GENERALIZING THE PROPERTY OF LOCALITY: ATOMIC/COATOMIC OPERATORS AND APPLICATIONS

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Dedicated to Isaak Veniaminovich Shragin

Abstract. In this paper we strudy the so-called *atomic* and *coatomic* operators introduced in [7] and generalizing the classical notion of a local operator between ideal function spaces. In particular, we discover characteristic properties of such operators, which can serve as their new definitions. These properties are intrinsic in the sense that they are independent of a particular σ-homomorphism of the underlying σ-algebrae and are based on purely measure-theoretic notions of *memory* and *comemory* of an operator, which are also studied in details in the paper. We also prove some new results on analytic properties of atomic, coatomic and local operators. For the reader's convenience some of the known results regarding such operators that are exploited in the present paper are also provided without proofs. In the last section, we show that the study of strong periodic in law solutions to a stochastic functional differential equation can be put, under rather general assumptions, into the framework of atomic operators. This result can serve as an additional strong motivation for introducing and studying atomic and coatomic operators in their most general form.

1. Introduction

Ordinary and partial differential equations from the point of view of operator theory are just functional equations involving only two types of operators: differential operators and Nemytskiĭ (composition) operators. Defined correctly over specific function spaces (say, Sobolev spaces for differential operators and Lebesgue or other ideal function spaces for Nemytskiĭ operators), they possess a common property of *locality*. Roughly speaking, an operator is local, if an image of a function near each point is determined by the values of this function near the same point (see [11] and [20] for the precise definitions). In fact, it is well-known that many properties of, say, ordinary differential equations (see e.g. [11]) stem just from locality of operators involved.

In his paper [17] I. Shragin proposed another, extended definition of locality, where he replaced neighborhoods of points by generic measurable sets. It can easily be shown that this definition covers the cases studied in [11] and [20] related to differential equations. On the other hand, Shragin's general concept of locality can also be applied in a variety of other cases, like stochastic analysis (see e.g. [15]) generalized Orlicz spaces [17] and some others.

Date: May 15, 2005.

1991 *Mathematics Subject Classification.* 47B38, 47A67, 34K05.

The research of the first and the second authors has been supported by the Giladi and Kamea programs of the Ministry of Science of the state of Israel.

There is however a lot of physical models which cannot be reduced to ODEs or PDEs being inherently nonlocal. Many of them can still be considered in a sense "almost local" since they have many similar features. Among such models one should mention first of all retarded differential equations or in general functional-differential equations (FDEs) with deviating argument [3]. Such models, except local operators (usually just differential and Nemytskiĭ ones) involve also some other objects with similar properties, like, for instance, linear inner superposition (shift) operators. In order to include such operators together with local ones in the framework of some general theory which would explain the respective similarities, many generalizations of local operators have been introduced. Here we study, following the recent paper [7], two such extensions of the notion of a local operator in ideal function spaces. Both classes of operators we deal with include, of course, local operators and are closed with respect to compositions. The first class of operators, called *atomic*, contains also all the linear shifts. The second one, in certain sense dual to the first one, called *coatomic*, includes in particular operators of conditional expectation. The study of atomic operators, besides obvious applications to FDEs with deviating argument, was also inspired by stochastic applications. For instance, following the idea of [4] it has been shown in [7] that atomic operators arise naturally in the problem of finding periodic (in distribution) solutions to stochastic differential equations. Moreover, many problems for stochastic dynamical systems can be reduced to the study of atomic operators (see e.g. [2] and references therein).

Both classes of operators dealt in this paper are defined on the basis of the notion of the *memory* of an operator introduced in [7]. Roughly speaking, a memory is a piece of information about the preimages the operator is able to "remember" given a piece of information about images. This notion most naturally arises just from the definition of a local operator [17]. In fact, a local operator "remembers" the behaviour of a preimage near each point and "reconstructs" using this information the image "near" the same point. Most of the common properties of local, atomic and coatomic operators come from just measure-theoretic structure of their memory. In this paper, upon describing the notion of the memory, we also study some its basic generic properties. Further, we give a survey of some analytic properties of atomic and coatomic operators comparing them with those of local operators, and put a particular attention on representation results for atomic operators which can be useful in stochastic analysis. Finally, we show, generalizing some results in [4], that finding a periodic in law solution to a stochastic functional differential equation is equivalent to a fixed point problem for a certain atomic operator.

2. Notation and preliminaries

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two measure spaces, and $\Sigma_1^0 \subset \Sigma_1$, $\Sigma_2^0 \subset \Sigma_2$ be the σ-ideals of μ_1- and μ_2-nullsets respectively. We denote by $\tilde{\Sigma}_i := \Sigma_i/\Sigma_i^0$, $i = 1, 2$ the respective *measure algebrae* (see § 42 of [18]). The elements of $\tilde{\Sigma}_i$ (i.e. the equivalence classes of sets) will be denoted $\tilde{e}_i$ or $[e_i]$, $i = 1, 2$. Further on we will however frequently abuse the notation and identify the elements of the measure algebrae $\tilde{\Sigma}_i$ with the elements of the respective original σ-algebrae of sets Σ_i. Also, in the sequel all the equalities will be understood up to a nullset, i.e. in the almost everywhere sense.

A measure space (Ω, Σ, μ) will be called standard, when Ω is a Polish space, μ is a finite Borel measure and Σ is either the Borel σ-algebra or its completion with respect to μ. A map $F\colon\ \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ is called a *σ-homomorphism*, if $F(\Omega_1) = \Omega_2$, $F(\Omega_1 \backslash e) = \Omega_2 \backslash F(e)$ whenever $e \in \tilde{\Sigma}_1$ and

$$F\left(\bigsqcup_{i=1}^{\infty} e_i\right) = \bigsqcup_{i=1}^{\infty} F(e_i),$$

for any pairwise disjoint collection of $\{e_i\}_{i=1}^{\infty} \subset \tilde{\Sigma}_1$, where $\sqcup$ stands for the disjoint union. Every (Σ_2, Σ_1)-measurable map $g\colon\ \Omega_2 \to \Omega_1$ satisfying

$$\mu_2(g^{-1}(e_1)) = 0 \text{ when } \mu_1(e_1) = 0 \tag{1}$$

generates a σ-homomorphism according to the formula $F(\tilde{e}_1) := [g^{-1}(e_1)]$.

All the measure spaces we will be dealing with in the sequel are assumed to be complete, and, for the sake of simplicity, the measures will be supposed finite. Further, the notation $L^p(\Omega, \Sigma, \mu; \mathcal{X})$, where $\mathcal{X}$ is a separable Banach space, will stand, as usual, for the classical Lebesgue space of $\mathcal{X}$-valued functions measurable with respect to Σ and μ-summable with power p (if $p \in (0, +\infty)$) or μ-essentially bounded (if $p = +\infty$). These spaces are silently assumed to be equipped with their strong topologies. If $\mathcal{X}$ is a separable metric space, then $L^0(\Omega, \Sigma, \mu; \mathcal{X})$ stands for the metric space of $\mathcal{X}$-valued functions measurable with respect to Σ equipped with the topology of convergence in measure.

3. Memory and comemory of an operator

Let $X_i := L^{p_i}(\Omega_i, \Sigma_i, \mu_i; \mathcal{X}_i)$, $p_i \geq 0$, $i = 1, 2$. Consider an operator $T\colon\ X_1 \to X_2$. We recall now the following crucial concept of *memory* and the related concept of *comemory* introduced in [7].

Definition 3.1. *We call the* **memory** *of an operator $T\colon\ X_1 \to X_2$ on a set $e_2 \in \Sigma_2$ the family of all possible $\tilde{e}_1 \in \tilde{\Sigma}_1$ such that for any $x, y \in X_1$ satisfying $x\,|_{e_1} = y\,|_{e_1}$ it follows that $T(x)\,|_{e_2} = T(y)\,|_{e_2}$. In other words,*

$$\mathit{Mem}_T(\tilde{e}_2) := \left\{ \tilde{e}_1 \in \tilde{\Sigma}_1 \ :\ x\,|_{e_1} = y\,|_{e_1} \ \Rightarrow\ T(x)\,|_{e_2} = T(y)\,|_{e_2} \right\},$$

Similarly, the **comemory** *of the operator T on a set $e_1 \in \Sigma_1$ is the family*

$$\mathit{Comem}_T(\tilde{e}_1) := \left\{ \tilde{e}_2 \in \tilde{\Sigma}_2 \ :\ x\,|_{e_1} = y\,|_{e_1} \ \Rightarrow\ T(x)\,|_{e_2} = T(y)\,|_{e_2} \right\}.$$

Recall that according to our convention all the equalities in the above definition should be understood in almost everywhere sense.

It is clear from the definitions that

$$\tilde{e}_1 \in \mathrm{Mem}_T(\tilde{e}_2), \text{ if and only if } \tilde{e}_2 \in \mathrm{Comem}_T(\tilde{e}_1).$$

Example 3.1. Let $\mathcal{X} = \mathcal{X}_1 = \mathcal{X}_2$. Define a *shift* (inner superposition) operator $T_g\colon\ L^0(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}) \to L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X})$ by the formula

$$(T_g x)(\omega_2) := x(g(\omega_2)) \tag{2}$$

where $g\colon\ \Omega_2 \to \Omega_1$ is a given (Σ_2, Σ_1)-measurable function. For this operator to be well-defined on the classes of measurable functions we require

$$e_1 \in \Sigma_1,\, \mu_1(e_1) = 0 \Rightarrow \mu_2(g^{-1}(e_1)) = 0. \tag{3}$$

Then

$$\mathrm{Mem}_{T_g}(\tilde{e}_2) = \{\tilde{e}_1 \in \tilde{\Sigma}_1 \,:\, e_1 \supset g(e_2)\}.$$

Example 3.2. Let $\Omega \subset \mathbb{R}^n$ be a compact set supplied with the n-dimensional Lebesgue measure μ, Σ being the respective σ-algebra of measurable subsets of Ω. We define an operator $T\colon L^1(\Omega) \to L^1(\Omega)$ by the formula

$$(Tx)(\omega) := \int_\Omega x(s)\,ds \cdot \mathbf{1}(\omega).$$

Then

$$Mem_T(\tilde{\mathcal{E}}) = \begin{cases} \{\tilde{\Omega}\}, & \mu(\mathcal{E}) \neq 0, \\ \tilde{\Sigma}, & \mu(\mathcal{E}) = 0, \end{cases}$$

Below we enlist some obvious properties of memory and comemory.

Proposition 3.1. *For every operator $T\colon X_1 \to X_2$ and for all $e_1 \in \Sigma_1$ the following holds:*

(i) *if $\tilde{e}_2 \in Comem_T(\tilde{e}_1)$ and $e_2' \subset e_2$, $e_2' \in \Sigma_2$, then $\tilde{e}_2' \in Comem_T(\tilde{e}_1)$. In particular, $\tilde{\emptyset} \in Comem_T(\tilde{e}_1)$;*
(ii) *$Comem_T(\tilde{\Omega}_1) = \tilde{\Omega}_2$;*
(iii) *$Comem_T(\tilde{e}_1)$ is closed under at most countable unions of its elements;*
(iv) *$e_1 \subset e_1'$ implies $Comem_T(\tilde{e}_1) \subset Comem_T(\tilde{e}_1')$;*
(v) *the family $Comem_T(\tilde{e}_1)$ contains a maximum element (called "the maximum comemory") with respect to the inclusion.*

Remark. In other terms (see § 3 of [18]), the conditions (i) and (ii) mean that for all $e_1 \in \Sigma_1$ the family $\mathrm{Comem}_T(\tilde{e}_1)$ is a σ-*ideal*.

proof: Since (i)-(iv) are immediate, we concentrate only on the proof of (v). Let

$$q := \sup\{\mu_2(e_2) \,:\, \tilde{e}_2 \in \mathrm{Comem}_T(\tilde{e}_1)\}.$$

There is a sequence $\{\tilde{e}_2^\nu\} \subset \mathrm{Comem}_T(\tilde{e}_1)$ such that $\mu_2(e_2^\nu) \to q$ as $\nu \to \infty$. Define

$$E_2 := \bigcup_\nu e_2^\nu$$

and observe that $\tilde{E}_2 \in \mathrm{Comem}_T(\tilde{e}_1)$ according to (iii). Suppose now that there is a $\tilde{E}_2' \in \mathrm{Comem}_T(\tilde{e}_1)$ such that $\mu_2(E_2' \backslash E_2) \neq 0$. Then $D_2 := E_2' \cup E_2 \in \mathrm{Comem}_T(\tilde{e}_1)$, while

$$\mu_2(D_2) = \mu_2(E_2) + \mu_2(E_2' \backslash E_2) > q,$$

and this contradiction shows the statement. $\square$

Below we enlist some similar properties of *memory*.

Proposition 3.2. *For every operator $T\colon X_1 \to X_2$ and for all $e_2 \in \Sigma_2$ the following holds:*

(i) *if $\tilde{e}_1 \in Mem_T(\tilde{e}_2)$ and $e_1 \subset e_1'$, $e_1' \in \Sigma_1$, then $\tilde{e}_1' \in Mem_T(\tilde{e}_2)$. In particular, $\tilde{\Omega}_1 \in Mem_T(\tilde{e}_2)$;*
(ii) *$Mem_T(\tilde{e}_2)$ is closed under finite intersections of its elements.*

proof: (i) is straightforward. To prove (ii), assume $\tilde{e}_1 \in \mathrm{Mem}_T(\tilde{e}_2)$, $\tilde{e}_1' \in \mathrm{Mem}_T(\tilde{e}_2)$, and let $\{u, v\} \subset X_1$ be an arbitrary pair of functions satisfying

$$u\,\big|_{e_1 \cap e_1'} = v\big|_{e_1 \cap e_1'}.$$

Define now $z \in X_1$ by the formula

$$z(\omega) := \begin{cases} v(\omega), & \omega \in e_1, \\ u(\omega), & \omega \in \Omega_1 \setminus e_1. \end{cases}$$

Then $Tz\,|_{e_2} = Tu|_{e_2}$ and $Tz\,|_{e_2} = Tv|_{e_2}$, hence $Tu\,|_{e_2} = Tv|_{e_2}$, or, in other words, $\tilde{e}_1 \cap \tilde{e}_1' \in \mathrm{Mem}_T(\tilde{e}_2)$, which concludes the proof. $\square$

REMARK. Similarly to the case of comemory, the above statement asserts (see § 3 of [18]), that the for all $e_2 \in \Sigma_2$ the family $\mathrm{Mem}_T(\tilde{e}_2)$ is a *filter*.

Note that $\mathrm{Mem}_T(\tilde{e}_2)$ needs not to be closed under countable intersection of its elements and to contain a minimum element with respect to the inclusion (i.e. it is, generally speaking, not a σ-*filter*), as the example below shows.

Example 3.3. Let (Ω, Σ, μ) be as in the example 3.2. We define an operator T: $L^\infty(\Omega) \to L^\infty(\Omega)$ by the formula

$$(Tx)(\omega) := \limsup_{r \to 0^+} \frac{1}{\mu(B_r(x_0))} \int_{B_r(x_0)} x(s)\,ds \cdot 1(\omega),$$

where $B_r(x_0) \subset \mathbb{R}^n$ stands for the ball with radius $r > 0$ centered at $x_0 \in \operatorname{int}\Omega$. This operator is nonlinear, bounded, but discontinuous. One has, obviously, $\tilde{B}_r(x_0) \in \mathrm{Mem}_T(\tilde{\Omega})$ for all $r > 0$ small enough, but

$$[\{x_0\}] = \tilde{\emptyset} \notin \mathrm{Mem}_T(\tilde{\Omega}).$$

Note that the above example was only possible because the operator was taken to be discontinuous in measure. On the other hand, the following statement is valid.

Proposition 3.3. *For every continuous operator*

$$T \,:\, L^0(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1) \to L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_2)$$

and for all $e_2 \in \Sigma_2$ the following holds:

(i) *$Mem_T(\tilde{e}_2)$ is closed under at most countable intersections of its elements (and therefore, is a σ-filter);*

(ii) *$Mem_T(\tilde{e}_2)$ contains a minimum element (called the minimum memory) with respect to the inclusion.*

PROOF: To prove (i), consider a sequence $\{e_1^\nu\} \subset \Sigma_1$ with $\tilde{e}_1^\nu \in \mathrm{Mem}_T(\tilde{e}_2)$. According to (ii) of proposition 3.2, one may suppose without loss of generality that $e_1^{\nu+1} \subset e_1^\nu$ for all $\nu \in \mathbb{N}$. Let $e_1 := \cap_\nu e_1^\nu$ and assume that

$$x_1\,|_{e_1} = x_2|_{e_1}\,.$$

Construct a sequence $\{x_2^\nu\}$ in $L^0(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1)$ so that

$$x_2^\nu\,|_{e_1^\nu} = x_1\big|_{e_1^\nu} \text{ and } x_2^\nu\,|_{\Omega_1 \setminus e_1^\nu} = x_2\big|_{\Omega_1 \setminus e_1^\nu}\,.$$

One has then $x_2^\nu \to x_2$ in measure and therefore $T(x_2^\nu) \to T(x_2)$ in measure. This together with $(Tx_2^\nu)\,|_{e_2} = (Tx_1)|_{e_2}$ implies

$$(Tx_2)\,|_{e_2} = (Tx_1)|_{e_2}$$

when $\mu_2(e_2) \neq 0$. If e_2 is a nullset, the assertion is trivial.

To prove (ii) we adapt the main idea of the proof of existence of *the maximum comemory* (see proposition 3.1). Set

$$q := \inf\{\mu_1(e_1) \,:\, \tilde{e}_1 \in \mathrm{Mem}_T((\tilde{e}_2)\}$$

and choose a sequence $\{\tilde{e}_1^\nu\} \subset \mathrm{Mem}_T(\tilde{e}_2)$ such that $\mu_1(e_1^\nu) \to q$ as $\nu \to \infty$. Then

$$e_1 := \bigcap_\nu e_1^\nu \in \mathrm{Mem}_T(\tilde{e}_2)$$

due to (i). Supposing the existence of a $\tilde{e}_1' \in \mathrm{Mem}_T(\tilde{e}_2)$ with $\mu_1(e_1 \backslash e_1') \neq 0$ we would have $d_1 := e_1' \cap e_1 \in \mathrm{Mem}_T(\tilde{e}_2)$, while

$$\mu_1(d_1) = \mu_1(e_1) - \mu_1(e_1 \backslash e_1') < q,$$

leading to a contradiction. □

Example 3.4. The minimum memory, i.e. the minimum element of the set

$$\mathrm{Mem}_{T_g}(\tilde{e}_2) = \{\tilde{e}_1 \in \tilde{\Sigma}_1 \,:\, e_1 \supset g(e_2)\}$$

in the example 3.1 is given by the unique element $\tilde{E} \in \tilde{\Sigma}_2$ such that $E \supset g(e_2)$ and $\mu_2(E)$ is equal to the outer measure of the (not necessarily measurable) set $g(e_2)$.

4. Local operators

We are able to observe now that the classical definition of a *local* operator due to I. Shragin [17] can easily be translated in terms of the above introduced concepts of memory (or of comemory). In fact, with the notation $X_i = L^0(\Omega, \Sigma, \mu; \mathcal{X}_i)$, $i = 1, 2$ we get the following definition.

Definition 4.1. *An operator $T\colon X_1 \to X_2$ is called* **local***, if*

$$\tilde{e} \in \mathit{Mem}_T(\tilde{e})$$

for all $e \in \Sigma$, that is, if from $x\,|_e = y\,|_e$ for $x, y \in X_1$ follows $T(x)\,|_e = T(y)\,|_e$.

Obviously, the class of local operators is closed under compositions.

The most well-known example of a local operator is a Nemystkiĭ operator.

Example 4.1. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be separable metric spaces, $f\colon \Omega \times \mathcal{X}_1 \to \mathcal{X}_2$ be a sup-measurable function (i.e. $f(\cdot, x(\cdot))$ is μ-measurable whenever $x(\cdot)$ is μ-measurable). Then the Nemytskiĭ operator $N\colon\ L^0(\Omega, \Sigma, \mu; \mathcal{X}_1) \to L^0(\Omega, \Sigma, \mu; \mathcal{X}_2)$ (commonly known also under the name of the *superposition operator* [1]), defined by

$$(Nx)(\omega) := f(\omega, x(\omega))$$

is local. If $f\colon \Omega \times \mathcal{X}_1 \to \mathcal{X}_2$ is a Carathéodory function (i.e. $f(\omega, \cdot)$ is continuous for μ-almost every $\omega \in \Omega$ and $f(\cdot, x)$ is μ-measurable for all $x \in \mathbb{R}$), then the Nemytskiĭ operator N becomes continuous in measure (i.e. as an operator in L^0).

5. Atomic and coatomic operators

Let $X_i := L^0(\Omega_i, \Sigma_i; \mathcal{X}_i)$, $i = 1, 2$. Consider an arbitrary operator $T\colon X_1 \to X_2$. We introduce now, following [7], the notion of an atomic operator between spaces of measurable functions.

Definition 5.1. *An operator $T\colon\ X_1 \to X_2$ is called* **atomic** *with respect to a σ-homomorphism $F\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$, if*

$$F(\tilde{e}_1) \in \mathit{Comem}_T(\tilde{e}_1)$$

for all $\tilde{e}_1 \in \tilde{\Sigma}_1$.

Normally the reference to the particular σ-homomorphism will be omitted further on, if it is unnecessary.

We would like to emphasize that although the above definition has been given for a rather general case of possilbly even discontinuous operators, in applications one deals almost exclusively with continuous operators. It is this latter case that in the sequel we therefore will be mostly interested in.

We first try to study the structure of the memory of atomic operators. For this purpose for every $\tilde{e}_1 \in \tilde{\Sigma}_1$ define

$$F_T(\tilde{e}_1) := \max \mathrm{Comem}_T(\tilde{e}_1) \setminus \max \mathrm{Comem}_T(\tilde{\emptyset}).$$

The following proposition is valid.

Proposition 5.1. *For every operator $T\colon X_1 \to X_2$ one has*

(i) *$\tilde{e}_1^1 \cap \tilde{e}_1^2 = \emptyset$ implies $F_T(\tilde{e}_1^1) \cap F_T(\tilde{e}_1^2) = \emptyset$, i.e. F_T is disjointness preserving;*
(ii) *$\cup F_T(\tilde{e}_1^i) \subset F_T(\cup \tilde{e}_1^i)$ whenever $\{\tilde{e}_1^i\} \subset \tilde{\Sigma}_1$.*

PROOF: Since (ii) is immediate from the definition of maximum comemory, we concentrate on showing (i). Consider arbitrary couple of functions $\{x_1^1, x_1^2\} \subset X_1$. Define

$$x_1^{12} := x_1^1 \cdot 1_{e_1^1} + x_1^2 \cdot 1_{\Omega_1 \setminus e_1^1}.$$

Since $x_1^{12} = x_1^1$ over e_1^1, one has $Tx_1^{12} = Tx_1^1$ over $F_T(\tilde{e}_1^1)$. At the same time since $x_1^{12} = x_1^2$ over e_1^2, one has $Tx_1^{12} = Tx_1^2$ over $F_T(\tilde{e}_1^2)$. Now, if there is a $\tilde{e}_2 \neq \emptyset$, $\tilde{e}_2 \subset F_T(\tilde{e}_1^1) \cap F_T(\tilde{e}_1^2)$, then one would have $Tx_1^1 = Tx_1^2$ over e_2, which in view of arbitrariness of x_1^1 and x_1^2 means that Tx_1 is independent of $x_1 \in X_1$ over e_2, and hence $\tilde{e}_2 \subset \mathrm{Comem}_T(\tilde{\emptyset})$, which contradicts the definition of F_T. $\square$

We may claim now the following assertion which can serve as an intrinsic definition of an atomic operator in terms of the structure of its comemory.

Proposition 5.2. *Suppose that for a continuous operator $T\colon X_1 \to X_2$ one has $F_T(\tilde{\Sigma}_1) \neq \tilde{\emptyset}$. Then the following statements are equivalent:*

(i) *for every $\{\tilde{e}_1^1, \tilde{e}_1^2\} \subset \tilde{\Sigma}_1$, $\tilde{e}_1^1 \cap \tilde{e}_1^2 = \emptyset$, one has*

$$F_T(\tilde{e}_1^1 \sqcup \tilde{e}_1^2) = F_T(\tilde{e}_1^1) \sqcup F_T(\tilde{e}_1^2);$$

(ii) *$F_T(\tilde{\Sigma}_1)$ is a σ-algebra, while $F_T\colon \tilde{\Sigma}_1 \to F_T(\tilde{\Sigma}_1)$ is a σ-homomorphism;*
(iii) *the operator T is atomic.*

REMARK. If $F_T(\tilde{\Sigma}_1) = \tilde{\emptyset}$, then for all $\tilde{e}_1 \in \tilde{\Sigma}_1$ one has

$$\max \mathrm{Comem}_T(\tilde{e}_1) = \max \mathrm{Comem}_T(\tilde{\emptyset}).$$

In this case the operator T is atomic, if and only if

$$\max \mathrm{Comem}_T(\tilde{\emptyset}) = \tilde{\Omega}_2,$$

that is, when T is a constant operator (i.e. maps all X_1 just to one function).

PROOF: The implications $(i) \Leftarrow (ii) \Rightarrow (iii)$ are obvious. To show $(i) \Rightarrow (ii)$, it is enough to prove

$$\bigsqcup_{i \in \mathbb{N}} F_T(\tilde{e}_1^i) = F_T\left(\bigsqcup_{i \in \mathbb{N}} \tilde{e}_1^i\right). \tag{4}$$

But according to proposition 5.1, one clearly has

$$\bigsqcup_{i\in\mathbb{N}} F_T(\tilde{e}_1^i) \subset F_T\left(\bigsqcup_{i\in\mathbb{N}} \tilde{e}_1^i\right). \tag{5}$$

To verify the reverse inclusion, we observe that due to the validity of (i) one has

$$F_T\left(\bigsqcup_{i\in\mathbb{N}} \tilde{e}_1^i\right) = \left(\bigsqcup_{i\leq k} F_T(\tilde{e}_1^i)\right) \sqcup F_T\left(\bigsqcup_{i>k} \tilde{e}_1^i\right)$$

for every $k \in \mathbb{N}$. Hence, setting

$$\tilde{e}_2^0 := F_T\left(\bigsqcup_{i\in\mathbb{N}} \tilde{e}_1^i\right) \setminus \bigsqcup_{i\in\mathbb{N}} F_T(\tilde{e}_1^i),$$

we obtain

$$\tilde{e}_2^0 \subset F_T\left(\bigsqcup_{i>k} \tilde{e}_1^i\right)$$

for every $k \in \mathbb{N}$. This means that for each $k \in \mathbb{N}$ one has

$$\bigsqcup_{i>k} \tilde{e}_1^i \in \mathrm{Mem}_T(\tilde{e}_2^0),$$

and hence in view of continuity of T according to the proposition 3.3(i) one has

$$\emptyset = \bigcap_{k\in\mathbb{N}} \bigsqcup_{i>k} \tilde{e}_1^i \in \mathrm{Mem}_T(\tilde{e}_2^0).$$

The latter relationship means in other words that

$$\tilde{e}_2^0 \subset \max \mathrm{Comem}_T(\tilde{\emptyset})$$

which would contradict the definition of $\tilde{e}_2^0$ unless $\tilde{e}_2^0 = \emptyset$. Therefore,

$$F_T\left(\bigsqcup_{i\in\mathbb{N}} \tilde{e}_1^i\right) \setminus \bigsqcup_{i\in\mathbb{N}} F_T(\tilde{e}_1^i) = \emptyset$$

which together with (5) completes the proof of (4) and hence shows $(i) \Rightarrow (ii)$.

It remains to show $(iii) \Rightarrow (i)$. For this purpose suppose that (iii) is valid, while (i) is not, namely, that

$$\tilde{e}_2 := F_T(\tilde{e}_1^1 \sqcup \tilde{e}_1^2) \setminus \left(F_T(\tilde{e}_1^1) \sqcup F_T(\tilde{e}_1^2)\right) \neq \emptyset$$

for some $\{\tilde{e}_1^1, \tilde{e}_1^2\} \subset \tilde{\Sigma}_1$. Denote

$$\begin{array}{rcl} F'(\tilde{e}_1) & := & F(\tilde{e}_1) \setminus \max \mathrm{Comem}_T(\tilde{\emptyset}), \\ \tilde{\Omega}_1' & := & \tilde{\Omega}_1 \setminus \max \mathrm{Comem}_T(\tilde{\emptyset}). \end{array}$$

One has clearly

$$\begin{array}{rcl} F'(\tilde{e}_1^1) & \subset & F_T(\tilde{e}_1^1), \\ F'(\tilde{e}_1^2) & \subset & F_T(\tilde{e}_1^2), \\ F'(\tilde{\Omega}_1 \setminus (\tilde{e}_1^1 \sqcup \tilde{e}_1^2)) & \subset & F_T(\tilde{\Omega}_1 \setminus (\tilde{e}_1^1 \sqcup \tilde{e}_1^2)). \end{array}$$

Minding that

$$\tilde{\Omega}_1' = F'(\tilde{\Omega}_1) = F'(\tilde{e}_1^1) \sqcup F'(\tilde{e}_1^2) \sqcup F'(\tilde{\Omega}_1 \setminus (\tilde{e}_1^1 \sqcup \tilde{e}_1^2)),$$

together with the proposition 5.1, one arrives at the relationship

$$\tilde{\Omega}_1' \subset F_T(\tilde{e}_1^1) \sqcup F_T(\tilde{e}_1^2) \sqcup F_T(\tilde{\Omega}_1 \setminus (\tilde{e}_1^1 \sqcup \tilde{e}_1^2)).$$

Since $F_T(\tilde{e}_1) \subset \tilde{\Omega}_1'$ for all $\tilde{e}_1 \in \tilde{\Sigma}_1$, then the latter inclusion is in fact an equality, namely,

$$\tilde{\Omega}_1' = F_T(\tilde{e}_1^1) \sqcup F_T(\tilde{e}_1^2) \sqcup F_T(\tilde{\Omega}_1 \setminus (\tilde{e}_1^1 \sqcup \tilde{e}_1^2)).$$

Therefore, one must have $\tilde{e}_2 \subset F_T(\tilde{\Omega}_1 \setminus (\tilde{e}_1^1 \sqcup \tilde{e}_1^2))$ but on the other hand

$$F_T(\tilde{\Omega}_1 \setminus (\tilde{e}_1^1 \sqcup \tilde{e}_1^2)) \cap F_T(\tilde{e}_1^1 \sqcup \tilde{e}_1^2) = \emptyset$$

according to proposition 5.1(i), which implies $\tilde{e}_2 \not\subset F_T(\tilde{\Omega}_1 \setminus (\tilde{e}_1^1 \sqcup \tilde{e}_1^2))$ since $\tilde{e}_2 \neq \emptyset$. The latter contradiction concludes the proof. □

We introduce now another interesting class of operators also pointed out in [7]. For this purpose consider the following definition.

Definition 5.2. *An operator $T\colon X_1 \to X_2$ is called* **coatomic** *with respect to a σ-homomorphism $\Phi\colon \tilde{\Sigma}_2 \to \tilde{\Sigma}_1$, if*

$$\Phi(\tilde{e}_2) \in Mem_T(\tilde{e}_2)$$

for all $\tilde{e}_2 \in \tilde{\Sigma}_2$.

As in the case of atomic operators, we will usually omit the reference to the particular σ-homomorphism, if it is unnecessary.

The proposition below gives an intrinsic characterization of coatomic operators in terms of the structure of its comemory.

Proposition 5.3. *For every operator $T\colon X_1 \to X_2$ the following statements are equivalent:*

(i) *there is a σ-homomorphism $\Phi\colon \tilde{\Sigma}_2 \to \tilde{\Sigma}_1$, such that for every $\tilde{e}_2 \in \tilde{\Sigma}_2$ satisfying $\tilde{e}_2 \cap \max Comem_T(\tilde{\emptyset}) = \emptyset$ one has*

$$\tilde{e}_2 \subset F_T(\Phi(\tilde{e}_2));$$

(ii) *the operator T is coatomic.*

PROOF: To show $(ii) \Rightarrow (i)$, let $\tilde{e}_2 \in \tilde{\Sigma}_2$ satisfy $\tilde{e}_2 \cap \max \mathrm{Comem}_T(\tilde{\emptyset}) = \emptyset$. Then $\Phi(\tilde{e}_2) \in \mathrm{Mem}_T(\tilde{e}_2)$ means $\tilde{e}_2 \in \mathrm{Comem}_T(\Phi(\tilde{e}_2))$, and therefore

$$\tilde{e}_2 \subset \max \mathrm{Comem}_T(\Phi(\tilde{e}_2)).$$

The latter implies $\tilde{e}_2 \subset F_T(\Phi(\tilde{e}_2))$ according to the choice of $\tilde{e}_2$, thus proving that $(ii) \Rightarrow (i)$.

Assume now (i) be valid and consider an arbitrary $\tilde{e}_2 \in \tilde{\Sigma}_2$. One has then

$$\tilde{e}_2 \setminus \max \mathrm{Comem}_T(\tilde{\emptyset}) \subset F_T(\Phi(\tilde{e}_2 \setminus \max \mathrm{Comem}_T(\tilde{\emptyset}))),$$

which implies

$$\tilde{e}_2 \setminus \max \mathrm{Comem}_T(\tilde{\emptyset}) \in \mathrm{Comem}_T(\Phi(\tilde{e}_2 \setminus \max \mathrm{Comem}_T(\tilde{\emptyset}))),$$

or, in other words,

$$\Phi(\tilde{e}_2 \setminus \max \mathrm{Comem}_T(\tilde{\emptyset})) \in \mathrm{Mem}_T(\tilde{e}_2 \setminus \max \mathrm{Comem}_T(\tilde{\emptyset})).$$

But since over $\max \mathrm{Comem}_T(\tilde{\emptyset})$ the function Tx_1 is independent of $x_1 \in X_1$, then the latter relationship implies

$$\Phi(\tilde{e}_2 \setminus \max \mathrm{Comem}_T(\tilde{\emptyset})) \in \mathrm{Mem}_T(\tilde{e}_2).$$

Minding now that
$$\Phi(\tilde{e}_2 \setminus \max \mathrm{Comem}_T(\tilde{\emptyset})) \subset \Phi(\tilde{e}_2),$$
together with proposition 3.2(i), one gets
$$\Phi(\tilde{e}_2) \in \mathrm{Mem}_T(\tilde{e}_2),$$
which proves $(i) \Rightarrow (ii)$. $\square$

It is worth emphasizing that the definitions of atomic and coatomic operators provided in [7] are formally slightly different from those above. Namely, the respective notions were defined based on the requirements on the existence of certain nullset preserving σ-homomorphisms between the *original* σ-algebrae rather than between the respective measure algebrae. We recall that a σ-homomorphism F: $\Sigma_1 \to \Sigma_2$ is called nullset preserving, if $\mu_2(F(e_1)) = 0$ whenever $\mu_1(e_1) = 0$. In particular, in [7] the operator T: $X_1 \to X_2$ was called atomic with respect to the nullset preserving σ-homomorphism F: $\Sigma_1 \to \Sigma_2$, if $[F(e_1)] \in \mathrm{Comem}_T(\tilde{e}_1)$. However, this definition is obviously equivalent to the above one. In fact, every nullset-preserving σ-homomorphism F: $\Sigma_1 \to \Sigma_2$ between the original σ-algebrae generates a σ-homomorphism $\tilde{F}$ of the respective measure algebrae accorging to the formula $\tilde{F}(\tilde{e}_1) := [F(e_1)]$. On the other hand, if $\tilde{F}$: $\tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ is a σ-homomorphism between the respective measure algebrae, then the formula $F(e_1) := \pi(\tilde{F}(\tilde{e}_1))$, where π: $\tilde{\Sigma}_2 \to \Sigma_2$ is the lifting map satisfying $[\pi(\tilde{e}_2)] = \tilde{e}_2$, defines a nullset-preserving σ-homomorphism F: $\Sigma_1 \to \Sigma_2$ between the original σ-algebrae satisfying $[F(e_1)] := \tilde{F}(\tilde{e}_1)$. The lifting map π having the announced property exists due to the von Neumann-Maharam lifting theorem (theorem 4.4 in [10]) once the measure space $(\Omega_2, \Sigma_2, \mu_2)$ is complete. The same remark refers also to the definition of a coatomic operator.

It is also worth mentioning that completely analogous definitions can be given for operators defined only over some space $L^p(\Omega_1, \Sigma_1; \mathcal{X}_1)$ (rather than over the whole $L^0(\Omega_1, \Sigma_1; \mathcal{X}_1)$).

Both of the above introduced classes clearly contain that of local operators. In fact, the local operator is just the operator atomic (or coatomic) with respect to the identity σ-homomorphism. However, both classes are strictly wider than the class of local operators. For instance, every shift operator T_g (see example 3.1) is in fact atomic with respect to the σ-homomorphism generated by the function g. Moreover, since all the introduced classes are closed under compositions, then every composition of a Nemytskiĭ operator and a shift is atomic.

Observe now that a notion of a *coatomic* operator is in certain sense dual to the notion of an *atomic* operator. This assertion can be made precise by the following proposition from [7].

Proposition 5.4. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be reflexive separable Banach spaces. A linear bounded operator T: $L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$, $1 \le p, q < +\infty$, is coatomic (resp. atomic), if and only if its adjoint T': $L^{q'}(\Omega_2; \mathcal{X}_2') \to L^{p'}(\Omega_1; \mathcal{X}_1')$ is atomic (resp. coatomic).*

The above statement shows in fact that the class of coatomic operators is also strictly wider than that of local operators. In particular, it shows that this class contains all the linear *conditional expectation* operators (see e.g. chapter XI of [5]): the latter are dual to the natural inclusion operators which are clearly atomic.

We also think it worth mentioning that the classes of atomic and coatomic operators cannot coincide. In fact, it is an easy corollary of J. von Neumann-R. Sikorski theorem on representation of σ-homomorphisms (theorem 32.3 in [18]) that when $(\Omega_2, \Sigma_2, \mu_2)$ is a standard measure space, then a shift operator T_g, which is automatically atomic, can be coatomic only if the function $g\colon \Omega_2 \to \Omega_1$ is μ_2-equivalent to a bijection.

Analytic properties of atomic and coatomic operators have been studied in a detailed manner in [7]. Here we only enlist the most important of them.

Theorem 5.1 (Boundedness and continuity)**.** *Assume $\mathcal{X}_1$ and $\mathcal{X}_2$ be separable Banach spaces, while $1 \le p, q < \infty$. If a continuous in measure atomic operator $T\colon L^0(\Omega_1; \mathcal{X}_1) \to L^0(\Omega_2; \mathcal{X}_2)$ continuously maps $L^p(\Omega_1; \mathcal{X}_1)$ into $L^q(\Omega_2; \mathcal{X}_2)$, then T sends bounded sets of L^p into bounded sets of L^q, provided μ_2 is a nonatomic measure.*

If a coatomic operator $T\colon L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$, where $1 \le p, q < \infty$, maps L^p-convergent sequences into measure convergent ones, then it is continuous as an operator between these two spaces, provided that both μ_1 and μ_2 are nonatomic measures.

Moreover, it has been shown that in general no better relationships between acting, boundedness and continuity properties of atomic and coatomic operators between Lebesgue spaces hold.

The properties of noncompactness and weak continuity are very similar both to those of Nemytskiĭ operators [1] and to those of inner superposition (shift) operators [8].

Theorem 5.2 (Noncompactness)**.** *Assume $0 \le p, q \le \infty$ and $\mathcal{X}_i$, $i = 1, 2$ to be separable Banach spaces. Then a nonconstant atomic (resp. coatomic) operator $T\colon L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$ is not compact, provided μ_1 (resp. μ_2) is a nonatomic measure.*

Theorem 5.3 (Weak continuity)**.** *Let $1 \le p, q < \infty$ and $\mathcal{X}_i$, $i = 1, 2$ be separable Banach spaces. Then an atomic (resp. coatomic) operator $T\colon L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$ is weakly continuous, if and only if T is affine, i.e. $T(\cdot) - T(0)$ is a linear bounded operator, provided μ_1 (resp. μ_2) is a nonatomic measure.*

We mention also a nice convergence property of atomic operators.

Theorem 5.4 (Convergence)**.** *Assume that $0 < p < \infty$, $0 \le q \le \infty$, Σ_1 is countably generated and $\mathcal{X}_i$, $i = 1, 2$, are separable Banach spaces. Let a sequence of atomic operators $T_\nu\colon L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$, converge strongly (pointwise) to an operator T, which maps L^p-convergent sequences into measure convergent ones. Then T is atomic.*

It is worth also remarking that for local operators a much better convergence property holds. Namely, one has the following simple assertion.

Proposition 5.5. *Let $1 \le p, q < \infty$, $\mathcal{X}_i$, $i = 1, 2$, be separable Banach spaces while the local operators $T_\nu\colon L^p(\Omega; \mathcal{X}_1) \to L^q(\Omega; \mathcal{X}_2)$ converge weakly to an operator $T\colon L^p(\Omega; \mathcal{X}_1) \to L^q(\Omega; \mathcal{X}_2)$ in the sense that*

$$T_\nu u \rightharpoonup Tu \text{ weakly in } L^q(\Omega; \mathcal{X}_2)$$

for every $u \in L^p(\Omega; \mathcal{X}_1)$. Then T is a local operator.

PROOF: Consider arbitrary functions u_1, $u_2 \in L^p(\Omega;\mathcal{X}_1)$ such that $u_1(\omega) = u_2(\omega)$ for μ-a. e. $\omega \in e$. for some set $e \in \Sigma$ with $\mu(e) > 0$. Then for every $u_2' \in L^{q'}(e;\mathcal{X}_2') \subset (L^q(e;\mathcal{X}_2))'$ one has

$$\begin{aligned} 0 &= \int_\Omega < (T_\nu(u_1)(\omega) - T_\nu(u_2)(\omega)), u_2'(\omega) > \, d\mu(\omega) \\ &\to \int_\Omega < (T(u_1)(\omega) - T(u_2)(\omega)), u_2'(\omega) > \, d\mu(\omega), \end{aligned}$$

where $< \cdot, \cdot >$ stands for the pairing between $\mathcal{X}_2$ and $\mathcal{X}_2'$. This means

$$\int_\Omega < (T(u_1)(\omega) - T(u_2)(\omega)), u_2'(\omega) > \, d\mu(\omega) = 0, \tag{6}$$

and hence $T(u_1)(\omega) = T(u_2)(\omega)$ for μ-a.e. $\omega \in e$. In fact, otherwise there would exist such an $e' \subset e$ such that

$$|T(u_1)(\omega) - T(u_2)(\omega)| \geq \alpha > 0$$

for some $\alpha > 0$ and for μ-a.e. $\omega \in e'$, where $|\cdot|$ stands for the norm in $\mathcal{X}_2$. Then according to lemma 5.1 below there is a $v_2' \in L^{q'}(e;\mathcal{X}_2')$ such that

$$< (T_\nu(u_1)(\omega) - T_\nu(u_2)(\omega)), v_2'(\omega) > 0$$

for μ-a.e. $\omega \in e'$. Taking then $u_2' := 1_{e'} v_2'$, we obtain thus the contradiction with (6). $\square$

Lemma 5.1. *Let $1 \leq q < +\infty$ and $\mathcal{X}$ be a separable Banach space. If $f \in L^q(\Omega,\mu;\mathcal{X})$ is such that*

$$|f(\omega)| \geq \alpha > 0 \textit{ for } \mu - a.e.\ \omega \in e \subset \Omega,$$

then there is a $u' \in L^{q'}(\Omega,\mu;\mathcal{X}')$ such that

$$< f(\omega), u'(\omega) >> 0 \textit{ for } \mu - a.e.\ \omega \in e,$$

where $|\cdot|$ stands for the norm in $\mathcal{X}$ and $< \cdot, \cdot >$ stands for the pairing between $\mathcal{X}$ and $\mathcal{X}'$.

PROOF: By the Hahn-Banach theorem for every $x \in \mathcal{X}$ there is a $x' \in \mathcal{X}'$ such that $< x, x' >= |x|$ and $|x'|_{\mathcal{X}'} = 1$, $|\cdot|_{\mathcal{X}'}$ standing for the norm in $\mathcal{X}'$. Therefore, if $z \in L^q(\Omega,\mu;\mathcal{X})$ takes countable number of values, such that

$$|z(\omega)| \geq 2\alpha/3 > 0 \text{ for } \mu - \text{a.e. } \omega \in e \tag{7}$$

then there is a measurable $\mathcal{X}'$-valued function z' with countable number of values such that

$$< z(\omega), z'(\omega) >= |z(\omega)| \geq 2\alpha/3 > 0 \text{ for } \mu - \text{a.e. } \omega \in e,$$

while $|z(\omega)| = 1_e(\omega)$ (and hence, $z' \in L^{q'}(\Omega,\mu;\mathcal{X}')$). Now, for a given $f \in L^q(\Omega,\mu;\mathcal{X})$ we find a $z \in L^q(\Omega,\mu;\mathcal{X})$ which takes countable number of values and satisfies

$$|f(\omega) - z(\omega)| \leq \alpha/3$$

μ-a.e. in Ω (the existence of such a z follows from separability of $\mathcal{X}$). For such a z the relationship (7) holds and therefore taking $u' := z'$ one has

$$< f(\omega), u'(\omega >=< z(\omega), z'(\omega) > + < f(\omega) - z(\omega), z'(\omega) >\geq 2\alpha/3 - \alpha/3 = \alpha/3 > 0$$

for μ-a.e. $\omega \in e$. $\square$

Note that such a nice property cannot hold for general atomic operators. In fact, the following example from [9] shows that a weak limit of a sequence of shifts not necessarily is an atomic operator.

Example 5.1. Let the functions $g^\nu\colon (0,1) \to (0,1)$, $\nu \in \mathbb{N}$, be defined by the relationship

$$g^\nu(t) := \begin{cases} \nu t, & 0 \le t < 1/\nu, \\ & \dots \\ \nu t - k, & k/\nu \le t < (k+1)/\nu, \qquad k = 1, \dots, \nu - 1. \\ & \dots \\ \nu t - \nu + 1, & (\nu-1)/\nu \le t \le 1, \end{cases}$$

Then it is easy to observe that the respective inner superposition operators $T_{g^\nu}\colon L^p(0,1) \to L^p(0,1)$, $1 < p < +\infty$, converge weakly to a limit operator $T\colon L^p \to L^p$ given by the formula

$$T:\ u \mapsto \int_0^1 u(t)\,dt.$$

This operator is obviously not atomic.

Furthermore, it has been shown in [7] that unlike atomicity, in general coatomicity is not preserved even under uniform operator limits. An exception is the linear operator case, namely, the limit of a uniformly convergent sequence of linear coatomic operators between Lebesgue spaces is still coatomic.

Linear atomic and coatomic operators also posess rather particular properties as the following statement shows.

Theorem 5.5. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be reflexive separable Banach spaces, and $1 \le p, q < \infty$, while both μ_1 and μ_2 are nonatomic measures. Then for a continuous linear operator $T\colon L^p(\Omega; \mathcal{X}_1) \to L^q(\Omega; \mathcal{X}_2)$ which is either atomic or coatomic the following assertions are equivalent:*

(i) *T is Fredholm (of arbitrary index);*
(ii) *T is continuously invertible.*

PROOF: Clearly, it sufficies to show $(i) \Rightarrow (ii)$. We recall that in view of proposition 4.2 from [7], if T is atomic (resp. coatomic), then its dual $T'\colon L^{q'}(\Omega; \mathcal{X}_1') \to L^{p'}(\Omega; \mathcal{X}_2')$ is coatomic (resp. atomic). Therefore, the proof will be concluded once we show that if T is a linear continuous atomic or coatomic operator, then $\operatorname{Ker} T \ne \{0\}$ implies $\operatorname{Ker} T$ is infinite-dimensional.

For this purpose assume first that T is a linear continuous operator atomic with respect to a σ-homomorphism $F\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ and satisfying

$$\operatorname{Ker} T \ni u \ne 0.$$

Since $T 1_{e_1} u = 1_{F(e_1)} T u$, then $1_{e_1} u \in \operatorname{Ker} T$ for all $e_1 \in \tilde{\Sigma}_1$. Observing now that the set $\{1_{e_1} : e_1 \in \tilde{\Sigma}_1 \cap \{u \ne 0\}\}$ contains in view of nonatomicity of μ_1. infinitely many linearly independent elements, and hence $\operatorname{Ker} T$ is infinite-dimensional.

If T is a linear continuous operator coatomic with respect to a σ-homomorphism $\Phi\colon \tilde{\Sigma}_2 \to \tilde{\Sigma}_1$ and satisfying

$$\operatorname{Ker} T \ni u \ne 0,$$

then one makes a similar reasoning. In fact, $T1_{\Phi(e_2)}u = 1_{e_2}Tu$, then $1_{\Phi(e_2)}u \in \operatorname{Ker} T$ for all $e_2 \in \tilde{\Sigma}_2$. Denote now $\Psi(e_2) := \Phi(e_2) \cap \{u \neq 0\}$. Clearly,

$$\Psi : \tilde{\Sigma}_2 \to \tilde{\Sigma}_1 \cap \{u \neq 0\}$$

is still a σ-homomorphism. Let μ_Ψ stand for the measure over $\tilde{\Sigma}_2$ defined by $\mu_\Psi(\tilde{e}_2) := \mu_1(\Psi(\tilde{e}_2))$. Since $\mu_\Psi << \mu_2$ and μ_2 is assumed to be a nonatomic measure, then μ_Ψ is also nonatomic. Therefore for every $k \in \mathbb{N}$ one can find $n = 2^k$ disjoint nonempty sets $\{e_2^1, \dots, e_2^n\} \subset \tilde{\Sigma}_2$ such that

$$\bigsqcup_{i=1}^{n} \Psi(e_2^i) = \{u \neq 0\} \text{ and } \mu_\Psi(e_2^i) = \mu_1(\Psi(e_2^i)) = \mu_1(\{u \neq 0\})/n.$$

Since the set $\{1_{\Phi(e_2^i)}\}_{i=1}^n$ is linearly independent, then $\operatorname{Ker} T$ is again infinite-dimensional, which finishes the proof. $\square$

We remark that the equivalence of Fredholm property and continuous invertibility was proven in [6] for weighted shift operators (which are a particular case of atomic operators), but only under a rather restrictive condition on the shift.

The above theorem 5.5 has a curious corollary for nonlinear atomic operators. To formulate it, recall, that a nonlinear operator between two Banach spaces is called Fredholm, if it is everywhere continuously Fréchet differentiable, and its Fréchet derivative is a Fredholm operator. Further, we need to keep in mind the following simple statement.

Lemma 5.2. *Let Σ_1 be countably generated, $\mathcal{X}_i$, $i = 1, 2$, be separable Banach spaces and an atomic operator $T\colon L^p(\Omega; \mathcal{X}_1) \to L^q(\Omega; \mathcal{X}_2)$, where $1 \leq p, q < +\infty$, be Fréchet differentiable. Then the Fréchet derivative DT_u at every $u \in L^p(\Omega; \mathcal{X}_1)$ of the operator T is an atomic operator.*

PROOF: One has

$$DT_u(v) = \lim_{\nu \to \infty} F_\nu(v), \text{ where } F_\nu(v) := \frac{T(u + v/\nu) - T(u)}{1/\nu}.$$

Since the operators $F_\nu\colon L^p(\Omega; \mathcal{X}_1) \to L^q(\Omega; \mathcal{X}_2)$ are obviously atomic in view of the atomicity of T, then applying the theorem 5.4 suffices to finish the proof. $\square$

With the help of the above lemma we may announce now the following easy corollary of theorem 5.5.

Proposition 5.6. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be separable reflexive Banach spaces, Σ_1 be countably generated and $1 \leq p, q < \infty$, while both μ_1 and μ_2 are nonatomic measures. Then every atomic Fredholm operator $T\colon L^p(\Omega; \mathcal{X}_1)) \to L^q(\Omega; \mathcal{X}_2)$ is locally invertible in the sense for every $u \in L^p(\Omega; \mathcal{X}_1)$ there exist an open neighborhood $U \subset L^p(\Omega; \mathcal{X}_1)$ of u and $V \subset L^q(\Omega; \mathcal{X}_2)$ of Tu such that T is a diffeomorphism between U and V.*

PROOF: By lemma 5.2 the Fréchet derivative $DT_u\colon L^p(\Omega; \mathcal{X}_1) \to L^q(\Omega; \mathcal{X}_2)$ of the operator T is a linear continuous atomic operator. Since it is supposed to be Fredholm, then, by theorem 5.5 it is continuously invertible. It suffices to refer now to the classical implicit function theorem to conclude the proof of the claim. $\square$

6. Representation of atomic operators

It has been already mentioned that every composition of a Nemytskiĭ operator and a shift is atomic. In [7] it is shown that in a sense a converse is true, namely, that every atomic operator between spaces of measurable functions is a composition of a local operator and a shift, and under some additional set-theoretic assumptions (e.g. the continuum hypothesis) even of a Nemytskiĭ operator and a shift. This justifies the introduced terminology: in fact, atomic operators can be regarded as nonlinear integral operators generated by a random atomic measure.

Such a representation however does not serve any practical purposes since the respective function generating the Nemytskiĭ operator is obtained in a nonconstructive way and can be nonmeasurable, even if the operator itself is continuous in measure.

We show now an example of a continuous in measure atomic operator arising from a stochastic application, which cannot be represented as a composition of a Nemytskiĭ operator generated by a Carathéodory function, and a shift operator.

Example 6.1. Consider a probability space $(\Omega, \Sigma, \mathbb{P})$, the standard Wiener process W_t, the Wiener shift $g := \theta_{-1}\colon \Omega \to \Omega$ inducing the isomorphism of the σ-subalgebrae Σ_0 and $\Sigma_1 := g^{-1}(\Sigma_0)$. Letting $\mathcal{X} := L^2(0,1)$, define the operator $T\colon L^0(\Omega, \Sigma_1, \mathbb{P}; \mathcal{X}) \to L^0(\Omega, \Sigma_1, \mathbb{P}; \mathcal{X})$ as the stochastic integration with respect to the Wiener process

$$(Tx)(\omega) := \int_0^{(\cdot)} x(s, g(\omega)) dW_s(\omega).$$

Note that we shifted the Σ_1-measurable integrand $x(t,\omega)$ with the help of g. In this way the stochastic process $x(s, g(\omega))$ becomes Σ_0-measurable, so that the stochastic integral is well-defined. The operator T is atomic since it is a composition of the stochastic integral (which is local) and the shift T_g. However, the stochastic integral cannot be represented by a Nemytskiĭ operator generated by a Carathéodory function. Otherwise, the stochastic integral could have been, by the Riesz representation theorem, reduced to the ordinary Lebesgue-Stieltjes integral, which is impossible.

Remark 6.1. *We demonstrate in the next section that stochastic analysis provides more examples of nontrivial atomic operators. We show, for instance, that to find a periodic in law solution to a stochastic functional differential equation one needs to solve a fixed point problem for a certain atomic operator (see also* [4] *for a similar discussion in the case of linear ordinary stochastic differential equations).*

It is important to know when atomic operators can be represented as a composition of a Nemytskiĭ operator generated by a Carathéodory function, and a shift. This knowledge would help, for instance, to prove existence of invariant measures of such operators in a much easier way.

To formulate the corresponding representation result we need the following auxiliary notion from [13].

Definition 6.1. *Let $\Sigma_1 \subset \Sigma_1'$ be σ-algebrae of subsets of Ω_1. Then $\tilde{\Sigma}_1$ is said to satisfy Ω-condition with respect to $\tilde{\Sigma}_1'$ (written $\tilde{\Sigma}_1 \in \Omega(\tilde{\Sigma}_1')$), if there is an at most countable cover of Ω_1 by pairwise disjoint sets $\Omega_1 = \sqcup_j \Omega_1^j$, $\Omega_1^j \in \Sigma_1'$, such that for each $j \in \mathbb{N}$ one has*

$$\Sigma_1 \cap \Omega_1^j / \Sigma_1^0 = \Sigma_1' \cap \Omega_1^j / \Sigma_1^0$$

(recall that Σ_1^0 stands for the σ-ideal of μ_1-nullsets).

Now we can state the following result, the proof of which will be contained in the forthcoming paper [19].

Theorem 6.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be standard measure spaces and $F\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ be a σ-homomorphism. Then any continuous operator $T\colon X_1 \to X_2$ atomic with respect to F can be represented as*

$$(Tu)(x) = f(x, u(g(x))) \text{ for } \mu_2 - a.e. \; x \in \Omega_2$$

for some Carathéodory function $f\colon \Omega_2 \times \mathcal{X}_1 \to \mathcal{X}_2$, a measurable function $g\colon \Omega_2 \to \Omega_1$ satisfying (1) and every $u \in X_1$, if and only if $F(\tilde{\Sigma}_1) \in \Omega(\Sigma_2)$.

For the particular case of local operators the analogous result has been proven in [13].

In view of the above theorem it is tempting to characterize the Ω-condition in some more convenient way. This can be done for standard measure spaces. For this purpose we need to recall the following definition from [6].

Definition 6.2. *We say that a measurable function $g\colon \Omega_2 \to \Omega_1$ is said to satisfy the ω-condition, if it satisfies (1), while there exists a disjoint at most countable covering of Ω_2 by measurable sets $\Omega_2 = \sqcup_j \Omega_2^j$ such that over each Ω_2^j the function is injective and the respective inverses $\gamma_j\colon g(\Omega_2^j) \to \Omega_2^j$, $\gamma_j \circ g|_{\Omega_2^j} = id$, satisfy (1).*

Then the following assertion is valid (see [19] for the proof and the detailed discussion).

Proposition 6.1. *Let (Ω, Σ', μ) be a standard measure space with nonatomic measure and $\Sigma \subset \Sigma'$. Then $\tilde{\Sigma} \in \Omega(\tilde{\Sigma}')$, if and only if there is a function $h\colon \Omega \to \Omega$ satisfying the ω-condition, such that $\tilde{\Sigma} = h^{-1}(\tilde{\Sigma}')$. Moreover, every measurable function $h\colon \Omega \to \Omega$ satisfying (1) such that $\tilde{\Sigma} = h^{-1}(\tilde{\Sigma}')$ satisfies also the ω-condition.*

7. Atomic operators and periodic solutions of stochastic differential equations with time lags

In this section we prove that periodic in law solutions to a stochastic functional differential equation are in one-to-one correspondence with fixed points of certain atomic operators which are naturally related to the equation. We use the following notation: $C := C([-r, 0], \mathbb{R}^d)$, $x_t(s) := x(t+s)$ while $s \in [-r, 0]$ and $t \geq 0$. We are also supposed given a complete probability space $(\Omega, \Sigma, \mathbb{P})$. Throughout this section we only use one fixed probability measure $\mathbb{P}$, so that we omit the letter $\mathbb{P}$ when we describe function spaces.

We study the stochastic functional differential equation (see [12] for the detailed definitions)

$$dx(t) = H(t, x_t)dt + G(t, x_t)dW(t), \tag{8}$$

where $t > 0$, with the initial condition

$$x_0 = \varphi \in C. \tag{9}$$

Here $W(t),\ t \geq 0$ is the m-dimensional Brownian motion, and $H : [0, \infty) \times C \to \mathbb{R}^d$, $G : [0, \infty) \times C \to \mathbb{R}^{d \times m}$ are two jointly continuous, α-periodic and globally Lipschitz in the second variable functionals, i.e.

$$|H(t, y_1) - H(t, y_2)| + |G(t, y_1) - G(t, y_2)| \leq L\|y_1 - y_2\|_C$$

for all $t \in [0, \infty)$ and $y_1, y_2 \in C$, so that existence and uniqueness of solutions holds for any initial function $\varphi \in C$ (see [12]) and each solution satisfies the usual measurability property (called adaptedness) with respect to the natural filtration $\Sigma_t := \sigma\{W(u)\} : \ 0 \le u \le t$.

This gives rise to a two-parameter family of mappings

$$U_\tau^\sigma : \ L^2(\Omega, \Sigma_\sigma; C) \to L^2(\Omega, \Sigma_\tau; C), \quad \tau \ge \sigma,$$

defined by

$$U_\tau^\sigma(\varphi) := {}^\varphi x_\tau^\sigma, \quad \varphi \in L^2(\Omega, \Sigma_\sigma; C), \tag{10}$$

where ${}^\varphi x^\sigma(t)$ satisfies

$${}^\varphi x^\sigma(t) = \begin{cases} \varphi(0) + \int_\sigma^t H(u, {}^\varphi x_u^\sigma)du + \int_\sigma^t G(u, {}^\varphi x_u^\sigma)dW(u), & t > \sigma \\ \varphi(t-\sigma), & \sigma - r \le t \le \sigma. \end{cases} \tag{11}$$

Clearly,

$$U_\tau^\sigma \circ U_\sigma^0 = U_\tau^0, \quad \sigma \le \tau \tag{12}$$

(see theorem II(2.2) from [12, p. 40] for details).

Our first result in this section justifies the property of locality for the operator U_τ^σ.

Theorem 7.1. *Assume that $H : [0,\infty) \times C \to \mathbb{R}^d, \ \ G : [0,\infty) \times C \to \mathbb{R}^{d\times m}$ are jointly continuous and globally Lipschitz in the second variable functionals. Then the solution flow U_τ^σ, defined in (10)-(11), satisfies the property of locality, which in this case reads as follows: for every $\varphi, \psi \in L^2(\Omega, \Sigma_\sigma; C)$ and $e \in \Sigma_\sigma$ the equality $\varphi|_e = \psi|_e$ a. s. implies $(U_\tau^\sigma \varphi)|_e = (U_\tau^\sigma \psi)|_e$ a. s.*

PROOF: Locality of the solution flow follows from the uniqueness of solutions and a simple trick described below. We first redefine the coefficients H and G by putting

$$\tilde{H}(t, x, \omega) = H(t, x)1_e(\omega), \quad \tilde{G}(t, x, \omega) = G(t, x)1_e(\omega). \tag{13}$$

Clearly, these functions are globally Lipschitz, with the constant L independent of ω. This gives the uniqueness of solutions (see again [12]) to the equation

$$dx(t) = \tilde{H}(t, x_t)dt + \tilde{G}(t, x_t)dW(t) \ \ (t > 0). \tag{14}$$

Using the notation in (10), we put $y(t) := {}^\varphi x^\sigma(t)1_e$ so that

$$y(t) = \begin{cases} \varphi(0)1_e + \int_\sigma^t H(u, {}^\varphi x_u^\sigma)du1_e + \int_\sigma^t G(u, {}^\varphi x_u^\sigma)dW(u)1_e, & t > \sigma \\ \varphi(t-\sigma)1_e, & \sigma - r \le t \le \sigma. \end{cases}$$

Clearly,

$$\int_\sigma^t H(u, {}^\varphi x_u^\sigma)du1_e = \int_\sigma^t \tilde{H}(u, y_u)du.$$

Using locality of Itô integrals we obtain also

$$\int_\sigma^t G(u, {}^\varphi x_u^\sigma)dW(u)1_e = \int_\sigma^t G(u, {}^\varphi x_u^\sigma)1_e dW(u) = \int_\sigma^t \tilde{G}(u, y_u)dW(u).$$

This explains why $y(t)$ satisfies (14) for $t > \sigma$. In addition, $y(t)$ satisfies the initial condition $y_\sigma = \varphi 1_e$.

In a similar way, one can show that the function $z(t) := {}^{\psi}x^{\sigma}(t)1_e$ satisfies (14) for $t > \sigma$ and the initial condition $y_\sigma = \psi 1_e$. Uniqueness of solutions implies $y(\tau) = z(\tau)$ a. s. or, in other words,

$$(U^{\sigma}_{\tau}\varphi)1_e = {}^{\varphi}x^{\sigma}(t)1_e = {}^{\psi}x^{\sigma}(t)1_e = (U^{\sigma}_{\tau}\psi)1_e \text{ a.s.}$$

This completes the proof of the theorem. □

Remark 7.1. *The result just proved shows that the property of locality is common for solution flows U^{σ}_{τ} associated with stochastic functional differential equations. On the other hand, it was shown by S.-E. A. Mohammed that these solution flows can be non-Carathèodory. He calls such equations and flows singular in contrast to Carathèodory flows that are regular. A precise definition of the regular flow says that for any τ, σ the function $F_{\tau,\sigma}(\omega, x) := U^{\sigma}_{\tau}(\omega)x$ should be Carathèodory and thus generate a local operator. Still, the class of local operators are wider, and the singular flows are examples of non-Carathèodory local operators. The difference between regular and singular flows is crucial in many studies. If one, for example, wish to use the Lyapunov exponents of the equation, then one necessarily needs a Carathèodory operator U^{σ}_{τ}. Otherwise, the Lyapunov exponents are not defined, and the asymptotic behavior of solutions can be quite erratic.*

S.-E. A. Mohammed described also some important classes of regular and singular flows. Roughly speaking, one obtains a Carathèodory operator U^{σ}_{τ} and hence a regular flow if the diffusion G contains no delay. The switching from regular to singular flows typically occurs when the the diffusion G becomes delayed. For example, a singular equation can be as simple as $dx(t) = x(t-h)dW(t)$.

Now we are going to explain what α-peridiocity of H and G implies for the flow U^{σ}_{τ} let us define the canonical Brownian shift $\theta : \mathbb{R} \times \Omega \to \Omega$ on the Wiener space Ω by

$$\theta(a, \omega)(u) := \omega(a+u) - \omega(a), \ u, a \in \mathbb{R}, \ \omega \in \Omega. \tag{15}$$

This map is invertible, measure-preserving and a.s. satisfies

$$W(t, \theta(a, \omega)) - W(u, \theta(a, \omega)) = W(t+a, \omega) - W(u+a, \omega) \tag{16}$$

for every a, t, u. Moreover, it is straightforward that

$$\theta(a, \Sigma_t) = \Sigma_{t-a} \tag{17}$$

up to a $\mathbb{P}$-null set.

It is well-known (see e.g. [2, p. 5]) that if an ordinary stochastic differential equation has time-independent coefficients and satisfies some additional regularity assumptions, then its solution flow $U(t, \omega)$, defined by $(U(t, \omega)x = x(t, \omega)$, where $x(t, \omega)$ is the solution with $x(0, \omega) = x$, satisfies the so-called "cocycle property", which reads as follows:

$$U(t+s, \omega) = U(t, \theta(s, \omega)) \circ U(s, \omega) \tag{18}$$

almost surely for all t, s (see e.g. formula (1.1.1) in [2]). If coefficients are α-periodic, we obtain a discrete version of the cocycle property (18) where an arbitrary s is replaced by the period α (see also formula (26) below). The aim of what follows to generalize this result to the case of stochastic functional differential equations (8).

As in what follows we only use α-periodic functions and stochastic processes, we can simplify our previous notation by putting $g = \theta^{-1}(\alpha, \cdot)$. Clearly, the formula (15) gives

$$W(t,\omega) - W(u,\omega) = W(t+\alpha, g\omega) - W(u+\alpha, g\omega) \tag{19}$$

a.s. for all t, where $g\omega := g(\omega)$. This can be generalized as follows

$$T_g\left(\int_{t_0}^{t} \Phi(v)dW(v+\alpha)\right) = \int_{t_0}^{t} T_g(\Phi(v))dW(v) \tag{20}$$

a.s. for every adapted stochastic process $\Phi = \Phi(v,\omega)$ which can be integrated (see e.g. [16]). Here T_g acts in the random variable: $(T_g y)(u,\omega) = (T_g)(y(u,\cdot))(\omega)$.

It is also evident that the shift T_g provides an isometry:

$$T_g :\ L^2(\Omega, \Sigma_{\sigma+\alpha}; C) \to L^2(\Omega, \Sigma_\sigma; C), \quad \sigma \geq 0.$$

The next result of this section describes a generalized cocycle property for stochastic functional differential equations with periodic coefficients.

Theorem 7.2. *Assume that $H : [0,\infty) \times C \to \mathbb{R}^d$, $G : [0,\infty) \times C \to \mathbb{R}^{d\times m}$ are jointly continuous, α-periodic and globally Lipschitz in the second variable functionals. Then the solution flow U_τ^σ defined in (10)-(11) satisfies the generalized cocycle property*

$$T_g \circ U_{\tau+\alpha}^{\sigma+\alpha} \circ T_g^{-1} = U_\tau^\sigma, \quad (\tau \geq \sigma \geq 0). \tag{21}$$

PROOF: Given an arbitrary $\varphi \in L^2(\Omega, \Sigma_\sigma; C)$ we simplify the notation in (10) by setting

$$y(t) := {}^{\varphi}x_\tau^\sigma = U_\tau^\sigma \varphi. \tag{22}$$

According to (11), one has

$$y(t) = \begin{cases} \varphi(0) + \displaystyle\int_\sigma^t H(u,y_u)u + \int_\sigma^t G(u,y_u)dW(u), & t > \sigma \\ \varphi(t-\sigma), & \sigma - r \leq t \leq \sigma. \end{cases} \tag{23}$$

In addition, we let $\psi := T_g^{-1}\varphi \in L^2(\Omega, \Sigma_\tau; C)$, $z(t) := {}^{\psi}x^{\sigma+\alpha}(t)$. Now, using (10) we obtain

$$U_{\tau+\alpha}^{\sigma+\alpha}(T_g^{-1}\varphi) = U_{\tau+\alpha}^{\sigma+\alpha}\psi = {}^{\psi}x_{\tau+\alpha}^{\sigma+\alpha} = z_{\tau+\alpha}. \tag{24}$$

Again due to (11), the function $z(t)$ satisfies

$$z(t) \quad = \quad \begin{cases} \psi(0) + \displaystyle\int_{\sigma+\alpha}^t H(u,z_u)du + \int_{\sigma+\alpha}^t G(u,z_u)dW(u), \\ \hfill t > \sigma + \alpha, \\ \psi(t-\sigma-\alpha), \\ \hfill \sigma + \alpha - r \leq t \leq \sigma + \alpha. \end{cases}$$

To prove the theorem we have to verify that

$$(T_g \circ U_{\tau+\alpha}^{\sigma+\alpha})\psi = U_\tau^\sigma \varphi$$

whenever $\tau \geq \sigma \geq 0$ for every $\varphi \in L^2(\Omega, \Sigma_\sigma; C)$, or equivalently, that

$$T_g z(t+\alpha) = y(t) \tag{25}$$

for every $t \in [-r, \infty)$, where we put $t := s + \tau$. To verify the last equality we denote $\bar{z}(t) = T_g z(t+\alpha)$. Our aim is to prove that $\bar{z}(t)$ satisfies the same equation and the same initial condition as $y(t)$. Minding the uniqueness of the solution we will hence conclude the proof.

To verify the claim concerning $\bar{z}(t)$ we first observe that

$$\bar{z}(t) = T_g z(t+\alpha) = \begin{cases} \varphi(0) + T_g \int_{\sigma+\alpha}^{t+\alpha} H(u, z_u)du + T_g \int_{\sigma+\alpha}^{t+\alpha} G(u, z_u)dW(u), & t > \sigma \\ \psi(t-\sigma), & \sigma - r \leq t \leq \sigma, \end{cases}$$

because $T_g \psi = \varphi$. Therefore, it is readily seen that $\bar{z}(t)$ and $y(t)$ satisfy the same initial condition.

On the other hand, since T_g is an isometry and $H(t,x)$ is independent of ω, Lipschitz in x and α-periodic in t, we have

$$\begin{aligned} T_g \left(\int_{\sigma+\alpha}^{t+\alpha} H(u, z_u)du \right)(s) &= \textstyle\int_{\sigma+\alpha}^{t+\alpha} H(u, T_g(z(u+s)))du \\ &= \int_{\sigma}^{t} H(v+\alpha, T_g(z(v+\alpha+s)))dv \\ &= \left(\int_{\sigma}^{t} H(v, \bar{z}_v)dv \right)(s) \end{aligned}$$

for every $s \in [-r, 0]$.

A similar argument, using in addition the property (20), gives

$$\begin{aligned} T_g \left(\int_{\sigma+\alpha}^{t+\alpha} G(u, z_u)dW(u) \right)(s) &= T_g \left(\int_{\sigma}^{t} G(v+\alpha, z(v+\alpha+s))dW(v+\alpha) \right) \\ &= = \int_{\sigma}^{t} G(v, T_g(z(v+\alpha+s)))dW(v) \\ &= \left(\int_{\sigma}^{t} G(v, \bar{z}_v)dW(v) \right)(s) \end{aligned}$$

for every $s \in [-r, 0]$. This shows that $\bar{z}(t)$ satisfies the equation (23), so that $\bar{z}(t) = y(t)$ for every $t \in [-r, \infty)$. Since $t = s + \tau$ and $s \in [-r, 0]$, this completes the proof of the theorem. $\square$

Remark 7.2. *The above theorem 7.2 describes the cocycle property for both singular and regular solution flows associated with stochastic functional differential equations. Roughly speaking, if the diffusion G contains delays then the generalized cocycle property (21) is valid, while the standard cocycle property (26), generally speaking, fails.*

To verify that for regular solution flows the formula (21) gives the standard cocycle property (for the case of periodic coefficients), we assume that U_τ^σ is regular, i.e. that for every τ, σ the function $F_{\tau,\sigma}(\omega, x) := U_\tau^\sigma(\omega)x$ is Carathèodory. In this case we may use $x \in \mathbb{R}^d$. Letting $s = 0$ we get that a.s.

$$\begin{aligned} U_\tau^0(\omega)x &= (T_g \circ U_{\tau+\alpha}^\alpha(\omega) \circ T_g^{-1})x(\omega) &= U_{\tau+\alpha}^\alpha(g\omega)x \\ &= (U_{\tau+\alpha}^0(g\omega) \circ (U_\alpha^0)^{-1}(g\omega))x. \end{aligned}$$

Recalling that $\theta(\alpha, \cdot) = g^{-1}$ and multiplying by U yields the cocycle property for the periodic case (compare with (18)):

$$U_{\tau+\alpha}^0(\omega) = U_\tau^0(\theta(\alpha, \omega)) \circ U_\alpha^0(\omega) \tag{26}$$

almost surely.

We find it important to stress again that without assuming the flow $U^0_\tau(\omega)x$ to be Caratheodory in (ω, x) we would not be able to obtain the standard cocycle property (26) from its generalized version (21).. Theorem 7.2 shows how one can adjust the standard cocycle property for non-Caratheodory U. The generalized cocycle property (21) is independent of particular properties of the coefficients F and G, while the standard cocycle property depends, on the contrary, heavily on F and G (especially on G).

We conclude this section with a result showing how to obtain periodic in law solutions to (8) with the help of atomic operators. For this purpose we need some definitions.

Definition 7.1. *A solution $x(t)$ of the stochastic functional differential equation (8) with α-periodic coefficients is called a strong periodic in law solution with the period α if*

$$x(t+\alpha, \omega) = x(t, \theta(\alpha, \omega)) \ \text{a. s.} \tag{27}$$

for all $t \in [-r, \infty)$. Here θ is the Brownian shift from (15).

Roughly speaking, this formula ensures that each deterministic characteristic of the stochastic process $x(t)$ (like expectation, distributions etc.) will be α-periodical. On the other hand, it is unrealistic to find a periodic solution in a proper sense (i.e. without the Brownian shift θ), as the Brownian motion $W(t)$ has periodic in law increments, only.

Definition 7.2. *The monodromy operator associated with the stochastic functional differential equation (8) with periodic coefficients is defined by*

$$T = T_g \circ U^0_\alpha, \tag{28}$$

where $g = \theta^{-1}(\alpha, \cdot)$.

Evidently, $T : L^2(\Omega, \Sigma_0; C) \to L^2(\Omega, \Sigma_0; C)$ as $T_g : L^2(\Omega, \Sigma_\alpha; C) \to L^2(\Omega, \Sigma_0; C)$. From discussions in the previous sections it immediately follows that T is atomic.

In the theorem below we will also use a (a bit simplified) notation from (11), namely, we set

$${}^\varphi x^0(t) := {}^\varphi x(t) \quad \text{and} \quad {}^\varphi x^0_t := {}^\varphi x_t,$$

where φ is the initial function (at $\sigma = 0$) for the solution ${}^\varphi x(t)$.

Theorem 7.3. *Assume that $H : [0, \infty) \times C \to \mathbb{R}^d$, $G : [0, \infty) \times C \to \mathbb{R}^{d \times m}$ are jointly continuous, globally Lipschitz in the second variable and α-periodic in t functionals. Then the following statements are equivalent:*

1) *${}^\varphi x(t)$ is a strong α-periodic in law solution to (8);*
2) *$\varphi \in L^2(\Omega, \Sigma_0; C)$ is a fixed point of the monodromy operator T associated with (8).*

PROOF: We first observe that (27) can be rewritten as follows:

$$x_{t+\alpha}(\omega) = x_t(\theta(\alpha, \omega)) \text{ a. s.} \tag{29}$$

for all $t \in [0, \infty)$. Assume first that ${}^\varphi x(t)$ is a strong α-periodic in law solution. Setting $t = 0$ and using $g = \theta^{-1}(\alpha, \cdot)$ we obtain $T_g({}^\varphi x_\alpha) = {}^\varphi x_0 = \varphi$ so that

$$T\varphi = (T_g \circ U^0_\alpha)\varphi = T_g({}^\varphi x_\alpha) = \varphi.$$

Conversely, if $\varphi \in L^2(\Omega, \Sigma_0; C)$ is a fixed point of the monodromy operator T, then using the generalized cocycle property (21) we obtain

$$\begin{array}{rclcl} T_g({}^{\varphi}x_{t+\alpha}) & = & (T_g \circ U^0_{t+\alpha})\varphi & = & (T_g \circ U^{\alpha}_{t+\alpha} \circ U^0_{\alpha})\varphi \\ & = & (T_g \circ U^{\alpha}_{t+\alpha} \circ T_g^{-1}) \circ (T_g \circ U^0_{\alpha})\varphi & = & U^0_t T\varphi \\ & = & U^0_t \varphi & = & {}^{\varphi}x_t \end{array}$$

almost surely for every $t \in [0, \infty)$. Hence the solution ${}^{\varphi}x_t$ is α-periodic in law. $\square$

Remark 7.3. *This result can easily be adjusted to the case when one replaces the standard Brownian shift in (15) by another measure preserving shift θ defined on Ω (or its extension) and satisfying (17) for some σ-algebras Σ_t, not necessarily generated by $W(t)$. In addition, the Brownian motion $W(t)$ should remain Brownian motion on the extended probability space and again satisfy (16). Such a situation occurs, for instance, when Σ_t is generated by a pair $(W(t), V(t))$, where $V(t)$ is another stochastic process, which is independent of $W(t)$.*

In the case of non-Brownian shift, a solution $x(t)$ to (8) satisfying (27) with the new θ is called a weak α-periodic in law solution. The corresponding monodromy operator T can be called the weak monodromy operator associated with the shift θ, while its fixed points will be then referred to as weak fixed points. Theorem 7.3 will then read as follows: ${}^{\varphi}x(t)$ is a weak α-periodic in law solution to (8) if and only if φ is a weak fixed point of the corresponding monodromy operator T (which of course will be atomic again).

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