

EQUILIBRIA OF CHARGED HYPERELASTIC SOLIDS

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ABSTRACT. We investigate equilibria of charged deformable materials via the minimization of an electroelastic energy. This features the coupling of elastic response and electrostatics by means of a capacity term, which is naturally defined in Eulerian coordinates. The ensuing electroelastic energy is then of mixed Lagrangian–Eulerian type. We prove that minimizers exist by investigating the continuity properties of the capacity terms under convergence of the deformations.

1. INTRODUCTION

The interaction of electric and mechanical effects in solids is crucial in different modeling situations and is at the basis of a variety of applications. We focus here on the description of the electromechanical equilibrium of a charged conductor. In the absence of an external electric field, this results from the interplay of mechanical and electrostatic response. The first favors specific deformations under the effect of given external mechanical loads, whereas the latter favors shapes of larger capacitance.

More specifically, we consider the case of a hyperelastic charged conductor, embedded in an insulating medium, which we also assume to be deformable. This setting is inspired to electroactive-polymer devices, featuring indeed conductive parts embedded in polymeric matrices [18]. Coated wires, printed circuit boards, and capacitive deformation sensors [60] also fit within this framework.

The actual configuration of the whole system of conductor and insulator is specified by its deformation $y: \Omega \rightarrow \mathbb{R}^d$ from the bounded reference configuration $\Omega \subset \mathbb{R}^d$, $d \geq 2$. More precisely, we indicate the reference configuration of the conductor by $\omega \subset \Omega$, see Figure 1, so that $y(\bar{\omega})$ indicates the actual position of the deformed conductor whereas $y(\Omega \setminus \bar{\omega})$ indicates the deformed insulator.

We assume that the conductor carries a given total charge Q . Its equilibrium results from a competition of mechanical and electric actions. On the one hand, the body may be subjected to mechanical loading, favoring specific deformations. On the other hand, the actual shape of the body determines its electrostatic potential with respect to the background potential. In particular, deformations y maximizing the *electric capacitance* of the deformed shape $y(\bar{\omega})$ are preferred. A first realization of this competition, is encoded

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in the following choice for the *electroelastic* stored energy of the system

$$\mathcal{F}_1(y) := \int_{\Omega} W(x, \nabla y(x)) \, dx + \frac{Q^2}{2 \text{Cap}(y(\bar{\omega}))}, \quad (1.1)$$

resulting indeed from the sum of the *elastic stored energy* and the *electrostatic potential*.

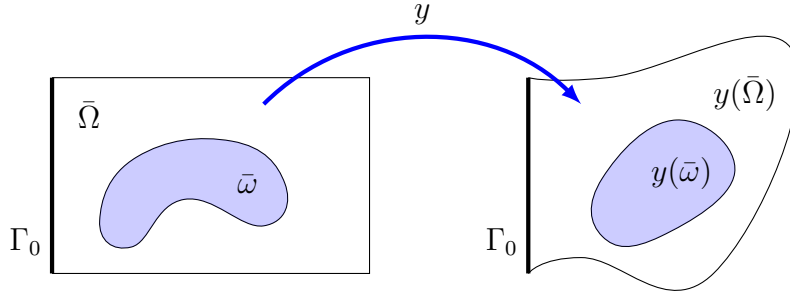


FIGURE 1. Setting of the problem.

In the expression above, $W: \Omega \times \mathbb{M}^{d \times d} \rightarrow [0, +\infty)$ is the elastic energy density of the medium, where $\mathbb{M}^{d \times d}$ indicates $(d \times d)$ -matrices. It is assumed to be a Carathéodory integrand and to satisfy the following assumptions:

$$\begin{aligned} \text{(Polyconvexity)} \quad & W(x, F) = \mathbb{W}(x, \mathcal{M}(F)) \quad \text{where } \mathbb{W}(x, \cdot) \text{ is convex} \\ & \text{in the minors } \mathcal{M}(F) \text{ of } F \in \mathbb{M}^{d \times d}, \forall x \in \Omega \end{aligned} \quad (1.2)$$

$$\text{(Growth)} \quad \text{there exists } c_W > 0, p > d, \text{ and } s > d - 1 \text{ such that}$$

$$\begin{cases} W(x, F) \geq c_W |F|^q + c_W \frac{|F|^{ds}}{\det F^s} - \frac{1}{c_W} & \text{if } \det F \geq 0, \\ W(x, F) = +\infty & \text{otherwise,} \end{cases} \quad (1.3)$$

$$\forall F \in \mathbb{M}^{d \times d}, \forall x \in \Omega.$$

In particular, the space-dependence $x \in \Omega \mapsto W(x, F)$ models the possibly very different elastic response of the conductor and the insulator. We refer to Section 3 below for a discussion on the meaning and role of the second term in the first line of (1.3). Let us mention that the above conditions (1.2)–(1.3) are compatible with *frame indifference*, namely, $W(\hat{R}F) = W(F)$ for every rotation $\hat{R} \in SO(d) := \{R \in \mathbb{M}^{d \times d} : R^{-1} = R^T \text{ and } \det R = 1\}$. Although not needed in our analysis, frame indifference is a crucial requirement from the modeling viewpoint. Note that no loads are assumed for the sake of notational simplicity. The case of nonvanishing loads can be treated as well.

As concerns the electric capacitance of the deformed body $y(\bar{\omega})$, the choice in (1.1) corresponds to that of *self-capacitance*, where the electrostatic potential is taken in relation with a far background, ideally at ∞ . From the modeling viewpoint, this corresponds to the case in which the complement of the deformed body $y(\bar{\omega})$ has negligible dielectric

response. In $d \geq 3$ dimensions, for all compact sets $E \subset \mathbb{R}^d$ one defines

$$\frac{1}{\text{Cap}(E)} = \min_{\mu} \int_E \int_E \frac{1}{\sigma_d \varepsilon_0 |x - y|^{d-2}} d\mu(x) d\mu(y)$$

where σ_d is the surface of the unit ball, ε_0 denotes the permittivity of vacuum, and the infimum is taken on nonnegative Borel measures μ with support in the compact set $E \subset \mathbb{R}^d$ and $\mu(E) = 1$. Note that such minimum exists and concentrates on the boundary ∂E . Following [38, Ch. 11.15], one can equivalently variationally reformulate the latter by letting the *capacity* of E be defined as

$$\text{Cap}(E) := \inf \left\{ \int_{\mathbb{R}^d} |\nabla v(\xi)|^2 d\xi : v \in L^{2^*}(\mathbb{R}^d) \text{ with } \nabla v \in L^2(\mathbb{R}^d; \mathbb{R}^d) \right. \\ \left. \text{and } v \geq 1 \text{ a.e. in a neighb. of } E \right\} \quad (1.4)$$

where $2^* = \frac{2d}{d-2}$, see [38, Ch. 4.3, 8.2, and 8.3].

The aim of this paper is to investigate minimizers of the electroelastic energy \mathcal{F}_1 on the set of *admissible deformations*

$$\mathcal{A} := \{y \in W^{1,q}(\Omega; \mathbb{R}^d) : y \text{ is a homeomorphism and } y = \text{id on } \Gamma_0\} \quad (1.5)$$

where the boundary portion $\Gamma_0 \subset \partial\Omega$ where the body is clamped is assumed to be nonempty and open in the topology of $\partial\Omega$. It is worth noting that the electroelastic energy \mathcal{F}_1 is of *mixed Eulerian-Lagrangian* type. Indeed, deformations are Lagrangian in nature, for they relate to the reference configuration, whereas the capacity term is Eulerian, as it depends on the actual shape $y(\bar{\omega})$ of the conductor.

We are also interested in a second, different situation where the reference electrostatic potential is that of the complement of the deformed conductor-insulator system $y(\bar{\Omega})$. This framework corresponds to the case when the complement $\mathbb{R}^d \setminus y(\bar{\Omega})$ is assumed to be conductive, which specifically refers to the setting of *capacitors*. Here, the relevant notion is that of *relative (or mutual) capacity* of two conductors E and $\mathbb{R}^d \setminus D$, where E is compact and D is open and contains E , which is specified as

$$\text{Cap}(E; D) := \inf \left\{ \int_D |\nabla v(\xi)|^2 d\xi : v \in W_0^{1,2}(D) \text{ and } v \geq 1 \text{ a.e. in a neighb. of } E \right\}. \quad (1.6)$$

In this setting, we consider the electroelastic energy

$$\mathcal{F}_2(y) := \int_{\Omega} W(x, \nabla y(x)) dx + \frac{Q^2}{2 \text{Cap}(y(\bar{\omega}); y(\Omega))},$$

again to be minimized on the class of admissible deformations \mathcal{A} .

Our main result reads as follows.

Theorem 1.1 (Existence of equilibria). *\mathcal{F}_1 and \mathcal{F}_2 admit minimizers in \mathcal{A} .*

The proof of Theorem 1.1 is given in Section 5 below and hinges upon two main ingredients: (1) a *closure* property of admissible deformations \mathcal{A} and (2) an *upper-semicontinuity* result for the capacity terms under the uniform convergence of the deformations. These two ingredients in particular allow to apply the Direct Method to \mathcal{F}_1 and \mathcal{F}_2 and secure the existence of minimizers.

For the sake of completeness, we also provide a *lower-semicontinuity* result for the capacity terms so that, ultimately,

$$\text{Cap}(y(\bar{\omega})) = \lim_{n \rightarrow +\infty} \text{Cap}(y^n(\bar{\omega})) \text{ and } \text{Cap}(y(\bar{\omega}); y(\Omega)) = \lim_{n \rightarrow +\infty} \text{Cap}(y^n(\bar{\omega}); y^n(\Omega)) \quad (1.7)$$

whenever $y: \Omega \rightarrow \mathbb{R}^d$ and $\{y^n\} \subset C^0(\bar{\Omega}; \mathbb{R}^d)$ are homeomorphisms such that $y^n \rightarrow y$ strongly in $C^0(\bar{\Omega}; \mathbb{R}^d)$. Note however that lower semicontinuity holds under some specific geometrical constraints on $y(\Omega)$. We refer to Propositions 4.5 and 6.1 below for the precise statements. The main technical tool for the proof is a detailed characterization of the monotonicity behavior of the capacity with respect to its arguments, cf. Proposition 4.4.

Before closing this introduction, let us remark that existence results in the setting of electroelastostatics are not new. The equilibrium of an electromagnetoelastic polyconvex material in void has been already investigated in the [54]. The main tool there is the careful use of A-quasiconvexity, related to relaxation under linear differential constraints. In this specific case, such constraints naturally correspond to the static Maxwell equations. A sufficient condition for the polyconvexity of isotropic electromagnetoelastic energy densities is given in [55]. In contrast with our setting, the formulation in [54, 55] is purely Lagrangian and no charge is considered.

The variational modelization in [18, 48, 49] moves along the same lines of [54], allowing charges and assuming the conductor to be surrounded by a polymeric matrix, as in our case. Let us note however, that the focus there is on modelization and simulation. In particular, no existence result for equilibria is provided.

In the series of papers [17, 45, 46, 47] the authors analyze the equilibrium shape of two-dimensional charged, perfectly conducting liquid drops. There, a variational energy of the type of \mathcal{F}_1 is studied, where nonetheless the elastic part is replaced by the perimeter of the liquid drop. Under different settings, existence for the corresponding minimization problem may hold or fail in different classes of shapes.

As already mentioned, our variational model is of mixed Eulerian–Lagrangian type, a class which has recently attracted attention due to its relevance in connection with multiphysics applications. Without any claim of completeness, let us recall that the mathematical analysis of mixed Eulerian–Lagrangian formulations have been considered in the modelization of defective crystals [8, 15], in the setting of nematic elastomers [3, 4], in dislocation-free finite plasticity [56, 33], in bulk-damage modeling [10], and in magnetostriction [53, 34]. Dimension reduction in nonlinear magnetoelasticity has been studied analytically and numerically in [36, 37, 39] under further restrictions on the Jacobian of elastic deformations. The membrane and Von Kármán regimes are the subject of [11] and [6], respectively. For energy functionals featuring both bulk and surface terms, as well as for refined phase-field models, we refer to [28], [35], and to the two recent contributions [19, 20].

Our paper is organized as follows. In Section 2, we introduce notation and recall results on Sobolev spaces with zero traces and on smooth approximations of sets. Section 3 discusses the properties of admissible deformations, as well as a connection with the theory of mappings with finite distortion. Section 4 analyzes upper semicontinuity of the capacity and Section 5 contains the proof of Theorem 1.1. Eventually, Section 6 completes our study of continuity properties for capacity terms and provides a discussion on the geometry of deformed sets, cf. Subsection 6.2.

2. NOTATION AND PRELIMINARIES

In this section, we collect definitions, notation, and preliminary results which will be used throughout the paper.

In the following, Ω is a nonempty, simply connected, bounded Lipschitz domain in \mathbb{R}^d , ω is a compactly contained subdomain of Ω , and Γ_0 is a subset of $\partial\Omega$ with $\mathcal{H}^{d-1}(\Gamma_0) > 0$, where \mathcal{H}^{d-1} stands for the $(d-1)$ -Hausdorff measure. By Sobolev embedding theorem, given $y \in W^{1,q}(\Omega, \mathbb{R}^d)$, $q > d$, we can consider its continuous up to the boundary representative $\tilde{y} \in C^0(\bar{\Omega}, \mathbb{R}^d)$. Therefore, the boundary condition $y|_{\Gamma_0} = \text{id}$ is interpreted as $\tilde{y}(x) = x$ for every $x \in \Gamma_0$.

Unless otherwise stated, throughout the paper we will use the symbol C to indicate any generic positive constant, possibly depending on data, and changing even within the same line.

2.1. Sobolev spaces with vanishing trace. In what follows, Sobolev functions vanishing at the boundary of the deformed set $y(\Omega)$ will turn out to be relevant. The boundary of the set $y(\Omega)$ may, in fact, show poor regularity. We hence need to introduce a characterization of Sobolev spaces with vanishing trace at the boundary of the set $y(\Omega)$.

Given a domain $D \subset \mathbb{R}^d$, different definitions of spaces with vanishing trace at the boundary ∂D can be considered. One possibility is letting $W_0^{1,2}(D)$ be the space defined as the closure of $C_0^\infty(D)$ in the $W^{1,2}$ -norm. An alternative is defining $\mathring{W}^{1,2}(D)$ to be the set of functions in $W^{1,2}(\mathbb{R}^d)$ that are equal to zero a.e. in $\mathbb{R}^d \setminus D$. It follows directly from these definitions that

$$W_0^{1,2}(D) \subset \mathring{W}^{1,2}(D). \quad (2.1)$$

The opposite inclusion, and hence the equality of these two spaces, holds for domains with C^0 boundaries. Note however that such continuity is difficult to ascertain a priori, for even the image of a smooth set via a homeomorphism might, in principle, have no C^0 boundary. We refer to [7] for a detailed discussion of this topic.

2.2. Smooth approximation of sets. The approximation theory from [2] entails that, for every open set $A \subset \mathbb{R}^d$, $\bar{A} \neq \mathbb{R}^d$, there exist a constant $\varepsilon_0 > 0$ and a collection of C^∞ -smooth open sets $\{A_\varepsilon\}_{0 < \varepsilon < \varepsilon_0}$ approximating A from inside in the following sense

$$\bigcup_{0 < \varepsilon < \varepsilon_0} A_\varepsilon = A \text{ and } \bar{A}_\varepsilon \subset A_{\varepsilon'} \text{ if } 0 < \varepsilon' < \varepsilon < \varepsilon_0.$$

These sets are classically defined by means of so-called *thinnings* of A , namely,

$$A_\varepsilon := \left\{ x \in \mathbb{R}^d : \widetilde{\text{dist}}(x, \mathbb{R}^d \setminus A) > \varepsilon \right\}, \quad (2.2)$$

where $\widetilde{\text{dist}}$ is a suitably regularized distance function, see [2, Remark 5.5].

Let $K \subset \mathbb{R}^d$ be a compact set with a nonempty interior. Consider the approximations B_ε of the open set $B := \mathbb{R}^d \setminus K$ as in (2.2). Clearly,

$$K^\varepsilon := \mathbb{R}^d \setminus B_\varepsilon = \left\{ x \in \mathbb{R}^d : \widetilde{\text{dist}}(x, K) \leq \varepsilon \right\} \quad (2.3)$$

are compact sets with C^∞ -boundary, approximating K from outside in the following sense

$$\bigcap_{0 < \varepsilon < \varepsilon_0} K^\varepsilon = K \text{ and } K^{\varepsilon'} \subset \text{int } K^\varepsilon \text{ if } 0 < \varepsilon' < \varepsilon < \varepsilon_0.$$

3. CLOSURE OF ADMISSIBLE DEFORMATIONS

We gather here some basic definitions and preliminaries from the setting of quasiconformal analysis and comment on the closure of the set \mathcal{A} of admissible deformations (1.5) under uniform energy bounds.

Definition 3.1 (Finite distortion). Let $f: \Omega \rightarrow \mathbb{R}^d$ be such that $f \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^d)$ and $\det \nabla f(x) \geq 0$ for almost every $x \in \Omega$. We say that f is a *mapping with finite distortion* if for almost every $x \in \Omega$ it holds that $\nabla f(x) = 0$ whenever $\det \nabla f(x) = 0$. The function

$$K_{f,p}(x) := \begin{cases} \frac{|\nabla f(x)|}{\det \nabla f(x)^{1/p}} & \text{if } 0 < \det \nabla f(x) < \infty, \\ 0 & \text{otherwise,} \end{cases} \quad (3.1)$$

is called the *outer distortion operator function* or *outer p -distortion* of f at $x \in \Omega$.

The special case $p = d$ in the definition above is one of the primary focuses of quasiconformal analysis and is analyzed in [27, 58]. If $K_{f,d} \in L^\infty(\Omega)$, Definition 3.1 corresponds to that of *quasiregular mappings*, also known as mappings with *bounded distortion*. The general case in which $1 \leq p < +\infty$, possibly with $p \neq d$ is addressed in [57] in connection with the study of the functional classes that preserve Sobolev mappings under change of variables. We refer to [51, 52] and to the monographs [24, 26] for overviews on the topics of mappings with bounded and finite distortion, respectively.

Definition 3.2 (Discrete and open maps). We say that a continuous mapping $f: D \rightarrow D'$ is *discrete* if $f^{-1}(y)$ is a discrete set for all $y \in D'$. If $f(U)$ is open for every open set $U \subset D$ we say that f is *open*.

The main properties of admissible deformations \mathcal{A} are collected in the next proposition.

Proposition 3.3 (Properties and closure of \mathcal{A}). *Let $y: \Omega \rightarrow \mathbb{R}^d$ be such that $y|_{\Gamma_0} = \text{id}$ and $\mathcal{F}_i(y) < \infty$, either for $i = 1$ or $i = 2$. Then,*

- (i) y has finite distortion;
- (ii) $K_{y,d} \in L^{ds}(\Omega)$;
- (iii) $\det \nabla y > 0$ a.e. in Ω ;
- (iv) y is continuous, open, and discrete;
- (v) y satisfies the Lusin \mathcal{N} and \mathcal{N}^{-1} conditions.

If, in addition, y is a weak limit of $W^{1,q}$ -homeomorphisms, then

- (vi) y is injective a.e., both in the image and in its domain;
- (vii) y is a homeomorphism.

In particular, if $y^n \in \mathcal{A}$ converge weakly to y in $W^{1,q}(\Omega; \mathbb{R}^d)$, then $y \in \mathcal{A}$ as well.

Proof. Properties (i) and (ii) follow immediately from (1.3). The proof of property (iii) for $W^{1,d}$ -mappings with $K_{y,d} \in L^{ds}(\Omega)$, $s > d - 1$, may be found in [32, Theorem 1.1]. Continuity, openness, and discreteness for mappings with bounded distortion have been obtained in the seminal paper [50]. Concerning mappings with finite distortion, it was shown in [58, Theorem 2.3] that $W_{\text{loc}}^{1,d}$ -mappings with finite distortion have continuous representatives. Now, since y is non-constant by $y|_{\Gamma_0} = \text{id}$, then due to [40], (i) and (ii) imply that y is open and discrete, i.e., Property (iv) holds. Property (v) is a consequence of [58, Proposition 2.4], see [41]. We refer the reader to [24], where all the aforementioned results and their consequences are discussed.

Property (vi) for limits of Sobolev homeomorphisms follows by [5], see also [44, Lemma 3.4] and [43, Theorem 10]. Finally, the Lusin \mathcal{N} -property, a.e. injectivity, and openness guarantee that y is a homeomorphism, see, for example, [19, Lemma 3.3]. \square

4. CAPACITY: MAIN PROPERTIES AND UPPER SEMICONTINUITY

Variational capacity is one of the main tools in nonlinear potential theory, see [23]. It delivers an essential understanding of the pointwise behavior in the Sobolev setting, for it measures, roughly speaking, the size of exceptional sets associated to Sobolev functions. We refer the interested reader to [23, Chapter 2] and [9] for a thorough discussion of the notion of capacity, as well as to [23, Chapter 4] for an overview on fine properties of Sobolev functions. We recall some basic properties below.

Definition 4.1. Let $F, E \subset \mathbb{R}^d$ be compact with $E \subset D \subset \mathbb{R}^d$ and D open. The *capacity* of F and the *capacity of E relative to D* are defined by

$$\text{Cap}(F) := \inf \left\{ \int_{\mathbb{R}^d} |\nabla v(\xi)|^2 \, d\xi : v \in \mathcal{C}_1(F) \right\}, \quad (4.1)$$

$$\text{Cap}(E; D) := \inf \left\{ \int_D |\nabla v(\xi)|^2 \, d\xi : v \in \mathcal{C}_2(E; D) \right\}, \quad (4.2)$$

where

$$\begin{aligned} \mathcal{C}_1(F) &:= \{ v \in L^{2^*}(\mathbb{R}^d) : \nabla v \in L^2(\mathbb{R}^d; \mathbb{R}^d), v \geq 1 \text{ a.e. in a neighb. of } F \}, \\ \mathcal{C}_2(E; D) &:= \{ v \in W_0^{1,2}(D) : v \geq 1 \text{ a.e. in a neighb. of } E \}. \end{aligned}$$

Functions in $\mathcal{C}_1(F)$ or $\mathcal{C}_2(E; D)$ are called *capacity test functions*. We say that a property holds *quasieverywhere* (*q.e.*), if it holds everywhere except from a set of zero capacity, and that a function is *quasicontinuous* on D if its discontinuity set in D has zero capacity.

The next proposition collects some basic properties of Sobolev functions related to the notion of capacity (see [23, Chapter 4]).

Proposition 4.2 (Fine properties of Sobolev functions). *Let $D \subset \mathbb{R}^d$ be an open set.*

- (i) *A function $v \in W_0^{1,2}(D)$ has a quasicontinuous representative \tilde{v} , uniquely defined q.e.*
- (ii) *Every strongly convergent sequence in $W^{1,2}(\mathbb{R}^d)$ admits a q.e. convergent subsequence in \mathbb{R}^d .*
- (iii) *A function $u \in W^{1,2}(D)$ belongs to $W_0^{1,2}(D)$ if and only if its quasicontinuous representative \tilde{u} is the restriction to D of a quasicontinuous map satisfying $\tilde{u} = 0$ q.e. on $\mathbb{R}^d \setminus D$.*

The notion of relative capacity can be equivalently reformulated as follows.

Proposition 4.3 (Equivalent formulations). *Let $F, E \subset \mathbb{R}^d$ be compact with $E \subset D \subset \mathbb{R}^d$ with D open and bounded. Then*

$$\text{Cap}(F) = \inf \left\{ \int_{\mathbb{R}^d} |\nabla v(\xi)|^2 d\xi : v \in \tilde{\mathcal{C}}_1(F) \right\}, \quad (4.3)$$

$$\text{Cap}(E; D) = \inf \left\{ \int_D |\nabla v(\xi)|^2 d\xi : v \in \tilde{\mathcal{C}}_2(E; D) \right\}, \quad (4.4)$$

where

$$\tilde{\mathcal{C}}_1(F) := \{v \in L^{2^*}(\mathbb{R}^d) : \nabla v \in L^2(\mathbb{R}^d; \mathbb{R}^d),$$

and its quasicontinuous representative \tilde{v} is such that $\tilde{v} \geq 1$ q.e. in $F\}$,

$$\tilde{\mathcal{C}}_2(E; D) := \{v \in W_0^{1,2}(D)$$

and its quasicontinuous representative \tilde{v} is such that $\tilde{v} \geq 1$ q.e. in $E\}$.

The behavior of capacity with respect to set inclusion is encoded in the next proposition.

Proposition 4.4 (Monotonicity properties of the capacity). *Let $F, F_k, E, E_k \subset \mathbb{R}^d$ be compact and D_k be bounded and open for every $k \in \mathbb{N}$. The following monotonicity properties hold:*

- (i) *If $F_1 \subset F_2$ and $E_1 \subset E_2 \subset D$, then $\text{Cap}(F_1) \leq \text{Cap}(F_2)$ and $\text{Cap}(E_1; D) \leq \text{Cap}(E_2; D)$.*
- (ii) *If $E \subset D_1 \subset D_2$, then $\text{Cap}(E; D_2) \leq \text{Cap}(E; D_1)$.*
- (iii) *If $E = \bigcap_{k=1}^{+\infty} E_k$ with $E_{k+1} \subset E_k \subset D$ and E_k is compact for every $k \in \mathbb{N}$, then*

$$\begin{aligned} \text{Cap}(E) &= \lim_{k \rightarrow \infty} \text{Cap}(E_k) = \inf_{k \rightarrow \infty} \text{Cap}(E_k), \\ \text{Cap}(E; D) &= \lim_{k \rightarrow \infty} \text{Cap}(E_k; D) = \inf_{k \rightarrow \infty} \text{Cap}(E_k; D). \end{aligned}$$

- (iv) *If $E = \bigcup_{k=1}^{+\infty} E_k$ with $E_k \subset E_{k+1} \subset E \subset D$ for every $k \in \mathbb{N}$, then*

$$\begin{aligned} \text{Cap}(E) &= \lim_{k \rightarrow \infty} \text{Cap}(E_k) = \sup_{k \rightarrow \infty} \text{Cap}(E_k), \\ \text{Cap}(E; D) &= \lim_{k \rightarrow \infty} \text{Cap}(E_k; D) = \sup_{k \rightarrow \infty} \text{Cap}(E_k; D). \end{aligned}$$

(v) If $D = \bigcup_{k=1}^{+\infty} D_k$ and E is compact, with $E \subset D_k \subset\subset D_{k+1} \subset\subset D$ for every $k \in \mathbb{N}$, then

$$\text{Cap}(E; D) = \lim_{k \rightarrow \infty} \text{Cap}(E; D_k) = \inf_{k \in \mathbb{N}} \text{Cap}(E; D_k).$$

Proof. The proof of Properties 4.4 (i), 4.4 (iii) and 4.4 (iv) can be found in [23, Theorem 2.2] and [9, Propositions 3.1, 4.1, 4.5]. Note that the proof in [23, Theorem 2.2] is performed for the relative capacity but that the case of the capacity follows by the same argument. Property 4.4 (ii) follows directly from the definition of capacity.

In order to prove Property 4.4 (v), we first notice that the inequality $\text{Cap}(E; D) \leq \inf_{k \in \mathbb{N}} \text{Cap}(E; D_k)$ follows directly from Property 4.4 (ii). Thus, it suffices to prove the opposite inequality. For convenience of the reader, we subdivide the proof into two steps.

STEP 1: assume that ∂E is of class C^∞ . Let $\varepsilon > 0$, and consider a map $u \in W_0^{1,2}(D)$ with $u \geq 1$ quasi everywhere on E , and such that

$$\text{Cap}(E; D) \geq \int_D |\nabla u|^2 dx - \varepsilon. \quad (4.5)$$

By possibly replacing u with $\bar{u} := \min\{u, 1\}$, we can assume that $u \equiv 1$ quasi everywhere on E . Let now $\eta \in C_c^\infty(D)$ be such that $\eta \equiv 1$ on \bar{E} . Note that such cut-off function exists because $|\partial E| = 0$. Set $v := u - \eta$. By definition, $v \equiv 0$ quasi everywhere on E . Thus, we can find a sequence $\{v_n\} \subset C_c^\infty(D \setminus \bar{E})$ satisfying the following properties:

$$\|v_n - (u - \eta)\|_{H^1(D)} < \varepsilon, \quad (4.6)$$

$$\text{supp}(v_n + \eta) \subset D_{k_n}, \quad (4.7)$$

for a suitable subsequence $\{D_{k_n}\} \subset \{D_k\}$. Consider now the maps $u_n := v_n + \eta$. We have that

$$\|u_n - u\|_{H^1(D)} = \|v_n + \eta - u\|_{H^1(D)} < \varepsilon, \quad (4.8)$$

$$u_n \in C_c^\infty(D) \quad \text{for every } n \in \mathbb{N}, \quad (4.9)$$

$$u_n \equiv 1 \quad \text{quasi everywhere on } E. \quad (4.10)$$

Therefore, $\{u_n\} \subset \tilde{\mathcal{A}}(E; D)$, where $\tilde{\mathcal{A}}(E; D)$ is the class in Definition (4.4). Additionally, by (4.5) and (4.8),

$$\begin{aligned} \text{Cap}(E; D) &\geq \int_D |\nabla u|^2 dx - \varepsilon \geq \int_D |\nabla u_n|^2 dx - 2\varepsilon \geq \int_{D_{k_n}} |\nabla u_n|^2 dx - 2\varepsilon \\ &\geq \text{Cap}(E; D_{k_n}) - 2\varepsilon. \end{aligned} \quad (4.11)$$

Due to the arbitrariness of $\varepsilon > 0$, this yields Property 4.4 (v) in the case of smooth sets E .

STEP 2: Let now E be an arbitrary compact subset of D . Arguing as in Subsection 2.2, we can find a sequence of smooth compact sets E_m , approximating E from outside. In view of Property 4.4 (iii), we have that

$$\text{Cap}(E; D) = \inf \left\{ \text{Cap}(E_m; D) : E_m \text{ is compact, } \partial E_m \text{ is } C^\infty, \right.$$

$$E_{m+1} \subset \text{int } E_m, E = \bigcap_{m \in \mathbb{N}} E_m \left. \vphantom{\bigcap} \right\}.$$

Fix $\varepsilon > 0$ and let $m(\varepsilon) \in \mathbb{N}$ be such that

$$\text{Cap}(E; D) \geq \text{Cap}(E_{m(\varepsilon)}; D) - \varepsilon.$$

By (4.11) and Property 4.4 (i), we deduce the existence of an index $k(m, \varepsilon)$ such that

$$\begin{aligned} \text{Cap}(E; D) &\geq \text{Cap}(E_{m(\varepsilon)}; D) - \varepsilon \geq \text{Cap}(E_{m(\varepsilon)}; D_{k(m, \varepsilon)}) - 3\varepsilon \geq \text{Cap}(E; D_{k(m, \varepsilon)}) - 3\varepsilon \\ &\geq \inf_{k \in \mathbb{N}} \text{Cap}(E; D_k) - 3\varepsilon. \end{aligned}$$

Given that ε is arbitrary, this completes the proof of Property 4.4 (v). \square

4.1. Upper semicontinuity of the capacity. This section is devoted to the proof of the lower semicontinuity of the capacity terms in \mathcal{F}_1 and \mathcal{F}_2 . This in particular rests upon the upper semicontinuity of the capacity and the relative capacity.

Proposition 4.5 (Upper semicontinuity). *Let $y: \bar{\Omega} \rightarrow \mathbb{R}^d$ and $\{y_n\} \subset C^0(\bar{\Omega}; \mathbb{R}^d)$ be homeomorphisms such that $y^n \rightarrow y$ strongly in $C^0(\bar{\Omega}; \mathbb{R}^d)$. Then,*

$$\limsup_{n \rightarrow +\infty} \text{Cap}(y^n(\bar{\omega})) \leq \text{Cap}(y(\bar{\omega})), \quad (4.12)$$

$$\limsup_{n \rightarrow +\infty} \text{Cap}(y^n(\bar{\omega}); y^n(\Omega)) \leq \text{Cap}(y(\bar{\omega}); y(\Omega)). \quad (4.13)$$

Proof. Since the maps $\{y^n\}$ are homeomorphisms, we have that $y^n(\bar{\omega})$ is compact and $y^n(\Omega)$ is a domain for every $n \in \mathbb{N}$. Let $\{E_m\}$ be a sequence of C^∞ compact sets, approximating $y(\bar{\omega})$ from outside (see Figure 2 below). The existence of such approximating sets is discussed in Subsection 2.2. By the uniform convergence of the sequence $\{y^n\}$ we deduce that $y(\bar{\omega}) \cup y^n(\bar{\omega}) \subset \text{int } E_m$ for $n \in \mathbb{N}$ large enough, and for every $m \in \mathbb{N}$. Hence, Property 4.4 (i) entails that

$$\limsup_{n \rightarrow +\infty} \text{Cap}(y^n(\bar{\omega})) \leq \text{Cap}(E_m) \quad \forall m \in \mathbb{N}.$$

By taking $m \rightarrow +\infty$ and using Property 4.4 (ii) we get (4.12).

Let now $\{D_\ell\}$ be a sequence of C^∞ open sets, approximating $y(\Omega)$ from inside (see again Figure 2). Again, the existence of such approximating sets is discussed in Subsection 2.2. By the uniform convergence we have that $D_\ell \subset y(\Omega) \cap y^n(\Omega)$ for $n \in \mathbb{N}$ large enough.

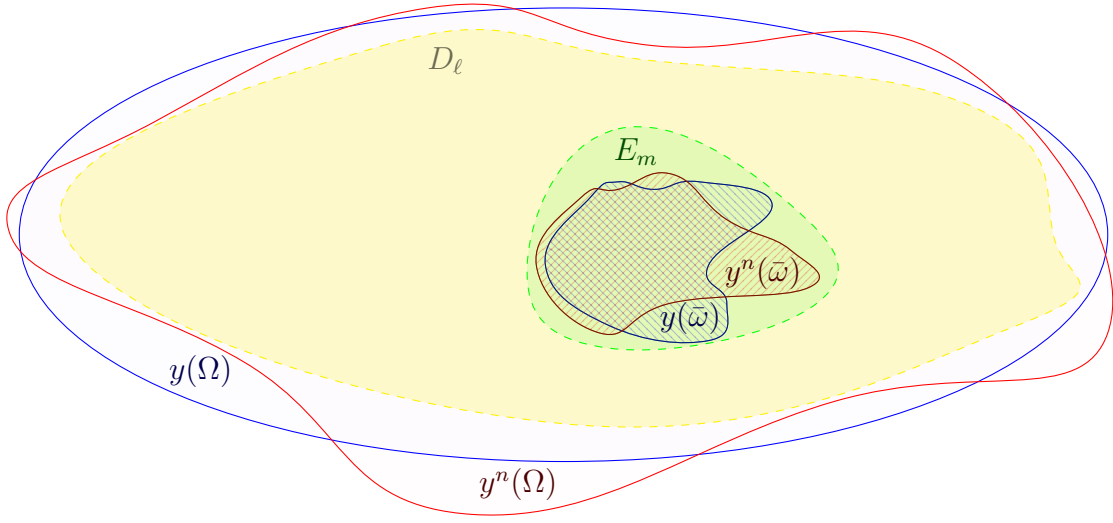
Then, from Properties 4.4 (i) and 4.4 (ii) we deduce that

$$\text{Cap}(y^n(\bar{\omega}); y^n(\Omega)) \leq \text{Cap}(E_m; y^n(\Omega)) \leq \text{Cap}(E_m; D_\ell)$$

for n large enough, and for every m and ℓ . In particular,

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \text{Cap}(y^n(\bar{\omega}); y^n(\Omega)) &\leq \inf_{\ell} \inf_m \text{Cap}(E_m; D_\ell) \\ &= \inf_{\ell} \text{Cap}(y(\bar{\omega}); D_\ell) = \text{Cap}(y(\bar{\omega}); y(\Omega)), \end{aligned}$$

where the second-to-last equality follows by Property 4.4 (iii), and the last one by Property 4.4 (v). \square

FIGURE 2. Sets D_ℓ and E_m .

5. PROOF OF THEOREM 1.1

This section is devoted to the proof of our main result, Theorem 1.1. This follows from an application of the Direct Method.

Let $\{y_1^n\}_{n \in \mathbb{N}}$, $\{y_2^n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ be minimizing sequences for the functionals \mathcal{F}_1 and \mathcal{F}_2 , respectively. The compactness of the sequences $\{y_1^n\}_{n \in \mathbb{N}}$ and $\{y_2^n\}_{n \in \mathbb{N}}$ follows directly from (1.3), and from the observation that

$$\liminf_{n \rightarrow +\infty} \mathcal{F}_1(y_1^n) \leq \mathcal{F}_1(\text{id}) = \int_{\Omega} W(x, \text{Id}) \, dx + \frac{Q^2}{2 \text{Cap}(\bar{\omega})} < +\infty,$$

$$\liminf_{n \rightarrow +\infty} \mathcal{F}_2(y_2^n) \leq \mathcal{F}_2(\text{id}) = \int_{\Omega} W(x, \text{Id}) \, dx + \frac{Q^2}{2 \text{Cap}(\bar{\omega}; \Omega)} < +\infty.$$

Hence, there exist $y_1, y_2 \in W^{1,q}(\Omega; \mathbb{R}^d)$ such that, up to extracting a not relabeled subsequence,

$$y_i^n \rightharpoonup y_i \text{ weakly in } W^{1,q}(\Omega; \mathbb{R}^d) \quad \text{for } i = 1, 2. \quad (5.1)$$

Then,

$$\int_{\Omega} |\nabla y_i|^q \, dx \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} |\nabla y_i^n|^q \, dx \quad \text{for } i = 1, 2. \quad (5.2)$$

Moreover, by (5.1), it follows that

$$\det \nabla y_i^n \rightharpoonup \det \nabla y_i \quad \text{weakly in } L^{q/d}(\Omega) \quad \text{for } i = 1, 2, \quad (5.3)$$

and so $\det \nabla y_i \geq 0$ almost everywhere in Ω , since $\det \nabla y_i^n \geq 0$ almost everywhere in Ω . Indeed, by (5.3) and Mazur's lemma, we find linear combinations d_i^n of $\det \nabla y_i^n$ such that $d_i^n \rightarrow \det \nabla y_i$ strongly in $L^{q/d}(\Omega)$, and $d_i^n \geq 0$ a.e. This yields that $\det \nabla y_i \geq 0$ almost everywhere in Ω .

Let now K_i be a weak limit of $K_{y_i^n, d}$ in $L^{ds}(\Omega)$. Then by [16] (see also [26, Theorem 8.10.1] and [59]), y_i has finite distortion and

$$\int_{\Omega} (K_{y_i, d}(x))^{ds} dx \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} (K_{y_i^n, d}(x))^{ds} dx. \quad (5.4)$$

Note also that neither $\{y_i^n\}$ nor y_i can be constant owing to the boundary conditions $y_i|_{\Gamma_0} = y_i^n|_{\Gamma_0} = \text{id}$ (see Subsection 6.2). Therefore, $y_i \in \mathcal{A}$ and satisfies Properties (i)–(vii) of Proposition 3.3.

In view of the Sobolev embedding theorem and the weak convergence (5.1), we may assume $y_i^n \rightarrow y_i$ strongly in $C^0(\bar{\Omega}, \mathbb{R}^n)$. Combining (1.3), (5.2), (5.4), and (4.12), we obtain

$$\begin{aligned} \mathcal{F}_1(y_1) &= \int_{\Omega} W(x, \nabla y_1(x)) dx + \frac{Q^2}{2 \text{Cap}(y_1(\bar{\omega}))} \\ &\leq \liminf_{n \rightarrow +\infty} \int_{\Omega} W(x, \nabla y_1^n(x)) dx + \frac{Q^2}{2 \limsup_{n \rightarrow +\infty} \text{Cap}(y_1^n(\bar{\omega}))} \\ &= \liminf_{n \rightarrow +\infty} \left(\int_{\Omega} W(x, \nabla y_1^n(x)) dx + \frac{Q^2}{2 \text{Cap}(y_1^n(\bar{\omega}))} \right) = \liminf_{n \rightarrow +\infty} \mathcal{F}_1(y_1^n) = \inf_{\mathcal{A}} \mathcal{F}_1 \end{aligned}$$

and, analogously,

$$\mathcal{F}_2(y_2) \leq \liminf_{n \rightarrow +\infty} \mathcal{F}_2(y_2^n) = \inf_{\mathcal{A}} \mathcal{F}_2,$$

so that the statement of Theorem 1.1 follows.

6. CONTINUITY OF THE CAPACITY AND GEOMETRY OF DEFORMED SETS

In this section we complete our study of continuity properties of the capacity by investigating its lower semicontinuity under some additional requirements on the geometry of the deformed configuration. Note that the proof of Theorem 1.1 does not rely on such lower semicontinuity.

6.1. Lower semicontinuity of the capacity.

Proposition 6.1 (Lower semicontinuity). *Let $y: \bar{\Omega} \rightarrow \mathbb{R}^d$ and $\{y^n\} \subset C^0(\bar{\Omega}; \mathbb{R}^d)$ be homeomorphisms such that $y^n \rightarrow y$ strongly in $C^0(\bar{\Omega}; \mathbb{R}^d)$. Then,*

$$\text{Cap}(y(\bar{\omega})) \leq \liminf_{n \rightarrow +\infty} \text{Cap}(y^n(\bar{\omega})). \quad (6.1)$$

Suppose additionally that y is such that $W_0^{1,2}(y(\Omega)) = \mathring{W}^{1,2}(y(\Omega))$. Then,

$$\text{Cap}(y(\bar{\omega}); y(\Omega)) \leq \liminf_{n \rightarrow +\infty} \text{Cap}(y^n(\bar{\omega}); y^n(\Omega)). \quad (6.2)$$

Proof. Let us start by checking (6.2). Without loss of generality, we may assume that that right-hand-side of (6.2) is finite.

By Proposition 4.3, for every $n \in \mathbb{N}$, we find $v^n \in W_0^{1,2}(y^n(\Omega))$ with quasicontinuous representative \tilde{v}^n , such that $\tilde{v}^n \geq 1$ q.e. on $y^n(\bar{\omega})$, and

$$\int_{y^n(\Omega)} |\nabla v^n(\xi)|^2 d\xi \leq \text{Cap}(y^n(\bar{\omega}); y^n(\Omega)) + \frac{1}{n}.$$

Let v_{ext}^n be an extension of v^n by 0 outside $y^n(\Omega)$, i.e., $v_{ext}^n \in W_0^{1,2}(\mathbb{R}^d)$ and $\tilde{v}_{ext}^n = 0$ a.e. on $\mathbb{R}^d \setminus y^n(\Omega)$. Then, for all $n \geq n_0$ it holds that

$$\int_{\mathbb{R}^d} |\nabla v_{ext}^n(\xi)|^2 d\xi = \int_{y^n(\Omega)} |\nabla v^n(\xi)|^2 d\xi.$$

Therefore $\{v_{ext}^n\}_{n \in \mathbb{N}} \subset W_0^{1,2}(\mathbb{R}^d)$ is bounded. Hence, there exists $v \in W_0^{1,2}(\mathbb{R}^d)$ such that, up to subsequence, $v_{ext}^n \rightharpoonup v$ weakly in $W_0^{1,2}(\mathbb{R}^d)$.

We proceed by showing that $v \in \tilde{\mathcal{C}}_2(y(\bar{\omega}), y(\Omega))$. By Mazur's lemma (see, e.g., [14, p. 6]) we find a sequence $\{u^n\} \subset W_0^{1,2}(\mathbb{R}^d)$ such that $u^n \rightarrow v$ strongly in $W_0^{1,2}(\mathbb{R}^d)$ with the property that u^n is a convex combination of $\{v_{ext}^n, v_{ext}^{n+1}, \dots\}$. In particular, denoting by $Y^n(\bar{\omega})$ the set $Y^n(\bar{\omega}) := \bigcap_{k=n}^{\infty} y^k(\bar{\omega})$, we have that the quasicontinuous representatives

$\{\tilde{u}^n\}$ associated to $\{u^n\}$ satisfy $\tilde{u}^n \geq 1$ q.e. on $\bigcap_{k=n}^{N_n} y^k(\bar{\omega}) \supset Y^n(\bar{\omega})$ and $\tilde{u}^n = 0$ q.e. on $\mathbb{R}^d \setminus \bigcup_{k=n}^{N_n} y^k(\Omega)$, for a suitable integer $N_n \geq n$.

In view of Property (ii) of Proposition 4.2 we infer that, up to subsequences,

$$u^n \rightarrow v \quad \text{q.e. on } \mathbb{R}^d. \quad (6.3)$$

Additionally,

$$|\chi_{Y^n(\bar{\omega})} u^n - \chi_{y(\bar{\omega})} v| \leq |\chi_{Y^n(\bar{\omega})} (u^n - v)| + |(\chi_{Y^n(\bar{\omega})} - \chi_{y(\bar{\omega})}) v|.$$

The first term converges to 0 as $n \rightarrow \infty$ due to the fact that $\|\chi_{Y^n(\bar{\omega})}\|_{L^\infty(\mathbb{R}^d)} \leq 1$, and by (6.3). The second term is infinitesimal owing to the uniform convergence of $\{y^n\}$. Thus,

$$\chi_{Y^n(\bar{\omega})} u^n \rightarrow \chi_{y(\bar{\omega})} v \quad \text{q.e. on } \mathbb{R}^d, \quad (6.4)$$

and hence, $v = 1$ q.e. on $y(\bar{\omega})$.

We now show that $v = 0$ a.e. in $\mathbb{R}^d \setminus y(\Omega)$. For any bounded measurable set $F \subset \mathbb{R}^d \setminus y(\Omega)$, we have

$$\left| \int_F v(\xi) d\xi \right| = \lim_{n \rightarrow \infty} \left| \int_F v_{ext}^n(\xi) d\xi \right| \leq \lim_{n \rightarrow \infty} \int_{y^n(\Omega) \setminus y(\Omega)} |v_{ext}^n(\xi)| d\xi = 0.$$

The last equality follows from the equiintegrability of $\{v_{ext}^n\}_{n \in \mathbb{N}}$, as well as from the fact that $|y^n(\Omega) \setminus y(\Omega)| \rightarrow 0$ (due to the uniform convergence of $\{y^n\}$). Therefore, we conclude that $v \in \dot{W}^{1,2}(y(\Omega))$. Since $\dot{W}^{1,2}(y(\Omega)) = W_0^{1,2}(y(\Omega))$, this yields that $v \in \tilde{\mathcal{A}}(y(\bar{\omega}), y(\Omega))$.

The lower semicontinuity of the capacity follows then from the chain of inequalities

$$\begin{aligned} \text{Cap}(y(\bar{\omega}); y(\Omega)) &\leq \int_{y(\Omega)} |\nabla v(\xi)|^2 \, d\xi \leq \int_{\mathbb{R}^d} |\nabla v(\xi)|^2 \, d\xi \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} |\nabla v_{ext}^n(\xi)|^2 \, d\xi \\ &= \liminf_{n \rightarrow \infty} \int_{y^n(\Omega)} |\nabla v^n(\xi)|^2 \, d\xi \leq \liminf_{n \rightarrow \infty} \text{Cap}(y^n(\bar{\omega}); y^n(\Omega)). \end{aligned}$$

The proof of (6.1) follows the same lines as above, with the simplification of not requiring extensions. In particular, we can find a sequence $\{v^n\}_{n \in \mathbb{N}} \subset \tilde{\mathcal{C}}_1(y^n(\bar{\omega}))$ with

$$\int_{\mathbb{R}^d} |v^n(\xi)| \, d\xi \leq \text{Cap}(y^n(\bar{\omega})) + \frac{1}{n}$$

such that, at least for a not relabeled subsequence, $\nabla v^n \rightharpoonup \nabla v$ weakly in $L^2(\mathbb{R}^d, \mathbb{R}^d)$ with $v \in \tilde{\mathcal{C}}_1(y(\bar{\omega}))$. We hence have that

$$\text{Cap}(y(\bar{\omega})) \leq \int_{\mathbb{R}^d} |\nabla v(\xi)|^2 \, d\xi \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} |\nabla v^n(\xi)|^2 \, d\xi = \liminf_{n \rightarrow \infty} \text{Cap}(y^n(\bar{\omega})). \quad \square$$

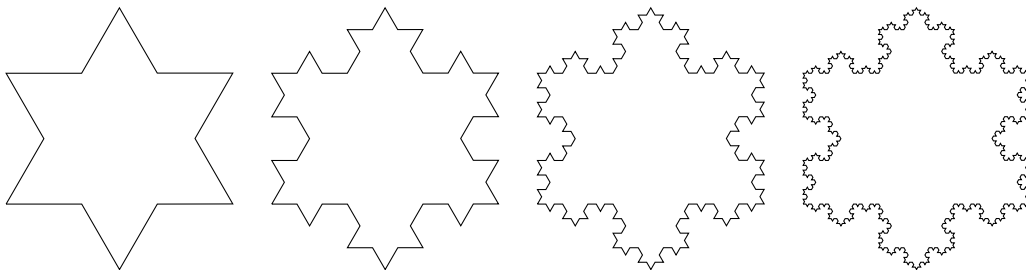
6.2. On the regularity of the deformed $y(\Omega)$. In case of the relative capacity $\text{Cap}(y(\bar{\omega}), y(\Omega))$, the lower semicontinuity result from Proposition 6.1 is conditional to the fact that $W_0^{1,2}(y(\Omega)) = \mathring{W}^{1,2}(y(\Omega))$, for this is needed in order to have that limiting maps v satisfy $v \in H_0^1(y(\Omega))$.

As already mentioned in Subsection 2.1, the two spaces above can be identified in case the boundary of the deformed set $y(\Omega)$ is C^0 . On the other end, even if y is a homeomorphism arising as uniform limit of homeomorphisms $\{y^n\}$ for which $y^n(\Omega)$ is a C^0 domain for every n , it is a priori not guaranteed that the boundary of $y(\Omega)$ will show such regularity.

An explicit counterexample is provided by the Koch snowflake \mathfrak{X} . This set does not have a C^0 -boundary and can be realized as image of a ball $B(0, 1)$ under a quasiconformal map $y: \mathbb{R}^d \rightarrow \mathbb{R}^d$. In dimension $d = 2$, this follows from Ahlfors' three point condition [1], see also [29, Theorem 2.7]; the case $d = 3$ is studied in [42]. On the other hand, y is the uniform limit of mappings $\{y^n\}$, such that $y^n(\Omega)$ is a C^0 -domain \mathfrak{X}_n (polyhedron) for every $n \in \mathbb{N}$, cf. Figure 3 below.

Note however that the lack of regularity for the boundary of the deformed set $y(\Omega)$ may be overcome by imposing additional constraints on the approximating deformations $\{y^n\}$, for instance, by requiring that the sets $y^n(\Omega)$ are Lipschitz with the same constant L for all $n \in \mathbb{N}$. A slightly weaker assumption would be to impose that all sets $y^n(\Omega)$ are uniformly regular: a bounded set $D \subset \mathbb{R}^d$ is called *regular* if there are positive constants b and r_0 such that for all $z \in \partial D$ and all $0 < r \leq r_0$ there holds

$$|B(z, r) \cap (\mathbb{R}^d \setminus D)| \geq b|B(z, r)|.$$

FIGURE 3. Approximation of the Koch snowflake \mathfrak{X}_j , $j = 1, 2, 3, 4$

In other words, a set is regular if the density of its complement is large enough. This holds, for example, for sets satisfying an outer cone condition. Recall the following characterization of Sobolev functions with zero traces, taken from [13, Theorems 4.1–4.2]. The first part of the statement below may be found in [12, Theorem V.3.4 and Remark V.3.5]. For the second part, under a cone property of D a proof can be found in [30] and [12, Theorem X.6.7 and Remark X.6.8], we also refer to [31] for the case of weaker integrability assumptions.

Lemma 6.2. *Let D be a bounded domain, $1 \leq p < \infty$, and for every $\xi \in \mathbb{R}^d$, let $d(\xi) := \text{dist}(\xi, \mathbb{R}^d \setminus D)$. If $u/d \in L^p(D)$ and $u \in W^{1,p}(D)$, then $u \in W_0^{1,p}(D)$.*

If, instead, D is a bounded regular domain and $1 < p < \infty$, then $u \in W_0^{1,p}(D)$ if and only if $u/d \in L^1(D)$ and $u \in W^{1,p}(D)$.

The next lemma follows from [22, Theorem 3.3] and [21, Proposition 1].

Lemma 6.3. *Let D be a bounded regular domain in \mathbb{R}^d and $1 < p < \infty$. Then there exists a constant C , depending only on p , d , b and r_0 , such that the inequality*

$$\left\| \frac{u}{d} \right\|_{L^p(D)} \leq C \|\nabla u\|_{L^p(D)} \quad (6.5)$$

holds for all $u \in W_0^{1,p}(D)$.

In the next lemma, we show that the condition of regularity of deformed domains is closed under weak Sobolev convergence.

Lemma 6.4. *Let $y, y^n \in W^{1,q}(\Omega; \mathbb{R}^d)$, $q > d$, be homeomorphisms such that $y^n \rightharpoonup y$ weakly in $W^{1,q}(\Omega; \mathbb{R}^d)$, and $y^n(\Omega)$ is a regular domain with constants b and r_0 for every $n \in \mathbb{N}$. Then $y(\Omega)$ is also a regular domain.*

Proof. From classical set theory there holds $A \cap B \supset (C \cap D) \setminus ((C \setminus A) \cup (D \setminus B))$, and hence, $|A \cap B| \geq |C \cap D| - |C \setminus A| - |D \setminus B|$ for every collection of sets A, B, C , and D . By Sobolev embedding theorems we can assume that $y^n \rightarrow y$ strongly in $C^0(\bar{\Omega}, \mathbb{R}^d)$. Let $z \in \partial y(\Omega)$, and let $\{z_n\}$ be a sequence of points such that $z^n \in \partial y^n(\Omega)$ for every $n \in \mathbb{N}$, and $z^n \rightarrow z$. Then, from the uniform regularity of the sets $y^n(\Omega)$ we find

$$\begin{aligned} |B(z, r) \cap (\mathbb{R}^d \setminus y(\Omega))| &\geq |B(z^n, r) \cap (\mathbb{R}^d \setminus y^n(\Omega))| - |B(z^n, r) \setminus B(z, r)| - |y(\Omega) \setminus y^n(\Omega)| \\ &\geq b|B(z^n, r)| - |B(z^n, r) \setminus B(z, r)| - |y(\Omega) \setminus y^n(\Omega)|. \end{aligned}$$

Now fix $\varepsilon > 0$. For $n_0 \in \mathbb{N}$ large enough there holds $|B(z^n, r) \setminus B(z, r)| \leq \varepsilon$ and $|y(\Omega) \setminus y^n(\Omega)| \leq |\partial y(\Omega)| + \varepsilon$ for every $n \geq n_0$. Hence,

$$|B(z, r) \cap (\mathbb{R}^d \setminus y(\Omega))| \geq b|B(z^n, r)| - |\partial y(\Omega)| - 2\varepsilon,$$

for every $n \geq n_0$. Since $y \in W^{1,q}(\Omega; \mathbb{R}^d)$ and Ω is a bounded Lipschitz domain, there exists an extension $\tilde{y} \in W^{1,q}(\tilde{\Omega}; \mathbb{R}^d)$, where $\tilde{\Omega} \supset \bar{\Omega}$ is a domain with smooth boundary, such that, by identifying the maps with their continuous representative, $\tilde{y}|_{\tilde{\Omega}} = y|_{\bar{\Omega}}$. Since y is a homeomorphism, arguing as in [33, Lemma 3.1] we have that $|\partial y(\Omega)| = |y(\partial\Omega)| = |\tilde{y}(\partial\Omega)| = 0$ as $|\partial\Omega| = 0$ and \tilde{y} satisfies the Lusin \mathcal{N} -condition, see [41]. Owing to the arbitrariness of ε , passing to the limit as $n \rightarrow +\infty$ we obtain

$$|B(z, r) \cap (\mathbb{R}^d \setminus y(\Omega))| \geq b|B(z, r)|,$$

which in turn yields that $y(\Omega)$ is regular with constants b and r_0 . \square

Combining Proposition 6.1 and Lemmas 6.2–6.4, we are in the position of presenting a lower semicontinuity result for the capacity under no a priori condition on the regularity of the boundary of $y(\Omega)$ but under uniform regularity of $\{y^n(\Omega)\}$.

Proposition 6.5. *Let $y, y^n \in W^{1,q}(\Omega; \mathbb{R}^d)$, $q > d$, be homeomorphisms such that $y^n \rightharpoonup y$ weakly in $W^{1,q}(\Omega; \mathbb{R}^d)$, and $y^n(\Omega)$ is a regular domain with constants b and r_0 for every $n \in \mathbb{N}$. Then (6.2) holds.*

Proof. In view of Lemma 6.2, following the proof of Proposition 6.1, it is enough to show that v/d is bounded in $L^2(y(\Omega))$. First, from Lemma 6.3, we obtain the uniform bound

$$\int_{y^n(\Omega)} \left| \frac{v^n(\xi)}{d^n(\xi)} \right|^2 d\xi \leq C \int_{y^n(\Omega)} |\nabla v^n|^2 d\xi \leq C, \quad (6.6)$$

for a constant C independent of n .

Moreover, up to subsequences,

$$\frac{v^n(\xi)}{d^n(\xi)} \chi_{y^n(\Omega)}(\xi) \rightarrow \frac{v(\xi)}{d(\xi)} \chi_{y(\Omega)}(\xi) \quad \text{a.e. in } \mathbb{R}^d. \quad (6.7)$$

Indeed, the almost everywhere convergence of $\{v^n\}$ to v follows from Sobolev embeddings, whereas the pointwise convergence of $\{d^n\}$ to d results from the uniform convergence of y^n to y . Convergence (6.7) follows then directly for $\xi \in y(\Omega)$, as $d^n(\xi), d(\xi) > \alpha > 0$ for n large enough. Analogously, if $\xi \in \mathbb{R}^d \setminus y(\Omega)$, then $\xi \in \mathbb{R}^d \setminus y^n(\Omega)$ for n big enough, and so both sides of (6.7) are equal to 0. Arguing as in the proof of Lemma 6.4, we find that $\partial y(\Omega)$ has measure zero, which completes the proof of (6.7).

From the pointwise convergence (6.7) and from the boundedness in (6.6) we conclude that $\frac{v}{d} \chi_{y(\Omega)}$ is the weak limit of $\frac{v^n}{d^n} \chi_{y^n(\Omega)}$ in $L^2(\mathbb{R}^d)$ (see, for example, [25, Theorem 13.44]). Thus,

$$\int_{y(\Omega)} \left| \frac{v(\xi)}{d(\xi)} \right|^2 d\xi \leq \liminf_{n \rightarrow \infty} \int_{y^n(\Omega)} \left| \frac{v^n(\xi)}{d^n(\xi)} \right|^2 d\xi \leq C,$$

which in turn yields the thesis. \square

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