}\cdot z_x(\Gamma_r) m_{r\sharp}\mu(dx) \\ &= \int_{A_r} -D(x)\cdot z_{D(x)}(\Gamma_r)\,\mu(dx) \\ &= \int_{A_0} -D(x)\cdot z_{D(x)}(\Gamma_r)\,\mu(dx) + \int_{A_r\setminus A_0} -D(x)\cdot z_{D(x)}(\Gamma_r)\,\mu(dx). \end{aligned}$$

The latter term in the last line tends to 0 with r because the integrand is bounded by 2 and the set on which we integrate converges to the empty set. The former indeed is equal to

$$\int_{S^1} -\delta^*(v|\Gamma_r)\nu(dv) \to F(\Gamma)$$

where the convergence relies on the fact that Hausdorff convergence implies pointwise convergence of the support functions δ^* (see, for instance, [7]). The Γ−liminf inequality is then proved.

Let us pass to the Γ−limsup inequality. For every fixed Γ such that $F(\Gamma) < +\infty$, we have to find a sequence $(\Gamma_r)_r \to \Gamma$ such that $\limsup F_r(\Gamma_r) \leq F(\Gamma)$. For each r it is sufficient to rotate Γ so that its intersection with the boundary of the unit ball becomes x_r instead of x and to perform an omothety around x_r in order to satisfy the length constraint. We have hence a sequence of sets Γ_r such that $F_r(\Gamma_r)$ is finite and given by the integral expression in (2.23), for which it holds $\Gamma_r \to \Gamma$ in the Hausdorff distance. This convergence is true thanks to $x_r \to \bar{x}$ and to the fact that convergence holds also for the ratios of the omotheties, which are prescribed by the length constraints. It remains to estimate $F_r(\Gamma_r)$. For each couple of point x, z we have

$$|x-z|-|x| = |x|\left(\sqrt{1-\frac{2x\cdot z}{|x|^2}+\frac{|z|^2}{|x|^2}}-1\right) \le -D(x)\cdot z + \frac{|z|^2}{2|x|}.$$

So we may write

$$\begin{aligned} F_r(\Gamma_r) &\le \int_{A_r/r} \left(|x-z_x(\Gamma_r)|-|x|\right)\, m_{r\sharp}\mu(dx) \\ &\le \int_{A_r/r}\left(-D(x)\cdot z_{D(x)}(\Gamma_r)+\frac{|z_{D(x)}(\Gamma_r)|^2}{2|x|}\right) m_{r\sharp}\mu(dx) \\ &= \int_{A_0} -D(x)\cdot z_{D(x)}(\Gamma_r)\,\mu(dx) + \int_{A_r\setminus A_0} -D(x)\cdot z_{D(x)}(\Gamma_r)\,\mu(dx) \\ &\quad + r\int_{A_r}\frac{|z_{D(x)}(\Gamma_r)|^2}{2|x|}\mu(dx). \end{aligned}$$

In the last sum, the first term yields $F(\Gamma)$ in the limit, while the second and the third tend to zero. □

2.4 Iterated estimates for small diameters

We show here that, if the diameter of the transported set to a certain point of Σ is sufficiently small, the measure $\psi(B_r)$, for balls centred around that point, can be estimated by r itself.

Lemma 2.10. *There exists a constant C such that, given $x_0 \in \Sigma$, and r_0 chosen as usual, if we set $k = diam(t^{-1}(B_{r_0}))$, for all $r \le r_0/2$ it holds*

$$\psi(B_r) \le Ck\left(r+\psi(B_{2r})\right). \tag{2.25}$$

Proof. Consider a point of Σ, which we will call 0, together with two balls around it, of radii r and $2r \le r_0$ respectively. Let x_1, x_2 be the intersection points of Σ and the boundary of the biggest ball. We can for each segment $\overline{0x_i}$ consider the Hausdorff distance $\mathrm{d_H}(\Sigma^i_{2r}, \overline{0x_i})$ between it and the corresponding branch of Σ and the distance $\mathrm{d_H}(\Sigma_{2r}, \overline{0x_1}\cup\overline{0x_2})$. We have a set K in which Σ_{2r} is contained, i.e. the set obtained by fattening the two segments by a quantity equal to the latter distance. We may estimate, using each branch of Σ_{2r} as Γ in (2.10),

$$\mathrm{d_H}(\Sigma_{2r}, \overline{0x_1} \cup \overline{0x_2}) \le \max_{i=1,2} \mathrm{d_H}(\Sigma_{2r}^i, \overline{0x_i}) \le Cr\psi(B_{2r}).$$

Consider now the set $K' = K \cap B_r$ and its convex hull K''. Since we want to estimate the area of the set transported to B_r it is sufficient to estimate the area of the set of points which are closer to K'' than to the points x_i. Moreover, being k greater than the diameter of $t^{-1}(B_r)$, we can replace this set by its intersection with B_k. We include this set in the union of

- two stripes T_1, T_2 which are $2r$ wide and each T_i is ortogonal to $\overline{0x_i}$ and has the points 0 and x_i on its boundary,
- a sector E of amplitude $180° - \widehat{x_1 0 x_2}$ starting from 0, delimited by the boundaries of the stripes T_i passing through the origin,
- four small sectors $C_{i,j}$, each of them delimited by the boundary of the stripe T_i passing through x_i and the axis of the segment $\overline{x_i y_{i,j}}$, where the points $y_{i,j}$ are the corner points of the boundary of K'' near x_i.

The amplitude of these last sectors is the same of the angle $0\hat{x}_i y_{i,j}$, which can be estimated by $C\psi(B_{2r})$ thanks to the estimate on the Hausdorff distance. We know that also the amplitude α of the sector E can be estimated the same. In fact, we can consider in (2.11) $\Gamma = \Sigma_{2r}$, $x = x_1$ and $y = x_2$. We obtain

$$2r \le \mathcal{H}^1(\Sigma_{2r}) \le 2r \cos\left(\frac{\alpha}{2}\right)\left(1 + C\psi(\Sigma_{2r})^2\right),$$

and, dividing by $2r$ and using $(\cos\beta)^{-1} \ge 1 + c\beta^2$, which is true for $\beta \le \pi$, we get

$$\alpha^2 \le C\psi(\Sigma_{2r})^2.$$

Being μ a measure with an L^∞ density, it is enough to estimate the areas of T_i, E and $C_{i,j}$ intersected with B_k, and we obtain

$$\psi(B_r) \le C\left(kr + k^2\psi(B_{2r})\right).$$

For simplicity we will estimate k^2 by Ck. $\square$

The interest in the estimate (2.25) is that we can iterate it, especially when we have small diameters of the transported sets.

Theorem 2.11. *Suppose that there exists $r_1 < r_0/2$ such that $k = diam(t^{-1}(B(x_0, r_1))) < 1/(2C)$. Then, for all $r < r_1$ we have an estimate like*

$$\psi(B_r) \leq \frac{Ckr}{1-2Ck} + \left(\frac{2r}{r_1}\right)^{\log_2 1/Ck}. \tag{2.26}$$

Proof. Fixed $r < r_1$ we can find an integer h such that $r_1/2 < r2^h \leq r_1$. Iterating (2.25) we obtain

$$\psi(B_r) \leq Ckr \sum_{i=0}^{h-1} (2Ck)^i + (Ck)^h \psi(B_{r_1}) \leq \frac{Ckr}{1-2Ck} + \left(\frac{2r}{r_1}\right)^{\log_2 1/Ck}.$$

□

Notice that, due to the semicontinuous behaviour of the diameter of the transported set, saying that there exists a small r_1 such that $diam(t^{-1}(B_{r_1})) < 1/(2C)$ is the same as saying that $diam(t^{-1}(\{x_0\})) < 1/(2C)$.

Notice also the following useful consequence:

Corollary 2.12. *If $x_0 \in \Sigma$ and $diam(t^{-1}(\{x_0\})) < 1/(2C)$, then $\psi(\{x_0\}) = 0$, i.e. all atoms of ψ have transported sets with large diameter.*

Proof. Just use (2.26) and $\psi(\{x_0\}) = \lim_r \psi(B_r)$. □

3 Blow-up limits

3.1 triple junctions

We make here use of the previous section's tools to establish the expected result regarding singular points of Σ. For simplicity we will always centre our analysis in a point x_0 supposed to be the origin.

Theorem 3.1. *Suppose $0 \in \Sigma$ is a triple junction: then there exists the limit as $r \to 0$ of Σ_r/r in the Hausdorff distance and it is composed by the union of three rays with $120°$ angles.*

Proof. Thanks to Lemma 2.7 (which states that the limits up to subsequences are shaped like the union of three rays angled 120°) we just need to show the uniqueness of those limits. By means of Remark 3, it is enough to achieve

$$\int_0^{r_0} \frac{\psi(B(0,r))}{r} dr < +\infty. \tag{3.1}$$

We will show that, for small r, it holds $\psi(B(0,r)) \leq Cr^2$, thus achieving the goal.

Let us consider small values of r, such that the angles between the points of intersection of Σ with the boundary of $B(0,3r)$ are all smaller than 130° (we know that for small r this happens, otherwise we could produce a subsequence having a limit different from the admissible ones). Consider now a point x at a distance $|x| = cr$ from the origin: we want to show that, if c is great enough, it is not possible to have $x \in t^{-1}(B(0,r))$. Supposing on the contrary that x is transported to $B(0,r)$, we gain that no point of Σ is contained in the ball centred in x and of radius $r(c-1)$. In particular no point of Σ may lay on the part of $\partial B(0,3r)$ contained in such a ball. Yet, the amplitude of such arc depends only on c and, as c increases to infinity, tends to $2\arccos(1/3) > 130°$. This would mean that, for big c, we would have an arc of 130° on $\partial B(0,3r)$ without any of the three points of intersection with Σ, which is a contradiction. So there exists a constant c_0 such that $t^{-1}(B(0,r)) \subset B(0,c_0 r)$ and this, since $\mu \in L^\infty$, completes the proof. $\square$

Remark 4. Notice that (3.1) remains true also in the case where the measure μ, instead of being L^∞, is simply L^p with $p > 1$, because in this case we can use Holder inequality to get an estimate like

$$\psi(B(0,r)) \leq Cr^{2-2/p},$$

and this is sufficient for the convergence of the desired integral.

3.2 Endpoints

Here we will state an analogous theorem concerning existence of the limit near an endpoint of Σ, giving also a caracterization of the direction of the ray we find as a limit. We use the same notation as in the $\Gamma-$convergence subsection, from which this theorem arises.

Theorem 3.2. *If* 0 *is an endpoint of* Σ *the limit of* Σ_r/r *in the Hausdorff distance as* $r \to 0$ *exists and is given by a single ray from the origin in the direction of* $-\bar{v}$, *where* $\bar{v}$ *is given by*

$$\bar{v} = \int_{S^1} v\,\nu(dv)$$

Proof. Thanks to Lemma 2.7 it is enough to determine the direction of the rays that can be possible limits of subsequences. To do this we use the $\Gamma-$convergence result provided in Lemma 2.9. In fact every set K limit of a subsequence of Σ_r/r intersecting ∂B_1 in a point $\bar{x}$ must be the set maximizing $\int_{S^1} \delta^*(v|\Gamma)\nu(dv)$ among all sets Γ compact, connected, passing through $\bar{x}$ and such that $\mathcal{H}^1(\Gamma) \leq 1$. This maximizer is always a segment of unit length directed from $\bar{x}$ according to the vector $\bar{v}$. But we also know that $0 \in K$ and the only possible position for $\bar{x}$ so that the maximing set passes through the origin is $\bar{x} = -\lambda\bar{v}$. Then $\bar{x}$ is uniquely determined and the limit of Σ_r/r exists. $\square$

3.3 Ordinary points

Our next step is establishing existence of the same limit in the four cases in which we will divide the general case of an ordinary point. In fact we will classify these points x_0 according to the shape of the set $T(x_0) = \{x \in \Omega : d(x,\Sigma) = |x - x_0|\}$. This set coincides up to negligible sets with $t^{-1}(x_0)$. Moreover, $T(x_0)$ is always a convex set, thus endowed with its own entire dimension: it may be $0, 1$ or 2. The four cases will be given by

1. $T(x_0) = \{x_0\}$, i.e. dimension 0;
2. $T(x_0)$ is a segment starting in x_0 (a subcase of dimension 1);
3. $T(x_0)$ is a segment having x_0 in its relative interior (the other subcase of dimension 1);
4. $T(x_0)$ is two-dimensional (i.e. with non empty interior).

Let us start from the easiest of the four cases:

Theorem 3.3. *Suppose* 0 *is an ordinary point of type* 3*: then there exists the limit of* Σ_r/r *as* $r \to 0$ *in the Hausdorff distance and it is the diameter composed by the two unit rays orthogonal to the segment* $T(0)$.

Proof. No optimality of Σ is here required: just notice that Σ is contained in the complement of two suitable balls tangent in 0 to the segment orthogonal to $T(0)$. □

Now we move to a case just a little more complicated:

Theorem 3.4. *Suppose* 0 *is an ordinary point of type* 2*: then there exists the limit of* Σ_r/r *as* $r \to 0$ *in the Hausdorff distance and it is the diameter composed by the two unit rays orthogonal to the segment* $T(0)$*.*

Proof. Now we can only ensure that Σ stays outside a single ball tangent in 0 to the segment orthogonal to $T(0)$: this is enough to say that, provided a limit of a subsequence is a diameter, it must be the diameter orthogonal to $T(0)$. But every limit of subsequences here is a diameter, thanks to Lemma 2.8, since $\psi(\{0\}) = \mu(T(0)) = 0$. So the limits are uniquely determined and this makes the limit exists. □

Our next case uses something more, because here $T(0)$ gives no information on the possible limit:

Theorem 3.5. *Suppose* 0 *is an ordinary point of type* 1*: then there exists the limit of* Σ_r/r *as* $r \to 0$ *in the Hausdorff distance and it is a diameter.*

Proof. By using Lemma 2.8 on subsequences we know that any limit point in the Hausdorff distance has to be a diameter and, to identify it, it is enough to show that the function θ (with respect to any of the two branches of Σ going out from 0) has a limit. As usual, we will look for the inequality $\int_0^{r_0} \frac{\psi(B(0,r))}{r} dr < +\infty$, required by Remark 3. Here we can use the result valid when $diam(t^{-1}(0))$ is small , given by Theorem 2.11, (we have actually a vanishing diameter) to establish an estimate like $\psi(B_r) \leq Cr$ for small r. This gives the convergence of the integrand and the proof is achieved. □

The last case requires something more, that we will state as another Γ-convergence lemma. This time we will use the fact that $\Sigma \cap B_r$ minimizes, among all sets Γ sharing with it the same two intersections with ∂B_r, the functional

$$\int_{A_r} d(x,\Gamma)\,\mu(dx) + P\left(\mathcal{H}^1(\Sigma_r) - \mathcal{H}^1(\Gamma)\right), \tag{3.2}$$

where the quantity $P(\varepsilon)$ is defined, for $\varepsilon < 0$, as the increase in the functional if we cut away a curve of length $-\varepsilon > 0$ starting from a given endpoint in Σ (it is in fact a penalization if Γ is too long), while for $\varepsilon > 0$ it is the diminution (a negative quantity) of the functional if we add a straight line segment ε-long starting from the same endpoint, in the direction of the tangent vector in it (which exists and coincides with the direction of $\bar{v}(x_0)$, thanks to Theorem 3.2). We give now an estimate precise of the term P, in term of the saved/lost length ε (if we save length we have $\varepsilon > 0$ and $P < 0$, and viceversa). Let v_0 be the unit vector in the direction of $\bar{v}(x_0)$, that we know is the tangent vector to Σ in the endpoint that we call x_0. If $\varepsilon > 0$ we can estimate

$$\begin{aligned} P(\varepsilon) &\leq \int_{A_0(x_0)-x_0} (|x - \varepsilon v_0| - |x|)\ \mu(dx) \\ &\leq \int_{A_0(x_0)-x_0} \left(-\varepsilon v_0 \cdot \frac{x}{|x|} + \frac{\varepsilon^2}{2|x|} \right) \mu(dx) = -\varepsilon v_0 \cdot \bar{v}(x_0) + o(\varepsilon); \end{aligned}$$

if, on the other hand $\varepsilon < 0$ we have

$$P(\varepsilon) \leq \int_{A_{r_\varepsilon}(x_0)} (|x - w_\varepsilon| - d(x, \Sigma))\ \mu(dx),$$

where w_ε is the point of Σ situated after an arc Γ_ε of length $|\varepsilon|$ starting form the extremal point 0 and $r_\varepsilon = diam(\Gamma_\varepsilon)$ (for small ε it holds $r_\varepsilon = |w_\varepsilon - x_0|$). We may go on with the estimation with

$$\begin{aligned} P(\varepsilon) &\leq \int_{A_0(x_0)} (|x - w_\varepsilon| - |x - x_0|)\, \mu(dx) + \int_{A_{r_\varepsilon(x_0)} \setminus A_0(x_0)} (|x - w_\varepsilon| - d(x, \Sigma))\, \mu(dx) \\ &\leq -(w_\varepsilon - x_0) \cdot \bar{v}(x_0) + o(\varepsilon) + r_\varepsilon \mu(A_{r_\varepsilon}(x_0) \setminus A_0(x_0)). \end{aligned}$$

For small ε it is clear that $r_\varepsilon = |w_\varepsilon - x_0| \leq |\varepsilon|$ and moreover we have $(w_\varepsilon - x_0)/r_\varepsilon = -v_0 - \delta_\varepsilon$ with $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. We can then estimate again

$$P(\varepsilon) \leq r_\varepsilon v_0 \cdot \bar{v}(x_0) + r_\varepsilon \delta_\varepsilon \cdot \bar{v}(x_0) + o(\varepsilon) + o(r_\varepsilon) \leq -\varepsilon v_0 \cdot \bar{v}(x_0) + o(\varepsilon).$$

Notice that such estimates can be used in fact to get a precise quantitative version of part of the proof Lemma 2.2.

So the in the minimization problem given by (3.2) it's still true that Σ_r minimizes if we replace the real penalization by a function given by $P(\varepsilon) = -c\varepsilon + o(\varepsilon)$, where $c = v_0 \cdot \bar{v}(x_0) = |\bar{v}(x_0)|$. We may also require

$$P(\varepsilon) \geq -c\varepsilon, \tag{3.3}$$

because the minimization is preserved if we make bigger the value of the functional on sets different from the minimizer: in this case no matter if the value of P is made bigger outside 0 (it's the same reason for which we have only given estimate from above of the real penalization). We now rescale the functionals as before, obtaining that Σ_r/r minimizes

$$F_r(\Gamma)=\begin{cases}\int_{A_r/r}(d(x,\Gamma)-|x|)\ m_{r\sharp}\mu(dx)+\frac{1}{r}P(r(l_r-\mathcal{H}^1(\Gamma))) & \text{if } x_r^1,x_r^2\in\Gamma,\\ +\infty & \text{otherwise,}\end{cases}$$

where $l_r=\mathcal{H}^1(\Sigma_r)/r\to 2$ (as usual, it is a consequence of (2.12)) and x_r^1 and x_r^2 are the points in which Σ_r/r intersects the boundary of the unit ball.

Lemma 3.6. *Let F denote the functional given by*

$$F(\Gamma)=\begin{cases}\int_{S^1}-\delta^*(v|\Gamma)\nu(dv)-c(2-\mathcal{H}^1(\Gamma)) & \textit{if } x^1,x^2\in\Gamma\\ +\infty & \textit{otherwise.}\end{cases}$$

Then $F_r\xrightarrow{\Gamma}F$ with respect to the Hausdorff convergence on the space X of compact connected sets contained in a fixed large closed ball, provided $x_r^i\to x^i$ for $i=1,2$.

Proof. The proof is close to that of Lemma (2.9): for the $\Gamma-$liminf inequality fix a Γ and an approaching sequence $(\Gamma_r)_r$ and use the same estimate to deal with the integral term of the functionals F_r and F. For the penalization term, thanks to (3.3), we have

$$-c(1-\mathcal{H}^1(\Gamma))\leq\liminf_r\frac{1}{r}P(r(l_r-\mathcal{H}^1(\Gamma_r))).$$

For the proof of $\Gamma-$limsup inequality it is sufficient to build a sequence $(\Gamma_r)_r$ such that it converges to Γ, the points x_r^i belong to Γ_r and we have $\mathcal{H}^1(\Gamma_r)\to\mathcal{H}^1(\Gamma)$: the convergence of the last term follows then from the asymptotic behaviour near 0 of the function P and the first can be estimated the same as in the proof of Lemma (2.9). To obtain such a sequence it is sufficient to apply to Γ an affine transformation sending x^i to x_r^i. The convergence $x_r^i\to x^i$ implies the convergences we need. □

We can now state the last theorem regarding existence of the limit.

Theorem 3.7. *Suppose* 0 *is an ordinary point of type* 4*: then there exists the limit of* Σ_r/r *as* $r \to 0$ *in the Hausdorff distance and it is a corner composed by two unit rays.*

Proof. Being X a compact metric space (see, for instance, [3]), a a consequence of our previous $\Gamma-$convergence result (Lemma 3.6), all limits of Σ_r/r, that we know must be the union of two segments (Lemma 2.7), must minimize the functional appearing in Lemma 3.6. We will now try to identify those pair of radii that may be minimizers, exactly as in the proof of Theorem 3.2, in order to have uniqueness of the limits and then the existence of the limit.

Let us consider a ball in which the vertical ray directed upwards is given by the vector $\bar{v}$. We want to show the existence of the limit of Σ_r/r, so we must identify the possible limits of subsequences as a unique one. We stress that this is strongly different from saying that the functional F has a unique minimizer: for every converging subsequence we have a different functional F, depending on the limit points x^i. What we want to do is show that there exists just one possible choice of x^i, $i = 1, 2$ so that the corner composed by the rays arriving in such two points minimizes the corresponding functional. We may identify the points x^i by means of the angles α, β between the corresponding rays and the horizontal line. Notice that if we have some two rays as a limit of subsequence, the set A_0 has to be contained in the sector having 0 has a vertex and the normal vector to the rays as boundary directions. This implies in particular that $\alpha, \beta \geq 0$.

Consider now the ellipse having x^i as focuses and 2 as the length of the greater axe. The centre 0 lies on it. The tangent direction to the ellipse is not horizontal unless $\alpha = \beta$. Any Γ consisting by two segments joining, in order, x^1, y and x^2, where y lies on such an ellipse can be used as a variation to K (being K the corner we're taking into consideration as limit of Σ_r/r) and provides the same value as K to the length-penalizing term. Yet, if y has a positive component in the direction of $\bar{v}$, the integral term turns out to be strictly lesser. This shows that only $\alpha = \beta$ is possible.

We are now going to perform variations in which we move the vertex of the corner up or down to a certain value y of the $\bar{v}-$component. The value of F on the set Γ obtained in such a way can be estimated by

$$\int_{S^1} v \cdot y v_0 \, \nu(dv) - c(2 - \mathcal{H}^1(\Gamma)) = -y|\bar{v}| + c\mathcal{H}^1(\Gamma) - 2c,$$

where v_0 is the unit vector in the direction of $\bar{v}$. By $\bar{v}$ we mean the vector calculated at 0, while we denote by $\bar{v}(x_0)$ the one obtained at the endpoint x_0. Notice that $c = |\bar{v}(x_0)|$. We have $\mathcal{H}^1(\Gamma) = 2\sqrt{\cos^2\alpha + (y - \sin\alpha)^2}$ and we may write at the first order in y:

$$F(\Gamma) = -y|\bar{v}| + cy\sin\alpha + o(y).$$

Optimality of K (i.e. $y = 0$) gives so necessarily $|\bar{v}| = c\sin\alpha$, and this completes the determination of α. $\square$

4 Something more on regularity

We present in this section a regularity result, as a by subproduct of our previous analysis.

Theorem 4.1. *Let γ be an arc length parametrization of a subset $\Sigma_1 \subset \Sigma$ consisting of a simple curve with no triple junction nor endpoint in its relative interior, such that it holds $k = \sup_{x\in\Sigma_1} diam(t^{-1}\{x\}) < 1/(2C)$. Then $\gamma \in C^{1,1}$ and it holds*

$$|\gamma''| \leq \frac{Ck}{1-2Ck}.$$

Proof. Notice that the condition on the diameters of the transported sets prevents Σ_1 to contain atoms, thanks to Corollary 2.12. So, writing Σ_1, if necessary, as a countable union of subsets, we can suppose that it is compactly contained in the complement of triple junctions, endpoints and atoms with mass larger than $(2C)^{-1}$.

Because of semicontinuity, for every point $x \in \Sigma_1$ it will exist a ball $B(x, r_1)$ such that $diam(t^{-1}(B(x, r_1))) < 1/(2C)$. We can also suppose $r_1 < r_0[\Sigma_1]$ (the radius defined in Theorem 2.5). Then for every $y \in B(x, r_1/2)$ it holds $diam(t^{-1}(B(y, r_1/2))) < 1/(2C)$. This means that we can use estimate (2.26) in all these points. By using also theorem 2.6 we can then say that, whenever $y_1 = \gamma(t_1)$ and $y_2 = \gamma(t_2)$ are points in such a neighbourhood at distance r, we can estimate

$$\Delta\theta(y_1, y_2) \leq \frac{Ckr}{1-2Ck} + r^\alpha C(r_1),$$

where $\Delta\theta(y_1, y_2)$ is the angle between the tangent vector to Σ in y_1 and the segment $\overline{y_1y_2}$ (such an angle can be estimated by the variation of the function

θ) and α is an exponent greater than 1. By writing the same inequality interchanging the role of y_1 and y_2, summing up, and taking into account that γ is an arc length parametrization, so that all derivatives are unit vector determined only by the direction of the tangent vector, we get

$$|\gamma'(t_1) - \gamma'(t_2)| \leq \frac{Ckr}{1-2Ck} + r^\alpha C(r_1). \tag{4.1}$$

Taking into account that $r = |y_1 - y_2| \leq |t_1 - t_2|$, this implies that γ is locally $C^{1,1}$, and so it has almost everywhere a second derivative. Passing to the limit in (4.1) we get

$$|\gamma''(t)| \leq \frac{Ck}{1-2Ck}$$

for almost every t.

$\square$

Let us have a look to some consequences. First of all we see that the situation analysed in theorem 3.5 is in fact impossible to be found.

Corollary 4.2. *No ordinary point x_0 in Σ is such that $T(x_0) = \{x_0\}$.*

Proof. Just notice, that, thanks to Theorem 4.1, in a neighbourhood of such a point we should have a $C^{1,1}$ curve. But for $\gamma \in C^{1,1}$ in every point of the curve we have a positive radius ball to which γ is tangent from outside in the considered point. This ensures the existence of some more points, different from x_0, which are transported to x_0. $\square$

Next consequence deals with triple junctions and can be considered a quite complete answer to the question about them posed in [4]. We will state it in the form of an all-inclusive theorem.

Theorem 4.3. *Suppose that $x_0 \in \Sigma$ is a triple junction: then the three branches of Σ starting from it are parametrized by arc length by $C^{1,1}$ curves at least in a neighbourhood of x_0 and have tangent vector in x_0 which form three $120°$ angles.*

Proof. Just use previously proved results (Theorem 3.1) and notice that, due to $t^{-1}(B_r) \subset B_{c_0 r}$, we have $diam(t^{-1}(x_0)) = 0$, which is enough for Theorem 4.1 and local $C^{1,1}$ regularity.

$\square$

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References

[1] W. K. Allard and F. J. Almgren, The structure of Stationary One Dimesional Varifolds with Positive Density. *Invent. Math.* (34), no. 2, 83–97, 1976.

[2] L. Ambrosio, Lecture Notes on Optimal Transport Problems, in *Mathematical Aspects of Evolving Interfaces*, 1–52, Springer Verlag, Berlin, 2003.

[3] L. Ambrosio and P. Tilli *Topics on Analysis in Metric Spaces*, Oxford University Press, 2004.

[4] G. Buttazzo, E. Oudet and E. Stepanov, Optimal Transportation Problems with Free Dirichlet Regions. *Progress in Nonlinear Diff. Equations and their Applications* (51), 41–65, Birkhauser, 2002.

[5] G. Buttazzo and E. Stepanov, Optimal Transportation Networks as Free Dirichlet Regions for the Monge-Kantorovich Problem. *Ann. Scuola Norm. Super. Pisa Cl. Sci.* (2), no. 4, 631–678, 2003.

[6] G. Buttazzo and E. Stepanov, Minimization Problems for Average Distance Functionals, 2004. To appear in *Calculus of Variations: Topics from the Mathematical Heritage of Ennio De Giorgi* D. Pallara (ed.), Quaderni di Matematica, series edited by Dipartimento di Matematica, Seconda Università di Napoli, available at cvgmt.sns.it.

[7] C. Castaing and M. Valadier *Convex Analysis and Measurable Multifunction*, Springer Verlag, Berlin, 1977.