EXISTENCE RESULTS FOR A MORPHOELASTIC MODEL

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Abstract. We present some existence results for three-dimensional quasistatic morphoelasticity. The state of the growing body is described by its deformation and the underlying growth tensor and is ruled by the interplay of hyperelastic energy minimization and growth dynamics. By introducing a regularization in the model, we prove that solutions can be obtained as limits of time-discrete solutions, built by means of an exponential-update scheme. By further allowing the dependence of growth dynamics on an additional scalar field, to be interpreted as a nutrient or inhibitor, we formulate an optimal control problem and prove existence of optimal controls and states. Eventually, we tackle the existence of coupled morphoelastic and nutrient solutions, when the latter is allowed to diffuse and interact with the growing body.

1. Introduction

Morphoelasticity describes the growth of an elastic body and finds its main application in the context of biological systems. Here, growth is often a central aspect and is driven by a variety of phenomena acting at different scales. Below, we limit ourselves in summarizing some key modeling issues, referring to the recent monograph [11] for a thorough introduction to the topic and additional material.

The description of the mechanical response of a growing body can be simplified by restricting the attention to the macroscopic level of continua. Assume to be given a nonempty, open, simply connected and bounded set $\Omega \subset \mathbb{R}^3$ with smooth boundary, to be interpreted as the *reference configuration* of the body. At all times $t \in [0,T]$, $T>0$, the *deformation* of the body will be denoted by $y(t) : \Omega \to \mathbb{R}^3$. Classical morphoelastic models postulate the *multiplicative decomposition* of the deformation gradient $\nabla y(t)$ into an *elastic strain tensor* $F_{\rm el} \in \mathbb{R}^{3\times 3}$, related to stresses, and a *growth tensor* $G \in \mathbb{R}^{3\times 3}$, specifying the growth dynamics, namely,

$$\nabla y(t) = F_{\rm el}(t) G(t).$$

In case $G(t)$ is compatible, namely, if $G(t) = \nabla y_{\rm gr}(t)$ for some given *growth deformation* $y_{\rm gr}(t) : \Omega \to \mathbb{R}^3$, one can prove that $F_{\rm el}(t)$ is compatible as well and the latter multiplicative decomposition corresponds to the classical chain rule applied to the composition $y(t) = y_{\rm el}(t) \circ y_{\rm gr}(t)$. Here, $y_{\rm el}(t)$ can be interpreted as the elastic deformation of the *evolved* configuration $y_{\rm gr}(t,\Omega)$. We however do not assume compatibility here, for this would limit the applicability of the theory, see [11, Sec. 12.5].

The *state* of the morphoelastic system is hence determined by the pair $(y(t), G(t))$ for $t \in [0,T]$. Its evolution in time is governed by the interplay between the mechanical equilibrium and the growth process. As the time scales of mechanical equilibration and of growth usually differ by orders of magnitude, inertial effects can be assumed to be negligible and one resorts to the quasistatic

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approximation of the equilibrium system

$$\operatorname{div} P(t) + f(t) = 0 \quad \text{in} \quad [0,T] \times \Omega, \tag{1.1}$$

$$y(t) = \mathrm{id} \quad \text{in} \quad [0,T] \times \Gamma_{\mathrm{D}}, \tag{1.2}$$

$$P(t)n = g(t) \quad \text{in} \quad [0,T] \times \Gamma_{\mathrm{N}}. \tag{1.3}$$

The tensor $P(t)$ above is the *first Piola-Kirchhoff* stress. We assume the body to be *hyperelastic*, so that its elastic state is determined by the elastic energy density $W = W(F_{\mathrm{el}}) = W(\nabla y(t) G^{-1}(t))$. In particular, $P(t)$ in (1.1) is given by

$$P(t) = \det G(t)\, \mathrm{D}W(\nabla y(t) G^{-1}(t)) G^{-\top}(t). \tag{1.4}$$

The quasistatic equilibrium system features a time-dependent body force $f(t) : \Omega \to \mathbb{R}^3$ as well as a time-dependent surface traction $g(t) : \Gamma_{\mathrm{N}} \to \mathbb{R}^3$, to be imposed on the Neumann part Γ_{N} of the boundary $\partial\Omega$. In addition, the body is clamped at $\Gamma_{\mathrm{D}} \subset \partial\Omega$, where $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$.

The evolution of the growth tensor $G(t)$ is specified via the space-parametrized ODE in rate form

$$G'(t) G^{-1}(t) = M(t)$$

where the prime stands for partial time differentiation. The constitutive choice for the *growth rate* M reflects the combination of different effects driving the evolution and we refer the reader to [7, 11] for a discussion on the many possibilities. In all generality, M can depend on time $t \in [0,T]$, referential position $x \in \Omega$, and actual position $y(t,x)$, modeling indeed nonhomogeneous growth conditions in time and space. Growth may also be influenced by the state of the system, namely, by $G(t)$ and by the deformation gradient $\nabla y(t)$. In addition, the stress $P(t)$ is known to be possibly driving growth in some applications [16].

In the following, we hence resort in focusing on a some reduced evolution model by prescribing

$$G'(t) G^{-1}(t) = M(G(t), \nabla y(t)) \quad \text{in} \quad [0,T] \times \Omega, \tag{1.5}$$

$$G(0) = G^0 \quad \text{in} \quad \Omega \tag{1.6}$$

In the latter, all nonhomogeneities are neglected for the sake of simplicity. Note however that these could be considered as well, at the price of some additional notational intricacy. Notably, the dependence on the stress $P(t)$ can be accounted for in (1.5) by means of the dependence on the tensors $G(t)$ and $\nabla y(t)$, by implicitly assuming (1.4). Note in passing that the actual dependence of M on stress or strain is still debated [1].

Research in morphoelasticity has been up to now primarily devoted to clarifying the mechanical setting and to deriving numerical simulations. In this respect, we refer the reader to the recent [6, 8, 14] and [2, 9, 17]. To the best of our knowledge, an existence theory for solutions of the nonlinear morphoelastic evolution system (1.1)-(1.6) is still unavailable.

In this paper, we move first steps in this direction, by focusing on some nonlocally relaxed versions of the growth-dynamics rule (1.5). Indeed, the analysis of problem (1.1)-(1.6) requires formulating first a time-discrete version of the system, introducing suitable piecewise continuous and piecewise affine interpolants of the key quantities, and eventually passing to the limit as the width of the time step tends to zero. In particular, a key point is the limit passage in the nonlinear rate M, which we will assume to be Lipschitz continuous with respect to its variables. This in turn calls for some time-compactness of the interpolants of ∇y, which however is not to be expected in the quasistatic framework of (1.1)-(1.3). In fact, the best one can hope for from minimality is a uniform Sobolev bound, see (3.19) later on. Consequently, the analysis of (1.5) would soon grind to a halt. We hence propose to introduce a regularization of ∇y in the dependence of M. This is achieved by replacing (1.5) by

$$G'(t) G^{-1}(t) = M(G(t), (K\nabla y)(t)) \quad \text{in} \quad [0,T] \times \Omega \tag{1.7}$$

where $(K\nabla y)(t)$ is defined as a space and time convolution as

$$(K\nabla y)(t,x) = \int_0^t \int_{\mathbb{R}^3} \kappa(t-s)\phi(x-z)\nabla y(s,z)\,\mathrm{d}z\,\mathrm{d}t \quad \forall\,(t,x) \in [0,T]\times\Omega. \tag{1.8}$$

Note that, here and in the following, ∇y is tacitly intended to be trivially extended to zero in $\mathbb{R}^3\setminus\Omega$ whenever needed. This regularization serves the mathematical purpose of allowing a satisfactory existence theory, at the price of a minor modification to the model. In fact, the presence of the convolution term may be justified from the modeling viewpoint as of introducing a nonlocal in space and time dependence of the rate M on the deformation gradient. In particular, the presence of the time-convolution kernel κ induces a second time scale into the problem, which may be interpreted as a time relaxation. Note that a similar effect would have been achieved by considering viscoelastic dynamics instead.

Our first main result is an existence theory for a variational formulation on the regularized *morphoelastic evolution problem* (1.1)-(1.3), (1.6)-(1.7). As already mentioned, this is achieved via a time-discretization argument. A crucial observation here is that the sign of the determinant of $G(t)$ is preserved along the evolution, as an effect of the nonlinear structure of (1.7). This conservation is crucial, for it guarantees that the growth process is nondegenerate and locally orientation preserving. Correspondingly, we resort to a time-discrete scheme of exponential type, reproducing this sign conservation at the discrete level.

The structure of the morphoelastic evolution problem is reminiscent of the quasistatic evolution problem in creep inelasticity, the difference being that in this latter setting M is taken to be the variation with respect to G of the total energy functional. As such, in creep inelasticity M is directly related to W. This entails conservation of energy, at least formally, which in turn provides the fundamental a priori estimate. The present situation is different, for energy cannot be expected to be conserved along the growing process, as we are not including all energy exchanges in the description. In particular, no relation is imposed between the elastic-energy density W and the growth-rate function M, the specification of the latter being usually just phenomenological. As a consequence, we have to obtain a priori estimates otherwise.

Note that most existence theories in multidimensional inelasticity at finite strains hinge on the presence of higher-order gradients in the internal variable G, here to be interpreted as inelastic strain [12, 19, 20, 22] (see [18, 25], however, where no gradient is involved). We avoid such higher-order terms here, still allowing some nonlocal effect in space via the convolution term $K\nabla y$.

Our existence result can be compared with the one in [21]. There, a similar model to (1.1)-(1.6) is introduced in the frame of rate-dependent viscoplasticity and proved to admit solutions via a time-discretization and passage to the limit procedure. The existence theory in [21] is however quite different from ours. At first, the analysis in [21] hinges upon assuming a variational origin of the flow rule, which is not avaliable here. Secondly, a gradient term in G is considered, whereas our model is local in G. Thirdly, the solution notion in [21] is variational, making the regularization of the occurrence ∇y in the flow rule unnecessary. Eventually, the time-discretization scheme in [21] is the classical variational one, while we consider an *exponential* variant instead, see (3.4).

As a second existence result, we consider an additional dependence of M from an external field $\mu(t)$, which we assume to be scalar for definiteness. Namely, we replace (1.7) by

$$G'(t)G^{-1}(t) = M(G(t),(K\nabla y)(t),\mu(t)) \quad \text{in} \quad [0,T]\times\Omega. \tag{1.9}$$

The field $\mu(t):\Omega\to\mathbb{R}$ can be interpreted as the concentration of a nutrient (or an inhibitor), influencing the growth rate. In Section 4 we analyze an optimal control problem, where $\mu(t)$ acts as a control and drives the trajectory $t\mapsto(y(t),G(t))$ to minimize a target functional, possibly of the form

$$J(y,G) = \int_\Omega i(G(T))\,\mathrm{d}x + \int_0^T \int_\Omega j(y,G)\,\mathrm{d}x\,\mathrm{d}t.$$

Here, the functions i and j are suitably lower semicontinuous and favor specific deformations and growth tensors. In particular, the choice $i(G) = \det G$, $j = 0$ represents volume minimization whereas $j(y,G) = |y - y_{\rm target}|^2$, $i = 0$ corresponds to the possible attainment of a given target deformation $y_{\rm target}$.

Eventually, we couple the evolution of the state $(y(t), G(t))$ with that of the scalar field $\mu(t)$ by additionally specifying its evolution as

$$\mu'(t) - \nu\Delta\mu(t) = h(t) - H((\kappa * y)(t)) \quad \text{in} \ \ [0,T]\times\Omega, \tag{1.10}$$

$$\mu(t) = \mu_{\rm D}(t) \quad \text{in} \ \ [0,T]\times\partial\Omega, \tag{1.11}$$

$$\mu(0) = \mu^0 \quad \text{in} \ \ \Omega. \tag{1.12}$$

Here, $\nu > 0$ and $h(t) : \Omega \to \mathbb{R}$ plays the role of a given source. The term $H(\kappa * y)$ instead models the consumption of $\mu(t)$ during the growth process, which is indeed assumed to depend on the actual position of the body, again mollified by a time-convolution compactifying term. In particular, the triplet $(y(t), G(t), \mu(t))$ describes a system, where growth is influenced by the field μ which diffuses and is consumed during growth. The coupling of the quasistatic equilibrium (1.1)-(1.3), the growth dynamic (1.6), (1.9), and the nutrient dynamic (1.10)-(1.12) gives rise to a *nutrient-morphoelastic evolution problem*, which is variationally reformulated and proved to admit solutions in Section 5 below. Prototypical phenomena encoded by the above system are those in which diffusion happens on a much slower time scale with respect to that of mechanical equilibration. We hence keep track here of viscous effects in the nutrient dynamics. For completeness, we mention that an alternative modeling choice would be to replace (1.10) by a quasistatic evolution of the nutrient as well. This latter scenario could still be included in our analysis at the mathematical price of introducing a further nonlocality in the dependence of M on μ.

The paper is organized as follows. In Section 2 we introduce the precise mathematical framework and state our main results. Section 3 is devoted to the proof of Theorem 2.2, in which concentration of nutrients is neglected. This latter dependence is accounted for in the control problem formulated in Theorem 2.4 whose proof is the subject of Section 4. Eventually, the full nutrient-morphoelastic evolution problem is analyzed in Section 5.

2. Setting and main results

We devote this section to making assumptions precise and stating our existence results. As anticipated in the Introduction, we let the reference configuration of the body $\Omega \subset \mathbb{R}^3$ be nonempty, open, simply connected, bounded, and smooth and $\Gamma_{\rm N}$, $\Gamma_{\rm D} \subset \partial\Omega$, with $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$ open in the topology of $\partial\Omega$ and disjoint, $\Gamma_{\rm D} \neq \emptyset$, and $\overline{\Gamma}_{\rm D} \cup \overline{\Gamma}_{\rm N} = \partial\Omega$.

Throughout the paper, $GL_+(3)$ and $SO(3)$ denote the general linear group and the set of proper rotations, i.e.,

$$GL_+(3) = \{A \in \mathbb{R}^{3\times3} : \det A > 0\}, \quad SO(3) = \{A \in \mathbb{R}^{3\times3} : \det A = 1,\ A^\top A = \mathrm{Id}\},$$

where $\top$ denotes transposition and Id is the identity 2-tensor. Given the 2-tensors $A, B \in \mathbb{R}^{3\times3}$ and the 3-tensors $C, D \in \mathbb{R}^{3\times3\times3}$ we classically define $A : B, C : D \in \mathbb{R}$ and $C : B, B : C \in \mathbb{R}^3$ as (summation convention) $A : B := A_{ij}B_{ij}$, $C : D := C_{ijk}D_{ijk}$, $(C : B)_i := C_{ijk}B_{jk}$, and $(B : C)_i := B_{jk}C_{jki}$, respectively. The space of 2-tensors $\mathbb{R}^{3\times3}$ is endowed with the natural scalar product $A : B := \mathrm{tr}\,(A^\top B)$, where $\mathrm{tr}\,(A) := A_{ii}$ and corresponding norm $|A|^2 := A : A$. We note that this norm is submultiplicative, i.e., $|AB| \le |A||B|$. Similarly, we define the norm $|C|^2 := C : C$, the partial transposition $(C^t)_{ijk} := C_{jik}$, and the products $(CB)_{ijk} := C_{ijl}B_{lk}$ and $(BC)_{ijk} := B_{il}C_{ljk}$ so that $|CB|, |BC| \leq |B||C|$.

Furthermore, for $A, B : \Omega \to \mathbb{R}^{3\times 3}$ differentiable, the gradient ∇A is a 3-tensor reading $(\nabla A)_{ijk} := A_{ij,k}$ and it holds that $\nabla(AB) = (B^\top \nabla A^\top)^t + A\nabla B$.

We will denote by id the identity map $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}^3$. Given a time-dependent map $\psi(t)$, we indicate by $\psi'(t)$ its (possibly partial) derivative with respect to time.

In all of the following, we assume to be given

$$p > 3$$

and denote by $q = p/(p-1) < 3/2$ the corresponding conjugate exponent. Let us define the class of admissible deformations $\mathcal{Y}$ and admissible growth-tensor fields $\mathcal{G}_\infty$ as

$$\mathcal{Y} := \{y \in W^{1,p}(\Omega;\mathbb{R}^3) : \det \nabla y > 0 \ \text{ a.e. in } \Omega,\ y = \mathrm{id} \ \text{ on } \Gamma_{\mathrm{D}}\}, \tag{2.1}$$

$$\mathcal{G}_\infty := \{G \in W^{1,\infty}(\Omega;\mathbb{R}^{3\times 3}) : \det G > 0 \ \text{ a.e. in } \Omega\}. \tag{2.2}$$

The prescription on the a.e. positivity of $\det G$ in $\mathcal{G}_\infty$ is intended to guarantee that G is not degenerate and is orientation preserving.

For a given growth tensor field $G(t) \in \mathcal{G}_\infty$, the variational formulation of the quasistatic equilibrium system (1.1)-(1.3) corresponds to the minimization on $\mathcal{Y}$ of the *total elastic energy*

$$\mathcal{E}(y, G(t)) := \int_\Omega W(\nabla y(x) G^{-1}(t,x)) \det G(t,x)\, \mathrm{d}x - \langle \ell(t), y\rangle, \tag{2.3}$$

where we have indicated by $\ell(t) \in (W^{1,p}(\Omega;\mathbb{R}^3))'$ (dual) the *generalized load*

$$\langle \ell(t), y\rangle := \int_\Omega f(t)\cdot y\, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g(t)\cdot y\, \mathrm{d}\mathcal{H}^2.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(W^{1,p}(\Omega;\mathbb{R}^3))'$ and $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. The explicit occurrence of $\det G(t,x)$ in the total elastic energy is a consequence of the fact that the integration is taken with respect to the pre-growth reference configuration Ω [17, 24].

Above, $W : \mathbb{R}^{3\times 3} \to [0,\infty]$ denotes the *elastic energy density*. We assume that $W \in C^1(GL_+(3))$, $W \equiv \infty$ on $\mathbb{R}^{3\times 3} \setminus GL_+(3)$ and that it satisfies the following standard hypotheses:

(H1) (polyconvexity) $\exists \widehat{W} : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R} \to [0,\infty]$ convex and such that

$$W(A) = \widehat{W}(A, \mathrm{cof}\, A, \det A) \quad \forall\, A \in GL_+(3).$$

(H2) (coercivity and control) $\exists c_1,\ c_2 > 0$ such that

$$W(A) \geq c_1|A|^p - \frac{1}{c_1} \quad \text{and} \quad |A^\top \partial_A W(A)| \leq c_2(W(A)+1) \quad \forall\, A \in GL_+(3).$$

The second assumption in **(H2)** prescribes the controllability of the *Mandel tensor* $A^\top \partial_A W(A)$ via the energy [4, 5] and turns out to be particularly relevant in connection with finite-strain elastoplasticity [10, 19, 23]. Note that assumptions **(H1)-(H2)** are compatible with frame-indifference, namely,

$$W(RA) = W(A) \quad \forall\, R \in SO(3),\ \forall\, A \in GL_+(3).$$

We will check below that $\det G > 0$ for all times. As W is unbounded out of $GL_+(3)$ only, we hence have that $\det \nabla y > 0$ for a.e. times, as soon as the energy is finite.

Concerning body forces and traction we assume

(H3) $f \in W^{1,1}(0,T; L^1(\Omega;\mathbb{R}^3))$ and $g \in W^{1,1}(0,T; L^1(\Gamma_N;\mathbb{R}^3))$.

Note in particular, that **(H3)** entails

$$\ell \in W^{1,1}(0,T;(W^{1,p}(\Omega;\mathbb{R}^3))').$$

Let us recall that the evolution of $G(t)$ is driven by relation (1.7), featuring the convolution term $K\nabla y$. To this effect, we specify

(H4) $\kappa \in W^{1,1}(0,T)$ and $\phi \in W^{1,q}(\mathbb{R}^3)$

and define the operator K as

$$(K\psi)(t,x) := (\kappa * (\phi \star \psi))(t,x) \quad \forall\, \psi \in L^1((0,T)\times\mathbb{R}^3;\mathbb{R}^{3\times 3}),$$

where $*$ and $\star$ denote the standard convolution products on $(0,t)$ and $\mathbb{R}^3$, respectively. Namely,

$$(\kappa * \psi)(t,\cdot) := \int_0^t \kappa(t-s)\psi(s,\cdot)\,\mathrm{d}s \qquad \text{for } t\in(0,T)$$

and

$$(\phi \star \psi)(\cdot,x) := \int_{\mathbb{R}^3} \phi(x-z)\psi(\cdot,z)\,\mathrm{d}z \qquad \text{for } z\in\mathbb{R}^3.$$

By applying K to (components of) functions defined on Ω only, we actually consider the corresponding trivial extensions to zero to the whole $\mathbb{R}^3$, without introducing new notation. As regards the initial values, we assume

(H5) $G^0 \in \mathcal{G}_\infty$, $\det G^0 \geq \delta$ a.e. for some $\delta>0$, and $y^0 \in \arg\min_{\mathcal{Y}}\mathcal{E}(\cdot,G^0)$.

Note that we will prove in Lemma 3.2 below that such a minimizer y^0 exists for all $G^0\in\mathcal{G}_\infty$.

We will work under the following regularity of the growth-rate function

(H6) $M \in W^{1,\infty}(\mathbb{R}^{3\times3}\times\mathbb{R}^{3\times3};\mathbb{R}^{3\times3})$.

From assumptions **(H5)** and **(H6)**, for all $t\mapsto G(t)$ solving (1.7) it follows that $\det G(t)>0$ a.e. in Ω, $\forall\, t\in[0,T]$. Indeed, by the Jacobi formula and equation (1.7) we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\det G(t) = \det G(t)\,\mathrm{tr}\,M\big(G(t),(K\nabla y)(t)\big).$$

Solving this ODE gives

$$\det G(t) = \det G^0\,\exp\left(\int_0^t \mathrm{tr}\,M\big(G(s),(K\nabla y)(s)\big)\,\mathrm{d}s\right). \tag{2.4}$$

Using **(H5)** and estimating the integrand above yields

$$\det G(t) \geq \det G^0\,\exp\big(-3T\,\|M\|_{L^\infty}\big) > 0 \qquad \text{a.e. in } \Omega,\ \forall\, t\in[0,T],$$

where, here and in the rest of the paper, we use the short-hand $\|\cdot\|_{L^\infty}$ to identify any L^∞ norm, in this case $\|\cdot\|_{L^\infty(\mathbb{R}^{3\times3}\times\mathbb{R}^{3\times3};\mathbb{R}^{3\times3})}$. This lower bound on $\det G(t)$ will turn out crucial in combination with the coercivity in **(H2)** in order to prove the coercivity of the total elastic energy $\mathcal{E}$.

Definition 2.1 (Morphoelastic solution)**.** *We say that $(y,G):[0,T]\to\mathcal{Y}\times\mathcal{G}_\infty$ is a* morphoelastic solution *if*

$$y(t) \in \arg\min\nolimits_{y\in\mathcal{Y}}\mathcal{E}(y,G(t)) \quad \textit{for a.e. } t\in(0,T), \tag{2.5}$$

$$G'(t)G^{-1}(t) = M(G(t),(K\nabla y)(t)) \quad \textit{a.e. in } \Omega,\ \textit{for a.e. } t\in(0,T), \tag{2.6}$$

$$(y(0),G(0)) = (y^0,G^0) \quad \textit{a.e. in } \Omega. \tag{2.7}$$

Our basic existence result is the following.

Theorem 2.2 (Morphoelastic existence)**.** *Under assumptions* **(H1)**-**(H6)** *there exists a morphoelastic solution* $(y,G)\in L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3))\times L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^{3\times3}))\cap W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3}))$.

The proof of Theorem 2.2 is in Section 3 below.

Let us now turn to the case where the growth dynamics is influenced by the given nutrient field $\mu : [0,T] \times \Omega \to \mathbb{R}$. To this aim, the growth-rate function M has to be modified by including an additional dependence on the nutrient field μ. Also in this extended case, we assume M to be Lipschitz continuous, namely, we modify **(H6)** as

(H7) $M \in W^{1,\infty}(\mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R}; \mathbb{R}^{3\times 3})$.

Correspondingly, we specify the class of admissible growth tensor fields as

$$\mathcal{G}_p := \{G \in W^{1,p}(\Omega; \mathbb{R}^{3\times 3}) : \det G > 0 \quad \text{a.e. in } \Omega\}.$$

One can define the following.

Definition 2.3 (Nutrient-driven morphoelastic solution). *Assume to be given $\mu \in L^p(0,T; W^{1,p}(\Omega))$. We say that $(y, G) : [0,T] \to \mathcal{Y} \times \mathcal{G}_p$ is a* nutrient-driven morphoelastic solution given μ *if*

$$y(t) \in \arg\min_{y\in\mathcal{Y}} \mathcal{E}(y, G(t)) \quad \textit{for a.e. } t \in (0,T), \tag{2.8}$$

$$G'(t)G^{-1}(t) = M(G(t), (K\nabla y)(t), \mu(t)) \quad \textit{a.e. in } \Omega, \ \textit{for a.e. } t \in (0,T), \tag{2.9}$$

$$(y(0), G(0)) = (y^0, G^0) \quad \textit{a.e. in } \Omega. \tag{2.10}$$

In Section 4 we check that the existence result of Theorem 2.2 can be readily extended to include the nutrient-driven case. In particular, one can define a possibly set-valued solution operator

$$\begin{aligned} S : &\, L^p(0,T; W^{1,p}(\Omega)) \\ &\to L^\infty(0,T; W^{1,p}(\Omega;\mathbb{R}^3)) \times W^{1,\infty}(0,T; L^\infty(\Omega;\mathbb{R}^{3\times 3})) \cap W^{1,p}(0,T; W^{1,p}(\Omega;\mathbb{R}^{3\times 3})) \end{aligned}$$

defining the set $S(\mu)$ of all nutrient-driven morphoelastic solutions (y, G) given μ, according to Definition 2.3. One can hence use the solution operator S to specify the optimal control problem

$$\min_{\mu\in\mathcal{A}} \big\{ J(y, G, \mu) \ : \ (y, G) \in S(\mu) \big\}. \tag{2.11}$$

Here, $\mathcal{A} \subset L^p(0,T; W^{1,p}(\Omega))$ is the set of admissible controls μ. We assume that

(H8) $\mathcal{A}$ is bounded in $L^p(0,T; W^{1,p}(\Omega))$ and compact in $L^1((0,T)\times\Omega)$.
(H9) $J : L^\infty(0,T; W^{1,p}(\Omega;\mathbb{R}^3)) \times C([0,T]; L^\infty(\Omega;\mathbb{R}^{3\times 3})) \times L^p(0,T; W^{1,p}(\Omega)) \to [0,\infty]$ is lower semicontinuous with respect to the corresponding weak* topology.

As already mentioned in the Introduction, this assumption on J allows flexibility with respect to the possible choices for J. These include, in particular,

$$J(y, G, \mu) = \beta_1 \int_\Omega \det G(T)\,\mathrm{d}x + \beta_2 \int_0^T \int_\Omega |y - y_{\mathrm{target}}|^p \,\mathrm{d}x\,\mathrm{d}t + \beta_3 \int_0^T \int_\Omega |\mu|^p \,\mathrm{d}x\,\mathrm{d}t,$$

which, together with incompressibility (i.e., $\det \nabla y = 1$), would correspond to a weighted combination (β_1, β_2, $\beta_3 \geq 0$) of final volume minimization, attainment of a given *target* deformation $y_{\mathrm{target}} \in L^\infty(0,T, W^{1,p}(\Omega;\mathbb{R}^3))$, and minimization of the amount of provided nutrient (in connection with tumor growth, one may think here of a chemotherapy drug).

Solutions to the optimal control problem (2.11) are optimal controls μ^* and corresponding optimal pairs $(y^*, G^*) \in S(\mu^*)$. Our next result guarantees that these exist.

Theorem 2.4 (Existence of optimal controls). *Under assumptions* **(H1)**-**(H5)**, **(H7)**-**(H9)** *the solution operator S is well-defined and the optimal control problem* (2.11) *admits a solution (y^*, G^*, μ^*).*

Theorem 2.4 is proved in Section 4.

A further extension of the model corresponds to considering the driving nutrient field to be unknown and to evolve together with the mechanical variables, as effect of system (1.10)-(1.12). To this aim, we introduce the class of admissible nutrient concentrations as

$$\mathcal{M} := W^{2,p}(\Omega)$$

and qualify boundary and initial data as

(H10) $\mu_{\rm D} \in W^{1,\infty}(0,T;L^p(\Omega)) \cap L^\infty(0,T;W^{2,p}(\Omega))$ and $\mu^0 \in \mathcal{M}$.

The nutrient source and consumption terms in the right-hand side of equation (1.10) are provided via

(H11) $h \in L^\infty(0,T;L^p(\Omega))$ and $H \in W^{1,\infty}(\mathbb{R}^3)$.

Note in particular that we have $H(\kappa * y) \in L^\infty(0,T;L^p(\Omega))$ whenever $y \in L^1(0,T;L^p(\Omega;\mathbb{R}^3))$. We are now ready to define our concept of solution of the fully coupled system.

Definition 2.5 (Nutrient-morphoelastic solution). *We say that* $(y,G,\mu) : [0,T] \to \mathcal{Y} \times \mathcal{G}_p \times \mathcal{M}$ *is a* nutrient-morphoelastic solution *if*

$$y(t) \in \arg\min_{y\in\mathcal{Y}} \mathcal{E}(y,G(t)) \quad \textit{for a.e.} \quad t \in (0,T), \tag{2.12}$$

$$G'(t)G^{-1}(t) = M(G(t),(K\nabla y)(t),\mu(t)) \quad \textit{a.e. in} \quad \Omega, \quad \textit{for a.e.} \quad t \in (0,T), \tag{2.13}$$

$$\mu'(t) - \nu\Delta\mu(t) = h(t) - H((\kappa * y)(t)) \quad \textit{a.e. in} \quad \Omega, \quad \textit{for a.e.} \quad t \in (0,T), \tag{2.14}$$

$$\mu(t) = \mu_{\rm D}(t) \quad \textit{a.e. on } \partial\Omega, \quad \textit{for a.e.} \quad t \in (0,T), \tag{2.15}$$

$$(y(0),G(0),\mu(0)) = (y^0,G^0,\mu^0) \quad \textit{a.e. in } \Omega. \tag{2.16}$$

We are eventually in the position of presenting an existence result for nutrient-morphoelastic solutions.

Theorem 2.6 (Nutrient-morphoelastic existence). *Under assumptions* **(H1)**-**(H5)**, **(H7)**, *and* **(H10)**-**(H11)** *there exists a nutrient-morphoelastic solution* $(y,G,\mu) \in L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3)) \times W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3})) \cap W^{1,p}(0,T;W^{1,p}(\Omega;\mathbb{R}^{3\times3})) \times W^{1,\infty}(0,T;L^p(\Omega)) \cap L^\infty(0,T;W^{2,p}(\Omega))$.

Theorem 2.6 is proved in Section 5.

3. Proof of Theorem 1: Morphoelastic existence

This section is devoted to prove the existence of morphoelastic solutions, namely, trajectories $t \in [0,T] \mapsto (y(t),G(t)) \in \mathcal{Y} \times \mathcal{G}_\infty$ fulfilling (2.5)-(2.7). We argue by time-discretization: we obtain time-discrete solutions, prove a-priori estimates for the piecewise affine and backward piecewise constant time-discrete interpolants, and eventually pass to the limit as the time step converges to zero. For convenience of the reader, each of the above steps is associated to a corresponding subsection.

In order to shorten notation, from here on we use the symbols $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{W^{1,\infty}}$ to indicate generic L^∞ and $W^{1,\infty}$ norms, without explicitly specifying dependencies.

3.1. Time discretization.

We consider a uniform partition $\{0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T\}$, $N \in \mathbb{N}$, $t_i = i\tau$, $\tau > 0$, of the time interval $[0,T]$. Given any vector $\{z_i\}_{i=0}^N$, we will denote by $\hat{z}_\tau$ and $\bar{z}_\tau$ the corresponding piecewise affine and backward piecewise constant interpolants associated to the partition. Namely,

$$\hat{z}_\tau(0) := z_0, \quad \hat{z}_\tau(t) := \alpha_i(t)z_i + (1-\alpha_i(t))z_{i-1}, \tag{3.1}$$

$$\bar{z}_\tau(0) := z_0, \quad \bar{z}_\tau(t) := z_i \qquad \text{for } t \in ((i-1)\tau, i\tau], \ i = 1,\ldots,N, \tag{3.2}$$

where $\alpha_i(t) := (t-(i-1)\tau)/\tau$ for $t \in ((i-1)\tau, i\tau]$, $i = 1, \dots, N$. Setting $\ell_i := \ell(t_i)$ for $i = 0, \dots, N$, we obtain a discrete solution $\{(y_i, G_i)\}_{i=1}^N \in \mathcal{Y}^N \times \mathcal{G}_\infty^N$ by recursively solving

$$y_i \in \operatorname{argmin}_{y \in \mathcal{Y}} \left\{ \int_\Omega W(\nabla y\, G_i^{-1}) \det G_i \, \mathrm{d}x - \langle \ell_i, y \rangle \right\}, \tag{3.3}$$

$$G_i = \exp\left(\tau\, M(G_{i-1}, (K_\tau \nabla y)_{i-1})\right) G_{i-1} \quad \text{a.e. in } \Omega, \tag{3.4}$$

for $i = 1, \dots, N$, starting from the initial data $(y_0, G_0) = (y^0, G^0) \in \mathcal{Y} \times \mathcal{G}_\infty$ with $\det G^0 \geq \delta > 0$ a.e. As already mentioned, this *exponential-update* scheme is designed to reproduce at the discrete level the nonlinear geometry of the differential system (2.6), in particular the nondegeneracy property (2.4).

In (3.4), the operator $(K_\tau \nabla y)_{i-1}$ is given by

$$(K_\tau \nabla y)_{i-1}(x) := (\kappa *_\tau (\phi \star \nabla y))_{i-1}(x) \quad \text{for a.e. } x \in \Omega, \tag{3.5}$$

where the time discrete convolution $*_\tau$ is defined as

$$(\kappa *_\tau \nabla y)_{i-1} := \sum_{j=0}^{i-1} \tau\, \kappa_j\, \nabla y_{i-1-j} \quad \text{for } i = 1, \dots, N$$

with $\kappa_i := \kappa(t_i)$, $i = 0, \dots, N$, see [27]. In order to prove the existence of discrete solutions, we start by considering the equilibrium problem in the following lemma.

Lemma 3.1 (Equilibrium problem). *Under assumptions* **(H1)**–**(H2)** *for every* $G \in L^\infty(\Omega; \mathbb{R}^{3\times 3})$ *with* $\det G \geq \eta > 0$ *a.e. for some* $\eta > 0$*, and for every* $\ell \in (W^{1,p}(\Omega; \mathbb{R}^3))'$ *there exists* $y \in \mathcal{Y}$ *solving*

$$\int_\Omega W(\nabla y\, G^{-1}) \det G \, \mathrm{d}x - \langle \ell, y \rangle \leq \int_\Omega W(\nabla \hat{y}\, G^{-1}) \det G \, \mathrm{d}x - \langle \ell, \hat{y} \rangle \quad \forall \hat{y} \in \mathcal{Y}. \tag{3.6}$$

Proof. Since $W \in C^1(GL_+(3))$, for all η, $g > 0$ we have that

$$\lambda(g, \eta) := \max\{W(A^{-1}) \ : \ A \in GL_+(3),\ |A| \leq g,\ \det A \geq \eta\} < \infty. \tag{3.7}$$

Recalling that $\mathrm{id} \in \mathcal{Y}$ we get

$$\begin{aligned} \inf_{y \in \mathcal{Y}} \left\{ \int_\Omega W(\nabla y\, G^{-1}) \det G \, \mathrm{d}x - \langle \ell, y \rangle \right\} &\leq \int_\Omega W(G^{-1}) \det G \, \mathrm{d}x - \langle \ell, \mathrm{id} \rangle \\ &\leq 6|\Omega| \lambda(\|G\|_{L^\infty}, \eta) \|G\|_{L^\infty}^3 + \|\ell\|_{(W^{1,p}(\Omega;\mathbb{R}^3))'} \|\mathrm{id}\|_{W^{1,p}(\Omega;\mathbb{R}^3)} < \infty. \end{aligned} \tag{3.8}$$

Owing to the coercivity from **(H2)**, every minimizing sequence $\{y_k\} \subset \mathcal{Y}$ fulfills

$$\begin{aligned} &\eta \|\nabla y_k\|_{L^p(\Omega;\mathbb{R}^{3\times3})}^p \leq \eta \|\nabla y_k\, G^{-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3})}^p \|G\|_{L^\infty}^p \leq \frac{\|G\|_{L^\infty}^p}{c_1} \int_\Omega W(\nabla y_k\, G^{-1}) \det G \, \mathrm{d}x + \frac{\eta |\Omega| \|G\|_{L^\infty}^p}{c_1^2} \\ &\quad \leq \frac{\|G\|_{L^\infty}^p}{c_1} \left(\int_\Omega W(G^{-1}) \det G \, \mathrm{d}x + \langle \ell, y_k - \mathrm{id} \rangle \right) + \frac{\eta |\Omega| \|G\|_{L^\infty}^p}{c_1^2} \\ &\quad \leq \frac{\|G\|_{L^\infty}^p}{c_1} \left(6|\Omega| \lambda(\|G\|_{L^\infty}, \eta) \|G\|_{L^\infty}^3 + \|\ell\|_{(W^{1,p}(\Omega;\mathbb{R}^3))'} \|y_k - \mathrm{id}\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \right) + \frac{\eta |\Omega| \|G\|_{L^\infty}^p}{c_1^2} \\ &\quad \leq C(1 + \|\nabla y_k\|_{L^p(\Omega;\mathbb{R}^{3\times3})}) \end{aligned}$$

for all $k \in \mathbb{N}$, where in the second-to-last inequality we have used (3.8), and where the last inequality follows by **(H3)** and the Poincaré inequality. The constant $C > 0$ depends on η, c_1, $|\Omega|$, $\|G\|_{L^\infty}$, $\lambda(\|G\|_{L^\infty}, \eta)$, and $\|\ell\|_{(W^{1,p}(\Omega;\mathbb{R}^3))'}$. In particular, by the definition of $\mathcal{Y}$ we infer that $\{y_k\}$ is a bounded sequence in $W^{1,p}(\Omega; \mathbb{R}^3)$, so that there exists $y \in \mathcal{Y}$ such that

$$y_k \rightharpoonup y \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^3) \tag{3.9}$$

for some not relabeled subsequence. Since $G^{-1} \in L^\infty(\Omega;\mathbb{R}^{3\times 3})$, we deduce that

$$\nabla y_k G^{-1} \rightharpoonup \nabla y\, G^{-1} \quad \text{in } L^p(\Omega;\mathbb{R}^{3\times 3}). \tag{3.10}$$

The weak continuity of the minors of ∇y_k [3] for $p > 3$ and (3.10) yield

$$\begin{aligned}
\operatorname{cof}(\nabla y_k\, G^{-1}) &\rightharpoonup \operatorname{cof}(\nabla y\, G^{-1}) \quad \text{in } L^{p/2}(\Omega;\mathbb{R}^{3\times 3}),\\
\det(\nabla y_k\, G^{-1}) &\rightharpoonup \det(\nabla y\, G^{-1}) \quad \text{in } L^{p/3}(\Omega),
\end{aligned}$$

which, combined with the polyconvexity from **(H1)**, imply

$$\int_\Omega W(\nabla y\, G^{-1}) \det G \,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega W(\nabla y_k\, G^{-1}) \det G \,\mathrm{d}x. \tag{3.11}$$

By (3.9) we have $\langle \ell, y_k\rangle \to \langle \ell, y\rangle$, which, together with (3.11), leads to the minimality (3.6). □

Owing to Lemma 3.1, for every $G^0 \in \mathcal{G}_\infty$ with $\det G^0 \ge \delta > 0$ a.e. there exists $y^0 \in \mathcal{Y}$ solving (3.3) for $i = 0$. The existence of discrete solutions is then guaranteed by the following lemma.

Lemma 3.2 (Discrete existence)**.** *Under assumptions* **(H1)**–**(H3)**, **(H5)**–**(H6)**, *let* $(y_0, G_0) = (y^0, G^0)$, *where* y^0 *solves* (3.6) *for* $G = G^0$ *and* $\ell = \ell_0$. *For every* $i = 1, \dots, N$, *there exists* $(y_i, G_i) \in \mathcal{Y} \times \mathcal{G}_\infty$ *solving* (3.3)–(3.4) *with* $\det G_i \ge \exp(-3\tau i \|M\|_{L^\infty})\delta$ *a.e.*

Proof. We proceed by induction on i. Assume that there exist $G_{i-1} \in \mathcal{G}_\infty$ with $\det G_{i-1} \ge \exp(-3\tau(i-1)\|M\|_{L^\infty})\delta$ a.e. and define G_i via position (3.4). Let us check that $G_i \in W^{1,\infty}(\Omega;\mathbb{R}^{3\times 3})$, and that $\det G_i \ge \exp(-3\tau i\|M\|_{L^\infty})\delta$ a.e. in Ω, for $i = 1, \dots, N$. By differentiating (3.4), we find

$$\nabla G_i = (G_{i-1}^\top \nabla \exp\big(\tau\, M(G_{i-1}, (K_\tau \nabla y)_{i-1})\big)^\top)^t + \exp\big(\tau\, M(G_{i-1}, (K_\tau \nabla y)_{i-1})\big)\nabla G_{i-1} \tag{3.12}$$

a.e. in Ω, for $i = 1, \dots, N$, where the above equality holds in the sense of distributions. In view of **(H6)** and by the fact that $G_{i-1} \in \mathcal{G}_\infty$, we infer the estimate

$$\begin{aligned}
\|\nabla G_i\|_{L^\infty} &\le \|G_{i-1}^\top \nabla \exp\big(\tau\, M(G_{i-1}, (K_\tau \nabla y)_{i-1})\big)^\top\|_{L^\infty}\\
&\quad + \|\exp\big(\tau\, M(G_{i-1}, (K_\tau \nabla y)_{i-1})\big)\nabla G_{i-1}\|_{L^\infty}\\
&\le \|G_{i-1}\|_{L^\infty}\|\nabla \exp\big(\tau\, M(G_{i-1}, (K_\tau \nabla y)_{i-1})\big)\|_{L^\infty}\\
&\quad + \|\exp\big(\tau\, M(G_{i-1}, (K_\tau \nabla y)_{i-1})\big)\|_{L^\infty}\|\nabla G_{i-1}\|_{L^\infty}\\
&\le C\left(1 + \|\nabla \exp\big(\tau\, M(G_{i-1}, (K_\tau \nabla y)_{i-1})\|_{L^\infty}\right)
\end{aligned}$$

with C depending on $\|M\|_{L^\infty}$ and $\|G_{i-1}\|_{W^{1,\infty}}$. Denoting by $D_1 M$ and $D_2 M$ the differentials of M with respect to its first and second matrix-valued variables, respectively, we obtain from the properties of the matrix exponential that

$$
\begin{aligned}
&|\nabla \exp\big(\tau\, M(G_{i-1}, (K_\tau \nabla y)_{i-1})\big)| \\
&\quad = \tau \Big| \int_0^1 \exp\Big((1-s)\tau\, M\big(G_{i-1}, (K_\tau \nabla y)_{i-1}\big)\Big)\, \nabla M(G_{i-1}, (K_\tau \nabla y)_{i-1}) \\
&\qquad\qquad \times \exp\Big(s\tau\, M\big(G_{i-1}, (K_\tau \nabla y)_{i-1}\big)\Big)\, \mathrm{d}s \Big| \\
&\quad \le \tau \int_0^1 \exp\Big((1-s)\tau \big|M\big(G_{i-1}, (K_\tau \nabla y)_{i-1}\big)\big|\Big)\, \exp\Big(s\tau \big|M\big(G_{i-1}, (K_\tau \nabla y)_{i-1}\big)\big|\Big)\, \mathrm{d}s \\
&\qquad\qquad \times \big|\nabla M\big(G_{i-1}, (K_\tau \nabla y)_{i-1}\big)\big| \\
&\quad = \tau \exp\Big(\tau \big|M\big(G_{i-1}, (K_\tau \nabla y)_{i-1}\big)\big|\Big)\, \Big| D_1 M\big(G_{i-1}, (K_\tau \nabla y)_{i-1}\big)\, \nabla G_{i-1} \\
&\qquad\qquad\qquad\qquad + D_2 M\big(G_{i-1}, (K_\tau \nabla y)_{i-1}\big)\, \nabla (K_\tau \nabla y)_{i-1} \Big| \\
&\quad \le 2\tau \exp\big(\tau \|M\|_{L^\infty}\big) \|M\|_{W^{1,\infty}} \Big(|\nabla G_{i-1}| + |\nabla (K_\tau \nabla y)_{i-1}|\Big),
\end{aligned} \tag{3.13}
$$

so that, using once again **(H6)** and the fact that $G_{i-1} \in \mathcal{G}_\infty$ we have that

$$
\|\nabla G_i\|_{L^\infty} \le C\tau(1 + \|\nabla (K_\tau \nabla y)_{i-1}\|_{L^\infty}) + C,
$$

where C depends on $\|M\|_{W^{1,\infty}}$, and $\|G_{i-1}\|_{W^{1,\infty}}$. Now, by (3.5) and **(H4)**,

$$
\|\nabla (K_\tau \nabla y)_{i-1}\|_{L^\infty} \le \sum_{j=0}^{i-1} \tau |\kappa_j| \|\nabla y_{i-1-j}\|_{L^p(\Omega;\mathbb{R}^{3\times3})} \|\nabla \phi\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)} \le C \tag{3.14}
$$

owing to the fact that $y_j \in \mathcal{Y}$ for $j = 0, \ldots, i-1$. This yields that $G_i \in W^{1,\infty}(\Omega; \mathbb{R}^{3\times3})$.

The lower bound on the determinant of G_i follows by induction, namely,

$$
\det G_i = \exp\big(\tau \operatorname{tr} M(G_{i-1}, (K_\tau \nabla y)_{i-1})\big) \det G_{i-1} \ge \exp\big(-3\tau \|M\|_{L^\infty}\big) \det G_{i-1} \tag{3.15}
$$

$$
\ge \exp\big(-3\tau \|M\|_{L^\infty}\big) \exp\big(-3\tau(i-1) \|M\|_{L^\infty}\big)\delta = \exp\big(-3\tau i \|M\|_{L^\infty}\big)\delta \tag{3.16}
$$

a.e. in Ω, for $i = 1, \ldots, N$.

We can hence conclude the proof by applying Lemma 3.1 for $G = G_i$ and $\ell = \ell_i$ in order to find a deformation $y_i \in \mathcal{Y}$ solving (3.3). □

3.2. **A-priori estimates.** Denoting by $\overline{(K_\tau \nabla y)}_\tau$ and $\widehat{(K_\tau \nabla y)}_\tau$ the backward piecewise constant and piecewise affine interpolants associated to $K_\tau \nabla y$, cf. (3.1), (3.2), and (3.5), the main result of this subsection is the following.

Proposition 3.3 (A-priori estimates)**.** *There exist $\tau^* \in (0,1)$ depending on $\|M\|_{L^\infty}$ such that, for every $\tau \in (0, \tau^*)$ we have*

$$
\|\widehat{G}_\tau\|_{W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3}))} \le C, \tag{3.17}
$$

$$
\det \overline{G}_\tau(t) \ge C \det G_0 \ge C\delta \quad \textit{a.e. in } \Omega, \ \forall\, t \in [0,T], \tag{3.18}
$$

$$
\|\overline{y}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3))} + \|\widehat{y}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3))} \le C, \tag{3.19}
$$

$$
\|\overline{G}_\tau\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^{3\times3}))} + \|\widehat{G}_\tau\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^{3\times3}))} \le C, \tag{3.20}
$$

$$
\|\overline{(K_\tau \nabla y)}_\tau\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^{3\times3}))} \le C, \tag{3.21}
$$

$$
\|\widehat{(K_\tau \nabla y)}_\tau\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^{3\times3}))} + \|\widehat{(K_\tau \nabla y)}_\tau\|_{W^{1,\infty}(0,T;L^\infty(\Omega;\ \mathbb{R}^{3\times3}))} \le C, \tag{3.22}
$$

where the positive constant C depends on c_1, Ω, $\|G^0\|_{W^{1,\infty}}$, $\|M\|_{W^{1,\infty}}$, $\|\kappa\|_{W^{1,1}(0,T)}$, $\|\phi\|_{W^{1,q}(\mathbb{R}^3)}$, $\|\ell\|_{W^{1,1}(0,T;(W^{1,p}(\Omega;\mathbb{R}^3))')}$, and T.

Proof. Within this proof, the symbol C stands for a positive constant, possibly depending on c_1, Ω, $\|G^0\|_{W^{1,\infty}}$, $\|M\|_{W^{1,\infty}}$, $\|\kappa\|_{W^{1,1}(0,T)}$, $\|\phi\|_{W^{1,q}(\mathbb{R}^3)}$, $\|\ell\|_{W^{1,1}(0,T;(W^{1,p}(\Omega;\mathbb{R}^3))')}$, and T but independent of τ. The actual value of C can change from line to line.

We first show estimate (3.17). We subtract G_{i-1} from both sides of (3.4), divide by τ, and contract with G_i. This yields

$$\frac{G_i - G_{i-1}}{\tau} : G_i = \frac{\Big(\exp\big(\tau\, M(G_{i-1},(K_\tau\nabla y)_{i-1})\big) - \mathrm{Id}\Big)G_{i-1}}{\tau} : G_i \tag{3.23}$$

a.e. in Ω, for $i = 1,\dots,N$. The left-hand side can be treated as

$$\frac{G_i - G_{i-1}}{\tau} : G_i = \frac{|G_i|^2}{2\tau} + \frac{|G_i - G_{i-1}|^2}{2\tau} - \frac{|G_{i-1}|^2}{2\tau},$$

whereas for the right-hand side of (3.23) we use the Cauchy-Schwarz and the Young inequalities, as well as the properties of the matrix exponential in order to get that

$$\begin{aligned}
&\frac{\Big(\exp\big(\tau\, M(G_{i-1},(K_\tau\nabla y)_{i-1})\big) - \mathrm{Id}\Big)G_{i-1}}{\tau} : G_i \\
&\quad\le \left|\frac{\exp\big(\tau\, M(G_{i-1},(K_\tau\nabla y)_{i-1})\big) - \mathrm{Id}}{\tau}\right| |G_{i-1}|\,|G_i| \\
&\quad\le \frac{\exp\big(\tau\,|M(G_{i-1},(K_\tau\nabla y)_{i-1})|\big) - 1}{\tau} |G_{i-1}|\,|G_i| \\
&\quad\le \frac{\exp\big(\tau\,\|M\|_{L^\infty}\big) - 1}{\tau}\left(\frac{|G_{i-1}|^2}{2} + \frac{|G_i|^2}{2}\right)
\end{aligned}$$

where we have also used the fact that $|\exp A - \mathrm{Id}| \le \exp(|A|) - 1$. Therefore, we have obtained that

$$\frac{|G_i|^2}{2\tau} + \frac{|G_i - G_{i-1}|^2}{2\tau} - \frac{|G_{i-1}|^2}{2\tau} \le \frac{\exp\big(\tau\,\|M\|_{L^\infty}\big) - 1}{\tau}\left(\frac{|G_{i-1}|^2}{2} + \frac{|G_i|^2}{2}\right).$$

Fix an integer $m \le N$. By multiplying by τ and summing up for $i = 1,\dots,m$, we deduce that

$$\frac{|G_m|^2}{2} - \frac{|G_0|^2}{2} \le \frac{1}{2}\big(\exp\big(\tau\,\|M\|_{L^\infty}\big) - 1\big)\big(|G_m|^2 + |G_0|^2\big) + \sum_{i=1}^{m-1}\big(\exp\big(\tau\,\|M\|_{L^\infty}\big) - 1\big)|G_i|^2.$$

Taking $\tau < (\log 2)/\|M\|_{L^\infty} =: \tau^*$ and applying the Discrete Gronwall Lemma, cf. [15, Proposition 2.2.1], we conclude that

$$|G_m| \le C, \quad \forall\, m = 1,\dots,N, \tag{3.24}$$

and hence

$$\widehat{G}_\tau \text{ and } \overline{G}_\tau \text{ are bounded in } L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{3\times 3})) \text{ independently of } \tau \in (0,\tau^*). \tag{3.25}$$

By subtracting G_{i-1} from both sides of (3.4), dividing by τ, and taking the norm, we get

$$\begin{aligned}
\left|\frac{G_i - G_{i-1}}{\tau}\right| &\le \left|\frac{\exp\big(\tau\, M(G_{i-1},(K_\tau\nabla y)_{i-1})\big) - \mathrm{Id}}{\tau}\right| |G_{i-1}| \\
&\le \frac{1}{\tau}\big(\exp\big(\tau\,\|M\|_{L^\infty}\big) - 1\big)|G_{i-1}| \quad \forall\, i = 1,\dots,N.
\end{aligned} \tag{3.26}$$

By (3.24), for $\tau < \tau^*$ we infer (3.17).

Next, we prove (3.18) and (3.19). Recalling the definition of $\mathcal{G}_\infty$, by subtracting G_{i-1} from both sides of (3.4), dividing by τ, taking the gradient, and contracting with ∇G_i, we obtain

$$\nabla\left(\frac{G_i - G_{i-1}}{\tau}\right) : \nabla G_i = \nabla(E_{i-1}G_{i-1}) : \nabla G_i \tag{3.27}$$

a.e. in Ω, for $i = 1, \dots, N$, where we have set

$$E_{i-1} := \frac{\exp\big(\tau\, M(G_{i-1}, (K_\tau \nabla y)_{i-1})\big) - \mathrm{Id}}{\tau}. \tag{3.28}$$

The left-hand side of (3.27) reads equivalently

$$\nabla\left(\frac{G_i - G_{i-1}}{\tau}\right) : \nabla G_i = \frac{|\nabla G_i|^2}{2\tau} + \frac{|\nabla G_i - \nabla G_{i-1}|^2}{2\tau} - \frac{|\nabla G_{i-1}|^2}{2\tau}, \tag{3.29}$$

while for the right-hand side, we compute

$$\begin{aligned}
\nabla(E_{i-1}G_{i-1}) : \nabla G_i &\le |\nabla(E_{i-1}G_{i-1})||\nabla G_i| = |(G_{i-1}^\top \nabla E_{i-1}^\top)^t + E_{i-1}\nabla G_{i-1}||\nabla G_i| \\
&\le \big(|G_{i-1}^\top \nabla E_{i-1}^\top| + |E_{i-1}\nabla G_{i-1}|\big)|\nabla G_i| \\
&\le \big(|G_{i-1}||\nabla E_{i-1}| + |E_{i-1}||\nabla G_{i-1}|\big)|\nabla G_i|.
\end{aligned} \tag{3.30}$$

From the properties of the matrix exponential, arguing as in (3.13), and from the estimate on the convolution (3.14) we deduce that

$$\begin{aligned}
&|\nabla E_{i-1}| \le 2\exp\big(\tau\|M\|_{L^\infty}\big)\|M\|_{W^{1,\infty}}\Big(|\nabla G_{i-1}| + |\nabla(K_\tau\nabla y)_{i-1}|\Big) \\
&\le 2\exp\big(\tau\|M\|_{L^\infty}\big)\|M\|_{W^{1,\infty}}\Big(|\nabla G_{i-1}| + \sum_{j=0}^{i-1}\tau|\kappa_j|\|\nabla y_{i-1-j}\|_{L^p(\Omega;\mathbb{R}^{3\times3})}\|\nabla\phi\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)}\Big).
\end{aligned} \tag{3.31}$$

On the other hand, by iterating (3.15), we find

$$\det G_m \ge \exp(-3\tau m\|M\|_{L^\infty})\det G_0 \ge \exp(-3T\|M\|_{L^\infty})\det G_0 \quad \forall\, m = 1, \dots, N,$$

which gives (3.18). Thus, **(H2)**, **(H6)**, (3.3), and the Poincaré inequality imply

$$\begin{aligned}
\|\nabla y_i\, G_i^{-1}\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})} &\le C\,(1 + \langle \ell_i, y_i\rangle) \le C\,\big(1 + \|\ell_i\|_{(W^{1,p}(\Omega;\mathbb{R}^3))'}\|y_i\|_{W^{1,p}(\Omega;\mathbb{R}^3)}\big) \\
&\le C\,\big(1 + \|\nabla y_i\|_{L^p(\Omega;\mathbb{R}^{3\times3})}\big) \\
&\le C\,\big(1 + \|\nabla y_i\, G_i^{-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3})}\|G_i\|_{L^\infty}\big)\,.
\end{aligned}$$

By using the Young Inequality and (3.25), we arrive at

$$\|\nabla y_i G_i^{-1}\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})} \le C\,\big(1 + \|G_i\|^q_{L^\infty}\big) \le C, \tag{3.32}$$

which in turn yields

$$\|\nabla y_i\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})} \le \|\nabla y_i\, G_i^{-1}\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})}\|G_i\|^p_{L^\infty} \le C \quad \forall\, i = 1, \dots, N. \tag{3.33}$$

In particular, we obtain the bound (3.19).

We now prove the bound (3.20). Going back to (3.31), from (3.19) and **(H4)** we infer the estimate

$$|\nabla E_{i-1}| \le C(1 + |\nabla G_{i-1}|).$$

On the other hand, the same computations as in (3.26) yield

$$|E_{i-1}| \le \frac{1}{\tau}\,(\exp(\tau\|M\|_{L^\infty}) - 1) \le C.$$

Therefore, by combining (3.27)–(3.30) and multiplying by τ, from (3.25) we conclude that

$$\frac{|\nabla G_i|^2}{2} + \frac{|\nabla G_i - \nabla G_{i-1}|^2}{2} - \frac{|\nabla G_{i-1}|^2}{2} \le C\tau\,(1 + |\nabla G_{i-1}|)\,|\nabla G_i|$$

for $i = 1, \dots, N$. Summing up for $i = 1, \dots, m$ for some $m = 1, \dots, N$ and using the Young Inequality we get

$$\frac{|\nabla G_m|^2}{2} - \frac{|\nabla G_0|^2}{2} \le \sum_{i=1}^m C\tau(1+|\nabla G_{i-1}|)|\nabla G_i| \le \sum_{i=1}^m C\tau(1+|\nabla G_{i-1}|^2) + \frac{1}{4}\sum_{i=1}^m \tau|\nabla G_i|^2.$$

We can hence apply the Discrete Gronwall Lemma and obtain $|\nabla G_m| \le C$ for all $m = 1, \dots, N$, which, together with (3.25), implies (3.20).

It remains to prove the a-priori bounds involving the discrete convolution term. From (3.14), (3.19), and **(H4)**, for every $i = 1, \dots, N$ there holds

$$\begin{aligned}\|\nabla(K_\tau \nabla y)_i\|_{L^\infty} &\le \sum_{j=0}^N \tau|\kappa_j|\|\nabla y_{i-j}\|_{L^p(\Omega;\mathbb{R}^{3\times 3})}\|\nabla\phi\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)}\\ &\le \tau(N+1)\|\kappa\|_{L^\infty(0,T)}\|\nabla\overline{y}_\tau\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^{3\times3}))}\|\nabla\phi\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)} \le C. \end{aligned} \tag{3.34}$$

Analogously,

$$\frac{(K_\tau\nabla y)_i(x) - (K_\tau\nabla y)_{i-1}(x)}{\tau} = \int_{\mathbb{R}^3} \phi(x-z)\left(\kappa_0\nabla y_i(z) + \sum_{j=1}^i(\kappa_j - \kappa_{j-1})\nabla y_{i-j}(z)\right)\mathrm{d}z$$

for a.e. $x \in \Omega$, so that

$$\left|\frac{(K_\tau\nabla y)_i - (K_\tau\nabla y)_{i-1}}{\tau}\right| \le (\|\kappa\|_{L^\infty(0,T)} + \|\kappa'\|_{L^1(0,T)})\|\nabla\overline{y}_\tau\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^{3\times3}))}\|\phi\|_{L^q(\mathbb{R}^3)}$$

a.e. in Ω. This, combined with (3.19) and **(H4)**, yields (3.21) and (3.22) and concludes the proof of the proposition. ☐

3.3. **Passage to the limit.** We are now in a position to pass to the limit as $\tau \to 0$. In view of (3.17), (3.19), and (3.20), we find $G \in W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3})) \cap L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^{3\times3}))$ and a limiting deformation $y \in L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3))$ such that, up to the extraction of not relabeled subsequences, there holds

$$\widehat{G}_\tau \overset{*}{\rightharpoonup} G \quad \text{in } W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3})) \cap L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^{3\times3})), \tag{3.35}$$

$$\overline{y}_\tau \overset{*}{\rightharpoonup} y \quad \text{in } L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3)). \tag{3.36}$$

The Aubin-Lions Lemma [26] yields

$$\widehat{G}_\tau \to G \quad \text{in } C([0,T];C^\alpha(\overline{\Omega};\mathbb{R}^{3\times3})) \quad \forall\,\alpha \in (0,1), \tag{3.37}$$

and, in particular,

$$G(0) = G_0. \tag{3.38}$$

From (3.1) and (3.2), we deduce the identity

$$|\overline{G}_\tau(t) - \widehat{G}_\tau(t)| = \tau(1-\alpha_i(t))|\widehat{G}'_\tau(t)| \quad \text{a.e. in } \Omega,$$

for all $t \in ((i-1)\tau, i\tau]$, $i = 1, \dots, N$, so that

$$|\overline{G}_\tau(t) - \widehat{G}_\tau(t)| \le \tau|\widehat{G}'_\tau(t)| \quad \text{a.e. in } \Omega,\ \forall\, t \in [0,T]. \tag{3.39}$$

Hence, (3.17), (3.20), and (3.37) yield

$$\overline{G}_\tau \to G \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^{3\times3}), \tag{3.40}$$

$$\overline{G}_\tau(t) \to G(t) \quad \text{in } L^\infty(\Omega;\mathbb{R}^{3\times3}),\ \forall t \in [0,T], \tag{3.41}$$

$$\overline{G}_\tau \overset{*}{\rightharpoonup} G \quad \text{in } L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^{3\times3})). \tag{3.42}$$

Moreover, by (3.19), we find $\xi \in L^\infty(0,T;L^p(\mathbb{R}^3;\mathbb{R}^{3\times 3}))$ such that, up to subsequences,

$$\nabla \overline{y}_\tau \overset{*}{\rightharpoonup} \xi \quad \text{in } L^\infty(0,T;L^p(\mathbb{R}^3;\mathbb{R}^{3\times 3}))$$

where $\nabla \overline{y}_\tau$ is the trivial extension to the whole $\mathbb{R}^3$.

We proceed by identifying ξ. Let $x_0 \in \Omega$ and $r > 0$ be such that $B_r(x_0) := \{x \in \mathbb{R}^3 \ : \ |x - x_0| < r\} \subset \Omega$. Then, for every $\psi \in L^1((0,T);\mathbb{R}^{3\times 3})$ we have

$$\int_0^T \int_{B_r(x_0)} \xi(t,x)\cdot\psi(t)\,\mathrm{d}x\,\mathrm{d}t = \lim_{\tau\to 0}\int_0^T \int_{B_r(x_0)} \nabla\overline{y}_\tau(t,x)\cdot\psi(t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{B_r(x_0)} \nabla y(t,x)\cdot\psi(t)\,\mathrm{d}x\,\mathrm{d}t,$$

where the last equality is due to (3.36). This yields $\int_{B_r(x_0)} \xi(t,x)\,\mathrm{d}x = \int_{B_r(x_0)} \nabla y(t,x)\,\mathrm{d}x$ for a.e. $t \in (0,T)$ and for every $B_r(x_0) \subset \Omega$. Namely, $\xi = \nabla y$ a.e. in $(0,T)\times\Omega$. Analogously, let $y_0 \in \Omega^c$ and let $s > 0$ be such that $B_s(y_0) \subset \Omega^c$. Then, for every $\psi \in L^1((0,T);\mathbb{R}^{3\times 3})$ there holds

$$\int_0^T \int_{B_s(y_0)} \xi(t,x)\cdot\psi(t)\,\mathrm{d}x\,\mathrm{d}t = \lim_{\tau\to 0}\int_0^T \int_{B_s(y_0)} \nabla\overline{y}_\tau(t,x)\cdot\psi(t)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

which implies $\xi = 0$ a.e. in $(0,T)\times\Omega^c$. Therefore, $\xi = \nabla y$.

Arguing as in (3.23), we deduce

$$\widehat{G}'_\tau(t) = \frac{\Big(\exp\big(\tau\, M(\overline{G}_\tau(t-\tau), \overline{(K_\tau\nabla y)}_\tau(t-\tau))\big) - \mathrm{Id}\Big)}{\tau}\,\overline{G}_\tau(t-\tau). \tag{3.43}$$

Owing to (3.21) and (3.22), the same argument as in the proof of (3.40) yields

$$\overline{(K_\tau\nabla y)}_\tau \to K\nabla y \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^{3\times 3}). \tag{3.44}$$

Therefore, by (3.35), (3.40), the regularity of M, and the fact that the exponential map is locally Lipschitz, we find that (y,G) solves (2.6).

We are hence left with proving that minimality (2.5) holds almost everywhere in time. To this aim, assume to be given $\hat{y} \in \mathcal{Y}$ and recall that $W(\nabla\hat{y}\,G^{-1}(t))\ \det G(t) \in L^\infty(\Omega)$ since $G(t) \in L^\infty(\Omega;\mathbb{R}^{3\times 3})$ and $\det G(t) \geq C\delta$ a.e. In particular, we have that

$$W(\nabla\hat{y}\,G^{-1}(t))\ \leq \lambda(\|G(t)\|_{L^\infty}, C\delta) \quad \text{a.e. in } \Omega\times(0,T) \tag{3.45}$$

where λ is defined in (3.7). We will make use of the following simplified version of [23, Lemma 4.1].

Lemma 3.4. *Under assumption* **(H2)** *there exist* c_3, $\epsilon > 0$ *such that*

$$|W(FH) - W(F)| \leq c_3(W(F)+1)|H - \mathrm{Id}| \quad \forall F,\ H \in GL_+(3),\ |H-\mathrm{Id}| \leq \epsilon. \tag{3.46}$$

Let $s \in (0,T)$. By choosing $F = \nabla\hat{y}\,G^{-1}(s)$ and $H = G(s)\overline{G}_\tau^{-1}(s)$ in (3.46) we find that

$$\begin{aligned}
&\int_\Omega W(\nabla\hat{y}\,\overline{G}_\tau^{-1}(s))\ \det\overline{G}_\tau(s)\,\mathrm{d}x - \int_\Omega W(\nabla\hat{y}\,G^{-1}(s))\ \det G(s)\,\mathrm{d}x \\
&\quad\leq \left(c_3\int_\Omega (W(\nabla\hat{y}\,G^{-1}(s))+1)\ \det\overline{G}_\tau(s)\,\mathrm{d}x\right)\|G(s)\overline{G}_\tau^{-1}(s) - \mathrm{Id}\|_{L^\infty} \\
&\qquad + \int_\Omega W(\nabla\hat{y}\,G^{-1}(s))\,\big(\det\overline{G}_\tau(s) - \det G(s)\big)\,\mathrm{d}x.
\end{aligned}$$

Owing to (3.45), this entails that $W(\nabla\hat{y}\overline{G}_\tau^{-1}(t))\ \det\overline{G}_\tau(t) \in L^1(\Omega)$ for all τ as well. Moreover, taking into account convergence (3.41), we have that

$$\limsup_{\tau\to 0}\left(\int_\Omega W(\nabla\hat{y}\,\overline{G}_\tau^{-1}(s))\ \det\overline{G}_\tau(s)\,\mathrm{d}x - \int_\Omega W(\nabla\hat{y}\,G^{-1}(s))\ \det G(s)\,\mathrm{d}x\right) = 0. \tag{3.47}$$

Fix now $t \in (0,T)$ and $\delta > 0$ small so that $t+\delta \in (0,T)$. From the discrete minimality (3.3), we deduce that

$$\int_t^{t+\delta}\int_\Omega W(\nabla \overline{y}_\tau(s)\,\overline{G}_\tau^{-1}(s))\,\det \overline{G}_\tau(s)\,\mathrm{d}x\,\mathrm{d}s - \int_t^{t+\delta}\langle \overline{\ell}_\tau(s), \overline{y}_\tau(s)\rangle\,\mathrm{d}s \le \int_t^{t+\delta}\int_\Omega W(\nabla \hat{y}\,\overline{G}_\tau^{-1}(s))\,\det \overline{G}_\tau(s)\,\mathrm{d}x\,\mathrm{d}s - \int_t^{t+\delta}\langle \tag{3.48}$$

We now aim at passing to the lim inf as $\tau \to 0$ first. Convergence (3.36) and the regularity of the applied loads implies that

$$\int_t^{t+\delta}\langle \overline{\ell}_\tau(s), \overline{y}_\tau(s)\rangle\,\mathrm{d}s \to \int_t^{t+\delta}\langle \ell(s), y(s)\rangle\,\mathrm{d}s \quad \text{and} \quad \int_t^{t+\delta}\langle \overline{\ell}_\tau(s), \hat{y}\rangle\,\mathrm{d}s \to \int_t^{t+\delta}\langle \ell(s), \hat{y}\rangle\,\mathrm{d}s.$$

By (3.32), there exists $e \in L^\infty(0,T;L^p(\Omega;\mathbb{R}^{3\times 3}))$ such that, for a not relabeled subsequence,

$$\nabla \overline{y}_\tau \overline{G}_\tau^{-1} \rightharpoonup^* e \quad \text{ in } L^\infty(0,T;L^p(\Omega;\mathbb{R}^{3\times 3})).$$

On the other hand, the convergences (3.40) and (3.36) yield that $e = \nabla y G^{-1}$. We can hence use convergence (3.47) and the polyconvexity from **(H1)** in order to pass to the lim inf in (3.48) and obtain that

$$\begin{aligned}\int_t^{t+\delta}\int_\Omega W(\nabla y(t)\,G^{-1}(s))\,\det G(s)\,\mathrm{d}x\,\mathrm{d}s - \int_t^{t+\delta}\langle \ell(s), y(s)\rangle\,\mathrm{d}s &\le \liminf_{\tau\to 0}\int_t^{t+\delta}\int_\Omega W(\nabla \hat{y}\,\overline{G}_\tau^{-1}(s))\,\det \overline{G}_\tau(s)\,\mathrm{d}x\,\mathrm{d}s - \int \\ &\overset{(3.47)}{\le} \int_t^{t+\delta}\int_\Omega W(\nabla \hat{y}\,G^{-1}(s))\,\det G(s)\,\mathrm{d}x\,\mathrm{d}s - \int_t^{t+\delta}\langle \ell(s), \hat{y}\rangle\,\mathrm{d}s. \end{aligned}\tag{3.49}$$

By applying again Lemma 3.4, this time for the choice $F = \nabla \hat{y}\,G^{-1}(t)$ and $H = G(t)G^{-1}(s)$ and using the time regularity of G and ℓ one proves that

$$\lim_{\delta\to 0}\left(\frac{1}{\delta}\int_t^{t+\delta}\int_\Omega W(\nabla \hat{y}\,G^{-1}(s))\,\det G(s)\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{\delta}\int_t^{t+\delta}\langle \ell(s), \hat{y}\rangle\,\mathrm{d}s\right) = \int_\Omega W(\nabla \hat{y}\,G^{-1}(t))\,\det G(t)\,\mathrm{d}x - \langle \ell(t), \hat{y}\rangle. \tag{3.50}$$

Estimate (3.49) implies that the function

$$t \in (0,T) \mapsto \int_\Omega W(\nabla y(t)\,G^{-1}(t))\,\det G(t)\,\mathrm{d}x - \langle \ell(t), y(t)\rangle \tag{3.51}$$

is integrable. Choose now $t \in (0,T)$ to be one of its Lebesgue points. By using (3.50) one can pass to the limit as $\delta \to 0$ in (3.49) and deduce that

$$\int_\Omega W(\nabla y(t)\,G^{-1}(t))\,\det G(t)\,\mathrm{d}x - \langle \ell(t), y(t)\rangle \le \int_\Omega W(\nabla \hat{y}\,G^{-1}(t))\,\det G(t)\,\mathrm{d}x - \langle \ell(t), \hat{y}\rangle.$$

As $\hat{y} \in \mathcal{Y}$ is arbitrary, minimality (2.5) follows.

4. Proof of Theorem 2: Optimal control

We now turn to the existence proof of optimal controls μ^* and optimal pairs $(y^*, G^*) \in S(\mu^*)$ solving problem (2.11).

The first step is to check that the solution operator S is well-defined. This amounts in proving the existence of nutrient-driven morphoelastic solutions for given $\mu \in L^p(0,T;W^{1,p}(\Omega))$, and can be ascertained by extending the argument of Theorem 2.2. Indeed, by resorting again to a time discretization with constant time step τ, starting from $(y_0, G_0) = (y^0, G^0) \in \mathcal{Y} \times \mathcal{G}_p$, one solves for $\{(y_i, G_i)\}_{i=1}^N \in \mathcal{Y}^N \times \mathcal{G}_p^N$ such that, for $i = 1, \dots, N$,

$$y_i \in \mathrm{argmin}_{y\in\mathcal{Y}}\left\{\int_\Omega W(\nabla y\,G_i^{-1})\,\det G_i\,\mathrm{d}x - \langle \ell_i, y\rangle\right\}, \tag{4.1}$$

$$G_i = \exp\big(\tau\,M(G_{i-1}, (K_\tau \nabla y)_{i-1}, \mu_i)\big)\,G_{i-1} \quad \text{ a.e. in } \Omega, \tag{4.2}$$

where now the additional datum μ_i is defined as

$$\mu_i := \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} \mu(t)\,\mathrm{d}t \in W^{1,p}(\Omega) \qquad \text{for } i = 1,\dots,N.$$

Note here that $\|\overline{\mu}_\tau\|_{L^p(0,T;W^{1,p}(\Omega))} \leq \|\mu\|_{L^p(0,T;W^{1,p}(\Omega))}$ and $\overline{\mu}_\tau \to \mu$ in $L^p(0,T;W^{1,p}(\Omega))$ as $\tau \to 0$. The existence of a solution to (4.1)-(4.2) can be obtained by simply adapting the argument of Lemma 3.2. The only modification is required in the estimate on ∇G_i, which now hinges on **(H7)** and reads

$$\begin{aligned}
\|\nabla G_i\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})} &\leq \|G_{i-1}\|_{L^\infty}\|\nabla \exp\big(\tau\, M(G_{i-1},(K_\tau\nabla y)_{i-1},\mu_i)\big)\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}\\
&\quad + \|\exp\big(\tau\, M(G_{i-1},(K_\tau\nabla y)_{i-1},\mu_i)\big)\|_{L^\infty}\|\nabla G_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}\\
&\leq \|G_{i-1}\|_{L^\infty}\|\nabla \exp\big(\tau\, M(G_{i-1},(K_\tau\nabla y)_{i-1},\mu_i)\big)\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}\\
&\quad + C\|G_{i-1}\|_{W^{1,p}(\Omega;\mathbb{R}^{3\times3})}.
\end{aligned}$$

Here and in the following, we use the symbol C to indicate a positive constant, possibly depending on data but not on μ nor on τ. The actual value of C can change from line to line.

By arguing as in (3.13), we get

$$|\nabla \exp\big(\tau\, M(G_{i-1},(K_\tau\nabla y)_{i-1},\mu_i)\big)| \leq 3\tau\exp\big(\tau\|M\|_{L^\infty}\big)\|M\|_{W^{1,\infty}}\Big(|\nabla G_{i-1}|+|\nabla(K_\tau\nabla y)_{i-1}|+|\nabla\mu_i|\Big),$$

so that, using once more **(H7)** we have

$$\begin{aligned}
&\|\nabla \exp\big(\tau\, M(G_{i-1},(K_\tau\nabla y)_{i-1},\mu_i)\big)\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}\\
&\quad\leq C\tau(\|G_{i-1}\|_{W^{1,p}(\Omega;\mathbb{R}^{3\times3})} + \|\nabla(K_\tau\nabla y)_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})} + \|\mu_i\|_{W^{1,p}(\Omega)}).
\end{aligned}$$

Owing to (3.5) and **(H4)** along with the fact that $y_j \in \mathcal{Y}$ for $j = 0,\dots,i-1$,

$$\|\nabla(K_\tau\nabla y)_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})} \leq \sum_{j=0}^{i-1}\tau|\kappa_j|\|\nabla y_{i-1-j}\|_{L^p(\Omega;\mathbb{R}^{3\times3})}\|\nabla\phi\|_{L^1(\mathbb{R}^3;\mathbb{R}^3)} \leq C, \tag{4.3}$$

which, together with the facts that $\overline{\mu}_\tau$ is bounded in $L^p(0,T;W^{1,p}(\Omega))$ independently of τ and that $G_{i-1} \in \mathcal{G}_p$, implies that $G_i \in W^{1,p}(\Omega;\mathbb{R}^{3\times3})$.

In view of passing to the limit in the time discretization, a priori estimates independent of τ have to be provided. The extra μ-dependence of the growth-rate function M has no influence on estimates (3.17)-(3.19) and (3.21)-(3.22), which can be readily obtained as in Proposition 3.3. As regards the estimate on $\overline{G}_\tau$, one is asked to deal with an extra term featuring $\nabla\mu_i$. In particular, subtracting G_{i-1} from both sides of (4.2), dividing by τ, and taking the gradient we find

$$\nabla\left(\frac{G_i - G_{i-1}}{\tau}\right) = \nabla(E_{i-1}G_{i-1}) = (G_{i-1}^\top\nabla E_{i-1}^\top)^t + E_{i-1}\nabla G_{i-1} \tag{4.4}$$

a.e. in Ω, for $i = 1,\dots,N$, where

$$E_{i-1} := \frac{\exp\big(\tau\, M(G_{i-1},(K_\tau\nabla y)_{i-1},\mu_i)\big) - \mathrm{Id}}{\tau}.$$

We can hence control the L^p norm as follows

$$\left\|\nabla\left(\frac{G_i - G_{i-1}}{\tau}\right)\right\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})} \leq \|G_{i-1}^\top\nabla E_{i-1}^\top\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})} + \|E_{i-1}\nabla G_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}. \tag{4.5}$$

From the properties of the matrix exponential, arguing as in (3.13), we get that

$$|\nabla E_{i-1}| \leq 3\exp\big(\tau\|M\|_{L^\infty}\big)\|M\|_{W^{1,\infty}}\Big(|\nabla G_{i-1}| + |\nabla(K_\tau\nabla y)_{i-1}| + |\nabla\mu_i|\Big)$$

and hence that

$$
\begin{aligned}
&\|\nabla E_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}\\
&\quad\le C\exp\big(\tau\|M\|_{L^\infty}\big)\|M\|_{W^{1,\infty}}\\
&\qquad\times\Big(\|\nabla G_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}+\|\nabla(K_\tau\nabla y)_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}+\|\nabla\mu_i\|_{L^p(\Omega;\mathbb{R}^3)}\Big)\\
&\quad\le C\exp\big(\tau\|M\|_{L^\infty}\big)\|M\|_{W^{1,\infty}}\\
&\qquad\times\Big(\|\nabla G_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}+\sum_{j=0}^{i-1}\tau|\kappa_j|\|\nabla y_{i-1-j}\|_{L^p(\Omega;\mathbb{R}^{3\times3})}\|\nabla\phi\|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}+\|\mu_i\|_{W^{1,p}(\Omega)}\Big)\\
&\quad\le C\left(1+\|\nabla G_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}+\|\mu_i\|_{W^{1,p}(\Omega)}\right),
\end{aligned}
\tag{4.6}
$$

for all $i=1,\dots,N$, where we have also used that $\|y_j\|_{W^{1,p}(\Omega;\mathbb{R}^3)}$ is bounded independently of τ, as well as that one can control

$$
\sum_{j=0}^{i-1}\tau|\kappa_j|\|\nabla y_{i-1-j}\|_{L^p(\Omega;\mathbb{R}^{3\times3})}\|\nabla\phi\|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}\le C.
\tag{4.7}
$$

On the other hand, similar calculations as in (3.26) shows that

$$
\|E_{i-1}\|_{L^\infty}\le\frac{1}{\tau}\left(\exp(\tau\|M\|_{L^\infty})-1\right)\le C.
$$

Going back to (4.5) and using the Hölder Inequality, (4.6), and the fact that $\|G_{i-1}\|_{L^\infty}$ is bounded independently of τ, we deduce that

$$
\begin{aligned}
&\left\|\nabla\left(\frac{G_i-G_{i-1}}{\tau}\right)\right\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}\\
&\quad\le\|G_{i-1}\|_{L^\infty}\|\nabla E_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}+\|E_{i-1}\|_{L^\infty}\|\nabla G_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}\\
&\quad\le C\left(1+\|\nabla G_{i-1}\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}+\|\mu_i\|_{W^{1,p}(\Omega)}\right)\\
&\quad\le C\left(1+\|\nabla G_0\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}+\sum_{j=1}^{i-1}\tau\left\|\nabla\left(\frac{G_j-G_{j-1}}{\tau}\right)\right\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}+\|\mu_i\|_{W^{1,p}(\Omega)}\right).
\end{aligned}
$$

By taking the p-power, applying the Discrete Gronwall Lemma, and recalling that $\overline{\mu}_\tau$ is bounded in $L^p(0,T;W^{1,p}(\Omega))$ independently of τ we conclude that $\nabla\widehat{G}'_\tau$ is bounded in $L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times3\times3}))$, independently of τ. Consequently, it follows that $\widehat{G}_\tau$ is bounded in $W^{1,p}(0,T;W^{1,p}(\Omega;\mathbb{R}^{3\times3}))$ and that $\overline{G}_\tau$ is bounded in $L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^{3\times3}))$, both independently of τ.

One can now extract not relabeled subsequences and pass to the limit as $\tau\to0$, following the very argument of Subsection 3.3. Note nonetheless that the convergence of $\widehat{G}_\tau$ and $\overline{G}_\tau$ is slightly weaker, namely,

$$
\begin{aligned}
&\widehat{G}_\tau\overset{*}{\rightharpoonup}G &&\text{in } W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3}))\cap W^{1,p}(0,T;W^{1,p}(\Omega;\mathbb{R}^{3\times3})), &(4.8)\\
&\widehat{G}_\tau\to G &&\text{in } C([0,T];C(\overline{\Omega};\mathbb{R}^{3\times3})), &(4.9)\\
&\overline{G}_\tau\to G &&\text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^{3\times3})\cap L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times3})), &(4.10)\\
&\overline{G}_\tau\overset{*}{\rightharpoonup}G &&\text{in } L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^{3\times3})), &(4.11)
\end{aligned}
$$

where we have also used the Aubin-Lions Lemma. As $\overline{\mu}_\tau\to\mu$ in $L^p(0,T;W^{1,p}(\Omega))$ and the growth-rate function M is Lipschitz continuous with respect to μ from **(H7)**, we readily check again that the limit (y,G) of time-discrete solutions is a nutrient-driven solution in the sense of Definition 2.3.

The above argument shows that the solution operator

$$
\begin{aligned}
&S : L^p(0,T;W^{1,p}(\Omega)) \\
&\quad \to L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3)) \times W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3})) \cap W^{1,p}(0,T;W^{1,p}(\Omega;\mathbb{R}^{3\times3}))
\end{aligned}
$$

representing the set $S(\mu)$ of all nutrient-driven solutions (y,G) given μ is well-defined and bounded.

In order to prove Theorem 2.4, let now $\mu_k \in \mathcal{A}$ and $(y_k,G_k) \in S(\mu_k)$ with

$$
J(y_k,G_k,\mu_k) \to \inf_{\mu\in\mathcal{A}} \big\{ J(y,G,\mu) \ : \ (y,G) \in S(\mu) \big\} \geq 0.
$$

Since $\mathcal{A}$ is bounded in $L^p(0,T;W^{1,p}(\Omega))$ and compact in $L^1((0,T)\times\Omega)$ by **(H8)**, one can find $\mu^* \in \mathcal{A}$ and pass to a not relabeled subsequence such that $\mu_k \to \mu^*$ a.e. Moreover, the boundedness of S implies that (y_k,G_k) are uniformly bounded in $L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3)) \times W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3})) \cap W^{1,p}(0,T;W^{1,p}(\Omega;\mathbb{R}^{3\times3}))$. Hence, by extracting again (without relabeling) we get that

$$
\begin{aligned}
y_k &\overset{*}{\rightharpoonup} y^* \quad \text{in } L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^{3\times3})),\\
G_k &\overset{*}{\rightharpoonup} G^* \quad \text{in } W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3})) \cap W^{1,p}(0,T;W^{1,p}(\Omega;\mathbb{R}^{3\times3})).
\end{aligned}
$$

The latter implies in particular that $G_k \to G^*$ in $C([0,T]\times\overline{\Omega};\mathbb{R}^{3\times3})$. Moreover, since $\det G_k$ is a.e. bounded below by a positive constant independently of k one has that

$$
(G_k)^{-1} \to (G^*)^{-1} \quad \text{in } C([0,T]\times\overline{\Omega};\mathbb{R}^{3\times3})),
$$

as well. These convergences are enough to pass to the limit in relations (2.8)-(2.10) and obtain that the limiting (y^*,G^*) belongs to $S(\mu^*)$. In particular, the almost-everywhere-in-time minimality (2.8) follows along the same lines as in the proof of Theorem 1, see Subsection 3.3.

On the other hand, due to the lower semicontinuity of J from **(H9)** we get that

$$
J(y^*,G^*,\mu^*) \leq \liminf_{k\to\infty} J(y_k,G_k,\mu_k) = \min_{\mu\in\mathcal{A}} \big\{ J(y,G,\mu) \ : \ (y,G) \in S(\mu) \big\}.
$$

In particular, μ^* is an optimal control and $(y^*,G^*) \in S(\mu^*)$ is the corresponding optimal state.

5. Proof of Theorem 3: Nutrient-morphoelastic existence

In this section, we prove the existence of nutrient-morphoelastic solutions, namely, trajectories $t \in [0,T] \mapsto (y(t),G(t),\mu(t)) \in \mathcal{Y}\times\mathcal{G}_p\times\mathcal{M}$ satisfying (2.12)-(2.16). Using again a time discretization with constant time step τ, starting from $(y_0,G_0,\mu_0) = (y^0,G^0,\mu^0) \in \mathcal{Y}\times\mathcal{G}_p\times\mathcal{M}$, we look for $\{(y_i,G_i,\mu_i)\}_{i=1}^N \in \mathcal{Y}^N \times \mathcal{G}_p^N \times \mathcal{M}^N$ fulfilling

$$
y_i \in \operatorname{argmin}_{y\in\mathcal{Y}} \left\{ \int_\Omega W(\nabla y\, G_i^{-1}) \det G_i \,\mathrm{d}x - \langle \ell_i, y\rangle \right\}, \tag{5.1}
$$

$$
G_i = \exp\big(\tau\, M(G_{i-1},(K_\tau \nabla y)_{i-1},\mu_{i-1})\big)\, G_{i-1} \quad \text{a.e. in } \Omega, \tag{5.2}
$$

$$
\frac{\mu_i-\mu_{i-1}}{\tau} - \nu\Delta\mu_i = h_i - H((\kappa *_\tau y)_{i-1}) \quad \text{a.e. in } \Omega, \tag{5.3}
$$

$$
\mu_i = \mu_{\mathrm{D},i} \quad \text{a.e. on } \partial\Omega \tag{5.4}
$$

for $i=1,\dots,N$. In (5.3)-(5.4), we have set

$$
h_i := \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} h(s)\,\mathrm{d}s \in L^p(\Omega) \quad \text{for } i=1,\dots,N,
$$

and

$$
\mu_{\mathrm{D},i} := \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} \mu_{\mathrm{D}}(s)\,\mathrm{d}s \in W^{2,p}(\Omega) \quad \text{for } i=1,\dots,N,
$$

and we recall that

$$(\kappa *_\tau y)_{i-1}(x) := \sum_{j=0}^{i-1} \tau\, \kappa_j\, y_{i-1-j}(x) \quad \text{for a.e. } x \in \Omega, \text{ for } i = 1, \dots, N,$$

where $\kappa_i := \kappa(t_i)$, $i = 0, \dots, N$.

Let us first prove that the scheme (5.1)-(5.4) admits a solution. Using the same arguments as in Lemma 3.2, it follows that for $i = 0, \dots, N-1$, given $G_i \in \mathcal{G}_p$, there exists $y_i \in \mathcal{Y}$ solving (5.1). By quite similar arguments to those in the beginning of Section 4, it is readily verified that for $i = 1, \dots, N$, given $(y_{i-1}, G_{i-1}, \mu_{i-1}) \in \mathcal{Y} \times \mathcal{G}_p \times \mathcal{M}$, there exists a solution $G_i \in \mathcal{G}_p$ to (5.2). It remains to check that for $i = 1, \dots, N$, given $(y_{i-1}, \mu_{i-1}) \in \mathcal{Y} \times \mathcal{M}$, there exists a solution $\mu_i \in \mathcal{M}$ to (5.3)-(5.4). Letting

$$F_i := \tau h_i - \tau H((\kappa *_\tau y)_{i-1}) + \mu_{i-1} \in L^p(\Omega),$$

an application of [13, Theorem 2.4.2.5] shows that the problem

$$\mu_i - \nu\tau\Delta\mu_i = F_i \ \text{ a.e. in } \Omega, \quad \mu_i = \mu_{\mathrm{D},i} \ \text{ a.e. on } \partial\Omega, \tag{5.5}$$

has a unique solution $\mu_i \in \mathcal{M}$.

We next perform some a-priori estimates. The additional dependence of the growth-rate function M on μ has no impact on the estimates (3.17)-(3.19) and (3.21)-(3.22). These estimates can be obtained as in Proposition 3.3. For the estimates on $\bar{\mu}_\tau$ and $\hat{\mu}_\tau$, we deduce by applying [13, Theorems 2.3.3.6 & 1.5.1.2] and using (5.5) that

$$\begin{aligned} \|\mu_i\|_{W^{2,p}(\Omega)} &\le C \left(\|\mu_i - \nu\tau\Delta\mu_i\|_{L^p(\Omega)} + \|\mu_i|_{\partial\Omega}\|_{W^{2-1/p,p}(\partial\Omega)} \right) \\ &\le C \left(\|F_i\|_{L^p(\Omega)} + \|\mu_{\mathrm{D},i}\|_{W^{2,p}(\Omega)} \right) \end{aligned} \tag{5.6}$$

for every $i = 1, \dots, N$. Here and in the following, the symbol C stands for a positive constant, possibly depending on the data but not on τ and varying from line to line. By letting $\tilde{\mu}_i := \mu_i - \mu_{\mathrm{D},i}$ and

$$\tilde{F}_i := h_i - H((\kappa *_\tau y)_{i-1}) - \frac{\mu_{\mathrm{D},i} - \mu_{\mathrm{D},i-1}}{\tau} + \nu\Delta\mu_{\mathrm{D},i} \in L^p(\Omega),$$

equations (5.3) and (5.4) read

$$\frac{\tilde{\mu}_i - \tilde{\mu}_{i-1}}{\tau} - \nu\Delta\tilde{\mu}_i = \tilde{F}_i \quad \text{a.e. in } \Omega, \tag{5.7}$$

$$\tilde{\mu}_i = 0 \quad \text{a.e. on } \partial\Omega, \tag{5.8}$$

for $i = 1, \dots, N$. We multiply (5.7) by the function $\tau|\tilde{\mu}_i|^{p-2}\tilde{\mu}_i$ and integrate over Ω to obtain that

$$\int_\Omega (\tilde{\mu}_i - \tilde{\mu}_{i-1})\, |\tilde{\mu}_i|^{p-2}\tilde{\mu}_i \,\mathrm{d}x - \nu\tau \int_\Omega (\Delta\tilde{\mu}_i)\, |\tilde{\mu}_i|^{p-2}\tilde{\mu}_i \,\mathrm{d}x = \tau \int_\Omega \tilde{F}_i\, |\tilde{\mu}_i|^{p-2}\tilde{\mu}_i \,\mathrm{d}x. \tag{5.9}$$

From the convexity of the map $\mu \mapsto |\mu|^p/p$ we obtain

$$\int_\Omega (\tilde{\mu}_i - \tilde{\mu}_{i-1})\, |\tilde{\mu}_i|^{p-2}\tilde{\mu}_i \,\mathrm{d}x \ge \frac{1}{p}\int_\Omega |\tilde{\mu}_i|^p \,\mathrm{d}x - \frac{1}{p}\int_\Omega |\tilde{\mu}_{i-1}|^p \,\mathrm{d}x. \tag{5.10}$$

On the other hand, by applying the Hölder and the Young Inequalities we have

$$\begin{aligned} \tau \int_\Omega \tilde{F}_i\, |\tilde{\mu}_i|^{p-2}\tilde{\mu}_i \,\mathrm{d}x &\le \tau \left(\int_\Omega |\tilde{F}_i|^p \,\mathrm{d}x \right)^{1/p} \left(\int_\Omega |\tilde{\mu}_i|^p \,\mathrm{d}x \right)^{1/q} \\ &\le \tau \left(\frac{1}{p}\int_\Omega |\tilde{F}_i|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega |\tilde{\mu}_i|^p \,\mathrm{d}x \right). \end{aligned} \tag{5.11}$$

Furthermore, by using the Green Formula and taking into account the boundary condition (5.8), we get

$$-\nu\tau \int_\Omega (\Delta\tilde{\mu}_i)\, |\tilde{\mu}_i|^{p-2}\tilde{\mu}_i \, \mathrm{d}x = \nu\tau(p-1)\int_\Omega |\tilde{\mu}_i|^{p-2}|\nabla\tilde{\mu}_i|^2 \, \mathrm{d}x. \tag{5.12}$$

Thus, by plugging (5.10)-(5.12) in (5.9), we arrive at

$$\frac{1}{p}\int_\Omega |\tilde{\mu}_i|^p \, \mathrm{d}x - \frac{1}{p}\int_\Omega |\tilde{\mu}_{i-1}|^p \, \mathrm{d}x + \nu\tau(p-1)\int_\Omega |\tilde{\mu}_i|^{p-2}|\nabla\tilde{\mu}_i|^2 \, \mathrm{d}x \le \frac{\tau}{p}\int_\Omega |\tilde{F}_i|^p \, \mathrm{d}x + \frac{\tau}{q}\int_\Omega |\tilde{\mu}_i|^p \, \mathrm{d}x.$$

Summing this inequality from $i = 1$ to $m \le N$ we infer that

$$\left(\frac{1}{p} - \frac{\tau}{q}\right)\|\tilde{\mu}_m\|^p_{L^p(\Omega)} \le \frac{1}{p}\|\tilde{\mu}_0\|^p_{L^p(\Omega)} + \frac{\tau}{p}\sum_{i=1}^{m}\|\tilde{F}_i\|^p_{L^p(\Omega)} + \frac{\tau}{q}\sum_{i=1}^{m-1}\|\tilde{\mu}_i\|^p_{L^p(\Omega)}.$$

Upon choosing τ small enough, by applying the Discrete Gronwall Lemma and owing to assumptions **(H10)** and **(H11)** we deduce that $\|\tilde{\mu}_m\|_{L^p(\Omega)} \le C$ for all $m = 1, \dots, N$. Hence, the definition of $\tilde{\mu}_m$, the reverse triangle inequality, and the fact that $\overline{\mu_{\mathrm{D}}}_{,\tau}$ is bounded in $L^\infty(0, T; W^{2,p}(\Omega))$ independently of τ, result in $\|\mu_m\|_{L^p(\Omega)} \le C$ for all $m = 1, \dots, N$. Now, inserting this estimate into (5.6) and using again **(H10)** and **(H11)**, we find that, for sufficiently small τ,

$$\|\mu_i\|_{W^{2,p}(\Omega)} \le C \quad \forall i = 1, \dots, N.$$

This proves that $\bar{\mu}_\tau$ and $\hat{\mu}_\tau$ are bounded in $L^\infty(0, T; W^{2,p}(\Omega))$ independently of τ, and, together with (5.3), **(H10)**, and **(H11)**, it also proves that $\hat{\mu}_\tau$ is bounded in $W^{1,\infty}(0, T; L^p(\Omega))$ independently of τ.

Using the fact that $\bar{\mu}_\tau$ is bounded in $L^\infty(0, T; W^{2,p}(\Omega))$ independently of τ, the same arguments from Section 4 entail that $\widehat{G}_\tau$ is bounded in $W^{1,p}(0, T; W^{1,p}(\Omega; \mathbb{R}^{3\times3}))$ and that $\overline{G}_\tau$ is bounded in $L^\infty(0, T; W^{1,p}(\Omega; \mathbb{R}^{3\times3}))$, both independently of τ.

We proceed to show some a-priori bounds for $\overline{(\kappa *_\tau y)}_\tau$ and $\widehat{(\kappa *_\tau y)}_\tau$. By (3.19) and **(H4)**, we have for every $i = 1, \dots, N$

$$\begin{aligned}\|(\kappa *_\tau y)_i\|_{L^p(\Omega;\mathbb{R}^3)} &\le \sum_{j=0}^{N}\tau|\kappa_j|\|y_{i-j}\|_{L^p(\Omega;\mathbb{R}^3)}\\ &\le \tau(N+1)\|\kappa\|_{L^\infty(0,T)}\|\overline{y}_\tau\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^3))} \le C\end{aligned}$$

and analogously

$$\|\nabla(\kappa *_\tau y)_i\|_{L^p(\Omega;\mathbb{R}^{3\times3})} \le \tau(N+1)\|\kappa\|_{L^\infty(0,T)}\|\nabla\overline{y}_\tau\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^{3\times3}))} \le C,$$

which imply that both $\overline{(\kappa *_\tau y)}_\tau$ and $\widehat{(\kappa *_\tau y)}_\tau$ are bounded in $L^\infty(0, T; W^{1,p}(\Omega; \mathbb{R}^3))$ independently of τ. Moreover for a.e. $x \in \Omega$,

$$\begin{aligned}\left|\frac{(\kappa *_\tau y)_i(x) - (\kappa *_\tau y)_{i-1}(x)}{\tau}\right| &= \left|\kappa_0 y_i(x) + \sum_{j=1}^{i}(\kappa_j - \kappa_{j-1})y_{i-j}(x)\right|\\ &\le \|\kappa\|_{L^\infty(0,T)}|y_i(x)| + \|\kappa'\|_{L^1(0,T)}|y_{i-j}(x)|,\end{aligned}$$

so that

$$\left\|\frac{(\kappa *_\tau y)_i - (\kappa *_\tau y)_{i-1}}{\tau}\right\|_{L^p(\Omega;\mathbb{R}^3)} \le C\left(\|\kappa\|_{L^\infty(0,T)} + \|\kappa'\|_{L^1(0,T)}\right)\|\overline{y}_\tau\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^3))}.$$

This, along with (3.19) and **(H4)**, yields that $\widehat{(\kappa *_\tau y)}_\tau'$ is bounded in $L^\infty(0, T; L^p(\Omega; \mathbb{R}^3))$ independently of τ.

We can now pass to the limit as $\tau \to 0$. Following the arguments in Sections 3.3 and 4, we can extract subsequences, without relabeling, in such a way that

$$
\begin{aligned}
\overline{y}_\tau &\overset{*}{\rightharpoonup} y && \text{in } L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3)),\\
\widehat{G}_\tau &\overset{*}{\rightharpoonup} G && \text{in } W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3}))\cap W^{1,p}(0,T;W^{1,p}(\Omega;\mathbb{R}^{3\times3})),\\
\overline{G}_\tau &\to G && \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^{3\times3})\cap L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times3})),\\
\overline{(K_\tau\nabla y)}_\tau &\to K\nabla y && \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^{3\times3})\cap L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times3})),
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{\mu}_\tau &\overset{*}{\rightharpoonup} \mu && \text{in } L^\infty(0,T;W^{2,p}(\Omega))\cap W^{1,\infty}(0,T;L^p(\Omega)),\\
\overline{\mu}_\tau &\to \mu && \text{in } L^\infty(0,T;L^p(\Omega)),\\
\overline{\mu}_\tau &\overset{*}{\rightharpoonup} \mu && \text{in } L^\infty(0,T;W^{2,p}(\Omega)),\\
\overline{(\kappa *_\tau y)}_\tau &\to \kappa * y && \text{in } L^\infty(0,T;L^p(\Omega;\mathbb{R}^3)).
\end{aligned}
$$

Since the functions M and H are Lipschitz continuous according to **(H7)** and **(H11)**, and since $\overline{h}_\tau \to h$ in $L^\infty(0,T;L^p(\Omega))$, we easily check that the limit (y,G,μ) is a nutrient-morphoelastic solution in the sense of Definition 2.5. Once again, the almost-everywhere-in-time minimality (2.12) follows by adapting the argument of Subsection 3.3.

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