Curves Homothetically Shrinking by Curvature

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ABSTRACT. Note for students with a proof of the fact that the circle is the only smooth embedded curve in the plane homothetically shrinking by curvature.
This result is due to Abresch and Langer [**1**] and, independently, to Epstein and Weinstein [**3**].

Let $\gamma \subset \mathbb{R}^2$ be a smooth connected embedded curve, fixing a reference point on the curve γ we have an arclength parameter s which gives a unit tangent vector field and a unit normal vector field ν, which is the counterclockwise rotation of $\pi/2$ in $\mathbb{R}^2$ of the vector τ. Then, the curvature is given by $k = \langle \partial_s \tau \,|\, \nu \rangle$.

If γ is homothetically shrinking around the origin of $\mathbb{R}^2$ when moving by curvature, it is easy to see that the normal projection of the position vector γ at every point must be proportional to the curvature *vector* $k\nu = \overline{k} = \partial^2_{ss}\gamma$ which gives the velocity of the evolving curve. That is, $\lambda\overline{k} + \langle \gamma \,|\, \nu \rangle = 0$ for a nonnegative constant λ. Dilating the curve by a factor $1/\sqrt{\lambda}$ we can assume that $\lambda = 1$. Multiplying then this equation for ν we get the characterizing equation $k + \langle \gamma \,|\, \nu \rangle = 0$.

THEOREM 1. *The only smooth complete embedded curves in $\mathbb{R}^2$ satisfying $k + \langle \gamma \,|\, \nu \rangle = 0$ are the lines through the origin and the unit circle.*

PROOF. The relation $k = -\langle \gamma \,|\, \nu \rangle$ implies the ODE for the curvature $k_s = k\langle \gamma \,|\, \tau \rangle$. Suppose that at some point $k = 0$, then also $k_s = 0$ at the same point, hence, by the uniqueness theorem for ODE's we conclude that k is identically zero and we are dealing with a line L which, as $\langle x \,|\, \nu \rangle = 0$ for every $x \in L$, must contain the origin of $\mathbb{R}^2$.
So we suppose that k is always nonzero and possibly reversing the orientation of the curve we assume that $k > 0$ at every point, that is, the curve is strictly convex.
Computing the derivative of $|\gamma|^2$,

$$\partial_s |\gamma|^2 = 2\langle \gamma \,|\, \tau \rangle = 2k_s/k = 2\partial_s \log k$$

we get $k = Ce^{|\gamma|^2/2}$ for some constant $C > 0$, it follows that k is bounded from below by $C > 0$.
We consider now a new coordinate $\theta = \arccos \langle e_1 \,|\, \nu \rangle$, this can be done for the whole curve as we know that this latter is convex (obviously, as for the arclength parameter s

it is only locally continuous, θ "jumps" after a complete round).
Differentiating with respect to the arclength parameter we have $\partial_s\theta = k$ and

$$k_\theta = k_s/k = \langle\gamma\,|\,\tau\rangle \qquad k_{\theta\theta} = \frac{\partial_s k_\theta}{k} = \frac{1 + k\langle\gamma\,|\,\nu\rangle}{k} = \frac{1}{k} - k\,. \tag{1}$$

Multiplying both sides of the last equation by $2k_\theta$ we get $\partial_\theta[k_\theta^2 + k^2 - \log k^2] = 0$, that is, the quantity $k_\theta^2 + k^2 - \log k^2$ is equal to some constant E along all the curve. Notice that such quantity E cannot be less than 1, moreover, if $E = 1$ we have that k must be constant and equal to one along the curve, which consequently must be the unit circle centered at the origin of $\mathbb{R}^2$.
When $E > 1$, it follows that k is uniformly bounded from above, hence recalling that $k = Ce^{|\gamma|^2/2}$, the image of the curve is contained in a ball of $\mathbb{R}^2$ and by the embeddedness and completeness hypotheses, the curve must be closed, simple and strictly convex, as $k > 0$ at every point.
We now suppose that γ is not the unit circle and we look at the critical points of the curvature k. Since $k_{\theta\theta} = \frac{1}{k} - k$, there holds that $k_{\theta\theta} \neq 0$ when $k_\theta = 0$, otherwise this second order ODE for k would imply $k_\theta = 0$ everywhere, hence $k = 1$ identically and we would be in the case of the unit circle. Thus, the critical points of the curvature are not degenerate, hence, by the compactness of the curve they are isolated and finite. Moreover, by looking at the equation for the curvature (1) we can see easily that $k_{\min} < 1$ and $k_{\max} > 1$.
Suppose now that $k(0) = k_{\max}$ and $k(\overline{\theta})$ is the first subsequent critical value for k, for some $\overline{\theta} > 0$. Then the curvature is strictly decreasing in the interval $[0, \overline{\theta}]$ and again by the second order ODE, the function k (hence also the curve, by integration) is symmetric with respect to $\theta = 0$ and $\theta = \overline{\theta}$. This clearly implies that $k(\overline{\theta})$ must be the minimum $k_{\min}$ of the curvature, as every critical point is not degenerate.

By the four vertex theorem **[5, 6]**, on every closed curve there are at least four critical points of k, as a consequence our curve is composed by at least four pieces like the one described above. Hence, since the curve is closed and embedded the curvature $k(\theta)$ must be a periodic function with period $T > 0$ not larger than π (since 2π is an obvious multiple of the period) and $\overline{\theta} = T/2$. More precisely, the period T must be $2\pi/n$ for some $n \geq 2$.

By a straightforward computation, starting by differentiating the equation $k_{\theta\theta} = \frac{1}{k} - k$, one gets $(k^2)_{\theta\theta\theta} + 4(k^2)_\theta = 4k_\theta/k$, then we compute

$$\begin{aligned}
4\int_0^{T/2} \sin 2\theta \,\frac{k_\theta}{k}\, d\theta &= \int_0^{T/2} \sin 2\theta \,[(k^2)_{\theta\theta\theta} + 4(k^2)_\theta]\, d\theta \\
&= \sin 2\theta (k^2)_{\theta\theta}\big|_0^{T/2} - 2\int_0^{T/2} \cos 2\theta (k^2)_{\theta\theta}\, d\theta + 4\int_0^{T/2} \sin 2\theta (k^2)_\theta\, d\theta \\
&= 2\sin T[k(T/2)k_{\theta\theta}(T/2) + k_\theta^2(T/2)] - 2\cos 2\theta (k^2)_\theta\big|_0^{T/2} \\
&\quad - 4\int_0^{T/2} \sin 2\theta (k^2)_\theta\, d\theta + 4\int_0^{T/2} \sin 2\theta (k^2)_\theta\, d\theta \\
&= 2\sin T[k(T/2)k_{\theta\theta}(T/2) + k_\theta^2(T/2)] \\
&\quad - 4\cos T k(T/2)k_\theta(T/2) + 4k(0)k_\theta(0)\,.
\end{aligned}$$

Now, since $k_\theta(0) = k_\theta(T/2) = 0$ and $k(T/2) = k_{\min}$, using the equation for the curvature $k_{\theta\theta} = 1/k - k$ we get

$$4\int_0^{T/2} \sin 2\theta \,\frac{k_\theta}{k}\, d\theta = 2\sin T(1 - k_{\min}^2)\,,$$

and this last term is nonnegative as $k_{\min} < 1$ and $0 < T \leq \pi$.
Looking at the left hand integral we see instead that the factor $\sin 2\theta$ is always nonnegative, as $T \leq \pi$ and k_θ is always nonpositive in the interval $[0, T/2]$, as we assumed that we were moving from the maximum $k_{\max}$ at $\theta = 0$ to the minimum $k_{\min}$ at $\theta = T/2$ without crossing any other critical point of k. This gives a contradiction so γ must be the unit circle. □

REMARK 2. The original proof of Abresch and Langer (or the one by Epstein and Weinstein) is different, actually this result is a consequence of their general classification theorem.
To my knowledge, this "shortcut" in the embedded case is due to Chou and Zhu [**2**, Proposition 2.3].

We discuss now a while the analysis in the case of an immersed–only, complete, smooth curve.
If we now do not assume the embeddedness, we have to deal either with a smooth, complete immersion of $\mathbb{S}^1$ or of $\mathbb{R}$ possibly with self–intersections. In the first case, the curve is closed and compact, in the second case we can see that the initial part of the analysis in the proof of the above theorem still holds, hence, either the curve is a line through the origin of $\mathbb{R}^2$ or the curvature is everywhere positive and bounded from above, which implies that the whole curve is bounded. Then, if we only assume an estimate of the length of the curve, by the completeness it follows that the curve is closed.

Hence, we concentrate only on closed curves. As we said, $k > 0$, the equations (1) hold and the quantity $k_\theta^2 + k^2 - \log k^2$ is constant along the curve, equal to some constant

E which must be larger than one (otherwise we are dealing with the unit circle). Again, the curve is symmetric with respect to the critical points of the curvature, which are all nondegenerate, isolated and finite. Hence, the curvature $k(\theta)$ is oscillating between its maximum and its minimum with some period $T > 0$. If we exclude the unit circle, such period must be an integer fraction (at least of a factor 2, by the four vertex theorem) of an integer multiple (at least 2, otherwise we are dealing with the unit circle) of 2π, that is, $T = 2m\pi/n$ with $n, m \geq 2$.

Notice that there are two parameters here around, the rotation number of the closed curve and the number of critical points of the curvature.

Suppose that $k_{\min} < k_{\max}$ are these two consecutive critical values of k, it follows that they are two distinct positive zeroes of the function $E + \log k^2 - k^2$ when $E > 1$ with $0 < k_{\min} < 1 < k_{\max}$.
We have that the change $\Delta\theta$ in the angle θ along the piece of curve delimited by two consecutive points where the curvature assumes the values $k_{\min}$ and $k_{\max}$, must be the semiperiod $T/2$. Then, the analysis reduces to understand what are the admissible T. Such quantity $\Delta\theta$ is given by the integral

$$I(E) = \int_{k_{\min}}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}}\,.$$

Abresch and Langer (and also Epstein and Weinstein) by studying the behavior of this integral were able to classify *all* the immersed closed curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle \gamma \,|\, \nu \rangle = 0$. These form a family of curves indexed by two parameters called *Abresch–Langer curves*, see [**1**] for a detailed description.
We now state and partially prove the main properties of the integral $I(E)$ needed in such analysis.
It should be noticed that, by the discussion about the period T, the last statement in the next proposition implies Theorem 1.

PROPOSITION 3. *The function $I : (1, +\infty) \to \mathbb{R}$ satisfies*

(1) $\lim_{E\to 1^+} I(E) = \pi/\sqrt{2}$,
(2) $\lim_{E\to +\infty} I(E) = \pi/2$,
(3) *$I(E)$ is monotone nonincreasing.*

As a consequence $I(E) > \pi/2$.

PROOF. Notice that the study of the quantity $I(E)$ is equivalent to the study of the semiperiod for the one–dimensional Hamiltonian system with Hamiltonian function given by $H(k_\theta, k) = (k_\theta^2 + k^2 - \log k^2)/2$.

(1) The global minimum $1/2$ of the strictly convex potential $V(k) = (k^2 - \log k^2)/2$ is assumed at $k = 1$ and the limiting value for the period of the Hamiltonian system when $E \to 1^+$ is equal to the period of the corresponding linearized system (see [**4**, Chapter 12]). The linearized Hamiltonian is $H_L(\hat{k}_\theta, \hat{k}) = \hat{k}_\theta^2/2 + \hat{k}^2 + 1/2$ for the new variable $\hat{k} = k - 1$, which gives the equation $\hat{k}_{\theta\theta} = -2\hat{k}$ for $\hat{k}$. As any solution of this last ODE is clearly $\sqrt{2}\pi$–periodic, its semiperiod is equal to $\pi/\sqrt{2}$.

(2) As $0 < k_{\min} < 1 < k_{\max}$ for $E > 1$, we can write

$$I(E) = \int_{k_{\min}}^{1} \frac{dk}{\sqrt{E - k^2 + \log k^2}} + \int_{1}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}} = I_-(E) + I_+(E)\,.$$

We want to prove that $\lim_{E\to+\infty} I_-(E) = 0$ and $\lim_{E\to+\infty} I_+(E) = \pi/2$.
Introducing the variable $w = k/k_{\min}$ the first integral becomes

$$I_-(E) = k_{\min} \int_1^{1/k_{\min}} \frac{dw}{\sqrt{k_{\min}^2(1-w^2) + \log w^2}}\,.$$

Notice that, given a real number $0 < \alpha < 1$, it is always possible to find $\widetilde{k}(\alpha)$ such that $|k_{\min}(1-w^2)| \leq \alpha |\log w^2|$ with $w \in [1, 1/k_{\min}]$ and $k_{\min} \leq \widetilde{k}$. Fixing then such an α, we have

$$\begin{aligned}
0 \leq I_-(E) \;&\leq \frac{k_{\min}}{\sqrt{1-\alpha}} \int_1^{1/k_{\min}} \frac{dw}{\sqrt{2\log w}} \\
&\leq \frac{k_{\min}}{\sqrt{1-\alpha}} \Big(\int_1^{n} \frac{dw}{\sqrt{2\log w}} + \int_n^{1/\sqrt{k_{\min}}} \frac{dw}{\sqrt{2\log w}} + \int_{1/\sqrt{k_{\min}}}^{1/k_{\min}} \frac{dw}{\sqrt{2\log w}} \Big) \\
&\leq k_{\min}(C_1 + C_2/\sqrt{k_{\min}} + o_{k_{\min}}(1)/k_{\min})\,,
\end{aligned}$$

hence, the claim on $I_-(E)$ follows.
Regarding $I_+(E)$, we proceed in a similar way by changing again the integration variable to $w = k/k_{\max}$. In this way we obtain

$$\begin{aligned}
\lim_{E\to+\infty} I_+(E) &= \lim_{E\to+\infty} \int_{1/k_{\max}}^{1} \frac{dw}{\sqrt{1 - w^2 + \frac{2\log w}{k_{\max}^2}}} \\
&= \lim_{E\to+\infty} \int_0^1 \chi_{[1/k_{\max},1]} \frac{dw}{\sqrt{1 - w^2 + \frac{2\log w}{k_{\max}^2}}} \\
&= \pi/2\,,
\end{aligned}$$

where in the last equality we applied the dominated convergence theorem.

(3) See the original paper of Abresch and Langer **[1]** or the general result by Zevin and Pinsky in **[7]**. □

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