$C^{1,\alpha}$-RECTIFIABILITY IN LOW CODIMENSION IN HEISENBERG GROUPS

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ABSTRACT. A natural higher order notion of $C^{1,\alpha}$-rectifiability for any $0 < \alpha \leq 1$ is introduced for subsets of Heisenberg groups $\mathbb{H}^n$ in terms of covering a set almost everywhere with a countable union of $(\mathbf{C}_H^{1,\alpha}, \mathbb{H})$-regular surfaces. Using this we prove a geometric characterisation of $C^{1,\alpha}$-rectifiable sets of low-codimension in Heisenberg groups $\mathbb{H}^n$ in terms of an almost everywhere existence of suitable approximate tangent paraboloids.

MSC (2010): 28A75 (primary); 43A80, 53C17 (secondary).
Keywords: rectifiability, approximate tangent paraboloids, Heisenberg group.

1. Introduction

Rectifiable sets are focal to studies in geometric measure theory and admit various applications in several branches of mathematical analysis. Interest in such sets arises mainly for their geometric, measure-theoretic, and analytic properties which include a notion of (approximate) tangent spaces defined almost everywhere, a version of the area and coarea formulas (see [1] and [20]), and a framework for studying boundedness of a class of singular integral operators (see, e.g., [6–8]).

In metric spaces, particularly Carnot groups, the definition of rectifiability diverges along several, not necessarily equivalent, directions (see, e.g., [2, 14, 16, 24]). The original definition by Federer [11, Section 3.2.14] is in terms of composing a set with countably many Lipschitz images of subsets of the Euclidean space $\mathbb{R}^n$. This is adopted in [1] and shown to be inappropriate in general metric spaces considering even the basic setting of the Heisenberg group $\mathbb{H}^1$. In [22], Mattila et al. defined rectifiability in the Heisenberg group $\mathbb{H}^n$ considering a countable union of C^1 $\mathbb{H}$-regular surfaces. This is related to the approach of using notions of regular surfaces in the sense of Franchi, Serapioni and Serra Cassano (see, e.g., [15, 17, 18]). Several results can be found on characterizations and basic properties of rectifiable sets in Euclidean spaces and general metric spaces (see, e.g., [1, 9, 11, 12, 21, 22]). A well-known characterization in the Heisenberg group $\mathbb{H}^n$ is in terms of the a.e. existence of the approximate tangent spaces [22]. This is in the Spirit of the Euclidean analogue which is in terms of an almost everywhere existence of approximate tangent planes (see, e.g., [21, Corollary 15.16]).

A missing piece in the study of rectifiability in metric spaces is the natural notion of higher order rectifiability which can be defined in terms of essentially composing a set with countably many objects of higher order regularity defined in an appropriate sense. Motivated by the seminal work of Anzellotti and Serapioni [3] in the Euclidean setting, our goal in this article is to initiate progress along this line in the metric setting of Heisenberg groups. We introduce a notion of $C^{1,\alpha}$-rectifiability for any $0 < \alpha \leq 1$ defined in terms of composing a set with countably many $(C_H^{1,\alpha}, \mathbb{H})$-regular surfaces (refer to Definitions 2.8 and 2.11). Using this, we address the problem of characterisation of $C^{1,\alpha}$-rectifiable sets in a metric setting. An interesting, and perhaps gratifying, discovery is the fact that analogous geometric criterion of *approximate tangent paraboloids* as in the Euclidean characterisation of $C^{1,\alpha}$-rectifiable sets, $0 < \alpha \leq 1$ (see [10, 25]) applies in the setting of low-codimensional sets of the Heisenberg groups $\mathbb{H}^n$

Throughout the paper we write k and k_m as the *dimension* and *metric dimension* respectively, $\mathcal{G}(\mathbb{H}^n, k)$ is the Grassmannian of k-dimensional subgroups (see Definition 2.16) and denote by $Q_\alpha(p, V, \lambda)$ the α-paraboloid centered at the point p with base V and dilation parameter λ (see Definition 2.20 for more details). We state the main results of this paper:

Theorem 1.1. *Fix $\alpha \in (0,1]$ and $n < k \leq 2n$. Let $E \subset \mathbb{H}^n$ be a $\mathcal{H}^{k_m}$-measurable set with $\mathcal{H}^{k_m}(E) < \infty$ such that for $\mathcal{H}^{k_m}$-a.e. $p \in E$ there are $V_p \in \mathcal{G}(\mathbb{H}^n, k)$ and $\lambda > 0$ such that*

$$\lim_{r \to 0^+} \frac{1}{r^{k_m}} \mathcal{H}^{k_m}\left(E \cap B(p,r) \setminus Q_\alpha(p, V_p, \lambda)\right) = 0. \tag{1}$$

Assume, in addition, that for $\mathcal{H}^{k_m}$-a.e. $p \in E$ there holds

$$\Theta_*^{k_m}(E, p) > 0.$$

Then E is $C^{1,\alpha}$-rectifiable in the sense of Definition 2.11.

Next, we prove that the opposite implication is also true. As above, we are in the low-codimensional setting so $n < k \leq 2n$.

Proposition 1.2. *If $E \subset \mathbb{H}^n$ is a $C^{1,\alpha}$-rectifiable set with $\mathcal{H}^{k_m}(E) < \infty$, then for $\mathcal{H}^{k_m}$-a.e. $p \in E$ there exist $V_p \in \mathcal{G}(\mathbb{H}^n, k)$ and $\lambda = \lambda_p > 0$ such that*

$$\lim_{r \to 0^+} \frac{1}{r^{k_m}} \mathcal{H}^{k_m}\left(E \cap B(p,r) \setminus Q_\alpha(p, V_p, \lambda)\right) = 0.$$

The proof of Theorem 1.1 draws technique from [10, Lemma 3.5] by first recovering the Holder regularity of the distribution of subgroups as in Lemma 3.1 using the density conditions. We remark that, unlike an analogous result in the Euclidean setting (see [10, Theorem 1.1]) where the positive lower density condition is recovered from the approximate tangent paraboloid condition, in our setting, this is not so direct and we thus impose the explicit requirement. This is the same scenario in the characterisation of k-rectifiable sets of low codimension in $\mathbb{H}^n$ (see [22, Theorem 3.15]) where the positive lower density condition is required and is asked if such condition can be removed. The strategy of proof is similar to that of [22] via a standard decomposition argument using the density conditions and a selection of horizontal vectorfields corresponding to horizontal complement of the distribution of vertical subgroups. Further, application of the Holder regularity result and an inclusion in paraboloids establishes the convergence of the sequence of Whitney functions from which the conclusion follows via standard approximation results and the Whitney extension theorem.

It is interesting to notice that most of the technical points exploited to prove Theorem 1.1 can be extended to a general Carnot group without much effort. Indeed, the structure of the Heisenberg group $\mathbb{H}^n$ only plays a fundamental role in Proposition 2.17 and, consequently, in the Hölder-continuity result (Lemma 3.2). That said, in Remark 3.5 we will show that our main result can be extended in codimension one to any Carnot group $\mathbb{G}$ and briefly discuss codimension ≥ 2.

The structure of the paper is the following. In Section 2 we briefly recall the main properties of the Heisenberg group $\mathbb{H}^n$, the definition of Hausdorff measure with some useful density results and Pansu differentiability for general Carnot groups. Next, in Section 2.1, we exploit them to finally introduce the notion of low codimensional $C^{1,\alpha}$-rectifiability that will be used throughout the paper.

In Section 2.3, following [22], the intrinsic Grassmannian is defined and characterized (Proposition 2.17). To conclude the preliminaries, in Section 2.4, we prove several technical results concerning α-paraboloids and cylinders (Definition 2.20) that will be used for the proof of the main theorem.

In Section 3 we give a proof to Proposition 1.2 as well as other technical results (Lemma 3.3) which will be used in the next section to prove Theorem 1.1. Finally, in Section 3.2 we briefly discuss how our main result can be extended to general Carnot groups in codimension one.

2. Preliminaries

We shall restrict to essential notions of our space. The interested reader is invited to the references [4, 5] on Carnot groups, and in particular on Heisenberg groups.

The Heisenberg group $\mathbb{H}^n$ is the simplest Carnot group whose Lie algebra $\mathfrak{h}^n$ has a step two stratification; more precisely, we have

$$\mathfrak{h}^n = \mathfrak{h}_1 \oplus \mathfrak{h}_2,$$

where $\mathfrak{h}_1 = \operatorname{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ and $\mathfrak{h}_2 = \operatorname{span}\{T\}$ with commutators

$$[X_i, Y_j] = \delta_{ij} T \quad \text{and} \quad [X_i, X_j] = [Y_i, Y_j] = 0.$$

The vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$ span a vector subbundle of the tangent vector bundle $\mathrm{T}\mathbb{H}^n$, the so-called horizontal vector bundle $\mathrm{H}\mathbb{H}^n$. Via exponential coordinates $\mathbb{H}^n$ can be identified with $\mathbb{R}^{2n+1}$ and we may express the group law using the Baker-Campbell-Hausdorff formula as follows:

$$p \cdot q := \left(p' + q', p_{2n+1} + q_{2n+1} - 2 \sum_{i=1}^{n} (p_i q_{i+n} - p_{i+n} q_i) \right),$$

where $p' := (p_1, \cdots, p_{2n})$. The inverse of p is given by

$$p^{-1} = (-p', -p_{2n+1}),$$

and $e = 0$ is the identity of $\mathbb{H}^n$. The center of $\mathbb{H}^n$ is the subgroup

$$\mathbb{T} := \{p = (0, \dots, 0, p_{2n+1})\}.$$

For any $q \in \mathbb{H}^n$ and $r > 0$, we denote by $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ the left translation $p \mapsto q \cdot p =: \tau_q(p)$ and by $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ the dilation

$$p \mapsto \left(rp', r^2 p_{2n+1}\right) =: \delta_r p.$$

We denote by $\|\cdot\|$ the homogeneous (with respect to dilations) norm and by d the metric given, respectively, by

$$\|p\| := d(p, e) := \max\left\{\left\|p'\right\|_{\mathbb{R}^{2n}}, |p_{2n+1}|^{1/2}\right\},$$

where $\|\cdot\|_{\mathbb{R}^{2n}}$ denotes the standard euclidean norm, and

$$d(p, q) = d\left(q^{-1}p, e\right) = \left\|q^{-1}p\right\|$$

for all $p, q \in \mathbb{H}^n$. We conclude this section by recalling the definition of the Hausdorff measure in metric spaces and some density results.

Definition 2.1. *Let $E \subset \mathbb{H}^n$ and $k \in (0, \infty)$. The* k-dimensional Hausdorff measure *$\mathcal{H}^k$ of E is defined by setting*

$$\mathcal{H}^k_d(E) := \sup_{\delta > 0} \mathcal{H}^k_\delta(E),$$

where $\mathcal{H}^k_\delta(E) = \inf\left\{\sum_i 2^{-k}\mathrm{diam}(E_i)^k : E \subset \bigcup_i E_i, \mathrm{diam}(E_i) \le \delta\right\}$.

Given a $\mathcal{H}^k$-measurable subset $E \subset \mathbb{H}^n$ we define the corresponding upper and lower k-densities of E at $p \in \mathbb{H}^n$ as follows:

$$\Theta^{*k}(E, p) = \limsup_{r \to 0} \frac{\mathcal{H}^k(E \cap B(p, r))}{r^k} \quad \textit{and} \quad \Theta^k_*(E, p) = \liminf_{r \to 0} \frac{\mathcal{H}^k(E \cap B(p, r))}{r^k}.$$

We now recall the following standard density estimates for Hausdorff measures; see, for example, [11, 2.10.19].

Lemma 2.2. *Let $E \subset \mathbb{H}^n$ be $\mathcal{H}^k$-measurable with $\mathcal{H}^k(E) < +\infty$. Then*

(i) For $\mathcal{H}^k$-a.e. $p \in E$, it turns out that $2^{-k} \le \Theta^{*k}(E, p) \le 5^k$.

(ii) For $\mathcal{H}^k$-a.e. $p \in \mathbb{H}^n \setminus E$, it turns out that $\Theta^{*k}(E, p) = 0$.

Let Ω be an open subset of $\mathbb{H}^n$ (identified with $\mathbb{R}^{2n+1}$ as explained above) and $m \ge 0$ a nonnegative integer. Following [22, Section 2.2], we denote by $\mathbf{C}^m(\Omega)$ the space of real-valued functions which are m times continuously differentiable in the Euclidean sense. We further denote by $\mathbf{C}^m(\Omega, \mathrm{H}\mathbb{H}^n)$ the set of all C^m-sections of $\mathrm{H}\mathbb{H}^n$ construed in the sense of regularity between smooth manifolds.

Definition 2.3. *Let $f \in \mathbf{C}^1(\Omega)$. We define the horizontal gradient of f as*

$$\nabla_H f := (X_1 f, \ldots, X_n f, Y_1 f, \ldots, Y_n f)$$

or, equivalently, as the section of the horizontal bundle $\mathrm{H}\mathbb{H}^n$

$$\nabla_H f := \sum_{j=1}^n (X_j f)\, X_j + (Y_j f)\, Y_j,$$

with canonical coordinates $(X_1 f, \ldots, X_n f, Y_1 f, \ldots, Y_n f)$.

Definition 2.4. *Let $\mathcal{U} \subset \mathbb{H}^n$ and $f : \mathcal{U} \to \mathbb{R}$ a continuous function. We say that $f \in \mathbf{C}^1_H(\mathcal{U})$ if $\nabla_H f$ exists and is continuous in $\mathcal{U}$. Furthermore, if $\nabla_H f \in C^{0,\alpha}$ for some $0 < \alpha \leq 1$, then we say that $f \in \mathbf{C}^{1,\alpha}_H(\mathcal{U})$.*

We write $\left[\mathbf{C}^1_H(\mathcal{U})\right]^\ell$ as the set of ℓ-tuples $f = (f_1, \ldots, f_\ell)$ such that $f_i \in \mathbf{C}^1_H(\mathcal{U})$ for each $1 \leq i \leq \ell$. We define $\left[\mathbf{C}^{1,\alpha}_H(\mathcal{U})\right]^\ell$ in the same way.

Remark 2.5. The inclusion $\mathbf{C}^1(\mathcal{U}) \subset \mathbf{C}^1_H(\mathcal{U})$ is strict (see e.g., [15, Remark 5.9]).

We would like to introduce an intrinsic notion of differentiability in Carnot groups due to P. Pansu [23].

Definition 2.6. *Let $\mathbb{G}_1, \mathbb{G}_2$ be Carnot groups, with homogeneous norms $\|\cdot\|_1, \|\cdot\|_2$ and dilations $\delta^1_\lambda, \delta^2_\lambda$. We say that $L : \mathbb{G}_1 \to \mathbb{G}_2$ is H-linear, or is a homogeneous homomorphism, if L is a group homomorphism such that*

$$L\left(\delta^1_\lambda g\right) = \delta^2_\lambda L(g), \qquad \text{for all } g \in \mathbb{G}_1 \text{ and } \lambda > 0$$

Definition 2.7 (Pansu differentiability)**.** *Let $(\mathbb{G}_1, d_1)$ and $(\mathbb{G}_2, d_2)$ be Carnot groups and $\mathcal{A} \subset \mathbb{G}_1$. A function $f : \mathcal{A} \to \mathbb{G}_2$ is Pansu differentiable in $g \in \mathcal{A}$ if there is a H-linear map $L_g : \mathbb{G}_1 \to \mathbb{G}_2$ such that*

$$\frac{d_2\left(f(g)^{-1} \cdot f\left(g'\right), L_g\left(g^{-1} \cdot g'\right)\right)}{d_1\left(g, g'\right)} \to 0, \qquad \text{as } d_1\left(g, g'\right) \to 0, g' \in \mathcal{A}$$

The homogeneous homomorphism L_g is denoted $d_H f_g$ and is called the Pansu differential of f in g.

2.1. **$C^{1,\alpha}$-rectifiability in low codimension.** In [22, Proposition 2.20] it was proved that the metric dimension in $\mathbb{H}^n$ is given by

$$k_m = k + 1 \quad \text{if } n + 1 \leq k \leq 2n.$$

This tells us that the notion of *rectifiability via Lipschitz maps* is only interesting in low dimension ($k \leq n$). Indeed, any Lipschitz function $f : A \subset \mathbb{R}^k \to \mathbb{H}^n$ satisfies

$$\mathcal{H}^{k_m}\left(f(A)\right) = 0$$

whenever dimension and metric dimension are different (i.e., the case $k > n$). Therefore, we need to find a different notion of rectifiability.

The idea, looking at the Euclidean case, is to first introduce a notion of *regular surfaces* which is more fitting in our setting.

Definition 2.8. *Let $n + 1 \leq k \leq 2n$. A set $S \subset \mathbb{H}^n$ is a k-dimensional $(\mathbf{C}^{1,\alpha}_H, \mathbb{H})$-regular surface if for any $p \in S$ there are $\mathcal{U} \subseteq \mathbb{H}^n$ open and $f \in [\mathbf{C}^{1,\alpha}_H(\mathcal{U})]^{2n+1-k}$ satisfying*

(a) $d_H f_q$ is surjective at all $q \in \mathcal{U}$;

(b) $S \cap \mathcal{U} = \{q \in \mathcal{U} \,:\, f(q) = 0\}$.

The operator d_H is the *Pansu differential* and it is represented by the horizontal gradient $\nabla_H f$ introduced above. This definition (with $\mathbf{C}^{1,\alpha}_H$ replaced by $\mathbf{C}^1_H$) was already given in [22] so we refer the reader to that paper for more details.

Definition 2.9. *Fix $n+1 \le k \le 2n$. Let S be a k-dimensional $(\mathbf{C}_H^{1,\alpha}, \mathbb{H})$-regular surface and let f be as above. The tangent group to S at $p_0 \in S$, denoted as $T_{\mathbb{H}}S(p_0)$, is given by*

$$T_{\mathbb{H}}S(p_0) := \{p \in \mathbb{H} \,:\, d_H f_{p_0}(p) = 0\}\,.$$

The following characterization of $\mathbb{H}$-regular surfaces is an immediate consequence of the definition:

Proposition 2.10. *A set S is a k-dimensional $(\mathbf{C}_H^{1,\alpha}, \mathbb{H})$-regular surface if and only if S is locally the intersection of $(2n+1-k)$ 1-codimensional $(\mathbf{C}_H^{1,\alpha}, \mathbb{H})$-regular surfaces with linearly independent normal vectors.*

We recall that, for any open set $\Omega \subset \mathbb{H}^n$, the Taylor's expansion of a function $f \in C_H^{1,\alpha}(\Omega)$ based at the point $x_0 \in \Omega$ is given by (see [13, Theorem 1.42])

$$f(x) = f(x_0) + d_H f_{x_0}(x_0^{-1}x) + \mathcal{O}\left(d(x_0, x)^{1+\alpha}\right). \tag{2}$$

To conclude this introductory section, we can give the formal definition of $C^{1,\alpha}$-rectifiability for a subset of a homogeneous group.

Definition 2.11. *A measurable set $E \subset \mathbb{H}^n$ is $C^{1,\alpha}$-rectifiable if there are k-dimensional $(\mathbf{C}_H^{1,\alpha}, \mathbb{H})$-regular surfaces S_i, with $i \in \mathbb{N}$, such that*

$$\mathcal{H}^{k_m}\left(E \setminus \bigcup_{i\in\mathbb{N}} S_i\right) = 0,$$

where $k_m = k$ if $1 \le k \le n$ and $k_m = k+1$ if $n+1 \le k \le 2n$.

2.2. Whitney's extension theorem.

The following Whitney-type extension theorem was proved in [26, Theorem 4] for general Carnot groups but here stated for the Heisenberg groups $\mathbb{H}^n$.

Theorem 2.12 ($C^{1,\alpha}$-extension)**.** *Let F be a closed subset of $\mathbb{H}^n$, let $\alpha \in (0,1]$ and $f : F \to \mathbb{R}$, $g : F \to \mathrm{H}\mathbb{H}^n$ satisfying the following property: there exists a positive constant M such that*

- *(i) $|f(x)|, |g(x)| \le M$ on every compact subset of F;*
- *(ii) $|f(x) - f(y) - \langle g(x), \pi(y^{-1}x)\rangle| \le M d(x,y)^{1+\alpha}$ for every $x, y \in F$;*
- *(iii) $|g(x) - g(y)| \le M d(x,y)^{\alpha}$ for every $x, y \in F$;*

where $\langle \cdot,\cdot \rangle$ denotes the inner product in $\mathrm{H}\mathbb{H}^n$ (identified as an Euclidean space). Then there exists an extension $\tilde{f} : \mathbb{H}^n \to \mathbb{R}$, $\tilde{f} \in \mathbf{C}_H^{1,\alpha}(\mathbb{H}^n)$, such that

$$g(x) = \nabla_H \tilde{f}(x) \quad \text{for all } x \in F.$$

2.3. The intrinsic Grassmannian.

A subgroup $S \subset \mathbb{H}^n$ is a *homogeneous subgroup* if $\delta_r(S) \subseteq S$ for all $r > 0$, where δ_r is the intrinsic dilation defined by

$$\delta_r(p) = (rp_1, \dots, rp_{2n}, r^2 p_{2n+1}).$$

A homogeneous subgroup S is either *horizontal*, i.e. contained in $\exp(\mathfrak{h}_1)$, or *vertical*, i.e., it contains the center $\mathbb{T}$ of $\mathbb{H}^n$. We introduce the notation

$$d(p, S) := \inf_{s\in S} d(p,s) = \inf_{s\in S} \|p^{-1}s\|.$$

Notice also that horizontal subgroups are commutative while vertical subgroups are non-commutative and *normal* in $\mathbb{H}^n$.

Definition 2.13. *Two homogeneous subgroups S and T of $\mathbb{H}^n$ are* complementary subgroups *in $\mathbb{H}^n$ if $S \cap T = \{0\}$ and $\mathbb{H}^n = T \cdot S$. If, in addition, T is normal we say that $\mathbb{H}^n$ is the* semidirect *product of S and T and write $\mathbb{H}^n = T \rtimes S$.*

If $\mathbb{H}^n$ is the semidirect product of homogeneous subgroups S and T, then we can define unique projections $\pi_S : \mathbb{H}^n \to S$ and $\pi_T : \mathbb{H}^n \to T$ such that

$$\mathrm{id}_{\mathbb{H}^n} = \pi_T \cdot \pi_S.$$

Furthermore, if T is normal in $\mathbb{H}^n$, then the following algebraic equalities hold:

$$\begin{aligned}
&\pi_T(p^{-1}) = \pi_S^{-1}(p) \cdot \pi_T^{-1}(p) \cdot \pi_S(p), \quad \pi_S(p^{-1}) = \pi_S^{-1}(p),\\
&\pi_T(\delta_\lambda p) = \delta_\lambda \pi_T(p), \quad \pi_S(\delta_\lambda p) = \delta_\lambda \pi_S(p),\\
&\pi_T(p \cdot q) = \pi_T(p) \cdot \pi_S(p) \cdot \pi_T(q) \cdot \pi_S^{-1}(p), \quad \pi_S(p \cdot q) = \pi_S(p) \cdot \pi_S(q).
\end{aligned}$$

Proposition 2.14. *If $\mathbb{H}^n = T \rtimes S$ as above, then the projections π_S and π_T are continuous, π_S is a h-homomorphism and there is $c(S,T) := c > 0$ such that*

$$\begin{aligned}
&c\|\pi_S(p)\| \le d(p,T) \le \|\pi_S(p)\|,\\
&c\|\pi_S^{-1}(p) \cdot \pi_T(p) \cdot \pi_S(p)\| \le d(p,S) \le \|\pi_S^{-1}(p) \cdot \pi_T(p) \cdot \pi_S(p)\|,
\end{aligned}$$

holds for all $p \in \mathbb{H}^n$.

This result was proved in [22] for the Heisenberg group and generalized in [19] to all homogeneous groups.

Remark 2.15. In [19], we also proved that the constant c does not depend on S and T if we consider (for $1 \le k \le n$) a k-homogeneous subgroup S and write

$$\mathbb{H}^n = S^\perp \rtimes S,$$

where $S^\perp$ is the vertical subgroup defined as follows. If $S = \langle f_1, \dots, f_k\rangle$, we take

$$S^\perp = \langle f_1, \dots, f_k\rangle^{\perp_{H^1}} \oplus \langle e_{2n+1}\rangle,$$

where $\perp_{H^1}$ denotes the orthogonal in the horizontal layer of $\mathbb{H}^n$ with respect to the fixed scalar product. In this case, we denote by $c_{\mathbb{H}}$ the universal constant.

We are now ready to introduce the notion of *intrinsic Grassmannian* as in [22].

Definition 2.16. *A k-homogeneous subgroup S belongs to the k-Grassmannian $\mathcal{G}(\mathbb{H}^n, k)$ if there exists a $(2n+1-k)$-subgroup T such that $\mathbb{H}^n = T \cdot S$. Moreover, the union*

$$\mathcal{G}(\mathbb{H}^n) := \bigcup_{k=0}^{2n+1} \mathcal{G}(\mathbb{H}^n, k)$$

is often referred to as the intrinsic Grassmannian *of $\mathbb{H}^n$.*

Proposition 2.17. *The trivial subgroups $\{e\}$ and $\mathbb{H}^n$ are the unique elements of $\mathcal{G}(\mathbb{H}^n, 0)$ and $\mathcal{G}(\mathbb{H}^n, 2n+1)$ respectively and*

(i) for $1 \le k \le n$, $\mathcal{G}(\mathbb{H}^n, k)$ coincides with the set of all horizontal k-homogeneous subgroups;

(ii) for $n+1 \le k \le 2n$, $\mathcal{G}(\mathbb{H}^n, k)$ coincides with the set of all vertical k-homogeneous subgroups.

Furthermore, any vertical subgroup T with linear dimension in $\{1, \dots, n\}$ is not an element of the intrinsic Grassmannian of $\mathbb{H}^n$.

This result was proved in [22, Proposition 2.17]. Notice that in the Carnot setting this result is no longer true since the identity

$$\mathcal{G}(\mathbb{G}, k) = \{\text{vertical } k\text{-homogeneous subgroups}\}$$

does not hold for any k if we do not put additional assumptions on $\mathbb{G}$ or to the possible values of k.

Remark 2.18. If S is a $(\mathbf{C}^{1,\alpha}_H, \mathbb{H})$-regular surface, then $T_{\mathbb{H}}S(p_0) \in \mathcal{G}(\mathbb{H}^n)$.

Remark 2.19. The Grassmannian $\mathcal{G}(\mathbb{H}^n)$ is a subset of the Euclidean counterpart (that is, in $\mathbb{R}^{2n+1}$) and is endowed with the same topology. Moreover, $\mathcal{G}(\mathbb{H}^n, k)$ is a compact metric space with respect to the distance

$$\rho(S_1, S_2) := \max_{\|x\|=1} d\left(\pi_{S_1}(x), \pi_{S_2}(x)\right).$$

2.4. **Intersection lemma.** The main objects we deal with in this paper are α-paraboloids and cylinders, so we first recall the definitions in this setting.

Definition 2.20. *Fix $\alpha \in (0,1]$, $\lambda, \eta > 0$ and $r > 0$. Fix moreover $x \in \mathbb{H}^n$ and $S \in \mathcal{G}(\mathbb{H}^n)$. The α-paraboloid centered at $x \in \mathbb{H}^n$ with base S and parameter λ is defined as*

$$Q_\alpha(x, S, \lambda) = \left\{ y \in \mathbb{H}^n \,:\, d(x^{-1}y, S) \le \lambda d(x,y)^{1+\alpha} \right\},$$

while the cylinder with axis S and parameter η is given by

$$\mathcal{C}(S, \eta) := \{y \in \mathbb{R}^n : d(y, S) < \eta\}.$$

Throughout the paper we will mainly consider sets of the form $\mathcal{C}(S, \lambda r^{1+\alpha}) \cap B(x, r)$.

Definition 2.21. *Let $E \subset \mathbb{H}^n$ be $\mathcal{H}^{k_m}$-measurable and $\alpha \in (0,1]$. We say that a homogeneous subgroup V_p, of dimension k and metric dimension k_m, is an* approximate tangent paraboloid *to E at p if $\Theta^{*k_m}(E, p) > 0$ and*

$$\lim_{r \to 0} r^{-k_m} \mathcal{H}^{k_m}\left(E \cap B(p, r) \setminus Q_\alpha(p, V_p, \lambda)\right) = 0 \quad \text{for all } \lambda > 0.$$

We write $\operatorname{apPar}^{k_m}_{\mathbb{H}}(E, p)$ for the set of all approximate tangent paraboloids to E at p and, if there is only one, we denote it by V_p.

The following result gives the analogous relation between cylinders and paraboloids as in [10, Lemma 2.3] in the Euclidean setting.

Lemma 2.22. *Let $S \in \mathcal{G}(\mathbb{H}^n, k)$ and $r_0 > 0$ be fixed. Suppose that for every $r < r_0$*

$$\mathcal{H}^k\left(E \cap B(x, r) \setminus \mathcal{C}(S, \lambda r^{1+\alpha})\right) \le \epsilon r^k.$$

Then for all $r < r_0$ we have

$$\mathcal{H}^k\left(E \cap B(x, r) \setminus Q_\alpha(x, S, \lambda')\right) \le \frac{\epsilon}{1 - 2^{-k}} r^k,$$

where $\lambda' := 4^{1+\alpha}\lambda$.

Proof. A straightforward verification gives that

$$\left(B(x,r)\setminus B(x,\tfrac{r}{2})\right)\setminus Q_\alpha(x,V,\lambda')\subset B(x,r)\setminus \mathcal{C}(V,\lambda r^{1+\alpha})$$

As a consequence we obtain

$$\begin{aligned}B(x,r)\setminus Q_\alpha(x,V,\lambda') &= \bigcup_{j\in\mathbb{N}}\left(B(x,\tfrac{r}{2^j})\setminus B(x,\tfrac{r}{2^{j+1}})\right)\setminus Q_\alpha(x,V,\lambda')\\ &\subset \bigcup_{j\in\mathbb{N}} B(x,\tfrac{r}{2^j})\setminus \mathcal{C}\left(V,\lambda\left(\tfrac{r}{2^j}\right)^{1+\alpha}\right).\end{aligned}$$

Using the assumption we obtain that

$$\mathcal{H}^k\left(E\cap B(x,r)\setminus Q_\alpha(x,V,\lambda')\right)\le \sum_{j\in\mathbb{N}}\varepsilon\left(\frac{r}{2^j}\right)^k=\frac{\varepsilon}{1-2^{-k}}r^k. \qquad \square$$

Using the Taylor's expansion (2), it is not difficult to prove the following (repeating similar arguments as in [22, Lemma 2.28]).

Lemma 2.23. *Fix $n<k\le 2n$. Let $S\subset\mathbb{H}^n$ be a k-dimensional $\mathbb{H}$-regular $C^{1,\alpha}$-surface and $x\in S$. Then there exist $\lambda>0$ and $r_0=r_0(S,x)>0$ such that*

$$S\cap B(x,r_0)\subset Q_\alpha(x,T_{\mathbb{H}}S(x),\lambda). \tag{3}$$

Proof. By Definitions 2.8 and 2.9 we have that there are $r_0>0$ and $f\in\left[\mathbf{C}^{1,\alpha}_H(\mathcal{U})\right]^{2n+1-k}$ such that $d_Hf_x:\mathbb{H}^n\to\mathbb{R}^{2n+1-k}$ is surjective and

$$S\cap B(x,r)=\{p: f(p)=0\}\,,\quad T_{\mathbb{H}}S(x)=\ker(d_Hf_x).$$

For any $p\in S\cap B(x,r_0)$, from (2) we have that

$$\left\|d_Hf_x(x^{-1}p)\right\|_{\mathbb{R}^{2n+1-k}}=\mathcal{O}\left(d(x,p)^{1+\alpha}\right). \tag{4}$$

By H-linearity of d_Hf_x there is $c=c(x,f)>0$ such that

$$\left\|d_Hf_x(x^{-1}p)\right\|_{\mathbb{R}^{2n+1-k}}\ge c\,d\left(x^{-1}p,T_{\mathbb{H}}S(x)\right). \tag{5}$$

Indeed, if $L:\mathbb{H}^n\to\mathbb{R}^{2n+1-k}$ is H-linear, $\ker(L)$ is a vertical subgroup and we have by the intrinsic decomposition that there exists V horizontal such that $\ker(L)\cdot V=\mathbb{H}^n$. Then $L:V\to\mathbb{R}^{2n+1-k}$ is injective and hence there exists $c>0$ such that

$$\|L(v)\|_{\mathbb{R}^{2n+1-k}}\ge c\|v\|,$$

for all $v\in V$. Now by (4) and (5) we find $\lambda>0$ such that (3) holds. $\square$

We now prove that vertical subgroups in the Grassmannian have horizontal complements that can be chosen in a Lipschitz-continuous way.

Definition 2.24. *If $\nu\in\mathfrak{h}_1$ we denote by $\mathbb{N}(\nu)$ the 1-codimensional normal subgroup orthogonal to ν, that is,*

$$\mathbb{N}(\nu):=\{x\in\mathcal{H}^n\ :\ \langle\nu,\pi(x)\rangle=0\},$$

where π is the projection of $\mathcal{H}^n$ onto the horizontal layer $\mathfrak{h}_1$ defined by setting

$$\pi(x):=\sum_{i=1}^n(x_iX_i+x_{n+i}Y_i).$$

Lemma 2.25. *Given $T \in \mathcal{G}(\mathbb{H}^n, k)$ with $n < k \leq 2n$, we can always find unit vectors $\nu_1, \dots, \nu_{2n+1-k} \in \mathfrak{h}_1$, Lipschitz-continuously depending on T, such that*

$$S := \exp\left(\operatorname{span}\{\nu_1, \dots, \nu_{2n+1-k}\}\right)$$

is a horizontal complement of T. Furthermore $T = \cap_{j=1}^{2n+1-k} \mathbb{N}(\nu_j)$ and for all $p \in \mathbb{H}^n$ and all $\alpha \in (0,1]$ the following inclusion holds:

$$Q_\alpha(p, T, \lambda) \subseteq \bigcap_{j=1}^{2n+1-k} Q_\alpha(p, \mathbb{N}(\nu_j), \lambda). \tag{6}$$

Proof. Using [17, Lemma 3.26] we obtain a horizontal subgroup S complementary to T. Moreover, from the construction in the proof of that lemma it follows that S depends Lipschitz continuously on T (indeed, as observed in the proof of [22, Lemma 2.32], this is a continuous dependence and, by linearity argument in the construction, this is equivalent to Lipschitz-continuity). Hence, we can choose $\nu_1, \dots, \nu_{2n+1-k}$ as an orthonormal basis of S depending on T in a Lipschitz way.

Now denote by $\mathfrak{t} \subset \mathfrak{h}$ the Lie algebra of T and $\mathfrak{h}^i := \operatorname{Span}\{\mathfrak{t}, \nu_1, \dots, \nu_{i-1}, \nu_{i+1}, \dots, \nu_{2n+1-k}\}$. Then we have that $\mathbb{N}(\nu_i) = \exp(\mathfrak{h}^i) \in \mathcal{G}(\mathbb{H}^n, 2n)$ and $T = \cap_i \mathbb{N}(\nu_i)$. Now, since $d(p, \mathbb{N}(\nu_i)) \leq d(p, T)$ we have that $Q_\alpha(p, T, \lambda) \subset Q_\alpha(p, \mathbb{N}(\nu_i), \lambda)$ for all $i = 1, \dots, 2n+1-k$, and hence (6) follows. □

The following is a basic result with proof which closely follows an Euclidean version that can be found in [10, Lemma 2.1].

Lemma 2.26. *Let $S, T \in \mathcal{G}(\mathbb{H}^n, k)$ with $n < k \leq 2n$ and set $\vartheta := \rho(S, T)$. Then there are $Z \in \mathcal{G}(\mathbb{H}^n, k-1)$ and $\ell > 0$ such that for any positive number η we have*

$$\mathcal{C}(S, \eta) \cap \mathcal{C}(T, \eta) \subseteq \mathcal{C}\left(Z, \frac{3n\eta}{\ell\vartheta}\right).$$

Proof. First, we claim that there is $e \in T^\perp$ with $\|e\| = 1$ such that there holds $\|\pi_{S^\perp}(e)\| = \ell\vartheta$. Indeed, an application of the triangular inequality gives

$$\vartheta \geq \rho(S^\perp, T^\perp) - C_T - C_S \geq \sup_{\substack{t \in T^\perp \\ \|t\|=1}} \|\pi_{S^\perp}(t)\| - C_T - C_S,$$

where

$$C_T = \max_{\|x\|=1} d(\pi_T(x), \pi_{T^\perp}(x)) \quad \text{and} \quad C_S = \max_{\|x\|=1} d(\pi_S(x), \pi_{S^\perp}(x)).$$

It follows that

$$\vartheta + C_T + C_S \geq \sup_{\substack{t \in T^\perp \\ \|t\|=1}} \|\pi_{S^\perp}(t)\| > 0$$

so there must be $L_1, L_2 > 0$ such that

$$L_1\vartheta \geq \sup_{\substack{t \in T^\perp \\ \|t\|=1}} \|\pi_{S^\perp}(t)\| \geq L_2\vartheta.$$

By compactness, we can find $e \in T^\perp$ and $\ell \in [L_2, L_1]$ such that

$$\|\pi_{S^\perp}(e)\| = \ell\vartheta,$$

and this concludes the proof of the claim. Now consider an orthonormal basis $e_{k+1}, \dots, e_{2n+1}$ of $S^\perp$ and define

$$Z := \operatorname{span}\{e, e_{k+1}, \dots, e_{2n+1}\}^\perp.$$

It is easy to verify that $\dim(Z) = k-1$ and for any $x \in \mathcal{C}(S,\eta) \cap \mathcal{C}(T,\eta)$ we have

$$\begin{cases} \|x^{-1} \cdot e_i\| \leq \eta & \text{for } i = k+1, \dots, 2n+1, \\ \|x^{-1} \cdot e\| \leq \eta. \end{cases}.$$

We now set $e' := \frac{\pi_S(e)}{\|\pi_S(e)\|}$ and consider the resulting orthonormal basis $\{e', e_{k+1}, \dots, e_{2n+1}\}$ of $Z^\perp$. Then for $x \in \mathcal{C}(S,\eta) \cap \mathcal{C}(T,\eta)$, using the triangle inequality, we have

$$\begin{aligned} \|x^{-1} \cdot e'\| &= \frac{1}{\|\pi_S(e)\|} \|x^{-1} \cdot \pi_S(e)\| \\ &= \frac{1}{\ell\vartheta} \left\| x^{-1} \cdot \left(e - \sum_{i=k+1}^{2n+1} (e^{-1} \cdot e_i) e_i \right) \right\| \\ &\leq \frac{1}{\ell\vartheta} (2n+1-k+1)\eta. \end{aligned}$$

We finally infer that

$$\begin{aligned} \|\pi_{Z^\perp}(x)\| &\leq \|x^{-1} \cdot e'\| + \sum_{i=k+1}^{2n+1} \|x^{-1} \cdot e_i\| \\ &\leq \frac{1}{\ell\vartheta}(2n+1-k+1)\eta + \sum_{i=k+1}^{2n+1} \|x^{-1} \cdot e_i\| \\ &\leq \frac{1}{\ell\vartheta}(2(2n+1) - 2k + 1)\eta \leq \frac{3n}{\ell\vartheta}\eta, \end{aligned}$$

and this concludes the proof. □

3. Proof of the main results

The goal of this section is to prove Theorem 1.1 and Proposition 1.2. For some of the key results below we adopted techniques similar to the ones used in [10, Section 3] and [22, Section 3] respectively.

Proof of Proposition 1.2. Let $E \subset \mathbb{H}^n$ be $C^{1,\alpha}$-rectifiable and let $\{\Gamma_i\}_{i \in \mathbb{N}}$ be the family of $C^{1,\alpha}$-regular $\mathbb{H}$-surfaces such that

$$\mathcal{H}^k \left(E \setminus \bigcup_{i \in \mathbb{N}} \Gamma_i \right) = 0.$$

In particular E is $\mathcal{H}^k$-rectifiable so (see [22, Theorem 3.15]) for $\mathcal{H}^k$-a.e. $x \in E$ there exists an approximate tangent subgroup $T_x \in \mathcal{G}(\mathbb{H}^n, k)$ and $\Theta^k_*(E,x) > 0$. For each $i \in \mathbb{N}$ denote by E_i the set $E \cap \Gamma_i$; by standard density properties (e.g., [22, Lemma 3.6]) for $\mathcal{H}^k$-a.e. $x \in E_i$ we have that

$$\Theta^k(E \setminus E_i, x) = 0. \tag{7}$$

By Lemma 2.23, for some $\lambda > 0$ we have that $E_i \cap B(x,r) \setminus Q_\alpha(x, T_x, \lambda) = \emptyset$ for every $x \in E_i$ and r small enough. The conclusion follows by the latter and (7). $\square$

For the sake of simplicity, in the next technical result we will use the following notation for cylinders:

$$C_\alpha^r(x) := \mathcal{C}(V_x, \lambda r^{1+\alpha}),$$

where $\lambda > 0$ does not appear in the left-hand side because it is fixed.

Lemma 3.1. *Let $E \subset \mathbb{H}^n$, take $n < k \le 2n$ and let $M, \lambda, \delta > 0$, $0 < \alpha, r \le 1$ be fixed. Suppose that for every $z \in E$ and for every $s > 0$ we have*

$$\mathcal{H}^{k_m}\left(E \cap B(z,s)\right) \le M s^{k_m}. \tag{8}$$

Consider any two points x, y such that $d(x,y) \le r$ and $V_x, V_y \in \mathcal{G}(\mathbb{H}^n, k)$ satisfying

$$\begin{cases} \mathcal{H}^{k_m}\left(E \cap B(x,r)\right) \ge \delta r^{k_m} \\ \mathcal{H}^{k_m}\left(E \cap B(y,r)\right) \ge \delta r^{k_m} \end{cases} \tag{9}$$

and

$$\begin{cases} \mathcal{H}^{k_m}\left(E \cap B(x,2r) \setminus C_\alpha^r(x)\right) \le \varepsilon r^{k_m} \\ \mathcal{H}^{k_m}\left(E \cap B(y,2r) \setminus C_\alpha^r(y)\right) \le \varepsilon r^{k_m} \end{cases} \tag{10}$$

where $\varepsilon \le \frac{\delta}{4}$. Then there exists a positive constant $C := C(n, \delta, M, \lambda)$ such that

$$\rho(V_x, V_y) \le C r^\alpha.$$

Proof. The proof is by a contradiction. Suppose $\theta := \rho(V_x, V_y) > C r^\alpha$, with $C = C(n, \delta, M, \lambda) > 0$ to be chosen later. First we observe that by the assumptions (9) and (10) we obtain that

$$\mathcal{H}^k(E \cap C_\alpha^r(x) \cap B(x,r)) \ge (\delta - \varepsilon) r^k.$$

Furthermore, using that

$$E \cap C_\alpha^r(x) \cap B(x,r) \setminus C_\alpha^r(y) \subseteq E \cap B(y,2r) \setminus C_\alpha^r(y),$$

it follows that $\mathcal{H}^k(E \cap C_\alpha^r(x) \cap B(x,r) \setminus C_\alpha^r(y)) \le \varepsilon r^k$ and hence

$$(\delta - \varepsilon) r^k \le \mathcal{H}^k\left(E \cap C_\alpha^r(x) \cap B(x,r)\right) \le \mathcal{H}^k\left(E \cap C_\alpha^r(x) \cap B(x,r) \cap C_\alpha^r(y)\right) + \varepsilon r^k.$$

Thus, by the assumption on ε, we obtain that

$$\mathcal{H}^{k_m}\left(E \cap C_\alpha^r(x) \cap C_\alpha^r(y) \cap B(x,r)\right) \ge \frac{\delta}{2} r^{k_m} \tag{11}$$

and, in particular, $B(x,r) \cap C_\alpha^r(x) \cap C_\alpha^r(y) \ne \emptyset$.

Now since V_x and V_y are vertical k-subgroups they intersect and hence by Lemma 2.26 we have that there exists a vertical $(k-1)$-subgroup Z such that

$$C_\alpha^r(x) \cap C_\alpha^r(y) \subset \mathcal{C}\left(Z, 4n\lambda r^{1+\alpha}/\ell\theta\right).$$

where $\ell > 0$ is as in Lemma 2.26. We set $\eta := 4n\lambda r^{1+\alpha}/\ell\theta$ and $E_\eta = E \cap B(x,r) \cap \mathcal{C}(Z, \eta)$. By the supposition on θ we infer $\eta \le r$. We claim that

$$\mathcal{H}^{k_m}(E_\eta) < \frac{4nC_1 M\lambda}{\ell C} r^{k_m}. \tag{12}$$

where $C_1 = C_1(n, k_m) > 0$. Indeed, clearly E_η can be covered with h balls of radius ηr and $h \le C_1 \eta^{-k_m+1}$, where C_1 depends only on n and k_m. Thus by (8) we have, using that $0 < r < 1$ and the assumption on θ,

$$\mathcal{H}^{k_m}(E_\eta) \le C_1 M \eta r^{k_m} = \frac{4nC_1 M\lambda r^{1+\alpha}}{\ell\theta} r^{k_m} < \frac{4nC_1 M\lambda}{\ell C} r^{k_m},$$

hence the claim (12). Now, using the above estimate together with (11) we obtain that

$$\frac{\delta}{2} r^{k_m} \le \mathcal{H}^{k_m}(E_\eta) < \frac{4nC_1 M\lambda}{\ell C} r^{k_m}.$$

Choosing $C = \frac{4n}{\delta\ell} \max\{C_1 M\lambda, \lambda\}$ the contradiction is immediate and the result follows. $\square$

Thanks to this result, Lemma 2.25 can be improved as in the following.

Lemma 3.2. *Let $n < k \le 2n$ and fix $M, \lambda, \delta > 0$, $0 < \alpha, r \le 1$. Let $U \subset \mathbb{H}^n$ be an open subset with diam$(U) > 2r$ and $E \subseteq U$ as in Lemma 3.1. Consider the mapping*

$$U \ni p \longmapsto V_p \in \mathcal{G}(\mathbb{H}, k)$$

and suppose, for every $p, q \in E$ with $d(p,q) < r$ and satisfying (9), that V_p and V_q satisfy (10). If we denote by

$$W_p = \operatorname{span}\left(\exp\{\nu_1(p), \dots, \nu_{2n+1-k}(p)\}\right)$$

the horizontal complement for $T = V_p$ in Lemma 2.25, then the mappings

$$p \longmapsto \nu_j(p), \quad j = 1, \dots, 2n+1-k$$

are α-Hölder continuous in some $U' \subset U$.

Proof. By Lemma 3.1 we have that for every $p, q \in E$ with $d(p,q) < r$,

$$\rho(V_p, V_q) \le C d(p,q)^\alpha$$

for some $C > 0$. On the other hand, in Lemma 2.25 we proved that

$$V_p \longmapsto \nu_j(p), \quad j = 1, \dots, 2n+1-k$$

are Lipschitz-continuous maps for $p \in \tilde{U}$, for some open set $\tilde{U} \subseteq U$, that is, there exists $c' > 0$ such that

$$d(\nu_j(p), \nu_j(q)) \le c' \rho(V_p, V_q)$$

holds for all $p, q \in \tilde{U}$. Putting the above inequalities together, we get

$$d(\nu_j(p), \nu_j(q)) \le \tilde{c} d(p,q)^\alpha,$$

for all $p, q \in \tilde{U} \cap E$, where $\tilde{c} := C \cdot c'$. Take $U' = \tilde{U} \cap E$ and the conclusion follows. $\square$

The following is a very useful separation result.

Lemma 3.3. *Let $\alpha \in (0,1]$, $\lambda > 0$, $p, q \in \mathbb{H}^n$ and $V_p, V_q \in \mathcal{G}(\mathbb{H}^n, k)$, $n < k \le 2n$, vertical subgroups satisfying the assumptions of Lemma 3.1. Then there exists $\lambda' > \lambda$ such that, if we further assume*

$$q \notin Q_\alpha(p, V_p, \lambda'),$$

then the following inclusion holds:

$$B\left(q, \frac{r}{2}\right) \cap C_\alpha^{r/2}(q) \subset B(p, 2r) \setminus Q_\alpha(p, V_p, \lambda),$$

where $r := d(p,q)$. *Moreover, for* λ' *we can take any value satisfying*

$$\lambda' \geq \frac{2^\alpha cC + (1 + 3^{1+\alpha})\lambda}{2^{1+\alpha}c}, \tag{13}$$

where c *and* C *are the positive constants given, respectively, in Proposition 2.14 and Lemma 3.1.*

Proof. Let $z \in B(q, r/2) \cap C_\alpha^{r/2}(q)$. Using the triangular inequality we have

$$d(p,z)^{1+\alpha} \leq [d(p,q) + d(q,z)]^{1+\alpha} \leq \left(\frac{3}{2}\right)^{1+\alpha} r^{1+\alpha},$$

so to prove that z does not belong to the paraboloid $Q_\alpha(p, V_p, \lambda)$ we only need to estimate from below $d(p^{-1}z, V_p)$. We start by noticing that

$$q \notin Q_\alpha(p, V_p, \lambda') \implies d(p^{-1}q, V_p) > \lambda' r^{1+\alpha}, \tag{14}$$

and, similarly, also that

$$z \in C_\alpha^{r/2}(q) \implies d(q^{-1}z, V_q) \leq \lambda \left(\frac{r}{2}\right)^{1+\alpha}. \tag{15}$$

Since V_p is a vertical homogeneous subgroups, by Proposition 2.14 we have

$$d(p^{-1}z, V_p) \geq c\|\pi_{V_p^\perp}(p^{-1}z)\|,$$

where c is a constant that depends on the dimension of p only. Now, taking into account that $V_p^\perp$ and $V_q^\perp$ are horizontal by definition, we have

$$\begin{aligned}
\|\pi_{V_p^\perp}(p^{-1}z)\| &= \left\|\pi_{V_p^\perp}(p^{-1}q) \cdot \pi_{V_q^\perp}(q^{-1}z) \cdot \pi^{-1}_{V_q^\perp}(q^{-1}z) \cdot \pi_{V_p^\perp}(q^{-1}z)\right\| \\
&\geq \|\pi_{V_p^\perp}(p^{-1}q)\| - \|\pi_{V_q^\perp}(q^{-1}z)\| - \|\pi^{-1}_{V_q^\perp}(q^{-1}z) \cdot \pi_{V_p^\perp}(q^{-1}z)\|.
\end{aligned}$$

Notice that for the last term we have the inequality

$$\begin{aligned}
\|\pi^{-1}_{V_q^\perp}(q^{-1}z) \cdot \pi_{V_p^\perp}(q^{-1}z)\| &\leq \rho(V_p, V_q) d(q,z) \\
&\leq Cr^\alpha \frac{r}{2} = \frac{C}{2} r^{1+\alpha},
\end{aligned}$$

where C is the constant given in Lemma 3.1. If we apply Proposition 2.14 to the first two terms we get the following inequalities:

$$\begin{aligned}
d(p^{-1}z, V_p) &\geq c\|\pi_{V_p^\perp}(p^{-1}z)\| \\
&\geq c\|\pi_{V_p^\perp}(p^{-1}q)\| - c\|\pi_{V_q^\perp}(q^{-1}z)\| - c\|\pi^{-1}_{V_q^\perp}(q^{-1}z) \cdot \pi_{V_p^\perp}(q^{-1}z)\| \\
&\geq cd(p^{-1}q, V_p) - d(q^{-1}z, V_q) - \frac{C}{2} cr^{1+\alpha},.
\end{aligned}$$

Using both (14) and (15) yields the inequality

$$d(p^{-1}z, V_p) > \left[c\lambda' - \lambda 2^{-1-\alpha} - c\frac{C}{2}\right] r^{1+\alpha}$$

and, as a consequence, the quantity

$$\left[c\lambda' - \lambda 2^{-1-\alpha} - c\frac{C}{2}\right] r^{1+\alpha} - \lambda\left(\frac{3}{2}\right)^{1+\alpha} r^{1+\alpha}$$

is nonnegative if we take $\lambda' > 0$ as in (13), concluding the proof. $\square$

3.1. **Proof of Theorem 1.1.** Following [22], we first prove uniqueness almost everywhere of approximate tangent paraboloids.

Proposition 3.4. *Let $E \subset \mathbb{H}^n$ be $\mathcal{H}^{k_m}$-measurable with $\mathcal{H}^{k_m}(E) < \infty$, and let A be the set of points of E for which there is an approximate tangent parabolid of dimension k and metric dimension k_m. Then the following holds:*

(a) A is $\mathcal{H}^{k_m}$-measurable;
(b) E has an unique approximate tangent paraboloid V_p at $\mathcal{H}^{k_m}$-a.e. $p \in A$;
(c) the mapping $A \ni p \mapsto V_p \in \mathcal{G}(\mathbb{H}^n, k)$ is measurable.

The proof of this result follows the same strategy of [22, Proposition 3.9] so we refer the reader to that paper for more details.

We are now ready to prove our main result. We follow closely the strategy in [22, Theorem 3.15] and point out the main differences in our case.

Proof of Theorem 1.1. First, since $\mathcal{H}^{k_m}(E) < \infty$, by a standard density estimate (see Lemma 2.2), $\Theta^{*k_m}(E,p) \le 5^{k_m}$ for $\mathcal{H}^{k_m}$-a.e. $x \in E$; we can assume

$$\mathcal{H}^{k_m}(E \cap B(p,r)) \le 7^{k_m} r^{k_m} \quad \text{for all } p \in E \text{ and } r > 0. \tag{16}$$

Using the positive lower density and condition (1), we have that for $\mathcal{H}^{k_m}$-a.e. $p \in E$ we can find $\ell(p) > 0$, $0 < r(p) \le 1$ and $V_p = \mathrm{apPar}^{k_m}_{\mathbb{H}}(E,p)$ such that

$$\mathcal{H}^{k_m}\left(E \cap B(p,r)\right) > \ell(p) r^{k_m} \quad \text{for all } 0 < r < r(p), \tag{17}$$

and, for some $\lambda = \lambda_p > 0$, we also have

$$\mathcal{H}^{k_m}\left(E \cap B(p,r) \setminus Q_\alpha(p, V_p, \lambda)\right) \le \varepsilon r^{k_m}$$

with $\varepsilon < \frac{1}{4^{k_m}+1}\ell(p)$. Moreover, since $B(p,r) \cap Q_\alpha(p, V_p, \lambda) \subset C^r_\alpha(p)$, it follows that

$$\mathcal{H}^{k_m}\left(E \cap B(p,r) \setminus C^r_\alpha(p)\right) \le \varepsilon r^{k_m}. \tag{18}$$

Consider for any $i \ge 1$ the set

$$E_i := \left\{ p \in E \ : \ \min\{r(p), \ell(p)\} > \frac{1}{i} \right\},$$

and denote by $E^* = \bigcup_{i \ge 1} E_i$. Then clearly $\mathcal{H}^{k_m}(E \setminus E^*) = 0$. Hence it suffices to prove the result for the set E_i for each $i \ge 1$.

Now recall that, by Lemma 2.25, for any $p \in E^*$, we can find $2n+1-k$ horizontal unit vectors $\nu_h(p)$ in the horizontal bundle $\mathrm{H}\mathbb{H}^n_p$ transversal to V_p such that

$$T_p := \exp\left(\mathrm{span}\{\nu_1(p), \dots, \nu_{2n+1-k}(p)\}\right)$$

is a horizontal subgroup of $\mathbb{H}^n$ satisfying $\mathbb{H}^n = V_p \cdot T_p$. Moreover, using the Lipschitz continuity part of Lemma 2.25 and Proposition 3.4, we have

$$E^* = \bigcup_{j\geq 1} F_j,$$

with $\mathcal{H}^{k_m}(F_j) < \infty$ and $\nu_h\big|_{F_j}$ is $\mathcal{H}^{k_m}$-measurable. As a consequence

$$\nu_h : E^* \to \mathrm{H}\mathbb{H}^n$$

is a measurable sections of $\mathrm{H}\mathbb{H}^n$ for each $1 \leq h \leq 2n+1-k$.

Now, following the strategy proposed in [22], we define for appropriate indices and for $p \in E^*$ the function

$$\rho_{i,h,j}(p) := \sup\left\{ \frac{|\langle \nu_h(p), \pi(p^{-1}q)\rangle|}{d(p,q)^{1+\alpha}} \ : \ q \in E_i,\ 0 < d(p,q) < \frac{1}{j} \right\},$$

where $\pi : \mathbb{H}^n \to \mathfrak{h}_1$ is the projection onto the first layer given by

$$\pi(q) := \sum_{j=1}^{n} (q_j X_j + \tilde{q}_j Y_j).$$

For any pair of points $p, q \in E_i$, applying Lemma 3.1 with $r = d(p,q)$ we obtain that the map $p \to V_p$ is α-Hölder when restricted to E_i, i.e.,

$$\rho(V_p, V_q) \leq C d(p,q)^{\alpha} \quad \text{for every } p, q \in E_i,$$

where $C := C(n, \ell(p), \lambda)$ is the constant as in Lemma 3.1. Notice that here we have used (16) to verify the assumption (8). We now claim that, up to taking a larger aperture $\lambda' > \lambda$, we have that

$$E_i \setminus Q_\alpha(p, V_p, \lambda') = \varnothing.$$

Suppose for a contradiction that this does not hold. Let $p, q \in E_i$ be such that the assumptions of Lemma 3.3 hold; then we have the inclusion

$$B\left(q, \frac{r}{2}\right) \cap C_\alpha^{r/2}(q) \subset B(p, 2r) \setminus Q_\alpha(p, V_p, \lambda)$$

assuming that $q \notin Q_\alpha(p, V_p, \lambda')$, with λ' satisfying (13). Consequently, using the estimates (17) and (18), we deduce that

$$\begin{aligned}(\ell(p) - \varepsilon)(r/2)^{k_m} &\leq \mathcal{H}^{k_m}\left(E_i \cap B\left(q, \frac{r}{2}\right) \cap C_\alpha^{r/2}(q)\right) \\ &\leq \mathcal{H}^{k_m}\left(E \cap B(p, 2r) \setminus Q_\alpha(p, V_p, \lambda)\right) \leq 2^{k_m}\varepsilon^{k_m},\end{aligned}$$

which gives a contradiction since $\varepsilon < \frac{1}{4^{k_m}+1}\ell(p)$. Hence the claim holds and by Lemma 2.25 it follows that

$$E_i \setminus Q_\alpha(p, \mathbb{N}(\nu_h(p)), \lambda') = \varnothing \quad \text{for all } 1 \leq h \leq 2n+1-k. \tag{19}$$

As a consequence of (19) we have that for all $i \geq 1$ and $1 \leq h \leq 2n+1-k$

$$\lim_{j\to\infty} \rho_{i,h,j}(p) = 0.$$

If we now apply Lusin theorem to each ν_h and Egoroff theorem to the sequence $(\rho_{i,h,j})_{j\in\mathbb{N}}$ we find that we can write

$$E_i = E_{i,0} \bigcup \left(\bigcup_{\beta\geq 1} K_{i,\beta} \right),$$

with $E_{i,0}$ $\mathcal{H}^{k_m}$-negligible, $K_{i,\beta}$ compact, $\nu_h\big|_{K_{i,\beta}}$ continuous and $\rho_{i,h,j}$ going to zero uniformly in $K_{i,\beta}$ with respect to j. Using Lemma 3.2 we obtain that

$$\nu_h\big|_{K_{i,\beta}}$$

is actually α-Hölder continuous. By applying the Whitney theorem (see Theorem 2.12) on $K_{i,\beta}$ we obtain the functions

$$f_{i,\beta,h} \in \mathbf{C}^{1,\alpha}_H(\mathbb{H}^n)$$

with $f_{i,\beta,h}\big|_{K_{i,\beta}} = 0$, $\nabla_H f_{i,\beta,h}\big|_{K_{i,\beta}} = \nu_h$ and $|\nabla_H f_{i,\beta,h}| \neq 0$ on $K_{i,\beta}$. The set

$$S_{i,\beta,h} := \{p \in \mathbb{H}^n \ : \ f_{i,\beta,h}(p) = 0, \ |\nabla_H f_{i,\beta,h}| \neq 0\}$$

is a 1-codimensional $(\mathbf{C}^{1,\alpha}_H, \mathbb{H})$-regular surface containing $K_{i,\beta}$ so we can consider the following intersection:

$$S_{i,\beta} := \bigcap_{h=1}^{2n+1-k} S_{i,\beta,h}.$$

By Proposition 2.10 we have that $S_{i,\beta}$ is a k-codimensional $(\mathbf{C}^{1,\alpha}_H, \mathbb{H})$-regular surface that contains the set $K_{i,\beta}$. Moreover, we have

$$E \subset E_0 \cup (\cup_{i\geq 1} \cup_{\beta\geq 1} S_{i,\beta}),$$

with $E_0 = (E \setminus E^*) \cup \cup_{i=1}^{\infty} E_{i,0}$ and $\mathcal{H}^{k_m}(E_0) = 0$. Hence E is $C^{1,\alpha}$-rectifiable. □

3.2. **Extension to Carnot groups.** Let $\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota$ be a Carnot group of dimension q and homogeneous dimension

$$Q = \sum_{j=1}^{\iota} j \cdot \dim(H^j).$$

In analogy with the Heisenberg case, we say that a homogeneous subgroup $T \subset \mathbb{G}$ of codimension $k \leq \dim(H^1)$ is *vertical* if

$$T = T_H \oplus H^2 \oplus \cdots \oplus H^\iota,$$

where $T_H \subset H^1$ has dimension $\dim(H^1) - k$. As a consequence, the definition of the Grassmannian (Definition 2.16) can be immediately extended, but there is an issue that we mentioned already in the paper: if S is a vertical subgroup of codimension at least 2, then there is no guarantee that a **horizontal** complement exists which, in turn, means that it may not even belong to the Grassmannian.

Remark 3.5. The issue discussed above does not apply to the particular case of codimension one since a **horizontal** complement always exists and is generated by a single element. Indeed, if we replace

$$k_m \text{ with } Q-1 \text{ and } k \text{ with } q-1,$$

then all technical results can be proved using the same argument, except for Lemma 3.2 which is trivial because the distance between the horizontal complements is given by the distance between the two generating vectors which can be chosen using the orthogonality condition.

Acknowledgements. The authors would like to thank Davide Vittone for pointing out the one-codimensional case for Carnot groups and also for valuable suggestions regarding the original draft.

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