An Inverse Problem for Two-Frequency Photon Transport in a Slab

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Abstract

An inverse problem of photon transport in a dusty medium with slab symmetry is studied. The problem consists in finding the unknown densities of two different kinds of dust from measurements of radiation intensities at two different frequencies. Under suitable assumptions, the problem is shown to have a unique solution. Some numerical experiments are also presented.

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1 Introduction

The subject of this article is an inverse photon transport problem, motivated by astrophysics, which consists in obtaining the unknown densities of two different kinds of materials present in a dusty medium (say, and interstellar cloud) by flux measurements of photons with two different frequencies $\nu_1 > \nu_2$ (say, UV and IR). Clearly, the description of photon transport in an interstellar cloud requires a three-dimensional transport equation in a rather complicated geometry (possibly stochastic, see e.g. Ref. [**?**]). Here, for the sake of simplicity, we shall set the problem in a space-homogeneous, slab geometry. To better understand this setting, one can think to a laboratory experiment in which a light beam crosses a uniform gaseous mixture of two dusts contained in a transparent box.

The mathematical model leads to a system of two stationary transport equations for the phase-space densities $f_1(x, \mu)$ and $f_2(x, \mu)$ of of photons with frequencies ν_1 and ν_2, respectively. Here, $x \in [0, l]$ is the position variable (l being the thickness of the slab) and $\mu \in (-1, 1)$ is the direction cosine. The stationary transport equations are assumed to have the following form, [**?**, **?**],

$$\mu \frac{\partial f_1}{\partial x}(x, \mu) + \Sigma_1 \, f_1(x, \mu) = \Sigma_{1\to 1} \int_{-1}^{1} p_{1\to 1}(\mu' \to \mu) \, f_1(x, \mu') \, d\mu' \tag{1a}$$

$$\begin{aligned} \mu \frac{\partial f_2}{\partial x}(x, \mu) + \Sigma_2 \, f_2(x, \mu) = \Sigma_{1\to 2} \int_{-1}^{1} p_{1\to 2}(\mu' \to \mu) \, f_1(x, \mu') \, d\mu' \\ + \Sigma_{2\to 2} \int_{-1}^{1} p_{2\to 2}(\mu' \to \mu) \, f_2(x, \mu') \, d\mu', \end{aligned} \tag{1b}$$

where, $\Sigma_{1\to 1} \geq 0$, $\Sigma_{1\to 2} \geq 0$, $\Sigma_{2\to 2} \geq 0$ are the scattering cross-sections, $p_{1\to 1} \geq 0$, $p_{1\to 2} \geq 0$, $p_{2\to 2} \geq 0$ are the scattering probability densities and

$$\Sigma_1 := \Sigma_{1\to 1} + \Sigma_{1\to 2} + \Sigma_{1,c}, \qquad \Sigma_2 := \Sigma_{2\to 2} + \Sigma_{2,c}, \tag{2}$$

are the total cross-sections ($\Sigma_{1,c} \geq 0$ and $\Sigma_{2,c} \geq 0$ are the capture cross-sections). Since we are assuming the medium to be space-homogeneous, then all the cross-sections and scattering probabilities are independent of x. Moreover, note that, since $\nu_1 > \nu_2$, the energy-increasing scattering $2 \to 1$ is not considered for obvious physical reasons.

Assuming the scattering to be number-conservative, we must have

$$\int_{-1}^{1} p_{i\to j}(\mu' \to \mu) \, d\mu = 1 \tag{3}$$

for all $\mu' \in [-1, 1]$ and for all $i, j = 1, 2$ excluding $i = 1$, $j = 2$, because $p_{2\to 1} \equiv 0$. Moreover, we assume that

$$p(\mu' \to \mu) = p(\mu \to \mu'). \tag{4}$$

Since we are considering a homogeneous dusty medium composed of two kinds of dust, with different physical properties, for $i, j = 1, 2$ we shall put

$$\Sigma_{i\to j} = \rho_1\,\sigma^1_{i\to j} + \rho_2\,\sigma^2_{i\to j}, \qquad \Sigma_{i,c} = \rho_1\,\sigma^1_{i,c} + \rho_2\,\sigma^2_{i,c}, \tag{5}$$

where $\rho_1 \geq 0$ and $\rho_2 \geq 0$ are the (constant) densities of the two dusts and the σ's are microscopic cross-sections. We assume the scattering properties to be identical for the two dusts and, therefore, the probabilities p's do not depend on the dust index. The importance of this assumption will be clear in Sec. **??**.

To complete our model, it remains to formulate suitable boundary conditions. We shall consider the following conditions of assigned inflow:

$$\begin{aligned} &f_1(0,\mu) = \varphi_1^+(\mu), \quad f_2(0,\mu) = \varphi_2^+(\mu), \quad \text{for } \mu \in (0,1) \\ &f_1(l,\mu) = \varphi_1^-(-\mu), \quad f_2(l,\mu) = \varphi_2^-(-\mu), \quad \text{for } \mu \in (-1,0), \end{aligned} \tag{6}$$

where $\varphi_1^\pm(\mu)$ and $\varphi_2^\pm(\mu)$ are known incoming photon distributions at both sides of the slab at the two frequencies (they are considered as function of $\mu \in [0,1]$ for future notation convenience).

The inverse problem consists in finding the unknown dust densities, ρ_1 and ρ_2, from the knowledge of the integrated right-outflows:

$$H_1 := \int_0^1 f_1(l,\mu)\mu\,d\mu, \qquad H_2 := \int_0^1 f_2(l,\mu)\mu\,d\mu. \tag{7}$$

An inverse problem similar to ours has been considered in Ref. [**?**], where the case of an interstellar cloud composed by a unique type of dust (without the assumption of slab symmetry) is considered, the inverse problem consisting in finding the value of the cross-section from measurements of the far field.

The present paper is organized as follows. In Section **??** we treat the direct problem in a fairly general setting. In Section **??** we setup the inverse problem and analyze it in the simplifying assumptions of no frequency-scattering and no back-scattering. In Sec. **??** some results of Sec. **??** is generalized to the back-scattering case. Finally, in Section **??** we report some numerical experiments.

2 The direct problem

In this section we study the direct problem (**??**) with boundary conditions (**??**). Thus, we are not concerned here with the decomposition of cross-sections with respect to the "dust index" (see eq. (**??**)).

The physical presentation of Sec. 1 motivates the following mathematical setting. Let us consider the Banach space

$$X^2 = X \times X, \quad X := L^1\left((0,l)\times(-1,1)\right), \tag{8}$$

whose elements are couples

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

of (equivalence classes of) Lebesgue-summable functions on the phase-space $(0, l) \times (-1, 1)$. The norm of X^2 is given by

$$\|\boldsymbol{f}\|_{X^2} = \|f_1\|_X + \|f_2\|_X = \int_0^l dx \int_{-1}^1 (|f_1(x, \mu)| + |f_2(x, \mu)|)\, d\mu. \tag{9a}$$

The positive cone $\left[X^2\right]^+$ of X^2, [**?**], is defined as

$$\left[X^2\right]^+ := \big\{\boldsymbol{f} \in X^2 \mid f_1(x, \mu) \geq 0,\ f_2(x, \mu) \geq 0, \\ \text{for a.e. } (x, \mu) \in (0, l) \times (-1, 1)\big\}. \tag{9b}$$

It is possible to interpret every element $\boldsymbol{f} \in \left[X^2\right]^+$ as a couple

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

of photon distributions in the slab. In this case, the norm $\|\boldsymbol{f}\|_{X^2}$ is interpreted as the total number of photons (ν_1-photons + ν_2-photons).

Let us now define the operators Σ and K acting on a generic $\boldsymbol{f} \in X^2$. The first one simply multiplies each component f_i of $\boldsymbol{f}$ by the corresponding cross-section Σ_i, i.e.

$$\Sigma := \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}, \tag{10}$$

The second one is a scattering operator:

$$K := \begin{pmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{pmatrix}, \tag{11a}$$

where, for $f \in X$ and $i, j = 1, 2$, we put

$$(K_{ij} f)(x, \mu) := \int_{-1}^1 k_{ij}(\mu, \mu')\, f(x, \mu')\, d\mu', \tag{11b}$$

and

$$k_{ij}(\mu, \mu') := \Sigma_{j \to i}\, p_{j \to i}(\mu' \to \mu), \tag{11c}$$

(with $k_{12}(\mu, \mu') \equiv 0$). From definition (**??**) we immediately have that

$$k_{ij} \in L^1([-1, 1] \times [-1, 1]), \qquad k_{ij} \geq 0, \quad i, j = 1, 2. \tag{12}$$

and, accordingly to (**??**),

$$\int_{-1}^1 k_{ij}(\mu, \mu')\, d\mu = \Sigma_{j \to i}, \qquad \mu' \in [-1, 1], \tag{13}$$

which imply that K is a bounded operator on X^2. Note that the stationary transport equation (**??**) can be concisely written as

$$\mu \frac{\partial}{\partial x} \boldsymbol{f} = -\Sigma \boldsymbol{f} + K \boldsymbol{f}. \tag{14}$$

Let us now define the operator B, acting in X^2, as follows:

$$B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}, \tag{15a}$$

where, for $i, j \in \{1, 2\}$ and $f \in X$, we put

$$(B_{ij} f)(x, \mu) := \begin{cases} \dfrac{1}{\mu} \displaystyle\int_0^x \mathrm{e}^{-\Sigma_i \frac{x-x'}{\mu}} (K_{ij} f)(x', \mu)\, dx', & \text{if } \mu \in (0, 1), \\ -\dfrac{1}{\mu} \displaystyle\int_x^l \mathrm{e}^{-\Sigma_i \frac{x-x'}{\mu}} (K_{ij} f)(x', \mu)\, dx', & \text{if } \mu \in (-1, 0). \end{cases} \tag{15b}$$

The "mild version" of system (**??**) with boundary conditions (**??**) (obtained by "solving" eqs. (**??**) as if the right-hand side terms were known) can be recast into the following form:

$$\boldsymbol{f} = B \boldsymbol{f} + \boldsymbol{q}, \tag{16}$$

where we put

$$\boldsymbol{q} := \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad q_i(x, \mu) := \begin{cases} \mathrm{e}^{-\Sigma_i \frac{x}{\mu}} \varphi_i^+(\mu), & \text{if } \mu \in (0, 1), \\ \mathrm{e}^{-\Sigma_i \frac{x-l}{\mu}} \varphi_i^-(-\mu), & \text{if } \mu \in (-1, 0). \end{cases} \tag{17}$$

Definition 2.1. *We call* mild solution *of system* (**??**) *with boundary conditions* (**??**), *a solution* $\boldsymbol{f} \in X^2$ *of eq.* (**??**).

In order to solve eq. (**??**), we need suitable assumptions on the inflow datum. To this aim we introduce the "inflow space"

$$Y^2 = Y \times Y, \qquad Y := L^1 \left((-1, 1),\, |\mu|\, d\mu \right), \tag{18a}$$

which is a Banach space with norm

$$\|\boldsymbol{\varphi}\|_{Y^2} = \|\varphi_1\|_Y + \|\varphi_2\|_Y = \int_{-1}^{1} \left(|\varphi_1(\mu)| + |\varphi_2(\mu)| \right) |\mu|\, d\mu, \tag{18b}$$

whose non-negative elements $\boldsymbol{\varphi} \in \left[Y^2 \right]^+$ can be interpreted as incoming flows of photons in the slab. Note that φ_i is defined by (**??**) and so $\boldsymbol{\varphi}$ takes into account the incoming flux φ_i^+ at $x = 0$ and the incoming flux φ_i^- at $x = l$.

Lemma 2.1. *Assume that* Σ_1, Σ_2 *are strictly positive and that* $\boldsymbol{\varphi} \in Y^2$, *where*

$$\boldsymbol{\varphi} := \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \qquad \varphi_i(\mu) := \begin{cases} \varphi_i^+(\mu), & \text{if } \mu \in (0,1), \\ \varphi_i^-(-\mu), & \text{if } \mu \in (-1,0). \end{cases} \tag{19}$$

Then $\boldsymbol{q} \in X^2$.

Proof. Let q_1 and q_2 be the two components of $\boldsymbol{q}$. For q_1, using (**??**) we can write

$$\int_0^l |q_1(x,\mu)|\, dx = \frac{|\mu|}{\Sigma_1}\left(1 - \mathrm{e}^{-\Sigma_1 \frac{l}{|\mu|}}\right)|\varphi_1(\mu)| \leq \frac{|\mu|}{\Sigma_1}|\varphi_1(\mu)|,$$

for all $\mu \in (-1,1)$. Thus, since $\varphi_1 \in Y$,

$$\|q_1\|_X = \int_{-1}^1 d\mu \int_0^l |q_1(x,\mu)|\, dx \leq \frac{1}{\Sigma_1}\int_{-1}^1 |\varphi_1(\mu)||\mu|\, d\mu < +\infty.$$

In the same manner we can prove that

$$\|q_2\|_X \leq \frac{1}{\Sigma_2}\int_{-1}^1 |\varphi_2(\mu)||\mu|\, d\mu < +\infty$$

and, therefore, $\boldsymbol{q} \in X^2$, with $\|\boldsymbol{q}\|_{X^2} \leq \max\{1/\Sigma_1, 1/\Sigma_2\}\, \|\boldsymbol{\varphi}\|_{Y^2}$. $\square$

Lemma 2.2. *The operator* B *is a bounded operator on* X^2. *Moreover, if* $\Sigma_{1,c} > 0$ *and* $\Sigma_{2,c} > 0$, *then* $I - B$ *is invertible and the inverse is given by*

$$(I-B)^{-1} = \begin{pmatrix} (I-B_{11})^{-1} & 0 \\ (I-B_{22})^{-1}B_{21}(I-B_{11})^{-1} & (I-B_{22})^{-1} \end{pmatrix}. \tag{20}$$

Proof. Let $f \in X$. For $\mu \in (-1,0)$ we easily obtain

$$\int_0^l |(B_{11}f)(x,\mu)|\, dx \leq \frac{1}{\Sigma_1}\int_0^l \left|(K_{11}f)(x',\mu)\right| dx' = \frac{\Sigma_{1\to 1}}{\Sigma_1}\int_0^l \left|f(x',\mu)\right| dx',$$

where the last equality is easily proved by using (**??**) and (**??**). In a similar way it can be proved that the same inequality holds also for $\mu \in (0,1)$. Thus we have

$$\|B_{11}f\|_X \leq \int_{-1}^1 \int_0^l |(B_{11}f)(x,\mu)|\, dx\, d\mu \leq \frac{\Sigma_{1\to 1}}{\Sigma_1}\|f\|_X .$$

In an analogous way we can prove that

$$\|B_{22}f\|_X \leq \frac{\Sigma_{2\to 2}}{\Sigma_2}\|f\|_X .$$

From our assumptions, the positive coefficients $\Sigma_{1\to1}/\Sigma_1$ and $\Sigma_{2\to2}/\Sigma_2$ are strictly less than 1 (see definition (**??**)) and, therefore, the operators B_{11} and B_{22} are strict contractions on X. Thus the inverse operators $(I - B_{11})^{-1}$ and $(I - B_{22})^{-1}$ exist. Since it is not difficult to prove that also B_{21} is a bounded operator on X, with

$$\|B_{21}f\|_X \leq \frac{\Sigma_{1\to2}}{\Sigma_2}\|f\|_X .$$

then we have that B is a bounded operator on X^2 and that $(I - B)^{-1}$ is also bounded. Finally, one can check directly that (**??**) holds. Note that $\Sigma_{1\to2}/\Sigma_2$ is not necessarily less than 1, but this does not affect the proof. □

As an immediate consequence of Lemma **??** we obtain the existence and uniqueness of a mild solution to system (**??**) with boundary conditions (**??**).

Proposition 2.1. *If $\boldsymbol{\varphi} \in Y^2$, $\Sigma_{1,c} > 0$ and $\Sigma_{2,c} > 0$, then the equation (**??**) has a unique solution $\boldsymbol{f} \in X^2$, given by $\boldsymbol{f} = (I - B)^{-1}\boldsymbol{q}$. Moreover, if $\boldsymbol{\varphi} \in \left[Y^2\right]^+$, then $\boldsymbol{f} \in \left[X^2\right]^+$.*

Proof. The first part of the theorem follows immediately from Lemma **??** and Lemma **??**. As far as the non-negativity is concerned, note that $\boldsymbol{\varphi} \in [Y^+]^2$ implies $\boldsymbol{q} \in [X^+]^2$ and that the operators B_{ij} preserve non-negativity. Then, since

$$(I - B_{ii})^{-1} = \sum_{k=0}^{\infty} B_{ii}^k, \qquad i = 1, 2,$$

we have that also $(I - B_{11})^{-1}$ and $(I - B_{22})^{-1}$ preserve non-negativity and so does B (from (**??**)). □

In our particular case, in which the medium is assumed to be homogeneous, the mild solution is indeed more regular, as it is stated in the following Proposition.

Proposition 2.2. *Under the assumptions of Lemma **??**, the mild solution $\boldsymbol{f}$ is also a strong solution, i.e. it has distributional derivative with respect to x for a.e. $\mu \in (-1, 1)$ and*

$$\mu\frac{\partial}{\partial x}\boldsymbol{f} \in X^2. \tag{21}$$

Proof. Let $\boldsymbol{f}$ be the mild solution to system (**??**) with boundary conditions (**??**), i.e. the solution to eq. (**??**). Note that we can take the (generalized) derivative with respect to x, for a.e. $\mu \in [-1, 1]$, of both sides of the identity

$$\boldsymbol{f}(x, \mu) = (B\boldsymbol{f})(x, \mu) + \boldsymbol{q}(x, \mu)$$

(holding almost everywhere). In fact, from definitions (**??**) and (**??**), for the first component and for $\mu \in (0,1)$ we have

$$\mu f_1(x,\mu) = \int_0^x \mathrm{e}^{-\Sigma_1 \frac{x-x'}{\mu}} \left(K_{11} f_1\right)(x',\mu)\, dx' + \mathrm{e}^{-\Sigma_1 \frac{x}{\mu}} \varphi_1^+(\mu)$$

and so we can explicitly compute the derivative with respect to x (for a.e. $\mu \in (0,1)$):

$$\mu \frac{\partial}{\partial x} f_1(x,\mu) = -\Sigma_1 f_1(x,\mu) + \int_{-1}^1 k_{11}(\mu,\mu')\, f_1(x,\mu')\, d\mu',$$

for a.e. $(x,\mu) \in (0,l) \times (0,1)$. Treating similarly the case of $\mu \in (-1,0)$ and the second component we arrive at

$$\mu \frac{\partial}{\partial x} \boldsymbol{f}(x,\mu) = -\Sigma \boldsymbol{f}(x,\mu) + (K\boldsymbol{f})(x,\mu),$$

for a.e. $(x,\mu) \in (0,l) \times (-1,1)$. Since, clearly, $\Sigma \boldsymbol{f}$ and $K\boldsymbol{f}$ belong to X^2, then (**??**) holds. □

3 The inverse problem

We now consider the inverse problem described in Sec. **??**. Thus, we reconsider the explicit dependence of the cross-sections on the two dust densities

$$\rho = (\rho_1, \rho_2) \in \Omega,$$

where

$$\Omega := \{\rho_1 \geq 0,\ \rho_2 \geq 0 \mid \rho_1 + \rho_2 > 0\}, \tag{22}$$

and rewrite (**??**) by using frequency indices:

$$\Sigma_{i\to j}(\rho) = \rho_1\, \sigma^1_{i\to j} + \rho_2\, \sigma^2_{i\to j}, \qquad \Sigma_{i,c}(\rho) = \rho_1\, \sigma^1_{i,c} + \rho_2\, \sigma^2_{i,c}, \tag{23}$$

for $i,j \in \{1,2\}$, $\rho \in \Omega$. Consequently, recalling definitions (**??**), (**??**), (**??**), (**??**), also the operators Σ, K, B and the function $\boldsymbol{q}$ have now to be thought as functions of ρ:

$$\Sigma = \Sigma(\rho), \qquad K = K(\rho), \qquad B = B(\rho), \qquad \boldsymbol{q} = \boldsymbol{q}(\rho), \qquad \boldsymbol{f} = \boldsymbol{f}(\rho).$$

Let us assume that $\boldsymbol{\varphi} \in Y^2$ and $\sigma_{i,c} > 0$ for $i = 1,2$. Thus, the hypothesis of Prop. **??** are satisfied if and only if $\rho \in \Omega$ and we can consider the solution $\boldsymbol{f}(\rho)$ of the direct problem, eq. (**??**), that we now rewrite putting in evidence the dependence on ρ:

$$\boldsymbol{f}(\rho) = \left[I - B(\rho)\right]^{-1} \boldsymbol{q}(\rho), \qquad \rho \in \Omega, \tag{24}$$

with $\boldsymbol{f}(\rho) = \boldsymbol{f}(\rho)(x,\mu)$. Since $\mu\frac{\partial}{\partial x}\boldsymbol{f}(\rho) \in X^2$ (see Prop. (**??**)), then the "out-trace" $\boldsymbol{f}(\rho)^{\mathrm{out}}$ of $\boldsymbol{f}(\rho)$

$$\boldsymbol{f}(\rho)^{\mathrm{out}}(\mu) := \begin{cases} \boldsymbol{f}(\rho)(0,\mu), & \text{if } \mu \in (-1,0), \\ \boldsymbol{f}(\rho)(l,\mu), & \text{if } \mu \in (0,1). \end{cases} \tag{25}$$

is well-defined and belongs to Y^2 (see Ref. [**?**]). The inverse problem consists in finding the unknown dust densities $\rho = (\rho_1, \rho_2)$ assuming that the integrated right-outflow

$$H(\rho) := \int_0^1 \boldsymbol{f}(\rho)^{\mathrm{out}}(\mu)\,\mu\,d\mu = \int_0^1 \boldsymbol{f}(\rho)(l,\mu)\,\mu\,d\mu, \tag{26}$$

as well as the inflow $\boldsymbol{\varphi}$ (given by eq. (**??**) and independent on ρ) are known. This amounts to inverting (on its range) the function $H : \Omega \to \mathbb{R}^2$,

$$H(\rho) = \begin{pmatrix} H_1(\rho_1,\rho_2) \\ H_2(\rho_1,\rho_2) \end{pmatrix}$$

given by

$$H(\rho) := \int_0^1 \left[(I - B(\rho))^{-1}\,\boldsymbol{q}(\rho) \right](l,\mu)\,\mu\,d\mu. \tag{27}$$

From now on we shall assume, without explicitly mentioning it, that the assumptions $\boldsymbol{\varphi} \in \left[Y^2\right]^+$ and $\sigma_{i,c} > 0$, $i = 1, 2$, hold.

3.1 Solution of the inverse problem in the case of no frequency-scattering

In order to obtain some simple result, let us now make the following assumptions:

A1) there is no frequency-scattering, i.e. $k_{12} = k_{21} = 0$;

A2) the inflow from the right vanishes, i.e. $\boldsymbol{\varphi}(\mu) = 0$ for $\mu \in (-1, 0)$;

A3) there is no "backward" scattering sending photons towards the left, i.e. the scattering kernels are such that $k_{ii}(\mu, \mu') = 0$, for $\mu \in (-1, 0)$ and $i = 1, 2$.

In Sect. **??**, however, we shall see that the assumptions **A2** and **A3** can be dropped. Under the above assumptions, the solution of the stationary problem (**??**)+(**??**) is identically equal to 0 for $\mu \in (-1, 0)$, and the stationary problem for $\mu \in (0, 1)$ now reads as follows:

$$\mu\frac{\partial f_i}{\partial x}(x,\mu) + \Sigma_i\, f_i(x,\mu) = \int_0^1 k_{ii}(\mu,\mu')\, f_i(x,\mu')\,d\mu', \tag{28a}$$

$$f_i(0,\mu) = \varphi_i(\mu), \tag{28b}$$

for $x \in (0,l)$, $\mu \in (0,1)$ and $i = 1,2$. Moreover, since from now on only k_{11} and k_{22} may be different from 0, for the sake of brevity in the following we shall put

$$k_i := k_{ii} = \Sigma_{i\to i}\, p_{i\to i} \qquad \text{and} \qquad p_i := p_{ii}.$$

Remark 3.1. The direct problem is now completely decoupled in frequency but, nevertheless, the underlying dependence on ρ of Σ_i and k_i couples the two equations as far as the inverse problem is concerned.

Note that our stationary problem (**??**) has the form of an "evolution" problem, with respect to space variable $x \in [0,l]$, from the inflow at $x = 0$ to the outflow at $x = l$. In order to study such "evolution" problem by means of semigroup theory, [**?**, **?**], we introduce a further Banach space:

$$Z^2 = Z \times Z, \qquad Z := L^1\left((0,1),\ \mu\, d\mu\right), \tag{29a}$$

whose elements are couples

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

of Lebesgue-summable functions of the unique, positive variable $\mu \in (0,1)$, with norm

$$\|\boldsymbol{f}\|_{Z^2} = \|f_1\|_Z + \|f_2\|_Z = \int_0^1 \left(|f_1(\mu)| + |f_2(\mu)|\right) \mu\, d\mu. \tag{29b}$$

Consider the operator $A : \mathcal{D}(A) = \mathcal{D} \times \mathcal{D} \subset Z^2 \to Z^2$, with

$$\mathcal{D} := \left\{ f \in Z \mid \mu^{-1} f \in Z \right\} = \left\{ f \in Z \ \mid\ f \in L^1(0,1) \right\}, \tag{30a}$$

defined by $A\boldsymbol{f} = (A_1 f_1, A_2 f_2)$, where, for $i = 1,2$,

$$\left(A_i f_i\right)(\mu) := -\frac{\Sigma_i}{\mu}\, f_i(\mu) + \frac{1}{\mu}\int_0^1 k_i(\mu,\mu')\, f_i(\mu')\, d\mu'. \tag{30b}$$

Note that (**??**) can be recast into an abstract Cauchy problem in the Banach space Z^2:

$$\begin{cases} \dfrac{d}{dx}\boldsymbol{f}(x) = A\boldsymbol{f}(x), \qquad 0 < x \le l, \\[2ex] \boldsymbol{f}(0) = \boldsymbol{\varphi}. \end{cases} \tag{31}$$

If we can prove that the operator A is the generator of a strongly continuous (and, clearly, diagonal) semigroup

$$\mathrm{e}^{xA} = \begin{pmatrix} \mathrm{e}^{xA_1} & 0 \\ 0 & \mathrm{e}^{xA_2} \end{pmatrix}, \qquad x \ge 0,$$

then the solution of (**??**) at $x = l$, i.e. the right-outflow, can be written as

$$\boldsymbol{f}^{\text{out}} = (f_1^{\text{out}}, f_2^{\text{out}}) = \mathrm{e}^{lA}\, \boldsymbol{\varphi} = (\mathrm{e}^{lA_1}\, \varphi_1, \mathrm{e}^{lA_2}\, \varphi_2) \tag{32}$$

(see Refs. [**?**, **?**]), assuming $\boldsymbol{\varphi} \in \mathcal{D}(A)$.

Theorem 3.1. *The operator A defined by (**??**) is the generator of a strongly continuous semigroup of contractions on the Banach space Z^2.*

Proof. First of all we remark that it is easy to see that $C_0^\infty\left((0,1)\right) \subset \mathcal{D}$, and so A is densely definite. Let us decompose each operator A_i into loss and gain terms, $A_i = L_i + G_i$, where

$$\left(L_i f\right)(\mu) := -\frac{\Sigma_i}{\mu}\, f(\mu), \qquad \left(G_i f\right)_i(\mu) := \frac{1}{\mu}\int_0^1 k_i(\mu,\mu')\, f(\mu')\, d\mu',$$

both defined on $f \in \mathcal{D}$. Clearly, the multiplicative operator L_i generates a C_0-semigroup of contractions on Z. Moreover, it easy to check that

$$\left\|G_i f\right\|_Z \leq \alpha_i \left\|L_i f\right\|_Z, \quad \text{for all } f \in \mathcal{D},$$

where $\alpha_i := \Sigma_{i\to i}/\Sigma_i$. Thus G_i is L_i-bounded, with L_i-bound strictly less than 1. For $f \in \mathcal{D}$, $\lambda > 0$ and $\tau \in [0,1]$ we have

$$\begin{aligned} &\left\|\lambda f - (L_i + \tau G_i) f\right\|_Z = \\ &\int_0^1 \left| \left(\lambda\mu + \Sigma_i\right) f_i(\mu) - \tau \Sigma_{i\to i}\int_0^1 p_i(\mu,\mu') f_i(\mu')\, d\mu' \right| d\mu \\ &\geq \int_0^1 \left(\lambda\mu + \Sigma_i\right)\left|f_i(\mu)\right| d\mu - \tau\Sigma_{i\to i}\int_0^1\int_0^1 p_i(\mu,\mu')\big|f_i(\mu')\big|\, d\mu'\, d\mu \\ &= \int_0^1 \left(\lambda\mu + \Sigma_i - \tau\Sigma_{i\to i}\right)\left|f_i(\mu)\right| d\mu \geq \lambda \left\|f\right\|_Z, \end{aligned}$$

since $\Sigma_i - \tau\Sigma_{i\to i} = \Sigma_{i,c} + (1-\tau)\Sigma_{i\to i} \geq 0$ for all $\tau \in [0,1]$. The preceding inequality implies that the operator $L_i + \tau G_i$ is dissipative. Hence, all the conditions of the unbounded perturbation theorem for contractive semigroups are satisfied and we can conclude that $A_i = L_i + G_i$ generates a strongly continuous semigroup of contractions on Z for $i = 1, 2$ (see Corollary 3.2 of Chapter 3.3 and Theorem 2.1 of Chapter 3.2 in Ref. [**?**]). Finally, it is not difficult to show that this implies that A generates a strongly continuous semigroup of contractions on Z^2. □

Now it is convenient to write separately the contributions of each type of dust. According to (**??**) we shall write

$$A_i(\rho) = \rho_1 A_i^1 + \rho_2 A_i^2, \qquad \rho = (\rho_1, \rho_2) \in \Omega$$

where each

$$A^j = \begin{pmatrix} A_1^j & 0 \\ 0 & A_2^j \end{pmatrix}$$

is defined on $\mathcal{D}(A)$ and, obviously, shares with A all the properties proved in Theorem **??** (in fact, $A^1 = A(\rho_1, 0)$ and $A^2 = A(0, \rho_2)$). Note that we have four operators A_i^j characterized by a frequency index $i = 1, 2$ and a dust index $j = 1, 2$. Formula (**??**) becomes

$$\boldsymbol{f}(\rho)^{\text{out}} = \left(f(\rho)_1^{\text{out}}, f(\rho)_2^{\text{out}}\right) = \mathrm{e}^{l\rho_1 A^1 + l\rho_2 A^2}\, \boldsymbol{\varphi}$$
$$= \left(\mathrm{e}^{l\rho_1 A_1^1 + l\rho_2 A_1^2}\varphi_1, \mathrm{e}^{l\rho_1 A_2^1 + l\rho_2 A_2^2}\varphi_2\right) \quad (33)$$

It is just a matter of some tedious calculations proving that, by virtue of assumption **A1** and of the fact that the probability functions p_i do not depend on the dust index, the operators A_i^1 and A_i^2 commute, i.e.

$$A_i^1 A_i^2 \boldsymbol{f} = A_i^2 A_i^1 \boldsymbol{f}, \quad \text{for } \boldsymbol{f} \in \mathcal{D}(A) \text{ and } i = 1, 2. \quad (34)$$

This implies, [**?**, **?**],

$$\mathrm{e}^{l\rho_1 A_i^1 + l\rho_2 A_i^2} = \mathrm{e}^{l\rho_1 A_i^1}\, \mathrm{e}^{l\rho_2 A_i^2} = \mathrm{e}^{l\rho_2 A_i^2}\, \mathrm{e}^{l\rho_1 A_i^1}, \qquad i = 1, 2, \quad (35)$$

and so it is easy to calculate the derivatives of the right-outflow with respect to $\rho \in \Omega$:

$$\frac{\partial f_i^{\text{out}}}{\partial \rho_j}(\rho) = l\, A_i^j\, \mathrm{e}^{lA_i(\rho)}\, \varphi_i, \qquad i, j = 1, 2. \quad (36)$$

These are derivatives in the topology of Z and, therefore, it can be proved that the integrated right-outflow $H = (H_1, H_2) : \Omega \to \mathbb{R}^2$,

$$H_i(\rho) := \int_0^1 f_i^{\text{out}}(\rho, \mu)\, \mu\, d\mu = \int_0^1 \left(\mathrm{e}^{lA_i(\rho)}\varphi_i\right)(\mu)\, \mu\, d\mu, \quad (37)$$

is continuously differentiable, with

$$\frac{\partial H_i}{\partial \rho_j}(\rho) = \int_0^1 \frac{\partial f_i^{\text{out}}}{\partial \rho_j}(\rho, \mu)\, \mu\, d\mu = l \int_0^1 \left(A_i^j \mathrm{e}^{lA_i(\rho)}\varphi_i\right)(\mu)\, \mu\, d\mu \quad (38)$$

for $i, j = 1, 2$. Thus, since

$$\left(A_i^j f_i\right)(\mu) = -\frac{\sigma_i^j}{\mu}\, f_i(\mu) + \frac{\sigma_{i\to i}^j}{\mu} \int_0^1 p_i(\mu, \mu')\, f_i(\mu')\, d\mu'$$

and so, from (**??**),

$$\int_0^1 \left(A_i^j f_i\right)(\mu)\, \mu\, d\mu = -\sigma_{i,c}^j \int_0^1 f_i(\mu)\, d\mu, \quad (39)$$

we can explicitly calculate the Jacobian determinant JH of H, which turns out to be

$$\det JH(\rho) = \det\left(\frac{\partial H_i}{\partial \rho_j}\right)(\rho) = l^2 \Psi_1(\rho)\Psi_2(\rho) \det\left(\sigma_{i,c}^j\right) \quad (40)$$

where

$$\Psi_i(\rho) := \int_0^1 f_i^{\text{out}}(\rho, \mu)\, d\mu = \int_0^1 \left(\mathrm{e}^{lA_i(\rho)}\varphi_i\right)(\mu)\, d\mu. \quad (41)$$

Thus, we can finally give a sufficient condition for the inverse problem to be solvable.

Theorem 3.2. *Under the assumptions* **A1**-**A3** *and*

A4) $\boldsymbol{\varphi} \in \left[Z^2\right]^+$ *and* $\varphi_1(\mu) > 0$ *for* $\mu \in S_1$, $\varphi_2(\mu) > 0$ *for* $\mu \in S_2$, *where* S_1 *and* S_2 *are two non-zero measure sets contained in* $(0,1)$*;*

A5) $\sigma^j_{i,c} > 0$ *for* $i, j \in \{1,2\}$ *and*

$$\det\left(\sigma^j_{i,c}\right) \neq 0, \tag{42}$$

the function $H : \Omega \to \mathbb{R}^2$ *is globally invertible on its range.*

Proof. Conditions **A4** and **A5** imply that the right-hand side of eq. (**??**) does not vanish. In fact, both the operators A^i and the semigroups e^{xA_i} have the property of mapping the set of functions that are positive on a non-zero measure set contained in $(0,1)$ into itself. For the same reason, from eq. (**??**) we obtain

$$\frac{\partial H_i}{\partial \rho_j}(\rho) < 0, \qquad \rho \in \Omega, \quad i,j = 1,2 \tag{43}$$

where $\partial H_i/\partial \rho_j$ has to be understood as a right-derivative if $\rho_{j'} = 0$, $j' \neq j$. Thus, H is locally invertible, continuous and, moreover, each component H_i is strictly monotonous with respect to each variable ρ_j. These facts can be used to prove directly that $H : \Omega \to \mathbb{R}^2$ is globally invertible on its range (see Ref. [**?**]). The global invertibility of H can also be deduced from the very general Hadamard-Caccioppoli Theorem [**?**]; in fact it is not difficult to prove that H satisfies the hypothesis of that theorem (in particular, H is a proper function i.e. the pre-image of a compact set is compact). □

Remark 3.2. Note that conditions **A4** and **A5** have a precise physical meaning: the observer must of course have something to measure for both frequencies (assumption **A4**) and the light absorption properties must actually discriminate the two types of dust (assumption **A5**).

Remark 3.3. In the no-scattering case (i.e. $K \equiv 0$), the solution of the direct problem con be written explicitly. This allows to prove that, in this case, the above conditions **A4** and **A5** are sufficient and also necessary.

3.2 Solution of the inverse problem with back-scattering

In this section we give a sketch of how the assumptions **A2** and **A3** of the previous section (decoupling between rightward and leftward particles) may be removed.

Let us just retain assumption **A1** (absence of frequency scattering); then the stationary transport problem (**??**) + (**??**) can be written as follows:

$$
\begin{aligned}
\frac{\partial f_i^+}{\partial x}(x,\mu) &= -\frac{\Sigma_i}{\mu} f_i^+(x,\mu) + \frac{1}{\mu}\int_0^1 k_i^{++}(\mu,\mu')\, f_i^+(x,\mu')\, d\mu' \\
&\qquad + \frac{1}{\mu}\int_0^1 k_i^{+-}(\mu,\mu')\, f_i^-(l-x,\mu')\, d\mu', \\
\frac{\partial f_i^-}{\partial x}(x,\mu) &= -\frac{\Sigma_i}{\mu} f_i^-(x,\mu) + \frac{1}{\mu}\int_0^1 k_i^{--}(\mu,\mu')\, f_i^-(x,\mu')\, d\mu' \\
&\qquad + \frac{1}{\mu}\int_0^1 k_i^{-+}(\mu,\mu')\, f_i^+(l-x,\mu')\, d\mu',
\end{aligned} \tag{44a}
$$

$$
f_i^+(0,\mu) = \varphi_i^+(\mu), \qquad f_i^-(0,\mu) = \varphi_i^-(\mu) \tag{44b}
$$

for $x \in (0,l)$, $\mu \in (0,1)$ and $i = 1,2$, where

$$
\begin{array}{ll}
f_i^+(x,\mu) := f_i(x,\mu), & f_i^-(x,\mu) := f_i(l-x,-\mu), \\
k_i^{++}(\mu,\mu') := k_i(\mu,\mu'), & k_i^{+-}(\mu,\mu') := k_i(\mu,-\mu'), \\
k_i^{-+}(\mu,\mu') := k_i(-\mu,\mu'), & k_i^{--}(\mu,\mu') := k_i(-\mu,-\mu'),
\end{array} \tag{45}
$$

for $\mu, \mu' \in (0,1)$, $i = 1,2$. In analogy with the previous section, this can be recast into an evolution problem in the Banach space

$$
Z^4 = Z \times Z \times Z \times Z
$$

(where Z is defined in (**??**)), whose non-negative column-vectors

$$
\boldsymbol{f} = (f_1^+, f_1^-, f_2^+, f_2^-)^T
$$

represent the distributions with respect to $\mu \in (0,1)$ of, respectively, rightward ν_1-photons, leftward ν_1-photons, rightward ν_2-photons and leftward ν_2-photons. Thus, (**??**) has the form

$$
\begin{cases}
\dfrac{d}{dx}\boldsymbol{f}(x) = A\boldsymbol{f}(x), \qquad x \in [0,l], \\
\boldsymbol{f}(0) = \boldsymbol{\varphi}.
\end{cases} \tag{46}
$$

where

$$
\boldsymbol{\varphi} := (\varphi_1^+, \varphi_1^-, \varphi_2^+, \varphi_2^-)^T
$$

and A is clearly defined by the right-hand sides of eqs. (**??**) on the domain

$$
\mathcal{D}(A) := \left\{ \boldsymbol{f} \in Z^4 \mid \mu^{-1}\boldsymbol{f} \in Z^4 \right\}. \tag{47}
$$

As in Section **??** (the computations are just a little bit more complicated) we can show that A generates a strongly continuous semigroup of contractions on Z^4 yielding the solution of (**??**)

$$
\boldsymbol{f}(x) = \mathrm{e}^{xA}\boldsymbol{\varphi}, \qquad x \geq 0. \tag{48}
$$

Moreover, the contribution of each type of dust can be put in evidence by writing, according to (**??**),

$$A(\rho) = \rho_1 A^1 + \rho_2 A^2, \qquad \rho = (\rho_1, \rho_2) \in \Omega$$

where each A^i is defined on $\mathcal{D}(A)$ and generates a strongly continuous semigroup e^{xA^i}. It is just a little more difficult showing that, also in this case, $A^1 A^2 \boldsymbol{f} = A^2 A^1 \boldsymbol{f}$ for all $\boldsymbol{f} \in \mathcal{D}(A)$, which implies

$$\mathrm{e}^{xA} = \mathrm{e}^{x(\rho_1 A^1 + \rho_2 A^2)} = \mathrm{e}^{x\rho_1 A^1} \mathrm{e}^{x\rho_2 A^2} = \mathrm{e}^{x\rho_2 A^2} \mathrm{e}^{x\rho_1 A^1}. \tag{49}$$

Now, the function that must be inverted in order to solve the inverse problem is the integrated right-outflow $H : \Omega \to \mathbb{R}^2$,

$$H(\rho) := \int_0^1 \left(\boldsymbol{f}(\rho)^{\mathrm{out}} \right)^+ (\mu)\, \mu\, d\mu = \int_0^1 \left(\mathrm{e}^{lA(\rho)} \boldsymbol{\varphi} \right)^+ (\mu)\, \mu\, d\mu \tag{50}$$

where, for every $\boldsymbol{g} = (g_1^+, g_1^-, g_2^+, g_2^-)^T \in Z^4$ we put

$$\boldsymbol{g}^+ := (g_1^+, g_2^+)^T$$

and we also stressed the dependence of $\boldsymbol{f}$ on ρ by writing $\boldsymbol{f}(\rho)$. By using (**??**), just as in Sec. **??**, we have that H is differentiable with

$$\frac{\partial}{\partial \rho_i} H(\rho) = \int_0^1 \frac{\partial}{\partial \rho_i} \left(\boldsymbol{f}(\rho)^{\mathrm{out}} \right)^+ (\mu)\, \mu\, d\mu = l \int_0^1 \left(A^i \mathrm{e}^{lA(\rho)}\, \boldsymbol{\varphi} \right)^+ (\mu)\, \mu\, d\mu. \tag{51}$$

For $\mu \in (0,1)$ and $i = 1, 2$, we put $\psi_i^+(\mu) := f_i^+(l, \mu)$ and

$$p_i^+(\mu) := \int_0^1 p_i^{++}(\mu', \mu)\, d\mu', \quad p_i^-(\mu) := \int_0^1 p_i^{+-}(\mu', \mu)\, d\mu'.$$

Thus we can write

$$\begin{aligned} \frac{\partial H_i}{\partial \rho_j}(\rho) &= -l\rho_j \int_0^1 \left\{ \sigma_i^j \psi_i^+(\mu) - \sigma_{i,\mathrm{sc}}^j \left[p_i^+(\mu) \psi_i^+(\mu) + p_i^-(\mu) \varphi_i^-(\mu) \right] \right\} d\mu \\ &= -l\rho_j \int_0^1 \left\{ \left[\sigma_{i,\mathrm{c}}^j + \left(1 - p_i^+(\mu)\right) \sigma_{i,\mathrm{sc}}^j \right] \psi_i^+(\mu) - \sigma_{i,\mathrm{sc}}^j\, p_i^-(\mu) \varphi_i^-(\mu) \right\} d\mu, \end{aligned} \tag{52}$$

where $\sigma_{i,\mathrm{sc}}^j := \sigma_{i \to i}^j$ denotes the microscopic scattering cross-section. Assuming that the scattering properties are the same for leftward and rightward photons we have $p_i(\mu, -\mu') = p_i(-\mu, \mu')$, i.e. $p_i^{+-}(\mu, \mu') = p_i^{-+}(\mu, \mu')$, which implies, recalling (**??**),

$$1 = \int_{-1}^1 p_i(\mu', \mu)\, d\mu' = p_i^+(\mu) + p_i^-(\mu).$$

In this case, (**??**) may be re-written as follows

$$\frac{\partial H_i}{\partial \rho_j}(\rho) = -l\rho_j \int_0^1 \left\{\sigma_{i,\mathrm{c}}^j\, \psi_i^+(\mu) + \sigma_{i,\mathrm{sc}}^j\, p_i^-(\mu)\big[\psi_i^+(\mu) - \varphi_i^-(\mu)\big]\right\} d\mu. \quad (53)$$

Note that φ_i^- is the inflow of leftward photons. Without entering in details, let us note that, as in Sec. **??**, under suitable conditions on the inflow data we have

$$\frac{\partial H_i}{\partial \rho_j}(\rho) < 0, \qquad \rho \in \Omega \quad (54)$$

and the condition

$$\det\left(\frac{\partial H_i}{\partial \rho_j}\right)(\rho) \neq 0, \qquad \rho \in \Omega, \quad (55)$$

implies the global invertibility of H. In particular, if we restore assumption **A2** and, therefore, we assume $\varphi_i^- = 0$, then (**??**) becomes

$$\frac{\partial H_i}{\partial \rho_j}(\rho) = -l\rho_j \int_0^1 \big[\sigma_{i,\mathrm{c}}^j + \sigma_{i,\mathrm{sc}}^j\, p_i^-(\mu)\big]\, \psi_i^+(\mu)\, d\mu \quad (56)$$

and we may state a theorem completely analogous to Theorem **??**.

4 Numerical experiments

In this final section we present some representative numerical experiments. Even though the values of physical quantities are chosen in accordance with the real ones (or, at least, with their current estimations), the reported examples are not intended to solve any "real" problem in interstellar physics, but only to show the validity of our approach. The used numerical schemes, as well, are not expected to be "optimal" in any respects.

For the sake of simplicity, we shall disregard the back-scattering and assume the forward-scattering to be isotropic ($p_i \equiv 1$). In this case, the direct problem (**??**) reads as follows:

$$\mu \frac{\partial f_i}{\partial x}(x,\mu) + \Sigma_i\, f_i(x,\mu) = \Sigma_{i,s} \int_0^1 f_i(x,\mu')\, d\mu', \quad (57a)$$

$$f_i(0,\mu) = \varphi_i(\mu), \quad (57b)$$

for $x \in (0,l)$, $\mu \in (0,1)$ and $i = 1,2$, with

$$\begin{aligned} \Sigma_{i,c} &= \rho_1\, \sigma_{i,c}^1 + \rho_2\, \sigma_{i,c}^2, \\ \Sigma_{i,s} &= \rho_1\, \sigma_{i,s}^1 + \rho_2\, \sigma_{i,s}^2, \\ \Sigma_i &= \rho_1\, \sigma_i^1 + \rho_2\, \sigma_i^2. \end{aligned} \quad (58)$$

Here, $\sigma^j_{i,c}$, $\sigma^j_{i,s}$ and $\sigma^j_i = \sigma^j_{i,c} + \sigma^j_{i,s}$ are, respectively, the absorption, scattering and total microscopic cross-section, i and j being, respectively, the frequency and dust indices. In the following we shall use a slab width $l = 10^{15}$ m, corresponding approximately to a tenth of light-year. The two frequency indices $j = 1, 2$, correspond to wavelengths $\lambda_1 = 0.1\,\mu$m and $\lambda_2 = 10\,\mu$m, respectively. For both frequencies we take the same inflow datum $\varphi_1 = \varphi_2 = \varphi$, which is assumed to be a Gaussian distribution peaked around the cosine $\mu = 1$ (see Figure 1). The function φ has been normalized in such a way that

$$\int_0^1 \varphi(\mu)\,\mu\,d\mu = 1,$$

which implies that the integrated right-outflows

$$H_i(\rho) := \int_0^1 (f_i(\rho))^{\text{out}}\,(\mu)\,\mu\,d\mu$$

will range from 0 (total absorption, $\rho_1, \rho_2 \to +\infty$) to 1 (empty space, $\rho_1, \rho_2 \to 0$).

In Figures 2 and 2 we show the image under the map $H = (H_1, H_2)$ of a square $\rho_{\min} \le \rho_1, \rho_2 \le \rho_{\max}$, for two different couples of dust grains. The coordinate lines correspond to a logarithmically spaced grid in the (ρ_1, ρ_2) space.

In Figure 2 we use $\rho_{\min} = 10^{-7}\,\text{m}^{-3}$, $\rho_{\max} = 10^{-1}\,\text{m}^{-3}$ (note that the typical number density of dust in the interstellar medium is about $10^{-6}\,\text{m}^{-3}$) and the dust grains are those described in Table 1. In Figure 3 we use $\rho_{\min} = 10^{-8}\,\text{m}^{-3}$, $\rho_{\max} = 5 \times 10^{-3}\,\text{m}^{-3}$ and the dust grains are those described in Table 2.

Note, in particular, that in Figure 1 the ultraviolet outflow H_1 is very sensitive to changes of ρ_1 whereas the infrared outflow is almost unaffected by it. This had to be expected, looking at values in Table 1, since the size of the graphite spherules is comparable with the ultraviolet wavelength λ_1.

To solve the inverse problem we used a bisection-like algorithm. This can be resumed in the following three steps:

1 - given the measured fluxes H^0_1 and H^0_2, a minimum and maximum value for the two densities ρ_1 and ρ_2 are fixed and the direct problem is solved in order to generate a grid like that of Figures 2 and 3;

2 - the algorithm tries to establish which grid-cell the measured value (H^0_1, H^0_2) belongs to;

3 - a refined grid is generated within this new cell.

Steps 2 and 3 are then repeated until the desired accuracy is reached. Of course, the most delicate step is n. 2, mostly because the location algorithm treats grid-cells approximately as parallelograms, which is often a very poor approximation. This can be fixed by choosing in a suitable way a number of "safety margin" parameters.

In Table 3 we show the results of six different simulations, performed with the dust data of Table 1: H_1^0 and H_2^0 are the "measured" data, plotted in Figure 4; ρ_1^0 and ρ_2^0 are the true values of the densities (expressed in m^{-3}); Δ_1 and Δ_2 are the relative errors between the true values of the densities and those predicted by the algorithm (not shown in the table, since up to the displayed number of digits they coincide with ρ_1^0 and ρ_2^0). Moreover, T is the computation time (on a standard PC), expressed in seconds. The positions of the measurements data in the (H_1, H_2)-space are shown in Figure 4.

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Dragoni-Barletti, **Table 1**

λ (μm)	Dust kind	radius (μm)	σ_c (μm^2)	σ_s (μm^2)
0.1	graphite	0.25	0.18	0.29
10.0	graphite	0.25	0.93 E-2	0.25 E-3
0.1	silicate	1.00	2.67	4.03
10.0	silicate	1.00	4.22	0.50

Caption of Table 1:

Optical properties of radius-0.25 μm graphite and radius-1 μm silicate spherical dust grains, with respect to $\lambda_1 = 0.1\,\mu$m (UV) and $\lambda_2 = 10\,\mu$m (IR) wavelengths. The value are taken from Ref. [**?**].

Dragoni-Barletti, **Table 2**

λ (μm)	Dust kind	radius (μm)	σ_c (μm^2)	σ_s (μm^2)
0.1	SiC	1.58	2.20	17.58
10.0	SiC	1.58	10.93	2.58
0.1	silicate	10.0	2.44 E+2	3.84 E+2
10.0	silicate	10.0	3.73 E+2	4.15 E+2

Caption of Table 2:

Optical properties of radius-1.58 μm silicon carbide (SiC) and radius-10 μm silicate spherical dust grains, with respect to $\lambda_1 = 0.1\,\mu$m (UV) and $\lambda_2 = 10\,\mu$m (IR) wavelengths. The values are taken from Ref. [**?**].

Dragoni-Barletti, **Table 3**

	H_1^0	H_2^0	ρ_1^0	ρ_2^0	Δ_1	Δ_2	T
1	9.664 E-1	9.962 E-1	1.50 E-4	5.00 E-7	1.5 E-4	1.3 E-4	9.4 E+0
2	8.379 E-1	7.927 E-1	1.00 E-5	5.00 E-5	4.0 E-4	1.9 E-4	2.8 E+1
3	3.996 E-1	6.136 E-1	2.00 E-3	1.00 E-4	4.3 E-4	2.8 E-4	4.4 E+1
4	4.195 E-2	5.652 E-1	1.00 E-2	1.00 E-4	2.0 E-5	7.6 E-4	4.3 E+2
5	9.857 E-3	9.197 E-3	1.25 E-3	1.00 E-3	1.6 E-4	1.9 E-16	4.8 E+2
6	1.446 E-1	9.102 E-1	7.00 E-3	5.00 E-6	1.0 E-4	9.1 E-5	4.8 E+2

Caption of Table 3:

Data relative to six solved inverse problem.

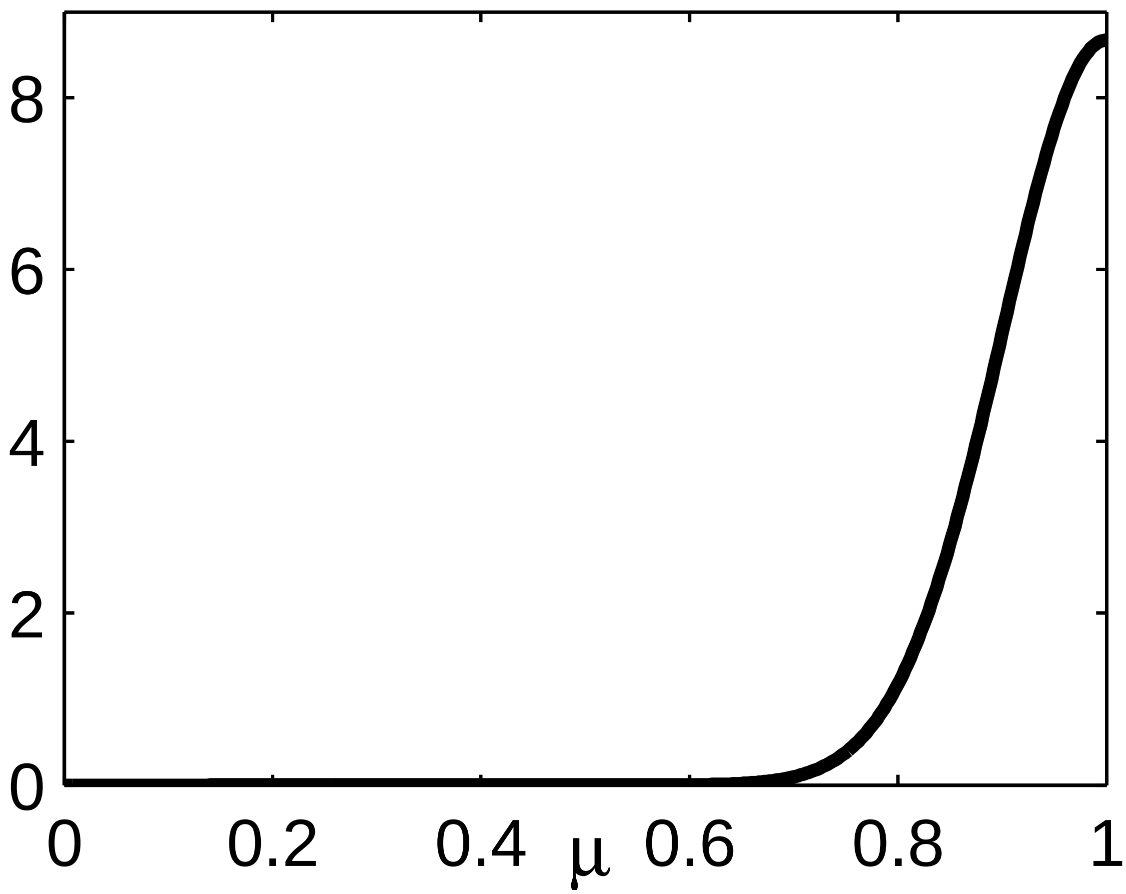

Figure 1: Inflow datum $\varphi(\mu)$ used in the computations.

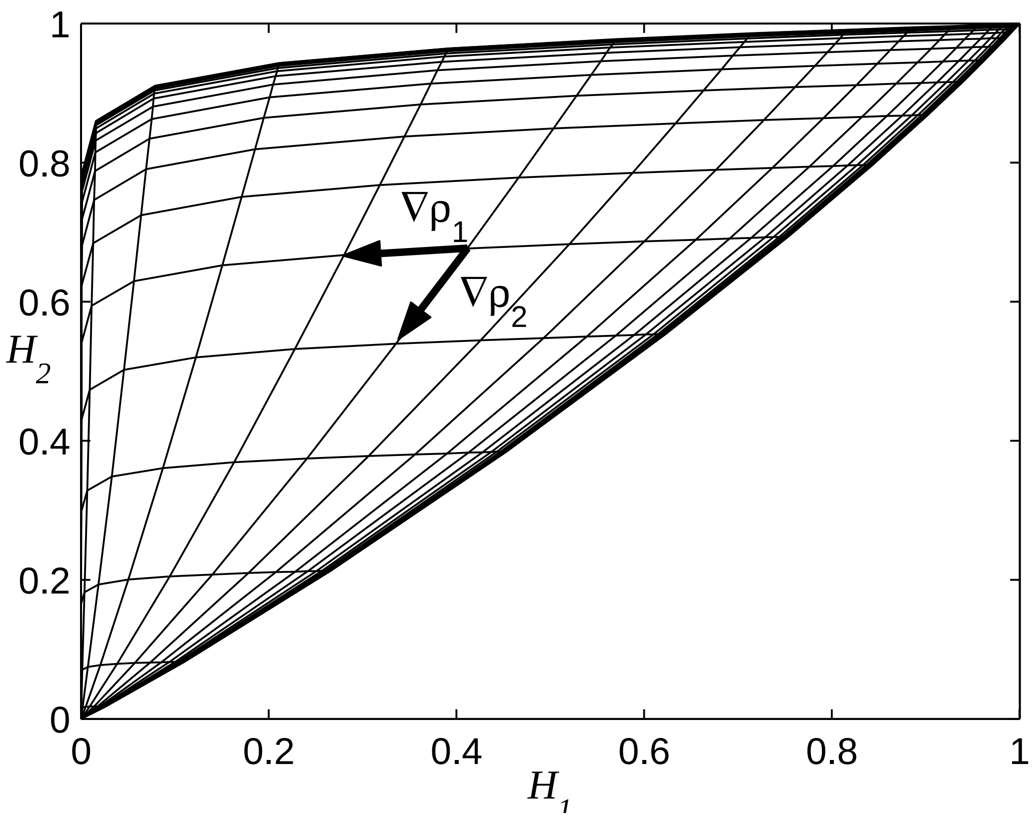

Figure 2: Image of the square $10^{-7}\,\mathrm{m}^{-3} \leq \rho_1, \rho_2 \leq 10^{-1}\,\mathrm{m}^{-3}$ under the map H, using the dust values of Table 1.

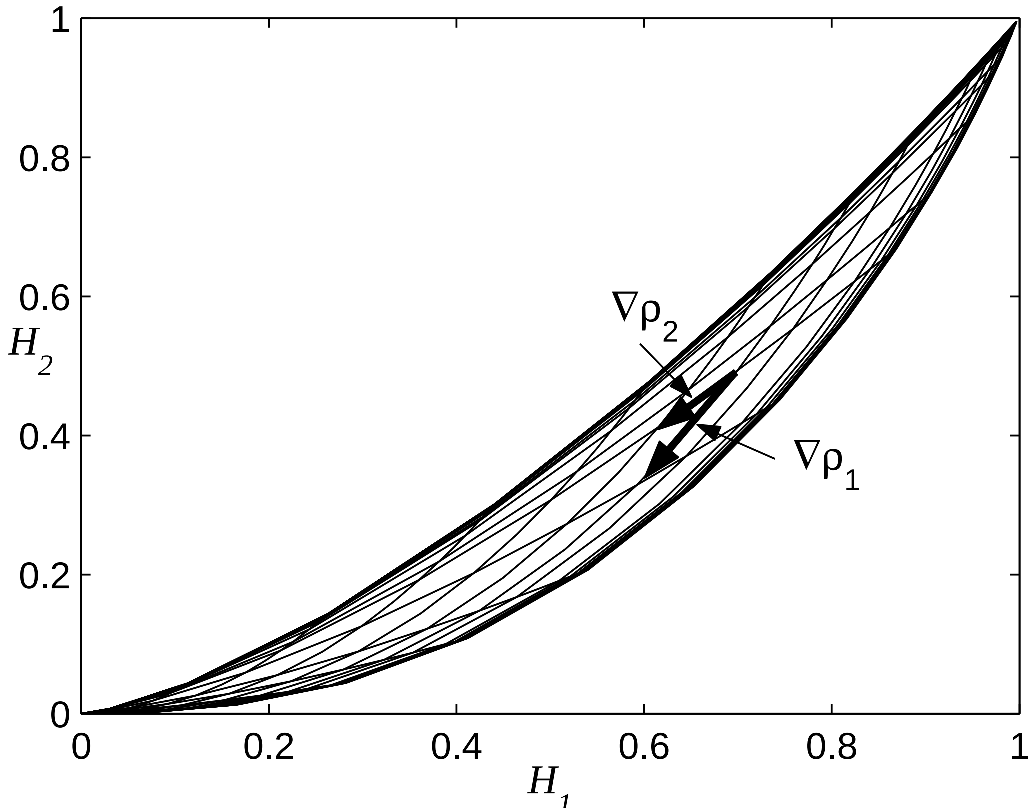

Figure 3: Image of the square $10^{-8}\,\mathrm{m}^{-3} \leq \rho_1, \rho_2 \leq 5 \times 10^{-3}\,\mathrm{m}^{-3}$ under the map H, using the dust values of Table 2.

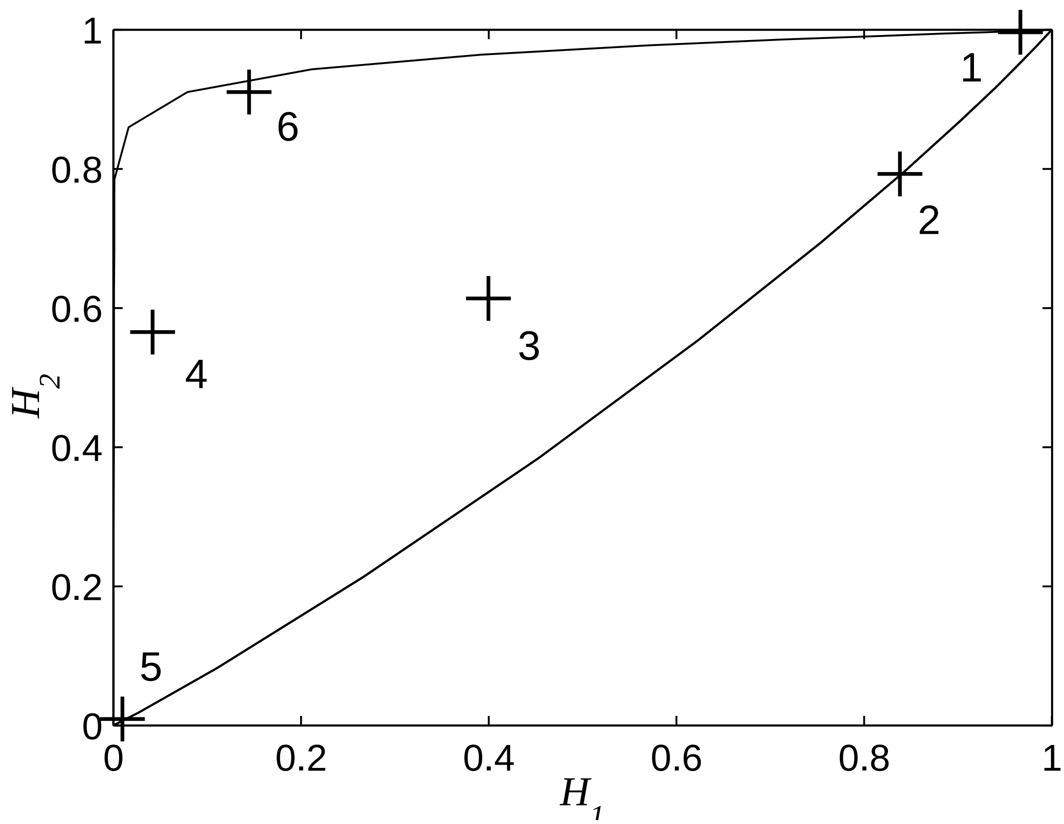

Figure 4: Positions of the six inflow data (H_1^0, H_2^0) used in the simulations reported in Table 3.