Autonomous Integral Functionals with Discontinuous Nonconvex Integrands: Lipschitz Regularity of Minimizers, DuBois-Reymond Necessary Conditions, and Hamilton-Jacobi Equations

Gianni Dal Maso* and Hélène Frankowska†

Abstract

This paper is devoted to the autonomous Lagrange problem of the calculus of variations with a discontinuous Lagrangian. We prove that every minimizer is Lipschitz continuous if the Lagrangian is coercive and locally bounded. The main difference with respect to the previous works in the literature is that we do not assume that the Lagrangian is convex in the velocity. We also show that, under some additional assumptions, the DuBois-Reymond necessary condition still holds in the discontinuous case. Finally, we apply these results to deduce that the value function of the Bolza problem is locally Lipschitz and satisfies (in a generalized sense) a Hamilton-Jacobi equation.

Key words. Discontinuous Lagrangians, nonconvex integrands, Lipschitz minimizers, DuBois-Reymond necessary conditions, Hamilton-Jacobi equations.

AMS Mathematics Subject Classification 2000: 49N60 (primary), 49K05, 49L25 (secondary).

*SISSA, via Beirut 2-4, 34014 Trieste, Italy
e-mail: dalmaso@sissa.it
†CNRS, CREA, Ecole Polytechnique, 1, Rue Descartes, 75005 Paris, France
e-mail: franko@poly.polytechnique.fr

1 Introduction

In this paper we study the Lipschitz continuity of the solutions to the Lagrange problem of the calculus of variations

$$\min\left\{\int_a^b L(y(t),y'(t))dt \mid y\in W^{1,1}(a,b;\mathbb{R}^n),\;\; y(a)=x_a,\;\; y(b)=x_b\right\}, \tag{1.1}$$

where the Lagrangian $L:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$ is a Borel function having a superlinear growth with respect to the second variable, i.e., there exists a function Θ: $\mathbb{R}^n\to\mathbb{R}_+$, with

$$\lim_{|u|\to\infty}\frac{\Theta(u)}{|u|}=+\infty, \tag{1.2}$$

such that

$$\forall\,(x,u)\in\mathbb{R}^n\times\mathbb{R}^n,\;\; L(x,u)\geq\Theta(u). \tag{1.3}$$

We assume also that L is bounded in a neighborhood of each point of $\mathbb{R}^n\times\{0\}$, i.e.,

$$\forall\,x_0\in\mathbb{R}^n,\ \exists\,M>0,\ \exists\,r>0,\ \forall\,(x,u)\in B(x_0,r)\times B(0,r),\ L(x,u)\leq M, \tag{1.4}$$

where $B(x_0,r)$ is the closed ball with center x_0 and radius r.

The existence of a solution to (1.1) is an easy consequence of the direct method of the calculus of variations when the functional

$$\mathcal{L}_a^b(y):=\int_a^b L(y(t),y'(t))dt \tag{1.5}$$

is sequentially weakly lower semicontinuous on $W^{1,1}(a,b;\mathbb{R}^n)$.

By the classical results of Olech [15] and Ioffe [13], a standard assumption for the semicontinuity of $\mathcal{L}_a^b$ is that L is lower semicontinuous on $\mathbb{R}^n\times\mathbb{R}^n$ and $L(x,\cdot)$ is convex on $\mathbb{R}^n$ for every $x\in\mathbb{R}^n$, but these conditions are not necessary for the lower semicontinuity of $\mathcal{L}_a^b$ (see, e.g., [12]).

Recently Amar, Bellettini and Venturini have proved in [1] that any integral functional of the form (1.5), satisfying suitable growth conditions, has a lower semicontinuous envelope $\overline{\mathcal{L}}_a^b$ that can be written as

$$\overline{\mathcal{L}}_a^b(y)=\int_a^b L^+(y(t),y'(t))dt\,, \tag{1.6}$$

where L^+ is an integrand depending on L (see (4.4)). If L is continuous, then L^+ coincides with the convexification L_0 of L with respect to u, but, if L is discontinuous, one can prove only that the function $L^+(x,\cdot)$ is convex for a.e. $x\in\mathbb{R}^n$, and there are examples where $L^+(\cdot,u)$ is not lower semicontinuous on $\mathbb{R}^n$.

This shows that there are problems of the form (1.1) which admit a solution even if L is not convex in u (nor lower semicontinuous in x), and provides a motivation for the study of the Lipschitz continuity of the solutions of (1.1) without convexity hypotheses.

If $L(x,\cdot)$ is convex for every $x \in \mathbb{R}^n$, it was proved by Ambrosio, Ascenzi, and Buttazzo in [2] that every minimizer of (1.1) is Lipschitz continuous. This kind of results goes back to Tonelli [17, 18] for smooth Lagrangians, and is the first step to prove, under some additional conditions on L, that all minimizers are smooth (see, e.g., [8, Section 2.6]). Note that, in general, when the Lagrangian is time dependent, the problem may have no Lipschitz minimizer (see [5] and [10]).

The aim of Section 2 of the present paper is to show that the convexity hypothesis can be removed from [2]. Assuming only (1.2), (1.3), and (1.4), we prove that all minimizers of (1.1) are still Lipschitz continuous (Theorem 2.1), and provide an estimate of the Lipschitz constant if, in addition, L is locally bounded (Theorem 2.2).

If L is continuous, then every minimizer y of (1.1) is also a minimizer of the same problem with L replaced by its convexification L_0 with respect to u, so that the Lipschitz continuity of y follows from [2]. But, if L is discontinuous, we can only say (under suitable growth conditions) that y is a minimizer of (1.1) with L replaced by L^+, and we know that $L^+(x,\cdot)$ is convex only for a.e. $x \in \mathbb{R}^n$. For this reason we can not apply the results of [2]. On the other hand, the proof of [2] is based on an extension of the DuBois-Reymond necessary condition, which is not always valid when $L(x,\cdot)$ is not convex. Therefore we need different arguments.

As in [2], we begin by proving (Lemma 2.3) that if y is a minimizer of (1.1), then the function $\psi(t) := t$ is a minimizer of the problem

$$\min\left\{\int_a^b f(t,\psi'(t))dt \mid \psi \in W^{1,1}(a,b),\ \ \psi(a)=a,\ \ \psi(b)=b\right\},$$

where

$$f(t,v) := \begin{cases} L(y(t), y'(t)/v)v & \text{if } \ v > \frac{1}{2}, \\ +\infty & \text{if } \ v \le \frac{1}{2}. \end{cases} \tag{1.7}$$

Then we show (Lemma 2.4) that $\psi(t) := t$ is a minimizer of the problem

$$\min\left\{\int_a^b f_0(t,\psi'(t))dt \mid \psi \in W^{1,1}(a,b),\ \ \psi(a)=a,\ \ \psi(b)=b\right\},$$

where $f_0 = \overline{\mathrm{co}}_v f$ is the lower semicontinuous convex envelope of f with respect to v. This implies (Lemma 2.6) that there exists a constant $c \in \mathbb{R}$ such that

$$d_v^l f_0(t,1) \le c \le d_v^r f_0(t,1) \quad \text{for a.e. } \ t \in [a,b],$$

where d_v^l and d_v^r denote the left and right derivatives with respect to v.

These inequalities, together with (1.2), (1.3), and (1.4), are used to obtain a bound on the Lipschitz constant of y (Theorem 2.1), which is locally uniform (with respect to the data of the problem) if L is locally bounded (Theorem 2.2).

In Section 3 we obtain some extensions of the DuBois-Reymond necessary condition. When $L(x, \cdot)$ is not convex this condition is not always satisfied, and we propose some additional assumptions on L, which hold true, for instance, when $L(x, \cdot)$ is semiconvex or differentiable. Under these assumptions we show (Theorems 3.2 and 3.6) that, if y is a minimizer, then there exists a constant $c \in \mathbb{R}$ such that

$$c \in L(y(t), y'(t)) - \langle \partial_u^- L(y(t), y'(t)), y'(t) \rangle \quad \text{for a.e. } t \in [a, b],$$

where $\partial_u^- L(y(t), y'(t))$ is the subdifferential of $L(y(t), \cdot)$ at $y'(t)$. More general results of this kind (Lemma 3.1 and Theorem 3.10) are obtained with different generalized gradients of L.

Finally, in Section 4 we apply the Lipschitz regularity of minimizers to study the value function of the Bolza problem:

$$\text{(1.8)} \quad V(t, x) := \inf \left\{ \int_0^t L(y(s), y'(s)) ds + \varphi(y(t)) \mid y \in W^{1,1}(0, t; \mathbb{R}^n),\ y(0) = x \right\},$$

where $\varphi: \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$, $\varphi \not\equiv +\infty$, and L is locally bounded, not necessarily convex with respect to the second variable.

Let $\mathbb{R}_+^\star := \{t \in \mathbb{R} \mid t > 0\}$ and let H be the Hamiltonian associated with L, defined by

$$\text{(1.9)} \qquad H(x, p) := \sup_{u \in \mathbb{R}^n} (\langle p, u \rangle - L(x, u)),$$

i.e., $H(x, \cdot)$ is the Legendre-Fenchel transform of $L(x, \cdot)$.

Assuming that for all $(t, x) \in \mathbb{R}_+^\star \times \mathbb{R}^n$ the infimum in (1.8) is attained, we prove that V is locally Lipschitz on $\mathbb{R}_+^\star \times \mathbb{R}^n$ (Theorem 4.4) and solves the Hamilton-Jacobi equation

$$\text{(1.10)} \qquad V_t + H(x, -V_x) = 0$$

in a generalized sense (Theorem 4.8). When φ is lower semicontinuous, we also provide a comparison result for lower semicontinuous subsolutions of (1.10), which characterizes the value function as the maximal lower semicontinuous subsolution of (1.10) (Theorem 4.9).

We conclude the paper with two results (Theorems 4.12 and 4.13) which show the relationships between minimizers of (1.8) and contingent derivatives of the value function.

2 Lipschitz Regularity of Minimizers

Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function, let $[a,b]$ be a bounded closed interval in $\mathbb{R}$, and let $y \in W^{1,1}(a,b;\mathbb{R}^n)$ be a function such that

$$(2.1) \qquad \int_a^b L(y(t), y'(t))dt \le \inf_{z \in S(y)} \int_a^b L(z(t), z'(t))dt < +\infty,$$

where $S(y) := \{z \in W^{1,1}(a,b;\mathbb{R}^n) \mid z(a) = y(a), \ \ z(b) = y(b)\}$.

The main results of this section are the following two theorems.

Theorem 2.1 *Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function which satisfies (1.2), (1.3), (1.4), and let $y \in W^{1,1}(a,b;\mathbb{R}^n)$ be a function which satisfies (2.1). Then y is Lipschitz continuous.*

When L is locally bounded on $\mathbb{R}^n \times \mathbb{R}^n$, we obtain a uniform estimate of the Lipschitz constant of every minimizer.

Theorem 2.2 *Let $\Theta: \mathbb{R}^n \to \mathbb{R}_+$ be a function satisfying (1.2), let $\Psi: \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function, and let A, B, α, $\beta > 0$. Then there exists a constant $K = K(\Theta, \Psi, A, B, \alpha, \beta)$ with the following property: if $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is any Borel function satisfying (1.3) and*

$$(2.2) \qquad \forall\, R > 0, \qquad \sup_{|x| \le R,\, |u| \le R} L(x,u) \le \Psi(R),$$

and $y \in W^{1,1}(a,b;\mathbb{R}^n)$ satisfies (2.1) and

$$(2.3) \qquad \inf_{a \le t \le b} |y(t)| \le A,$$

$$(2.4) \qquad \int_a^b L(y(t), y'(t))dt \le B,$$

$$(2.5) \qquad \alpha \le b - a \le \beta,$$

then y is Lipschitz continuous with Lipschitz constant bounded by K.

To prove Theorems 2.1 and 2.2 we need some technical lemmas.

Let us fix a function $y \in W^{1,1}(a,b;\mathbb{R}^n)$ which satisfies (2.1). As in [2] we use the auxiliary function $f: [a,b] \times \mathbb{R} \to [0, +\infty]$ defined by (1.7), which turns out to be $\mathcal{L}_1 \times \mathcal{B}_1$-measurable, where $\mathcal{L}_1$ and $\mathcal{B}_1$ denote the σ-algebras of Lebesgue measurable subsets of $[a,b]$ and of Borel subsets of $\mathbb{R}$, respectively. From (1.7) and (2.1) it follows that

$$(2.6) \qquad \int_a^b f(t,1)dt < +\infty.$$

The following lemma is well known (see, e.g., [8, p. 46]). We write the proof only to give a self-contained presentation of the arguments used to obtain Theorems 2.1 and 2.2.

Lemma 2.3 *We have*

$$\int_a^b f(t,1)dt \leq \int_a^b f(t,\psi'(t))dt \tag{2.7}$$

for every $\psi \in W^{1,1}(a,b)$ *such that* $\psi(a) = a$ *and* $\psi(b) = b$.

Proof — Let us fix $\psi \in W^{1,1}(a,b)$, with $\psi(a) = a$ and $\psi(b) = b$, such that the right hand side of (2.7) is finite. Then $\psi'(t) > \frac{1}{2}$ for almost all $t \in [a,b]$. Thus ψ is increasing and $|\psi(t) - \psi(s)| \geq \frac{1}{2}|t-s|$ for all $s,t \in [a,b]$. Therefore the inverse function $\psi^{-1}: [a,b] \to [a,b]$ is Lipschitz continuous with Lipschitz constant 2. These properties imply that

$$N \in \mathcal{L}_1, \quad |N| = 0 \Longleftrightarrow \psi(N) \in \mathcal{L}_1, \quad |\psi(N)| = 0, \tag{2.8}$$

where $|\cdot|$ denotes the Lebesgue measure, and that $z := y \circ \psi^{-1}$ belongs to $W^{1,1}(a,b;\mathbb{R}^n)$ and satisfies $z(a) = y(a)$ and $z(b) = y(b)$. Using (2.8) and the chain rule one proves easily that $z'(t) = y'(\psi^{-1}(t))/\psi'(\psi^{-1}(t))$ for a.e. $t \in [a,b]$. Thus, after the change of variables $s = \psi^{-1}(t)$, one gets

$$\int_a^b L(z(t),z'(t))dt = \int_a^b L(y(s),y'(s)/\psi'(s))\psi'(s)ds,$$

which, together with (2.1) and (1.7), yields (2.7). □

If $g: \mathbb{R} \to [0,+\infty]$ is an arbitrary function, its lower semicontinuous convex envelope $\overline{\mathrm{co}}\, g: \mathbb{R} \to [0,+\infty]$ is, by definition, the greatest lower semicontinuous convex function which is less than or equal to g. It is well known that the epigraph of $\overline{\mathrm{co}}\, g$ is the closed convex hull of the epigraph of g, and that, if $\overline{\mathrm{co}}\, g$ is finite in a neighborhood of some point $v \in \mathbb{R}$, then

$$\overline{\mathrm{co}}\, g(v) = \inf\{\lambda g(v_1) + (1-\lambda)g(v_2) \mid (\lambda, v_1, v_2) \in A(v)\}, \tag{2.9}$$

where $A(v)$ is the set of all triples $(\lambda, v_1, v_2) \in \mathbb{R}^3$ with $0 < \lambda < 1$, $g(v_1) < +\infty$, $g(v_2) < +\infty$, and $v = \lambda v_1 + (1-\lambda)v_2$.

Let us return to the function f defined in (1.7), and let $f_0 = \overline{\mathrm{co}}_v f$ be its lower semicontinuous convex envelope with respect to v. We observe that for every $t \in [a,b]$, the function $v \mapsto f_0(t,v)$ is continuous on $(\frac{1}{2},+\infty)$, since it is convex and finite on this set.

Let us prove that for every $v \in (\frac{1}{2},+\infty)$ the function $t \mapsto f_0(t,v)$ is Lebesgue measurable. Given $\gamma \in \mathbb{R}$, by (2.9) the set $\{t \in [a,b] \mid f_0(t,v) < \gamma\}$ is the projection onto $[a,b]$ of the set of all points $(t,\lambda,v_1,v_2) \in [a,b] \times (0,1) \times (\frac{1}{2},+\infty) \times (\frac{1}{2},+\infty)$ such that $v = \lambda v_1 + (1-\lambda)v_2$ and $\lambda f(t,v_1) + (1-\lambda)f(t,v_2) < \gamma$. As this set belongs to the σ-algebra

$\mathcal{L}_1 \times \mathcal{B}_1 \times \mathcal{B}_1 \times \mathcal{B}_1$, from the projection theorem (see, e.g., [4, Theorem 8.3.2]) we conclude that $\{t \in [a,b] \mid f_0(t,v) < \gamma\}$ is Lebesgue measurable. This proves that $t \to f_0(t,v)$ is Lebesgue measurable, and hence f_0 is a Carathéodory function on $[a,b] \times (\frac{1}{2}, +\infty)$.

The following lemma is usually proved when f is continuous in v, or satisfies some growth condition. We give here a detailed proof to show that we do not need any additional hypothesis.

Lemma 2.4 *We have*

$$\int_a^b f_0(t,1)dt \leq \int_a^b f(t,1)dt \leq \int_a^b f_0(t,\varphi(t))dt \tag{2.10}$$

for every $\varphi \in L^1(a,b)$ *with*

$$\int_a^b \varphi(t)dt = b - a. \tag{2.11}$$

In particular,

$$f(t,1) = f_0(t,1) \ \textit{for a.e. } t \in [a,b]. \tag{2.12}$$

Proof — The first inequality in (2.10) follows from the fact that $f_0(t,1) \leq f(t,1)$ for every $t \in [a,b]$. To prove the second inequality we argue by contradiction. Assume that there exists $\varphi \in L^1(a,b)$, satisfying (2.11), such that

$$\int_a^b f_0(t,\varphi(t))dt < \int_a^b f(t,1)dt. \tag{2.13}$$

As $f_0(t,v) = +\infty$ for $v < \frac{1}{2}$, from (2.6) and (2.13) we obtain that $\varphi(t) \geq \frac{1}{2}$ for a.e. $t \in [a,b]$. Changing, if needed, φ on a set of measure zero, we may assume that this inequality is satisfied for every $t \in [a,b]$. If we replace $\varphi(t)$ by $\frac{1}{2}\varphi(t) + \frac{1}{2}$, we obtain a new function, still denoted by $\varphi(t)$, which continues to fulfill (2.11) and (2.13) (by convexity), and, in addition, satisfies the improved inequality $\varphi(t) \geq \frac{3}{4}$ for every $t \in [a,b]$. As f_0 is a Carathéodory function on $[a,b] \times (\frac{1}{2}, +\infty)$, the function $t \mapsto f_0(t,\varphi(t))$ is measurable.

Let us fix $\varepsilon > 0$ such that

$$\int_a^b [f_0(t,\varphi(t)) + \varepsilon]dt \ < \ \int_a^b f(t,1)dt. \tag{2.14}$$

For every $t \in [a,b]$ we define $A_\varepsilon(t)$ as the set of all triples $(\lambda, v_1, v_2) \in \mathbb{R}^3$ such that $0 < \lambda < 1$, $v_1 > \frac{1}{2}$, $v_2 > \frac{1}{2}$, $\lambda v_1 + (1-\lambda)v_2 = \varphi(t)$, and

$$\lambda f(t,v_1) + (1-\lambda) f(t,v_2) < f_0(t,\varphi(t)) + \varepsilon.$$

By (2.9) this set is nonempty for every $t \in [a,b]$.

From the measurability properties of f and $t \mapsto f_0(t, \varphi(t))$ we deduce that the graph of the set-valued map $t \rightsquigarrow A_\varepsilon(t)$ belongs to $\mathcal{L}_1 \times \mathcal{B}_1 \times \mathcal{B}_1 \times \mathcal{B}_1$. By the Aumann–von Neumann selection theorem (see, e.g., [7, Theorem III.22]) there exist Lebesgue measurable functions $\mu, \varphi_1, \varphi_2$ on $[a, b]$ such that $(\mu(t), \varphi_1(t), \varphi_2(t)) \in A_\varepsilon(t)$ for every $t \in [a, b]$. From the definition of $A_\varepsilon(t)$ and from (2.14) we deduce that

$$\forall\, t \in [a,b], \quad 0 < \mu(t) < 1, \quad \varphi_1(t) > \frac{1}{2}, \quad \varphi_2(t) > \frac{1}{2}, \tag{2.15}$$

$$\forall\, t \in [a,b], \quad \mu(t)\varphi_1(t) + (1-\mu(t))\varphi_2(t) = \varphi(t), \tag{2.16}$$

$$\int_a^b [\mu(t) f(t, \varphi_1(t)) + (1-\mu(t)) f(t, \varphi_2(t))]dt < \int_a^b f(t,1)dt. \tag{2.17}$$

For every $k \geq 2$ let

$$E^{(k)} := \Big\{ t \in [a,b] \mid \mu(t) \in (\frac{1}{k+1}, \frac{1}{k}] \cup [\frac{k-1}{k}, \frac{k}{k+1}) \Big\}.$$

As $0 < \mu(t) < 1$, the interval $[a, b]$ is the union of the sets $E^{(k)}$, which are pairwise disjoint. As $\varphi \in L^1(a, b)$ and $f(\cdot, 1) \in L^1(a, b)$ by (2.6), from (2.16) and (2.17) we obtain that for every $k \geq 2$ the functions $\varphi_i(t)$ and $f(t, \varphi_i(t))$ belong to $L^1(E^{(k)})$.

By the Lyapunov theorem (see, e.g., [4, Theorem 8.6.3 and Proposition 8.6.2]) there exist two disjoint measurable sets $E_1^{(k)}$ and $E_2^{(k)}$, with $E_1^{(k)} \cup E_2^{(k)} = E^{(k)}$, such that

$$\sum_{i=1}^{2} \int_{E_i^{(k)}} \varphi_i(t)dt = \int_{E^{(k)}} [\mu(t)\varphi_1(t) + (1-\mu(t))\varphi_2(t)]dt, \tag{2.18}$$

$$\sum_{i=1}^{2} \int_{E_i^{(k)}} f(t, \varphi_i(t))dt = \int_{E^{(k)}} [\mu(t) f(t, \varphi_1(t)) + (1-\mu(t)) f(t, \varphi_2(t))]dt. \tag{2.19}$$

Let $E_1 := \bigcup_{k=2}^{\infty} E_1^{(k)}$ and $E_2 := \bigcup_{k=2}^{\infty} E_2^{(k)}$. By (2.11), (2.16), (2.17), (2.18), (2.19) we obtain

$$\int_{E_1} \varphi_1(t)dt + \int_{E_2} \varphi_2(t)dt = \int_a^b \varphi(t)dt = b - a, \tag{2.20}$$

$$\int_{E_1} f(t, \varphi_1(t))dt + \int_{E_2} f(t, \varphi_2(t))dt < \int_a^b f(t,1)dt. \tag{2.21}$$

Let $\varphi_3 \in L^1(a, b)$ be the function defined by $\varphi_3 := \varphi_1$ on E_1 and $\varphi_3 := \varphi_2$ on E_2, and let ψ be the primitive of φ_3 with $\psi(a) = a$. By (2.20) we have also $\psi(b) = b$, while (2.21) gives

$$\int_a^b f(t, \psi'(t))dt < \int_a^b f(t,1)dt,$$

which contradicts (2.7) and concludes the proof of (2.10).

As $f_0 \le f$ and $f(\cdot, 1) \in L^1(a, b)$ by (2.6), if we take $\varphi \equiv 1$ in (2.10) we get (2.12). $\square$

Remark 2.5 As $v \mapsto f_0(t, v)$ is convex and finite, for every $t \in [a, b]$ there exist the limits

$$d_v^l f_0(t,1) := \lim_{v \to 1-} \frac{f_0(t,v) - f_0(t,1)}{v-1} = \sup_{v<1} \frac{f_0(t,v) - f_0(t,1)}{v-1}, \tag{2.22}$$

$$d_v^r f_0(t,1) := \lim_{v \to 1+} \frac{f_0(t,v) - f_0(t,1)}{v-1} = \inf_{v>1} \frac{f_0(t,v) - f_0(t,1)}{v-1}, \tag{2.23}$$

and we have $-\infty < d_v^l f_0(t, 1) \le d_v^r f_0(t, 1) < +\infty$.

For the sake of completeness, we give now a new elementary proof of a particular case of Theorem 3.1 of [2].

Lemma 2.6 *There exists a constant $c \in \mathbb{R}$ such that*

$$d_v^l f_0(t,1) \le c \le d_v^r f_0(t,1) \quad \textit{for a.e. } t \in [a,b]. \tag{2.24}$$

Proof — We argue by contradiction. If (2.24) does not hold, then there exists $\alpha \in \mathbb{R}$ such that

$$\operatorname*{ess\,sup}_{t \in [a,b]} d_v^l f_0(t,1) > \alpha > \operatorname*{ess\,inf}_{t \in [a,b]} d_v^r f_0(t,1).$$

Then the sets

$$A_l := \Big\{ t \in [a,b] \mid d_v^l f_0(t,1) > \alpha \Big\} \quad \& \quad A_r := \{ t \in [a,b] \mid d_v^r f_0(t,1) < \alpha \}$$

are disjoint and have positive measure. By (2.22) for every $t \in A_l$ the set

$$E_l(t) := \Big\{ v \in (\tfrac{1}{2}, 1) \mid \frac{f_0(t,v) - f_0(t,1)}{v-1} > \alpha \Big\}$$

is nonempty. Since the graph of the set-valued map $t \rightsquigarrow E_l(t)$ belongs to $\mathcal{L}_1 \times \mathcal{B}_1$, by the Aumann–von Neumann selection theorem (see, e.g., [7, Theorem III.22]) there exists a measurable function $\delta_l \colon A_l \to (0, \frac{1}{2})$ such that

$$\forall\, t \in A_l, \quad f_0(t, 1 - \delta_l(t)) - f_0(t,1) < -\alpha \delta_l(t). \tag{2.25}$$

Similarly, using (2.23) we can prove that there exists a measurable function $\delta_r \colon A_r \to (0, \frac{1}{2})$ such that

$$\forall\, t \in A_r, \quad f_0(t, 1 + \delta_r(t)) - f_0(t,1) < \alpha \delta_r(t). \tag{2.26}$$

Let us define

$$c_l := \Big[\int_{A_l} \delta_l(t) dt \Big]^{-1} \quad \& \quad c_r := \Big[\int_{A_r} \delta_r(t) dt \Big]^{-1}, \tag{2.27}$$

and let $\varphi(t) := -c_l\delta_l(t)$ for $t \in A_l$, $\varphi(t) = c_r\delta_r(t)$ for $t \in A_r$, and $\varphi(t) = 0$ otherwise. Then by (2.27) we have $\int_a^b \varphi(t)dt = 0$, and, by (2.10), for every $\varepsilon > 0$ this implies

$$\int_a^b [f_0(t, 1 + \varepsilon\varphi(t)) - f_0(t, 1)]\, dt \geq 0,$$

which is equivalent to

$$\int_{A_l} [f_0(t, 1 - \varepsilon c_l\delta_l(t)) - f_0(t, 1)]\, dt + \int_{A_r} [f_0(t, 1 + \varepsilon c_r\delta_r(t)) - f_0(t, 1)]\, dt \geq 0. \tag{2.28}$$

By the monotonicity property of the difference quotient of a convex function, using (2.25) we obtain for $\varepsilon c_l < 1$

$$f_0(t, 1 - \varepsilon c_l\delta_l(t)) - f_0(t, 1) \leq \varepsilon c_l\, [f_0(t, 1 - \delta_l(t)) - f_0(t, 1)] < -\alpha\varepsilon c_l\delta_l(t) \tag{2.29}$$

for every $t \in A_l$. Similarly, for $\varepsilon c_r < 1$ we obtain, using (2.26),

$$f_0(t, 1 + \varepsilon c_r\delta_r(t)) - f_0(t, 1) \leq \varepsilon c_r\, [f_0(t, 1 + \delta_r(t)) - f_0(t, 1)] < \alpha\varepsilon c_r\delta_r(t) \tag{2.30}$$

for every $t \in A_r$. From (2.27)–(2.30) it follows that

$$0 < -\alpha\varepsilon c_l \int_{A_l} \delta_l(t)dt + \alpha\varepsilon c_r \int_{A_r} \delta_r(t)dt = 0.$$

This contradiction proves (2.24). $\square$

Proof of Theorem 2.1 — By (2.23) and (2.24) there exists a constant $c \in \mathbb{R}$ such that for a.e. $t \in [a, b]$ and every $\varepsilon \in (0, 1)$ we have

$$c \leq d_v^r f_0(t, 1) \leq \frac{f_0(t, 2 - \varepsilon) - f_0(t, 1)}{1 - \varepsilon},$$

hence $(1 - \varepsilon)c \leq f_0(t, 2 - \varepsilon) - f_0(t, 1)$, which implies

$$(1 - \varepsilon)c + \varepsilon f_0(t, 1) \leq f_0(t, 2 - \varepsilon) - (1 - \varepsilon)f_0(t, 1). \tag{2.31}$$

By convexity we have

$$f_0(t, 2 - \varepsilon) \leq \varepsilon f_0(t, 1/\varepsilon) + (1 - \varepsilon)f_0(t, 1), \tag{2.32}$$

so that we obtain from (2.31) and (2.32)

$$(1 - \varepsilon)c + \varepsilon f_0(t, 1) \leq \varepsilon f_0(t, 1/\varepsilon) \leq \varepsilon f(t, 1/\varepsilon). \tag{2.33}$$

By (1.3) and (1.7) for every $v > 0$ we have

$$f(t,v) \geq L(y(t), y'(t)/v)v \geq \Theta(y'(t)/v)v \geq \overline{\mathrm{co}}\,\Theta(y'(t)/v)v, \tag{2.34}$$

where $\overline{\mathrm{co}}\,\Theta$ is the lower semicontinuous convex envelope of Θ, which still satisfies (1.2). Since the function $v \mapsto \overline{\mathrm{co}}\,\Theta(y'(t)/v)v$ is convex for $v > 0$, from (2.34) we deduce that

$$\forall\, v > 0, \quad f_0(t,v) \geq \overline{\mathrm{co}}\,\Theta(y'(t)/v)v. \tag{2.35}$$

From (1.7), (2.33), and (2.35) we obtain

$$(1-\varepsilon)c + \varepsilon\, \overline{\mathrm{co}}\,\Theta(y'(t)) \leq L(y(t), \varepsilon y'(t)) \tag{2.36}$$

for a.e. $t \in [a,b]$ and every $\varepsilon \in (0,1)$.

Let us now fix $t \in [a,b]$ such that (2.36) holds and $|y'(t)| > 2$. For $\nu \in (0,1]$ let $\varepsilon(t) = \frac{\nu}{|y'(t)|} < \frac{1}{2}$. By (2.36) we have

$$\min\{c, 0\} + \nu \frac{\overline{\mathrm{co}}\,\Theta(y'(t))}{|y'(t)|} \leq L\left(y(t), \nu \frac{y'(t)}{|y'(t)|}\right). \tag{2.37}$$

Since $y \in W^{1,1}(a,b;\mathbb{R}^n)$, there exists $R > 0$ such that

$$\forall\, t \in [a,b], \quad |y(t)| \leq R. \tag{2.38}$$

Since $B(0,R)$ is compact, from (1.4) we know that

$$\exists\, M > 0,\ \exists\, r \in (0,1],\ \forall\, u \in B(0,r), \quad L(y(t),u) \leq M. \tag{2.39}$$

Choosing $\nu = r$, from (2.37), (2.38), and (2.39) we get

$$\min\{c,0\} + r \frac{\overline{\mathrm{co}}\,\Theta\,(y'(t))}{|y'(t)|} \leq M. \tag{2.40}$$

Since $\overline{\mathrm{co}}\,\Theta$ satisfies (1.2), by (2.40) there exists a constant $C = C(\Theta, c, r, M) \geq 2$, depending only on $\overline{\mathrm{co}}\,\Theta$, c, r, and M, such that for a.e. $t \in [a,b]$ with $|y'(t)| > 2$ we have

$$|y'(t)| \leq C. \tag{2.41}$$

As $C \geq 2$, inequality (2.41) holds also when $|y'(t)| \leq 2$. $\square$

Proof of Theorem 2.2 — By Lemma 2.6 there exists c such that

$$d_v^l f_0(t,1) \leq c \leq d_v^r f_0(t,1) \quad \text{for a.e. } t \in [a,b]. \tag{2.42}$$

By (1.3) and (2.4) we have

$$\int_a^b \Theta(y'(t))dt \le B, \tag{2.43}$$

which, by (1.2), gives

$$\int_a^b |y'(t)|dt \le M_1,$$

for a constant $M_1 = M_1(\Theta, B, \beta) > 0$. This inequality, together with (2.3), yields $|y(t)| \le R$ for every $t \in [a, b]$, with $R = R(\Theta, A, B, \beta) = A + M_1$.

We next provide an estimate of c from below. From (2.43) and (2.5) it follows that

$$\alpha \operatorname*{ess\,inf}_{t\in[a,b]} \Theta(y'(t)) \le \int_a^b \Theta(y'(t))dt \le B. \tag{2.44}$$

By (1.2) there exists $M_2 = M_2(\Theta, \alpha, B)$ such that for a set $\Omega_y \subset [a, b]$ of positive measure

$$\forall\, t \in \Omega_y, \;\; |y'(t)| \le M_2. \tag{2.45}$$

This implies that

$$\forall\, t \in \Omega_y, \;\; L(y(t), \tfrac{4}{3}y'(t)) \le \Psi(R + 2M_2). \tag{2.46}$$

Since by (2.22)

$$c \ge \sup_{v<1} \frac{f_0(t, v) - f_0(t, 1)}{v - 1}$$

and since $f_0(t, 1) \ge 0$ for almost all $t \in [a, b]$, we get, setting $v = \frac{3}{4}$,

$$c \ge -4f(t, \tfrac{3}{4}) = -3L(y(t), \tfrac{4}{3}y'(t)) \ge -3\Psi(R + 2M_2). \tag{2.47}$$

We now return to the last part of the proof of Theorem 2.1 with $\nu = r = 1$ and $M = \Psi(R + 1)$. As the constant C which appears in (2.41) depends on c in a decreasing way, and $c \ge -3\Psi(R + 2M_2)$, it is enough to set $K = C(\Theta, -3\Psi(R + 2M_2), 1, \Psi(R + 1))$. $\square$

3 DuBois-Reymond Necessary Conditions

Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function, let $y \in W^{1,1}(a, b; \mathbb{R}^n)$ be a function which satisfies (2.1), and let $g: [a, b] \times \mathbb{R} \to [0, +\infty]$ be the function defined by

$$g(t, v) := \begin{cases} L(y(t), vy'(t)) & \text{if } 0 < v < 2, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

By (1.7) we have $f(t,v) = g(t,\frac{1}{v})v$ for every $v \neq 0$. Let $g_0 := \overline{\mathrm{co}}_v g$ be the lower semicontinuous convex envelope of g with respect to v. As the functions $v \mapsto f_0(t,v)$ and $v \mapsto g_0(t,\frac{1}{v})v$ are lower semicontinuous and convex for $v > 0$, we deduce that

$$\forall\, t \in [a,b],\ v > 0,\quad f_0(t,v) = g_0\,(t, 1/v)\, v.$$

Therefore we obtain from (2.12)

$$g_0(t,1) = g(t,1) = L(y(t), y'(t)) \quad \text{for a.e. } t \in [a,b]. \tag{3.2}$$

Furthermore $t \mapsto g_0(t,v)$ is measurable. To prove this fact it is enough to adapt the arguments used for f_0 in the proof of Lemma 2.3.

Let us define $d_v^l g_0(t,1)$ and $d_v^r g_0(t,1)$ as in (2.22) and (2.23). It is easy to prove that

$$d_v^l f_0(t,1) = g_0(t,1) - d_v^r g_0(t,1) \;\;\&\;\; d_v^r f_0(t,1) = g_0(t,1) - d_v^l g_0(t,1).$$

Therefore, by (2.24) there exists a constant $c \in \mathbb{R}$ such that

$$d_v^l g_0(t,1) \le L(y(t), y'(t)) - c \le d_v^r g_0(t,1) \quad \text{for a.e. } t \in [a,b]. \tag{3.3}$$

Notice that if $u \mapsto L(y(t), u)$ is differentiable at $y'(t)$, with gradient $\nabla_u L(y(t), y'(t))$, then by (3.1) the function $v \mapsto g(t,v)$ has a derivative at $v = 1$ which is equal to $\langle \nabla_u L(y(t), y'(t)), y'(t)\rangle$. By (3.2) this implies

$$d_v^l g_0(t,1) = d_v^r g_0(t,1) = \left\langle \nabla_u L(y(t), y'(t)), y'(t) \right\rangle, \tag{3.4}$$

and from (3.3) and (3.4) we obtain the DuBois-Reymond necessary condition

$$L(y(t), y'(t)) - \left\langle \nabla_u L(y(t), y'(t)), y'(t) \right\rangle = c \;\text{ for a.e. } t \in [a,b].$$

Our aim is to derive similar results when $L(y(t), \cdot)$ is not differentiable. All our extensions of the DuBois-Reymond necessary condition (Theorems 3.2, 3.6, and 3.10) are based on the following lemma.

Lemma 3.1 *Let $L\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function, let $y \in W^{1,1}(a,b;\mathbb{R}^n)$ be a function which satisfies (2.1), and let $\psi\colon [a,b] \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function, with $\xi \mapsto \psi(t,\xi)$ convex and positively homogeneous of degree one, such that for a.e. $t \in [a,b]$*

$$-d_v^l g_0(t,1) \le \psi(t, -y'(t)) \quad \& \quad d_v^r g_0(t,1) \le \psi(t, y'(t)). \tag{3.5}$$

Then there exist a constant $c \in \mathbb{R}$ and a measurable function $p\colon [a,b] \to \mathbb{R}^n$ such that for a.e. $t \in [a,b]$

$$p(t) \in \partial_\xi \psi(t,0), \tag{3.6}$$

$$L(y(t), y'(t)) - \langle p(t), y'(t)\rangle = c, \tag{3.7}$$

where $\partial_\xi \psi(t,0)$ denotes the subdifferential of the convex function $\psi(t,\cdot)$ at 0.

Proof — Since $\psi(t,\cdot)$ is convex and positively homogeneous of degree one,

$$\forall\, \xi \in \mathbb{R}^n, \quad \max_{q\in\partial_\xi\psi(t,0)} \langle q,\xi\rangle = \psi(t,\xi). \tag{3.8}$$

By (3.3) and (3.5) there exists a constant c such that for a.e. $t\in[a,b]$

$$-\psi(t,-y'(y)) \le L(y(t),y'(t)) - c \le \psi(t,y'(t)),$$

so that (3.8) implies

$$\min_{q\in\partial_\xi\psi(t,0)} \langle q,y'(t)\rangle \le L(y(t),y'(t)) - c \le \max_{q\in\partial_\xi\psi(t,0)} \langle q,y'(t)\rangle.$$

Let us fix $t\in[a,b]$ such that these inequalities are satisfied. The set $\partial_\xi\psi(t,0)$ being convex, we deduce that for some $q\in\partial_\xi\psi(t,0)$ we have $\langle q,y'(t)\rangle = L(y(t),y'(t)) - c$. For every $t\in[a,b]$ let

$$B(t) := \{q\in\partial_\xi\psi(t,0) \mid \langle q,y'(t)\rangle = L(y(t),y'(t)) - c\}\,. \tag{3.9}$$

By the previous argument $B(t)\ne\emptyset$ for a.e. $t\in[a,b]$. The graph of the set-valued map $t\rightsquigarrow B(t)$ is the intersection of the sets B_1 and B_2 defined by

$$\begin{aligned} B_1 &:= \{(t,q)\in[a,b]\times\mathbb{R}^n \mid \langle q,y'(t)\rangle = L(y(t),y'(t)) - c\}\,, \\ B_2 &:= \{(t,q)\in[a,b]\times\mathbb{R}^n \mid \forall\,\xi\in\mathbb{R}^n,\ \ \langle q,\xi\rangle \le \psi(t,\xi)\}\,. \end{aligned}$$

Clearly B_1 and B_2 belong to $\mathcal{L}_1\times\mathcal{B}_n$, where $\mathcal{B}_n$ denotes the σ-algebra of all Borel subsets of $\mathbb{R}^n$. This implies that the graph of the set-valued map $t\rightsquigarrow B(t)$ defined by (3.9) belongs to $\mathcal{L}_1\times\mathcal{B}_n$, and by the Aumann-von Neumann selection theorem (see [7, Theorem III.22]), there exists a measurable function $p\colon[a,b]\to\mathbb{R}^n$ such that $p(t)\in B(t)$ for a.e. $t\in[a,b]$. Then (3.6) and (3.7) follow from (3.9). $\square$

Let $L_0 := \overline{\mathrm{co}}_u\, L$ be the lower semicontinuous convex envelope of L with respect to u. Then $(t,u)\mapsto L_0(y(t),u)$ is a Carathéodory function. This can be verified as in the case of f_0 (proof of Lemma 2.3), taking convex combinations of $n+1$ vectors.

Theorem 3.2 *Let $L\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$ be a Borel function and let $y\in W^{1,1}(a,b;\mathbb{R}^n)$ be a function which satisfies (2.1). Suppose that for a.e. $t\in[a,b]$*

$$L(y(t),y'(t)) = L_0(y(t),y'(t)), \tag{3.10}$$

$$-d_v^l g_0(t,1) \le d_u L_0(y(t),y'(t))(-y'(t)), \tag{3.11}$$

$$d_v^r g_0(t,1) \le d_u L_0(y(t),y'(t))(y'(t)), \tag{3.12}$$

where $d_u L_0(x,u)(\xi)$ denotes the directional derivative of the convex function $L_0(x,\cdot)$ at u in the direction ξ. Then there exist a constant $c \in I\!\!R$ and a measurable function $p: [a,b] \to I\!\!R^n$ such that a.e. $t \in [a,b]$

$$p(t) \in \partial_u L_0(y(t), y'(t)), \tag{3.13}$$

$$L_0(y(t), y'(t)) - \langle p(t), y'(t)\rangle = c. \tag{3.14}$$

Consequently,

$$\langle p(t), y'(t)\rangle - L(y(t), y'(t)) = \sup_{u \in I\!\!R^n} (\langle p(t), u\rangle - L(y(t), u)) = -c \tag{3.15}$$

for a.e. $t \in [a,b]$.

Remark 3.3 Since $g_0(t,v) \geq L_0(y(t), vy'(t))$ for a.e. $t \in [a,b]$ and for every $v \in I\!\!R$, if (3.10) holds, then by (3.2)

$$-d_v^l g_0(t,1) \geq d_u L_0(y(t), y'(t))(-y'(t)) \quad \& \quad d_v^r g_0(t,1) \geq d_u L_0(y(t), y'(t))(y'(t)),$$

so that (3.11) and (3.12) are actually equalities. Assumptions (3.10)–(3.12) are satisfied, for instance, if $g_0(t,v) = L_0(y(t), vy'(t))$ for a.e. $t \in [a,b]$ and every $v \in I\!\!R$.

Remark 3.4 If H is the Hamiltonian associated to L, defined in (1.9), then, by (3.15), $H(y(t), p(t)) = -c$ for almost all $t \in [a,b]$. The function p corresponds to the co-state of optimal control theory. In other words, we proved that the Hamiltonian is constant along the optimal trajectory/co-state pair (y,p). In the case of smooth Hamiltonians this is indeed a well known property of optimal trajectories of autonomous Bolza control problems.

Proof of Theorem 3.2 — The result follows from Lemma 3.1, taking $\psi(t,\xi) := d_u L_0(y(t), y'(t))(\xi)$. Indeed, the convexity of $L_0(y(t),\cdot)$ implies that $\psi(t,\cdot)$ is convex and $\partial_\xi \psi(t,0) = \partial_u L_0(y(t), y'(t))$ for a.e. $t \in [a,b]$. Equality (3.15) follows from (3.10), (3.13), and (3.14). □

To state further extensions of the DuBois-Reymond necessary condition, we need to recall several notions of generalized derivatives. Let $\varphi: I\!\!R^m \to I\!\!R \cup \{+\infty\}$. The subdifferential of φ at $x \in \mathrm{dom}(\varphi)$ is defined by

$$\partial^- \varphi(x) := \left\{ p \in I\!\!R^n \mid \liminf_{y \to x} \frac{\varphi(y) - \varphi(x) - \langle p, y-x\rangle}{|y-x|} \geq 0 \right\}. \tag{3.16}$$

An equivalent definition of subdifferential uses the lower contingent derivatives of φ defined by

$$\forall\ u \in I\!\!R^m, \quad D_\uparrow \varphi(x)(u) := \liminf_{\substack{h \to 0+ \\ v \to u}} \frac{\varphi(x+hv) - \varphi(x)}{h}. \tag{3.17}$$

Then,

$$p \in \partial^- \varphi(x) \iff \forall\, v \in \mathbb{R}^n,\ \langle p, v\rangle \le D_\uparrow \varphi(x)(v). \tag{3.18}$$

(see, e.g., [4]). The upper contingent derivative of φ at x is defined by

$$\forall\, u \in \mathbb{R}^m,\ D_\downarrow \varphi(x)(u) := \limsup_{\substack{h\to 0+\\ v\to u}} \frac{\varphi(x+hv)-\varphi(x)}{h}.$$

The superdifferential $\partial^+\varphi(x)$ of φ at x is defined by $\partial^+\varphi(x) := -\partial^-(-\varphi)(x)$ or, equivalently, by

$$p \in \partial^+ \varphi(x) \iff \forall\, v \in \mathbb{R}^n,\ \langle p, v\rangle \ge D_\downarrow \varphi(x)(v). \tag{3.19}$$

We use also the lower Dini directional derivative, defined by

$$\forall\, u \in \mathbb{R}^m,\ d^- \varphi(x)(u) := \liminf_{h\to 0+} \frac{\varphi(x+hu)-\varphi(x)}{h}. \tag{3.20}$$

Let us return to the Lagrangian L considered at the beginning of this section. Partial derivatives and partial differentials of L with respect to u are defined in the usual way: given $x \in \mathbb{R}^n$, we consider the function $\varphi(\cdot) := L(x,\cdot)$, and set $D_{\uparrow u}L(x,u) := D_\uparrow\varphi(u)$, $\partial_u^- L(x,u) := \partial^-\varphi(u)$, $D_{\downarrow u}L(x,u) := D_\downarrow\varphi(u)$, $\partial_u^+ L(x,u) := \partial^+\varphi(u)$, and $d_u^- L(x,u) := d^-\varphi(u)$.

Remark 3.5 By (3.1) and (3.2) for a.e. $t \in [a,b]$ and every $v \in (0,2)$ we have

$$g_0(t,v) \le g(t,v) = L(y(t), vy'(t)) \quad \& \quad g_0(t,1) = g(t,1) = L(y(t), y'(t)),$$

which implies

$$-d_v^l g_0(t,1) \le d_u^- L(y(t),y'(t))(-y'(t)) \quad \& \quad d_v^r g_0(t,1) \le d_u^- L(y(t),y'(t))(y'(t)). \tag{3.21}$$

Therefore the conclusions of Lemma 3.1 continue to hold if (3.5) is replaced by

$$d_u^- L(y(t),y'(t))(\pm y'(t)) \le \psi(t, \pm y'(t)) \tag{3.22}$$

for a.e. $t \in [a,b]$.

For every $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$ let $\xi \mapsto \overline{\mathrm{co}}\, D_{\uparrow u}L(x,u)(\xi)$ be the lower semicontinuous convex envelope of the function $\xi \mapsto D_{\uparrow u}L(x,u)(\xi)$.

Theorem 3.6 *Let $L: I\!\!R^n \times I\!\!R^n \to I\!\!R_+$ be a Borel function and let $y \in W^{1,1}(a,b;I\!\!R^n)$ be a function which satisfies (2.1). Suppose that for a.e. $t \in [a,b]$*

$$-d_v^l g_0(t,1) \leq \overline{\mathrm{co}}\, D_{\uparrow u} L(y(t),y'(t))(-y'(t)), \tag{3.23}$$

$$d_v^r g_0(t,1) \leq \overline{\mathrm{co}}\, D_{\uparrow u} L(y(t),y'(t))(y'(t)), \tag{3.24}$$

$$\forall\, \xi \in I\!\!R^n, \quad \overline{\mathrm{co}}\, D_{\uparrow u} L(y(t),y'(t))(\xi) \in I\!\!R. \tag{3.25}$$

Then there exist a constant $c \in I\!\!R$ and a measurable function $p: [a,b] \to I\!\!R^n$ such that

$$p(t) \in \partial_u^- L(y(t),y'(t)) \quad \textit{for a.e. } t \in [a,b], \tag{3.26}$$

$$L(y(t),y'(t)) - \langle p(t), y'(t)\rangle = c \quad \textit{for a.e. } t \in [a,b]. \tag{3.27}$$

Remark 3.7 By (3.21) inequalities (3.23) and (3.24) are satisfied if

$$d_u^- L(y(t),y'(t))(\pm y'(t)) \leq \overline{\mathrm{co}}\, D_{\uparrow u} L(y(t),y'(t))(\pm y'(t)),$$

for a.e. $t \in [a,b]$. This shows that (3.23)–(3.25) are always satisfied if $L(y(t),\cdot)$ is differentiable at $y'(t)$.

Remark 3.8 We recall that a function $\varphi: I\!\!R^n \to I\!\!R$ is called *semiconvex* if there exists $\omega: I\!\!R_+ \times \mathbf{R}_+ \to I\!\!R_+$, satisfying

$$\forall\, r \leq R,\ \forall\, s \leq S,\ \ \omega(r,s) \leq \omega(R,S)\ \ \&\ \ \lim_{s\to 0+} \omega(R,s) \ = \ 0,$$

such that for every $R > 0$, $\lambda \in [0,1]$, and all $x, y \in B(0,R)$

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y) + \lambda(1-\lambda)|x-y|\,\omega(R,|x-y|).$$

Observe that every convex function is semiconvex (with ω equal to zero). Furthermore, if $\varphi: I\!\!R^n \to \mathbf{R}$ is continuously differentiable, then it is semiconvex. Using standard arguments of convex analysis (see, e.g., [3, p. 25]) one can prove that every semiconvex function φ is locally Lipschitz. Furthermore, for every $v \in I\!\!R^n$

$$D_\uparrow \varphi(x)(v) = \lim_{h\to 0+} \frac{\varphi(x+hv) - \varphi(x)}{h},$$

and the function $v \mapsto D_\uparrow\varphi(x)(v)$ is convex (see, e.g., [6, Theorem 3.9]). These facts, together with Remark 3.7, show that assumptions (3.23)–(3.25) of Theorem 3.6 are always satisfied when $L(x,\cdot)$ is semiconvex.

Proof of Theorem 3.6 — Let us define $\psi(t,\xi) := \overline{\mathrm{co}}\, D_{\uparrow u} L(y(t),y'(t))(\xi)$. Using the projection theorem it is possible to check the measurability with respect to t. Notice that $\partial_\xi \psi(t,0) \subset \partial_u^- L(y(t),y'(t))$ for a.e. $t \in [a,b]$. Indeed, if $q \in \partial_\xi\psi(t,0)$, then $\langle q, \xi\rangle \leq \psi(t,\xi) \leq D_{\uparrow u} L(y(t),y'(t))(\xi)$ for every $\xi \in I\!\!R^n$, hence $q \in \partial_u^- L(y(t),y'(t))$ by (3.18). The conclusion follows then from Lemma 3.1. $\square$

Remark 3.9 Theorem 3.2 has stronger assumptions and stronger conclusions than Theorem 3.6. Indeed, as $L \geq L_0$, it follows from (3.10) that

$$d_u L_0(y(t), y'(t))(\xi) = D_{\uparrow u} L_0(y(t), y'(t))(\xi) \leq D_{\uparrow u} L(y(t), y'(t))(\xi) \tag{3.28}$$

for every $\xi \in \mathbb{R}^n$. Since $\xi \mapsto d_u L_0(y(t), y'(t))(\xi)$ is convex, we conclude that

$$d_u L_0(y(t), y'(t))(\pm y'(t)) \leq \overline{\mathrm{co}}\, D_{\uparrow u} L(y(t), y'(t))(\pm y'(t)).$$

This shows that (3.10)–(3.12) imply (3.23)–(3.25).

On the other hand, (3.18) and (3.28) yield $\partial_u^- L_0(y(t), y'(t)) \subset \partial_u^- L(y(t), y'(t))$ for a.e. $t \in [a, b]$. Therefore (3.10), (3.13), and (3.14) imply (3.26) and (3.27).

Theorem 3.10 *Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function and let $y \in W^{1,1}(a, b; \mathbb{R}^n)$ be a function which satisfies (2.1). Suppose that $L(x, \cdot)$ is locally Lipschitz continuous for every $x \in \mathbb{R}^n$. Then there exist a constant $c \in \mathbb{R}$ and a measurable function $p: [a, b] \to \mathbb{R}^n$ such that for a.e. $t \in [a, b]$*

$$p(t) \in \partial_u L(y(t), y'(t)), \tag{3.29}$$

$$L(y(t), y'(t)) - \langle p(t), y'(t) \rangle = c, \tag{3.30}$$

where $\partial_u L(x, u)$ denotes the Clarke generalized gradient of $L(x, \cdot)$ at u.

Proof — Let us define

$$\psi(t, \xi) = \limsup_{\substack{h \to 0+ \\ u \to y'(t)}} \frac{L(y(t), u + h\xi) - L(y(t), u)}{h}.$$

It is known that $\psi(t, \cdot)$ is convex and that $\partial_\xi \psi(t, 0)$ is the Clarke generalized gradient of $L(y(t), \cdot)$ at $y'(t)$ (see [9]). Since $d_u^- L(y(t), y'(t))(\xi) \leq \psi(t, \xi)$, the result follows from Lemma 3.1 and Remark 3.5. □.

Replacing subdifferential by superdifferential we get another extension of the DuBois-Reymond necessary condition, which is meaningful only at those points $t \in [a, b]$ for which $\partial_u^+ L(y(t), y'(t)) \neq \emptyset$.

Proposition 3.11 *Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function and let $y \in W^{1,1}(a, b; \mathbb{R}^n)$ be a function which satisfies (2.1). There exists a constant $c \in \mathbb{R}$ such that*

$$\forall\, p \in \partial_u^+ L(y(t), y'(t)), \quad L(y(t), y'(t)) - \langle p, y'(t) \rangle = c$$

for a.e. $t \in [a, b]$.

Proof — From (3.2) we have

$$d_v^r g_0(t,1) \le D_{\downarrow u} L(y(t), y'(t))(y'(t)),$$
$$d_v^l g_0(t,1) \ge -D_{\downarrow u} L(y(t), y'(t))(-y'(t)).$$

These inequalities and (3.3) imply that there exists a constant c such that for a.e. $t \in [a,b]$

$$-D_{\downarrow u} L(y(t), y'(t))(-y'(t)) \le L(y(t), y'(t)) - c \le D_{\downarrow u} L(y(t), y'(t))(y'(t)),$$

and we deduce from (3.19) that for all $p \in \partial_u^+ L(y(t), y'(t))$

$$\langle -p, -y'(t) \rangle \le L(y(t), y'(t)) - c \le \langle p, y'(t) \rangle ,$$

ending the proof. □

4 Hamilton-Jacobi Inequalities

Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ with $\varphi \not\equiv +\infty$. Given $T > 0$ and $y_0 \in \mathbb{R}^n$, let us consider the Bolza problem:

$$\min_y \int_0^T L(y(s), y'(s))ds + \varphi(y(T))$$

over all absolutely continuous functions $y \in W^{1,1}(0,T;\mathbb{R}^n)$ satisfying the initial condition $y(0) = y_0$. The dynamic programming approach associates with this problem the family of problems ($t \ge 0$, $x \in \mathbb{R}^n$):

$$\min_y \int_0^t L(y(s), y'(s))ds + \varphi(y(t))$$

over all absolutely continuous functions $y \in W^{1,1}(0,t;\mathbb{R}^n)$ satisfying $y(0) = x$. The corresponding value function $V\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by (1.8).

Proposition 4.1 *Let $L\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function and let $\varphi\colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ with $\varphi \not\equiv +\infty$. Then $V(0,x) = \varphi(x)$ for every $x \in \mathbb{R}^n$. Furthermore, if L satisfies (1.2) and (1.3), and φ is lower semicontinuous, then*

$$\liminf_{\substack{t \to 0+ \\ x \to x_0}} V(t,x) \ge \varphi(x_0) \tag{4.1}$$

for every $x_0 \in \mathbb{R}^n$.

Proof — Fix $x_0 \in \mathbb{R}^n$ and let $t_i \to 0+$, $x_i \to x_0$ be such that

$$\liminf_{\substack{t \to 0+ \\ x \to x_0}} V(t,x) = \lim_{i \to \infty} V(t_i, x_i).$$

If the above limit is infinite, then our claim follows. If this limit is finite, then we consider $y_i \in W^{1,1}(0, t_i; \mathbb{R}^n)$ such that $y_i(0) = x_i$ and

$$\int_0^{t_i} L(y_i(s), y_i'(s))ds + \varphi(y_i(t_i)) \leq V(t_i, x_i) + \frac{1}{i}.$$

By (1.3), since $\varphi \geq 0$, for some $M > 0$ we have $\int_0^{t_i} \Theta(y_i'(s))ds \leq M$ for every i. Setting $y_i'(s) = 0$ for $s \in (t_i, 1]$, we deduce from (1.2) that the functions y_i' are equiintegrable and therefore the functions y_i are equicontinuous. Since $t_i \to 0+$ and $y_i(0) = x_i \to x_0$, we get $y_i(t_i) \to x_0$. On the other hand, since $L \geq 0$, we have $\varphi(y_i(t_i)) \leq V(t_i, x_i) + \frac{1}{i}$. Taking the lower limit and using the lower semicontinuity of φ we conclude the proof. □

We recall that $\mathbb{R}_+^\star := \{t \in \mathbb{R} \mid t > 0\}$. In this section we often assume the following hypotheses:

(**H1**) for every $(t,x) \in \mathbb{R}_+^\star \times \mathbb{R}^n$ the infimum in (1.8) is attained,

(**H2**) L is locally bounded and satisfies (1.2) and (1.3).

It is easy to see that (**H2**) implies that $0 \leq V(t,x) < \infty$ for all $(t,x) \in \mathbb{R}_+^\star \times \mathbb{R}^n$.

Remark 4.2 If for every $t > 0$ the functional

$$\mathcal{L}_0^t(y) := \int_0^t L(y(s), y'(s))ds \tag{4.2}$$

is sequentially weakly lower semicontinuous on $W^{1,1}(0,t;\mathbb{R}^n)$ and φ is lower semicontinuous, then from (1.2) and (1.3) it follows that (**H1**) is satisfied. Furthermore, arguing as in [11, Proof of Proposition 3.1], we can show that in this case V is lower semicontinuous on $\mathbb{R}_+ \times \mathbb{R}^n$.

Lemma 4.3 *Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function and let $\varphi: \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ with $\varphi \not\equiv +\infty$. Assume that* (**H1**) *and* (**H2**) *are satisfied. Then, given $(t_0, x_0) \in \mathbb{R}_+^\star \times \mathbb{R}^n$ and $0 < \delta < t_0$, there exists $r > 0$ such that for all $(t,x) \in B((t_0,x_0),\delta)$ every minimizer $y(\cdot;t,x)$ of (1.8) is r-Lipschitz.*

Proof — Consider $y_0 \in \mathbb{R}^n$, with $\varphi(y_0) < +\infty$, and set $z(s) = x + \frac{s}{t}(y_0 - x)$. If $y(\cdot; t, x)$ is a minimizer of (1.8), we obtain

$$\int_0^t L(y(s;t,x), y'(s;t,x))ds \leq \varphi(y_0) + t \sup_{s\in[0,t]} L(z(s), (y_0 - x)/t).$$

Since L is locally bounded, for every $0 < \delta < t_0$ there exists a constant $M_\delta > 0$ such that

$$\sup_{s\in[0,t]} L(z(s), (y_0 - x)/t) \leq M_\delta$$

for every $(t, x) \in B((t_0, x_0), \delta)$. The conclusion follows now from Theorem 2.2. □

Theorem 4.4 *Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function and let $\varphi: \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ with $\varphi \not\equiv +\infty$. Assume that* (**H1**) *and* (**H2**) *are satisfied. Then V is locally Lipschitz on $\mathbb{R}_+^\star \times \mathbb{R}^n$. If φ is lower semicontinuous on $\mathbb{R}^n$, then V is lower semicontinuous on $\mathbb{R}_+ \times \mathbb{R}^n$.*

Proof — The Lipschitz continuity is proved in [11, Corollary 3.4]. For the reader's convenience we repeat here the proof.

Fix $(t_0, x_0) \in \mathbb{R}_+^\star \times \mathbb{R}^n$. By Lemma 4.3, there exist $r > 0$ and $\delta > 0$ such that for all $(t, x) \in B((t_0, x_0), \delta)$ every minimizer $y(\cdot; t, x)$ of (1.8) is r-Lipschitz. We may assume that $5\delta < t_0$. Let (t_1, x_1) and (t_2, x_2) be two distinct points of $B((t_0, x_0), \delta)$, let $h_1 := |t_1 - t_2| + |x_1 - x_2|$, and $s_1 := h_1 - t_1 + t_2$. Let $u_1 \in \mathbb{R}^n$ be such that $y(s_1; t_2, x_2) = x_1 + h_1 u_1$. Then $0 < h_1 < t_1$, $0 \leq s_1 \leq 2h_1$, and

$$|u_1| \leq \frac{|y(s_1; t_2, x_2) - x_2|}{h_1} + \frac{|x_2 - x_1|}{h_1} \leq 2r + 1. \tag{4.3}$$

Let $y_1: [0, t_1] \mapsto \mathbb{R}^n$ be the function defined by

$$y_1(s) = \begin{cases} x_1 + s u_1 & \text{if } 0 \leq s \leq h_1, \\ y(s - t_1 + t_2; t_2, x_2) & \text{if } h_1 \leq s \leq t_1. \end{cases}$$

Then

$$V(t_1, x_1) \leq \int_0^{t_1} L(y_1(s), y_1'(s))ds + \varphi(y_1(t_1)) =$$

$$= \int_0^{h_1} L(x_1 + s u_1, u_1)ds + \int_{s_1}^{t_2} L(y(s; t_2, x_2), y'(s; t_2, x_2))ds + \varphi(y(t_2; t_2, x_2)).$$

As $s_1 = h_1 - t_1 + t_2 \geq 0$ and $L \geq 0$, we obtain

$$V(t_1, x_1) \leq \int_0^{h_1} L(x_1 + s u_1, u_1)ds + V(t_2, x_2).$$

Since L is locally bounded, it follows from (4.3) that there exists a constant M, depending only on L, t_0, x_0, δ, and r, such that

$$V(t_1,x_1) - V(t_2,x_2) \le Mh_1 = M(|t_1-t_2| + |x_1-x_2|).$$

Exchanging the roles of (t_1,x_1) and (t_2,x_2) we get that V in M-Lipschitz on $B((t_0,x_0),\delta)$.

If φ is lower semicontinuous, then V is lower semicontinuous at all points of $\{0\}\times \mathbb{R}^n$ by Proposition 4.1. The lower semicontinuity on $\mathbb{R}^*_+ \times \mathbb{R}^n$ is a consequence of the local Lipschitz continuity. $\square$

We recall the definition of the function $L^+(x,u)$ used in [11] to study Hamilton-Jacobi equation (1.10) arising from a discontinuous Lagrangian:

$$\text{(4.4)} \quad L^+(x,u) := \limsup_{h\to 0+} \frac{1}{h} \inf_y \left\{ \int_{-h}^{0} L(y(s),y'(s))ds \mid y(-h) = x - hu,\ y(0) = x \right\}.$$

Similarly we define the function $L^-(x,u)$ by

$$\text{(4.5)} \qquad L^-(x,u) := \liminf_{h\to 0+} \frac{1}{h} \inf_y \left\{ \int_{0}^{h} L(y(s),y'(s))ds \mid y(h) = x + hu,\ y(0) = x \right\}.$$

Proposition 4.5 *Assume that L is locally bounded. If $u_h \to u$ as $h\to 0+$, then*

$$\text{(4.6)} \quad L^+(x,u) = \limsup_{h\to 0+} \frac{1}{h} \inf_y \left\{ \int_{-h}^{0} L(y(s),y'(s))ds \mid y(-h) = x - hu_h,\ y(0) = x \right\},$$

$$\text{(4.7)} \qquad L^-(x,u) = \liminf_{h\to 0+} \frac{1}{h} \inf_y \left\{ \int_{0}^{h} L(y(s),y'(s))ds \mid y(h) = x + hu_h,\ y(0) = x \right\}.$$

Proof — The following proof is a slight modification of the proof of [11, Proposition 3.6]. Let us fix (x,u) and u_h as in the statement of the proposition, and let $\overline{L}(x,u)$ be the right hand side of (4.6). We want to show that $\overline{L}(x,u) \le L^+(x,u)$. For every $h>0$ let $\varepsilon_h = |u_h - u|$ and let y_h be such that $y_h(-(1-\varepsilon_h)h) = x-(1-\varepsilon_h)hu$, $y_h(0)=x$, and

$$\int_{-(1-\varepsilon_h)h}^{0} L(y_h(s),y_h'(s))ds - h^2 \le$$

$$\le \inf_y \left\{ \int_{-(1-\varepsilon_h)h}^{0} L(y(s),y'(s))ds : y(-(1-\varepsilon_h)h) = x-(1-\varepsilon_h)hu,\ y(0) = x \right\}.$$

We extend y_h on the interval $[-h, -(1-\varepsilon_h)h]$ by the affine function satisfying $y_h(-h) = x - hu_h$ and $y_h(-(1-\varepsilon_h)h) = x-(1-\varepsilon_h)hu$. Since on this interval the derivative of y_h is equal to $(u_h - (1-\varepsilon_h)u)/\varepsilon_h$, which is uniformly bounded, we deduce that for some $M > 0$ and all $h > 0$,

$$\int_{-h}^{0} L(y_h(s), y_h'(s))ds \le \int_{-(1-\varepsilon_h)h}^{0} L(y_h(s), y_h'(s))ds + \int_{-h}^{-(1-\varepsilon_h)h} M ds.$$

Dividing by h and taking the upper limit when $h \to 0+$ we get $\overline{L}(x,u) \le L^+(x,u)$. The opposite inequality can be proved in the same way. The proof of (4.7) is similar. □

Remark 4.6 From the previous proposition it follows that, if $h_i \to 0+$, $u_i \to u$, and $y_i \in W^{1,1}(0,h_i)$ satisfies $y_i(0) = x$ and $y(h_i) = x + h_i u_i$, then

$$L^-(x,u) \le \liminf_{i\to\infty} \frac{1}{h_i} \int_0^{h_i} L(y_i(s), y_i'(s))ds. \tag{4.8}$$

Proposition 4.7 *Assume that L is locally bounded. For every $y \in W^{1,1}(0,t;\mathbb{R}^n)$ we have $L^+(y(s),y'(s)) \le L(y(s),y'(s))$ and $L^-(y(s),y'(s)) \le L(y(s),y'(s))$ for a.e. $s \in [0,t]$. If y is a minimizer of (1.8), then $L^+(y(s),y'(s)) = L^-(y(s),y'(s)) = L(y(s),y'(s))$ for a.e. $s \in [0,t]$.*

Proof — Assume first $y \in W^{1,\infty}(0,t;\mathbb{R}^n)$. Since L is locally bounded, the function $s \mapsto \psi(s) := \int_0^s L(y(\tau),y'(\tau))d\tau$ is absolutely continuous. Let $s \in [0,t]$ be such that both $\psi'(s)$ and $y'(s)$ do exist and $\psi'(s) = L(y(s),y'(s))$. Since $u_h = (y(s)-y(s-h))/h$ converges to $y'(s)$ as $h \to 0+$, from Proposition 4.5 we obtain

$$L^+(y(s),y'(s)) \le \lim_{h\to 0+} \frac{1}{h}\int_{s-h}^{s} L(y(\tau),y'(\tau))d\tau = \psi'(s) = L(y(s),y'(s)),$$

which concludes the proof of the inequality $L^+(y(s),y'(s)) \le L(y(s),y'(s))$ when y is Lipschitz.

If $y \in W^{1,1}(0,t;\mathbb{R}^n)$, we can apply a Lusin type approximation theorem for Sobolev functions (see, e.g., [19, Theorem 3.10.5]), which asserts that for every $\varepsilon > 0$ there exist $y_\varepsilon \in W^{1,\infty}(0,t;\mathbb{R}^n)$ and an open set U_ε such that $|U_\varepsilon| < \varepsilon$ and $y_\varepsilon(s) = y(s)$ for all $s \in [0,t] \setminus U_\varepsilon$. As $y_\varepsilon'(s) = y'(s)$ for a.e. $s \in [0,t]\setminus U_\varepsilon$, we obtain that $L^+(y(s),y'(s)) \le L(y(s),y'(s))$ and $L^-(y(s),y'(s)) \le L(y(s),y'(s))$ for a.e. $s \in [0,t]\setminus U_\varepsilon$. Since ε is arbitrary, these inequalities hold for a.e. $s \in [0,t]$.

If y is a minimizer of (1.8), then for every $s \in (0,t)$ and every $h \in (0,s)$ we have

$$\int_{s-h}^{s} L(y(\tau),y'(\tau))d\tau = \inf_z \left\{ \int_{-h}^{0} L(z(\tau),z'(\tau))d\tau \mid z(-h) = y(s-h),\ z(0) = y(s) \right\}. \tag{4.9}$$

Let us fix a Lebesgue point $s \in (0,t)$ for the function $L(y(s),y'(s))$ such that $(y(s) - y(s-h))/h \to y'(s)$ as $h \to 0+$. If we divide both sides of (4.9) by h, the left hand side tends to $L(y(s),y'(s))$ while the right hand side tends to $L^+(y(s),y'(s))$ thanks to Proposition 4.5, applied with $x := y(s)$, $u := y'(s)$, and $u_h := (y(s)-y(s-h))/h$. Therefore $L^+(y(s),y'(s)) = L(y(s),y'(s))$ for a.e. $s \in [0,t]$. The proof for $L^-(y(s),y'(s))$ is similar. □

Let us define

$$H^+(x,p) := \sup_{u\in \mathbb{R}^n} \left(\langle p,u\rangle - L^+(x,u)\right), \tag{4.10}$$

$$H^-(x,p) := \sup_{u\in \mathbb{R}^n} \left(\langle p,u\rangle - L^-(x,u)\right). \tag{4.11}$$

Theorem 4.8 *Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function and let $\varphi: \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ with $\varphi \not\equiv +\infty$. Assume that* (**H1**) *and* (**H2**) *are satisfied. Then the value function V satisfies the following two inequalities:*

$$\forall\ (t,x) \in \mathbb{R}_+^\star \times \mathbb{R}^n,\ \ \exists\ u \in \mathbb{R}^n,\ \ D_\uparrow V(t,x)(-1,u) \le -L^-(x,u), \tag{4.12}$$

$$\forall\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,\ \forall\ u \in \mathbb{R}^n,\ \ D_\downarrow V(t,x)(1,-u) \le L^+(x,u). \tag{4.13}$$

Consequently, V is a supersolution to (1.10) on $\mathbb{R}_+^\star \times \mathbb{R}^n$ when H is replaced by H^-, i.e.,

$$\forall\ (t,x) \in \mathbb{R}_+^\star \times \mathbb{R}^n,\ \ \forall\ (p_t,p_x) \in \partial^- V(t,x),\ \ p_t + H^-(x,-p_x) \ge 0, \tag{4.14}$$

and V is a subsolution to (1.10) on $\mathbb{R}_+ \times \mathbb{R}^n$ when H is replaced by H^+, i.e.,

$$\forall\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,\ \forall\ (p_t,p_x) \in \partial^- V(t,x),\ \ p_t + H^+(x,-p_x) \le 0. \tag{4.15}$$

Proof — Let $t > 0$, $x \in \mathbb{R}^n$, and let y be a minimizer of (1.8). By Theorem 2.1 $y(\cdot)$ is Lipschitz. By minimality for all $0 < h \le t$ we have

$$V(t,x) = V(t-h,y(h)) + \int_0^h L(y(s),y'(s))ds.$$

Consider $h_i \to 0+$ such that for some $u \in \mathbb{R}^n$, $u_i := (y(h_i)-x)/h_i \to u$. Then $y(h_i) = x + h_i u_i$, and (4.8) yields

$$D_\uparrow V(t,x)(-1,u) \le -\limsup_{i\to\infty} \frac{1}{h_i}\int_0^{h_i} L(y(s),y'(s))ds \le -L^-(x,u),$$

which proves (4.12).

Let $(p_t,p_x) \in \partial^- V(t,x)$. Then, by (3.18),

$$-p_t + \langle p_x,u\rangle \le D_\uparrow V(t,x)(-1,u),$$

hence $p_t + \langle -p_x, u\rangle - L^-(x,u) \geq 0$. By (4.11) this inequality gives (4.14).

To prove inequality (4.13), we fix any $u \in \mathbb{R}^n$ and let $h_i \to 0+$, $u_i \to u$. From the definition of V it follows that

$$V(t+h_i, x-h_iu_i) - V(t,x) \leq$$
$$\leq \inf_y \left\{ \int_{-h_i}^0 L(y(s), y'(s))ds \mid y(-h_i) = x - h_iu_i, y(0) = x \right\}.$$

Then we divide by h_i and pass to the upper limit as $i \to \infty$. Taking (4.6) into account we obtain (4.13). To prove (4.15) it is enough to apply (3.18), (4.13), and (4.10). $\square$

Theorem 4.9 *Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function, let $\varphi: \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function with $\varphi \not\equiv +\infty$, and let $W: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function which satisfies the initial condition $W(0,\cdot) = \varphi$. Assume that* (**H1**) *and* (**H2**) *are satisfied. If W is a subsolution of the Hamilton-Jacobi equation (1.10), in the sense that*

$$\forall\ (t,x) \in \mathrm{dom}(W),\ \forall\ u \in \mathbb{R}^n,\ \ D_\downarrow W(t,x)(1,-u) \leq L^+(x,u), \tag{4.16}$$

then $W \leq V$ on $\mathbb{R}_+ \times \mathbb{R}^n$. Therefore the value function V is the greatest lower semicontinuous function W which satisfies inequality (4.16) and the initial condition $W(0,\cdot) = \varphi$.

We shall use the following well known lemma (see, e.g., [16, Chapter 5, Section 2, Exercise 3]). For completeness we give here an elementary proof. To simplify notation, in the case of functions f of one real variable the lower Dini derivative $d^-f(t)(1)$, defined by (3.20), is denoted by $d^-f(t)$.

Lemma 4.10 *Let $[a,b]$ be a bounded closed interval in $\mathbb{R}$ and let $f: [a,b] \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function such that $d^-f(t) \leq 0$ for every $t \in [a,b)$ with $f(t) < +\infty$. Then $f(b) \leq f(a)$.*

Proof — For every $\varepsilon > 0$ let us consider the lower semicontinuous function $f_\varepsilon(t) := f(t) - \varepsilon t$, which satisfies the stronger inequality $d^-f_\varepsilon(t) \leq -\varepsilon$ for every $t \in [a,b)$ with $f_\varepsilon(t) < +\infty$. We claim that $f_\varepsilon(b) \leq f_\varepsilon(a)$. If this inequality is not satisfied for some $\varepsilon > 0$, then the infimum of f_ε is attained at some $t_\varepsilon \in [a,b)$ and we have $f_\varepsilon(t_\varepsilon) < +\infty$. Since $d^-f_\varepsilon(t_\varepsilon) \leq -\varepsilon$, there exists $s_\varepsilon \in (t_\varepsilon, b)$ such that $f_\varepsilon(s_\varepsilon) < f_\varepsilon(t_\varepsilon)$, contradicting the minimality of t_ε and proving our claim. Taking the limit in the inequality $f_\varepsilon(b) \leq f_\varepsilon(a)$ as $\varepsilon \to 0+$ we conclude the proof. $\square$

An alternative proof of the following corollary can be found in [14, Chapter 6, Exercise 10].

Corollary 4.11 *Let $[a,b]$ be a bounded closed interval in $\mathbb{R}$ and let $f\colon [a,b] \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function with $f(a) < +\infty$. Suppose that there exists a constant $M \in \mathbb{R}$ such that $d^- f(t) \le M$ for every $t \in [a,b)$ with $f(t) < +\infty$. Then $f(t) < +\infty$ for every $t \in [a,b]$. Suppose, in addition, that for some $g \in L^1(a,b)$ we have $d^- f(t) \le g(t)$ for a.e. $t \in [a,b]$. Then*

$$f(b) - f(a) \le \int_a^b g(t)dt. \tag{4.17}$$

Proof — By Lemma 4.10 the function $f_M(t) := f(t) - Mt$ is nonincreasing. Therefore $f(t) \le f(a) + M(t-a) < +\infty$ for every $t \in [a,b]$. If $d^- f(t) \le g(t)$ for a.e. $t \in [a,b]$, then $f'_M(t) \le g(t) - M$ for a.e. $t \in [a,b]$, and the Lebesgue theorem on derivatives of monotone functions yields

$$f_M(b) - f_M(a) \le \int_a^b f'_M(t)dt \le \int_a^b (g(t) - M)dt,$$

which implies (4.17). □

Proof of Theorem 4.9 — Let us fix $t > 0$, $x \in \mathbb{R}^n$, and let y be a minimizer of (1.8). It is Lipschitz continuous by Theorem 2.1. Let us define $\gamma(s) := W(s, y(t-s))$. Then γ is lower semicontinuous on $[0,t]$ and $\gamma(0) = \varphi(y(t)) < +\infty$. Let us fix $s \in [0,t)$ with $\gamma(s) < +\infty$. Consider a sequence $h_i \to 0+$ such that

$$d^- \gamma(s) = \lim_{i\to\infty} \frac{\gamma(s+h_i) - \gamma(s)}{h_i}. \tag{4.18}$$

We can write

$$\gamma(s+h_i) - \gamma(s) = W(s+h_i, y(t-s) - h_i u_i) - W(s, y(t-s)), \tag{4.19}$$

with $u_i := -(y(t-s-h_i) - y(t-s))/h_i$. Passing to a subsequence we may assume that u_i converges in $\mathbb{R}^n$ to some vector u, whose norm is bounded by the Lipschitz constant of y. From (4.16), (4.18), and (4.19) it follows that

$$d^- \gamma(s) \le L^+(y(t-s), u). \tag{4.20}$$

Since the function L^+ is locally bounded, we conclude that there exists a constant M such that $d^- \gamma(s) \le M$ for every $s \in [0,t)$ with $\gamma(s) < +\infty$. By Corollary 4.11 this implies that $\gamma(s) < +\infty$ for every $s \in [0,t]$.

If the derivative $y'(t-s)$ exists, then $u = y'(t-s)$. Therefore (4.20) gives $d^- \gamma(s) \le L^+(y(t-s), y'(t-s))$ for a.e. $s \in [0,t]$, which, together with Proposition 4.7, yields $d^- \gamma(s) \le L(y(t-s), y'(t-s))$ for a.e. $s \in [0,t]$. By Corollary 4.11 we obtain

$$\gamma(t) \le \gamma(0) + \int_0^t L(y(t-s), y'(t-s))ds,$$

which is equivalent to

$$W(t,x) \le \varphi(y(t)) + \int_0^t L(y(s), y'(s))ds.$$

Since the right hand side is equal to $V(t,x)$, we have proved that $W(t,x) \le V(t,x)$.

The last assertion of the theorem follows now from Theorems 4.4 and 4.8. $\square$

We conclude this section with some results which connect the minimizers of (1.8) with the contingent derivatives of the value function.

Theorem 4.12 *Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function and let $\varphi: \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ with $\varphi \not\equiv +\infty$. Assume that* (**H1**) *and* (**H2**) *are satisfied. If y is a minimizer of (1.8), then*

$$D_\uparrow V(t-s, y(s))(-1, y'(s)) = D_\downarrow V(t-s, y(s))(-1, y'(s)) = -L(y(s), y'(s)), \tag{4.21}$$

$$D_\uparrow V(t-s, y(s))(1, -y'(s)) = D_\downarrow V(t-s, y(s))(1, -y'(s)) = L(y(s), y'(s)) \tag{4.22}$$

for almost all $s \in [0,t]$.

Proof — Since V is locally Lipschitz on $\mathbb{R}_+^\star \times \mathbb{R}^n$ by Theorem 4.4, the function $\gamma(s) := V(t-s, y(s))$ is locally absolutely continuous on $[0,t)$. Fix $s \in (0,t)$ such that the derivatives $\gamma'(s)$ and $y'(s)$ exist. Let us prove that

$$D_\uparrow V(t-s, y(s))(-1, y'(s)) = D_\downarrow V(t-s, y(s))(-1, y'(s)) = \gamma'(s), \tag{4.23}$$

$$D_\uparrow V(t-s, y(s))(1, -y'(s)) = D_\downarrow V(t-s, y(s))(1, -y'(s)) = -\gamma'(s). \tag{4.24}$$

Let $h_i \to 0+$ and $u_i \to y'(s)$ such that

$$D_\downarrow V(t-s, y(s))(-1, y'(s)) = \lim_{i\to\infty} \frac{V(t-s-h_i, y(s)+h_i u_i) - V(t-s, y(s))}{h_i}. \tag{4.25}$$

As V is Lipschitz near $(t-s, y(s))$, there exists a constant $M > 0$ such that

$$\begin{aligned} |V(t-s-h_i, y(s+h_i)) - V(t-s-h_i, y(s)+h_i u_i)| \le \\ \le M|y(s+h_i) - y(s) - h_i y'(s)| + M h_i |u_i - y'(s)| \end{aligned} \tag{4.26}$$

for i large enough. From (4.25) and (4.26) we obtain the second equality in (4.23). The other equalities in (4.23) and (4.24) can be obtained in the same way.

On the other hand, by minimality, for all small $h > 0$ we have

$$V(t-s-h, y(s+h)) = V(t-s, y(s)) - \int_s^{s+h} L(y(\tau), y'(\tau))d\tau. \tag{4.27}$$

Since y is Lipschitz by Theorem 2.1, the function $s \mapsto L(y(s), y'(s))$ is bounded. By the Lebesgue theorem (4.27) implies that $\gamma'(s) = -L(y(s), y'(s))$ for a.e. $s \in [0,t]$. The conclusion follows now from (4.23) and (4.24). $\square$

Theorem 4.13 *Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function and let $\varphi: \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function with $\varphi \not\equiv +\infty$. Assume that* (**H1**) *and* (**H2**) *are satisfied, and let V be the value function given by (1.8). Define the set-valued maps*

$$F(t,x) := \{u \in \mathbb{R}^n \mid D_\uparrow V(t,x)(-1,u) \leq -L(x,u)\},$$
$$G(t,x) := \{u \in \mathbb{R}^n \mid D_\downarrow V(t,x)(1,-u) \geq L(x,u)\}.$$

Given $t > 0$, $x \in \mathbb{R}^n$, and $y \in W^{1,1}(0,t;\mathbb{R}^n)$ with $y(0) = x$, the following conditions are equivalent:

(a) y is a minimizer of (1.8),

(b) $y'(s) \in F(t-s, y(s))$ for a.e. $s \in [0,t]$,

(c) $y'(s) \in G(t-s, y(s))$ for a.e. $s \in [0,t]$.

Proof — If y is a minimizer if (1.8), then $y'(s) \in F(t-s, y(s))$ and $y'(s) \in G(t-s, y(s))$ for a.e. $s \in (0,t)$ by (4.21) and (4.22).

If y solves the differential inclusion $y'(s) \in F(t-s, y(s))$ for a.e. $s \in [0,t]$, we define $\gamma(s) := V(t-s, y(s))$. Since V is locally Lipschitz on $\mathbb{R}_+^* \times \mathbb{R}^n$, the function γ is locally absolutely continuous on $[0,t)$. Using (4.23) and the definition of $F(t-s, y(s))$, we obtain that $L(y(s), y'(s)) \leq -\gamma'(s)$ for almost all $s \in [0,t]$, which implies that $L(y(s), y'(s))$ is integrable on $[0, t-\varepsilon]$ for every $\varepsilon > 0$. By integrating we obtain

$$\gamma(t-\varepsilon) + \int_0^{t-\varepsilon} L(y(s), y'(s))ds \leq \gamma(0).$$

As $\gamma(t-\varepsilon) = V(\varepsilon, y(t-\varepsilon))$ and $\gamma(0) = V(t,x)$, taking the lower limit as $\varepsilon \to 0$ and using Proposition 4.1 we get

$$\varphi(y(t)) + \int_0^t L(y(s), y'(s))ds \leq V(t,x).$$

Consequently, y is a minimizer of (1.8).

If y solves the differential inclusion $y'(s) \in G(t-s, y(s))$ for a.e. $s \in [0,t]$, we repeat the same proof, replacing (4.23) with (4.24). □

Acknowledgements. The work of Gianni Dal Maso is part of the European Research Training Network "Homogenization and Multiple Scales" under contract HPRN-2000-00109, and of the Project "Calculus of Variations", supported by SISSA and by the Italian Ministry of Education, University, and Research. Gianni Dal Maso wishes to thank the hospitality and support of the University of Paris-Dauphine for two visits in May 2000 and January 2001, during which the main results of this paper have been obtained.

The work of Hélène Frankowska is part of the European Research Training Network "Evolution Equations for Deterministic and Stochastic Control Systems", RTN-2002-12. Hélène Frankowska wishes to thank SISSA for the hospitality in the summer 2002.

References

[1] AMAR M., BELLETTINI G. & VENTURINI S. (1998) *Integral representation of functionals defined on curves of $W^{1,p}$*, Proc. Roy. Soc. Edinburgh Sect. A **128**, 193-217.

[2] AMBROSIO L., ASCENZI O. & BUTTAZZO G. (1989) *Lipschitz regularity for minimizers of integral functionals with highly discontinuous integrands*, J. Math. Anal. Appl. **142**, 301-316.

[3] AUBIN J.-P. (1993) OPTIMA AND EQUILIBRIA, Grad. Texts in Math. 140, Springer-Verlag, Berlin.

[4] AUBIN J.-P. & FRANKOWSKA H. (1990) SET-VALUED ANALYSIS, Birkhäuser, Boston.

[5] BALL J. & MIZEL V.J. (1990) *One dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation*, Arch. Rational. Mech. Anal. **90** (1985), 325-388.

[6] CANNARSA P. & FRANKOWSKA H. (1991) *Some characterizations of optimal trajectories in control theory*, SIAM J. on Control and Optimization, 29, 1322-1347.

[7] CASTAING C. & VALADIER M. (1977) CONVEX ANALYSIS AND MEASURABLE MUTIFUNCTIONS, Springer-Verlag, Berlin.

[8] CESARI L. (1983) OPTIMIZATION THEORY AND APPLICATIONS. PROBLEMS WITH ORDINARY DIFFERENTIAL EQUATIONS, Appl. Math. 17, Springer-Verlag, Berlin.

[9] CLARKE F.H. (1983) OPTIMIZATION AND NONSMOOTH ANALYSIS, Wiley-Interscience, New York.

[10] CLARKE F.H. & VINTER R.B. (1985) *Regularity properties of solutions to the basic problem in the calculus of variations*, Trans. Amer. Math. Soc. **289**, 73-98.

[11] DAL MASO G. & FRANKOWSKA (2000) *Value functions for Bolza problems with discontinuous Lagrangians and Hamilton-Jacobi inequalities*, ESAIM Control Optim. Calc. Var. **5**, 369-394.

[12] DE GIORGI E., BUTTAZZO G. & DAL MASO G. (1983) *On the lower semicontinuity of certain integral functionals*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **74**, 274-282.

[13] IOFFE A.D. (1977) *On lower semicontinuity of integral functionals*, SIAM J. Control Optim. **15**, 521-521 and 991-1000.

[14] LETTA G. (1976) TEORIA ELEMENTARE DELL'INTEGRAZIONE, Boringhieri, Torino.

[15] OLECH C. (1976) *Weak lower semicontinuity of integral functionals*, J. Optim. Theory Appl. **19**, 3-16.

[16] ROYDEN H.L. (1969) REAL ANALYSIS, Collier Macmillan, New York.

[17] TONELLI L. (1915) *Sur une méthôde directe du calcul des variations*, Rend. Circ. Mat. Palermo **39**, 223-264.

[18] TONELLI L. (1921) FONDAMENTI DI CALCOLO DELLE VARIAZIONI, Vol. 1, 2, Zanichelli, Bologna.

[19] ZIEMER W.P. (1989) WEAKLY DIFFERENTIABLE FUNCTIONS, Springer-Verlag, Berlin.