CONNECTING MEASURES BY MEANS OF BRANCHED TRANSPORTATION NETWORKS AT FINITE COST

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Abstract. The paper studies the couples of finite Borel measures φ_0 and φ_1 with compact supports in $\mathbb{R}^n$ which can be transported to each other at a finite W^α cost, where

$$W^\alpha(\varphi_0, \varphi_1) := \inf\{\mathbb{M}^\alpha(T) \,:\, \partial T = \varphi_0 - \varphi_1\}, \qquad \alpha \in [0,1],$$

the infimum being taken over real normal currents of finite mass, and $\mathbb{M}^\alpha(T)$ standing for the α-mass of T. Besides the class of α-irrigable measures (i.e. measures which can be transported to a Dirac measure with the appropriate total mass at a finite W^α cost), two other important classes of measures are studied, which are called in the paper *purely α-non-irrigable* and *marginally α-non-irrigable*, and are in a certain sense complementary to each other. For instance, purely α-non-irrigable and Ahlfors-regular measures are, roughly speaking, those having sufficiently high dimension. One shows that for φ_0 to be transported to φ_1 at finite W^α cost their naturally defined purely α-non-irrigable parts have to coincide.

1. Introduction

If φ_0 and φ_1 are finite nonnegative Borel measures with compact supports in $\mathbb{R}^n$, then fixed $\alpha \in [0,1]$ one defines the α-cost of transporting measure φ_0 to the measure φ_1 by the formula

$$W^\alpha(\varphi_0, \varphi_1) := \inf\{\mathbb{M}^\alpha(T) \,:\, \partial T = \varphi_0 - \varphi_1\},$$

where the infimum is taken over all real flat chains of finite mass, and $\mathbb{M}^\alpha(T)$ stands for the α-mass of the flat chain T. Recently there is a lot of interest to the study of both W^α costs and the optimal flat chains T transporting measure φ_0 to φ_1 and providing such a cost (i.e. providing the minimum in the above relationship). Such optimization problems arise quite naturally in modeling tree-like structures. For instance, in discrete settings the respective costs and their minimizers were introduced and studied for communication networks in [9], for pipelines [5] and drainage networks in [10]. They are quite natural in fluid mechanics (and hence also in the traffic flow models based on the latter) when modeling the flow of liquids in tubes subject to Poiseuille's law which implies the increase of resistance as the tube becomes thinner [6, 11, 2, 3, 4, 7, 15]. In continuous settings such models were introduced in [18] and in a different though equivalent formulations in [11] and in [2] (the equivalence of various formulations has been proven in [4]). For the study of the regularity properties of flat chains T optimal with respect to W^α cost the interested reader is referred to [11, 19, 3, 4, 7, 15].

Date: December 20, 2007.

1991 *Mathematics Subject Classification.* 49Q10 (primary); 49Q15, 28A75 (secondary).

A lot of attention has been dedicated recently to the structure of the class of measures with the topology introduced by the above cost. In fact, in the case $\alpha = 1$ the above formula gives the classical Kantorovich-Rubinstein distance between the measures, while for $\alpha \in (1-1/n, 1]$, it is a distance topologically (but not metrically) equivalent to the Kantorovich-Rubinstein distance [12]. When $\alpha \leq 1-1/n$, then W^α has all the properties of the distance except that it can take infinite values, and it gives a stronger topology than that of the Kantorovich-Rubinstein distance. In this case an interesting question is which couples of measures can be transported to each other at a finite W^α cost. In particular, many papers were dedicated to the study of a particular case of this problem, namely, characterizing probability measures φ which can be transported to (or from) a point measure δ at a finite W^α cost (it is customary to call such measures *α-irrigable*) [11, 7]. In this paper we go a bit further addressing the general question. Namely, we point out, besides the class of α-irrigable measures, also two other important classes of measures, which we call *purely α-non-irrigable* and *marginally α-non-irrigable*, which are in a natural sense complementary to each other. Sharpening the results of [11, 7], we will see that purely α-non-irrigable measures can be characterized, similarly to α-irrigable measures, in terms of the dimension of a measure (Proposition 4.16). We emphasize however that we need to use here a notion of dimension different from the ones used in [7]. For instance, purely α-non-irrigable and Ahlfors-regular measures are, roughly speaking, those having sufficiently high dimension (Proposition 4.17). We will show further that for a purely α-non-irrigable measure φ_0 the inequality $W^\alpha(\varphi_0, \varphi_1) < +\infty$ holds only when $\varphi_0 = \varphi_1$ (Theorem 5.4), and for generic measures φ_0, φ_1 in order to be transported to each other at finite W^α cost their naturally defined purely α-non-irrigable parts have to coincide (Theorem 5.8).

2. Notation and preliminaries

If φ and ψ are signed Borel measure over a metric space X, will say that $\varphi \leq \psi$, if $\varphi(e) \leq \psi(e)$ for all Borel sets $e \subset X$. It is customary to call the measure φ positive, if $\varphi \geq 0$ and $\varphi \neq 0$. Further, we set

$$\begin{aligned}(\phi \wedge \psi)(e) &:= \inf\{\phi(e') + \psi(e \setminus e') \colon e' \subset e, e' \in \mathscr{B}(X)\},\\ (\phi \vee \psi)(e) &:= \sup\{\phi(e') + \psi(e \setminus e') \colon e' \subset e, e' \in \mathscr{B}(X)\}\end{aligned}$$

whenever $e \subset X$ is a Borel set and $\mathscr{B}(X)$ stands for the Borel σ-algebra of X. It is easy to show (see [1, Chapter 1] for details), that $\phi\wedge\psi$ (resp. $\phi\vee\psi$) is the maximum signed Borel measure μ over X satisfying $\mu(e) \leq \psi(e) \wedge \phi(e)$ for all Borel $e \subset X$ (resp. minimum signed Borel measure μ over X satisfying $\mu(e) \geq \psi(e) \vee \phi(e)$ for all Borel $e \subset X$).

If φ is a signed Borel measure, we denote by $\varphi^\pm$ its positive and negative parts respectively, namely,

$$\varphi^+ := \varphi \vee 0, \qquad \varphi^- := -(\varphi \wedge 0) = (-\varphi)^+.$$

We have then

$$\begin{aligned}(\varphi \wedge \psi)(e) &= \varphi(e) + \inf\{\psi(e') - \varphi(e') \colon e' \subset e, e' \in \mathscr{B}(X)\}\\ &= \varphi(e) - \sup\{\varphi(e') - \psi(e') \colon e' \subset e, e' \in \mathscr{B}(X)\}\\ &= \varphi(e) - ((\varphi - \psi) \vee 0)(e),\end{aligned}$$

which implies

$$(\varphi - \psi)^+ = \varphi - \varphi \wedge \psi, \tag{2.1}$$

and therefore also $(\varphi-\psi)^- = \psi - \varphi\wedge\psi$. By a similar calculation it is immediate to prove that

$$\tilde{\varphi}\wedge(\varphi-\psi) = \varphi\wedge(\tilde{\varphi}+\psi) - \psi, \tag{2.2}$$

where $\tilde{\varphi}$, φ and ψ are signed Borel measures.

If $\mathcal{A}$ is a bounded sequence of signed Borel measures on X, then we denote by $\sup\{\varphi : \varphi\in\mathcal{A}\}$ the least Borel measure greater than all the measures from $\mathcal{A}$, so that, in particular, $\varphi\vee\psi = \sup\{\varphi,\psi\}$.

For $x\in X$ we denote by δ_x the Dirac measure (of unit mass) concentrated in x. We will also often denote by δ a finite positive atomic measure of not necessarily unit mass concentrated on a single point (i.e. $\delta = M\delta_x$ for some $M>0$ and $x\in X$), whereas the total mass of this measure will be always clear from the context and the support is either clear from the context or not important.

Finally, for a number $p\in[1,+\infty]$ we denote by p' its conjugate defined by $1/p+1/p'=1$.

3. One-dimensional flat chains and transports

Throughout the paper we will extensively use the notion of currents and flat chains, for which the interested reader is referred to [13, 16]. The terminology and notation we use will be essentially borrowed from these references. Here we only recall rather briefly some basic facts we will use in the sequel. In what follows we will be always dealing with real one-dimensional currents with compact supports in $\mathbb{R}^n$. Real k-dimensional currents are, roughly speaking, dual objects to smooth differential k-forms. If T is a real current, for every open $U\subset\mathbb{R}^n$ we set

$$\mu_T(U) := \sup\{T(\omega) : \operatorname{supp}\omega\subset U, \|\omega\|_{L^\infty}\le 1\}.$$

We set also $\mathbb{M}(T) := \mu_T(\mathbb{R}^n)$ and call this quantity the *mass* of T. It is well known that when $\mathbb{M}(T)<+\infty$, then μ_T defines a finite Radon measure and T is representable by integration as $T=\tau_T\wedge\mu_T$ for some unit simple k-vector field τ_T, in the sense that

$$T(\omega) = \int_{\mathbb{R}^n}\langle\tau_T(x),\omega(x)\rangle\, d\mu_T(x)$$

for every regular differential k-form ω. We say that T is a *normal current*, if $\mathbb{M}(T)<+\infty$ and $\mathbb{M}(\partial T)<+\infty$. When μ_T is concentrated in a countably $(\mathcal{H}^k,k)$-rectifiable set $\Sigma\subset\mathbb{R}^n$, then one automatically has $\mu_T = \theta\mathcal{H}^k\llcorner\Sigma$ for some $\theta\in L^1(\mathcal{H}^k\llcorner\Sigma)$ (called *multiplicity* or *density* of T). In this case the current T is called *rectifiable*, and for a fixed $\alpha\in[0,1]$ one defines its α-mass by the formula

$$\mathbb{M}^\alpha(T) := \int_\Sigma |\theta|^\alpha\, d\mathcal{H}^k,$$

so that $\mathbb{M}^1(T)=\mathbb{M}(T)$. For general normal currents, and even for a larger class of currents called *flat chains* [13, 16], one defines $\mathbb{M}^\alpha$ by a standard relaxation procedure with respect to the *flat norm topology* of currents.

We will need to introduce the following basic constructions from [14]. First notice that one can introduce a natural order relation on currents of finite mass by defining $S\le T$, where T and S are k-dimensional currents, whenever

$$\mathbb{M}(T-S)+\mathbb{M}(S)\le\mathbb{M}(T)$$

(in this case one says that S is a *subcurrent* of T). It is easy to show that the above relation is in fact a partial order. Further, for flat chains of finite mass (in particular, for normal currents) one has by lemma 3.7 from [14] that $S \leq T$, then $\mathbb{M}^\alpha(S) \leq \mathbb{M}^\alpha(T)$ for all $\alpha \in [0,1]$. Finally, the current is called acyclic, if it has no subcurrent C with $\partial C = 0$.

We will also need the representation of real one-dimensional normal currents in terms of measures over a set of Lipschitz paths. Two Lipschitz-continuous paths $\hat\theta_1$, $\hat\theta_2$: $[0,1] \to \mathbb{R}^n$ will be considered equivalent, if there is a continuous surjective increasing function (usually called "reparameterization") ϕ: $[0,1] \to [0,1]$ such that $\hat\theta_1(t) = \hat\theta_2(\phi(t))$ for all $t \in [0,1]$. Let then Θ stand for the set of equivalence classes of Lipschitz-continuous paths. In this way each $\theta \in \Theta$ can be clearly identified with some directed rectifiable curve. In the sequel we will frequently slightly abuse the language, identifying the elements of Θ (i.e. directed rectifiable curves) with their parameterizations (i.e. Lipschitz-continuous paths parameterizing such curves), and even with the respective traces, when it cannot lead to a confusion. We consider the set Θ to be equipped with the distance

(3.1)

$$d_\Theta(\theta_1, \theta_2) := \inf\left\{ \max_{t\in[0,1]} |\hat\theta_1(t) - \hat\theta_2(t)| \,:\, \hat\theta_i \text{ parameterization of } \theta_i, i = 1,2 \right\},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. To each $\theta \in \Theta$ we associate the integral one-dimensional current $[\![\theta]\!]$ defined by the formula

$$[\![\theta]\!](\omega) := \int_0^1 \langle \dot\theta(t), \omega(\theta(t))\rangle\, dt$$

(note that the latter integral does not depend on the parameterization of θ so it is well defined on equivalence classes $\theta \in \Theta$). Finally, we will call *transport* every finite Borel measure η on Θ, and we will denote

$$\eta(i) := (t_i)_\# \eta, \quad t_i(\theta) := \theta(i), \qquad i = 0,1.$$

We will need the following representation theorem for currents in terms of transports, which summarizes theorems 6.2 and 6.3 from [14].

Theorem 3.1. *Let η be a transport satisfying*

$$\int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta) < +\infty.$$

Then there exists a normal one-dimensional current $T = T_\eta$ on $\mathbb{R}^n$ defined by the formula

$$T_\eta(\omega) := \int_\Theta [\![\theta]\!](\omega)\, d\eta(\theta), \tag{3.2}$$

where ω is an arbitrary smooth on 1-form. Moreover,

$$\partial T = \eta(1) - \eta(0).$$

In particular, if $\eta(1) \wedge \eta(0) = 0$, then

$$(\partial T)^+ = \eta(1), \qquad (\partial T)^- = \eta(0).$$

Furthermore, for all Borel sets $e \subset \mathbb{R}^n$ one has

$$\mu_T(e) \leq \int_\Theta \mathbb{M}([\![\theta]\!] \llcorner e)\, d\eta(\theta). \tag{3.3}$$

Vice versa, given a one-dimensional acyclic real normal current T with compact support, there exists a Borel measure η over Θ such that $T = T_\eta$ as defined by (3.2) *and*

$$\mathbb{M}(T) = \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta), \tag{3.4}$$

and hence in particular,

$$\mu_T(e) = \int_\Theta \mathbb{M}([\![\theta]\!]\llcorner e)\, d\eta(\theta)$$

for all Borel sets $e \subset \mathbb{R}^n$. Moreover, one can choose an η so that

$$\eta(1) = (\partial T)^+, \quad \eta(0) = (\partial T)^-, \tag{3.5}$$

where $(\partial T)^\pm$ are the positive and the negative part of the measure ∂T respectively, while η-a.e. $\theta \in \Theta$ is an arc.

We will need in the sequel the following simple lemmata.

Lemma 3.2. *Let η be a transport satisfying* (3.4) *for a one-dimensional real normal current $T = T_\eta$ with compact support, and let $\tilde\eta$ be another transport such that $\tilde\eta \leq \eta$. Then for $\tilde T := T_{\tilde\eta}$ one has $\tilde T \leq T$ and $\mathbb{M}(\tilde T) = \int_\Theta \mathbb{M}([\![\theta]\!])\, d\tilde\eta(\theta)$.*

Proof. Let $\eta' := \eta - \tilde\eta$ and $T' := T_{\eta'}$. One has by first part of Theorem 3.1 that

$$\mathbb{M}(\tilde T) \leq \int_\Theta \mathbb{M}([\![\theta]\!])\, d\tilde\eta(\theta), \qquad \mathbb{M}(T') \leq \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta'(\theta),$$

and therefore, minding that $T = \tilde T + T'$, we get

$$\begin{aligned}\mathbb{M}(T) \leq \mathbb{M}(\tilde T) + \mathbb{M}(T') &\leq \int_\Theta \mathbb{M}([\![\theta]\!])\, d\tilde\eta(\theta) + \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta'(\theta)\\ &= \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta) = \mathbb{M}(T),\end{aligned}$$

which implies that all the above inequalities are actually equalities and hence the thesis follows. □

Lemma 3.3. *Let η be a transport and $\varphi \leq \eta(0)$ (resp. $\varphi \leq \eta(1)$). Then there exists a transport $\tilde\eta \leq \eta$ such that $\tilde\eta(0) = \varphi$ (resp. $\tilde\eta(1) = \varphi$).*

Proof. It is enough to consider the case $\varphi \leq \eta(0)$, since the remaining case is completely symmetric. Disintegrating then η with respect to the partition of Θ given by the map $t_0 \colon \Theta \to \mathbb{R}^n$, we get $\eta = \eta(0) \otimes \eta_x$, where $x \in \mathbb{R}^n \mapsto \eta_x$ is a Borel function, the values of which are probability measures concentrated on $t_0^{-1}(x)$ for each $x \in \mathbb{R}^n$. Therefore, the definition $\tilde\eta := \varphi \otimes \eta_x$ gives the desired construction. □

We will also prove the following lemma regarding the estimate of the α-mass of a flat chain in terms of α-masses of its slices. For the slices we use the notation from [8, § 4.2.1, 4.3.1]. Namely, if $u \colon \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz-continuous function, we denote by $\langle T, u, r\rangle$ the slice of the k-dimensional normal current T in the level set $\{x \in \mathbb{R}^n : u(x) = r\}$, which are $(k-1)$-dimensional currents defined for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$, where $\mathcal{L}^1$ stands for the one-dimensional Lebesgue measure.

Lemma 3.4. *For every k-dimensional flat chain of finite mass one has*

$$\int_{\mathbb{R}} \mathbb{M}^\alpha(\langle T, u, r\rangle)\, d\mathcal{L}^1(r) \leq \mathrm{Lip}(u)\mathbb{M}^\alpha(T),$$

where $\mathrm{Lip}(u)$ *stands for the Lipschitz constant of* $u\colon \mathbb{R}^n \to \mathbb{R}$.

Proof. For the case $\alpha = 1$ the claim is given by [8, § 4.2.1]. In the case $\alpha \in [0, 1)$, it is enough to consider the case $\mathbb{M}^\alpha(T) < +\infty$, which implies that T is rectifiable according to rectifiability theorem due to White [17, theorem 8.1], i.e. $T = \theta[\![\Sigma]\!]$, where $\Sigma \subset \mathbb{R}^n$ is some $(\mathcal{H}^k, k)$-rectifiable set, and $\theta \in L^1(\Sigma; \mathcal{H}^k)$ is a positive multiplicity. By [8, § 4.3.8], one gets for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$ the relationship

$$\langle \theta^\alpha [\![\Sigma]\!], u, r\rangle = \theta^\alpha [\![\Sigma \cap u^{-1}(r)]\!]$$

with the appropriate orientation of the latter, and therefore,

$$\mathbb{M}^\alpha(\langle \theta[\![\Sigma]\!], u, r\rangle) = \mathbb{M}(\langle \theta^\alpha [\![\Sigma]\!], u, r\rangle).$$

This implies

$$\begin{aligned}\int_{\mathbb{R}} \mathbb{M}^\alpha(\langle T, u, r\rangle)\, d\mathcal{L}^1(r) = \int_{\mathbb{R}} \mathbb{M}(\langle \theta^\alpha [\![\Sigma]\!], u, r\rangle)\, d\mathcal{L}^1(r) &\leq \mathrm{Lip}(u)\mathbb{M}(\theta^\alpha [\![\Sigma]\!]) \\ &= \mathrm{Lip}(u)\mathbb{M}^\alpha(T),\end{aligned}$$

concluding the proof. □

4. Irrigability properties of measures

We introduce now the following notions.

Definition 4.1. *The finite nonnegative Borel measure φ with compact support in $\mathbb{R}^n$ is called*

- (i) α-irrigable, *if* $W^\alpha(\varphi, \delta) < +\infty$,
- (ii) purely α-non-irrigable, *if every positive Borel measure $\varphi' \leq \varphi$ is not α-irrigable,*
- (iii) marginally α-non-irrigable, *if it is not α-irrigable but there is no purely α-non-irrigable positive measure $\varphi' \leq \varphi$.*

Remark 4.2. The finite nonnegative Borel measure φ is α-irrigable, if and only if $W^\alpha(\varphi, \psi) < +\infty$ for some (and hence for all) atomic measures ψ concentrated on a finite set of points.

Remark 4.3. If the finite nonnegative Borel measure φ is α-irrigable, then it is β-irrigable for all $\beta \in [\alpha, 1]$. In fact, α-irrigability is equivalent to the existence of a one-dimensional real normal current T satisfying $\partial T = \varphi' - \delta$ and $\mathbb{M}^\alpha(T) < +\infty$, which can also be considered acyclic since every normal current has an acyclic subcurrent with the same boundary (proposition 3.12 from [14]). Minding that for acyclic normal currents

$$\mathbb{M}^\beta(T) \leq \frac{1}{2^{\beta-\alpha}} \mathbb{M}^\alpha(T)\mathbb{M}(\partial T)^{\beta-\alpha}$$

for all $\beta \in [\alpha, 1]$ according to theorem 7.4 from [14], we get $\mathbb{M}^\beta(T) < +\infty$ which means β-irrigability of φ.

Remark 4.4. For each $\alpha \in [0,1]$ one has that every finite nonnegative Borel measure φ concentrated on a countable set of points is either α-irrigable or marginally α-non-irrigable. In fact, if φ is not α-irrigable, and $\varphi' \leq \varphi$ is a positive measure, then φ' is also concentrated on an at most countable set of points, and hence any measure $\tilde{\varphi} \leq \varphi'$ concentrated on a finite set of points is α-irrigable, which means that φ' is not purely α-non-irrigable.

Lemma 4.5. *If φ is α-irrigable, then so is φ', whenever $\varphi' \leq \varphi$. Therefore, if φ' is not α-irrigable, then so if φ, whenever $\varphi' \leq \varphi$.*

Proof. If φ is α-irrigable, then there exists a one-dimensional real normal current T satisfying $\partial T = \varphi' - \delta$ and $\mathbb{M}^\alpha(T) < +\infty$. Since every normal current has an acyclic subcurrent with the same boundary (proposition 3.12 from [14]), we may consider without loss of generality T to be acyclic. Letting then η to be a transport satisfying conditions of the second part of Theorem 3.1, we get $\eta(0) = \varphi$. If $\varphi' \leq \varphi$, then by Lemma 3.3 there exists a transport $\eta' \leq \eta$ such that $\eta'(0) = \varphi'$, while by Lemma 3.2 one has $T_{\eta'} \leq T$, hence $\mathbb{M}^\alpha(T_{\eta'}) \leq \mathbb{M}^\alpha(T) < +\infty$, which concludes the proof. □

It is worth noticing that the converse assertions to that of the above Lemma 4.5 are not true. In fact, if $\varphi = \mathcal{L}^n \llcorner B_1(0) + \delta_0$, then φ is not α-irrigable for $\alpha < 1/n'$, since so is the Lebesgue measure $\mathcal{L}^n \llcorner B_1(0) \leq \varphi$ (see, e.g. [7], or Proposition 4.16 below). On the other hand, $\delta_0 \leq \varphi$ and is clearly α-irrigable for all $\alpha \in [0,1]$. This example is also interesting because it provides a measure which belongs to neither of the three classes pointed out (because $\mathcal{L}^n \llcorner B_1(0)$ is purely α-non-irrigable for $\alpha < 1/n'$ by Proposition 4.16).

Note that if φ is a marginally α-non-irrigable measure, then there always exists a positive α-irrigable measure $\varphi' \leq \varphi$. The following examples show the existence of marginally α-non-irrigable measures.

Example 4.6. Let $Q_l \subset \mathbb{R}^k \subset \mathbb{R}^n$ stand for the k-dimensional cube of sidelength l centered at the origin, where we consider $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. If we divide each side of the unit cube Q_1 in N equal intervals, which gives the division of the cube in N^k equal sub-cubes, and let $\{x_i\}_{i=1}^{N^k}$ be the set of centers of these sub-cubes, then defining the probability measure

$$\varphi_N := \frac{1}{N^k} \sum_{i=1}^{N^k} \delta_{x_i},$$

we get

$$W^\alpha(\varphi_N, \delta_0) \geq \left(\frac{1}{N^k}\right)^\alpha \frac{1}{2N} N^k = \frac{1}{2} N^{k-1-k\alpha}.$$

In fact, if $T = \sum_j \theta_j [\![\Sigma_j]\!]$ is a real polyhedral chain (i.e. a "weighted directed graph" with edges $[\![\Sigma_j]\!]$ and weights θ_j) such that

$$W^\alpha(\varphi_N, \delta_0) = \mathbb{M}^\alpha(T),$$

then clearly $\theta_j \geq 1/N^k$, while to estimate $\sum_j \mathcal{H}^1(\Sigma_j)$ from below it is enough to notice that the paths followed by each of the N^k single masses for a distance of at least $1/2N$ while being transported by T to the origin, are pairwise disjoint.

Now choose a sequence of natural numbers $\{N_j\}$ such that $\lim_{j\to\infty} N_j^\varepsilon/2^j = +\infty$ for all $\varepsilon > 0$ and set

$$\varphi := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi_{N_j}.$$

Then φ is an atomic probability measure concentrated on a countable subset of the unit cube Q_1, while since $\varphi \geq \frac{1}{2^j}\varphi_{N_j}$ for all $j \in \mathbb{N}$, then

$$W^\alpha(\varphi, \delta_0) \geq W^\alpha(\frac{1}{2^j}\varphi_{N_j}, \frac{1}{2^j}\delta_0) \geq \frac{1}{2^{\alpha j}}\frac{1}{2}N_j^{k-1-k\alpha},$$

and therefore $W^\alpha(\varphi, \delta_0) = +\infty$ unless $k - 1 - k\alpha < 0$, i.e. $\alpha \geq 1/k'$. Hence φ is not α-irrigable for $\alpha < 1/k'$, which means in view of Remark 4.4 that it is marginally α-non-irrigable for such α.

Remark 4.7. By taking in the construction of the above Example 4.6 each measure φ_{N_j} to be concentrated on the cube $Q_{1/j}$ rather than on Q_1, we would obtain a marginally α-non-irrigable measure φ, which has an additional property that $\operatorname{supp}\varphi$ is countable.

Remark 4.8. Again slightly modifying the construction of the above Example 4.6 one could have obtained a marginally α-non-irrigable measure φ which is not necessarily zero-dimensional. For instance, this happens once one uses in this construction the measures

$$\varphi_N := \frac{1}{N^k}\sum_{i=1}^{N^k} \mathcal{H}^1 \llcorner \Delta_i,$$

where $\Delta_i := \{x_i\} \times [0,1] \times \{0\}^{n-k-1}$ are segments of unit length centered at $x_i \subset \mathbb{R}^k$ and perpendicular to $\mathbb{R}^k \subset \mathbb{R}^n$. In fact, each of the measures φ_N is still α-irrigable, even for $\alpha = 0$, while the respective sum φ is not α-irrigable for sufficiently small α, since otherwise supposing the existence of an optimal current T transporting φ to δ_0 at finite W^α cost, we would have that its projection to $\mathbb{R}^k$ transports the measure from Example 4.6 to δ_0 at finite W^α cost.

We now show another way of constructing a marginally α-non-irrigable measure.

Example 4.9. Let $\{z_k\}$ be a sequence of rational points in a square, and φ be any atomic measure concentrated on $\{z_k\}$. Then $W^\alpha(\varphi, \delta) = +\infty$, if $\alpha = 0$.

To prove this assertion we may assume without loss of generality that δ is also concentrated on one of the points z_k. Let Σ_k be Steiner minimal tree (i.e. connected set of minimal length) containing all z_i, $i = 1, \ldots, k$. By Lemma 4.10 below we may assume that Σ_k is closed. Moreover, if $W^\alpha(\varphi, \delta) < +\infty$, then there is a flat chain T of finite mass such that $\mathbb{M}^\alpha(T) = W^\alpha(\varphi, \delta) < +\infty$, $\partial T = \varphi - \delta$. By rectifiablity theorem, $T = \theta[\![\Sigma_T]\!]$ with some rectifiable $\Sigma \subset \mathbb{R}^n$, $\mathcal{H}^1(\Sigma_T) \leq C$. Using the equivalent expression of the transportation cost W^α from [2] which uses the transports η rather than currents, and the results from [4], it is not difficult to see that Σ_T may be assumed connected. Therefore, $\mathcal{H}^1(\Sigma_k) \leq C$.

Now, since all Σ_k are connected compact sets, then, up to a subsequence, $\Sigma_k \to \Sigma$ in the sense of Hausdorff, while $\{z_k\} \subset \Sigma$, and hence Σ is equal to the whole square (since it is closed) which is in contradiction with the estimate $\mathcal{H}^1(\Sigma) \leq C$ valid in view of the Golab theorem.

Lemma 4.10. *Let $\Sigma \subset \mathbb{R}^n$ be connected. Then $\mathcal{H}^1(\Sigma) = \mathcal{H}^1(\bar{\Sigma})$.*

Proof. It suffices to prove the claim for the case $\mathcal{H}^1(\Sigma) < +\infty$. Let $\mu := \mathcal{H}^1 \llcorner \Sigma$. One has

$$\mu(B_r(x)) = \mathcal{H}^1(\Sigma \cap B_r(x)) \geq r$$

for all $x \in \Sigma$ and $r \leq \operatorname{diam} \Sigma/2$, in view of connectedness of Σ. Since by assumption the measure μ is finite, then the above estimate holds also for all $x \in \bar{\Sigma}$. In fact, if $x \in \bar{\Sigma}$ and $x_k \to x$, $x_k \in \Sigma$, then $B_r(x_k) \to B_r(x)$ in the sense of Hausdorff, and hence,

$$\mu(B_r(x)) \geq \limsup_k \mu(B_r(x_k)).$$

One has now $\Theta_1^*(\mu, x) \geq 1/2$ for all $x \in \bar{\Sigma}$, which implies $\mu \geq 1/2\mathcal{H}^1 \llcorner \bar{\Sigma}$. Therefore,

$$0 = \mu(\bar{\Sigma} \setminus \Sigma) \geq \frac{1}{2}\mathcal{H}^1 \llcorner \bar{\Sigma}(\bar{\Sigma} \setminus \Sigma) = \frac{1}{2}\mathcal{H}^1(\bar{\Sigma} \setminus \Sigma),$$

which concludes the proof. □

We will also need the following assertions.

Lemma 4.11. *If $\mathcal{A}$ is a bounded nonempty family of not α-irrigable (resp. purely α-non-irrigable, marginally α-non-irrigable) finite positive Borel measures, then so is*

$$\varphi' = \sup\{\varphi \,:\, \varphi \in \mathcal{A}\}.$$

Proof. If all the measures from $\mathcal{A}$ are not α-irrigable, then the non-irrigability of φ' follows immediately from Lemma 4.5. Further, if all the measures from $\mathcal{A}$ are purely α-non-irrigable (resp. marginally α-non-irrigable), then considering an arbitrary positive measure $\psi \leq \varphi'$, we have that $\psi \wedge \tilde{\varphi} \neq 0$ for some $\tilde{\varphi} \in \mathcal{A}$ (since otherwise one would have $\varphi' - \psi \geq \varphi$ for all $\varphi \in \mathcal{A}$, which would contradict the definition of φ'). Therefore, ψ is not α-irrigable (resp. not purely α-non-irrigable), because otherwise $\psi \wedge \tilde{\varphi}$ would be α-irrigable (resp. purely α-non-irrigable) contrary to the assumption on $\tilde{\varphi}$. This shows the pure α-non-irrigability (resp. marginal α-non-irrigability) of φ' since ψ is arbitrary. □

Lemma 4.12. *A sum of an arbitrary family of finite positive and not α-irrigable (resp. purely α-non-irrigable, marginally α-non-irrigable) Borel measures is still not α-irrigable (resp. purely α-non-irrigable, marginally α-non-irrigable), if it is finite.*

Proof. In view of Lemma 4.11 it is enough to prove the assertion for finite sums, and hence, by induction, to sums of two measures. Now, if φ_1 and φ_2 are not α-irrigable, then neither is $\varphi_1 + \varphi_2$ by Lemma 4.5. Finally, assume that φ_1 and φ_2 are purely (resp. marginally) α-non-irrigable. It remains to show that so is $\varphi_1 + \varphi_2$. Arguing by contradiction, we suppose that there is a positive α-irrigable (resp. purely α-non-irrigable) measure $\varphi \leq \varphi_1 + \varphi_2$. We have then

$$(\varphi - \varphi_1)^+ = \varphi - \varphi \wedge \varphi_1 \leq \varphi_1 - \varphi \wedge \varphi_1 + \varphi_2 = (\varphi - \varphi_1)^- + \varphi_2.$$

Since

$$(\varphi - \varphi_1)^+ \wedge (\varphi - \varphi_1)^- = 0,$$

we have

$$(\varphi - \varphi_1)^+ \leq \varphi_2,$$

which implies that either $(\varphi - \varphi_1)^+ = 0$, or $(\varphi - \varphi_1)^+$ is positive but not α-irrigable (resp. not purely α-non-irrigable). In the first case one has then $\varphi \leq \varphi_1$, which means that φ is not α-irrigable (resp. not purely α-non-irrigable) contrary to the assumption. In the second case, if we supposed that φ is α-irrigable, then minding that $(\varphi-\varphi_1)^+ \leq \varphi$, we get that φ is not α-irrigable by Lemma 4.5, contradicting the assumption. If on the contrary we supposed φ to be purely α-non-irrigable, and we obtained that $(\varphi-\varphi_1)^+$ is not purely α-non-irrigable, then there is a positive α-irrigable measure $\hat{\varphi} \leq (\varphi-\varphi_1)^+ \leq \varphi$ which again contradicts the assumption on φ and hence concludes the proof. □

Remark 4.13. Repeating almost word-to-word the proofs of the above Lemmata 4.11 and 4.12, one observes that any sum, as well as the supremum of any bounded sequence of α-irrigable measures, is either α-irrigable of marginally α-non-irrigable.

Remark 4.14. It is easy to show that whenever φ is either α-irrigable of marginally α-non-irrigable, then

$$\varphi = \sup\{\psi \,:\, \psi \leq \varphi \text{ is } \alpha\text{-irrigable}\}.$$

In fact, the assertion is immediate for an α-irrigable measure φ. If φ is marginally α-non-irrigable, then denoting by φ' the above supremum, we see that $\varphi-\varphi' \leq \varphi$, and hence, either $\varphi=\varphi'$, or there is a positive α-irrigable measure $\tilde{\varphi} \leq \varphi-\varphi'$. But the latter case is impossible since otherwise one would have, minding that a sum of two α-irrigable measures is still α-irrigable, that $\psi+\tilde{\varphi} \leq \varphi'$ whenever $\psi \leq \varphi$ is α-irrigable, hence $\varphi'+\tilde{\varphi} \leq \varphi'$, which would mean $\tilde{\varphi}=0$ contrary to the assumption.

We recall the following well-known notion.

Definition 4.15. *The Hausdorff dimension* $\dim\varphi$ *of the Borel measure* φ *over* $\mathbb{R}^n$ *is defined as*

$$\dim \varphi := \sup\{\beta \colon \varphi \ll \mathcal{H}^\beta\},$$

where $\mathcal{H}^\beta$ *stands for the* β*-dimensional Hausdorff measure.*

For instance, if $\varphi \ll \mathcal{L}^n$, then $\dim \varphi = n$.

We provide now two statements on characterization of irrigability properties of a measure by means of a dimension which sharpen the similar results from [7]. It is important to emphasize that the notion of dimension we use here (i.e. that given by Definition 4.15 above) is different from the notions used in [7] which evaluate dimensions of the set where φ is concentrated. For instance, if $\varphi = \mathcal{L}^n \llcorner B_1(0) + \delta_0$, then $\dim\varphi = 0$ according to Definition 4.15, while all the definitions used in [7] attribute dimension n to the above measure.

Proposition 4.16. *Every finite positive Borel measure* φ *with compact support in* $\mathbb{R}^n$ *and* $\dim\varphi > (1/\alpha)'$ *is purely* α*-non-irrigable.*

Proof. Suppose the contrary, i.e. the existence of an α-irrigable measure $\varphi' \leq \varphi$, $\varphi' \neq 0$. Since $\varphi' \ll \varphi$, then by definition of the dimension one has

$$\dim\varphi' \geq \dim\varphi > (1/\alpha)'. \tag{4.1}$$

Since φ' is α-irrigable, then by theorem 1.2 from [7], one has that φ' is concentrated on a set of zero $\mathcal{H}^k$ measure, where $k := (1/\alpha)'$. Together with (4.1) this gives $\varphi'=0$, which is the desired contradiction concluding the proof. □

Proposition 4.17. *Let* φ *be a positive Borel measure with compact support in* $\mathbb{R}^n$ *and Ahlfors-regular in dimension* $\beta \geq 1$*, in the sense that there exist two constants*

$c > 0$ and $C > 0$ such that for every positive $\rho < \operatorname{diam} \operatorname{supp} \varphi$ and for every $x \in \operatorname{supp} \varphi$ one has

$$c\rho^\beta \leq \varphi(B_\rho(x)) \leq C\rho^\beta. \tag{4.2}$$

Then φ is purely α-non-irrigable, if $\beta \geq (1/\alpha)'$, and α-irrigable otherwise.

Proof. Clearly, $\dim \varphi = \beta$, so that when $\beta > (1/\alpha)'$ the claim follows from Proposition 4.16. In the case $\beta < (1/\alpha)'$ one has α-irrigability of φ in view of corollary 1.6 from [7]. Finally, if $\beta = (1/\alpha)'$ and $\varphi' \leq \varphi$, then

$$\varphi'(B_\rho(x)) \leq \varphi(B_\rho(x)) \leq C\rho^\beta$$

for all $x \in \operatorname{supp} \varphi$, hence for all $x \in \operatorname{supp} \varphi'$. Therefore, by lemma 1.4 from [7], if A is any set on which φ' is concentrated, one has $\mathcal{H}^\beta(A) > 0$. On the other hand, by theorem 1.2 from [7] in order that the measure φ' be α-irrigable, the condition $\mathcal{H}^{(1/\alpha)'}(A) = 0$ must hold. This concludes the proof of the claim. □

5. Characterization of couples of measures which can be transported at finite cost

We first prove the following simple though crucial lemma.

Lemma 5.1. *Let φ_0 and φ_1 be finite Borel measures with compact supports in $\mathbb{R}^n$ such that $\varphi_0 \wedge \varphi_1 = 0$ and $\varphi_0, \varphi_1 \neq 0$. Then there exist two positive Borel measures φ_0', φ_1' satisfying $\varphi_0' \leq \varphi_0$ and $\varphi_1' \leq \varphi_1$, such that*

(i) $\operatorname{supp} \varphi_0' \cap \operatorname{supp} \varphi_1' = \emptyset$,
(ii) $\varphi_0'(\mathbb{R}^n) = \varphi_1'(\mathbb{R}^n) > 0$;
(iii) $W^\alpha(\varphi_0', \varphi_1') \leq W^\alpha(\varphi_0, \varphi_1)$ *(so that, in particular, if $W^\alpha(\varphi_0, \varphi_1) < +\infty$, then $W^\alpha(\varphi_0', \varphi_1') < +\infty$).*

Proof. For $R > 0$ define

$$\Theta_R := \{\theta \in \Theta \colon |\theta(1) - \theta(0)| \geq R\}.$$

Consider a normal one-dimensional real current T satisfying $\partial T = \varphi_0 - \varphi_1$ and

$$\mathbb{M}^\alpha(T) = W^\alpha(\varphi_0, \varphi_1).$$

Further, let η be a transport such that $T = T_\eta$ constructed according to the second part of Theorem 3.1. Finally, set $\eta_R := \eta \llcorner \Theta_R$. Clearly, for some $R > 0$ we have $\eta_R \neq 0$ since otherwise η is concentrated on the set of paths θ such that $\theta(0) = \theta(1)$ and hence we would have $\varphi_0 = \eta(0) = \eta(1) = \varphi_1$.

We now choose $r \leq R/3$ and consider an $x_0 \in \mathbb{R}^n$ such that

$$\varphi_0' := \eta_R(0) \llcorner B_r(x_0) \neq 0$$

(such a point exists since otherwise we would have $\eta_R(0) = 0$). Define

$$\eta' := \eta_R \llcorner \{\theta \colon \theta(0) \in B_r(x_0)\}$$

so that we have $\varphi_0' = \eta'(0)$ and set $\varphi_1' := \eta'(1)$. Then $\varphi_0'(\mathbb{R}^n) = \varphi_1'(\mathbb{R}^n) > 0$, that is, (ii) holds. Now, clearly φ_0' is concentrated in some set $e_0 \subset B_r(x_0)$, while φ_1' is concentrated in some set e_1 such that whenever $y_1 \in e_1$ one has $|y_1 - y_0| \geq R$ for some $y_0 \in e_0$. Hence,

$$\operatorname{dist}(\operatorname{supp} \varphi_0', \operatorname{supp} \varphi_1') \geq R - 2r \geq R/3 > 0,$$

so that (i) is satisfied. Finally, (iii) follows from the estimate

$$W^\alpha(\varphi_0', \varphi_1') \le \mathbb{M}^\alpha(T_{\eta'}) \le \mathbb{M}^\alpha(T_\eta) = W^\alpha(\varphi_0, \varphi_1),$$

where the second and the third inequality is due to Lemma 3.2 (which is applied in view of the inequality $\eta' \le \eta$). □

Remark 5.2. It is worth remarking (though we will not use it in the sequel) that by a slightly more careful construction one could have chosen, given an arbitrary $C > 0$, the measures φ_0', φ_1' so as to satisfy additionally

$$\operatorname{diam} \operatorname{supp} \varphi_0' \vee \operatorname{diam} \operatorname{supp} \varphi_1' \le C \operatorname{dist}(\operatorname{supp} \varphi_0', \operatorname{supp} \varphi_1').$$

Lemma 5.3. *If φ_0, φ_1 are finite Borel measures with compact supports in $\mathbb{R}^n$ satisfying*

$$\operatorname{supp} \varphi_0 \cap \operatorname{supp} \varphi_1 = \emptyset,$$

and $W^\alpha(\varphi_0, \varphi_1) < +\infty$. Then both φ_0 and φ_1 are α-irrigable measures.

Proof. Consider a normal one-dimensional real current T satisfying $\partial T = \varphi_1 - \varphi_0$ and

$$\mathbb{M}^\alpha(T) = W^\alpha(\varphi_0, \varphi_1) < +\infty.$$

Clearly, without loss of generality we may consider T to be supported over the convex hull $K := \operatorname{co}(\operatorname{supp} \varphi_0 \cup \operatorname{supp} \varphi_1)$ (otherwise, projecting T to K would not increase $\mathbb{M}^\alpha$, and would not change the boundary). We let $u(x) := \operatorname{dist}(x, \operatorname{supp} \varphi_0)$. By assumption one has the existence of some $t_2 > 0$ such that

$$\operatorname{supp} \varphi_1 \subset \{x \colon u(x) > t_2\}.$$

We take an arbitrary $t_1 \in (0, t_2)$. Since

$$\int_{t_1}^{t_2} \mathbb{M}^\beta(\langle T, u, t\rangle)\, dt \le \mathbb{M}^\beta(T \llcorner \{x \colon t_1 < u(x) < t_2\}) \le \mathbb{M}^\beta(T)$$

for all $\beta \in [0, 1]$ by Lemma 3.4, then

$$\mathbb{M}^\beta(\langle T, u, t\rangle) < +\infty \text{ for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}.$$

Fix now a $t \in [t_1, t_2]$ such that $\mathbb{M}^\alpha(\langle T, u, t\rangle) < +\infty$ and $\mathbb{M}(\langle T, u, t\rangle) < +\infty$. This means that the zero-dimensional flat chain of finite mass (i.e. finite Borel measure) $\varphi := \langle T, u, t\rangle$ is concentrated on an at most countable set of points $\{y_k\} \subset \{u = t\} \cap K$ and satisfies

$$\varphi = \sum_k c_k \delta_{y_k}, \text{ where } \sum_k c_k^\alpha < +\infty.$$

Therefore, φ is α-irrigable. It remains to observe that

$$W^\alpha(\varphi_0, \varphi_1) = \mathbb{M}^\alpha(T) = \mathbb{M}^\alpha(T \llcorner \{u \le t\}) + \mathbb{M}^\alpha(T \llcorner \{u \ge t\}). \tag{5.1}$$

In fact, by the rectifiability theorem due to White [17, theorem 8.1] the current T is rectifiable, namely, $T = \theta[\![\Sigma]\!]$, where $\Sigma \subset \mathbb{R}^n$ is some $(\mathcal{H}^1, 1)$-rectifiable set, and $\theta \in L^1(\Sigma; \mathcal{H}^1)$ is a positive multiplicity, while the assumption on t means that $\mathcal{H}^1(\Sigma \cap \{u = t\}) = 0$. On the other hand,

$$\partial T \llcorner \{u \le t\} = \varphi - \varphi_0, \qquad \partial T \llcorner \{u \ge t\} = \varphi_1 - \varphi,$$

so that (5.1) implies

$$W^\alpha(\varphi_0, \varphi_1) \ge W^\alpha(\varphi_0, \varphi) + W^\alpha(\varphi, \varphi_1).$$

Thus $W^\alpha(\varphi_0,\varphi) < +\infty$ and $W^\alpha(\varphi,\varphi_1) < +\infty$, which means that both φ_0 and φ_1 are α-irrigable measures. $\square$

We are now in a position to prove the following result.

Theorem 5.4. *Suppose that φ_0 is purely α-non-irrigable. Then for every measure $\varphi_1 \geq 0$ with $\varphi_1(\mathbb{R}^n) = \varphi_0(\mathbb{R}^n)$ one has $W^\alpha(\varphi_0,\varphi_1) < +\infty$, if and only if $\varphi_0 = \varphi_1$.*

Proof. Clearly if $\varphi_0 = \varphi_1$ we have $W^\alpha(\varphi_0,\varphi_1) = 0 < +\infty$. To prove the reverse implication we suppose by contradiction that $\varphi_0 \neq \varphi_1$. Define

$$\varphi_0' := \varphi_0 - \varphi_0 \wedge \varphi_1, \qquad \varphi_1' := \varphi_1 - \varphi_0 \wedge \varphi_1.$$

Clearly $\varphi_0' \wedge \varphi_1' = 0$, and since $\varphi_0 \neq \varphi_1$, then φ_0' and φ_1' are nonzero. Further, $\varphi_0' \leq \varphi_0$, so that φ_0' is purely α-non-irrigable. Applying Lemma 5.1 to the measures φ_0' and φ_1' we get smaller positive measures φ_0'' and φ_1'' such that φ_0'' (and in fact, also φ_1'') is α-irrigable by Lemma 5.3, and this contradiction concludes the proof. $\square$

Corollary 5.5. *Let φ be a finite positive Borel measure with Hausdorff dimension $\dim\varphi > (1/\alpha)'$ or Ahlfors-regular measure in dimension $\beta \geq (1/\alpha)'$. Then, $W^\alpha(\varphi,\psi) < +\infty$, if and only if $\varphi = \psi$.*

Proof. Combine Theorem 5.4 with Proposition 4.16 and Proposition 4.17. $\square$

Below we provide an example of not α-irrigable measures φ_0 and φ_1, such that $0 < W^\alpha(\varphi_0,\varphi_1) < +\infty$. As we will see, this is typical for marginally α-non-irrigable measures.

Example 5.6. Let φ_0 be any marginally α-non-irrigable measure concentrated on a countable set of points $\{z_k\}$ (e.g. a measure from Example 4.6 or Example 4.9), that is,

$$\varphi_0 := \sum_k c_k \delta_{z_k}$$

for some sequence $\{c_k\} \subset \mathbb{R}^+$. Let $\{z_k'\}$ be the sequence "shifted" rightwards by the formula $z_k' := z_k + (l_k, 0, \ldots, 0)$, where $l_k > 0$ are chosen in such a way that

$$\sum_k c_k^\alpha l_k < +\infty. \tag{5.2}$$

Let $\varphi_1 := \sum_k c_k \delta_{z_k'}$. One has then that the normal current $T := \sum_k c_k [\![z_k, z_k']\!]$ shifting each z_k to z_k' along a line segment parallel to the first coordinate axis, satisfies

$$M^\alpha(T) \leq \sum_k c_k^\alpha l_k < +\infty$$

by (5.2), and hence $W^\alpha(\varphi_0,\varphi_1) \leq M^\alpha(T) < +\infty$. On the other hand, φ_1 is also not α-irrigable (hence it is marginally α-non-irrigable by Remark 4.4), since otherwise one would have α-irrigability of φ_0 by triangle inequality for W^α.

We provide now the following simple statement on decomposition of measures.

Proposition 5.7. *For every $\alpha \in [0,1]$ every finite Borel measure φ can be uniquely represented as*

$$\varphi = \tilde\varphi + \varphi',$$

where $\tilde\varphi$ is purely α-non-irrigable and φ' is either α-irrigable or marginally α-non-irrigable. One can compute $\tilde\varphi$ and φ' according to the formulae

$$\tilde\varphi := \sup\{\psi \,:\, \psi \leq \varphi \text{ is purely } \alpha\text{-non-irrigable}\}, \quad \varphi' := \varphi - \tilde\varphi. \tag{5.3}$$

Proof. If $\tilde{\varphi}$ and φ' are given by (5.3), then $\tilde{\varphi}$ is purely α-non-irrigable by Lemma 4.11. Then one cannot have a purely α-non-irrigable and positive $\hat{\varphi} \leq \varphi'$, because otherwise $\tilde{\varphi} + \hat{\varphi}$ would be still purely α-non-irrigable and greater than $\tilde{\varphi}$, which would contradict the definition of $\tilde{\varphi}$. In other words, φ' is either α-irrigable or marginally α-non-irrigable. To show the uniqueness of the decomposition, suppose that

$$\varphi = \tilde{\varphi}_1 + \varphi'_1,$$

where $\tilde{\varphi}_1$ is purely α-non-irrigable and φ'_1 is either α-irrigable or marginally α-non-irrigable. Then $\tilde{\varphi}_1 \leq \tilde{\varphi}$ in view of the definition (5.3) of the latter. If $\tilde{\varphi} - \tilde{\varphi}_1 \neq 0$, then the latter measure is purely α-non-irrigable, since $\tilde{\varphi} - \tilde{\varphi}_1 \leq \tilde{\varphi}$. It remains to note that

$$(\tilde{\varphi} - \tilde{\varphi}_1) + \varphi' = \varphi'_1,$$

hence $\tilde{\varphi} - \tilde{\varphi}_1 \leq \varphi'_1$ contradicting the assumption that the latter be either α-irrigable or marginally α-non-irrigable. Therefore, $\tilde{\varphi}_1 = \tilde{\varphi}$, and hence, $\varphi'_1 = \varphi'$. □

In view of the above decomposition result, it is natural to call $\tilde{\varphi}$ the purely α-non-irrigable part of φ. We may announce now the following result.

Theorem 5.8. *Let φ_0 and φ_1 be finite Borel measures of equal total mass with compact supports in $\mathbb{R}^n$, and let $\tilde{\varphi}_i$ be the purely α-non-irrigable parts of φ_i in the sense of Proposition 5.7, and $\varphi'_i := \varphi_i - \tilde{\varphi}_i$, $i = 0, 1$. Then one has $W^\alpha(\varphi_0, \varphi_1) < +\infty$, if and only if $\tilde{\varphi}_0 = \tilde{\varphi}_1$ and $W^\alpha(\varphi'_0, \varphi'_1) < +\infty$.*

Proof. It is obvious that $\tilde{\varphi}_0 = \tilde{\varphi}_1$ and $W^\alpha(\varphi'_0, \varphi'_1) < +\infty$ implies $W^\alpha(\varphi_0, \varphi_1) < +\infty$. We prove therefore the reverse implication. If $W^\alpha(\varphi_0, \varphi_1) < +\infty$, then there exists a one-dimensional real normal current T satisfying $\partial T = \varphi_0 - \varphi_1$ and $\mathbb{M}^\alpha(T) < +\infty$. Since every normal current has an acyclic subcurrent with the same boundary (proposition 3.12 from [14]), we may consider without loss of generality T to be acyclic. Letting then η to be a transport satisfying conditions of the second part of Theorem 3.1, we get

$$\eta(0) = (\varphi_0 - \varphi_1)^+ = \varphi_0 - \varphi_0 \wedge \varphi_1.$$

If $\hat{\varphi}_0$ is an arbitrary purely α-non-irrigable measure satisfying $\hat{\varphi}_0 \leq \varphi_0$, then by Lemma 3.3 there exists a transport $\hat{\eta}_0 \leq \eta$ such that

$$\hat{\eta}_0(0) = \hat{\varphi}_0 \wedge (\varphi_0 - \varphi_0 \wedge \varphi_1),$$

while by Lemma 3.2 one has $T_{\hat{\eta}_0} \leq T$, hence $\mathbb{M}^\alpha(T_{\hat{\eta}_0}) \leq \mathbb{M}^\alpha(T) < +\infty$. This means

$$W^\alpha(\hat{\varphi}_0 \wedge (\varphi_0 - \varphi_0 \wedge \varphi_1), \hat{\eta}_0(1)) < +\infty,$$

and hence, since $\hat{\varphi}_0 \wedge (\varphi_0 - \varphi_0 \wedge \varphi_1)$ is purely α-non-irrigable, then

$$\hat{\varphi}_0 \wedge (\varphi_0 - \varphi_0 \wedge \varphi_1) = \hat{\eta}_0(1)$$

according to Theorem 5.4. On the other hand,

$$\hat{\varphi}_0 \wedge (\varphi_0 - \varphi_0 \wedge \varphi_1) = \hat{\eta}_0(1) \leq \eta(1) \leq \varphi_1 - \varphi_0 \wedge \varphi_1,$$

which means that $\hat{\varphi}_0 \wedge (\varphi_0 - \varphi_0 \wedge \varphi_1) = 0$, and thus, by Lemma 5.12 below applied with $\tilde{\varphi} := \tilde{\varphi}_0$, $\varphi := \varphi_0$ and $\psi := \varphi_0 \wedge \varphi_1$, we get $\hat{\varphi}_0 \leq \varphi_0 \wedge \varphi_1$.

In a completely symmetric way one gets that whenever $\hat{\varphi}_1 \leq \varphi_1$ is purely α-non-irrigable, then $\hat{\varphi}_1 \leq \varphi_0 \wedge \varphi_1$. Thus one has

$$\tilde{\varphi}_i := \sup\left\{\psi \,:\, \psi \leq \varphi_0 \wedge \varphi_1 \text{ is purely } \alpha\text{-non-irrigable}\right\}, \qquad i = 0, 1,$$

and hence $\tilde{\varphi}_0 = \tilde{\varphi}_1$. Finally, one has

$$W^\alpha(\varphi_0', \varphi_1') = W^\alpha(\varphi_0 - \tilde{\varphi}_0, \varphi_1 - \tilde{\varphi}_1) = W^\alpha(\varphi_0, \varphi_1) < +\infty,$$

concluding the proof. □

Remark 5.9. It is easy to observe now that when φ_0 is α-irrigable (resp. purely α-non-irrigable, marginally α-non-irrigable), then $W^\alpha(\varphi_0, \varphi_1) < +\infty$ implies the same property for φ_1. In fact, it is clear from definition and the triangle inequality for W^α in the case of an α-irrigable φ_0, while the case of purely α-non-irrigable φ_0 is provided by Theorem 5.4. In the remaining case of marginally α-non-irrigable φ_0 one has that, in terms of the above Theorem 5.8, $\tilde{\varphi}_0 = 0$ in view of the definition of a marginally α-non-irrigable measure, and hence $\tilde{\varphi}_1 = 0$. Therefore, for every measure $\hat{\varphi}_1 \leq \hat{\varphi}$ one has the existence of an α-irrigable measure $\varphi_1' \leq \hat{\varphi}_1$, which means, in other words, that φ_1 is marginally α-non-irrigable.

Corollary 5.10. *Let φ_0 and φ_1 be finite Borel measures of equal total mass with compact supports in $\mathbb{R}^n$, and $\varphi_0 \wedge \varphi_1 = 0$. Then $W^\alpha(\varphi_0, \varphi_1) < +\infty$ implies that φ_0 and φ_1 are either both α-irrigable or both marginally α-non-irrigable.*

Proof. Combine Theorem 5.8 with Remark 5.9. □

Remark 5.11. From Lemma 5.3 one gets that $W^\alpha(\varphi_0, \varphi_1) < +\infty$ implies

$$\operatorname{supp} \varphi_0 \cap \operatorname{supp} \varphi_1 \neq \emptyset$$

whenever φ_0 (hence also φ_1) is marginally α-non-irrigable.

The following lemma has been used in the proof of Theorem 5.8.

Lemma 5.12. *Let φ, $\tilde{\varphi}$ and ψ be finite positive Borel measures satisfying $\tilde{\varphi} \leq \varphi$, $\psi \leq \varphi$ and $\tilde{\varphi} \wedge (\varphi - \psi) = 0$. Then $\tilde{\varphi} \leq \psi$.*

Proof. By assumption one has using (2.2)

$$\tilde{\varphi} \wedge (\varphi - \psi) = \varphi \wedge (\tilde{\varphi} + \psi) - \psi = 0,$$

hence

$$\varphi \wedge (\tilde{\varphi} + \psi) = \psi.$$

But $\tilde{\varphi} \leq \tilde{\varphi} + \psi$, which combined with the assumption $\tilde{\varphi} \leq \varphi$ gives

$$\tilde{\varphi} \leq \varphi \wedge (\tilde{\varphi} + \psi) = \psi,$$

concluding the proof. □

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford mathematical monographs. Oxford University Press, Oxford, 2000.

[2] M. Bernot, V. Caselles, and J.-M. Morel. Traffic plans. *Publ. Mat.*, 49(2):417–451, 2005.

[3] M. Bernot, V. Caselles, and J.-M. Morel. Are there infinite irrigation trees? *J. Mathematical Fluid Mechanics*, 2006.

[4] M. Bernot, V. Caselles, and J.-M. Morel. The structure of branched transportation networks. *Preprint*, 2006.

[5] S. Bhaskaran and F.J.M. Salzborn. Optimal design of gas pipeline networks. *J. Oper. Res. Society*, 30:1047–1060, 1979.

[6] V. Caselles and J.-M. Morel. Irrigation. In *Variational methods for discontinuous structures*, volume 51 of *Progr. Nonlinear Differential Equations Appl.*, pages 81–90. Birkhäuser, Basel, 2002.

[7] G. Devillanova and S. Solimini. On the dimension of an irrigable measure. *Preprint*, 2006.

[8] H. Federer. *Geometric Measure Theory.* Springer-Verlag, 1969.
[9] E.N. Gilbert. Minimum cost communication networks. *Bell System Tech. J.*, 46:2209–2227, 1967.
[10] D.H. Lee. Low cost drainage networks. *Networks*, 6:351–371, 1976.
[11] F. Maddalena, S. Solimini, and J.-M. Morel. A variational model of irrigation patterns. *Interfaces Free Bound.*, 5(4):391–415, 2003.
[12] J.-M. Morel and F. Santambrogio. Comparison of distances between measures. *Appl. Math. Lett.*, 2006.
[13] F. Morgan. *Geometric measure theory. A Beginner's guide.* Academic Press, 1987.
[14] E. Paolini and E. Stepanov. Optimal transportation networks as flat chains. *Interfaces Free Bound.*, 8:393–436, 2006.
[15] F. Santambrogio. Optimal channel networks, landscape function and branched transport. *Preprint.*, 2006.
[16] L. Simon. *Lectures on geometric measure theory.* Proceedings of the Centre for Mathematical Analysis, Australian National University. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.
[17] B. White. Rectifiablity of flat chains. *Annals of Math.*, 150:165–184, 1999.
[18] Q. Xia. Optimal paths related to transport problems. *Communications in Contemporary Math.*, 5(2):251–279, 2003.
[19] Q. Xia. Interior regularity of optimal transport paths. *Calc. Var. Partial Diff. Equations*, 20(3):283–299, 2004.

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