A Hölder continuity result for a class of obstacle problems under non standard growth conditions

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We prove $C^{0,\alpha}$ regularity for minimizers u of functionals with $p(x)$-growth of the type

$$\mathcal{F}(w,\Omega) := \int_\Omega f(x,w(x),Dw(x))\,dx,$$

in the class $K := \{w \in W^{1,p(x)}(\Omega;\mathbb{R}) : w \geq \psi\}$, where the exponent function $p : \Omega \to (1,\infty)$ is assumed to be continuous with a modulus of continuity satisfying

$$\limsup_{\rho\to 0} \omega(\rho)\log\left(\frac{1}{\rho}\right) < +\infty,$$

and $1 < \gamma_1 \leq p(x) \leq \gamma_2 < +\infty$. Moreover, $\psi \in W^{1,1}_{\text{loc}}(\Omega)$ is a given obstacle function, whose gradient $D\psi$ belongs to a Morrey space $L^{q,\lambda}_{\text{loc}}(\Omega)$ with $n-\gamma_1 < \lambda < n$ and $q > \gamma_2$. We do not assume any quantitative continuity of the integrand function f.

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1 Introduction

The aim of this paper is to prove Hölder continuity for local minimizers of integral functionals of the type

$$\mathcal{F}(u,\Omega) := \int_\Omega f(x,u(x),Du(x))\,dx, \tag{1.1}$$

in the class $K := \{u \in W^{1,p(x)}(\Omega,\mathbb{R}) :\ u \geq \psi\}$, where ψ is a fixed obstacle function, Ω a bounded open set in $\mathbb{R}^n$ and $f : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ a Carathéodory function satisfying a non-standard growth condition of the type

$$L^{-1}|z|^{p(x)} \leq f(x,\xi,z) \leq L(\mu^2+|z|^2)^{p(x)/2}, \tag{1.2}$$

whenever $x \in \Omega$, $\xi \in \mathbb{R}$, $z \in \mathbb{R}^n$; here, $\mu \in [0,1], L \geq 1$ and the exponent function $p : \Omega \to (1,\infty)$ is continuous with modulus of continuity ω satisfying

$$\limsup_{\rho\to 0} \omega(\rho)\log\left(\frac{1}{\rho}\right) < +\infty. \tag{1.3}$$

We note that in order to prove $C^{0,\alpha}$ regularity of (local) minimizers u (provided there exists one) of the above mentioned problem, we need neither quasiconvexity of the functional $\mathcal{F}$ nor any quantified continuity assumption on the integrand function f. On the other hand we have to impose that the obstacle function ψ lies in an appropriate Morrey space $L^{q,\lambda}_{\text{loc}}(\Omega)$ (which in particular includes that ψ itself is Hölder continuous).

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Under the before mentioned hypotheses the proof of the $C^{0,\alpha}$-regularity result is based on the fact that Ekeland's variational principle (see [9]) provides a function v which is near to u (our original minimizer) with respect to the distance in $W^{1,p(x)}$ on a suitable ball $B_R \subset \Omega$. Moreover, it turns out that v is a local quasi minimizer (see Definition 2.3) of a much simpler $p(x)$-growth functional of the type

$$w \mapsto \int_{B_R} \left(|Dw|^{p(x)} + \tfrac{H(R)}{R^n} + 1 \right) \, dx.$$

Here, the quantity $H(R)$ depends on the radius R of the ball and the obstacle function ψ (see (5.6)). At this stage we use arguments employing Ekeland's variational principle in a way similar to [7, 17, 18]. It is worth to mention that the missing continuity properties of the funtional $\mathcal{F}$ do not allow a freezing procedure in our problem. Therefore we apply Ekeland's variational principle directly in the spaces $W^{1,p(x)}$. These spaces are known to be reflexive Banach spaces in the case that $1 < \inf_{B_R} p \leq \sup_{B_R} p < +\infty$. We refer the reader to [25], [26] and [27] for a more detailed discussion of properties of generalized Sobolev spaces. Finally, rescaling the problem to the unit ball in a way that the rescaled functional does not depend on the obstacle ψ itself, allows us to apply arguments of De Giorgi type for generalized $p(x)$-growth conditions to prove Hölder continuity for the reference function v. This procedure is only possible since we assume a certain Morrey-space condition for the obstacle. This assumption can be exploited in order to show that the perturbation term involving H behaves like a certain power of the involved radius. In conclusion, comparison via Ekeland's principle provides the desired result for u. Note here that already in [15] and [16], De Giorgi classes of generalized type for $p(x)$-growth conditions were introduced and Hölder continuity was shown for quasi minimizers of functionals $\int f(x, u, Du)\, dx$ with $p(x)$-growth.

Regularity properties of minimizers of functionals and solutions of equations and systems with $p(x)$-growth have been discussed in a number of papers within the past ten years (for $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity see for example [1, 2, 3, 4, 6, 10, 24] and the generalization to higher order systems in [22, 23]). They became of more and more interest since they represent a borderline case between standard p growth and $p - q$ growth conditions (which were studied for example in [13, 14]). At this stage we would like to remark that the above introduced continuity assumption are the weakest ones in the following sense: Zhikov showed in [29] that condition (1.3) is sufficient to achieve higher integrability of minimizers, and on the other hand the failure of (1.3) in general causes the loss of any type of regularity of minimizers.

Hölder continuity for obstacle problems with standard growth was already shown in [5] and [11]. Basically, the proof presented here in the $p(x)$-growth situation is according to the proof in [11]. Nevertheless many difficulties come up due to the variable growth exponent. Even if 'freezing' as it was done in most of the proofs of Hölder continuity for variable growth problems, is – due to missing continuity assumptions on the integrand function f – not possible in the present situation, the quantified continuity of p expressed by (1.3) allows to control the distance of maximal and minimal exponents on suitably small balls. In turn, this localization procedure allows to establish De Giorgi type estimates in the variable growth situation (see also [15]).

The Hölder continuity result of the present paper is used to show $C^{0,\alpha}$ regularity with a quantified Hölder exponent α in the situation where the modulus of continuity ω of the exponent function p fulfills the stronger condition $\omega(\rho) \log \frac{1}{\rho} \to 0$ when $\rho \to 0$. This is done by the authors in [12].

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2 Notation and statements

In the sequel Ω will denote an open bounded domain in $\mathbb{R}^n$ and $B(x, R)$ the open ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. We will possibly use in the following also the notation $B_R(x)$ to indicate the ball $B(x, R)$. If u is an integrable function defined on $B(x, R)$, we will set

$$(u)_{x,R} = \fint_{B(x,R)} u(x) dx = \frac{1}{\omega_n R^n} \int_{B(x,R)} u(x) dx,$$

where ω_n is the Lebesgue measure of $B(0, 1)$. We shall also adopt the convention of writing B_R and $(u)_R$ instead of $B(x, R)$ and $(u)_{x,R}$ respectively, when the center will not be relevant or it is clear from the context;

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moreover, unless otherwise stated, all balls considered will have the same center. Finally the letter c will freely denote a constant, not necessarily the same in any two occurrences, while only the relevant dependences will be highlighted.

We start with the following definition.

Definition 2.1 A function u is said to belong to the generalized Sobolev space $W^{1,p(x)}(\Omega;\mathbb{R})$ if $u \in L^{p(x)}(\Omega;\mathbb{R})$ and the distributional gradient $Du \in L^{p(x)}(\Omega;\mathbb{R}^n)$. Here the generalized Lebesgue space $L^{p(x)}(\Omega;\mathbb{R})$ is defined as the space of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

This is a Banach space equipped with the Luxemburg norm

$$||f||_{L^{p(x)}(\Omega;\mathbb{R})} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

This definition can be extended in a straightforward way to the case of vector-valued functions.

Next, we will set

$$\mathcal{F}(u,\mathcal{A}) := \int_{\mathcal{A}} f(x,u(x),Du(x))dx$$

for all $u \in W^{1,1}_{\text{loc}}(\Omega)$ and for all $\mathcal{A} \subset \Omega$.

We adopt the following notion of local minimizer and local $\mathcal{Q}$ minimizer:

Definition 2.2 We say that a function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is a local minimizer of the functional (1.1) if $|Du(x)|^{p(x)} \in L^1_{\text{loc}}(\Omega)$ and

$$\int_{\text{spt}\,\varphi} f(x,u(x),Du(x))dx \leq \int_{\text{spt}\,\varphi} f(x,u(x)+\varphi(x),Du(x)+D\varphi(x))dx$$

for all $\varphi \in W^{1,1}_0(\Omega)$ with compact support in Ω.

Definition 2.3 We say that a function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is a local $\mathcal{Q}$ minimizer of the functional (1.1) with $\mathcal{Q} \geq 1$, if for all $v \in W^{1,1}_{\text{loc}}(\Omega)$ we have

$$\mathcal{F}(u,H) \leq \mathcal{Q}\,\mathcal{F}(v,H),$$

where we set $H := \text{spt}(u-v) \Subset \Omega$.

We shall consider the following growth condition, with $L \geq 1$:

$$L^{-1}(\mu^2+|z|^2)^{p(x)/2} \leq f(x,\xi,z) \leq L(\mu^2+|z|^2)^{p(x)/2}. \tag{H1}$$

Moreover let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing continuous function, vanishing at zero, which represents the modulus of continuity of p :

$$|p(x)-p(y)| \leq \omega(|x-y|). \tag{H2}$$

We will always assume that ω satisfies the following condition:

$$\limsup_{R\to 0} \omega(R)\log\left(\frac{1}{R}\right) < +\infty\ ; \tag{2.1}$$

thus in particular, without loss of generality, we may assume that

$$\omega(R) \leq L|\log R|^{-1} \tag{2.2}$$

for all $R < 1$.

No differentiability is assumed on f with respect to x or with respect to z.

Since all our results are local in nature, without loss of generality we shall suppose that

$$1 < \gamma_1 \le p(x) \le \gamma_2 \qquad \forall x \in \Omega\ , \tag{2.3}$$

and

$$\int_\Omega |Du(x)|^{p(x)} dx < +\infty\ . \tag{2.4}$$

Finally we set

$$K := \{u \in W^{1,p(x)}(\Omega;\mathbb{R}) :\ u \ge \psi\}, \tag{2.5}$$

where $\psi \in W^{1,p(x)}(\Omega;\mathbb{R})$ is a fixed function.
Now we recall the definition of Morrey and Campanato spaces (see for example [20]).

Definition 2.4 (Morrey spaces).
Let Ω be an open and bounded subset of $\mathbb{R}^n$, let $1 \le p < +\infty$ and $\lambda \ge 0$. By $L^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that, if we set $\Omega(x_0,\rho) := \Omega \cap B(x_0,\rho)$, we get

$$||u||_{L^{p,\lambda}(\Omega)} := \left\{ \sup_{x_0\in\Omega,\ 0<\rho<\ \mathrm{diam}(\Omega)} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x)|^p dx \right\}^{1/p} < +\infty.$$

It is easy to see that $||u||_{L^{p,\lambda}(\Omega)}$ is a norm respect to which $L^{p,\lambda}(\Omega)$ is a Banach space.

Definition 2.5 (Campanato spaces).
Let Ω be an open and bounded subset of $\mathbb{R}^n$, let $p \ge 1$ and $\lambda \ge 0$. By $\mathcal{L}^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that, if we set $\Omega(x_0,\rho) := \Omega \cap B(x_0,\rho)$, we get

$$[u]_{p,\lambda} = \left\{ \sup_{x_0\in\Omega,\ 0<\rho<\ \mathrm{diam}(\Omega)} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x) - (u)_{x_0,\rho}|^p\, dx \right\}^{1/p} < +\infty,$$

where

$$(u)_{x_0,\rho} := \frac{1}{|\Omega(x_0,\rho)|} \int_{\Omega(x_0,\rho)} u(x)\, dx$$

is the average of u in $\Omega(x_0,\rho)$.
Also in this case it is not difficult to show that $\mathcal{L}^{p,\lambda}(\Omega)$ is a Banach space equipped with the norm

$$||u||_{\mathcal{L}^{p,\lambda}(\Omega)} = ||u||_{L^p(\Omega)} + [u]_{p,\lambda}.$$

Remark 2.6 The local variants $L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ and $\mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega)$ are defined in a standard way

$$u \in L^{p,\lambda}_{\mathrm{loc}}(\Omega) \ \Leftrightarrow\ u \in L^{p,\lambda}(\Omega') \quad \forall\, \Omega' \Subset \Omega$$

$$u \in \mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega) \ \Leftrightarrow\ u \in \mathcal{L}^{p,\lambda}(\Omega') \quad \forall\, \Omega' \Subset \Omega.$$

The main result of the paper is the following

Theorem 2.7 *Let $u \in W^{1,p(x)}(\Omega)$ be a local minimizer of the functional* (1.1) *in the class* (2.5)*, where $\psi \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a given obstacle function fulfilling*

$$D\psi \in L^{q,\lambda}_{\mathrm{loc}}(\Omega), \tag{2.6}$$

with $q = \gamma_2\, \tilde q$ for some $\tilde q > 1$ and $n - \gamma_1 < \lambda < n$, where γ_1 and γ_2 have been introduced in (2.3)*. Suppose moreover that the Lagrangian f satisfies the growth condition* (H1) *and the function p fulfills assumptions* (H2) *and* (2.1)*. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.*

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The strategy of the proof of Theorem 2.7, which is given in Chapter 5, is the following: due to (2.1), localizing allows us to control the difference of maximal and minimal exponents in a ball B_R with suitable small radius. Ekeland's variational principle then provides a function v on the ball B_R, which is near to u with respect to the distance in $W^{1,p(x)}(B_R)$ and a free minimizer of a suitably modified functional (5.11) (see Section 5.2) . It turns out (see Section 5.3) that the minimality of v translates into $\mathcal{Q}$ minimality of the functional (5.12) with a constant $\mathcal{Q}$ which depends only on L and the global bounds γ_1 and γ_2 of p. Lemma 4.4, which takes into consideration the fact, that the Morrey condition (2.6) allows to rescale the problem in such a way that the dependency on the obstacle turns into a radius power, takes use of the De Giorgi type estimates which are shown in Lemma 4.2 for a rescaled functional on the unit ball. Finally, exploiting the comparison via Ekeland, the control of the oscillations of v (see (5.14)) carries over to the function u and therefore provides the desired Hölder continuity.

3 Some known results

The interest of Campanato's spaces lies mainly in the following result which will be used in the next sections.

Theorem 3.1 *Let Ω be a bounded open Lipschitz domain of $\mathbb{R}^n$, and let $n < \lambda < n+p$. Then the space $\mathcal{L}^{p,\lambda}(\Omega)$ is isomorphic to $\mathcal{C}^{0,\alpha}(\bar\Omega)$ with $\alpha = \frac{\lambda-n}{p}$. We also remark that, using Poincaré inequality, we have that, for a weakly differentiable function v, if $Dv \in L^{p,\lambda}(\Omega)$, then $v \in \mathcal{L}^{p,p+\lambda}(\Omega)$.*

Remark 3.2 Theorem 3.1 also holds for a larger class of domains (see [20], Sect. 2.3).

The following well known results will be needed at several stages of the proof of our main theorem.

Lemma 3.3 ([20], Lemma 7.1)
Let $a > 0$ and let $\{\chi_i\}$ be a sequence of real positive numbers, such that

$$\chi_{i+1} \le \mathcal{C}\,\mathcal{B}^i\,\chi_i^{1+a}$$

with $\mathcal{C} > 0$ and $\mathcal{B} > 1$. If

$$\chi_0 \le \mathcal{C}^{-\frac{1}{a}}\,\mathcal{B}^{-\frac{1}{a^2}}, \tag{3.1}$$

we have

$$\chi_i \le \mathcal{B}^{-\frac{i}{a}}\,\chi_0$$

and hence in particular

$$\lim_{i\to\infty} \chi_i = 0.$$

Lemma 3.4 ([20], Lemma 7.3)
Let $\varphi(t)$ be a positive function, and assume that there exists a constant q and a number τ, $0 < \tau < 1$ such that for every $R < R_0$

$$\varphi(\tau\,R) \le \tau^\delta\,\varphi(R) + B\,R^\beta$$

with $0 < \beta < \delta$, and

$$\varphi(t) \le q\,\varphi(\tau^k\,R)$$

for every t in the interval $(\tau^{k+1}\,R, \tau^k\,R)$ (in particular this inequality holds with $q = 1$ if φ is non-decreasing).
Then for every $\rho < R \le R_0$ we have

$$\varphi(\rho) \le C\,\left\{\left(\frac{\rho}{R}\right)^\beta\,\varphi(R) + B\,\rho^\beta\right\},$$

where C is a constant depending only on q, τ, δ and β.

We also present the following variant of the previous lemma.

Lemma 3.5 *Let $\Phi(t)$ be a nonnegative and nondecreasing function. Suppose that*

$$\Phi(\rho) \le A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \varepsilon\right]\Phi(R) + BR^{\beta},$$

for all $\rho \le R \le R_0$, with A, B, α, β nonnegative constants, $\beta < \alpha$. Then there exists a constant $\varepsilon_0 \equiv \varepsilon_0(A, \alpha, \beta)$ such that if $\varepsilon < \varepsilon_0$, for all $\rho \le R \le R_0$, then

$$\Phi(\rho) \le c\left[\left(\frac{\rho}{R}\right)^{\beta}\Phi(R) + B\rho^{\beta}\right],$$

where c is a constant depending on α, β, A, but independent of B.

We quote finally a higher integrability result for functionals of type (1.1). This result can be found in [12].

Lemma 3.6 *Let $\mathcal{O}$ be an open subset of Ω, let $u \in W^{1,1}_{\mathrm{loc}}(\mathcal{O})$ be a local minimizer in the class* (2.5) *of the functional* (1.1) *with $f : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying* (H1)*, with the exponent function p satisfying* (H2), (2.1) *and* (2.3) *and with ψ fulfilling condition* (2.6)*. Moreover suppose that*

$$\int_{\mathcal{O}} |Du(x)|^{p(x)} dx \le M_1$$

for some constant M_1. Then, there exist two positive constants c_0, δ depending on n, $\tilde{q}$, γ_1, γ_2, L, M_1, where $\tilde{q}$ is the quantity appearing in condition (2.6)*, such that, if $B_R \Subset \mathcal{O}$, then*

$$\left(\fint_{B_{R/2}} |Du(x)|^{p(x)(1+\delta)} dx\right)^{1/(1+\delta)} \le c_0 \fint_{B_R} |Du(x)|^{p(x)} dx + c_0\left(\fint_{B_R} \left(|D\psi(x)|^{p(x)(1+\delta)} + 1\right) dx\right)^{1/(1+\delta)}. \tag{3.2}$$

The proof of our main result takes use of a well known result from Ekeland (see [9]), which we quote here.

Lemma 3.7 *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{G} : \mathcal{X} \to (-\infty, +\infty]$ a lower semicontinouos funcional for which there holds $\inf_{\mathcal{X}} \mathcal{G} < \infty$. For given ε let $v \in \mathcal{X}$ be such that $\mathcal{G}(v) \le \inf_{\mathcal{X}} \mathcal{G} + \varepsilon$. Then there exists $\bar{v} \in \mathcal{X}$ such that*

$$\begin{aligned} d(\bar{v}, v) &\le 1, \\ \mathcal{G}(\bar{v}) &\le \mathcal{G}(v), \\ \mathcal{G}(\bar{v}) &\le \mathcal{G}(w) + \varepsilon d(w, \bar{v}), \quad \text{for all } w \in \mathcal{X}. \end{aligned}$$

4 Preliminaries

In this section we provide the main components for the proof of Theorem 2.7. Let us remark at this point that we will especially have a careful look at the dependencies of the constants involved on the exponent function $p(x)$. In particular we have to make sure that the constants do not depend on local bounds of p, denoted by p_1 and p_2, but only on the global bounds γ_1 and γ_2 on Ω. When necessary and not directly obvious, we will therefore give the detailed arguments for the replacements of p_1 and p_2 by γ_1 and γ_2 in the appearing constants.

4.1 A refined iteration lemma

We start by proving the following result which is a generalization of [20], Lemma 6.1.

Lemma 4.1 *Let $0 < \theta < 1, A > 0, B \ge 0, 1 < \gamma_1 \le p(x) \le \gamma_2 < +\infty$ and let $f \ge 0$ be a bounded function satisfying*

$$f(t) \le \theta f(s) + A\int_{B_R} \left|\frac{h(x)}{s-t}\right|^{p(x)} dx + B, \tag{4.1}$$

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for all $r \le t < s \le R$, where $h \in L^{p(x)}(B_R)$. Then there exists a constant $C \equiv C(\theta, \gamma_2)$ such that

$$f(r) \le C \left[A \int_{B_R} \left| \frac{h(x)}{R-r} \right|^{p(x)} dx + B \right].$$

Proof. Let $0 < \tau < 1$ be such that

$$\tau^{-\gamma_2}\theta < 1, \quad \text{i.e. } 1 > \tau^{\gamma_2} > \theta.$$

This means $\tau \equiv \tau(\theta, \gamma_2)$. Furthermore we define

$$t_0 := r, \quad t_{n+1} := t_n + (1-\tau)\tau^n (R-r) \le R.$$

Exploiting iteratively (4.1) we obtain

$$\begin{aligned} f(r) &= f(t_0) \\ &\le \theta f(t_1) + A \int_{B_R} \left| \frac{h(x)}{t_1 - t_0} \right|^{p(x)} dx + B \\ &\le \dots \\ &\le \theta^n f(t_n) + \sum_{j=0}^{n-1} \theta^j \left[A \int_{B_R} \left| \frac{h(x)}{t_{j+1} - t_j} \right|^{p(x)} dx + B \right] \\ &\le \theta^n f(t_n) + \sum_{j=0}^{n-1} \theta^j \left[A \int_{B_R} \left| \frac{h(x)}{(1-\tau)(R-r)} \right|^{p(x)} \tau^{-jp(x)}\, dx + B \right] \\ &=: (1). \end{aligned}$$

Using the fact that $\tau^{-jp(x)} \le \tau^{-j\gamma_2}$ we estimate

$$\begin{aligned} (1) &\le \theta^n f(t_n) + \sum_{j=0}^{n-1} \left(\theta \tau^{-\gamma_2}\right)^j \frac{A}{(1-\tau)^{\gamma_2}} \int_{B_R} \left| \frac{h(x)}{R-r} \right|^{p(x)} dx + B \sum_{j=0}^{n-1} \theta^j \\ &\le \theta^n f(t_n) + \frac{1}{1 - \theta\tau^{-\gamma_2}} \frac{A}{(1-\tau)^{\gamma_2}} \int_{B_R} \left| \frac{h(x)}{R-r} \right|^{p(x)} dx + \frac{B}{1-\theta} \\ &= \theta^n f(t_n) + A \frac{(1-\tau)^{-\gamma_2}}{1 - \theta\tau^{-\gamma_2}} \int_{B_R} \left| \frac{h(x)}{R-r} \right|^{p(x)} dx + \frac{B}{1-\theta}. \end{aligned}$$

Now take

$$C := \max\left\{ \frac{1}{1-\theta}, \frac{(1-\tau)^{-\gamma_2}}{1-\theta\tau^{-\gamma_2}} \right\}$$

and pass to the limit $n \to \infty$. This means $C \equiv C(\tau, \theta, \gamma_2) \equiv C(\theta, \gamma_2)$. □

4.2 A priori estimate for a reference problem

The following a priori estimate for a free $\mathcal{Q}$ minimizer of a rescaled problem to the unit ball will play a central role in our proof. Later we will apply this result to our situation, determining u and the exponent function p in an appropriate way.

Lemma 4.2 *Let $u \in W^{1,p(x)}(B_1(0))$ be a local $\mathcal{Q}$ minimizer of the functional*

$$\tilde{\mathcal{F}}(w, B_1(0)) := \int_{B_1(0)} \left(|Dw|^{p(y)} + 1 \right) dy, \tag{4.2}$$

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where $p : B_1(0) \to (1, \infty)$ *is a uniformly continuous function, whose modulus of continuity fulfills the condition* (2.1)*. Suppose furthermore that there exist two constants* p_1*,* p_2 *such that*

$$p_1 := \min_{B_1(0)} p(y) \leq \; p(y) \leq \; \max_{B_1(0)} p(y) =: p_2, \quad \textit{and} \quad p_2 - p_1 \leq \frac{1}{n}, \tag{4.3}$$

and for the constants p_1*,* p_2 *there holds* $p_1, p_2 \in [\gamma_1, \gamma_2]$ *with* $1 < \gamma_1 \leq \gamma_2 < +\infty$*. Suppose moreover that there exists a constant* M *such that*

$$\int_{B_1(0)} |Du|^{p(y)}\, dy \leq \; M. \tag{4.4}$$

Then u *is locally bounded and satisfies the estimates*

$$\sup_{B_{1/2}(0)} |u| \leq c \left[\left(\int_{B_1(0)} |u(y)|^{p_2}\, dy \right)^{1/p_2} + 1 \right], \tag{4.5}$$

with $c \equiv c(n, \gamma_1, \gamma_2, \mathcal{Q})$ *and*

$$\sup_{B_{1/2}(0)} u \leq \; c \left(\int_{B_1(0)} u_+^{p_2}\, dy \right)^{1/p_2} |A_{0,1}|^{\frac{\alpha}{p_2}} + 1, \tag{4.6}$$

for some suitable $\alpha > 0$*, where* $u_+(y) := \max\{u(y), 0\}$ *and moreover* $A_{0,1} = \{x \in B_1(0) : \; u(x) > 0\}$*; finally* $c \equiv c(n, M, \gamma_1, \gamma_2, \mathcal{Q})$*.*

Remark 4.3 Estimates (4.5) and (4.6) can be proved (possibly with a different constant c) with p_2 inside the integral replaced by $p(y)$ and $1/p_2$ outside replaced by $1/p_1$. The reason for these choices of the exponents will be clearer in Proposition 4.12 (see in particular (4.54), (4.56) and (4.57)).

Proof.

First step: De Giorgi type estimates. For any $k \in \mathbb{R}$ we define the sets

$$A_{k,\sigma} = \{x \in B_\sigma(0) : u(x) > k\}, \quad B_{k,\sigma} := \{x \in B_\sigma(0) : u(x) < k\}.$$

We claim that for any $k \in \mathbb{R}$, u satisfies the inequalities

$$\int_{A_{k,\sigma}} |Du(y)|^{p(y)}\, dy \leq \; c_1 \int_{A_{k,\tau}} \left| \frac{u(y) - k}{\tau - \sigma} \right|^{p(y)} dy + c_2\, |A_{k,\tau}|, \tag{4.7}$$

and

$$\int_{B_{k,\sigma}} |Du(y)|^{p(y)}\, dy \leq \; c_1 \int_{B_{k,\tau}} \left| \frac{u(y) - k}{\tau - \sigma} \right|^{p(y)} dy + c_2\, |B_{k,\tau}|, \tag{4.8}$$

for any $1/2 \leq \sigma < \tau \leq 1$, and with $c_1, c_2 \equiv c_1, c_2(\mathcal{Q}, \gamma_1, \gamma_2)$.

Indeed, for $1/2 \leq \sigma \leq s < t \leq \tau \leq 1$ let $\eta \in C_0^\infty(B_1(0))$ with $\operatorname{spt} \eta \subset B_t, \eta \equiv 1$ on $B_s(0)$, $|D\eta| \leq \frac{2}{t-s}$ be a standard cut-off function. We set $z(y) := u(y) - \eta\, \tilde{w}(y)$, where $\tilde{w}(y) := \max\{u(y) - k, 0\}$. Testing the $\mathcal{Q}$ minimality we therefore obtain

$$\begin{aligned}
\int_{A_{k,s}} |Du(y)|^{p(y)}\, dy \;&\leq\; \int_{A_{k,t}} |Du(y)|^{p(y)}\, dy \\
&\leq\; \mathcal{Q} \int_{A_{k,t}} \left[|Dz(y)|^{p(y)} + 1 \right] dy \\
&=\; \mathcal{Q} \int_{A_{k,t}} \left[|(1 - \eta)\, Du - D\eta(u - k)|^{p(y)} + 1 \right] dy
\end{aligned}$$

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$$\leq \quad \mathcal{Q}\left[c\int_{A_{k,t}\setminus A_{k,s}}|Du|^{p(y)}\,dy+c\int_{A_{k,t}}\left|\frac{u-k}{t-s}\right|^{p(y)}\,dy+|A_{k,t}|\right],$$

with $c \equiv c(\gamma_1,\gamma_2)$. Adding on both sides of the inequality the quantity $c\,\mathcal{Q}\int_{A_{k,s}}|Du(y)|^{p(y)}\,dx$ and dividing the resulting inequality by $1+c\,\mathcal{Q}$, we obtain

$$\int_{A_{k,s}}|Du(y)|^{p(y)}\,dy \leq \theta\int_{A_{k,t}}|Du(y)|^{p(y)}\,dy + c\int_{A_{k,\tau}}\left|\frac{u-k}{t-s}\right|^{p(y)}\,dy + c|A_{k,\tau}|,$$

with $\theta \equiv \frac{c\mathcal{Q}}{1+c\mathcal{Q}} < 1$, $c \equiv c(\mathcal{Q},\gamma_1,\gamma_2)$. Now we apply Lemma 4.1, taking into account assumption (4.4) with the particular choices $f(t) := \int_{A_{k,t}}|Du(y)|^{p(y)}\,dy$, θ as above, $A = c$, $B = c|A_{k,\tau}|$ to deduce (4.7) with $c_1, c_2 \equiv c_1, c_2(\mathcal{Q},\gamma_1,\gamma_2)$.

To prove (4.8), we remark that the function $-u$ is also a $\mathcal{Q}$ minimizer of the functional (4.2), so that we may argue in exactly the same way as above to deduce (4.7) for $-u$ (where also in the set $A_{k,s}$ we may replace u by $-u$), which easily translates into (4.8).

Second step: Boundedness of u: estimate (4.5).

Our aim is now to prove (4.5). Therefore we start by showing

$$\sup_{B_{1/2}(0)} u \leq c\left[\left(\int_{B_1(0)} u_+^{p_2}\,dy\right)^{1/p_2}+1\right], \tag{4.9}$$

where $u_+(y) := \max\{u(y),0\}$.

For $1/2 \leq \rho < r \leq 1$ let η be a function of class $C_0^\infty(B_{\frac{\rho+r}{2}})$ with $\eta \equiv 1$ on B_ρ and $|D\eta| \leq \frac{4}{r-\rho}$. Denoting by $p_1^* := \frac{np_1}{n-p_1}$ the Sobolev conjugate of p_1, we introduce the quantities

$$\varepsilon := 1-\frac{p_2}{p_1^*} = \frac{p_2}{n}-\frac{(p_2-p_1)}{p_1}, \qquad \beta := \varepsilon+\frac{p_2}{p_1} = 1+\frac{p_2}{n}.$$

Due to assumption (4.3), we have $p_2 \leq p_1^*$. Introducing the quantity

$$\Phi_{k,\rho} := \int_{A_{k,\rho}}(u-k)^{p_2}\,dy,$$

we now show that for arbitrary $h<k$ there holds

$$\Phi_{k,\rho} \leq c\,\Phi_{h,r}^{\beta}\left(\frac{1}{|k-h|^{p_2}}\right)^{\varepsilon}\left[\frac{1}{|r-\rho|^{p_2}}+\frac{1}{|k-h|^{p_2}}\right]^{p_2/p_1}. \tag{4.10}$$

In a first step, setting $\zeta = \eta(u-k)_+$ for any $k \in \mathbb{R}$, we deduce by Hölder's and Sobolev-Poincaré's inequality

$$\begin{aligned}
\int_{A_{k,\rho}}(u-k)^{p_2}\,dy &= \int_{A_{k,\rho}}\zeta^{p_2}\,dy \leq \int_{A_{k,r}}\zeta^{p_2}\,dy\\
&\leq c\left[\int_{A_{k,r}}\zeta^{p_1^*}\,dy\right]^{\frac{p_2}{p_1^*}}|A_{k,r}|^{\varepsilon}\\
&\leq c\,|A_{k,r}|^{\varepsilon}\left[\int_{A_{k,r}}|D\zeta|^{p_1}\,dy\right]^{\frac{p_2}{p_1}}\\
&\leq c\,|A_{k,r}|^{\varepsilon}\left[\int_{A_{k,r}}(|D\zeta|^{p(y)}+1)\,dy\right]^{\frac{p_2}{p_1}}
\end{aligned}$$

$$\leq \quad c\left[\int_{A_{k,\frac{\rho+r}{2}}} |Du|^{p(y)}\,dy + \int_{A_{k,\frac{\rho+r}{2}}} \left|\frac{u-k}{r-\rho}\right|^{p(y)}\,dy + |A_{k,r}|\right]^{\frac{p_2}{p_1}} |A_{k,r}|^{\varepsilon},$$

with $c \equiv c(n,\gamma_1,\gamma_2,\mathcal{Q})$. Implementing (4.7) in the previous estimate, we obtain for any $k \in \mathbb{R}$

$$\begin{aligned}\int_{A_{k,\rho}} (u-k)^{p_2}\,dy \quad &\leq \quad c\,|A_{k,r}|^{\varepsilon}\left[\int_{A_{k,r}} \left|\frac{u-k}{r-\rho}\right|^{p(y)}\,dy\right]^{\frac{p_2}{p_1}} + c\,|A_{k,r}|^{\beta} \\ &\leq \quad c\,|A_{k,r}|^{\varepsilon}\left[\int_{A_{k,r}} \left|\frac{u-k}{r-\rho}\right|^{p_2}\,dy\right]^{\frac{p_2}{p_1}} + c\,|A_{k,r}|^{\beta},\end{aligned} \tag{4.11}$$

with $c \equiv c(n,\gamma_1,\gamma_2,\mathcal{Q})$.

Next, we remark that for $h < k$ we have

$$|A_{k,r}| \leq \int_{A_{h,r}} \left|\frac{u-h}{k-h}\right|^{p_2}\,dy, \tag{4.12}$$

since $u-h > k-h$ on $A_{k,r}$, and moreover we deduce

$$\int_{A_{k,r}} (u-k)^{p_2}\,dy \leq \int_{A_{k,r}} (u-h)^{p_2}\,dy \leq \int_{A_{h,r}} (u-h)^{p_2}\,dy. \tag{4.13}$$

Introducing these relations in (4.11) we obtain

$$\begin{aligned}\Phi_{k,\rho} \quad &\leq \quad c\left(\int_{A_{h,r}} \left|\frac{u-h}{k-h}\right|^{p_2}\,dy\right)^{\varepsilon}\left(\int_{A_{h,r}} \left|\frac{u-h}{r-\rho}\right|^{p_2}\,dy\right)^{\frac{p_2}{p_1}} + c\left(\int_{A_{h,r}} \left|\frac{u-h}{k-h}\right|^{p_2}\,dy\right)^{\beta} \\ &\leq \quad c\,\Phi_{h,r}^{\beta}\left(\frac{1}{|k-h|^{p_2}}\right)^{\varepsilon}\left[\frac{1}{(r-\rho)^{p_2}} + \frac{1}{(k-h)^{p_2}}\right]^{\frac{p_2}{p_1}},\end{aligned}$$

with $c \equiv c(n,\gamma_1,\gamma_2,\mathcal{Q})$, the desired estimate (4.10).

Our aim is now to deduce a decay estimate for the quantity $\Phi_{k,\rho}$ to decreasing levels k on balls of increasing radii ρ. For this purpose we will take use of Lemma 3.3. Let us define the sequences of levels and radii

$$k_i = 2\,d\,(1-2^{-i-1}), \quad \rho_i = \frac{1}{2}\,(1+2^{-i}),$$

and the quantity

$$\chi_i := d^{-p_2}\,\Phi_{k_i,\rho_i} = d^{-p_2}\int_{A_{k_i,\rho_i}} (u-k_i)^{p_2}\,dy,$$

where $d \geq 1$ is a constant that will be chosen later. First, we note that

$$k_{i+1} - k_i = \frac{d}{2}\,2^{-i}, \qquad \rho_i - \rho_{i+1} = \frac{1}{4}\,2^{-i}.$$

Exploiting (4.10) with the choices $k = k_{i+1}, h = k_i, \rho = r_{i+1}, r = r_i$ and the fact that $d \geq 1$, we obtain

$$\begin{aligned}\chi_{i+1} &= d^{-p_2}\,\Phi_{k_{i+1},\rho_{i+1}} \\ &\leq cd^{-p_2}\,\Phi_{k_i,\rho_i}^{\beta}\left(d^{-1}2^{i+1}\right)^{p_2\varepsilon}\left[(4\cdot 2^i)^{p_2} + \left(d^{-1}2^{i+1}\right)^{p_2}\right]^{p_2/p_1} \\ &= c\,\chi_i^{\beta}\,d^{-p_2(1-\beta)}(d/2)^{-p_2\varepsilon}2^{ip_2\varepsilon}\left[4^{p_2}2^{p_2 i} + (d/2)^{-p_2}\,2^{ip_2}\right]^{p_2/p_1} \\ &\leq c\,d^{\frac{p_2}{p_1}(p_2-p_1)}2^{ip_2\beta}\chi_i^{\beta},\end{aligned} \tag{4.14}$$

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with $c \equiv c(n, \gamma_1, \gamma_2, \mathcal{Q})$.

Next we show that with the choice

$$d := 1 + \mathcal{A} \left(\int_{B_1(0)} u_+^{p_2} \, dy \right)^{1/p_2} , \tag{4.15}$$

where we determine the quantity $\mathcal{A}$ a bit later, the hypotheses of Lemma 3.3 are fulfilled for the sequence $(\chi_i)_i$.

To see this, let us first note that, since $u \in W^{1,p(x)}(B_1(0))$ and by the assumption $p_2 \leq p_1^*$, via Sobolev-Poincaré's inequality, we may conclude that $u \in L^{p_2}(B_1(0))$ and

$$\int_{B_1(0)} |u|^{p_2} \, dy \leq c(M).$$

This allows us to estimate

$$\int_{B_1(0)} u_+^{p_2} \, dy \leq \int_{B_1(0)} |u|^{p_2} \, dy \leq c(M),$$

and therefore

$$d^{\frac{p_2}{p_1}(p_2 - p_1)} \leq c(M, p_1, p_2) \left(1 + \mathcal{A}^{\frac{p_2}{p_1}(p_2 - p_1)} \right).$$

Consequently, (4.14) writes as

$$\chi_{i+1} \leq \tilde{c}(n, M, \gamma_1, \gamma_2, \mathcal{Q}) \left(1 + \mathcal{A}^{\frac{p_2}{p_1}(p_2 - p_1)} \right) 2^{ip_2\beta} \chi_i^{\beta}.$$

On the other hand, the choice of d and the fact that $d \geq 1$ immediately give

$$\chi_0 = d^{-p_2} \int_{A_{d,1}} (u - d)^{p_2} \, dy \leq \mathcal{A}^{-p_2} \left(\int_{B_1(0)} u_+^{p_2} \, dy \right)^{-1} \int_{A_{d,1}} (u - d)^{p_2} \, dy \leq \mathcal{A}^{-p_2}.$$

We apply Lemma 3.3 with the choices $\mathcal{B} \equiv 2^{\beta p_2} > 1$, $\mathcal{C} \equiv \tilde{c} \left(1 + \mathcal{A}^{\frac{p_2}{p_1}(p_2 - p_1)} \right) > 0$, $a \equiv \beta - 1 = \frac{p_2}{n}$. To guarantee that the condition $\chi_0 \leq \mathcal{C}^{-1/a} \mathcal{B}^{-1/a^2}$ is satisfied, we have to choose the quantity $\mathcal{A}$ in such a way that

$$\mathcal{A}^{p_2(\beta - 1)} = \mathcal{B}^{1/(\beta - 1)} \tilde{c} \left(1 + \mathcal{A}^{\frac{p_2}{p_1}(p_2 - p_1)} \right). \tag{4.16}$$

Note that, since $\beta - 1 \equiv \frac{p_2}{n}$ and by assumption $p_2 - p_1 < 1/n$, we always have that $p_2(\beta - 1) > \frac{p_2}{p_1}(p_2 - p_1)$, which guarantees that equation (4.16) has a unique solution $0 < \mathcal{A} \equiv \mathcal{A}(n, M, \gamma_1, \gamma_2, p_1, p_2, \mathcal{Q}) < +\infty$. Moreover we remark that with our global bounds γ_1, γ_2 for p we have that $p_2(\beta - 1) = p_2^2/n \in [\gamma_1^2/n, \gamma_2^2/n]$ and $p_2/p_1(p_2 - p_1) \in [0, \gamma_2/\gamma_1(\gamma_2 - \gamma_1)]$. Furthermore the solution $\mathcal{A}$ of equation (4.16) depends continuously on the parameters p_1 and p_2.

Lemma 3.3 now gives

$$\lim_{i \to \infty} \chi_i = 0,$$

which, noting that $\lim_{i \to \infty} \phi_1 = 1/2$ and $\lim_{i \to \infty} k_i = 2d$, directly translates into $\left| A_{2d,1/2} \right| = 0$ and therefore

$$\sup_{B_{1/2}(0)} u \leq 2d.$$

Taking into account the choice of d in (4.15), we end up with

$$\sup_{B_{1/2}(0)} u \leq c \left[\left(\int_{B_1(0)} u_+^{p_2} \, dy \right)^{1/p_2} + 1 \right],$$

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with $c \equiv c(n, \gamma_1, \gamma_2, \mathcal{A}, M, \mathcal{Q})$. At this point the above argumentation for the dependence of $\mathcal{A}$ on the bounds p_1 and p_2 allows us to conclude that the constant c can be replaced by a constant which depends only on the global bounds γ_1 and γ_2 instead of p_1 and p_2. (Note that c is a continuous function in p_1 and p_2 on a compact set $\{(p_1, p_2) \in [\gamma_1, \gamma_2]^2 : \, p_2 \geq p_1\}$.

An argument similar to the preceding one with the function $-u$, using (4.8) instead of (4.7) yields

$$\sup_{B_{1/2}(0)} (-u) \leq c \left[\left(\int_{B_1(0)} (-u)_+^{p_2} \, dy \right)^{1/p_2} + 1 \right]. \tag{4.17}$$

Therefore putting together (4.9) and (4.17), we finally deduce the desired estimate (4.5). □

Third step: Boundedness of u: estimate (4.6).

Starting again from (4.11), taking into account (4.12) and (4.13) and recalling that for any $h < k$ and $\rho \leq r$ we have $|A_{k,\rho}| \leq |A_{h,r}|$, we deduce instead of (4.10) the slightly different estimate

$$\Phi_{k,\rho} \leq c(n, \gamma_1, \gamma_2, \mathcal{Q}) \Phi_{h,r}^{p_2/p_1} |A_{h,r}|^{\varepsilon} \left[\frac{1}{|r-\rho|^{p_2}} + \frac{1}{|k-h|^{p_2}} \right]^{p_2/p_1}. \tag{4.18}$$

We observe that

$$\varepsilon - \tilde{\beta} = \frac{p_2}{n} - \tilde{\alpha}$$

where we can choose suitable $\tilde{\beta}, \tilde{\alpha}$ such that

$$\tilde{\beta} > \frac{(p_2 - 1)(p_2 - p_1)}{p_1} \qquad \tilde{\alpha} < \frac{p_2(p_2 - p_1)}{p_1}.$$

Taking into account that $|A_{k,\rho}| \leq |A_{h,r}|$ so that $|A_{k,\rho}|^{\tilde{\alpha}} \leq |A_{h,r}|^{\tilde{\alpha}}$, we deduce from (4.18)

$$\Phi_{k,\rho} \, |A_{k,\rho}|^{\tilde{\alpha}} \leq c \, \Phi_{h,r}^{p_2/p_1} \, |A_{h,r}|^{\tilde{\beta}+p_1/n} \left[\frac{1}{|r-\rho|^{p_2}} + \frac{1}{|k-h|^{p_2}} \right]^{p_2/p_1}.$$

At this point, taking into account that

$$\frac{p_2}{n} \geq \left(\tilde{\beta} + \frac{p_2}{p_1} \right) \tilde{\alpha}$$

we deduce that

$$|A_{h,r}|^{\tilde{\beta}+\frac{p_1}{n}} \leq |A_{h,r}|^{\tilde{\beta}} \, |A_{h,r}|^{(\tilde{\beta}+\frac{p_2}{p_1})\,\tilde{\alpha}} \leq \Phi_{h,r}^{\tilde{\beta}} \, |A_{h,r}|^{(\tilde{\beta}+\frac{p_2}{p_1})\,\tilde{\alpha}} \left[\frac{1}{|r-\rho|^{p_2}} + \frac{1}{|k-h|^{p_2}} \right]^{p_2/p_1}.$$

Therefore

$$\Phi_{k,\rho} \, |A_{k,\rho}|^{\tilde{\alpha}} \leq C \, \Phi_{h,r}^{\tilde{\beta}+\frac{p_2}{p_1}} \, \frac{1}{|k-h|^{p_2\,\tilde{\beta}}} \, |A_{h,r}|^{\left(\tilde{\beta}+\frac{p_2}{p_1}\right)\,\tilde{\alpha}} \left[\frac{1}{|r-\rho|^{p_2}} + \frac{1}{|k-h|^{p_2}} \right]^{p_2/p_1}.$$

Setting

$$\tilde{\Phi}_{k,t} := \Phi_{k,t} \, |A_{k,t}|^{\tilde{\alpha}},$$

we have

$$\tilde{\Phi}_{k,t} \leq C \, \tilde{\Phi}_{k,t}^{\tilde{\beta}+\frac{p_2}{p_1}} \left[\frac{1}{|r-\rho|^{p_2}} + \frac{1}{|k-h|^{p_2}} \right]^{p_2/p_1} \frac{1}{|k-h|^{p_2\,\tilde{\beta}}}. \tag{4.19}$$

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To apply Lemma 3.3 we take $d \geq 1$ which we determine later and set

$$k_i := d(1-2^{-i}), \qquad r_i := \frac{1}{2}(1+2^{-i}).$$

and therefore have that

$$r_{i+1} - r_i = \frac{1}{4}2^{-i}, \qquad k_{i+1} - k_i = \frac{d}{2}2^{-i}.$$

Rewriting (4.19) with $\rho := r_{i+1}, r := r_i, k := k_{i+1}, h := k_i$ and $\chi_i := d^{-p_2}\tilde{\Phi}_{k_i,r_i}$ and exploiting again the fact that $d \geq 1$, we deduce

$$\begin{aligned}
\chi_{i+1} &= d^{-p_2}\,\tilde{\Phi}_{k_i,r_i} \\
&\leq c\,d^{-p_2}\,\tilde{\Phi}_{k_i,r_i}^{\tilde{\beta}+\frac{p_2}{p_1}}\left[\frac{1}{(r_{i+1}-r_i)^{p_2}} + \frac{1}{(k_{i+1}-k_i)^{p_2}}\right]^{\frac{p_2}{p_1}} \frac{1}{(k_{i+1}-k_i)^{p_2\,\tilde{\beta}}} \\
&\leq c\,d^{-p_2}\left[\frac{4^{p_2}}{2^{-ip_2}} + \frac{2^{p_2}}{d^{p_2}2^{-ip_2}}\right]^{\frac{p_2}{p_1}} \frac{2^{p_2\tilde{\beta}}}{d^{p_2\tilde{\beta}}2^{-ip_2\tilde{\beta}}} d^{p_2(\tilde{\beta}+\frac{p_2}{p_1})}\chi_i^{\tilde{\beta}+\frac{p_2}{p_1}} \\
&\leq c(n,\gamma_1,\gamma_2,\tilde{\beta})d^{\frac{p_2}{p_1}(p_2-p_1)}2^{ip_2(\tilde{\beta}+\frac{p_2}{p_1})}\chi_i^{\tilde{\beta}+\frac{p_2}{p_1}}.
\end{aligned}$$

We now choose

$$d := 1 + \tilde{\mathcal{A}}\left(\int_{A_{0,1}} u^{p_2}\,dy\right)^{1/p_2} |A_{0,1}|^{\tilde{\beta}/p_2}, \tag{4.20}$$

where $\tilde{\mathcal{A}}$ will be fixed a bit later. Analoguously to the preceding argumentation we observe that

$$d \leq 1 + \tilde{\mathcal{A}}\left(\int_{B_1(0)} u_+^{p_2}\,dy\right)^{1/p_2} |B_1(0)|^{\tilde{\beta}/p_2} \leq c(n,M,p_2,\tilde{\beta})(1+\tilde{\mathcal{A}}),$$

and therefore

$$d^{\frac{p_2}{p_1}(p_2-p_1)} \leq \bar{c}\left(1+\tilde{\mathcal{A}}^{\frac{p_2}{p_1}(p_2-p_1)}\right),$$

with $\bar{c} \equiv \bar{c}(n,M,\gamma_1,\gamma_2,\tilde{\beta})$. Moreover, with (4.20) we have

$$\chi_0 = d^{-p_2}\int_{A_{k_0,r_0}} (u-k_0)^{p_2}\,dy|A_{k_0,r_0}|^{\tilde{\beta}} = d^{-p_2}|A_{0,1}|^{\tilde{\beta}}\int_{A_{0,1}} u^{p_2}\,dy \leq \tilde{\mathcal{A}}^{-p_2}.$$

We set $\mathcal{B} \equiv 2^{p_2(\tilde{\beta}+p_2/p_1)}, \mathcal{C} \equiv \bar{c}\left(1+\tilde{\mathcal{A}}^{p_2/p_1(p_2-p_1)}\right)$, $a \equiv \tilde{\beta}+p_2/p_1-1$. The assumptions of Lemma 3.3 are satisfied, if we choose $\tilde{\mathcal{A}}$ in such a way that

$$\tilde{\mathcal{A}}^{\tilde{\beta}+\frac{p_2}{p_1}-1} = \bar{c}\mathcal{B}^{(\tilde{\beta}+\frac{p_2}{p_1}-1)^{-1}}\left(1+\tilde{\mathcal{A}}^{\frac{p_2}{p_1}(p_2-p_1)}\right).$$

(we note that $\tilde{\alpha}$ and $\tilde{\beta}$ can be chosen in such a way that the preceding equation has a unique solution $0 < \tilde{\mathcal{A}} \equiv \tilde{\mathcal{A}}(n,M,\gamma_1,\gamma_2,p_1,p_2,\mathcal{Q}) < +\infty$).

By Lemma 3.3 we conclude that

$$\lim_{i\to\infty}\chi_i = 0$$

and as

$$\lim_{i\to\infty} r_i = \frac{1}{2}, \qquad \lim_{i\to\infty} k_i = d,$$

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we have that $|A_{d,1/2}| = 0$, that is, using (4.20)

$$\sup_{B_{1/2}(0)} u \leq d = c \left(\int_{A_{0,1}} u^{p_2} \, dy \right)^{1/p_2} |A_{0,1}|^{\frac{\alpha}{p_2}} + 1,$$

i.e. (4.6) with $c \equiv c(n, M, \gamma_1, \gamma_2, \mathcal{A}, \mathcal{Q})$. Again we remark here that by a compactness argument, together with the notes above, concerning the parameter $\tilde{\mathcal{A}}$, we conclude that the constant c may be replaced by a constant $c \equiv c(n, M, \gamma_1, \gamma_2, \mathcal{Q})$. □

4.3 A priori hölder continuity result.

Let $B_R \equiv B(x_0, R) \subset \Omega$ and let $p : B_R \to (1, \infty)$ be a uniformly continuous function which modulus of continuity fulfills condition (2.1) (and therefore in particular (2.2)); suppose that there exist constants γ_1, γ_2 such that

$$\gamma_1 \leq p_1 := \min_{B_R} p(x) \leq p(x) \leq p_2 := \max_{B_R}(x) \leq \gamma_2. \tag{4.21}$$

We set

$$H(R) = L \int_{B_R} (|D\psi|^{p(x)} + 1) \, dx, \qquad K := K(R) := \left(1 + \frac{H(R)}{R^n} \right)^{1/p_1},$$

where L appears in (2.2) and with ψ given function satisfying (2.6). We moreover set

$$\gamma := \frac{\lambda - n + p_2}{p_2} > 0. \tag{4.22}$$

Let us note that by (2.2) we may choose the radius $R \leq R_1 \equiv R_1(n, \lambda, \gamma_1, \gamma_2, L, \omega(\cdot)) > 0$ so small that

$$R \leq \left(\frac{1}{L} \right)^{\frac{1}{n-\lambda}}, \qquad \left(\frac{2}{L} R^\lambda \right)^{\frac{p_2 - p_1}{p_1}} \leq 1. \tag{4.23}$$

Since $H(R) \leq \frac{R^\lambda}{L}$, we immediately deduce

$$K = \left(1 + \frac{H(R)}{R^n} \right)^{1/p_1} \leq \left(1 + \frac{1}{L} R^{\lambda - n} \right)^{1/p_1} \leq \left(\frac{2}{L} \right)^{1/p_1} R^{\frac{\lambda - n}{p_1}}. \tag{4.24}$$

On the other hand, by the continuity of p and (2.2) we obtain

$$K^{p_2 - p_1} \leq \left(\frac{2}{L} \right)^{\frac{p_2 - p_1}{p_1}} \left(R^{\lambda - n} \right)^{\frac{p_2 - p_1}{p_1}} \leq \left(R^{-n(p_2 - p_1)} \right)^{1/p_1} \leq \left(R^{-n\omega(2R)} \right)^{1/p_1} \leq \bar{c}(n, L, \gamma_1, \gamma_2, \omega(\cdot)),$$

which yields

$$K^{p_2} \geq K^{p(x)} \geq K^{p_1} \geq \frac{1}{\bar{c}} K^{p_2} \geq \frac{1}{\bar{c}} K^{p(x)}. \tag{4.25}$$

Let now $v \in W^{1,p(x)}_{\mathrm{loc}}(B_R)$ be a free local $\mathcal{Q}$ minimizer of the functional

$$w \mapsto \int_{B_R} \left(|Dw|^{p(x)} + \frac{H(R)}{R^n} + 1 \right) dx. \tag{4.26}$$

The aim of this section is to prove that v is locally Hölder continuous and provide a useful decay estimate (namely (4.52)). This is the key to the proof of our main theorem, since it will turn out that the minimizer u of the original problem in the obstacle class (2.5) is in fact a local free $\mathcal{Q}$ minimizer of the functional above.

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The proof of the Hölder continuity of v is carried out by means of different steps, namely Lemmas 4.4, 4.9, 4.10 and Propositions 4.11, 4.12. Throughout the section we will assume that there exists a constant M such that

$$\int_{B_R} |Dv|^{p(x)}\, dx \le M. \tag{4.27}$$

In a first step we show that a rescaled version for the function v is a local free $\tilde{\mathcal{Q}}$ minimizer of the functional (4.2) and therefore locally bounded, satisfying estimates in spirit of (4.5).

Lemma 4.4 *Let $v \in W^{1,p(x)}_{\mathrm{loc}}(B_R)$ be a local $\mathcal{Q}$ minimizer of the functional* (4.26)*. Then v satisfies the following estimates*

$$\sup_{B_{R/2}} v \le c \left(\fint_{B_R} v_+^{p_2}\, dx \right)^{1/p_2} + c\, R^{\gamma}, \tag{4.28}$$

$$\sup_{B_{R/2}} (-v) \le c \left(\fint_{B_R} (-v)_+^{p_2}\, dx \right)^{1/p_2} + c\, R^{\gamma}, \tag{4.29}$$

with $c \equiv c(n, \gamma_1, \gamma_2, \mathcal{Q})$ and additionally

$$\sup_{B_{R/2}} v \le c \left(\fint_{B_R} (v-\kappa_0)_+^{p_2}\, dx \right)^{\frac{1}{p_2}} \left| \frac{A_{\kappa_0,R}}{R^n} \right|^{\frac{\tilde{\alpha}}{p_2}} + R^{\gamma} + \kappa_0, \tag{4.30}$$

for some suitable $\tilde{\alpha}$, for all $\kappa_0 \le \sup_{B_R} v$ and with $c \equiv c(n, L, \gamma_1, \gamma_2, \mathcal{Q})$.

Remark 4.5 Estimates (4.28) and (4.29) still hold if v and $-v$ are replaced respectively by $v-\kappa_0$ and $\kappa_0 - v$, for any $\inf_{B_R} v \le \kappa_0 \le \sup_{B_R} v$. For the justification on these restrictions on κ_0 see Remark 4.6.

Remark 4.6 We just would like to focus our attention on the restriction on κ_0 in (4.30). It is clear that (4.30) is interesting only for $\kappa_0 \le \sup_{B_R} v$, because otherwise we would have $(v-\kappa_0)_+ = 0$. On the other hand this makes sense as, by (4.28), if v is a local $\mathcal{Q}$ minimizer of the functional (4.26) then v is locally bounded. A similar argument justifies Remark 4.5. These restrictions on κ_0 will be used later (see (4.53)).

Proof.

First step: Rescaling the problem. We set

$$\tilde{v}(y) := \frac{1}{K\,R} v(x_0 + Ry), \tag{4.31}$$

and now show that $\tilde{v}$ is a local $\tilde{\mathcal{Q}}$ minimizer of the functional (4.2), with $\tilde{\mathcal{Q}} \equiv \tilde{\mathcal{Q}}(\mathcal{Q}, n, L, \gamma_1, \gamma_2)$ and $\tilde{p}(y) := p(x_0 + Ry)$.

Therefore let $\tilde{\varphi} \in C_c^{\infty}(B_1(0))$. Then $\varphi(x) := K\,R\,\tilde{\varphi}(\frac{x-x_0}{R}) \in C_c^{\infty}(B_R(x_0))$. By the $\mathcal{Q}$ minimality of v, also using (4.25), we obtain (denoting $\tilde{S} := \mathrm{spt}(\tilde{\varphi})$ and $S := \mathrm{spt}(\varphi)$):

$$\begin{aligned}
\int_{\tilde{S}} \left(|D\tilde{v}(y)|^{\tilde{p}(y)} + 1 \right) dy &= \int_{\tilde{S}} \left(\left| \frac{Dv(x_0+Ry)}{K} \right|^{p(x_0+Ry)} + 1 \right) dy \\
&= \frac{1}{R^n} \int_S \left(\left| \frac{Dv(x)}{K} \right|^{p(x)} + 1 \right) dx \\
&\overset{(4.25)}{\le} \frac{1}{R^n K^{p_1}} \int_S \left(|Dv(x)|^{p(x)} + K^{p_1} \right) dx \\
&\le \frac{1}{R^n K^{p_1}} \mathcal{Q} \int_S \left(|Dv(x) + D\varphi(x)|^{p(x)} + K^{p_1} \right) dx \qquad (4.32) \\
&= \frac{1}{R^n} \mathcal{Q} \int_S \left(\frac{|Dv(x) + D\varphi(x)|^{p(x)}}{K^{p_1}} + 1 \right) dx
\end{aligned}$$

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$$\begin{aligned}&\overset{(4.25)}{\leq} \quad \frac{1}{R^n}\mathcal{Q}\bar{c}\int_S\left(\left|\frac{Dv(x)+D\varphi(x)}{K}\right|^{p(x)}+1\right)\,dx\\ &= \quad \mathcal{Q}\bar{c}\int_{\tilde{S}}\left(|D\tilde{v}(y)+D\tilde{\varphi}(y)|^{\tilde{p}(y)}+1\right)\,dy,\end{aligned}$$

which yields the desired estimate with $\tilde{\mathcal{Q}} \equiv \tilde{\mathcal{Q}}(\mathcal{Q}, n, L, \gamma_1, \gamma_2) := \mathcal{Q}\bar{c}$.

Second step: Sup– estimates. The above estimate shows, that $\tilde{v}$ is a local $\tilde{\mathcal{Q}}$ minimizer of the functional (4.2), where the exponent function p is replaced by the rescaled function $\tilde{p}$. On the other hand, since $\tilde{p}_2 = p_2$ and $\tilde{p}_1 = p_1$, assumption (4.3) is satisfied, if we take the radius $R \leq R_2 \equiv R_2(n, \omega(\cdot))$ so small that $p_2 - p_1 \leq \frac{1}{n}$. Moreover, by (4.27) also (4.4) is satisfied. This allows us to apply Lemma 4.2 which yields, that estimates (4.5), (4.5) and especially (4.9) and (4.17) hold for the function $\tilde{v}$.

The desired estimates (4.28) and (4.29) now follow by rescaling. Noting that

$$\sup_{y\in B_{1/2}(0)}\tilde{v}(y) = \sup_{Ry\in B_{R/2}(0)}\frac{v(x_0+Ry)}{KR} = \frac{1}{KR}\sup_{x_0+Ry\in B_{R/2}(x_0)}v(x_0+Ry) = \frac{1}{KR}\sup_{x\in B_{R/2}}v(x).$$

and on the other hand that

$$\left[\int_{B_1(0)}\tilde{v}_+^{\tilde{p}_2}\,dy\right]^{1/\tilde{p}_2} = \left[\int_{B_R}\left(\frac{v_+}{KR}\right)^{p_2}\frac{1}{R^n}\,dx\right]^{1/p_2} = \left(\frac{1}{KR}\right)\left[\fint_{B_R}v_+^{p_2}\,dx\right]^{1/p_2},$$

finally multiplying by $K\,R$ and taking use of (2.2), (2.6), (4.22) and (4.25), we obtain

$$K\,R \leq c\,R^{\frac{\lambda-n}{p_1}+1} = c\,R^{\frac{\lambda-n+p_1}{p_1}} = c\,R^{\frac{p_2}{p_1}\gamma}\,R^{\frac{p_1-p_2}{p_1}} \leq c\,R^{\gamma},$$

with $c \equiv c(\gamma_1, \gamma_2, L)$, from what we deduce (4.28).

Estimate (4.30) can be achieved via (4.6) by a similar argument, taking into account that $|A(0,R)| = R^n\,|A(0,1)|$ and then writing $v - \kappa_0$ instead of v. □

Remark 4.7 We note that the ability of rescaling the $\mathcal{Q}$ minimizer v of (4.26) in such a way that one obtains a $\mathcal{Q}$ minimizer $\tilde{v}$ of the functional (4.2), which is completely independent of the obstacle, is mainly due to the fact that the obstacle lies in an appropriate Morrey space (see the argumentation for (4.25)).

Remark 4.8 If in Lemma 4.4 we replace (4.31) by

$$\tilde{v}(y) := \frac{1}{K(r)\,r}v(x_0+ry).$$

for any r such that $B_r \Subset B_R$, we have that (4.28), (4.29) and (4.30) still hold with R replaced by r; this because in (4.32) we only used the definition of local $\mathcal{Q}$ minimizer which can be applied also in this new situation as if $\varphi \in C_c^\infty(B_r(x_0))$ with $B_r \Subset B_R$, then in particular $\varphi \in C_c^\infty(B_R(x_0))$.

Lemma 4.9 *Let $v \in W^{1,p(x)}_{\mathrm{loc}}(B_R)$ be a local $\mathcal{Q}$ minimizer of the functional* (4.26)*, satisfying* (4.27)*. Then for every couple of balls $B_\rho \subset B_r \Subset B_R$ having the same center x_0 and for every $k \in \mathbb{R}$ v fulfills the following two estimates*

$$\int_{A_{k,\rho}}|Dv|^{p(x)}\,dx \leq c\left[\int_{A_{k,r}}\left|\frac{v-k}{r-\rho}\right|^{p(x)}dx + r^\lambda\right], \tag{4.33}$$

and

$$\int_{B_{k,\rho}}|Dv|^{p(x)}\,dx \leq c\left[\int_{B_{k,r}}\left|\frac{v-k}{r-\rho}\right|^{p(x)}dx + r^\lambda\right], \tag{4.34}$$

with $c \equiv c(\mathcal{Q}, \gamma_1, \gamma_2)$, where we set

$$A_{k,r} = \{x \in B_r : v(x) > k\}, \qquad B_{k,r} = \{x \in B_r : v(x) < k\}.$$

Proof. We use an argument similar to the one employed to achieve (4.7) and (4.8). For $\rho \leq t < s \leq r$, let $\eta \in C_c^\infty(B_r)$ with $\operatorname{spt}\eta \subset B_s$, $\eta \equiv 1$ on B_t, $|D\eta| \leq \frac{2}{s-t}$ be a standard cut-off function. We set $z(x) := v(x) - \eta\, w(x)$, where $w(x) := \max\{v(x) - k, 0\}$. Testing the $\mathcal{Q}$ minimality we obtain

$$\begin{aligned}\int_{A_{k,t}} |Dv|^{p(x)}\,dx &\leq \int_{A_{k,s}} \left(\mu^2 + |Dv|^2\right)^{p(x)/2}\,dx \\ &\leq \mathcal{Q}\int_{A_{k,s}} \left(\mu^2 + |Dz|^2\right)^{p(x)/2}\,dx + (\mathcal{Q}-1)\int_{A_{k,s}} \left(\frac{H(s)}{s^n} + 1\right)\,dx.\end{aligned}$$

The second integral we handle, using (2.6), as follows

$$\begin{aligned}\int_{A_{k,s}} \left(\frac{H(s)}{s^n} + 1\right)\,dx &\leq \int_{A_{k,s}} Ls^{-n}\left(\int_{B_s} |D\psi|^{p(y)} + 1\,dy\right)\,dx + |B_s| \\ &\leq s^{-n}L\int_{B_s}\int_{B_s} |D\psi|^{p(y)}\,dy\,dx + \int_{B_s}\int_{B_s} s^{-n}L\,dy\,dx + |B_s| \\ &\leq cs^\lambda + Ls^n + \omega_n\,s^n \leq cs^\lambda.\end{aligned}$$

We estimate the first integral as follows:

$$\begin{aligned}&\int_{A_{k,s}} \left(\mu^2 + |Dz|^2\right)^{p(x)/2}\,dx \\ &\qquad= \int_{A_{k,s}} \left(\mu^2 + |(1-\eta)Dv - D\eta(v-k)|^2\right)^{p(x)/2}\,dx \\ &\qquad\leq \left[c\int_{A_{k,s}\setminus A_{k,t}} \left(\mu^2 + |Dv|^2\right)^{p(x)/2}\,dx + c\int_{A_{k,s}} \left|\frac{v-k}{s-t}\right|^{p(x)}\,dx + |A_{k,s}|\right] \\ &\qquad\leq \left[c\int_{A_{k,s}\setminus A_{k,t}} |Dv|^{p(x)}\,dx + c\int_{A_{k,s}} \left|\frac{v-k}{s-t}\right|^{p(x)}\,dx\right].\end{aligned}$$

Putting these estimates together we deduce

$$\int_{A_{k,t}} |Dv|^{p(x)}\,dx \leq \tilde{c}\mathcal{Q}\left[\int_{A_{k,s}\setminus A_{k,t}} |Dv|^{p(x)}\,dx + \int_{A_{k,s}} \left|\frac{v-k}{s-t}\right|^{p(x)}\,dx\right] + cs^\lambda,$$

with $c \equiv c(n, \gamma_1, \gamma_2)$ and $c \equiv c(n, \mathcal{Q}, ||D\psi||_{L^{q,\lambda}(B_R)})$. Now, adding on both sides of the inequality the term $\mathcal{Q}\tilde{c}\int_{A_{k,t}} |Dv|^{p(x)}\,dx$ and dividing the resulting inequality by $1 + \mathcal{Q}\tilde{c}$, we obtain

$$\int_{A_{k,t}} |Dv|^{p(x)}\,dx \leq \theta\int_{A_{k,s}} |Dv|^{p(x)}\,dx + c\int_{A_{k,s}} \left|\frac{v(x)-k}{s-t}\right|^{p(x)}\,dx + c\,r^\lambda,$$

for any $\rho \leq t < s \leq r$ with $\theta \equiv \frac{\mathcal{Q}\tilde{c}}{1+\mathcal{Q}\tilde{c}} < 1$, $c \equiv c(n, \mathcal{Q}, \gamma_1, \gamma_2)$ and $\theta \equiv \theta(n, \mathcal{Q}, \gamma_1, \gamma_2)$. Therefore, taking into account (4.27), Lemma 4.1 provides the desired inequality (4.33).

On the other hand, repeating exactly the same argument for the function $-v$, which is also a $\mathcal{Q}$ minimizer of the functional (4.26), we end up with (4.34). □

Let us introduce some additional notation which we will use for the rest of this section. For a given radius r and a function v we define

$$M(r) := \sup_{B_r} v, \qquad m(r) := \inf_{B_r} v, \tag{4.35}$$

and

$$\operatorname{osc}(v, r) := \max_{B_r} v - \min_{B_r} v. \tag{4.36}$$

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Moreover for a given integer $i \in \mathbb{N}$ we define the quantity

$$k_i := M(4r) - 2^{-i-1}\mathrm{osc}(v, 4r). \tag{4.37}$$

The following Lemma is a rather technical one, which will be useful for the proof of Proposition 4.11.

Lemma 4.10 *Let $v \in W^{1,p(x)}_{\mathrm{loc}}(B_R)$ be a local $\mathcal{Q}$ minimizer of the functional* (4.26) *and let $\kappa_0 = \frac{1}{2}(M(4r) + m(4r))$ for some $B_{4r} \Subset B_R$. Assume that*

$$|A_{\kappa_0,2r}| \leq \gamma_0 \, |B_{2r}| \qquad \text{for some } \gamma_0 < 1. \tag{4.38}$$

If for an integer ν it holds that

$$\mathrm{osc}\,(v, 4r) \geq 2^{\nu+1} \, r^{\gamma}, \tag{4.39}$$

where $\gamma \equiv \frac{\lambda-n+p_2}{p_2}$, as introduced in (4.22)*, then there holds*

$$|A_{k_\nu,2r}| \leq c_\nu \, \nu^{-\frac{n(p_1-1)}{p_1(n-1)}} \, r^n, \tag{4.40}$$

with $c_\nu \equiv c_\nu(\gamma_1, \gamma_2, L, \mathcal{Q})$.

Proof. In a first step of the proof, let us define for arbitrary $\kappa_0 < h < k$ the function

$$w(x) = \begin{cases} (k-h) & \text{if} \quad v \geq k \\ (v-h) & \text{if} \quad h < v < k \\ 0 & \text{if} \quad v \leq k. \end{cases}$$

Since $w = 0$ in $B_{2r} \setminus A_{\kappa_0,r}$ and $|B_{2r} \setminus A_{\kappa_0,2r}| \geq (1-\gamma_0)\,|B_{2r}|$, due to (4.38), Sobolev's inequality provides

$$\left(\int_{B_{2r}} w^{\frac{n}{n-1}} \, dx \right)^{1-\frac{1}{n}} \leq c \int_{B_{2r}} |Dw| \, dx = \int_{\Delta} |Dw| \, dx = c \int_{\Delta} |Dv| \, dx,$$

where we set $\Delta = A_{h,2r} \setminus A_{k,2r}$. We therefore have

$$(k-h)\,|A_{k,2r}|^{1-\frac{1}{n}} \leq \left(\int_{B_{2r}} w^{\frac{n}{n-1}} \, dx \right)^{1-\frac{1}{n}} \leq c\,|\Delta|^{1-\frac{1}{p_1}} \left(\int_{A_{h,2r}} |Dv|^{p_1} \, dx \right)^{\frac{1}{p_1}}. \tag{4.41}$$

On the other hand, applying Lemma 4.9, estimate (4.33), additionally noting that $v - k \leq M(4r) - h$ on the set $A_{k,4r}$ we deduce (recalling also that $h < k$ and $\lambda < n$)

$$\begin{aligned} \int_{A_{k,2r}} |Dv|^{p_1} \, dx &\leq \int_{A_{k,2r}} (|Dv|^{p(x)} + 1) \, dx \\ &\leq c \int_{A_{k,4r}} \left| \frac{v-k}{r} \right|^{p(x)} dx + c\, r^{\lambda} \\ &\leq c\, r^{n-p_2} \, (M(4r) - h)^{p_1} + c\, r^{n-p_2} \, r^{\lambda-n+p_2} \\ &\leq c\, r^{n-p_2} \, \left[(M(4r)-h)^{p_1} + r^{\gamma\, p_2} \right], \end{aligned} \tag{4.42}$$

with $c \equiv c(\gamma_1, \gamma_2, \mathcal{Q}, L)$, where we assumed that $M(4r) - h \leq 1$, due to the fact that $h > \kappa_0$.

For $h \leq k \leq k_\nu$, we have, using (4.37) and (4.39)

$$M(4r) - h \geq M(4r) - k_\nu \geq r^{\gamma}, \tag{4.43}$$

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and hence, combining (4.41), (4.42) and (4.39), we end up with

$$\begin{aligned}(k-h)\,|A_{k,2r}|^{\frac{n-1}{n}} &\le c\,|\Delta|^{\frac{p_1-1}{p_1}}\, r^{\frac{n-p_2}{p_1}}\, \left[(M(4r)-h)^{p_1} + r^{\gamma\, p_2}\right]^{\frac{1}{p_1}} \\ &\overset{(4.43)}{\le} c\,|\Delta|^{\frac{p_1-1}{p_1}}\, r^{\frac{n-p_1}{p_1}}\, r^{\frac{p_1-p_2}{p_1}}\,(M(4r)-h).\end{aligned} \tag{4.44}$$

Exploiting the continuity (2.2), and therefore

$$r^{p_1-p_2} \le r^{-\omega(r)} \le c(L),$$

the preceding inequality simplifies to

$$(k-h)\,|A_{k,2r}|^{\frac{n-1}{n}} \le c\,|\Delta|^{\frac{p_1-1}{p_1}}\, r^{\frac{n-p_1}{p_1}}\,(M(4r)-h). \tag{4.45}$$

In a second step, we apply estimate (4.45) to the levels

$$k = k_i = M(4r) - 2^{-i-1}\,\mathrm{osc}\,(v,4r), \qquad h = k_{i-1}.$$

Noting that

$$k_i - k_{i-1} = 2^{-i}\,\frac{1}{2}\,\mathrm{osc}\,(v,4r) \qquad M(4r)-h = M(4r)-k_{i-1} = 2^{-i}\,\mathrm{osc}\,(v,4r),$$

and

$$\Delta_i = A_{k_{i-1},2r} \setminus A_{k_i,2r},$$

and raising both sides of estimate (4.45) to the power $\frac{p_1}{p_1-1}$, we obtain (we recall that $k_i \le k_\nu$, so $|A_{k_\nu,2r}| \le |A_{k_i,2r}|$)

$$|A_{k_\nu,2r}|^{\frac{p_1(n-1)}{n(p_1-1)}} \le |A_{k_i,2r}|^{\frac{p_1(n-1)}{n(p_1-1)}} \le c\,r^{\frac{n-p_1}{p_1-1}}\,|\Delta_i|.$$

Summing up the preceding estimate for $i = 1\ldots\nu$ and taking into account that, since $k_0 = \kappa_0$, there holds

$$\sum_{i=1}^{\nu} |\Delta_i| = |\{x \in B_{2r} :\ k_0 < v(x) \le k_\nu\}| \le |A_{\kappa_0,2r}|,$$

we obtain

$$\nu\,|A_{k_\nu,r}|^{\frac{p_1(n-1)}{n(p_1-1)}} \le c\,r^{\frac{n-p_1}{p_1-1}}\,|A_{\kappa_0,r}| \overset{(4.38)}{\le} c\,r^{\frac{p_1(n-1)}{p_1-1}}$$

and the desired inequality follows, with $c \equiv c(\gamma_1,\gamma_2,L,\mathcal{Q})$. $\square$

The following proposition is the key to the proof of Hölder continuity of the function v. It provides a quantitative estimate for the oscillations of v on shrinking balls, which will turn out to be the key for the quantitative estimates of Proposition 4.12.

Proposition 4.11 *Let $v \in W^{1,p(x)}_{\mathrm{loc}}(B_R)$ be a local $\mathcal{Q}$ minimizer of the functional* (4.26)*. Then v is locally Hölder continuous in B_R and there exists $0 < \alpha < \gamma < 1$ (where γ has been introduced in* (4.22)*) such that the following estimate holds*

$$\mathrm{osc}(v,\rho) \le c\left\{\left(\frac{\rho}{r}\right)^{\alpha} \mathrm{osc}(v,r) + \rho^{\alpha}\right\} \tag{4.46}$$

for every $\rho < r < R/4$, with $c \equiv c(n,L,\gamma_1,\gamma_2,\mathcal{Q})$.

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Proof. Let $0 < r < R/4$ and, as in the lemma before $\kappa_0 = \frac{1}{2}(M(4r) + m(4r))$. We may assume without loss of generality that

$$|A_{\kappa_0,2r}| \leq \frac{1}{2}\,|B_{2r}|, \tag{4.47}$$

since otherwise we would have $|B_{\kappa_0,2r}| = |B_{2r}| - |A_{\kappa_0,2r}| \leq \frac{1}{2}\,|B_{2r}|$ which translates into (4.47) if we replace v by $-v$. Recalling the definition of k_i in (4.37), for any given integer ν we have $k_\nu > \kappa_0$.

First we observe that due to Lemma 4.4, v is locally bounded and satisfies estimate (4.30), which we may write with R replaced by r and κ_0 replaced by k_ν:

$$\sup_{B_{r/2}}(v - k_\nu) \leq c\left[\left(\fint_{B_{2r}}(v - k_\nu)_+^{p_2}\,dx\right)^{1/p_2}\left|\frac{A_{k_\nu,2r}}{r^n}\right|^{\frac{\tilde{\alpha}}{p_2}} + r^\gamma\right].$$

We note at this point, that the replacement of R by r is justified via Lemma 4.8. On the other hand, since we may guarantee that $k_\nu \leq M(2r)$, estimate (4.30) holds, if we replace the ball B_r on the right hand side of the inequality by the ball of doubled radius B_{2r}.

Estimating the integral on the right hand side of the preceding inequality

$$\fint_{B_{2r}}(v - k_\nu)_+^{p_2}\,dx = cr^n\int_{A_{k_\nu,2r}}(v - k_\nu)^{p_2}\,dx \leq c\,\sup_{B_{2r}}(v - k_\nu)|A_{k_\nu,2r}|^{1/p_2},$$

we deduce

$$\sup_{B_{r/2}}(v - k_\nu) \leq \tilde{c}_\nu\,\sup_{B_{2r}}(v - k_\nu)\left|\frac{A_{k_\nu,2r}}{r^n}\right|^{\frac{\tilde{\alpha}+1}{p_2}} + r^\gamma, \tag{4.48}$$

with $\tilde{c}_\nu \equiv \tilde{c}_\nu(n, L, \gamma_1, \gamma_2, \mathcal{Q})$.

Let us choose now the integer ν in such a way that

$$\tilde{c}_\nu\, c_\nu^{\frac{\tilde{\alpha}+1}{p_2}}\,\nu^{-\frac{n(p_1-1)}{p_1(n-1)}\,\frac{\tilde{\alpha}+1}{p_2}} \leq \frac{1}{2},$$

where c_ν is the constant appearing in (4.40).

In the case that

$$\mathrm{osc}(v, 2r) \geq 2^{\nu+1}\,r^\gamma, \tag{4.49}$$

we are in the situation to apply Lemma 4.10, estimate (4.40) and (4.48) to conclude

$$\begin{aligned} M\left(\frac{r}{2}\right) - k_\nu &= \sup_{B_{r/2}}(v - k_\nu) \leq \tilde{c}_\nu\sup_{B_{2r}}(v - k_\nu)\left|\frac{A_{k_\nu,2r}}{r^n}\right|^{\frac{\tilde{\alpha}+1}{p_2}} + r^\gamma \\ &\leq \frac{1}{2}\,(M(4r) - k_\nu) + r^\gamma. \end{aligned}$$

Subtracting on both sides of the inequality the quantity $m\left(\frac{r}{2}\right)$ and using the fact that $m\left(\frac{r}{2}\right) \geq m(2r)$, we have, together with the definition of k_ν

$$\begin{aligned} \mathrm{osc}\left(v, \frac{r}{2}\right) &\leq k_\nu + \frac{1}{2}\,(M(4r) - k_\nu) + r^\gamma - m\left(\frac{r}{2}\right) \\ &= M(4r) - \frac{1}{2^{\nu+1}}\,\mathrm{osc}\,(v, 4r) + \frac{1}{2}\,\frac{1}{2^{\nu+1}}\,\mathrm{osc}\,(v, 4r) - m\left(\frac{r}{2}\right) + r^\gamma \\ &= \mathrm{osc}\,(v, 4r)\left[1 - \frac{1}{2^{\nu+1}} + \frac{1}{2^{\nu+2}}\right] + r^\gamma \\ &= \mathrm{osc}(v, 4r)\left(1 - \frac{1}{2^{\nu+2}}\right) + r^\gamma. \end{aligned} \tag{4.50}$$

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In conclusion, either the function $\operatorname{osc}(v,r)$ satisfies the above relation (4.49), wich implies (4.50), or else there holds

$$\operatorname{osc}(v,4r) \leq 2^{\nu+1}\, r^{\gamma}.$$

In any case we have

$$\operatorname{osc}\left(v,\frac{r}{2}\right) \leq \left(1-\frac{1}{2^{\nu+2}}\right) \operatorname{osc}(v,4r) + 2^{\nu+1}\, r^{\gamma},$$

for any radius $0<r<R/4$. Applying Lemma 3.4 with the choices $\tau = 1/8$ and $\delta = \log_\tau\left(1-\frac{1}{2^{\nu+2}}\right)$, moreover setting $\alpha := \min\{\delta,\gamma\}$ and requiring in particular that $0<\alpha<1$, we obtain (4.46). □

Proposition 4.12 *Let $v \in W^{1,p(x)}_{\mathrm{loc}}(B_R)$ be a local $\mathcal{Q}$ minimizer of the functional* (4.26)*. Then, for every $\rho<R$ we have*

$$\int_{B_\rho} |v-(v)_\rho|^{p_2}\, dx \leq c\left(\frac{\rho}{R}\right)^{n+p_2\,\alpha} \int_{B_R} |v-(v)_R|^{p_2}\, dx + c\,\rho^{n+p_2\,\alpha} \tag{4.51}$$

with $c \equiv c(n,\gamma_1,\gamma_2,L,\mathcal{Q})$, and

$$\int_{B_\rho} |Dv|^{p(x)}\, dx \leq c\left(\frac{\rho}{R}\right)^{n-p_2+p_2\,\alpha} \int_{B_R} |Dv|^{p(x)}\, dx + c\,\rho^{n-p_2+p_2\,\alpha}, \tag{4.52}$$

with $c \equiv c(n,\gamma_1,\gamma_2,L,\mathcal{Q},M)$ and where $0<\alpha<1$ is the constant appearing in (4.46)*.*

Proof. We shall prove (4.51) first. Let us write (4.28) and (4.29) with $R/2 = r$ and v and $-v$ replaced by $v-(v)_r$ and $(v)_r - v$ respectively (this is possible due to Remarks 4.5 – 4.8, together with the fact that $\inf_{B_r} v \leq (u)_r \leq \sup_{B_r} v$). Summing both sides of the inequalities obtained

$$\operatorname{osc}(v,r) \leq c\left(\fint_{B_{2r}} |v-(v)_r|^{p_2}\, dx\right)^{1/p_2} + c\,r^{\gamma}. \tag{4.53}$$

Now we first remark that

$$\begin{aligned}
\left[\fint_{B_{2r}} |v-(v)_r|^{p_2}\, dx\right]^{1/p_2} &\leq c\left[\fint_{B_{2r}} |v-(v)_{2r}|^{p_2}\, dx + |(v)_{2r}-(v)_r|^{p_2}\right]^{1/p_2} \\
&\leq c\left[\fint_{B_{2r}} |v-(v)_{2r}|^{p_2}\, dx\right]^{1/p_2} + c\left|(v)_{2r} - \frac{1}{r^n}\int_{B_r} v(y)\, dy\right| \\
&\leq c\left[\fint_{B_{2r}} |v-(v)_{2r}|^{p_2}\, dx\right]^{1/p_2} + c\left|\frac{1}{r^n}\int_{B_r} (v-(v)_{2r})\, dy\right| \\
&\leq c\left[\fint_{B_{2r}} |v-(v)_{2r}|^{p_2}\, dx\right]^{1/p_2},
\end{aligned} \tag{4.54}$$

with $c \equiv c(\gamma_1,\gamma_2)$. Therefore we have

$$\operatorname{osc}(v,r) \leq c\left(\fint_{B_{2r}} |v-(v)_{2r}|^{p_2}\, dx\right)^{1/p_2} + c\,r^{\gamma}. \tag{4.55}$$

On the other hand, for any $\rho<r$

$$\fint_{B_\rho} |v-(v)_\rho|^{p_2}\, dx \leq \operatorname{osc}(v,\rho)^{p_2}; \tag{4.56}$$

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hence, taking into account (4.46), we get for $\rho < r = R/2$ (we recall that $\alpha < \gamma$)

$$
\begin{aligned}
\fint_{B_\rho} |v-(v)_\rho|^{p_2}\,dx &\leq \operatorname{osc}(v,\rho)^{p_2} \\
&\leq c\left\{\left(\frac{\rho}{r}\right)^{\alpha}\operatorname{osc}(v,r)+\rho^{\alpha}\right\}^{p_2} \\
&\leq c\left\{\left(\frac{\rho}{r}\right)^{p_2\,\alpha}\operatorname{osc}(v,r)^{p_2}+\rho^{p_2\,\alpha}\right\} \\
&\overset{(4.55)}{\leq} c\left(\frac{\rho}{R}\right)^{p_2\,\alpha}\fint_{B_R}|v-(v)_R|^{p_2}\,dx + c\,\rho^{p_2\,\alpha},
\end{aligned}
\tag{4.57}
$$

with $c \equiv c(n,\gamma_1,\gamma_2,L,\mathcal{Q})$, where in the last line we used the fact that $r = R/2$. Therefore we obtained exactly (4.51). A simple argument shows that (4.51) holds, with a different choice of the constants, for any $\rho < R$.

Concerning (4.52), we first state a Caccioppoli type inequality for v

$$
\int_{B_\rho}|Dv|^{p(x)}\,dx \leq c\int_{B_{2\rho}}\left|\frac{v-(v)_{2\rho}}{\rho}\right|^{p(x)}dx + c\,\rho^{\lambda};
\tag{4.58}
$$

it is not difficult to see that (4.58) can be obtained using an argument similar to the one employed in Lemma 4.9.

On the other hand, due to assumption (2.2) and the localization we have $p_2 \leq p_1^*$ and therefore

$$
R^{n\left(1-\frac{p_2}{p_1^*}\right)} = R^{p_2}\,R^{n\frac{p_1-p_2}{p_1}} \leq c(L)\,R^{p_2}.
\tag{4.59}
$$

Finally, the Sobolev Poincaré inequality yields

$$
\begin{aligned}
\int_{B_R}(v-(v)_R)^{p(x)}\,dx &\leq c\int_{B_R}[(v-(v)_R)+1]^{p_2}\,dx \\
&\leq c\left(\int_{B_R}[(v-(v)_R)+1]^{p_1^*}\,dx\right)^{p_2/p_1^*} R^{n\left(1-\frac{p_2}{p_1^*}\right)} \\
&\overset{(4.59)}{\leq} c\,R^{p_2}\left(\int_{B_R}|Dv|^{p_1}\,dx\right)^{\frac{p_2}{p_1}} \\
&\leq c\,R^{p_2}\left[\int_{B_R}(|Dv|^{p(x)}+1)\,dx\right]^{\frac{p_2}{p_1}} \\
&\leq c(M)\,R^{p_2}\int_{B_R}(|Dv|^{p(x)}+1)\,dx,
\end{aligned}
\tag{4.60}
$$

where the constant M appears in (4.27).

Therefore summing up we may deduce, this time for $\rho < R/4$

$$
\begin{aligned}
\int_{B_\rho}|Dv|^{p(x)}\,dx &\overset{(4.58)}{\leq} c\int_{B_{2\rho}}\left|\frac{v-(v)_{2\rho}}{\rho}\right|^{p(x)}dx + c\,\rho^{\lambda} \\
&\leq c\int_{B_{2\rho}}\left|\frac{v-(v)_{2\rho}}{\rho}\right|^{p_2}+1\,dx + c\,\rho^{\lambda} \\
&= \frac{c}{\rho^{p_2}}\int_{B_{2\rho}}|v-(v)_{2\rho}|^{p_2}\,dx + c\,\rho^{\lambda} \\
&\overset{(4.57)}{\leq} \frac{c}{\rho^{p_2}}\left[\left(\frac{\rho}{R}\right)^{n+p_2\,\alpha}\int_{B_R}|v-(v)_R|^{p_2}\,dx + c\,\rho^{n+p_2\,\alpha}\right] + c\,\rho^{\lambda} \\
&\overset{(4.60)}{\leq} c\left(\frac{\rho}{R}\right)^{n-p_2+p_2\,\alpha}\int_{B_R}|Dv|^{p(x)}\,dx + c\,\rho^{n-p_2+p_2\,\alpha},
\end{aligned}
$$

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taking into account that, by definition of α, we have that $n - p_2 + p_2\,\alpha \leq \lambda$. Once more a simple argument shows that (4.52) holds, with a different choice of the constants, for any $\rho < R$. This finishes the proof. $\square$

We finally prove an up-to-the-boundary higher integrability result for the function v, which will be needed later for the comparison of v and the original minimizer u.

Proposition 4.13 *Let v be a local Q-minimizer of the functional* (4.26) *in the Dirichlet class $\{v \in u + W_0^{1,p(x)}(B_R)\}$, for some $u \in W^{1,p(x)}(B_R)$, where the function ψ fulfills the assumption* (2.6). *If moreover $|Du|^{p(x)} \in L^{1+\delta}$ for some $\delta > 0$, then there exist $\varepsilon \equiv \varepsilon(n, \gamma_1, \gamma_2, L) \in (0, \delta)$ and $c \equiv c(n, \gamma_1, \gamma_2, L)$ such that*

$$\begin{aligned} &\left(\fint_{B_R} |Dv|^{p(x)(1+\varepsilon)}\, dx \right)^{\frac{1}{(1+\varepsilon)}} \\ &\qquad \leq c \fint_{B_R} |Dv|^{p(x)}\, dx + \left(\fint_{B_R} |Du|^{p(x)(1+\delta)} + |D\psi|^{p(x)(1+\delta)} + 1\, dx \right)^{\frac{1}{1+\delta}} . \end{aligned} \tag{4.61}$$

Proof.

Case 1: interior situation. Let $0 < \rho < R$ and $x_0 \in B_R$ be an interior point such that $B_\rho(x_0) \subset B_R$. Let $t, s \in \mathbb{R}$ with $\frac{\rho}{2} < t < s < \rho$. Let $\eta \in C_c^\infty(B_\rho), 0 \leq \eta \leq 1$ be a cut-off function with $\eta \equiv 1$ on B_t, $\eta \equiv 0$ outside B_s and $|D\eta| \leq \frac{2}{|s-t|}$. We define the function $z := v - \eta(v - (v)_\rho)$. Testing the Q-minimality of v we deduce

$$\begin{aligned} \int_{B_t} |Dv|^{p(x)}\, dx &\leq \int_{B_s} (\mu^2 + |Dv|^2)^{p(x)/2}\, dx \\ &\leq Q \int_{B_s} (\mu^2 + |Dz|^2)^{p(x)/2}\, dx + Q\, L \int_{B_s} (|D\psi|^{p(x)} + 1)\, dx \\ &\leq Q \int_{B_s} (\mu^2 + |(1-\eta)\, Dv - D\eta(v - (v)_\rho)|^2)^{p(x)/2} + c \int_{B_s} (|D\psi|^{p(x)} + 1)\, dx \\ &\leq \bar{c} \int_{B_s \setminus B_t} |Dv|^{p(x)}\, dx + c \int_{B_s} \left| \frac{v - (v)_\rho}{s - t} \right|^{p(x)} dx + c \int_{B_s} (|D\psi|^{p(x)} + 1)\, dx, \end{aligned}$$

where $\bar{c} \equiv \bar{c}(Q, \gamma_1, \gamma_2)$.

Now, “filling the hole” and applying Lemma 4.1 we deduce the following

$$\fint_{B_{\rho/2}} |Dv|^{p(x)}\, dx \leq c \fint_{B_\rho} \left| \frac{v - (v)_\rho}{\rho} \right|^{p(x)} dx + c \int_{B_s} (|D\psi|^{p(x)} + 1)\, dx.$$

At this point, by Sobolev-Poincaré inequality, there exists $\chi < 1$ such that

$$\begin{aligned} \fint_{B_\rho} \left| \frac{v - (v)_\rho}{\rho} \right|^{p(x)} dx &\leq 1 + \fint_{B_\rho} \left| \frac{v - (v)_\rho}{\rho} \right|^{p_2} dx \\ &\leq 1 + c \left(\int_{B_\rho} (1 + |Dv|^{p(x)})\, dx \right)^{\frac{p_2 - p_1}{p_1\, \chi}} \rho^{-\frac{(p_2 - p_1)\, n}{p_1\, \chi}} \left(\fint_{B_\rho} |Dv|^{p_1\, \chi}\, dx \right)^{1/\chi} \\ &\leq c(\tilde{M}) \left(\fint_{B_\rho} |Dv|^{p(x)\, \chi}\, dx \right)^{1/\chi} + c. \end{aligned}$$

Therefore summing up we have the following reverse Hölder inequality

$$\fint_{B_{\rho/2}} |Dv|^{p(x)}\, dx \leq c_1 \Big(\fint_{B_\rho} |Dv|^{p(x)\, \chi}\, dx \Big)^{1/\chi} + c_2 \fint_{B_\rho} \Big(|D\psi|^{p(x)} + 1 \Big)\, dx, \tag{4.62}$$

for some suitable $\chi < 1$, $c_1, c_2 \equiv c_1, c_2(n, \gamma_1, \gamma_2, L)$.

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Case 2: situation at the boundary. We consider a point $x_0 \in \partial B_R$ and $0 < \rho < R$. Using the same cut-off function as before, we define $z := v - \eta(v-u)$.

On ∂B_R we have $z = u$ which yields $z \in u + W_0^{1,p(x)}(B_R)$. Defining $B_t^+ := B_t(x_0) \cap B_R$ and testing the Q-minimality for v, we obtain in exactly the same way as before

$$
\begin{aligned}
\int_{B_t^+} |Dv|^{p(x)}\,dx \quad &\leq \quad c\left[\int_{B_s^+\setminus B_t^+} |Dv|^{p(x)}\,dx + \int_{B_s^+} |Du|^{p(x)}\,dx \right.\\
&\qquad \left. + \int_{B_s^+} \left|\frac{v-u}{s-t}\right|^{p(x)} dx + \int_{B_\rho^+} (|D\psi|^{p(x)}+1)\,dx\right].
\end{aligned}
$$

Again “filling the hole” and using Lemma 4.1, we obtain

$$
\int_{B_{\rho/2}^+} |Dv|^{p(x)}\,dx \leq c\left[\int_{B_\rho^+} \left|\frac{v-u}{\rho}\right|^{p(x)} dx + \int_{B_\rho^+} |Du|^{p(x)}\,dx + \int_{B_\rho^+} \left(|D\psi|^{p(x)}+1\right)\,dx\right].
$$

Defining

$$
\bar{w} := \begin{cases} v-u & \text{on} \quad B_\rho^+ \\ 0 & \text{on} \quad B_\rho^- := B_\rho \setminus B_\rho^+ \end{cases}
$$

and applying Sobolev-Poincaré’s inequality in the version of [28, Corollary 4.5.3] (note that $|B_\rho^-| \geq 1/2|B_\rho|$) we deduce

$$
\begin{aligned}
\int_{B_\rho^+} |v-u|^{p(x)}\,dx \quad &\leq \quad \int_{B_\rho^+} (|v-u|^{p_2}+1)\,dx \\
&= \quad \int_{B_\rho} (|w|^{p_2}+1)\,dx \\
&\leq \quad c(n,\gamma_2)\frac{|B_\rho|}{|B_\rho^-|}\left(\int_{B_\rho} |Dw|^{\frac{np_2}{n+p_2}}\,dx\right)^{\frac{n+p_2}{n}} + \rho^n.
\end{aligned}
$$

We define

$$
\chi := \frac{np_2}{(n+p_2)\,p_1};
$$

we observe that due to the localization, it is possible to take $\chi < 1$. Now

$$
\begin{aligned}
\left(\int_{B_\rho} |Dw|^{p_1\,\chi}\,dx\right)^{\frac{1}{\chi}\,\frac{p_2}{p_1}} \quad &= \quad \left(\int_{B_\rho} |Dw|^{p_1\,\chi}\,dx\right)^{\frac{1}{\chi}} \left(\int_{B_\rho} (|Dw|^{p(x)}+1)\,dx\right)^{\frac{p_2-p_1}{\chi\,p_1}} \\
&\leq \quad c(\tilde{M})\left(\int_{B_\rho} |Dw|^{p_1\,\chi}\,dx\right)^{1/\chi} \\
&= \quad c\left(\int_{B_\rho^+} (|Dv-Du|^{p(x)\,\chi}+1)\,dx\right)^{1/\chi}.
\end{aligned}
$$

On the other hand, taking mean values we have first of all

$$
\begin{aligned}
\fint_{B_\rho^+} \left|\frac{v-u}{\rho}\right|^{p(x)} dx \quad &\leq \quad \frac{1}{\rho^{n\,p_2}}\int_{B_\rho^+} |v-u|^{p(x)}\,dx \\
&\leq \quad c\,\rho^{(p_1-p_2)\left[\frac{n}{p_2}+1\right]}\left(\fint_{B_\rho^+} |D(v-u)|^{p(x)\,\chi}\,dx\right)^{1/\chi}
\end{aligned}
$$

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$$\leq \quad c(L) \left(\fint_{B_\rho^+} |D(v-u)|^{p(x)\,\chi}\, dx \right)^{1/\chi}$$

that brings

$$\fint_{B_{\rho/2}^+} |Dv|^{p(x)}\, dx \leq c \left[\left(\fint_{B_\rho^+} |Dv|^{p(x)\chi}\, dx \right)^{1/\chi} + \fint_{B_\rho^+} \Big(|Du|^{p(x)} + |D\psi|^{p(x)} + 1 \Big)\, dx \right], \tag{4.63}$$

with $c \equiv c(n, \gamma_1, \gamma_2, L)$.

Note that (4.62) holds for any $B_\rho \subset B_R$ and (4.63) for any $0 < \rho \leq R$. Therefore we can apply the global version of Gehring's Lemma [8, Theorem 2.4], with the functions $g := |Dv|^{p(x)\chi}$, $f := (|Du|^{p(x)} + |D\psi|^{p(x)} + 1)^\chi$ to deduce the desired result. □

Remark. Note that the dependency of the higher integrability exponent ε and the constants coming up in Gehring's Lemma on the exponent χ can be replaced by dependencies on the global bounds γ_1 and γ_2 for p_2 and p_1. For a detailed discussion of this we refer the reader to [21].

5 Proof of Theorem 2.7

In this section we prove Hölder continuity for the function u. Therefore we start with localization of the problem.

5.1 First step: Localization.

We first note that in view of Lemma 3.6 we may find an exponent $\delta \equiv \delta(n, \tilde{q}, \gamma_1, \gamma_2, L) > 0$ such that

$$\int_{\Omega'} |Du|^{p(x)(1+\delta)}\, dx < +\infty.$$

Since our results are local in nature we may assume that

$$\int_{\Omega} |Du|^{p(x)(1+\delta)}\, dx < +\infty. \tag{5.1}$$

Without loss of generality let δ be so small that $1 + \delta \leq \tilde{q}$, where $\tilde{q}$ is the quantity in (2.6).

Let R_M be a maximal radius such that there holds $\omega(8R_M) \leq \delta/4$ and $B_R \subset \Omega$ a ball with radius $R \leq R_M$. We define

$$p_2 := \max\{p(x) : x \in \overline{B_R}\}, \quad p_1 := \min\{p(x) : x \in \overline{B_R}\}. \tag{5.2}$$

By the continuity of p we therefore deduce

$$\begin{aligned} p_2 - p_1 &\leq \omega(R) \leq \delta/4; \\ p_2(1 + \delta/4) &\leq p(x)(1 + \delta/4 + \omega(R)) \leq p(x)(1 + \delta). \end{aligned} \tag{5.3}$$

Furthermore we note that the localization together with the bound (2.2) for the modulus of continuity provides for any $R \leq 8R_M \leq 1$:

$$R^{-n\omega(R)} \leq \exp(nL) = c(n, L), \qquad R^{-\frac{n\omega(R)}{1+\omega(R)}} \leq c(n, L). \tag{5.4}$$

Additionally, in view of (2.4) we may assume that there exists $M < +\infty$ such that

$$\int_{\Omega} |Du(x)|^{p(x)}\, dx \leq M. \tag{5.5}$$

5.2 Second step: Comparison via Ekeland.

Let us consider the functional $\mathcal{F}$ on the ball B_R. We will show that the minimizer u in the obstacle class $K_0 \equiv \{w \in u + W_0^{1,1}(B_R) :\ w \geq \psi\}$ is in a certain sense 'near' to the infimum among all functions in the free class $V \equiv w \in u + W_0^{1,1}(B_R)$. We argue as follows.

For fixed $\delta > 0$ we choose $u_\delta \in V$ such that

$$\mathcal{F}[u_\delta, B_R] \leq \inf_{w \in V} \mathcal{F}[w, B_R] + \delta R^n.$$

Since $u_\delta \in V$ is not necessarily an element of the obstacle class K_0, we define $w_\delta := \max\{u_\delta, \psi\}$ and the set $\Sigma := \{x \in \mathbb{R}^n :\ u_\delta \geq \psi\}$. By the minimality of u we have

$$\begin{aligned}
\mathcal{F}[u, B_R] &\leq \mathcal{F}[w_\delta, B_R] \\
&= \mathcal{F}[u_\delta, \Sigma] + \mathcal{F}[\psi, B_R \setminus \Sigma] \\
&\leq \mathcal{F}[u_\delta, B_R] + \mathcal{F}[\psi, B_R] \\
&\leq \inf_{w \in V} \mathcal{F}[w, B_R] + \delta R^n + L \left[\int_{B_R} |D\psi|^{p(x)}\, dx + R^n \right]
\end{aligned}$$

Letting $\delta \to 0$ we obtain with the definition

$$H(R) := L \left[\int_{B_R} |D\psi|^{p(x)}\, dx + R^n \right], \tag{5.6}$$

the estimate

$$\mathcal{F}[u, B_R] \leq \inf_{w \in V} \mathcal{F}[w, B_R] + H(R). \tag{5.7}$$

Let us now introduce the distance

$$d(v_1, v_2) := \int_{B_R} H(R)^{-\frac{1}{p(x)}} R^{-n\left(1 - \frac{1}{p(x)}\right)} |Dv_1(x) - Dv_2(x)|\, dx, \tag{5.8}$$

and note that (V, d) is a complete metric space and $\mathcal{F}$ is a lower semi-continuous functional in the space (V, d).

Since $\inf_{w \in V} \mathcal{F}[w, B_R] > -\infty$ we may apply Ekeland's variational principle (Lemma 3.7) which provides a function $v \in V$ such that

$$\int_{B_R} H(R)^{-\frac{1}{p(x)}} R^{-n\left(1 - \frac{1}{p(x)}\right)} |Du(x) - Dv(x)|\, dx \leq 1, \tag{5.9}$$

$$\mathcal{F}[v, B_R] \leq \mathcal{F}[u, B_R], \tag{5.10}$$

and v is a minimizer in the class V of the functional

$$w \mapsto \mathcal{F}[w, B_R] + \int_{B_R} \left(H(R) R^{-n} \right)^{\frac{p(x)-1}{p(x)}} |Dw(x) - Dv(x)|\, dx. \tag{5.11}$$

5.3 Third step: $\mathcal{Q}$ minimality.

Let us show now that the function v is an element of the class $u + W_0^{1,p(x)}(B_R)$ and v is a local $\mathcal{Q}$ minimizer of the functional

$$w \mapsto \int_{B_R} \left[|Dw|^{p(x)} + \tfrac{H(R)}{R^n} + 1 \right] dx, \tag{5.12}$$

with $\mathcal{Q} \equiv \mathcal{Q}(L)$.

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To see this let $\varphi \in W_0^{1,p(x)}(B_R)$ be arbitrary. Exploiting the minimality of v we deduce

$$\mathcal{F}[v, \text{spt}\varphi] \le \mathcal{F}[v+\varphi, \text{spt}\varphi] + \int_{\text{spt}\varphi} \left(H(R)R^{-n}\right)^{\frac{p(x)-1}{p(x)}} |D\varphi| \, dx.$$

At this point, Young's inequality provides

$$\left(H(R)R^{-n}\right)^{\frac{p(x)-1}{p(x)}} |D\varphi| \le \varepsilon |D\varphi|^{p(x)} + c(\varepsilon, p(x))H(R)R^{-n},$$

where, using the explicit form of the constant in Young's inequality, the fact that $(p(x)-1)/p(x) \le (\gamma_2 - 1)/\gamma_2$ and assuming $\varepsilon < 1 - 1/\gamma_2$ one can easily see that $c(\varepsilon, p(x))$ in the above estimate may be replaced by a constant $c(\varepsilon, \gamma_1, \gamma_2)$. This allows us to deduce

$$\begin{aligned}
\mathcal{F}[v, \text{spt}\varphi] &\le \mathcal{F}[v+\varphi, \text{spt}\varphi] + \varepsilon \int_{\text{spt}\varphi} |D\varphi|^{p(x)} \, dx + c(\varepsilon, \gamma_1, \gamma_2) \frac{H(R)}{R^n} |\text{spt}\varphi| \\
&\le \mathcal{F}[v+\varphi, \text{spt}\varphi] + \frac{\varepsilon}{2^{\gamma_1 - 1}} \left(\int_{\text{spt}\varphi} |Dv|^{p(x)} \, dx + \int_{\text{spt}\varphi} |Dv + D\varphi|^{p(x)} \, dx \right) + c \frac{H(R)}{R^n} |\text{spt}\varphi|.
\end{aligned}$$

Using the growth condition (H1), choosing $\varepsilon = 2^{\gamma_1 - 2}$ we deduce

$$\begin{aligned}
&\int_{\text{spt}\varphi} \left(|Dv|^{p(x)} + \frac{H(R)}{R^n} + 1 \right) dx \\
&\qquad \le L\mathcal{F}[v, \text{spt}\varphi] + |\text{spt}\varphi| \left(\frac{H(R)}{R^n} + 1 \right) \\
&\qquad \le L\mathcal{F}[v+\varphi, \text{spt}\varphi] + |\text{spt}\varphi| \left(\frac{H(R)}{R^n} + 1 \right) + \frac{1}{2} \int_{\text{spt}\varphi} |Dv|^{p(x)} \, dx \\
&\qquad\quad + \frac{1}{2} \int_{\text{spt}\varphi} |Dv + D\varphi|^{p(x)} \, dx + c(\gamma_1, \gamma_2) \frac{H(R)}{R^n} |\text{spt}\varphi| \\
&\qquad \le L^2 \int_{\text{spt}\varphi} |Dv + D\varphi|^{p(x)} \, dx + c(L, \gamma_1, \gamma_2) \left(\frac{H(R)}{R^n} + 1 \right) |\text{spt}\varphi| + \frac{1}{2} \int_{\text{spt}\varphi} |Dv|^{p(x)} \, dx.
\end{aligned}$$

Absorbing the last term of the preceding inequality on the left hand side we obtain the desired $\mathcal{Q}$ minimality with $\mathcal{Q} \equiv \mathcal{Q}(L, \gamma_1, \gamma_2) > 1$.

5.4 Fourth step: Up-to-the-boundary higher integrability

By (5.1) and since v is a $\mathcal{Q}$ minimizer of the functional (5.12), Proposition 4.13 provides an exponent $\varepsilon \equiv \varepsilon(n, \gamma_1, \gamma_2, L) \in (0, \delta)$ such that

$$\begin{aligned}
&\left(\fint_{B_R} |Dv|^{p(x)(1+\varepsilon)} \, dx \right)^{\frac{1}{(1+\varepsilon)}} \\
&\qquad \le c \fint_{B_R} |Dv|^{p(x)} \, dx + \left(\fint_{B_R} |Du|^{p(x)(1+\delta)} + |D\psi|^{p(x)(1+\delta)} + 1 \, dx \right)^{\frac{1}{1+\delta}},
\end{aligned} \tag{5.13}$$

with a constant $c \equiv c(n, L, \gamma_1, \gamma_2)$.

5.5 Fifth step: Hölder continuity for the function v.

Since $v \in u + W_0^{1,p(x)}(B_R)$ is a local $\mathcal{Q}$ minimizer of the functional (5.12), taking into account that (4.27) is fulfilled via (5.5) we may apply Proposition 4.12 to conclude that

$$\int_{B_\rho} |Dv|^{p(x)} \, dx \le c \left(\frac{\rho}{R} \right)^{n - p_2 + p_2 \alpha} \int_{B_R} |Dv|^{p(x)} \, dx + c \, \rho^{n - p_2 + p_2 \alpha}; \tag{5.14}$$

for any $\rho < R$, with $c \equiv c(n, \gamma_1, \gamma_2, L, \mathcal{Q}, M)$. We remark that, due to the choice of α, we have that $n - p_2 + p_2 \alpha < \lambda$.

5.6 Sixth step: Comparison and Conclusion.

Now using (5.9) and the fact that

$$H(R) \leq c(||D\psi||_{L^{q,\lambda}}, n, L) R^{\lambda},$$

we deduce

$$\begin{aligned}
&\fint_{B_{R/2}} |Du - Dv|\,dx \\
&\quad \leq c(n) \int_{B_R} R^{-n} H(R)^{-\frac{1}{p(x)}}\, R^{-n\left(1-\frac{1}{p(x)}\right)} |Du - Dv|\, H(R)^{\frac{1}{p(x)}}\, R^{n\left(1-\frac{1}{p(x)}\right)}\,dx \\
&\quad \leq c\, R^{\frac{\lambda-n}{p_2}} \int_{B_R} R^{-n} H(R)^{-\frac{1}{p(x)}}\, R^{-n\left(1-\frac{1}{p(x)}\right)} |Du - Dv|\,dx \\
&\quad \leq c\, R^{\frac{\lambda-n}{p_2}}.
\end{aligned} \tag{5.15}$$

We proceed by a standard interpolation argument: choosing $\theta \in (0,1)$ such that $\theta/s + 1 - \theta = 1/p_2$, where we set $s := p_2(1+\varepsilon/4)$ and recalling that $s \in (p_2, p_2(1+\delta/4))$, we exploit higher integrability for the function u (Lemma 3.2) and the up-to-the-boundary higher integrability for v (Lemma 4.61) to deduce

$$\begin{aligned}
&\int_{B_{R/2}} |Du - Dv|^{p(x)}\,dx \\
&\quad \leq \int_{B_{R/2}} (|Du - Dv|^{p_2} + 1)\,dx \\
&\quad \leq c\, R^n \left(\fint_{B_{R/2}} |Du - Dv|^s\,dx\right)^{\frac{\theta\, p_2}{s}} \left(\fint_{B_{R/2}} |Du - Dv|\,dx\right)^{(1-\theta)\,p_2} + c\, R^n \\
&\quad \overset{(5.15)}{\leq} c\, R^n \left[R^{\frac{\lambda-n}{p_2}}\right]^{(1-\theta)p_2} \left[\left(\fint_{B_{R/2}} |Du|^s\,dx\right)^{\frac{\theta\, p_2}{s}} + \left(\fint_{B_{R/2}} |Dv|^s\,dx\right)^{\frac{\theta\, p_2}{s}}\right] + c\, R^n \\
&\quad \leq c\, R^n\, R^{(\lambda-n)\,(1-\theta)} \left[\left(\fint_{B_{R/2}} |Du|^{p_2(1+\delta/4)}\,dx\right)^{\frac{\theta}{1+\delta/4}} + \left(\fint_{B_{R/2}} |Dv|^{p_2(1+\varepsilon/4)}\,dx\right)^{\frac{\theta}{1+\varepsilon/4}}\right] \\
&\qquad + c\, R^n \\
&\quad \overset{(4.61)}{\leq} c\, R^{n\,\theta}\, R^{\lambda(1-\theta)} \left[\left(\fint_{B_{R/2}} |Du|^{p_2(1+\delta/4)}\,dx\right)^{\frac{\theta}{1+\delta/4}} + \left(\fint_{B_{R/2}} |Dv|^{p(x)}\,dx\right)^{\theta}\right. \\
&\qquad \left. + \left(\fint_{B_{R/2}} |Du|^{p(x)(1+\delta/4)}\,dx\right)^{\frac{\theta}{1+\delta/4}} + \left(\fint_{B_{R/2}} (1 + |D\psi|^{p_2(1+\delta/4)})\,dx\right)^{\frac{\theta}{1+\delta/4}}\right] + c\, R^n \\
&\quad \overset{(5.3),(5.10)}{\leq} c\, R^{n\,\theta}\, R^{\lambda(1-\theta)} \left[\left(\fint_{B_{R/2}} (1 + |Du|^{p(x)(1+\delta/4+\omega(R))})\,dx\right)^{\frac{\theta}{1+\delta/4}}\right. \\
&\qquad \left. + \left(\fint_{B_{R/2}} |Du|^{p(x)}\,dx\right)^{\theta} + \left(1 + \fint_{B_{R/2}} |D\psi|^{p(x)(1+\delta)}\,dx\right)^{\theta}\right] + c\, R^n \\
&\quad \overset{(3.2)}{\leq} c\, R^{n\theta}\, R^{\lambda(1-\theta)} \left[\left(\fint_{B_R} (1 + |Du|^{p(x)})\,dx\right)^{\frac{\theta\,(1+\delta/4+\omega(R))}{1+\delta/4}}\right. \\
&\qquad \left. + \left(\fint_{B_{R/2}} |Du|^{p(x)}\,dx\right)^{\theta} + \left(\fint_{B_R} (1 + |D\psi|^{p(x)(1+\delta)}\,dx\right)^{\theta}\right] + c\, R^n
\end{aligned}$$

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$$
\begin{aligned}
&\leq c\,R^{n\theta}\,R^{\lambda(1-\theta)}\left[\left(R^{-n\frac{\theta\,\omega(R)}{1+\delta/4}}+1\right)\left(\fint_{B_R}(1+|Du|^{p(x)})\,dx\right)\right.\\
&\qquad\left.+\left(1+\fint_{B_R}|D\psi|^{p(x)(1+\delta)}\,dx\right)^{\theta}\right]+R^n\\
&\overset{(2.6)}{\leq} c\,R^{n\theta}\,R^{\lambda(1-\theta)}\left\{\left[\fint_{B_R}(1+|Du|^{p(x)})\,dx\right]^{\theta}+R^{(\lambda-n)\theta}\right\}+c\,R^n\\
&\leq c\,R^{\lambda(1-\theta)}\left(\int_{B_R}|Du|^{p(x)}\,dx+R^{\lambda}\right)^{\theta}+c\,R^n,
\end{aligned}
$$

with $c \equiv c(n,\gamma_1,\gamma_2,L,||D\psi||_{L^{q,\lambda}})$. Now we choose $\beta > 0$ small enough such that

$$\lambda-\beta\,\frac{\theta}{1-\theta}>n-p_2+p_1\,\alpha, \tag{5.16}$$

where α appears in (5.14). This is possible since, due to the choice of α we have $\lambda > n-p_2+p_1\,\alpha$. We set moreover

$$\tau := \min\left\{\lambda-\beta\,\frac{\theta}{1-\theta},\beta\right\}>0. \tag{5.17}$$

Therefore we obtain

$$
\begin{aligned}
\int_{B_{R/2}}|Du-Dv|^{p(x)}\,dx &\leq c\,R^{\lambda(1-\theta)}\left(\int_{B_R}|Du|^{p(x)}\,dx+R^{\lambda}\right)^{\theta}+c\,R^n\\
&\leq c\,R^{\lambda(1-\theta)-\beta\,\theta}\,R^{\beta\,\theta}\left[\int_{B_R}|Du|^{p(x)}\,dx+R^{\lambda}\right]^{\theta}+c\,R^n\\
&\leq c\left[R^{\lambda-\beta\,\frac{\theta}{1-\theta}}\right]^{1-\theta}\left[R^{\beta}\int_{B_R}|Du|^{p(x)}\,dx+R^{\lambda+\beta}\right]^{\theta}+c\,R^n\\
&\leq c\left[R^{\lambda-\beta\,\frac{\theta}{1-\theta}}\int_{B_R}|Du|^{p(x)}\,dx+R^{\lambda-\beta\,\frac{\theta}{1-\theta}}\right]^{1-\theta}\\
&\qquad\times\left[R^{\beta}\int_{B_R}|Du|^{p(x)}\,dx+R^{\lambda+\beta}\right]^{\theta}+c\,R^n\\
&\overset{(5.17),(5.16)}{\leq} c\,R^{\tau}\int_{B_R}|Du|^{p(x)}\,dx+c\,R^{n-p_2+p_1\,\alpha}.
\end{aligned}
\tag{5.18}
$$

The comparison estimate (5.18) together with the reference estimate (5.14) for v, (5.10) and the growth condition allow us to estimate

$$
\begin{aligned}
\int_{B_\rho}|Du|^{p(x)} &\leq c\int_{B_\rho}|Dv|^{p(x)}\,dx+c\int_{B_\rho}|Du-Dv|^{p(x)}\,dx\\
&\leq c\left[\left(\frac{\rho}{R}\right)^{n-p_2+p_2\,\alpha}+R^{\tau}\right]\int_{B_R}|Du|^{p(x)}\,dx+c\,R^{n-p_2+p_2\,\alpha}.
\end{aligned}
\tag{5.19}
$$

Then estimate (5.19) holds for any radii $0<\rho\leq R\leq R_M$. Let $\varepsilon_0 \equiv \varepsilon_0(n,M,L,\gamma_1,\gamma_2,\lambda,\alpha)$ be the quantity provided by Lemma 3.5. We can find a radius $R_1>0$ so small that $R^{\tau}<\varepsilon_0$ for any $0<R\leq R_1$ and thus we have $R_1 \equiv R_1(n,\gamma_1,\gamma_2,L,M,\omega,\lambda,\alpha)$. Now Lemma 3.5 yields

$$\int_{B_\rho}|Du(x)|^{p(x)}\,dx \leq c\,\rho^{n-p_2+p_2\,\alpha}\leq c\,\rho^{n-p_1+p_1\,\alpha},$$

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with $c \equiv c(n, M, L, \gamma_1, \gamma_2, \lambda, \alpha)$, whenever $0 < \rho < R_1$. Since we have $\gamma_1 \le p_1 \le p_2 \le \gamma_2$, we deduce by a standard covering argument and by Poincaré's inequality that

$$u \in \mathcal{L}^{\gamma_1,\xi}_{\text{loc}}(\Omega),$$

with $\xi = n + \gamma_1\, \alpha$; thus Theorem 3.1 allows us to conclude that $u \in \mathcal{C}^{0,\alpha}_{\text{loc}}(\Omega)$. This finishes the proof. $\square$

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