# Classification of metric measure spaces and their ends using $p$-harmonic functions

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Dedicated to the memory of Seppo Rickman.

**Abstract**. By seeing whether a Liouville type theorem holds for positive, bounded, and/or finite energy $p$-harmonic and $p$-quasiharmonic functions, we classify proper metric spaces equipped with a locally doubling measure supporting a local $p$-Poincaré inequality. Similar classifications have earlier been obtained for Riemann surfaces and Riemannian manifolds.

We study the inclusions between these classes of metric measure spaces, and their relationship to the $p$-hyperbolicity of the metric space and its ends. In particular, we characterize spaces that carry nonconstant $p$-harmonic functions with finite energy as spaces having at least two well-separated $p$-hyperbolic sequences. We also show that every such space $X$ has a function $f \notin L^p(X) + \mathbf{R}$ with finite $p$-energy.

*Key words and phrases*: classification of metric measure spaces, doubling measure, end at infinity, finite energy, $p$-hyperbolic sequence, Liouville theorem, $p$-harmonic function, Poincaré inequality, $p$-parabolic, quasiharmonic function, quasiminimizer.

Mathematics Subject Classification (2020): Primary: 31E05; Secondary: 30L10, 31C45, 35J92, 46E36.

## 1. Introduction

The classical Liouville theorem states that every bounded holomorphic function in the whole plane is constant. A similar statement is true for harmonic and $p$-harmonic functions in $\mathbf{R}^n$, $1 < p < \infty$.

In the 1960s, Riemann surfaces were classified according to existence of global analytic or harmonic functions in various classes (bounded, positive and finite-energy), which culminated in the 1970 monograph by Sario and Nakai [55]. Together with Wang and Chung, they extended this classification to Riemannian manifolds in the monograph [56] from 1977. Holopainen [35] extended this classification further to $p$-harmonic functions on Riemannian manifolds in 1990, see also Kilpeläinen [43, Theorem 1.8] for some similar results for Euclidean domains. Subsequently, in the 1990s, first-order analysis on metric spaces began to be studied and it has since seen a growing interest. Our main aim is to obtain a similar classification of metric spaces as in the monographs mentioned above. We refer to later sections for the definitions.

*Throughout the paper, except for Sections 2 and 9, we assume that $1 < p < \infty$ and that $X$ is an unbounded proper connected metric space equipped with a locally doubling measure $\mu$ supporting a local $p$-Poincaré inequality.*

**Definition 1.1.** We say that $X$ belongs to the *Liouville type class*
$O^p_{HP}$ if every *positive* $p$-harmonic function on $X$ is constant;
$O^p_{HB}$ if every *bounded* $p$-harmonic function on $X$ is constant;
$O^p_{HD}$ if every $p$-harmonic function on $X$ with *finite energy* is constant;
$O^p_{HBD}$ if every *bounded* $p$-harmonic function on $X$ with *finite energy* is constant.
The corresponding classes $O^p_{QP}$, $O^p_{QB}$, $O^p_{QD}$ and $O^p_{QBD}$ for quasiharmonic functions (where the dependence on $p$ is implicit) are defined similarly. Moreover, we say that $X \in O^p_{\mathrm{par}}$ if $X$ is *$p$-parabolic* in the sense of Definition 4.2.

Our classification result can be summarized as follows.

**Theorem 1.2.** *We have the following inclusions*:

$$\begin{array}{ccccccccc} O^p_{HP} & \subsetneq & O^p_{HB} & \subset & O^p_{HBD} & = & O^p_{HD} & \supsetneq & O^p_{\mathrm{par}} \\ \cup & & \cup & & \| & & \| & & \\ O^p_{QP} & \subsetneq & O^p_{QB} & \subset & O^p_{QBD} & = & O^p_{QD}. & & \end{array}$$

*Moreover, $O^p_{QB} \setminus O^p_{HP}$, $O^p_{QP} \setminus O^p_{\mathrm{par}}$ and $O^2_{HBD} \setminus O^2_{HB}$ are nonempty.*

Some of these inclusions are of course trivial. In the setting of orientable Riemannian manifolds, it was shown in Sario–Nakai–Wang–Chung [56] (for $p = 2$) and Holopainen [35] (for general $p$) that

$$O^p_{\mathrm{par}} \subsetneq O^p_{HP} \subsetneq O^p_{HB} \subset O^p_{HBD} = O^p_{HD} \quad \text{and} \quad O^2_{HB} \subsetneq O^2_{HBD}. \tag{1.1}$$

(Whenever we discuss manifolds we implicitly assume that they are connected and have dimension $\geq 2$.)

A class of functions called "quasiharmonic" was also considered in [56]. However, those functions are solutions to $\Delta u = 1$, while our quasiharmonic functions are continuous quasiminimizers of the $p$-energy. Such quasiminimizers were introduced in Giaquinta–Giusti [23], [24] as a unified treatment of variational inequalities, elliptic partial differential equations and quasiregular mappings, see [13] and [16] for further discussion and references.

Since Riemann, planar Euclidean domains have been classified using conformal mappings: two planar domains belong to the same category if there is a conformal mapping between them. One of the motivations for studying the classes in (1.1) is that some of them are conformally invariant on Riemann surfaces, when $p = 2$. Consequently, two conformally equivalent Riemann surfaces either both belong to such a class or neither belongs to that class.

For higher-dimensional Euclidean domains and $p \neq 2$, conformal mappings are too rigid, and instead quasiconformal or quasisymmetric mappings are used. For $n$-dimensional Riemannian manifolds, $n$-harmonicity and $n$-parabolicity are conformal invariants.

The theory of quasiconformal mappings was extended to metric measure spaces in Heinonen–Koskela [31], [32], see also Heinonen–Koskela–Shanmugalingam–Tyson [33, Section 9]. Quasiconformal mappings do not preserve harmonic or $p$-harmonic functions, but they do preserve quasiharmonic functions (with $p = Q$) in proper connected spaces with a uniformly locally Ahlfors $Q$-regular measure supporting a uniformly local $Q$-Poincaré inequality, see Korte–Marola–Shanmugalingam [48, Theorem 4.1] and also Heinonen–Kilpeläinen–Martio [30, Corollary 4.7]. Quasiconformal mappings between such spaces therefore preserve the classes $O^Q_{QB}$ and $O^Q_{QP}$, and, by [33, Theorem 9.10], also $O^Q_{QD}$ and $O^Q_{QBD}$. Hence it is natural to include quasiharmonic Liouville type classes in our study. The existence of nonconstant global quasiharmonic functions on one (but not the other) space therefore gives a convenient way of checking whether two metric measure spaces can be quasiconformally equivalent. As far as we know, even for $p = 2$ and in the setting of Riemann surfaces, it is not known whether the classes $O^2_{HP}$ and $O^2_{HB}$ are quasiconformally invariant, see Sario–Nakai [55, p. 7]. On the other hand, it was noted already therein that $O^2_{HD}$ is quasiconformally invariant in that setting.

For complete Riemannian manifolds, the case $p = 2$ is also related to the Brownian motion: 2-parabolicity is equivalent to the fact that almost surely the Brownian motion starting from a compact set $K$ will intersect each neighborhood of $K$ infinitely often, see Grigor′yan [27, Theorem 5.1]. Thus the classification of metric measure spaces as in Theorem 1.2 has roots in the theory of Brownian motion, in complex dynamics (see [53, Theorem 0.1]), and in the study of quasiconformal maps.

A natural way of distinguishing between different spaces and manifolds is through their ends at infinity. For instance, (unweighted) $\mathbf{R}^n$ has one end if $n \geq 2$ and this end is $p$-hyperbolic if and only if $1 < p < n$. For $n = 1$, $\mathbf{R}$ has two ends which are both $p$-parabolic. An end, or a space, is $p$-hyperbolic if it is not $p$-parabolic, see Definition 4.2.

We show that if $X$ has two $p$-hyperbolic ends, then $X \notin O^p_{HBD}$. The converse is not true as explained in Example 8.5, but using the new concept of *$p$-hyperbolic sequences* we are able to give the following characterization.

**Theorem 1.3.** *$X \notin O^p_{HBD}$ if and only if there are two disjoint $p$-hyperbolic sequences $\{F_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ which are well-separated in the sense that the $p$-modulus of the family $\Gamma(F_1, G_1)$ of all curves from $F_1$ to $G_1$ satisfies*

$$\operatorname{Mod}_p(\Gamma(F_1, G_1)) < \infty.$$

*In this case, $X$ is also $p$-hyperbolic, i.e. $O^p_{\mathrm{par}} \subset O^p_{HBD}$.*

*In particular, $X \notin O^p_{HBD}$ if $X$ has two $p$-hyperbolic ends.*

As (unweighted) $\mathbf{R}^n \in O^p_{HP} \subset O^p_{HBD}$ for all $1 < p < \infty$ and $n \geq 1$, but is $p$-parabolic only for $p \geq n$, we see that $O^p_{\mathrm{par}} \subsetneq O^p_{HBD}$ (cf. Theorem 1.2).

For $p = 2$, similar characterizations of the bounded and finite-energy Liouville theorems (i.e. of $X \in O^2_{HB}$ resp. $X \in O^2_{HD}$) by means of well-separated massive and/or hyperbolic sets were obtained for Riemannian manifolds, see Grigor′yan [25, Proposition 1 and Theorem 2], [27, Theorem 13.10 (b)] and the references therein. In the setting of Gromov hyperbolic spaces with uniformly local assumptions (of doubling and $p$-Poincaré inequality), the validity of the finite-energy Liouville theorem for $p$-harmonic functions (i.e. $X \in O^p_{HD}$) was characterized using uniformization in Björn–Björn–Shanmugalingam [14, Theorem 10.5]. See Remark 6.3 for how our results in this paper improves upon that.

Hyperbolic sequences can be seen as subsets of the hyperbolic parts of the "boundary at $\infty$" of the metric space $X$. For simply connected complete Riemannian manifolds $M$ of negative sectional curvature, such a "boundary at $\infty$", $M(\infty)$, was introduced by Eberlein–O'Neill [20] and identified with the "sphere at $\infty$". If $M$, in addition, has negatively pinched sectional curvature $-b^2 \leq K \leq -a^2 < 0$, then it is possible to solve the asymptotic Dirichlet problem with any continuous boundary data on the sphere at infinity. This follows from Choi [18, Theorems 4.5 and 4.7] and Anderson [3] for $p = 2$, and has been generalized to $p > 1$ by Pansu [54] and Holopainen [37, Theorem 2.1], see also the discussion in [37, p. 3394]. These existence results imply that $M \notin O^p_{HB}$, but do not address the existence of $p$-harmonic functions with finite energy. On Gromov hyperbolic spaces, the above solvability of the Dirichlet problem at infinity was deduced in Holopainen–Lang–Vähäkangas [39, Theorem 6.2] for $p > 1$, under various additional assumptions.

Choi [18, Definition 5.1] considers ends on a finitely connected complete 2-dimensional Riemannian manifold with sectional curvature $K \le -a^2 < 0$ and shows that if the surface is orientable with $\int K = -\infty$, then it carries many nonconstant bounded harmonic functions, see [18, Theorem 5.13 and Corollary 5.14]. The Dirichlet problem for $p$-harmonic functions in unbounded domains with ends towards infinity was solved in Björn–Björn–Li [9, Theorems 6.6, 7.6 and 7.7] in the setting of Ahlfors $Q$-regular spaces under certain assumptions on $Q$, $p$ and the measure. The notions of parabolic and hyperbolic ends have also been used in the study of some other partial differential equations, see for instance Korolkov–Losev [47] for the case of the stationary Schrödinger equation.

Under global assumptions, the Liouville theorem (Theorem 3.2) for positive quasiharmonic functions on metric spaces was obtained in Kinnunen–Shanmugalingam [45]. In [13], we proved the so-called finite-energy Liouville theorem for noncomplete spaces with global assumptions under various additional assumptions. We can now deduce the Liouville theorem for finite-energy quasiharmonic functions without those additional assumptions, as a direct consequence of our identity $O^p_{QBD} = O^p_{QD}$ (and Theorem 3.2) provided that $X$ is complete, see Corollary 6.2. Moreover, using tools from Björn–Björn [7] we are able to lift this also to noncomplete spaces, see Theorem 9.1. The weighted real line and Example 8.1 show that the finite-energy Liouville theorem fails if the global assumptions are relaxed to uniformly local ones.

The following theorem shows that the Liouville type class $O^p_{HD}$ is related to the question of whether every function with finite energy on $X$ can be written as a global Sobolev function plus a constant, i.e. whether $D^p(X) = N^{1,p}(X) + \mathbf{R}$. this is the case for $p = 2$, then the classical theory of Dirichlet forms and the associated spectral decomposition can be extended to the Dirichlet space $D^2(X)$ of functions with finite energy, where the associated Dirichlet form is in terms of the Cheeger differential structure as in Koskela–Rajala–Shanmugalingam [50].

**Theorem 1.4.** *If $X \notin O^p_{HD}$, then $D^p(X) \ne N^{1,p}(X) + \mathbf{R}$.*

Example 7.1 shows that the converse fails. However, if $X \in O^p_{HD}$ and $X$ supports a global $(p,p)$-Sobolev inequality (in addition to our standing assumptions), then $D^p(X) = N^{1,p}(X) + \mathbf{R}$, see Proposition 7.3.

The rest of this paper is structured as follows. Definitions of the concepts related to the function spaces studied in this paper are given in Section 2, and the concepts regarding $p$-harmonicity and related useful tools are given in Section 3. Section 4 is devoted to the definitions of $p$-hyperbolic ends and $p$-hyperbolic sequences in metric measure spaces and a brief discussion of them. In Section 5 we prove the existence of nonconstant $p$-harmonic functions with finite energy under the assumption that the metric measure space has at least two distinct $p$-hyperbolic sequences. We follow this up by a discussion of classification of metric measure spaces in Section 6. In this section we also provide the proofs of Theorems 1.2 and 1.3. The third main theorem of the paper, Theorem 1.4, is proved in Section 7. In that section the converse of Theorem 1.4 is also discussed. Section 8 is devoted to providing examples that illustrate the sharpness of the results given in the paper. Example 8.3 is also essential when deducing most of the noninclusions in Theorem 1.2. Finally, Section 9 provides an extension of the finite-energy Liouville theorem to the noncomplete setting.

*Acknowledgement.* Parts of this research project were conducted during 2017 and 2018 when N. S. was a guest professor at Linköping University, partially funded by the Knut and Alice Wallenberg Foundation, and during the parts of 2019 when A. B. and J. B. were Taft Scholars at the University of Cincinnati. The authors would like to thank these institutions for their kind support and hospitality. A. B. and J. B. were partially supported by the Swedish Research Council grants 2016-03424 and 2020-04011 resp. 621-2014-3974 and 2018-04106. N. S. was partially supported by the National Science Foundation (U.S.A.) grants DMS-1500440 and DMS-1800161.

## 2. Preliminaries

We assume throughout the paper that $X$ is a metric space equipped with a metric $d$ and a positive complete Borel measure $\mu$ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. In this section we also assume that $1 \le p < \infty$. For proofs of the facts stated in this section we refer the reader to Björn–Björn [5] and Heinonen–Koskela–Shanmugalingam–Tyson [34].

A notion critical to this paper is that of $p$-modulus of families of curves in $X$. A *curve* is a continuous mapping from an interval. We will only consider locally rectifiable curves, and they can always be parameterized by their arc length $ds$.

**Definition 2.1.** Let $\Gamma$ be a family of locally rectifiable curves in $X$. The *$p$-modulus* of $\Gamma$ is the number

$$\operatorname{Mod}_p(\Gamma) := \inf_\rho \int_X \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions $\rho$ on $X$ such that $\int_\gamma \rho \, ds \ge 1$ for each $\gamma \in \Gamma$.

From now on, unless otherwise said, all our curves will be nonconstant, compact and *rectifiable*, i.e. of finite length. We follow Heinonen and Koskela [32] in introducing upper gradients (in [32] they are referred to as very weak gradients).

**Definition 2.2.** A nonnegative Borel function $g$ on $X$ is an *upper gradient* of an extended real-valued function $u$ on $X$ if for all curves $\gamma : [0, l_\gamma] \to X$,

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \le \int_\gamma g \, ds, \tag{2.1}$$

where we follow the convention that the left-hand side is $\infty$ whenever at least one of the terms therein is infinite. If $g$ is a nonnegative measurable function on $X$ and if (2.1) holds for $p$-almost every curve, then $g$ is a *$p$-weak upper gradient* of $u$. A property holds for *$p$-almost every curve* if it fails only for a curve family with zero $p$-modulus.

The notion of $p$-weak upper gradients was introduced in Koskela–MacManus [49]. It was also shown therein that if $g \in L^p_{\rm loc}(X)$ is a $p$-weak upper gradient of $u$, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of $f$ such that $\|g_j - g\|_{L^p(X)} \to 0$.

If $u$ has an upper gradient in $L^p_{\rm loc}(X)$, then it has a *minimal $p$-weak upper gradient* $g_u \in L^p_{\rm loc}(X)$ in the sense that $g_u \le g$ a.e. for every $p$-weak upper gradient $g \in L^p_{\rm loc}(X)$ of $u$, see Shanmugalingam [58]. The minimal $p$-weak upper gradient is well defined up to a set of measure zero in the cone of nonnegative functions in $L^p_{\rm loc}(X)$. Moreover, $g_u = g_v$ a.e. in the set $\{x \in X : u(x) = v(x)\}$, in particular $g_{\min\{u,c\}} = g_u \chi_{\{u<c\}}$ for $c \in \mathbf{R}$. Note also that a modification of an upper gradient on a Borel set of measure zero need not yield an upper gradient, but a modification of a $p$-weak upper gradient on a set of measure zero still yields a $p$-weak upper gradient.

Following Shanmugalingam [57], we define a version of Sobolev spaces on $X$.

**Definition 2.3.** For a measurable function $u : X \to [-\infty, \infty]$, let

$$\|u\|_{N^{1,p}(X)} = \biggl( \int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \biggr)^{1/p},$$

where the infimum is taken over all upper gradients $g$ of $u$. The *pre-Newtonian space* on $X$ is

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}.$$

The *Newtonian space* $N^{1,p}(X)/{\sim}$, where $f \sim h$ if and only if $\|f-h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see [57]. We are also interested in the homogeneous version of Sobolev spaces. The *Dirichlet space* $D^p(X)$ is the collection of all measurable functions on $X$ that have an upper gradient in $L^p(X)$.

We say that $u \in N^{1,p}_{\rm loc}(X)$ if for every $x \in X$ there exists $r_x$ such that $u \in N^{1,p}(B(x,r_x))$. The local spaces $L^p_{\rm loc}(X)$ and $D^p_{\rm loc}(X)$ are defined similarly. Note that if $X$ supports a local $p$-Poincaré inequality (as in Definition 2.7 below) then it follows by truncations and Fatou's lemma that $N^{1,p}_{\rm loc}(X) = D^p_{\rm loc}(X)$.

In this paper we assume that functions in the above function spaces $N^{1,p}_{\rm loc}(X)$, $D^p_{\rm loc}(X)$ are defined everywhere (with values in $[-\infty, \infty]$), not just up to an equivalence class in the corresponding function space.

For a measurable set $E \subset X$, the space $N^{1,p}(E)$ is defined by considering $(E, d|_E, \mu|_E)$ as a metric space in its own right. The spaces $N^{1,p}_{\rm loc}(E)$, $L^p(E)$, $L^p_{\rm loc}(E)$, $D^p(E)$ and $D^p_{\rm loc}(E)$ are defined similarly.

**Definition 2.4.** The (Sobolev) *capacity* of a set $E \subset X$ is the number

$$C_p^X(E) = C_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on $E$.

A property is said to hold *quasieverywhere* (q.e.) if the set of all points in $X$ at which the property fails has $C_p$-capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, if $u, v \in N^{1,p}_{\rm loc}(X)$ and $u = v$ a.e., then $u = v$ q.e.

**Definition 2.5.** The (Dirichlet) capacity of the pair $(E, F)$ of disjoint sets in $X$ is

$$\operatorname{cap}_{D^p}(E, F) = \int_X g_u^p \, d\mu,$$

where the infimum is taken over all functions $u \in D^p(X)$ with $u \ge 1$ on $E$ and $u \le 0$ on $F$.

The following equality was proved for compact sets in Kallunki–Shanmugalingam [42]. Since we need it for general closed sets, we provide a short proof. Here and later we let $\Gamma(E, F)$ be the collection of all curves in $X$ with one end point in $E$ and the other in $F$.

**Lemma 2.6.** *Let $E$ and $F$ be disjoint closed subsets of $X$. Then*

$$\operatorname{Mod}_p(\Gamma(E,F)) = \operatorname{cap}_{D^p}(E,F).$$

*Proof.* Let $v \in D^p(X)$ be admissible for $\operatorname{cap}_{D^p}(E,F)$. Then every upper gradient $g$ of $v$ is admissible for $\operatorname{Mod}_p(\Gamma(E,F))$ and hence

$$\operatorname{Mod}_p(\Gamma(E,F)) \le \int_X g^p \, d\mu.$$

Taking infimum over all upper gradients $g$ of $v$ and then taking infimum over all $v$ admissible for $\operatorname{cap}_{D^p}(E,F)$ proves one inequality in the lemma.

Conversely, let $\rho \in L^p(X)$ be admissible for $\operatorname{Mod}_p(\Gamma(E,F))$ and consider the function

$$u(x) := \min\biggl\{1, \inf_\gamma \int_\gamma \rho\,ds\biggr\},$$

with the infimum taken over all curves (including constant curves) $\gamma$ connecting $x$ to $F$. By Björn–Björn–Shanmugalingam [11, Lemma 3.1], $u$ has $\rho$ as an upper gradient, $u=0$ on $F$ and $u=1$ on $E$. Since $\rho \in L^p(X)$, we infer from Rogovin–Rogovin–Järvenpää–Järvenpää–Shanmugalingam [41, Corollary 1.10] that $u$ is measurable and thus $u \in D^p(X)$. It follows that

$$\operatorname{cap}_{D^p}(E,F) \le \int_X \rho^p\,d\mu,$$

and taking infimum over all $\rho \in L^p(X)$ admissible for $\operatorname{Mod}_p(\Gamma(E,F))$ concludes the proof. □

As in Björn–Björn [6], we define the following local versions of the notions of doubling measures and Poincaré inequality.

**Definition 2.7.** We say that the measure $\mu$ is *doubling within a ball* $B_0$ if there is a *doubling constant* $C>0$ (depending on $B_0$) such that for all balls $B=B(x,r):=\{y \in X : d(y,x)<r\} \subset B_0$,

$$\mu(2B) \le C\mu(B),$$

where $\lambda B = B(x,\lambda r)$.

Similarly, the *$p$-Poincaré inequality holds within a ball* $B_0$ if there are constants $C>0$ and $\lambda \ge 1$ (both depending on $B_0$) such that for all balls $B \subset B_0$, all integrable functions $u$ on $\lambda B$, and all upper gradients $g$ of $u$ in $\lambda B$,

$$\fint_B |u-u_B|\,d\mu \le C r_B \biggl(\fint_{\lambda B} g^p\,d\mu\biggr)^{1/p}, \tag{2.2}$$

where $u_B := \fint_B u\,d\mu := \int_B u\,d\mu/\mu(B)$ and $r_B$ is the radius of $B$.

Each of these properties is called *local* if for every $x \in X$ there is some $r>0$ (depending on $x$) such that the property holds within $B(x,r)$. The property is called *uniformly local* if $r$, $C$ and $\lambda$ are independent of $x$. If it holds within every ball $B(x_0,r_0)$ in $X$ with $C$ and $\lambda$ independent of $x_0$ and $r_0$, then it is called *global*.

# 3. Quasiharmonic and $p$-harmonic functions

*From now on, except for Section 9, we assume that $X$ is an unbounded proper connected metric space. We also assume that $1<p<\infty$, that $\mu$ is locally doubling and supports a local $p$-Poincaré inequality, and that $\Omega \subset X$ is an open set.*

A metric space $X$ is *proper* if every closed bounded set in $X$ is compact. It follows that $X$ is complete. Moreover, Proposition 1.2 and Theorem 1.3 in Björn–Björn [6] imply that under the above assumptions, the doubling property and $p$-Poincaré inequality actually hold within every ball in $X$.

**Definition 3.1.** A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *quasiminimizer* in $\Omega$ if there exists $Q_u \ge 1$ such that

$$\int_{\varphi\ne 0} g_u^p\,d\mu \le Q_u \int_{\varphi \ne 0} g_{u+\varphi}^p\,d\mu \tag{3.1}$$

for all $\varphi \in N_0^{1,p}(\Omega)$, where

$$N_0^{1,p}(\Omega) := \{\varphi|_\Omega : \varphi \in N^{1,p}(X) \text{ and } \varphi = 0 \text{ on } X \setminus \Omega\}.$$

A *quasiharmonic function* is a continuous quasiminimizer.

If $Q_u = 1$ in (3.1), then $u$ is a *minimizer*, and if it is in addition continuous, then it is a *p-harmonic function*.

Functions from $N_0^{1,p}(\Omega)$ can be extended by zero in $X \setminus \Omega$ and we will regard them in that sense if needed.

Note that the property of being a quasiminimizer depends on the index $p$ even though we have refrained from making that explicit in the notation. The integrals in (3.1) can be infinite but then they are infinite simultaneously. Under our assumptions, locally Lipschitz functions are dense in $N_{\mathrm{loc}}^{1,p}(\Omega)$, see [6, Theorem 8.4]. It therefore follows from Björn–Björn–Shanmugalingam [11, Theorem 5.7] (or [5, Theorem 5.45]) that Lipschitz functions with compact support in $\Omega$ are dense in $N_0^{1,p}(\Omega)$. Hence, the definition of quasiminimizers can equivalently be based on such compactly supported Lipschitz test functions. The integration in (3.1) can moreover equivalently be over $\operatorname{supp} \varphi$ rather than the set where $\varphi \neq 0$, see Björn [4, Proposition 3.2]. Note also that $N_0^{1,p}(X) = N^{1,p}(X)$, which has consequences for globally defined quasiminimizers on $X$.

Any quasiminimizer can be modified on a set of capacity zero so that it becomes locally Hölder continuous. This follows from the results in Kinnunen–Shanmugalingam [45, p. 417]. The assumptions therein are different from ours, but see Björn–Björn [6, Theorem 10.2 and the discussion around it] for how those results apply under the local assumptions considered here. Such a continuous representative is called a *quasiharmonic* function or, for $Q_u = 1$, a *p-harmonic function*.

The Liouville theorem given below follows from the Harnack inequality proved in [45, Corollary 7.3] or Björn–Marola [15, Corollary 9.4].

**Theorem 3.2.** *Assume that $\mu$ is globally doubling and supports a global $p$-Poincaré inequality. If $u$ is a positive quasiharmonic function on $X$, then it is constant. In particular, $X \in O_{QP}^p \subset O_{HP}^p$.*

The following lemma will be convenient when proving Theorem 1.4.

**Lemma 3.3.** *If $u \in N^{1,p}(X)$ is quasiharmonic on $X$, then it is constant.*

*Proof.* Let $u$ be quasiharmonic on $X$. If $u \in N^{1,p}(X) = N_0^{1,p}(X)$, then testing (3.1) with $-u \in N_0^{1,p}(X)$ yields

$$\int_{u\neq 0} g_u^p \, d\mu \le Q_u \int_{u \neq 0} g_{u-u}^p \, d\mu = 0.$$

This together with the local $p$-Poincaré inequality shows that $u$ is locally a.e.-constant, and as $u$ is continuous and $X$ connected, $u$ is constant. □

The following lemma about convergence of $p$-harmonic functions is a useful tool.

**Lemma 3.4.** *Let $\Omega_j$ be open sets such that $\Omega_j \subset \Omega_{j+1}$, $j = 1, 2, \dots$, and $X = \bigcup_{j=1}^\infty \Omega_j$. Assume that $u_j \in D^p(X)$ is $p$-harmonic in $\Omega_j$ and that there is a constant $M$ such that for all $j = 1, 2, \dots$,*

$$|u_j| \le M \text{ in } X \quad \text{and} \quad \|g_{u_j}\|_{L^p(X)} \le M.$$

*Then there are (finite) convex combinations*

$$\hat{u}_j = \sum_{k=j}^{N_j} \tilde{\lambda}_{k,j} u_k, \quad \text{with } \tilde{\lambda}_{j,k} \ge 0 \text{ and } \sum_{k=j}^{N_j} \tilde{\lambda}_{j,k} = 1,$$

*of the sequence* $\{u_j\}_{j=1}^\infty$*, which converge locally uniformly in* $X$ *to a function* $u \in D^p(X)$ *that is* $p$*-harmonic in* $X$*, satisfies* $|u| \le M$ *and moreover*

$$\|g_{\hat{u}_j} - g_u\|_{L^p(X)} \to 0, \quad \text{as } j \to \infty. \tag{3.2}$$

*Proof.* Theorem 5.4 in [6] implies that for every ball $B_0 \subset X$, there is some $1 \le q < p$ such that a $q$-Poincaré inequality holds within this ball in the sense of Definition 2.7. This better Poincaré inequality allows us to apply the continuity and convergence results for $p$-harmonic functions from Kinnunen–Shanmugalingam [45] and Shanmugalingam [59], see also the discussion in [6, Section 10].

More precisely, by [45, Proposition 3.3 and Theorem 5.2] and the fact that $|u_j| \le M$ on $X$, the (tail of the) sequence $\{u_j\}_{j=1}^\infty$ is equi(Hölder)-continuous on every ball in $X$, see also [5, Theorem 8.14]. Thus an appeal to the Ascoli theorem and the Harnack convergence principle ([59, Theorem 1.2] or [5, Theorem 9.37]), together with a Cantor diagonalization argument, yields a subsequence, also denoted $\{u_j\}_{j=1}^\infty$, that converges uniformly on balls in $X$ to a function $u$ that is $p$-harmonic in $X$. Note that $u \in N^{1,p}(B)$ for every ball $B$ and that $|u| \le M$ on $X$. It remains to prove (3.2).

Since the sequence $\{g_{u_j}\}_{j=1}^\infty$ is bounded in $L^p(X)$, we can use the reflexivity of $L^p(X)$ to extract a subsequence, still denoted $\{g_{u_j}\}_{j=1}^\infty$, that converges weakly to a nonnegative function $g \in L^p(X)$. Mazur's lemma (applied iteratively to the subsequences $\{g_{u_j}\}_{j=k}^\infty$) then provides us with a sequence of convex combinations

$$g_k = \sum_{j=k}^{N(k)} \lambda_{j,k} g_{u_j}, \quad \text{with } \lambda_{j,k} \ge 0 \text{ and } \sum_{j=k}^{N(k)} \lambda_{j,k} = 1,$$

such that $\|g_k - g\|_{L^p(X)} \le 2^{-k}$. Let $\hat{g} = g + \sum_{k=1}^\infty |g_k - g|$. Then $\hat{g} \in L^p(X)$ and $g_k \le \hat{g}$ in $X$ for all $k = 1, 2, \ldots$ .

Note that the functions $g_k$ are $p$-weak upper gradients of the corresponding convex combinations

$$v_k = \sum_{j=k}^{N(k)} \lambda_{j,k} u_j.$$

Hence $g_{v_k} \le g_k$ a.e. in $X$ and

$$\|g_{v_k}\|_{L^p(X)} \le \|g_k\|_{L^p(X)} \le \sum_{j=k}^{N(k)} \lambda_{j,k} \|g_{u_k}\|_{L^p(X)} \le M. \tag{3.3}$$

Next, choose an increasing sequence of balls $B_j$, so that $X = \bigcup_{j=1}^\infty B_j$. The sequence $\{v_k\}_{k=1}^\infty$ satisfies $|v_k| \le M$ on $X$. In view of (3.3), it is therefore bounded in $N^{1,p}(\overline{B}_j)$ for every $j = 1, 2, \ldots$ . Since $\overline{B}_j$ is a complete doubling metric space by Björn–Björn [6, Propositions 1.2 and 3.4], it follows from Ambrosio–Colombo–Di Marino [2, Corollary 41] that the Newtonian space $N^{1,p}(\overline{B}_j)/\sim$ is reflexive (where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$). Thus, using weakly converging subsequences and Mazur's lemma again (this time for subsequences in $N^{1,p}(\overline{B}_j)$), for each $j = 1, 2, \ldots$ we can find a further convex combination

$$\hat{u}_j = \sum_{k=j}^{\widehat{N}(j)} \hat{\lambda}_{j,k} v_k, \quad \text{with } \hat{\lambda}_{j,k} \ge 0 \text{ and } \sum_{k=j}^{\widehat{N}(j)} \hat{\lambda}_{j,k} = 1,$$

such that $\|\hat{u}_j - u\|_{N^{1,p}(\overline{B}_j)} \le 2^{-j}$. In particular, $\|g_{\hat{u}_j - u}\|_{L^p(B_j)} \le 2^{-j}$. As

$$g_u \le g_{\hat{u}_j} + g_{u - \hat{u}_j} \quad \text{and} \quad g_{\hat{u}_j} \le g_u + g_{\hat{u}_j - u},$$

we consequently have $\|g_{\hat{u}_j} - g_u\|_{L^p(B_j)} \le 2^{-j}$. In particular, $g_{\hat{u}_j} \to g_u$ in $L^p(B)$ for each ball $B$ and a.e. in $X$, as $j \to \infty$.

Note that the sequence $\{\hat{u}_j\}_{j=1}^\infty$ (being a convex combination of locally uniformly converging functions) also converges locally uniformly in $X$ to $u$.

Now, since $g_{v_k} \le g_k \le \hat{g}$ a.e. in $X$, we conclude that also

$$g_{\hat{u}_j} \le \sum_{k=j}^{\widehat{N}(j)} \hat{\lambda}_{j,k} g_{v_k} \le \hat{g} \in L^p(X).$$

The Lebesgue dominated convergence theorem therefore implies that $g_{\hat{u}_j} \to g_u$ in $L^p(X)$, which concludes the proof. □

We will use solutions of the Dirichlet problem and more precisely so-called $p$-harmonic extensions, which we define next, following Hansevi [28, Definition 4.6].

**Definition 3.5.** Assume that $C_p(X \setminus \Omega) > 0$. Let $f \in D^p(X)$. Then the *$p$-harmonic extension* $H_\Omega f$ of $f$ in $\Omega$ is the unique $p$-harmonic function in $\Omega$ such that $f - H_\Omega f \in D^p_0(\Omega)$, where

$$D^p_0(\Omega) := \{\varphi|_\Omega : \varphi \in D^p(X) \text{ and } \varphi = 0 \text{ on } X \setminus \Omega\}.$$

We also let $H_\Omega f = f$ on $X \setminus \Omega$ to get a globally defined function when needed.

The $p$-harmonic extension exists and is unique by Hansevi [28, Theorem 4.4]. If $\Omega$ is bounded and $f \in N^{1,p}(X)$, then the definition of $H_\Omega f$ coincides with other definitions in the literature, such as in Shanmugalingam [58, Theorem 5.6], Björn–Björn–Shanmugalingam [10, Definition 3.3] and [5, Definition 8.31]. The existence, uniqueness and other properties of $H_\Omega f$ in bounded sets were obtained in these references.

The following relation between $\operatorname{cap}_{D^p}$ and harmonic extensions in unbounded sets will be useful.

**Proposition 3.6.** *Let $F_0$ and $F_1$ be two disjoint closed sets with $\operatorname{cap}_{D^p}(F_0, F_1) < \infty$. Then there is $f \in D^p(X)$ such that $f = j$ on $F_j$, $j = 0, 1$. Moreover for any such $f$,*

$$\operatorname{cap}_{D^p}(F_0, F_1) = \int_X g^p_{H_\Omega f} \, d\mu, \quad \textit{where } \Omega = X \setminus (F_0 \cup F_1). \tag{3.4}$$

*Proof.* As $\operatorname{cap}_{D^p}(F_0, F_1) < \infty$, the existence of such a function $f$ is immediate. The definition of the harmonic extension in [28, Definition 4.6] shows that $H_\Omega f$ solves the minimization problem in the definition of $\operatorname{cap}_{D^p}(F_0, F_1)$, i.e. it satisfies (3.4). □

**Remark 3.7.** Since we consider $p$-harmonic functions on unbounded sets in this paper, results from Hansevi [28], [29] will be of primary importance here. We therefore comment on how the assumptions therein compare with ours.

In [28], $X$ is assumed to be proper, connected and supporting a global $(p, p)$-Poincaré inequality (with an averaged $L^p$-norm also on the left-hand side of (2.2)). However, the only use of the Poincaré inequality in the existence theorem [28, Theorem 3.4], the comparison principle [28, Lemma 3.6] and the convergence theorem for obstacle problems [29, Theorem 3.2] is through Maz′ya's inequality on a sequence of balls (on p. 98 and again on p. 102) with no need of uniform control of the constants. Therefore, it is enough to require that $\mu$ supports a $(p, p)$-Poincaré inequality on all sufficiently large balls, see the proof of Maz′ya's inequality in [5, Theorem 5.53]. Under our standing assumptions, such a $(p, p)$-Poincaré inequality on balls (with constants depending on the ball) follows from Björn–Björn [6, Theorems 1.3 and 5.1].

The inner regularity results in [28, Theorem 4.4] and the tools from Kinnunen–Shanmugalingam [45], Shanmugalingam [59], Kinnunen–Martio [44] and Björn–Björn [5], used in [28] and [29] are of local nature and therefore hold under our local assumptions. Note that since $X$ is assumed to be connected and proper, the local doubling property and Poincaré inequality self-improve so that they actually hold within every ball $B_0 \subset X$ (with constants depending on $B_0$), which is enough for such local regularity results, see the discussion in [6, Section 10].

In particular, the resolutivity and uniqueness results for Perron solutions with continuous boundary data on unbounded $p$-parabolic sets from [29, Section 7] are available under our assumptions and will be used later.

## 4. Hyperbolic ends and hyperbolic sequences

The theory of ends was originally developed to study the classification of Riemann surfaces, as in Sario–Nakai [55]. Heuristically, for us an end represents a point of $X$ at $\infty$. For example, if $X$ is homeomorphic to $\mathbf{S}^1 \times \mathbf{R}$, then in our sense it has two ends. However, note that if the metric on $X$ is such that at least one of the ends is hyperbolic and rotationally invariant, then from a geometric group theoretic point of view this end contains a copy of $\mathbf{S}^1$. In this paper we still consider this end as one point at $\infty$.

**Definition 4.1.** We say that a sequence $\{F_n\}_{n=1}^\infty$ is a *chain* at $\infty$ of $X$ (called a chain of $X$ for simplicity) if there is a point $x_0 \in X$ and a strictly increasing sequence of radii $R_n \to \infty$ such that $F_n$ is a component of $X \setminus B(x_0, R_n)$ and $F_{n+1} \subset F_n$.

Two chains $\{F_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ at $\infty$ are said to be *equivalent* if for each positive integer $k$ there are $n_k$ and $m_k$ such that $F_{n_k} \subset G_k$ and $G_{m_k} \subset F_k$. This equivalence relationship partitions the class of all chains of $X$ into pairwise disjoint equivalence classes, called *ends* of $X$.

From Kline–Lindquist–Shanmugalingam [46, Lemma 5.11] we know that the choice of $x_0$ does not play a central role in the construction of ends. Traditionally, an end of a manifold or a metric space $X$ is a sequence $\{F_n\}_{n=1}^\infty$ of connected sets that are components of complements of compact subsets $K_n \subset X$ such that $F_{n+1} \subset F_n$ for each $n$ and $X = \bigcup_{n=1}^\infty K_n$, see e.g. Choi [18, Definition 5.1]. For us it is more convenient to have ends made up of closed sets. Given our assumption that $X$ is proper, replacing $F_n$ with its closure merely gives an equivalent notion of ends.

The papers Grigor′yan [26], [27] and Holopainen [36] used different definitions of "ends", sufficient for their purposes. Since in this paper we will be discussing the possibility of a metric space having more than one end and even infinitely many ends, we need the precise terminology here.

The terminology we follow is adapted from Adamowicz–Björn–Björn–Shanmugalingam [1] and Estep [21]. The equivalence class that contains a chain $\{F_n\}_{n=1}^\infty$ was denoted $[F_n]$ in [21] and [46]. However, it was shown in [21] that if $\{F_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ are two chains at $\infty$ such that for each $k$ there is a positive integer $m_k$ with $F_{m_k} \subset G_k$, then the two chains are equivalent. Therefore in discussing an end, it suffices to discuss a chain at $\infty$ that represents the end. Hence from now on, we will also call a chain an end.

Recall the definitions of $\operatorname{Mod}_p$ and $\Gamma(E, F)$ from Section 2. Given an end $\{F_n\}_{n=1}^\infty$ and $E \subset X$, note that

$$\Gamma(E, F_{n+1}) \subset \Gamma(E, F_n).$$

As in Holopainen [36] and Holopainen–Koskela [38] we give the following definitions. On metric measure spaces, the study of $p$-parabolicity and $p$-hyperbolicity began in [38] and Holopainen–Shanmugalingam [40].

**Definition 4.2.** The end $\{F_n\}_{n=1}^\infty$ is *$p$-hyperbolic* if

$$\lim_{n\to\infty} \operatorname{Mod}_p(\Gamma(\overline{B(x_0,1)}, F_n)) > 0.$$

The space $X$ is *$p$-hyperbolic* if

$$\lim_{n\to\infty} \operatorname{Mod}_p(\Gamma(\overline{B(x_0,1)}, X \setminus B(x_0,n))) > 0.$$

We say that an end or $X$ is *$p$-parabolic* if it is not $p$-hyperbolic.

It follows directly from the definition that if $X$ has a $p$-hyperbolic end, then $X$ is a $p$-hyperbolic space. Conversely, if $X$ is a $p$-hyperbolic space with finitely many ends, then it has a $p$-hyperbolic end. Example 8.5 below shows that there is a $p$-hyperbolic space with infinitely many $p$-parabolic ends but no $p$-hyperbolic end.

**Remark 4.3.** Since the metric measure space $X$ is proper, we know from Shanmugalingam [60, Theorem 4.2] that if $\{F_n\}_{n=1}^\infty$ is an end of $X$, then it is $p$-hyperbolic if and only if

$$\operatorname{Mod}_p(\Gamma_{\mathrm{loc}}(\overline{B}, \{F_n\}_{n=1}^\infty)) > 0,$$

where $B = B(x_0,1)$ and $\Gamma_{\mathrm{loc}}(\overline{B}, \{F_n\}_{n=1}^\infty)$ is the collection of all locally rectifiable curves $\gamma$ starting in $\overline{B}$ and intersecting $F_n$ for each $n = 1,2,\ldots$. Note that

$$\Gamma_{\mathrm{loc}}(\overline{B}, \{F_n\}_{n=1}^\infty) = \bigcap_{n=1}^\infty \Gamma_{\mathrm{loc}}(\overline{B}, F_n) \quad \text{and} \quad \operatorname{Mod}_p(\Gamma_{\mathrm{loc}}(\overline{B}, F_n)) = \operatorname{Mod}_p(\Gamma(\overline{B}, F_n)),$$

where $\Gamma_{\mathrm{loc}}(\overline{B}, F_n)$ is the collection of all locally rectifiable curves in $X$ starting in $\overline{B}$ and intersecting $F_n$.

Euclidean spaces $\mathbf{R}^n$, $n \ge 2$, have exactly one end, and this end is $p$-parabolic if and only if $p \ge n$. Parabolicity can in many situations be characterized by volume growth conditions, see [8, Theorem 5.5], [19, Proposition 3.4], [27, Theorems 7.3 and 14.6], [36, Section 4] and [38, Theorem 1.7].

Our aim in this paper is to investigate when a metric measure space carries nonconstant $p$-(quasi)harmonic functions. For functions with finite energy, this property turns out to be closely related to the following notion, which extends the concepts given in Definition 4.2.

**Definition 4.4.** Let $x_0 \in X$. A sequence $\{F_n\}_{n=1}^\infty$ is a *$p$-hyperbolic sequence* if it is a decreasing sequence of nonempty closed sets such that:

(a) for each $r > 0$ there is $n > 0$ such that $B(x_0,r) \cap F_n = \varnothing$;

(b)
$$\lim_{n\to\infty} \operatorname{Mod}_p(\Gamma(\overline{B(x_0,1)}, F_n)) > 0. \tag{4.1}$$

Since $X$ is proper, it follows that (a) is equivalent to $\bigcap_{n=1}^\infty F_n = \varnothing$. This equivalence need not hold in nonproper spaces. It follows directly from the definitions that every $p$-hyperbolic end forms a $p$-hyperbolic sequence, and that the existence of a $p$-hyperbolic sequence implies that $X$ is $p$-hyperbolic. The following lemma shows that in Definitions 4.2 and 4.4, the ball $\overline{B(x_0,1)}$ can equivalently be replaced by any compact set with positive capacity, see also Holopainen–Shanmugalingam [40, Proof of Lemma 3.5].

**Lemma 4.5.** *Let $\{F_n\}_{n=1}^\infty$ be a decreasing sequence of closed sets in $X$ satisfying condition (a) of Definition 4.4. Also let $K_1$ and $K_2$ be compact sets with $C_p(K_j) > 0$, $j = 1, 2$. Then*

$$\lim_{n\to\infty} \operatorname{cap}_{D^p}(K_1, F_n) = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} \operatorname{cap}_{D^p}(K_2, F_n) = 0.$$

*Proof.* By replacing $K_1$ resp. $K_2$ by $K_1 \cup K_2$ we see that we may assume, without loss of generality, that $K_1 \subset K_2$. It follows from [6, Proposition 4.8 and Lemma 4.10] that there is a ball $B$ such that $K_2$ is contained in a rectifiably pathconnected component $G$ of $B$. Note that as $X$ is locally doubling and supports a local Poincaré inequality, it is locally quasiconvex, and hence $G$ is an open set.

As $K_1$ is compact, there is $f \in N^{1,p}(X)$ such that $f \equiv 0$ in $X \setminus G$ and $f \equiv 1$ on $K_1$. Let $v = H_{G\setminus K_1} f$ be the $p$-harmonic extension of $f$ in $G \setminus K_1$ as in Definition 3.5. We start by showing that $m := \inf_{K_2} v > 0$. If not, then the strong maximum principle (see [45] or [5, Theorem 8.13], together with [6, Section 10]) shows that $v \equiv 0$ in $G \setminus K_1$. Moreover, $v \equiv 0$ in $X \setminus G$, and since $v \equiv 1$ in $K_1$, we would have that $v = \chi_{K_1} \in N^{1,p}(X)$ and $g_v = 0$ a.e. The $p$-Poincaré inequality on $B$ then implies that $v$ is constant a.e. (and thus q.e.) in $B$, which contradicts $C_p(K_1) > 0$. Note from [6, Theorem 1.3] that our assumptions on $X$ imply the validity of a $p$-Poincaré inequality on arbitrary balls (with constants depending on the ball). Thus $m > 0$.

Let $n$ be large enough so that $F_n \cap B = \varnothing$. Let $u = H_{X\setminus(K_1\cup F_n)} f$ be the $p$-harmonic extension of $f$ in $X \setminus (K_1 \cup F_n)$. By Proposition 3.6,

$$\int_X g_u^p \, d\mu = \operatorname{cap}_{D^p}(K_1, F_n).$$

The comparison principle (see for example Hansevi [28, Lemma 3.6]) implies that $u \geq v$ in $B$. Hence $u \geq m$ in $K_2$, and thus $u/m$ is admissible for $\operatorname{cap}_{D^p}(K_2, F_n)$. Therefore

$$m^p \operatorname{cap}_{D^p}(K_2, F_n) \leq \int_X g_u^p \, d\mu = \operatorname{cap}_{D^p}(K_1, F_n) \leq \operatorname{cap}_{D^p}(K_2, F_n).$$

Since $m$ is independent of $n$, letting $n \to \infty$ concludes the proof. □

## 5. Existence of nonconstant $p$-harmonic functions with finite energy

**Theorem 5.1.** *Assume that there are two disjoint $p$-hyperbolic sequences $\{F_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ such that $\operatorname{Mod}_p(\Gamma(F_1, G_1)) < \infty$. Then $X$ supports a nonconstant bounded $p$-harmonic function with finite energy, i.e. $X \notin O^p_{HBD}$.*

Observe that $\operatorname{Mod}_p(\Gamma(F_1, G_1)) = \operatorname{cap}_{D^p}(F_1, G_1)$, by Lemma 2.6. Before proving Theorem 5.1, we first show how it implies the following important corollary. At the same time, Example 8.1 shows that Theorem 5.1 can be used also when $X$ only has one end, and that end is $p$-hyperbolic. The converse of Theorem 5.1 is proved in Proposition 6.4 below.

**Corollary 5.2.** *If $X$ has at least two $p$-hyperbolic ends, then $X$ supports a nonconstant bounded $p$-harmonic function with finite energy, i.e. $X \notin O^p_{HBD}$.*

*Proof.* A $p$-hyperbolic end is automatically a $p$-hyperbolic sequence. Denote the two $p$-hyperbolic ends as $\{F_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$, with $F_1 \cap G_1 = \varnothing$. We may also assume that they are created as in Definition 4.1 with the same strictly increasing sequence of radii $\{R_n\}_{n=1}^\infty$. Testing $\operatorname{Mod}_p(\Gamma(F_2, G_2))$ with

$$\rho = \frac{\chi_{B(x_0,R_2)}}{R_2 - R_1}$$

then shows that $\operatorname{Mod}_p(\Gamma(F_2, G_2)) < \infty$. Hence (after shifting indices), the corollary follows from Theorem 5.1. □

To prove Theorem 5.1 we first need to understand connectivity properties of curves between two $p$-hyperbolic sequences, or equivalently the relationship between the corresponding capacities. This will be the content of the following two lemmas.

**Lemma 5.3.** *Suppose that $\{F_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ are two disjoint $p$-hyperbolic sequences in $X$. Then*

$$\lim_{n\to\infty} \operatorname{cap}_{D^p}(F_n, G_n) > 0.$$

Recall that $\operatorname{cap}_{D^p}(F_n, G_n) = \operatorname{Mod}_p(\Gamma(F_n, G_n))$.

*Proof.* Because of the monotonicity of $\operatorname{cap}_{D^p}$ and choosing a subsequence if necessary, without loss of generality we may assume that $\operatorname{cap}_{D^p}(F_1, G_1) < \infty$. Let $B = B(x_0, 1)$. By Theorem 10.2 in Björn–Björn [6] there are positive constants $C$ and $\Lambda$ such that the weak Harnack inequality

$$\fint_B v\,d\mu \le C \inf_B v \tag{5.1}$$

holds for all nonnegative $p$-harmonic functions $v$ in $\Lambda B$. From the definition of $p$-hyperbolic sequences, we can find $N$ such that $\Lambda B \cap (F_N \cup G_N) = \varnothing$. Let $n \ge N$ be fixed but arbitrary. As $\operatorname{cap}_{D^p}(F_n, G_n) < \infty$, there is $f \in D^p(X)$ with $f = 0$ on $F_n$ and $f = 1$ on $G_n$. It follows from (4.1) and Lemma 2.6 that $C_p(F_n) > 0$. By Proposition 3.6

$$\operatorname{cap}_{D^p}(F_n, G_n) = \int_X g_u^p\,d\mu,$$

where $u = H_{X\setminus(F_n\cup G_n)} f$ is the $p$-harmonic extension of $f$ in $X \setminus (F_n \cup G_n)$. Let

$$m = \inf_B u \quad \text{and} \quad M = \sup_B u.$$

We distinguish two cases. If $\fint_B u\,d\mu \ge \frac{1}{2}$, then the weak Harnack inequality (5.1) implies that $2Cm \ge 1$ and hence any upper gradient of the function $2Cu$ is admissible for $\operatorname{Mod}_p(\Gamma(\overline{B}, F_n))$. Taking infimum over all such upper gradients implies that

$$\operatorname{Mod}_p(\Gamma(\overline{B}, F_n)) \le \int_X g_{2Cu}^p\,d\mu = (2C)^p \operatorname{cap}_{D^p}(F_n, G_n).$$

On the other hand, if $\fint_B u\,d\mu \le \frac{1}{2}$, then applying the weak Harnack inequality (5.1) to the $p$-harmonic function $1-u$, we see that $2C(1-M) \ge 1$. Thus, any upper gradient of the function $2C(1-u)$ is admissible for $\operatorname{Mod}_p(\Gamma(\overline{B}, G_n))$ and hence

$$\operatorname{Mod}_p(\Gamma(\overline{B}, G_n)) \le \int_X g_{2C(1-u)}^p\,d\mu = (2C)^p \operatorname{cap}_{D^p}(F_n, G_n).$$

Combining the above two inequalities, we have

$$\lim_{n\to\infty} \operatorname{cap}_{D^p}(F_n, G_n) \ge \frac{1}{(2C)^p} \lim_{n\to\infty} \min\{\operatorname{Mod}_p(\Gamma(\overline{B}, F_n)), \operatorname{Mod}_p(\Gamma(\overline{B}, G_n))\} > 0.$$

□

**Lemma 5.4.** *Suppose that $\{F_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ are two disjoint decreasing sequences of closed nonempty sets in $X$ satisfying (a) of Definition 4.4. Assume that* $\operatorname{Mod}_p(\Gamma(F_1, G_1)) < \infty$ *and that*

$$\lim_{n\to\infty} \operatorname{Mod}_p(\Gamma(F_n, G_n)) =: 2c_0 > 0.$$

*Then $\{F_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ are $p$-hyperbolic sequences.*

*Proof.* Since, by Lemma 2.6,

$$2c_0 \le \operatorname{cap}_{D^p}(F_1, G_1) < \infty,$$

there is a function $u \in D^p(X)$ such that $0 \le u \le 1$ on $X$, $u = 0$ on $F_1$ and $u = 1$ on $G_1$. Let $B$ be a sufficiently large ball such that

$$\int_{X\setminus B} g_u^p \, d\mu < c_0.$$

By changing $g_u$ on a set of zero measure, we can assume that it is a Borel function.

Let $n$ be large enough so that $F_n$ and $G_n$ are disjoint from $\overline{B}$. The curve family $\Gamma_n = \Gamma(F_n, G_n)$ can be written as the union $\Gamma' \cup \Gamma''$, where $\Gamma'$ contains the curves from $\Gamma_n$ passing through $\overline{B}$, while $\Gamma''$ consists of those curves from $\Gamma_n$ which avoid $\overline{B}$. By the choice of $u$, the function $\rho := g_u \chi_{X\setminus\overline{B}}$ is admissible for $\operatorname{Mod}_p(\Gamma'')$ and hence $\operatorname{Mod}_p(\Gamma'') < c_0$. Since every curve in $\Gamma'$ has a subcurve in $\Gamma(\overline{B}, F_n)$, it follows from [5, Lemma 1.34] or [34, p. 128] that

$$\operatorname{Mod}_p(\Gamma(\overline{B}, F_n)) \ge \operatorname{Mod}_p(\Gamma') \ge \operatorname{Mod}_p(\Gamma_n) - \operatorname{Mod}_p(\Gamma'') > c_0,$$

and similarly $\operatorname{Mod}_p(\Gamma(\overline{B}, G_n)) > c_0$. Thus, by Lemmas 2.6 and 4.5, $\{F_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ are $p$-hyperbolic sequences. □

*Proof of Theorem 5.1.* Since $\operatorname{cap}_{D^p}(F_1, G_1) = \operatorname{Mod}_p(\Gamma(F_1, G_1)) < \infty$ by Lemma 2.6, there is $f \in D^p(X)$ such that $f \equiv 0$ on $F_1$ and $f \equiv 1$ on $G_1$.

As in the proof of Lemma 5.3, for each $n \ge 1$, let $u_n = H_{X\setminus(F_n\cup G_n)} f$ be the $p$-harmonic extension of $f$ in $X \setminus (F_n \cup G_n)$. By Proposition 3.6,

$$\int_X g_{u_n}^p \, d\mu = \operatorname{cap}_{D^p}(F_n, G_n) \le \operatorname{cap}_{D^p}(F_1, G_1).$$

Lemma 3.4 provides us with a $p$-harmonic function $u$ on $X$ and convex combinations

$$v_n = \sum_{j=n}^{N_n} \lambda_{j,n} u_j, \quad \text{where } 0 \le \lambda_{j,n} \le 1 \text{ and } \sum_{j=n}^{N_n} \lambda_{j,n} = 1,$$

such that $v_n \to u$ locally uniformly and $\|g_{v_n} - g_u\|_{L^p(X)} \to 0$ as $n \to \infty$. Hence

$$\begin{aligned}\biggl(\int_X g_u^p \, d\mu\biggr)^{1/p} &= \lim_{n\to\infty} \biggl(\int_X g_{v_n}^p \, d\mu\biggr)^{1/p} \le \lim_{n\to\infty} \sum_{j=n}^{N_n} \lambda_{j,n} \biggl(\int_X g_{u_j}^p \, d\mu\biggr)^{1/p} \\ &\le \operatorname{cap}_{D^p}(F_1, G_1)^{1/p} < \infty,\end{aligned}$$

showing that $u$ is a bounded $p$-harmonic function in $X$ with finite energy.

Moreover, each $v_n$ is admissible for $\operatorname{cap}_{D^p}(F_{N_n}, G_{N_n})$, and so by Lemma 5.3,

$$\lim_{n\to\infty} \int_X g_{v_n}^p \, d\mu \ge \lim_{n\to\infty} \operatorname{cap}_{D^p}(F_{N_n}, G_{N_n}) > 0,$$

showing that $\int_X g_u^p \, d\mu > 0$ and so $u$ is nonconstant. □

# 6. Classification of metric measure spaces

Recall from Definition 3.1 and the equality $N^{1,p}_0(X) = N^{1,p}(X)$ that a function $u \in N^{1,p}_{\mathrm{loc}}(X)$ is quasiharmonic in $X$ if for each $\varphi \in N^{1,p}(X)$ we have

$$\int_{\varphi\neq 0} g_u^p \, d\mu \le Q_u \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu. \tag{6.1}$$

**Theorem 6.1.** *If $X$ supports a nonconstant quasiharmonic function with finite energy, then $X \notin O^p_{HBD}$. In particular,*

$$O^p_{HD} = O^p_{HBD} = O^p_{QD} = O^p_{QBD}.$$

A direct consequence of this result together with Theorem 3.2 is the following improvement of one of the main results in Björn–Björn–Shanmugalingam [13, Theorem 1.1] under the additional assumption that $X$ is complete (which under our standing assumptions follows from the properness of $X$). In Section 9 we explain how to obtain it for noncomplete spaces.

**Corollary 6.2.** *Assume that $\mu$ is globally doubling and supports a global $p$-Poincaré inequality. If $u \in D^p(X)$ is a quasiharmonic function on $X$ with finite energy, then it is constant. In particular, $X \in O^p_{QD}$.*

*Proof of Theorem 6.1.* We prove the contrapositive statement. So assume that $X \in O^p_{HBD}$ and that $u$ is a quasiharmonic function on $X$ with finite energy $\int_X g_u^p \, d\mu < \infty$. Our aim is to show that $u$ is constant, and we do so by showing that $g_u = 0$ a.e. in $X$.

Fix $x_0 \in X$ and let $B_j = B(x_0, j)$, $j = 1, 2, \ldots$. As $X$ is unbounded, $C_p(X \setminus B_j) > 0$. For each positive integer $k$ let $u_k = \min\{k, \max\{-k, u\}\}$. Let $v_{k,j} = H_{B_j} u_k$ be the $p$-harmonic extension of $u_k$ in $B_j$. Then $|v_{k,j}| \le k$ and

$$\int_X g_{v_{k,j}}^p \, d\mu = \int_{B_j} g_{v_{k,j}}^p \, d\mu + \int_{X\setminus B_j} g_{u_k}^p \, d\mu \le \int_X g_{u_k}^p \, d\mu < \infty.$$

Lemma 3.4 provides us with convex combinations $\hat{v}_{k,j}$ of the sequence $\{v_{k,j}\}_{j=1}^\infty$ which converge locally uniformly in $X$ to a bounded function $v_k \in D^p(X)$ that is $p$-harmonic in $X$, and moreover

$$\|g_{\hat{v}_{k,j}} - g_{v_k}\|_{L^p(X)} \to 0, \quad \text{as } j \to \infty.$$

As we have assumed that $X \in O^p_{HBD}$ (at the beginning of the proof), $v_k$ must be constant on $X$. Thus $g_{v_k} = 0$ and $g_{\hat{v}_{k,j}} \to 0$ in $L^p(X)$ as $j \to \infty$.

Since $\varphi_{k,j} := \hat{v}_{k,j} - u_k$ are convex combinations of functions in $N^{1,p}(X)$, we see that $\varphi_{k,j} \in N^{1,p}(X)$ and $g_{u+\varphi_{k,j}} \le g_{u-u_k} + g_{\hat{v}_{k,j}}$. The quasiminimizing property (6.1) of $u$ then implies that

$$\begin{aligned}
\int_X g_u^p \, d\mu &= \int_{\varphi_{k,j}\neq 0} g_u^p \, d\mu + \int_{\varphi_{k,j}=0} g_u^p \, d\mu \\
&\le Q_u \int_{\varphi_{k,j}\neq 0} g_{u+\varphi_{k,j}}^p \, d\mu + \int_{\varphi_{k,j}=0} g_{u+\varphi_{k,j}}^p \, d\mu \\
&\le Q_u \int_X (g_{u-u_k} + g_{\hat{v}_{k,j}})^p \, d\mu \\
&\le 2^p Q_u \biggl( \int_{|u|>k} g_u^p \, d\mu + \int_X g_{\hat{v}_{k,j}}^p \, d\mu \biggr),
\end{aligned}$$

where $Q_u$ is the quasiminimizing constant associated with $u$. Letting $j \to \infty$ and then $k \to \infty$ shows that $g_u = 0$ a.e. in $X$. From the local Poincaré inequality, the connectivity of $X$ and the continuity of $u$ we conclude that $u$ must be constant on $X$. $\square$

**Remark 6.3.** It follows directly from Theorem 6.1 that the following two equivalent conditions can be added to Theorem 10.5 in Björn–Björn–Shanmugalingam [14]:

(c) There exists a nonconstant bounded $p$-harmonic function on $(X, d, \mu)$ with finite $p$-energy.

(d) There exists a nonconstant quasiharmonic function on $(X, d, \mu)$ with finite $p$-energy.

Similar modifications can also be made in the conclusions in [14, Example 10.8].

We are now ready to state and prove the converse of Theorem 5.1.

**Proposition 6.4.** *If $X$ supports a nonconstant bounded $p$-harmonic function with finite energy, then there are two disjoint $p$-hyperbolic sequences $\{F_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ such that $\operatorname{Mod}_p(\Gamma(F_1, G_1)) < \infty$. In particular, $X$ is $p$-hyperbolic.*

To prove Proposition 6.4, we shall need the following definition, which extends the well-known notion of $p$-parabolic spaces to open subsets, see Proposition 6.6 below. For manifolds and $p = 2$, this definition appeared in Grigor′yan [25, Definition 3], [27, Section 14.1] and for metric spaces and $p > 1$ in Hansevi [29, Definition 4.1].

**Definition 6.5.** An unbounded open set $\Omega \subset X$ is *$p$-parabolic* if for each compact set $K \subset \Omega$ there exist functions $u_j \in N^{1,p}(\Omega)$ such that $u_j \geq 1$ on $K$ for all $j = 1, 2, ...$ and

$$\int_\Omega g_{u_j}^p \, d\mu \to 0 \quad \text{as } j \to \infty. \tag{6.2}$$

**Proposition 6.6.** *$X$ is $p$-parabolic in the sense of Definition 6.5 if and only if it is $p$-parabolic in the sense of Definition 4.2.*

*Proof.* If $X$ is $p$-hyperbolic in the sense of Definition 4.2, then fixing $x_0 \in X$ and $K := \overline{B(x_0, 1)}$, we know that

$$\lim_{n\to\infty} \operatorname{Mod}_p(\Gamma(K, X \setminus B_n)) =: c > 0,$$

where $B_n = B(x_0, n)$. Now suppose that there is a sequence $u_j \in N^{1,p}(X)$ as in Definition 6.5, related to the compact set $K$, and for each positive integer $n > 2$ let $\eta_n$ be a 1-Lipschitz function on $X$ such that $0 \leq \eta_n \leq 1$ on $X$, $\eta_n = 1$ on $\overline{B}_{n-1}$, and $\eta_n = 0$ outside $B_n$. Then $v_{n,j} := \eta_n u_j \in N^{1,p}(X)$ with $v_{n,j} = 1$ on the compact set $K$ and $v_{n,j} = 0$ outside $B_n$. It then follows from the definition of $p$-weak upper gradients that for $p$-almost every curve $\gamma \in \Gamma(K, X \setminus B_n)$ we have that $1 \leq \int_\gamma g_{v_{n,j}} \, ds$. Since

$$g_{v_{n,j}} \leq u_j \chi_{B_n \setminus B_{n-1}} + g_{u_j} \chi_{B_n},$$

we see that

$$\operatorname{Mod}_p(\Gamma(K, X \setminus B_n)) \leq 2^p \biggl( \int_{X \setminus B_{n-1}} |u_j|^p \, d\mu + \int_{B_n} g_{u_j}^p \, d\mu \biggr).$$

Letting $n \to \infty$ gives us that

$$0 < c \leq 2^p \int_X g_{u_j}^p \, d\mu,$$

which then forbids the sequence $u_j$ from satisfying (6.2), that is, $X$ cannot be $p$-parabolic in the sense of Definition 6.5.

Conversely, if $X$ is not $p$-parabolic in the sense of Definition 6.5, then there exist $c_0$ and a compact set $K_0 \subset X$ such that for every $u \in N^{1,p}(X)$ with $u \ge 1$ on $K_0$,

$$\int_X g_u^p \, d\mu \ge c_0 > 0.$$

In particular,

$$\lim_{n\to\infty} \operatorname{cap}_{D^p}(K_0, X \setminus B_n) \ge c_0,$$

which in combination with Lemmas 2.6 and 4.5 implies that

$$\lim_{n\to\infty} \operatorname{Mod}_p(\Gamma(\overline{B}_1, X \setminus B_n)) > 0,$$

that is, $X$ is $p$-hyperbolic in the sense of Definition 4.2. □

*Proof of Proposition* 6.4. Suppose that $u \in D^p(X)$ is a bounded nonconstant $p$-harmonic function $u$ on $X$ with finite energy. Without loss of generality we may assume that

$$\inf_X u = -1 \quad \text{and} \quad \sup_X u = 2.$$

Setting $\Omega = \{x : u(x) < 0\}$, choose a point $x_0 \in \Omega$. For $n = 1, 2, \dots$, let $F_n = \overline{\Omega} \setminus B(x_0, n)$. We shall show that the sequence $\{F_n\}_{n=1}^\infty$ is $p$-hyperbolic.

Assume not. Let $K \subset \Omega$ be an arbitrary compact set. Then by Lemmas 2.6 and 4.5, $\operatorname{cap}_{D^p}(K, F_n) = \operatorname{Mod}_p(\Gamma(K, F_n)) \to 0$ as $n \to \infty$. In particular, for sufficiently large $n$, there exist $u_n \in D^p(X)$ such that $u_n = 1$ in $K$, $u_n = 0$ in $F_n$, $0 \le u_n \le 1$ in $X$ and

$$\int_X g_{u_n}^p \, d\mu \to 0 \quad \text{as } n \to \infty.$$

Since $u_n|_\Omega$ has bounded support we see that $u_n \in N^{1,p}(\Omega)$. As $K$ was arbitrary, we conclude that $\Omega$ is $p$-parabolic in the sense of Definition 6.5. Since $u = 0$ on $\partial\Omega$, applying Corollary 7.7 in Hansevi [29] (see Remark 3.7) to the constant function $f \equiv 0$ then implies that $u \equiv 0$ in $\Omega$, which is a contradiction. Thus, the sequence $\{F_n\}_{n=1}^\infty$ is $p$-hyperbolic, and hence $X$ is $p$-hyperbolic.

Similarly, considering $\Omega' = \{x : u(x) > 1\}$ and $x_0' \in \Omega'$, we conclude that $G_n = \overline{\Omega}' \setminus B(x_0', n)$ also forms a $p$-hyperbolic sequence. Clearly, the two sequences are disjoint. Moreover, any upper gradient $g$ for $u$ is admissible for $\operatorname{Mod}_p(\Gamma(F_1, G_1))$ and hence $\operatorname{Mod}_p(\Gamma(F_1, G_1)) < \infty$. □

*Proof of Theorem* 1.3. One implication follows directly from Theorem 5.1, while the other (and the $p$-hyperbolicity) follows from Proposition 6.4. The last part (about $p$-hyperbolic ends) follows from Corollary 5.2. □

*Proof of Theorem* 1.2. The inclusions

$$\begin{array}{ccccc} O^p_{HP} & \subset & O^p_{HB} & \subset & O^p_{HBD} \\ \cup & & \cup & & \cup \\ O^p_{QP} & \subset & O^p_{QB} & \subset & O^p_{QBD} \end{array}$$

are trivial. That $O^p_{HD} = O^p_{HBD} = O^p_{QD} = O^p_{QBD}$ follows from Theorem 6.1, while the inclusion $O^p_{\mathrm{par}} \subset O^p_{HBD}$ follows from Theorem 1.3. We have thus shown all inclusions.

As (unweighted) $\mathbf{R}^n \in O^p_{QP} \subset O^p_{HBD}$ for all $1 < p < \infty$ and $n \ge 1$ (e.g. by Theorem 3.2), but is $p$-parabolic only for $p \ge n$, we see that $O^p_{\mathrm{par}} \subsetneq O^p_{HBD}$ and that $\mathbf{R}^n \in O^p_{QP} \setminus O^p_{\mathrm{par}}$ if $p < n$.

By Example 8.3 below, there is a measure $\mu$ on $\mathbf{R}$ (satisfying our standing assumptions) such that

$$(\mathbf{R}, \mu) \in O^p_{QB} \setminus O^p_{HP}.$$

It follows directly that

$$(\mathbf{R}, \mu) \in O^p_{HB} \setminus O^p_{HP}, \quad \text{and} \quad (\mathbf{R}, \mu) \in O^p_{QB} \setminus O^p_{QP}.$$

Finally, consider the Poincaré $n$-ball $B^n_\alpha$ as in Sario–Nakai–Wang–Chung [56, Section I.2.4], namely $B^n_\alpha = B(0,1) \subset \mathbf{R}^n$, $n \geq 3$, equipped with the Poincaré-type metric $ds_\alpha = (1-|x|^2)^\alpha \, dx$, $\alpha \leq -1$, and the corresponding Lebesgue measure. This makes $B^n_\alpha$ into an unbounded proper Riemannian manifold (and thus metric space) satisfying our standing assumptions. By [56, Lemma I.2.8 and I.2.9], $X \in O^2_{HD} \setminus O^2_{HB}$. □

## 7. $D^p(X) = N^{1,p}(X) + \mathbf{R}$

This section is devoted to Theorem 1.4, and we start with its proof. The rest of the section discusses the converse of Theorem 1.4.

*Proof of Theorem 1.4.* Since $X \notin O^p_{HD}$, there is a nonconstant $p$-harmonic function $u \in D^p(X)$. Suppose that there is some $c \in \mathbf{R}$ such that $u + c \in N^{1,p}(X)$. Then $u + c$ is also nonconstant and $p$-harmonic on $X$, but this is in contradiction with Lemma 3.3. □

The following example shows that $D^p(X) = N^{1,p}(X) + \mathbf{R}$ can fail even when $X \in O^p_{HD} = O^p_{QD}$.

**Example 7.1.** Let $X = \mathbf{R}^n$ (unweighted) with $p > n \geq 1$ and let

$$u(x) = \sum_{j=0}^{\infty} \bigl(1 - 2^{-j}|x - (4^j, 0, \dots, 0)|\bigr)_+.$$

Then both $\{x : u(x) = 0\}$ and $\{x : u(x) > \frac{1}{2}\}$ have infinite measure and thus $u \notin N^{1,p}(X) + \mathbf{R}$. However,

$$\int_{\mathbf{R}^n} g_u^p \, dx = \sum_{j=0}^{\infty} 2^{j(n-p)} \omega_n < \infty,$$

where $\omega_n$ is the volume of the unit ball in $\mathbf{R}^n$. Thus $u \in D^p(X)$.

Note that $X \in O^p_{QD}$ by Corollary 6.2, and even $X \in O^p_{QP}$ by Theorem 3.2.

In Proposition 7.3 below we show that the converse of Theorem 1.4 holds provided that $X$ supports the following global $(p,p)$-Sobolev inequality.

**Definition 7.2.** $X$ supports a *global $(p,p)$-Sobolev inequality* if there is a constant $C > 0$ such that

$$\int_X |u|^p \, d\mu \leq C \int_X g_u^p \, d\mu \tag{7.1}$$

whenever $u \in N^{1,p}(X)$.

One can equivalently require (7.1) to just hold for bounded $u \in N^{1,p}(X)$ with bounded support, see [34, Proposition 7.1.35]. If $X$ is a simply connected complete Riemannian manifold with sectional curvature $K \leq -a^2 < 0$, then it supports a global $(p,p)$-Poincaré inequality, see Holopainen–Lang–Vähäkangas [39, p. 129].

The global $(p,p)$-Sobolev inequality holds if and only if the *Rayleigh quotient*

$$R_p(X) := \inf_u \frac{\int_X g_u^p \, d\mu}{\int_X |u|^p \, d\mu} > 0,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ with $\|u\|_{N^{1,p}(X)} > 0$.

Classically, the Rayleigh quotient for $p = 2$ equals the first eigenvalue $\lambda_1$ (the bottom of the spectrum) of the 2-Laplacian. In the nonlinear case, the Rayleigh quotient is associated with a nonlinear eigenvalue problem that has even been studied on metric spaces (with upper gradients as here), see e.g. García Azorero–Peral Alonso [22] (on $\mathbf{R}^n$) and Latvala–Marola–Pere [51].

It was shown in Li–Wang [52, Theorem 1.4 (2)] that if $M$ is a complete 2-hyperbolic Riemannian manifold with $\lambda_1 > 0$, then the measure of balls centered at a point in $M$ grows at least exponentially with respect to the radius. A similar exponential volume growth was identified in Buckley–Koskela [17, Theorem 0.1 (2)] for proper $p$-hyperbolic metric measure spaces supporting a global $(p,p)$-Sobolev inequality.

**Proposition 7.3.** *If in addition to our standing assumptions on $X$, we know that $X \in O^p_{HD}$ and $X$ supports a global $(p,p)$-Sobolev inequality, then*

$$D^p(X) = N^{1,p}(X) + \mathbf{R}.$$

*Proof.* Let $f \in D^p(X)$, $x_0 \in X$ and set $B_j = B(x_0, j)$, $j = 1, 2, \dots$. As $X$ is unbounded, $C_p(X \setminus B_j) > 0$. For each positive integer $k$, let $v_k = H_{B_k} f$ be the $p$-harmonic extension of $f$ in $B_k$. By the global $(p,p)$-Sobolev inequality,

$$\begin{aligned}\int_X |f - v_k|^p \, d\mu &\le C \int_X g_{f-v_k}^p \, d\mu \\ &\le 2^p C \biggl( \int_X g_f^p \, d\mu + \int_X g_{v_k}^p \, d\mu \biggr) \le 2^{p+1} C \int_X g_f^p \, d\mu. \end{aligned} \tag{7.2}$$

Thus for each $j = 1, 2, \dots$,

$$\begin{aligned}\int_{B_j} |v_k|^p \, d\mu &\le 2^p \biggl( \int_{B_j} |f - v_k|^p \, d\mu + \int_{B_j} |f|^p \, d\mu \biggr) \\ &\le 2^{2p+1} C \biggl( \int_X g_f^p \, d\mu + \int_{B_j} |f|^p \, d\mu \biggr). \end{aligned}$$

As $N^{1,p}_{\rm loc}(X) = D^p_{\rm loc}(X)$ (see Section 2), and $X$ is proper, we see that $f \in N^{1,p}(B_j) \subset L^p(B_j)$, and so $\{v_k\}_{k=1}^\infty$ is a bounded sequence in $N^{1,p}(B_j)$.

Now an argument as in the proof of Lemma 3.4 shows that we have a sequence $\{\hat{v}_k\}_{k=1}^\infty$, of convex combinations of $\{v_k\}_{k=1}^\infty$, that converges in $N^{1,p}(B_j)$ for each $j$ to a function $v$ on $X$, with $g_v \in L^p(X)$. By Shanmugalingam [59, Theorem 1.1], the function $v$ is $p$-harmonic in each $B_j$ and thus in $X$. Since $X \in O^p_{HD}$, the function $v$ must be constant, say $v \equiv c$. As in (7.2),

$$\int_{B_j} |f - c|^p \, d\mu = \lim_{k \to \infty} \int_{B_j} |f - \hat{v}_k|^p \, d\mu \le \limsup_{k \to \infty} \int_X |f - \hat{v}_k|^p \, d\mu \le C' \int_X g_f^p \, d\mu.$$

After letting also $j \to \infty$, we see that $\int_X |f-c|^p \, d\mu$ is finite, that is, $f - c \in N^{1,p}(X)$. The converse inclusion is trivial. $\square$

## 8. Examples related to Liouville type classes

In this section we explore some examples that illustrate the (non)equality of the Liouville classes. As mentioned in the introduction, the Euclidean space $\mathbf{R}^n$ has only one end when $n \geq 2$, and that end is $p$-hyperbolic only when $1 \leq p < n$. On the other hand, $\mathbf{R}$ has two distinct ends.

We begin with an example showing that $\mathbf{R}^2$ can be equipped with a weight so that it has two well-separated $p$-hyperbolic sequences, even though it only has one end, cf. Theorem 1.3.

**Example 8.1.** Let $X = \mathbf{R}^2$ be equipped with the Euclidean distance and the measure $d\mu = w\,dx$, where

$$w(x) = e^{-\operatorname{dist}(x,A)} \quad \text{and} \quad A = \{x = (x_1, x_2) : |x_2| \leq |x_1|\}.$$

Also let $1 < p < 2$.

Observe that as $w$ is "uniformly almost constant" on every ball of radius 1, $\mu$ is uniformly locally doubling and supports a uniformly local 1-Poincaré inequality.

Even though $X$ only has one end, it is still possible to use Theorem 5.1 to show that $X$ supports a nonconstant bounded $p$-harmonic function with finite energy. Let $x_0 = (0, 0)$, $B = B(x_0, 1)$ and

$$\begin{aligned} F_n &= \{x = (x_1, x_2) \in A : x_1 \geq 2n\}, \\ G_n &= \{x = (x_1, x_2) \in A : x_1 \leq -2n\}, \quad n = 1, 2, \ldots. \end{aligned}$$

Now, by symmetry,

$$\begin{aligned} \operatorname{Mod}_p(\Gamma(\overline{B}, \mathbf{R}^2 \setminus (-2n, 2n)^2)) &\geq \operatorname{Mod}_p(\Gamma(\overline{B}, F_n)) \\ &\geq \tfrac{1}{4} \operatorname{Mod}_p^{\mathbf{R}^2}(\Gamma(\overline{B}, \mathbf{R}^2 \setminus (-2n, 2n)^2)), \end{aligned} \tag{8.1}$$

where $\operatorname{Mod}_p^{\mathbf{R}^2}$ denotes the standard $p$-modulus in unweighted $\mathbf{R}^2$ with respect to the Lebesgue measure. Since $1 < p < 2$, we know that unweighted $\mathbf{R}^2$ is $p$-hyperbolic and hence the right-hand side in (8.1) has a positive lower bound as $n \to \infty$. It follows that $X$ is $p$-hyperbolic and that $\{F_n\}_{n=1}^\infty$ is a $p$-hyperbolic sequence in $X$. Similarly $\{G_n\}_{n=1}^\infty$ is a $p$-hyperbolic sequence in $X$.

Next, every curve connecting $F_1$ to $G_1$ must pass through the strip

$$S := \{x \in \mathbf{R}^2 : |x_1| \leq \tfrac{1}{2}\},$$

whose characteristic function $\chi_S$ is thus admissible for the family $\Gamma(F_1, G_1)$ of such curves. A simple calculation then shows that

$$\operatorname{Mod}_p(\Gamma(F_1, G_1)) \leq \int_{\mathbf{R}^2} \chi_S \, d\mu < \infty.$$

Hence Theorem 5.1 is applicable and provides us with a bounded $p$-harmonic function in $X$ with finite energy.

Note that $\mathbf{R}^2$ equipped with the Lebesgue measure does not support any nonconstant bounded $p$-harmonic function. It therefore follows from Proposition 6.4 that $\operatorname{Mod}_p^{\mathbf{R}^2}(\Gamma(F_n, G_n)) = \infty$ for each positive integer $n$.

We have seen that spaces with at least two $p$-hyperbolic ends support nonconstant bounded $p$-harmonic functions with finite energy, while parabolic spaces do not. For spaces with only one end, which is $p$-hyperbolic, Example 8.1 and unweighted $\mathbf{R}^n$ with $1 < p < n$ show that they may or may not support nonconstant bounded $p$-harmonic functions with finite energy. A natural question is what happens in spaces with only one $p$-hyperbolic end and at least one $p$-parabolic end. The following examples and Proposition 8.4 show that both situations are possible.

**Example 8.2.** Consider

$$X = \{x = (x_1, x_2) \in \mathbf{R}^2 : x_2 \le 0\} \cup \big([-1, 1] \times (0, \infty)\big),$$

equipped with the Euclidean distance and the measure $e^{-\operatorname{dist}(x,A)}\,dx$, where

$$A = \{(x_1, x_2) : -|x_1| \le x_2 \le 0\} \cup \big([-1, 1] \times (0, \infty)\big).$$

Also let $1 < p < 2$. Similar to Example 8.1, we see that $X$ has one $p$-hyperbolic end and contains two disjoint $p$-hyperbolic sequences

$$F_n = \{x = (x_1, x_2) \in A : x_1 \ge 2n\} \quad \text{and} \quad G_n = \{x = (x_1, x_2) \in A : x_1 \le -2n\},$$

$n = 1, 2, \dots$, while the strip $[-1, 1] \times (0, \infty)$ forms a $p$-parabolic end. The uniformly local doubling property and a uniformly local 1-Poincaré inequality are also satisfied. Theorem 5.1 now implies that $X$ supports a nonconstant bounded $p$-harmonic function with finite energy.

In the following example we will see that when suitably equipped with a weight, $\mathbf{R}$ carries a nonconstant positive 2-harmonic function but no nonconstant bounded 2-harmonic function.

**Example 8.3.** Consider $\mathbf{R}$ equipped with the Euclidean distance and the weight

$$w(x) = \begin{cases} |x|^\alpha, & x \le -1, \\ 1, & x \ge -1, \end{cases}$$

for some fixed $\alpha > -1$. The measure $d\mu(x) = w(x)\,dx$ is uniformly locally doubling and supports a uniform local 1-Poincaré inequality. As pointed out above, $\mathbf{R}$ has two ends, denoted $\infty$ and $-\infty$. By Proposition 8.4 below, the end at $\infty$ is $p$-parabolic for each $p > 1$, while the end at $-\infty$ is $p$-hyperbolic if (and only if) $1 < p < 1 + \alpha$ (which then also requires $\alpha > 0$). Moreover, $(\mathbf{R}, \mu)$ is a $p$-hyperbolic space in this case, since it has a $p$-hyperbolic end. It thus follows, from Proposition 8.4 again, that

$$(\mathbf{R}, \mu) \in (O^p_{QB} \cap O^p_{QD}) \setminus (O^p_{HP} \cup O^p_{\mathrm{par}}) \quad \text{when } 1 < p < 1 + \alpha.$$

**Proposition 8.4.** *Consider the real line $\mathbf{R}$, equipped with the Euclidean distance and the measure $d\mu = w\,dx$, where $\mu$ is locally doubling and supports a local $p$-Poincaré inequality. Then the following are true.*

(a) *Each quasiharmonic function (with respect to $\mu$) on $\mathbf{R}$ is bounded if and only if it has finite energy.*

(b) *The end at $\infty$ is $p$-hyperbolic if and only if*

$$\int_0^\infty w^{1/(1-p)}\,dx < \infty. \tag{8.2}$$

(c) *The end at $-\infty$ is $p$-hyperbolic if and only if*

$$\int_{-\infty}^0 w^{1/(1-p)}\,dx < \infty. \tag{8.3}$$

(d) *The space $(\mathbf{R}, \mu) \in O^p_{\mathrm{par}}$ if and only if both (8.2) and (8.3) fail.*

(e) *If both (8.2) and (8.3) hold then there exists a nonconstant bounded global $p$-harmonic function with finite energy, i.e. $(\mathbf{R}, \mu) \notin O^p_{HBD}$.*

(f) *If (8.2) holds and (8.3) fails (or (8.2) fails and (8.3) holds), then*

$$(\mathbf{R}, \mu) \in (O^p_{QB} \cap O^p_{QD}) \setminus O^p_{HP}.$$

(g) *If both* (8.2) *and* (8.3) *fail then* $(\mathbf{R},\mu) \in O^p_{QP} \cap O^p_{QD}$.

Weights $\mu$ on $\mathbf{R}$ as above were characterized in Björn–Björn–Shanmugalingam [12, Theorem 1.2]. In particular it was shown that for each bounded interval $I$ there is a (global) $A_p$ weight $\widetilde{w}$ on $\mathbf{R}$ such that $\widetilde{w} = w$ on $I$.

*Proof.* (a) This follows from [13, Proposition 6.5].

(b) To see that the end at $\infty$ is $p$-hyperbolic when (8.2) holds, consider the family $\Gamma_R = \Gamma([-1,0],[R,\infty))$, $R>0$, and let $\rho$ be admissible for $\operatorname{Mod}_p(\Gamma_R)$. Since

$$1 \le \int_0^R \rho\,dx \le \biggl(\int_0^R \rho^p\,d\mu\biggr)^{1/p} \biggl(\int_0^R w^{1/(1-p)}\,dx\biggr)^{1/p} \le C\|\rho\|_{L^p(\mathbf{R},\mu)},$$

with $C$ independent of $R$, letting $R\to\infty$ shows that the end at $\infty$ is $p$-hyperbolic. On the other hand, if (8.2) fails then the function

$$\rho_R(t) := \frac{w^{1/(1-p)}(t)\chi_{[0,R]}(t)}{\int_0^R w^{1/(1-p)}\,dx}$$

is admissible for $\operatorname{Mod}_p(\Gamma_R)$ (as we may assume that $w$ is a Borel function), with

$$\int_{\mathbf{R}} \rho^p d\mu = \biggl(\int_0^R w^{1/(1-p)}\,dx\biggr)^{1-p} \to 0, \quad \text{as } R\to\infty,$$

and so the end at $\infty$ is $p$-parabolic. Thus (b) has been shown, (c) is shown similarly, and (d) follows immediately.

The remaining statements follow from [13, Theorems 1.2 and 1.3]. □

As the next example shows, when a metric space has infinitely many ends, the $p$-hyperbolicity of the space does not imply the existence of a $p$-hyperbolic end.

**Example 8.5.** The weighted infinite binary tree $X$ from [13, Example 7.2] is an example of a space that does not belong to $O^p_{HBD}$ for any $1<p<\infty$. By Theorem 1.3, $X$ is $p$-hyperbolic. It is equipped with the geodesic metric, giving each edge unit length. Each geodesic ray, emanating from the root $v_0$, defines an end at infinity and corresponds to exactly one point in the so-called visual boundary of $X$.

Fixing one such geodesic ray $\gamma$ from the root, the measure on $X$ is comparable to $2^{-\pi_\gamma(x)}\,dm(x)$, where $m$ is the 1-dimensional Lebesgue measure on each edge and $\pi_\gamma(x)$ is the closest point on $\gamma$ to $x$. Since the weight $w(x)=2^{-\pi_\gamma(x)}$ is nonincreasing along each geodesic ray, an argument as in Proposition 8.4 (b) together with Remark 4.3 tells us that the corresponding end must be $p$-parabolic. Thus, $X$ is a $p$-hyperbolic space having only $p$-parabolic ends. It is also uniformly locally doubling and supports a uniformly local 1-Poincaré inequality.

## 9. The finite-energy Liouville theorem in noncomplete spaces

Recall that the standing assumptions from Section 3 are not required in this section.

**Theorem 9.1.** *Assume that $X$ is a* (*not necessarily complete*) *metric space equipped with a globally doubling measure $\mu$ supporting a global $p$-Poincaré inequality, where $1<p<\infty$.*

*If $u \in D^p(X)$ is a quasiharmonic function on $X$ with finite energy, then it is constant.*

This shows that Theorem 1.1 in Björn–Björn–Shanmugalingam [13] holds even if none of the sufficient conditions (a)–(d) therein is satisfied.

As in Definition 3.1, a function $u \in N^{1,p}_{\mathrm{loc}}(X)$ is a *quasiminimizer* on the entire space $X$ if

$$\int_{\varphi \neq 0} g_u^p \, d\mu \le Q_u \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu \quad \text{for all } \varphi \in N^{1,p}(X),$$

and a *quasiharmonic function* is a continuous quasiminimizer. However, the definition of quasiminimizers on *strict subsets* of noncomplete spaces is more involved, see [13, Section 3] for such a definition and further discussion.

*Proof.* If $X$ is bounded, then this follows directly from [13, Theorem 1.1]. So we may assume that $X$ is unbounded. As in [7] we let $\widehat{X}$ be the completion of $X$. The metric $d$ extends directly to $\widehat{X}$ and we define the complete Borel regular measure $\hat{\mu}$ on $\widehat{X}$ by letting

$$\hat{\mu}(E) = \mu(E \cap X) \quad \text{for every Borel set } E \subset \widehat{X},$$

and then complete it, see [7, Corrigendum]. It follows from [7, Propositions 3.3 and 3.6] that $\hat{\mu}$ is globally doubling and supports a global $p$-Poincaré inequality on $\widehat{X}$, with the same doubling and Poincaré constants.

By [7, Theorem 4.1], there is $\hat{u} \in D^p(\widehat{X})$ such that $\hat{u} = u$ $C_p^X$-q.e. in $X$ and

$$g_{\hat{u},\widehat{X}} \le A_0 g_{u,X} \quad \text{a.e. in } X, \tag{9.1}$$

where $A_0$ only depends on $p$, the global doubling constant and both constants in the global $p$-Poincaré inequality. Let $\widehat{\varphi} \in N^{1,p}(\widehat{X})$ and $\varphi = \widehat{\varphi}|_X$. Then, $g_{u+\varphi,X} \le g_{\hat{u}+\widehat{\varphi},\widehat{X}}$ a.e. in $X$, and thus using (9.1),

$$\int_{\widehat{\varphi} \neq 0} g_{\hat{u},\widehat{X}}^p \, d\hat{\mu} \le A_0^p \int_{\varphi \neq 0} g_{u,X}^p \, d\mu \le A_0^p Q_u \int_{\varphi \neq 0} g_{u+\varphi,X}^p \, d\mu \le A_0^p Q_u \int_{\widehat{\varphi} \neq 0} g_{\hat{u}+\widehat{\varphi},X}^p \, d\mu.$$

Therefore $\hat{u}$ is a quasiminimizer on $\widehat{X}$, and hence has a continuous representative (see Section 3) that we can also call $\hat{u}$. By (9.1), we see that $\hat{u}$ has finite energy in $\widehat{X}$, and thus by Corollary 6.2, $\hat{u}$ is constant. As $u$ is continuous, it must also be constant. □

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