Quantitative estimates for parabolic optimal control problems under L^∞ and L^1 constraints in the ball: Quantifying parabolic isoperimetric inequalities

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Abstract

In this article, we present two different approaches for obtaining quantitative inequalities in the context of parabolic optimal control problems. Our model consists of a linearly controlled heat equation with Dirichlet boundary condition $(u_f)_t - \Delta u_f = f$, f being the control. We seek to maximise the functional $\mathcal{J}_T(f) := \frac{1}{2}\iint_{(0;T)\times\Omega} u_f^2$ or, for some $\varepsilon > 0$, $\mathcal{J}_T^\varepsilon(f) := \frac{1}{2}\iint_{(0;T)\times\Omega} u_f^2 + \varepsilon\int_\Omega u_f^2(T,\cdot)$ and to obtain quantitative estimates for these maximisation problems. We offer two approaches in the case where the domain Ω is a ball. In that case, if f satisfies L^1 and L^∞ constraints and does not depend on time, we propose a shape derivative approach that shows that, for any competitor $f = f(x)$ satisfying the same constraints, we have $\mathcal{J}_T(f^*) - \mathcal{J}_T(f) \gtrsim \|f - f^*\|^2_{L^1(\Omega)}$, f^* being the maximiser. Through our proof of this time-independent case, we also show how to obtain coercivity norms for shape hessians in such parabolic optimisation problems. We also consider the case where $f = f(t,x)$ satisfies a global L^∞ constraint and, for every $t \in (0;T)$, an L^1 constraint. In this case, assuming $\varepsilon > 0$, we prove an estimate of the form $\mathcal{J}_T^\varepsilon(f^*) - \mathcal{J}_T^\varepsilon(f) \gtrsim \int_0^T a_\varepsilon(t)\|f(t,\cdot) - f^*(t,\cdot)\|^2_{L^1(\Omega)}$ where $a_\varepsilon(t) > 0$ for any $t \in (0;T)$. The proof of this result relies on a uniform bathtub principle.

Keywords: Shape optimisation, Optimal control, Parabolic PDEs, Quantitative inequalities.

AMS classification: 49J15, 49Q10.

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1 Introduction

This Introduction is structured as follows: in Subsection 1.1, we present the scope of our article; in Subsection 1.1.1, we give an informal statement of our results while in Subsection 1.2.1 we give several bibliographical references on qualitative properties for optimal control problems, shape derivatives for parabolic problems and quantitative inequalities. In Subsection 1.3, we give basic information regarding the Schwarz rearrangement, which will be a key tool in our analysis, and we

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give bibliographical references for parabolic isoperimetric inequalities. In Subsection 1.2, we state our main results, Theorems II and III (Theorem I deals with the uniqueness of solutions to our optimal control problem and is also stated in this Section). In Subsection 1.4 we present the plan of our paper and, finally, in Subsection 1.5, we gather the notations we will use throughout the paper.

1.1 Scope of the article

1.1.1 Goal of this article: informal statement of the problems and of the results

In this article, our goal is to present two different approaches for obtaining *quantitative inequalities for optimal control problems*, which will also be dubbed *quantitative isoperimetric parabolic inequalities*. Before explaining how this fits in the growing field of qualitative questions in optimal control theory, let us vaguely state the type of results we wish to establish, and sketch the two approaches that will be put forth. By quantitative inequalities, we mean the following: we consider a controlled parabolic partial differential equation assuming the general form

$$u_t - \mathcal{L}u = f \text{ in } (0;T)\times\Omega, \tag{1.1}$$

$\mathcal{L}$ being an elliptic operator; this equation is supplemented with some initial condition and some boundary conditions. In this setting, f is the control and depends *a priori* both on time and space. It is assumed to satisfy some constraints, which will be taken into account by assuming that $f\in\mathscr{X}$, where $\mathscr{X}$ is some subset of a function space. The cost to be optimised is some functional $\mathcal{J}_T:\mathscr{X}\ni f\mapsto\mathcal{J}_T(f)$. The control problems reads

$$\boxed{\max_{f\in\mathscr{X}}\mathcal{J}_T(f).} \tag{1.2}$$

The quantitative inequality we aim at can take two different forms:

- **For time independent controls.** In the context where all controls $f\in\mathscr{X}$ write $f=f(x)$, and if the solution of (1.2) is some $\overline{f}$ (assumed to be unique for simplicity), the goal is to establish the following kind of estimate

$$\boxed{\forall f\in\mathscr{X}\,,\mathcal{J}_T(f)-\mathcal{J}_T(\overline{f})\leqslant -C(T)\|f-\overline{f}\|^2_{L^1(\Omega)}} \tag{1.3}$$

  for some constant $C(T)>0$. The right-hand side quantity is natural in the context of quantitative inequalities for shape optimisation problems [21] and optimal control problems [29], and is akin to the Fraenkel asymmetry. We refer to Subsection 1.2.1.

- **For time-dependent controls.** In the context where the controls are time dependent i.e. $f=f(t,x)$ and when the solution of (1.2) is some f^*, the goal is to establish something of the form

$$\boxed{\forall f\in\mathscr{X}\,,\mathcal{J}_T(f)-\mathcal{J}_T(f^*)\leqslant -\int_0^T\omega(s)\|f(s,\cdot)-f^*(s,\cdot)\|^2_{L^1(\Omega)}} \tag{1.4}$$

  for a function $\omega:[0;T]\to\mathbb{R}_+$ such that for any $s\in(0;T)$, $\omega(s)>0$. As will be explained more in detail in Subsection 1.2.1 and commented upon in the Conclusion, see Section 6.2, this is a stronger norm than the usual one.

To the best of our knowledge, neither type of quantitative estimates have been derived despite their natural interest.

Obviously, one can not expect to prove (1.3) or (1.4) for all optimal control problems. What we propose here is to *establish both these inequalities for a linearly controlled heat equation in the ball under* L^1 *and* L^∞ *constraints.* The main equation under consideration is set in $\Omega = \mathbb{B}(0;R)$ and writes

$$\begin{cases} \frac{\partial u_f}{\partial t} - \Delta u_f = f \text{ in } (0;T)\times\Omega\,, \\ u_f(t=0) = u^0 \geqslant 0 \text{ in } \Omega\,, \\ u_f(t,\cdot) = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.5}$$

We will also assume that the initial condition $u^0 \in \mathscr{C}^2(\Omega)\cap W_0^{1,2}(\Omega)$, $u^0 \geqslant 0$, which is fixed, is radially symmetric and non-increasing. In the time-independent case (when $f = f(x)$), the functional we seek to maximise is defined by

$$\mathcal{J}_T(f) := \frac{1}{2}\iint_{(0;T)\times\Omega} u_f^2(t,x)dxdt. \tag{1.6}$$

In the time dependent case (when $f = f(t,x)$), the functional we seek to maximise is

$$\mathcal{J}_T^\varepsilon(f) := \frac{1}{2}\iint_{(0;T)\times\Omega} u_f^2(t,x)dxdt + \frac{\varepsilon}{2}\int_\Omega u_f^2(T,x)dx \tag{1.7}$$

for some $\varepsilon > 0$. The main reason behind supplementing the functional with a final time term is to ensure the non-degeneracy of the switch function associated to the optimisation problem.

As a final comment, let us remark that the constant $C(T)$ appearing in (1.3) and the weight ω are constructed in a non-explicit way.

Remark 1. *Although we prove our results for maximisation of functionals, we believe the same strategies work for the minimisation of the functional. For both problems, both inequalities may have interesting consequences for inverse problems.*

Remark 2. *Obviously, if the optimal control* f^* *for the time-dependent case does not depend on time, which will be the case here,* (1.4) *implies* (1.3) *with* $C(T) = \int_0^T \omega(s)ds$*. However, the reason why we present two proofs is the possibility of generalising the methods used to prove* (1.3)*. In the conclusion, we explain why we believe this inequality can be extended to other types of control problems, such as bilinear control problems, or how, in general domains, technical assumptions on second order shape derivatives may enable one to derive it. In short, the proof of* (1.3) *relies on two properties of the control problem: the first one is shape derivatives, for which the trickiest part is to prove coercivity of second order derivatives in general domains; the second one is the convexity of the problem, which is something extremely general.*

The proof of (1.4) *is specific to the case of the ball and it is unclear whether or not it may be adapted to other domains. Indeed, the parabolic isoperimetric inequalities used in its proof [5, 6, 33] may not hold in general domains, in the sense that explicit characterisation of maximisers may not be attainable.*

1.2 Statement of the main results

Let $\Omega = \mathbb{B}(0;R)$ be a centred ball in dimension n. We assume that we are given an initial condition satisfying

$$u^0 \geqslant 0\,, u^0 \in \mathscr{C}^2(\Omega)\cap W_0^{1,2}(\Omega)\,, u^0 \text{ is radially symmetric and non-increasing.} \tag{1.8}$$

For a function $f \in L^2((0;T)\times\Omega)$ we consider the solution u_f of

$$\begin{cases} \frac{\partial u_f}{\partial t} - \Delta u_f = f \text{ in } (0;T)\times\Omega\,, \\ u_f(t=0) = u^0\,, \\ u_f(t,\cdot) = 0 \text{ in } (0;T)\times\partial\Omega. \end{cases} \tag{1.9}$$

The functional we wish to optimise in the time-independent case is

$$\mathcal{J}_T(f) := \frac{1}{2}\iint_{(0;T)\times\Omega} u_f^2(T,\cdot). \tag{J}$$

For a given L^1 constraint $V_0 \in (0;\mathrm{Vol}(\Omega))$, the sets of admissible controls are

$$\overline{\mathcal{M}}(\Omega) := \left\{ f \in L^\infty(\Omega)\,, 0 \leqslant f \leqslant 1\,, \int_\Omega f = V_0 \right\} \tag{$\overline{\mathbf{Adm}}$}$$

and

$$\mathcal{M}_T(\Omega) := \left\{ f \in L^\infty\left((0;T)\times\Omega\right) \text{ for a.e. } t \in (0;T),\ f(t,\cdot) \in \overline{\mathcal{M}}(\Omega). \right\} \tag{$\mathbf{Adm}_T$}$$

The first problem we address is

$$\boxed{\max_{f\in\overline{\mathcal{M}}(\Omega)} \mathcal{J}_T(f).} \tag{$\mathbf{I}_1$}$$

The second problem is set in the time-dependent case and the functional we seek to optimise is, for some $\varepsilon > 0$,

$$\mathcal{J}_T^\varepsilon(f) := \frac{1}{2}\iint_{(0;T)\times\Omega} u_f^2(T,\cdot) + \frac{\varepsilon}{2}\int_\Omega u_f^2(T,\cdot). \tag{1.10}$$

The second problem we address is

$$\boxed{\max_{f\in\mathcal{M}_T(\Omega)} \mathcal{J}_T^\varepsilon(f).} \tag{$\mathbf{I}_2^\varepsilon$}$$

Finally, we fix throughout the paper the notation

$$f^* := \mathbb{1}_{\mathbb{B}^*} \tag{1.11}$$

where $\mathbb{B}^*$ is the unique centred ball of volume V_0, and define u^* as the solution of (1.9) associated with $f \equiv f^*$.

Remark 3. *The existence of solutions of* ($\mathbf{I}_1$) *and* ($\mathbf{I}_2^\varepsilon$) *are easy consequence of the direct methods of the calculus of variations.*

As an easy corollary of [5, Theorem 3], f^* is a maximiser of both ($\mathbf{I}_1$) and ($\mathbf{I}_2^\varepsilon$). To prove our results, the first step is the following Theorem:

Theorem I. *f^* is the unique solution of* ($\mathbf{I}_1$) *and of* ($\mathbf{I}_2^\varepsilon$).

It is not the main goal of this paper, but the result is in itself interesting and relies on topological properties of some classes of functions defined *via* rearrangements. We refer to Section 3 for the proof.

Let us now pass to the two main results of this article. We choose to state first the time-dependent case, as the result holds without any restriction on the dimension. In this case, we need to take $\varepsilon > 0$. The reason behind this is technical, and amounts, to put it shortly, to forcing the switch function of the problem to be non-degenerate. We comment on this in the Conclusion.

Theorem II. *If $\varepsilon > 0$, there exists a constant $C(\varepsilon, T) > 0$ such that*

$$\forall f \in \mathcal{M}_T(\Omega)\,, \mathcal{J}_T^\varepsilon(f) - \mathcal{J}_T^\varepsilon(f^*) \leqslant -C(\varepsilon, T)\int_0^T \left(-\frac{\partial p_\varepsilon^*}{\partial \nu}\right)_{\partial \mathbb{B}^*} \|f(t,\cdot) - f^*\|^2_{L^1(\Omega)}. \tag{1.12}$$

In the estimate above p_ε^* solves

$$\begin{cases} \frac{\partial p_\varepsilon^*}{\partial t} + \Delta p_\varepsilon^* = -u^* \text{ in } (0;T)\times\Omega\,, \\ p_\varepsilon^*(T,\cdot) = \varepsilon u^*(T,\cdot)\,, \\ p_\varepsilon^*(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega \end{cases} \tag{1.13}$$

and is the switch function of ($\mathbf{I}_2^\varepsilon$).

Remark 4. *Actually, when $\varepsilon > 0$, we can even prove that there exists a constant $A(\varepsilon, T) > 0$ such that*

$$\forall t \in (0;T)\,, \left(-\frac{\partial p_\varepsilon^*}{\partial \nu}\right)_{\partial \mathbb{S}^*} \leqslant -A(\varepsilon, T). \tag{1.14}$$

We however choose to keep the partial derivative of the switch function as it seems to us to be a more precise result.

We then pass to the time-independent case, where the main innovation will be the use of shape derivatives. We include this result not only for the sake of completeness but also because this method seems, at this stage, generalisable to other domains, while it may not be the case for Theorem II.

Theorem III. *Assume $n = 2$.*

For any $T > 0$ there exists a constant $C(T) > 0$ such that

$$\forall f \in \overline{\mathcal{M}}(\Omega)\,, \mathcal{J}_T(f) - \mathcal{J}_T(f^*) \leqslant -C(T)\|f - f^*\|^2_{L^1(\Omega)}. \tag{1.15}$$

1.2.1 Bibliographical references

Let us now present the frameworks into which we think our present work fits.

Qualitative questions in optimal control problems The question of qualitative properties of optimal control problems has recently drawn a lot of attention. Indeed, in many situations, explicit computation of the optimal control is nearly impossible, and a line of research has emerged that deals with the question of knowing what optimal controls nearly look like, or whether or not these controls are (un)stable in a sense that has to be specified. Among all these qualitative queries, one may single out the following:

- **Insensitising controls.** The question of insensitising controls is a very natural one, and is a possible solution to the following question: given that it is often the case that one can not practically realise the exact control strategy and that some imperfections may arise, how can a *robust* control strategy be constructed? In that context, the goal is to find an *insensitising* optimal control. This question has been studied, for instance, in [1, 28] and, more recently, in [18].

- **The turnpike property.** The turnpike property states that, when dealing with time evolving optimal control problems, it is sometimes possible to actually find a *nearly static* optimal strategy or, in other words, that the optimal control remains *close*, in some sense that has to be quantified, to the solution of a stationary optimal control problem. First motivated by applications in economics [17], this field has been rapidly growing over the last decade and has found applications in many contexts (e.g. control of non-linear differential equations, control of the wave equation, control of semilinear heat equations, machine learning) [42, 13, 19, 26, 37, 39, 40, 43]. It has recently been derived for bilinear optimal control problems using quantitative inequalities for stationary optimisation problems [31].

Shape derivatives for time-evolving problems Our work presents what is to the best of our knowledge the first detailed analysis of a second order shape derivative for a time-evolving optimal control problems (in the sense that a coercivity norm for the second order shape derivative is obtained), albeit it deals with shape derivatives with respect to a subdomain. Although the literature devoted to time-evolving optimal control problems is scarce, we would like to point to [34] where a speed-method approach is presented, and to the recent preprint [11] where shape derivatives (with respect to the underlying domain Ω) are computed and used to obtain numerical simulations of a shape optimisation problem.

Quantitative inequalities The study of quantitative inequalities in shape optimisation problems is an enormous field. To mention a few works, we point to the seminal [21] for the quantitative isoperimetric inequality, and to [9] for quantitative spectral inequalities. Regarding quantitative inequalities for (stationary) control problems we refer to [8] for a quantitative inequality for the natural Dirichlet energy, to [12] for a quantitative spectral inequality (with respect to the potential) in $\mathbb{R}^n$ (both these works are done under L^p constraints), to [29] for a quantitative spectral inequality in the ball under L^1 and L^∞ constraints and to [31] for a generalisation of this inequality to other domains, and for an application to the turnpike property.

Let us comment on the type of estimates usually obtained: given a functional $\mathcal{F} : \Omega \mapsto \mathcal{F}(\Omega)$, a typical problem reads

$$\inf_{\Omega\,,\mathrm{Vol}(\Omega)=\overline{V}} \mathcal{F}(\Omega). \tag{1.16}$$

Let us assume that, up to a translation, the unique minimiser of this functional is a ball $\overline{\mathbb{B}}$ of volume $\overline{V}$ (this is the case when $\mathcal{F}(\Omega) = \mathrm{Per}(\Omega)$), then the inequality obtained in [21] reads: there exists a constant $C > 0$ such that, defining the Fraenkel asymmetry of Ω as

$$\mathcal{A}(\Omega) = \inf_{x\in\mathbb{R}^n} \mathrm{Vol}\left((x + \overline{\mathbb{B}})\Delta\Omega\right) \tag{1.17}$$

there holds

$$\mathcal{F}(\Omega) - \mathcal{F}(\overline{\mathbb{B}}) \geqslant C\mathcal{A}(\Omega)^2. \tag{1.18}$$

In the case of estimate (1.3), the coercivity obtained is akin to this measure of asymmetry if the maximiser $\overline{f}$ writes $\overline{f} = \mathbb{1}_{\overline{E}}$: by defining $\mathcal{J}_T(E) := \mathcal{J}_T(\mathbb{1}_E)$ with a slight abuse of notation, if we choose a competitor f of the form $f = \mathbb{1}_E$ then estimate (1.15) rewrites

$$\mathcal{J}_T(\overline{E}) - \mathcal{J}_T(E) \geqslant C(T)\,\mathrm{Vol}\left(\overline{E}\Delta E\right)^2. \tag{1.19}$$

On the other hand, (1.12) may seem more surprising. If, indeed, we assume that $f^*(t,x) = \mathbb{1}_{E^*(t)}(x)$ and if the competitor f is chosen to assume the form $f(t,x) = \mathbb{1}_{E(t)}(x)$ for some subset E of Ω then, seeing $\overline{E} := \cup_{t\in(0;T)}\{t\} \times E(t)$ and $\overline{E}^* := \cup_{t\in(0;T)}\{t\} \times E^*(t)$ as subsets of the cylindrical domain $(0;T)\times\Omega$, the "natural quantity" that one should obtain should be the squared asymmetry of $\overline{E}$ with respect to $\overline{E}^*$, that is, $\mathrm{Vol}(\overline{E}\Delta\overline{E}^*)^2$. The Jensen inequality enables to recover this discrepancy. This stronger norm may be a consequence of having chosen a volume constraint for every t. It is unclear at this stage whether or not replacing the constraint

$$\text{for a.e. } t\in(0;T), \int_\Omega f(t,\cdot)=V_0 \tag{1.20}$$

with a global constraint

$$\iint_{(0;T)\times\Omega} f=V_0 \tag{1.21}$$

would yield the coercivity norm

$$\left(\iint_{(0;T)\times\Omega}|f^*-f|\right)^2. \tag{1.22}$$

We refer to the Conclusion, Section 6.2.

1.3 Schwarz's rearrangement and isoperimetric inequalities for parabolic equations

In order to be able to comment on our results and methods of proof, we need to give the basic definition underlying most of our methods, that of Schwarz's rearrangement. The three books we refer to for a comprehensive introduction to rearrangements are [23, 25, 36]. Here, since we are already working in a ball $\Omega=\mathbb{B}(0;R)$, we only give the definitions for functions defined on the ball.

Definition 5. *For a function $\varphi\in L^2(\Omega)\,,\varphi\geqslant 0$, its Schwarz rearrangement is the unique radially symmetric non-increasing function $\varphi^\#:\Omega\to\mathbb{R}$ such that*

$$\forall t\in\mathbb{R}_+\,,\operatorname{Vol}\left(\{\varphi>t\}\right)=\operatorname{Vol}\left(\{\varphi^\#>t\}\right). \tag{1.23}$$

We define its one-dimensional counter part $\varphi^\dagger:[0;R]\to\mathbb{R}$ as

$$\varphi^\dagger(|x|):=\varphi^\#(x). \tag{1.24}$$

The first property is that the Schwarz rearrangement preserves all the Lebesgue norms:

$$\forall p\in(1;+\infty)\,,\forall u\in L^p(\Omega)\,,u\geqslant 0\,,\int_\Omega u^p=\int_\Omega (u^\#)^p. \tag{1.25}$$

Of great importance to us are two inequalities. The first one, the so-called Polyá-Szegö inequality asserts that

$$\forall\varphi\in W^{1,2}_0(\Omega)\,,\varphi^\#\in W^{1,2}_0(\Omega) \text{ and } \int_\Omega|\nabla\varphi^\#|^2\leqslant\int_\Omega|\nabla\varphi|^2. \tag{1.26}$$

The equality case in this equality was fully derived in [10] (see also [20]), and quantitative versions were given in [7, 14]. The second one is the Hardy-Littlewood inequality:

$$\forall f\,,g\in L^2(\Omega)\,,f\,,g\geqslant 0\,,\int_\Omega fg\leqslant\int_\Omega f^\# g^\#. \tag{1.27}$$

This inequality can be rewritten in the following form: for a.e. τ,

$$\int_{\{g>\tau\}} f\leqslant\int_{\{g^\#>\tau\}} f^\#. \tag{1.28}$$

A quantitative version of this inequality can be found in [15] (and [31] in a simpler case where smoothness of the involved function f is assumed). In Propositions 29 and 14, we give uniform versions of this quantitative inequality for families of functions.

Comparison principle for parabolic equations started with the work of Bandle [6], Vazquez [41], using the seminal ideas of Talenti [38], and were later extended in a series of works by Alvino, Lions and Trombetti [2, 5] and Rakotoson and Mossino [33]. By "comparison principle for parabolic equations" we mean results that enable one to compare the solution u of a parabolic equation of the form

$$\frac{\partial u}{\partial t} - \Delta u = f(t, \cdot) \tag{1.29}$$

with the solution v of the symmetrised equation

$$\frac{\partial v}{\partial v} - \Delta v = f^{\#}(t, \cdot). \tag{1.30}$$

Both equations are supplemented with Dirichlet boundary conditions, and we wilfully ignore first order terms. The correct comparison relation $\prec$ used for such comparisons is defined as:

$$f \prec g \text{ if and only if for any } r \in [0; R], \int_{\mathbb{B}(0;r)} f^{\#} \leqslant \int_{\mathbb{B}(0;r)} g, \tag{1.31}$$

and the typical result asserts that $u^{\#}(t, \cdot) \prec v(t, \cdot)$. In this paper, we will rely, for the uniqueness result, Theorem I, on the method of proof of [33], which enables more easily to encompass the equality case. We expand on their techniques in the proof of Theorem I, see Section 3.

1.4 Plan of the paper

This paper is structured as follows:

1. In Section 2 we gather several elementary information about the optimisation problems (adjoint, switch function, regularity of the solutions, convexity of the functionals).
2. In Section 3, we prove the uniqueness result stated in Theorem I.
3. Section 4 contains the proof of Theorem II and is independent of Section 5.
4. Section 5 corresponds to the proof of Theorem III. In it, we state our coercivity results for second order shape derivatives. This Section is independent of Section 4.
5. The Conclusion, Section 6, contains discussion about possible extensions, as well obstructions for generalising the results presented here.

1.5 Notational conventions

- For any $g \in L^2(\Omega)$, $g^{\#}$ denotes its Schwarz rearrangement and $g^{\dagger}$ its one-dimensional counterpart.
- $\mathbb{B}^* = \mathbb{B}(0; r^*)$ is the unique centred ball of volume V_0. In other words, it is the only centred ball satisfying $\mathbb{1}_{\mathbb{B}^*} \in \overline{\mathcal{M}}(\Omega)$.
- u^* is the solution of (1.9) associated with the static control $f \equiv f^* = \mathbb{1}_{B^*}$.
- For a function f that is discontinuous across a smooth hypersurface Σ with oriented normal ν, but continuous in $\Omega \backslash \Sigma$, the jump of f across Σ is

$$[\![f]\!]|_{\Sigma} := \lim_{t \to 0^+} \left(f(x + t\nu(x)) - f(x - t\nu(x)) \right). \tag{1.32}$$

2 Preliminary results

We gather here several results that will be used throughout the paper. We begin with some basic regularity estimates on the solutions of the equation.

Proposition 6. *For any $\alpha \in (0;1)$, there exists $M_\alpha > 0$ such that, for any $f \in \mathcal{M}_T(\Omega)$, we have the estimate*

$$\|u_f(t,\cdot)\|_{\mathscr{C}^{0,\alpha}((0;T)\times\Omega)} \leqslant M_\alpha. \tag{2.1}$$

Furthermore, for any $\alpha \in (0;1)$ and almost every $t \in (0;T)$, $u_f(t,\cdot) \in \mathscr{C}^{1,\alpha}(\Omega)$.

Proof of Proposition 6. For the first point, we use [27, Corollary 7.31, p.182] which ensures that for any $p > 1$,

$$\int_0^T \|u(t,\cdot)\|_{W^{2,p}(\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^p(\Omega)} \leqslant C(\|f\|_{L^\infty} + \|u^0\|_{\mathscr{C}^2}), \tag{2.2}$$

where C depends on the dimension, on p and on Ω. It thus follows that, in particular, for any $p \in (1;+\infty)$ there exists C_p such that

$$\|u\|_{W^{1,p}((0;T)\times\Omega)} \leqslant C_p. \tag{2.3}$$

It then suffices to apply the Sobolev embedding $W^{1,p}((0;T)\times\Omega) \hookrightarrow \mathscr{C}^{0,\alpha}(\Omega)$ for p large enough.

The second point follows from the fact that, from the same estimate, for any $p > 1$ and almost every $t \in (0;T)$, $u_f(t,\cdot) \in W^{2,p}(\Omega)$. The conclusion follows by the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow \mathscr{C}^{1,\alpha}(\Omega)$. □

We then provide structural information about the functionals which we seek to optimise. In Proposition 7, we establish convexity properties which will prove crucial while, in Proposition 8, we compute the adjoint and the switch function of the equation.

Proposition 7. *The map $\mathcal{J}_T : \overline{\mathcal{M}}(\Omega) \ni f \mapsto \mathcal{J}_T(f)$ is strictly convex. In the same way, for any $\varepsilon > 0$, the map $\mathcal{J}_T^\varepsilon : \mathcal{M}_T(\Omega) \ni f \mapsto \mathcal{J}_T^\varepsilon(f)$ is strictly convex.*

Proof of Proposition 7. We only prove the convexity of $\mathcal{J}_T$, the convexity of $\mathcal{J}_T^\varepsilon$ following along the same lines.

It follows from standard argument that the map $\overline{\mathcal{M}}(\Omega) \ni f \mapsto u_f$ is twice Gâteaux-differentiable. The convexity of the functional is equivalent to requiring that the second order Gâteaux derivative be non-negative. For any admissible perturbation h at f (that is, such that for every $t > 0$ small enough $f + th \in \overline{\mathcal{M}}(\Omega)$) the Gâteaux-derivative of u_f in the direction h, denoted by $\dot{u}_f$, solves

$$\begin{cases} \frac{\partial \dot{u}_f}{\partial t} - \Delta \dot{u}_f = h \text{ in } (0;T)\times\Omega\,, \\ \dot{u}_f = 0 \text{ on } (0;T)\times\partial\Omega\,, \\ \dot{u}_f(0,\cdot) \equiv 0. \end{cases} \tag{2.4}$$

From this equation on $\dot{u}_f$, we deduce that the Gâteaux-derivative of $\mathcal{J}_T$ at f in the direction h is given by

$$\dot{\mathcal{J}}_T(f)[h] = \iint_{(0;T)\times\Omega} \dot{u}_f u_f. \tag{2.5}$$

In the same way, the second order Gâteaux-derivative of u_f in the direction h, denoted by $\ddot{u}_f$, satisfies

$$\begin{cases} \frac{\partial \ddot{u}_f}{\partial t} - \Delta \ddot{u}_f = 0 \text{ in } (0;T)\times\Omega\,, \\ \ddot{u}_f = 0 \text{ on } (0;T)\times\partial\Omega\,, \\ \ddot{u}_f(0,\cdot) \equiv 0 \end{cases} \tag{2.6}$$

and thus

$$\ddot{u}_f = 0. \tag{2.7}$$

Furthermore, the second order Gâteaux-derivative of $\mathcal{J}_T$ in the direction h, denoted by $\ddot{\mathcal{J}}_T(f)[h,h]$, is given by

$$\ddot{\mathcal{J}}_T(f)[h,h] = \iint_{(0;T)\times\Omega} (\dot{u}_f)^2 + \iint_{(0;T)\times\Omega} \ddot{u}_f u_f = \iint_{(0;T)\times\Omega} (\dot{u}_f)^2 \geqslant 0 \tag{2.8}$$

and the last inequality is strict unless $h \equiv 0$. Since the second-order Gâteaux-derivative of the functional is non-negative, the functional is convex. □

This convexity property is one of the fundamental point to carry out the proof of Theorem III.

Proposition 8. *Let $f \in \overline{\mathcal{M}}(\Omega)$. Let p_f be the unique solution of*

$$\begin{cases} \frac{\partial p_f}{\partial t} + \Delta p_f = -u_f \text{ in } (0;T)\times\Omega, \\ p_f(T,\cdot) = 0\,, \\ p_f(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega. \end{cases} \tag{2.9}$$

Then for any $f \in \overline{\mathcal{M}}(\Omega)$ and any admissible perturbation h at f, the Gâteaux-derivative of $\mathcal{J}_T$ at f in the direction h is given by

$$\dot{\mathcal{J}}_T(f)[h] = \iint_{(0;T)\times\Omega} h(x)p_f(t,x)dtdx. \tag{2.10}$$

In the same way, let us consider a parameter $\varepsilon > 0$. Let $f \in \mathcal{M}_T(\Omega)$ and define $p_{\varepsilon,f}$ as the unique solution of

$$\begin{cases} \frac{\partial p_{\varepsilon,f}}{\partial t} + \Delta p_{\varepsilon,f} = -u_f \text{ in } (0;T)\times\Omega, \\ p_{\varepsilon,f}(T,\cdot) = \varepsilon u_f(T,\cdot)\,, \\ p_{\varepsilon,f}(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega. \end{cases} \tag{2.11}$$

Then for any $f \in \mathcal{M}_T(\Omega)$ and any admissible perturbation h at f, the Gâteaux-derivative of $\mathcal{J}_T^\varepsilon$ at f in the direction h is given by

$$\dot{\mathcal{J}}_T^\varepsilon(f)[h] = \iint_{(0;T)\times\Omega} h(t,x)p_f(t,x)dtdx. \tag{2.12}$$

p_f is dubbed the switch function for the functional $\mathcal{J}_T$, while $p_{\varepsilon,f}$ is dubbed the switch function for the functional $\mathcal{J}_T^\varepsilon$.

Proof of Proposition 8. We only prove this proposition in the case $f \in \overline{\mathcal{M}}(\Omega)$, the time-dependent case following along the same exact lines. Let us first note that, as a backward, linear heat equation, existence and uniqueness of a solution to (2.9) is guaranteed.

To get (2.10), we start from the expression (2.5) of the first order Gâteaux-derivative of the functional $\mathcal{J}_T$:

$$\dot{\mathcal{J}}_T(f)[h] = \iint_{(0;T)\times\Omega} \dot{u}_f u_f, \tag{2.13}$$

where $\dot{u}_f$ solves (2.4). If we multiply this equation by the solution p_f of (2.9) and integrate by parts, we get

$$\iint_{(0;T)\times\Omega} \dot{u}_f u_f = \iint_{(0;T)\times\Omega} hp_f. \tag{2.14}$$

Since $\dot{\mathcal{J}}_T(f)[h] = \iint_{(0;T)\times\Omega} \dot{u}_f u_f$, the conclusion follows.

□

We conclude this section with some information about the function u^* solution of (1.9) with $f \equiv f^*$.

Proposition 9. *The solution u^* of (1.9) with $f \equiv f^*$ is radially symmetric. Furthermore, for any $r \in (0;R)$ and any $t \in (0;T)$*

$$-\frac{\partial u^*}{\partial r}(t,r) > 0. \tag{2.15}$$

Proof of Proposition 9. The radial symmetry of the solution is immediate. In radial coordinates, and with a slight abuse of notation, u^* satisfies

$$\begin{cases} \frac{\partial u^*}{\partial t} - \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u^*}{\partial r}\right) = f^* \text{ in } (0;T)\times(0;R)\,, \\ u^*(t,R) = \frac{\partial u^*}{\partial r}(t,0) = 0 \text{ for any } t\,, \\ u^*(0,\cdot) = u^0. \end{cases} \tag{2.16}$$

From Proposition 6 above we can differentiate u^* with respect to r. Let us write

$$z := \frac{\partial u}{\partial r}. \tag{2.17}$$

It follows from (2.16) that z solves

$$\frac{\partial z}{\partial t} - \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial z}{\partial r}\right) = -\frac{(n-1)z}{r^2} \text{ in } (0;T)\times(0;R) \tag{2.18}$$

and that the following jump condition is satisfied at $r = r^*$:

$$\left[\!\!\left[\frac{\partial z}{\partial r}\right]\!\!\right](t,r^*) = -\left[\!\left[f^*\right]\!\right](r^*) = 1 > 0. \tag{2.19}$$

Since $u^* \geqslant 0$, the parabolic Hopf Lemma implies that for any $t > 0$,

$$z(t,R) < 0. \tag{2.20}$$

Differentiating (2.16) with respect to r and remembering that u^0 is non-increasing, we obtain that z solves

$$\begin{cases} \frac{\partial z}{\partial t} - \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial z}{\partial r}\right) = -\frac{(n-1)z}{r^2} \text{ in } (0;T)\times(0;R)\,, \\ z(t,R) < 0 \text{ for any } t\,, \\ z(t,0) = 0, \\ \left[\!\left[\frac{\partial z}{\partial r}\right]\!\right](t,r^*) = 1\,, t \in (0;T), \\ z(0,\cdot) \leqslant 0. \end{cases} \tag{2.21}$$

Let us then show that for any $(t,r) \in (0;T)\times(0;R)$

$$z(t,r) < 0. \tag{2.22}$$

First of all, multiplying (2.21) by the positive part z_+ of z we get (keeping in mind that $z_+(t,R) = 0$)

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^R r^{n-1}z_+(t,r)^2dr + \int_0^R r^{n-1}\left(\frac{\partial z_+}{\partial r}\right)^2 + (r^*)^{n-1}z_+(t,r^*) = -\int_0^R \frac{(n-1)z_+(t,r)^2}{r^{3-n}}dr \tag{2.23}$$

so that $z_+(t,\cdot) = 0$. As a consquence, $z \leqslant 0$. To argue that $z < 0$ in $(0;T)\times\Omega$, we follow the same procedure as for the strong parabolic maximum principle. If we first assume $z \leqslant 0$ satisfies an inequality rather than an equality, that is, that z satisfies

$$\frac{\partial z}{\partial t} - \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial z}{\partial r}\right) < -\frac{(n-1)z}{r^2}$$

then if by contradiction we assume that there exists (t_0, r_0), with $r_0 \in (0; R]$ and $t_0 \in (0; T)$ such that $z(t_0, r_0) = 0$, it follows that $r_0 < R$. By the jump condition, $r_0 \neq r^*$. Since $t_0 < T$, plugging the optimality conditions, the contradiction follows. To exclude the case $t_0 = T$ it suffices to consider the equation on $[0; T + \epsilon], \epsilon > 0$ and to carry out the same reasoning in $(0; T + \epsilon)$. To then pass from this case (strict inequality) to ours (the equality case), with

$$\frac{\partial z}{\partial t} - \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial z}{\partial r} \right) = -\frac{(n-1)z}{r^2}$$

it suffices to consider $z_\varepsilon(t, r) := z(t, r) - \varepsilon t$ and the conclusion follows from passing to the limit $\varepsilon \to 0^+$. □

3 Proof of Theorem I: Uniqueness of maximisers

Proof of Theorem I. It follows from [33] that f^* is a solution of ($\mathbf{I}_1$) and ($\mathbf{I}_2^\varepsilon$). The uniqueness property for ($\mathbf{I}_2^\varepsilon$) implies uniqueness for ($\mathbf{I}_1$) so we focus on the time-dependent case. Let us define u^* as the solution of (1.9) associated with $f \equiv f^*$. We consider another solution f of ($\mathbf{I}_2^\varepsilon$) and the solution u of (1.9) associated. By convexity of the functional we can assume that f is a characteristic function so that

$$f^\# = f^*. \tag{3.1}$$

We proceed along a series of claims. The first one is :

Claim 10. *If f solves ($\mathbf{I}_2^\varepsilon$) and if u is the associated solution of (1.9) then for almost every $t \in (0; T)$, there holds*

$$u^\#(t, \cdot) = u^*(t, \cdot). \tag{3.2}$$

Proof of Claim 10. It follows from [33] and the results recalled in the introduction that for almost every $t \in (0; T)$ we have

$$u^\#(t, \cdot) \prec u^*(t, \cdot). \tag{3.3}$$

The relation $\prec$ was defined in Equation (1.31). Thus, from [3, Proposition 2] we have that for almost every $t \in (0; T)$ we have

$$(u^\#)^2(t, \cdot) \prec (u^*)^2(t, \cdot). \tag{3.4}$$

Integrating this inequality in time and in space yields

$$\iint_{(0;T)\times\Omega} \left(u^\#\right)^2 \leqslant \iint_{(0;T)\times\Omega} (u^*)^2. \tag{3.5}$$

However, by equimeasurability of the Schwarz rearrangement (1.25) we have

$$\iint_{(0;T)\times\Omega} u^2 = \iint_{(0;T)\times\Omega} (u^\#)^2. \tag{3.6}$$

Since f is a maximiser of ($\mathbf{I}_2^\varepsilon$) it follows that equality holds for almost every t in

$$\int_\Omega \left(u^\#\right)^2 (t, \cdot) \leqslant \int_\Omega (u^*)^2 (t, \cdot). \tag{3.7}$$

Thus we have for almost every $t \in (0; T)$,

$$\int_\Omega \left(u^\#\right)^2 (t, \cdot) = \int_\Omega (u^*)^2 (t, \cdot). \tag{3.8}$$

Let us now introduce the set $\mathscr{K}(u^*)$ defined as

$$\mathscr{K}(u^*) = \left\{g \in L^2(\Omega)\,, g \prec u^*\right\}. \tag{3.9}$$

From [4] this is a compact (for the weak $L^\infty - *$ topology) and convex set whose set of extreme points is

$$\mathscr{C}(u^*) = \left\{g \in L^2(\Omega)\,, g^\# = u^*\right\}. \tag{3.10}$$

Since $x \mapsto x^2$ is strictly convex, the map $\mathscr{K}(v) \ni g \mapsto \int_\Omega g^2$ is strictly convex. Besides, once again because of the convexity of $x \mapsto x^2$, we have, for any $g \in \mathscr{K}(u^*)$,

$$(g^\#)^2 = (g^2)^\#.$$

As a consequence, the only solutions of the maximisation problem

$$\sup_{g \in \mathscr{K}(u^*)} \int_\Omega g^2 \tag{3.11}$$

are exactly the elements of $\mathscr{C}(u^*)$.

On the other hand, (3.8) states that $u(t,\cdot)$ is a solution of (3.11), so it follows that for almost every $t \in (0;T)$ there holds

$$u^\#(t,\cdot) = u^*(t,\cdot). \tag{3.12}$$

□

In particular, and this is the main point of this proof, the two following properties hold: first,

$$\text{If } f \text{ solves } (\mathbf{I}_2^\varepsilon) \text{ then for a.e. } t \in (0;T),\ u^\dagger(t,\cdot) = (u^*)^\dagger(t,\cdot). \tag{3.13}$$

Second, we have, as a consequence the following fact:

$$\text{If } f \text{ solves } (\mathbf{I}_2^\varepsilon) \text{ then for a.e. } t \in (0;T),\ \int_\Omega u(t,\cdot) = \int_\Omega u^\#(t,\cdot) = \int_\Omega u^*(t,\cdot). \tag{3.14}$$

We then prove that if f solves $(\mathbf{I}_2^\varepsilon)$, then all the level sets of u are balls.

Claim 11. *If f solves $(\mathbf{I}_2^\varepsilon)$, then all the level sets of u are balls.*

Proof of Claim 11. We follow the approach of [33]. We first recall [33, Theorem 1.2]: if $\varphi \in W^{1,2}((0;T), L^2(\Omega))$ then $\varphi^\# \in W^{1,2}((0;T), L^2(\Omega))$ and moreover there holds, if φ only has measure sets of measure zero,

$$\frac{\partial \varphi^\#}{\partial t}(t,s) = \frac{\partial w}{\partial s}(t,s) \tag{3.15}$$

where w is defined by

$$w(t,s) = \int_{\{\varphi(t,\cdot) \leqslant \varphi^\#(t,s)\}} \frac{\partial \varphi}{\partial t}. \tag{3.16}$$

We then consider (1.9). For any $\tau \in \mathbb{R}_+$, we multiply the equation by $(u-\tau)_+$ and integrate by parts in space. We obtain in a classical way

$$0 \leqslant -\frac{\partial}{\partial \tau} \int_{\{u>\tau\}} |\nabla u|^2(t,\cdot) = \int_{\{u>\tau\}} \left(f - \frac{\partial u}{\partial t}(t,\cdot)\right). \tag{3.17}$$

We write the repartition function of u as μ:

$$\mu(t,\tau) = \mathrm{Vol}\left(\{u(t,\cdot) > \tau\}\right). \tag{3.18}$$

By the isoperimetric inequality and the co-area formula, taking $S_n := n\,\mathrm{Vol}(\mathbb{B}(0;1))^{\frac{1}{n}}$, we obtain, as in [33],

$$S_n\mu(t,\tau)^{1-\frac{1}{n}} \leqslant \left(-\frac{\partial}{\partial\tau}\int_{\{u(t,\cdot)>\tau\}}|\nabla u|\right) \tag{3.19}$$

$$\leqslant \left(-\frac{\partial\mu}{\partial\tau}\right)^{\frac{1}{2}}\left(-\frac{\partial}{\partial\tau}\int_{\{u(t,\cdot)>\tau\}}|\nabla u|^2\right)^{\frac{1}{2}}. \tag{3.20}$$

This leads to

$$S_n\mu(t,\tau)^{1-\frac{1}{n}} \leqslant \left(-\frac{\partial\mu}{\partial\tau}\right)^{\frac{1}{2}}\left(-\frac{\partial}{\partial\tau}\int_{\{u(t,\cdot)>\tau\}}|\nabla u|^2\right)^{\frac{1}{2}} \tag{3.21}$$

$$\leqslant \left(-\frac{\partial\mu}{\partial\tau}\right)^{\frac{1}{2}}\left(\int_{\{u(t,\cdot)>\tau\}}\left(f-\frac{\partial u}{\partial t}(t,\cdot)\right)\right)^{\frac{1}{2}}. \tag{3.22}$$

Hence,

$$S_n^2\mu(t,\tau)^{2-\frac{2}{n}} \leqslant \left(-\frac{\partial\mu}{\partial\tau}\right)\int_{\{u(t,\cdot)>\tau\}}\left(f-\frac{\partial u}{\partial t}\right). \tag{3.23}$$

Here we recall that $\int_{\{u(t,\cdot)>\tau\}}\left(f-\frac{\partial u}{\partial t}\right)\geqslant 0$ by (3.17).

As is customary we use the Hardy-Littlewood inequality to obtain

$$\int_{\{u(t,\cdot)>\tau\}} f \leqslant \int_0^{\mu(t,\tau)} f^\dagger =: F(t,\mu(t,\tau)). \tag{3.24}$$

Let us now define

$$k(t,\tau) := \int_0^\tau u^\dagger(t,s)ds \tag{3.25}$$

and we obtain

$$\int_{\{u(t,\cdot)>\tau\}}\frac{\partial u}{\partial t} = \frac{\partial k}{\partial t}(t,\mu(t,\tau)). \tag{3.26}$$

As such, for some constant $c_n>0$,

$$1 \leqslant S_n^{-2}\left(-\frac{\partial\mu}{\partial\tau}\right)\mu(t,\tau)^{\frac{2}{n}-2}\left(F(t,\mu(t,\tau))-\frac{\partial k}{\partial t}(t,\mu(t,\tau))\right). \tag{3.27}$$

Integrating this equation between τ_0 and τ_1 for any $0\leqslant\tau_0\leqslant\tau_1$ yields

$$\tau_1-\tau_0 \leqslant S_n^{-2}\int_{\mu(t,\tau_0)}^{\mu(t,\tau_1)} s^{-2+\frac{2}{n}}\left(F(t,s)-\frac{\partial k}{\partial t}(t,s)\right)ds. \tag{3.28}$$

We hence get in a classical way [32] the following differential inequality

$$-\frac{\partial u^\dagger}{\partial\tau}(t,\tau) = -\frac{\partial^2 k}{\partial\tau^2}(t,\tau) \leqslant S_n^{-2}\tau^{-2+\frac{2}{n}}\left(F(t,\tau)-\frac{\partial k}{\partial t}(t,\tau)\right). \tag{3.29}$$

Let us now define

$$k_{u^*}(t,\tau) := \int_0^\tau \left(u^*\right)^\dagger(t,\cdot). \tag{3.30}$$

We recall that u^* is the solution of (1.9) associated with $f \equiv f^*$. Since f is radially symmetric and decreasing, all the equalities in the above reasoning carried for u hold for u^* with equalities instead of inequalities and k_{u^*} solves

$$\frac{\partial^2 k_{u^*}}{\partial \tau^2} + S_n^{-2}\tau^{-2+\frac{2}{n}} \frac{\partial k_{u^*}}{\partial t} = S_n^{-2}\tau^{-2+\frac{2}{n}} F(t,\tau). \tag{3.31}$$

Finally, we set $K = k - k_{u^*}$. From Equation (3.13), we have, for any $t \in (0;T)$ and any $s \in (0; \mathrm{Vol}(\Omega))$,

$$K(t,s) = 0. \tag{3.32}$$

Since $K \equiv 0$, every equality in the above reasoning must in fact be an equality. In particular, (3.19) is an equality, and hence all the level-sets of u are balls, which concludes the proof. $\square$

Remark 12. *It would be interesting to investigate whether or not using the quantitative isoperimetric inequality could lead to quantitative estimates, but it is not at this point clear how to do that. We refer to the Conclusion, Section 6.4.*

As is customary in the study of equality cases in Talenti-like inequalities, we need to check that the level sets are not just balls but rather concentric balls.

Claim 13. *If f solves ($\mathbf{I}_2^\varepsilon$) then the level sets of the associated solution u are concentric balls.*

Proof of Claim 13. The core idea of the proof is similar to [24]. Let us first consider the solution w of

$$\begin{cases} \frac{\partial w}{\partial t} + \Delta w = -1 \text{ in } (0;T)\times\Omega\,, \\ w = 0 \text{ on } (0;T)\times\Omega\,, \\ w(T,\cdot) = 0. \end{cases} \tag{3.33}$$

It follows from the same arguments as in the proof of Proposition 9 that w is radially symmetric and decreasing (for any $t < T$), and so we obtain by the Hardy-Littlewood inequality that for almost every $t \in (0;T)$,

$$\int_\Omega fw \leqslant \int_\Omega f^\# w = \int_\Omega f^* w. \tag{3.34}$$

However multiplying Equation (3.33) by u and integrating by parts both in time and space yields

$$\begin{aligned}
\iint_{(0;T)\times\Omega} fw &= \iint_{(0;T)\times\Omega} \left(\frac{\partial u}{\partial t} - \Delta u\right) w \\
&= -\int_\Omega wu^0 - \iint_{(0;T)\times\Omega} u\left(\frac{\partial w}{\partial t} + \Delta w\right) \\
&= -\int_\Omega wu^0 + \iint_{(0;T)\times\Omega} u \\
&= -\int_\Omega wu^0 + \iint_{(0;T)\times\Omega} u^* \text{ because of Claim 10} \\
&= \iint_{(0;T)\times\Omega} f^* w \text{ by the same computations with } u^* \text{ instead of } u.
\end{aligned}$$

However, and since w is radially symmetric and increasing, the Hardy-Littlewood inequality implies that for almost every $t \in (0;T)$ and almost every τ we have

$$\int_{\{w(t,\cdot)>\tau\}} f \leqslant \int_{\{w(t,\cdot)>\tau\}} f^\#. \tag{3.35}$$

Hence it follows that (3.35) must be an equality for almost every t. Thus since for almost every t the function w is symmetric and radially decreasing we get

$$\forall r \in (0;R)\,, \int_{\mathbb{B}(0;r)} f = \int_{\mathbb{B}(0;r)} f^*. \tag{3.36}$$

For the final step, let ϕ_1 be the first Dirichlet eigenvalue of the laplacian in Ω. It is standard to see that ϕ_1 is radially symmetric and decreasing. Introduce the solution ϕ of

$$\begin{cases} \frac{\partial \phi}{\partial t} + \Delta \phi = -\phi_1 \text{ in } (0;T) \times \Omega\,, \\ \phi = 0 \text{ on } (0;T) \times \Omega\,, \\ \phi(T, \cdot) = 0. \end{cases} \tag{3.37}$$

The function ϕ is radially symmetric and decreasing as well for any $t < T$. As a consequence, all level-sets of $\phi(t, \cdot)$ are level-sets of $w(t, \cdot)$ and conversely, from which we deduce that, for almost every $t \in (0;T)$ and almost every τ

$$\int_{\{\phi(t,\cdot)>\tau\}} f = \int_{\{\phi(t,\cdot)>\tau\}} f^{\#} = \int_{\{\phi(t,\cdot)>\tau\}} f^*. \tag{3.38}$$

This gives in turn

$$\int_\Omega f\phi(t, \cdot) = \int_\Omega f^* \phi(t, \cdot). \tag{3.39}$$

Multiplying (3.37) by u and integrating by parts gives in the same way

$$\iint_{(0;T)\times\Omega} u\phi_1 = \iint_{(0;T)\times\Omega} f\phi = \iint_{(0;T)\times\Omega} f^*\phi = \iint_{(0;T)\times\Omega} u^*\phi_1 = \iint_{(0;T)\times\Omega} u^{\#}\phi_1. \tag{3.40}$$

The last equality comes from (3.13).

Invoking the Hardy-Littlewood inequality we obtain in the same fashion that for almost every $t \in (0;T)$

$$\forall r \in (0;R)\,, \int_{\mathbb{B}(0;r)} u = \int_{\mathbb{B}(0;r)} u^{\#}. \tag{3.41}$$

It follows that $u = u^{\#}$ so that the conclusion is reached.

□

□

4 Proof of Theorem II

4.1 Plan of the proof and heuristics

This theorem relies on the following fact: assuming that we have a competitor f, to be compared with f^*, and defining, for every $t \in [0;T]$,

$$\delta(t) := \|f(t, \cdot) - f^*\|^2_{L^1(\Omega)} \tag{4.1}$$

we can set

$$\mathcal{M}_T(\Omega, \delta) = \left\{g \in \mathcal{M}_T(\Omega)\,, \text{ for a.e. } t \in [0;T]\,, \|g(t, \cdot) - f^*\|_{L^1(\Omega)} = \delta(t)\right\} \tag{4.2}$$

and replace f with the solution f_δ^* of

$$\max_{f\in\mathcal{M}_T(\Omega,\delta)} \mathcal{J}_T^\varepsilon(f). \tag{4.3}$$

That such a solution exists follows by the same argument as in Lemma 19 below (see the proof in Appendix A) but we can actually prove (and this is the part that is specific to Ω being a centred ball) that the solutions to (4.3) admits the following explicit description: let, for any $\overline{\delta}>0$, $\mathbb{A}_{\overline{\delta}}$ be defined, in radial coordinates, as

$$\mathbb{A}_{\overline{\delta}}=\{r<r^*-r_{\overline{\delta}}^-\}\sqcup\{r^*<r<r^*+r_{\overline{\delta}}^+\} \tag{4.4}$$

where $r_{\overline{\delta}}^-, r_{\overline{\delta}}^+$ are the unique parameters such that

$$\mathrm{Vol}(\mathbb{A}_{\overline{\delta}})=V_0\,,\mathrm{Vol}\left(\mathbb{A}_{\overline{\delta}}\Delta\mathbb{B}^*\right)=\overline{\delta}. \tag{4.5}$$

Then we will show (Proposition 16)

$$f_\delta: t\mapsto \mathbb{1}_{\mathbb{A}_{\delta(t)}} \tag{4.6}$$

is a solution of (4.3). Throughout the rest of this introduction to the proof, we keep the notation f_δ for this function.

Let us formally assume that

$$\int_0^T \delta(t)dt \ll 1 \tag{4.7}$$

and define, for any $\xi\in(0;1)$, $p_{\varepsilon,\xi}$ the adjoint state associated with $f(t)=f^*+\xi\,(f_\delta-f^*)$. By parabolic regularity, $p_{\varepsilon,\xi}$ should be a non-increasing function of r since the adjoint state p_ε^* associated to f^* is decreasing. By the mean-value theorem, there exists $\xi\in[0;1]$ such that

$$\mathcal{J}_T^\varepsilon(f_\delta)-\mathcal{J}_T^\varepsilon(f^*)=\iint_{(0;T)\times\Omega} p_{\varepsilon,\xi}\,(f_\delta-f^*)\,. \tag{4.8}$$

A natural step is then to try and apply the quantitative bathtub principle to this quantity: since $p_{\varepsilon,\xi}$ is a radially symmetric, non-increasing function of r, then for any $t\in(0;T)$, f^* is the only solution of

$$\sup_{f\in\overline{\mathcal{M}}(\Omega)}\int_\Omega f p_{\varepsilon,\xi}(t,\cdot). \tag{4.9}$$

The hope is then to prove that there exists a constant $C>0$ such that for any $t\in(0;T)$ there holds

$$\forall f\in\overline{\mathcal{M}}(\Omega)\,,\int_\Omega (f-f^*)p_{\varepsilon,\xi}\leqslant -C\left(-\frac{\partial p_{\varepsilon,\xi}}{\partial r}\right)(t,r^*)\|f(t,\cdot)-f^*\|_{L^1(\Omega)}^2. \tag{4.10}$$

However, the existence of such a uniform constant relies, in a crucial way, on ε: when $\varepsilon>0$, it is possible while, when $\varepsilon=0$, other difficulties may arise. The key difficulty is that when $\varepsilon>0$ we can guarantee that

$$\sup_{t\in[0;T]}\frac{\partial p_\varepsilon^*}{\partial r}(t,r^*)<0 \tag{4.11}$$

while for $\varepsilon=0$ we can only guarantee

$$\forall\tau>0\,,\exists\alpha(\tau)>0\,,\sup_{t\in[0;T-\tau]}\frac{\partial p_\varepsilon^*}{\partial r}(t,r^*)\leqslant-\alpha(\tau). \tag{4.12}$$

To give a synthetic presentation, we isolate the main tool of this proof in the following paragraph.

4.2 Uniform quantitative bathtub principle

Proposition 14. *Let $\beta > 0$ and consider a family of function $\{p_i\}_{i\in I} \in \mathscr{C}^{1,\beta}(\Omega)$ such that:*

1. *There exists $M > 0$ such that*
$$\sup_{i\in I} \|p_i\|_{\mathscr{C}^{1,\beta}} \leqslant M. \tag{4.13}$$
2. *For any $i \in I$, p_i is radially symmetric. Furthermore, there exists $\alpha > 0$ such that, for any $r \in [0; r^*]$,*
$$\forall i \in I\,, p_i(r) - p_i(r^*) \geqslant \alpha |r - r^*|. \tag{4.14}$$
We also assume that for any $i \in I$, p_i is decreasing in $(r^*; R)$. In particular, the unique level set of p_i of volume V_0 is $\mathbb{B}(0; r^*)$: there exists c_i such that
$$\mathbb{B}(0; r^*) = \{p_i > c_i\}\,, \partial\mathbb{B}(0; r^*) = \{p_i = c_i\}. \tag{4.15}$$
This in particular ensures that the minimum of p_i in $\mathbb{B}(0; r^*)$ is only achieved on $\partial\mathbb{B}(0; r^*)$. As another consequence, for this constant $\alpha > 0$, we have
$$\forall i \in I\,, -\frac{\partial p_i}{\partial r}(r^*) \geqslant \alpha > 0. \tag{4.16}$$

Then there exists a constant $\omega > 0$ such that
$$\forall f \in \mathcal{M}(\Omega)\,, \forall i \in I\,, \int_\Omega p_i(f^* - f) \geqslant \omega \left(-\frac{\partial p_i}{\partial r}(r^*)\right) \|f - f^*\|^2_{L^1(\Omega)}. \tag{4.17}$$

Proof of Proposition 14. Let us write $\mathscr{T} := \{p_i\}_{i\in I}$. We first note that the assumption ensure that for any $p \in \mathscr{T}$, f^* is the only solution of the problem
$$\sup_{f\in\overline{\mathcal{M}}(\Omega)} \int_\Omega fp. \tag{4.18}$$

We define
$$\mathscr{G} : \mathscr{T} \times \left(\overline{\mathcal{M}}(\Omega)\backslash\{f^*\}\right) \ni (p, f) \mapsto \frac{\int_\Omega p(f^* - f)}{-\frac{\partial p_i}{\partial r}(r^*)\|f - f^*\|^2_{L^1(\Omega)}} \tag{4.19}$$
and obviously proving (4.17) boils down to proving
$$\inf_{\mathscr{T}\times\left(\overline{\mathcal{M}}(\Omega)\backslash\{f^*\}\right)} \mathscr{G} > 0. \tag{4.20}$$

Let us consider a minimising sequence $\{p_k\,, f_k\} \in \left(\mathscr{T} \times \left(\overline{\mathcal{M}}(\Omega)\backslash\{f^*\}\right)\right)^{\mathbb{N}}$. Let us fix $\beta' \in (0; \beta)$. By (4.13) there exists $p_\infty \in \mathscr{C}^{1,\beta'}(\Omega)$ radially symmetric such that
$$p_k \underset{k\to\infty}{\to} p_\infty \text{ in } \mathscr{C}^{1,\beta'}(\Omega)\,, \tag{4.21}$$
and as a consequence we have
$$\|p_\infty\|_{\mathscr{C}^{1,\beta'}} = \lim_{k\to\infty} \|p_k\|_{\mathscr{C}^{1,\beta'}} \leqslant M \tag{4.22}$$
and (4.16) holds for p_∞. In the same way, and passing to the limit in (4.14), f^* is the only solution of
$$\sup_{f\in\overline{\mathcal{M}}(\Omega)} \int_\Omega p_\infty f. \tag{4.23}$$

Up to a subsequence we also have that there exists $f_\infty \in \overline{\mathcal{M}}(\Omega)$ such that

$$f_k \underset{k\to\infty}{\to} f_\infty \text{ weakly in } L^\infty - *. \tag{4.24}$$

We distinguish between two cases related to the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ defined by

$$\forall k \in \mathbb{N}\,, \delta_k := \|f_k - f^*\|_{L^1(\Omega)}. \tag{4.25}$$

The first case corresponds to the case where, up to a subsequence,

$$\delta_k \underset{k\to\infty}{\to} \delta_\infty > 0. \tag{4.26}$$

In that case, we define

$$\overline{\mathcal{M}}_{>\delta_\infty}(\Omega) := \left\{ f \in \overline{\mathcal{M}}(\Omega)\,, \|f - f^*\|_{L^1(\Omega)} \geqslant \frac{\delta_\infty}{2} \right\} \tag{4.27}$$

Following the same arguments as in [31, Proposition 22] we can see that the class $\overline{\mathcal{M}}_{>\delta_\infty}(\Omega)$ is closed under the weak $L^\infty - *$ convergence. Hence, it follows that

$$\|f_\infty - f^*\|_{L^1(\Omega)} \geqslant \frac{\delta_\infty}{2}. \tag{4.28}$$

This implies that

$$\lim_{k\to\infty} \mathscr{G}(p_k, f_k) \geqslant \frac{4}{\delta_\infty^2} \mathscr{G}(p_\infty, f_\infty) > 0 \tag{4.29}$$

since f^* is the only maximiser of $f \mapsto \int_\Omega f p_\infty$ in $\overline{\mathcal{M}}(\Omega)$.

The second case is the difficult one. We henceforth work under the assumption that

$$\delta_k \underset{k\to\infty}{\to} 0. \tag{4.30}$$

We introduce the sequence of variational problem

$$\forall k \in \mathbb{N}\,, \sup_{f\in\overline{\mathcal{M}}(\Omega)\,,\|f-f^*\|_{L^1(\Omega)}=\delta_k} \int_\Omega p_k f. \tag{4.31}$$

From the same arguments as in [31, Proposition 22] there exists a solution to this variational problem. Furthermore since $p_k^\# = p_k$ the function $\mathbb{1}_{\mathbb{A}_{\delta_k}}$ is a solution of this problem, where $\mathbb{A}_{\delta_k}$ is defined, in radial coordinates

$$\overline{\mathbb{A}}_{\delta_k} = \{r < r^* - r_{\delta_k}^-\} \sqcup \{r^* < r < r^* + r_{\delta_k}^+\} \tag{4.32}$$

and $r_{\delta_k}^-, r_{\delta_k}^+$ are the unique parameters such that

$$\mathrm{Vol}(\mathbb{A}_{\delta_k}) = V_0\,, \mathrm{Vol}(\mathbb{A}_{\delta_k} \Delta \mathbb{B}^*) = \delta_k. \tag{4.33}$$

Hence we assume that

$$f_k = \mathbb{1}_{\mathbb{A}_{\delta_k}}. \tag{4.34}$$

For a general $\overline{\delta} > 0$, we define $\mathbb{A}_{\overline{\delta}}$ in the same manner, that is,

$$\mathbb{A}_{\overline{\delta}} = \{r < r^* - r_{\overline{\delta}}^-\} \sqcup \{r^* < r < r^* + r_{\overline{\delta}}^+\} \tag{4.35}$$

where $r_{\overline{\delta}}^-, r_{\overline{\delta}}^+$ are the unique parameters such that

$$\text{Vol}(\mathbb{A}_{\overline{\delta}}) = V_0\,, \text{Vol}\left(\mathbb{A}_{\overline{\delta}}\Delta\mathbb{B}^*\right) = \overline{\delta}. \tag{4.36}$$

We also recall that we have, for the same exponent $\beta' > 0$,

$$\forall i \in I\,, \|p_i\|_{\mathscr{C}^{1,\beta'}} \leqslant M \tag{4.37}$$

as this will be a crucial point. Let us then prove the following claim:

Claim 15. *There exists $\delta_1 > 0$ and $\omega_0 > 0$ such that for any $0 \leqslant \delta \leqslant \delta_1$ there holds*

$$\forall p \in \mathscr{T} \cup \{p_\infty\}\,, \int_\Omega p(f^* - \mathbb{1}_{\mathbb{A}_\delta}) \geqslant \omega_0 \left(-\frac{\partial p}{\partial r}(r^*)\right) \|\mathbb{1}_{\mathbb{A}_\delta} - f^*\|^2_{L^1(\Omega)}. \tag{4.38}$$

Assuming this Claim holds it follows that for any k large enough we have

$$\mathscr{G}(p_k, f_k) \geqslant \omega_0, \tag{4.39}$$

hence leading to the required contradiction. It thus only remains to prove Claim 15:

Proof of Claim 15. Let us define, for any $\delta > 0$,

$$h_\delta := f^* - \mathbb{1}_{\mathbb{A}_\delta} = \mathbb{1}_{\{r^* - r_\delta^- < r < r^*\}} - \mathbb{1}_{\{r^* < r < r^* + r_\delta^+\}}. \tag{4.40}$$

The quantity we want to bound from below is

$$\int_\Omega h_\delta p_i. \tag{4.41}$$

First of all, explicit computations show that there exists a constant $c_0 = c_0(d, r^*)$ such that

$$r_\delta^+\,, r_\delta^- \underset{\delta\to 0}{\sim} c_0\delta. \tag{4.42}$$

We now write (4.42) in radial coordinates and obtain for any $p \in \mathscr{T} \cup \{p_\infty\}$

$$\frac{1}{(2\pi)^d}\int_\Omega h_\delta p = \int_{r^*-r_\delta^-}^{r^*} p(r)r^{n-1}dr - \int_{r^*}^{r^*+r_\delta^+} p(r)r^{n-1}dr.$$

Let us first notice that from (4.37) and (4.16), there exists $\overline{\varepsilon} > 0$ such that, for any $\delta \in (0; \overline{\varepsilon})$,

$$\forall p \in \mathscr{T} \cup \{p_\infty\}\,, \inf_{(r^*-\delta; r^*+\delta)} \left(-\frac{\partial p}{\partial r}\right) \geqslant -\frac{1}{2}\frac{\partial p}{\partial r}(r^*). \tag{4.43}$$

We now have thanks to the mean value theorem, that for any $r \in (0; R)$, there exists $y_{r-r^*} \in (r; r^*)$ or $(r^*; r)$ such that

$$p(r) = p(r^*) + p'(y_{r-r^*})(r - r^*). \tag{4.44}$$

As a consequence

$$\frac{1}{(2\pi)^n}\int_\Omega h_\delta p = p(r^*)\left(\int_{r^*-r_\delta^-}^{r^*} r^{n-1}dr - \int_{r^*}^{r^*+r_\delta^+} r^{n-1}dr\right) \tag{4.45}$$

$$+\left(\int_{r^*-r_\delta^-}^{r^*} r^{n-1}|r-r^*|(-p'(y_{r-r^*}))dr + \int_{r^*}^{r^*+r_\delta^+} r^{n-1}|r-r^*|(-p'(y_{r-r^*}))dr\right) \tag{4.46}$$

$$\geqslant p(r^*)\left(\int_{r^*-r_\delta^-}^{r^*} r^{n-1}dr - \int_{r^*}^{r^*+r_\delta^+} r^{n-1}dr\right) \tag{4.47}$$

$$-\frac{1}{2}\frac{\partial p}{\partial r}(r^*)\left(\int_{r^*-r_\delta^-}^{r^*} r^{n-1}|r-r^*|dr + \int_{r^*}^{r^*+r_\delta^+} r^{n-1}|r-r^*|dr\right) \text{ by (4.43)} \tag{4.48}$$

The right hand side of (4.47) is 0 because

$$\int_{r^*-r_\delta^-}^{r^*} r^{n-1}dr - \int_{r^*}^{r^*+r_\delta^+} r^{n-1}dr = \frac{1}{(2\pi)^n}\int_\Omega h_\delta = 0. \tag{4.49}$$

Furthermore by explicit computations we obtain

$$\int_{r^*-r_\delta^-}^{r^*} r^{n-1}|r-r^*|dr \underset{\delta\to 0}{\sim} C\delta^2,$$

and in the same way

$$\int_{r^*}^{r^*+r_\delta^+} r^{n-1}|r-r^*|dr \underset{\delta\to 0}{\sim} C\delta^2. \tag{4.50}$$

The conclusion follows immediately. □

This concludes the proof of the Proposition. □

We then present, in the following paragraph, the proof of the aforementioned Proposition 16 that deals with the characterisation of solutions of a penalised problem.

4.3 Characterisation of the solutions of an auxiliary problem

Let us consider a function $\delta : [0;T] \to [0;\mathrm{Vol}(\Omega)]$ and the class $\mathcal{M}_T(\Omega,\delta)$ defined in (4.2), as well as the function f_δ defined by (4.6).

Proposition 16. *For any $\varepsilon > 0$ and any positive function $\delta : [0;T] \to [0;\mathrm{Vol}(\Omega)]$, f_δ is a solution of the variational problem*

$$\max_{g\in\mathcal{M}_T(\Omega,\delta)} \mathcal{J}_T^\varepsilon(g). \tag{4.51}$$

Proof of Proposition 16. This is a straightforward adaptation of the proof of the parabolic isoperimetric inequality whose main steps were recalled in Section 3. Let us consider a function $g \in \mathcal{M}_T(\Omega,\delta)$ and u the associated solution of (1.9). With the same notations as in Section 3, proof of Theorem I we obtain

$$S_n^2\mu(t,\tau)^{2-\frac{2}{n}} \leqslant \left(-\frac{\partial\mu}{\partial\tau}\right)\int_{\{u(t,\cdot)>\tau\}}\left(g-\frac{\partial u}{\partial t}\right). \tag{4.52}$$

However, by the Hardy-Littlewood inequality, if we define $G(t,\mu(t,\tau)) := \int_0^{\mu(t,\tau)} \mathbb{1}_{\mathbb{A}_{\delta(t)}}^\dagger$ we obtain

$$\int_{\{u(t,\cdot)>\tau\}} g \leqslant G(t,\mu(t,\tau)). \tag{4.53}$$

This is a penalised version of the Hardy-Littlewood inequality: it is indeed straightforward to see that, for any function $\overline{g} \in \overline{\mathcal{M}}(\Omega, \delta(t)) = \{g \in \overline{\mathcal{M}}(\Omega)\,, \|g - f^*\|_{L^1(\Omega)} = \delta(t)\}$ and any measurable positive function ℓ, there holds

$$\int_\Omega \ell \overline{g} \leqslant \int_\Omega \mathbb{1}_{\mathbb{A}_{\delta(t)}} \ell^{\#}. \tag{4.54}$$

As a consequence, for some constant $c_n > 0$,

$$1 \leqslant S_n^{-2} \left(-\frac{\partial \mu}{\partial \tau}\right) \mu(t,\tau)^{\frac{2}{n}-2} \left(G(t, \mu(t,\tau)) - \frac{\partial k}{\partial t}(t, \mu(t,\tau))\right). \tag{4.55}$$

The rest of the proof follows along exactly the same lines.

□

4.4 Proof of Theorem II

In this subsection, we prove Theorem II with a fixed, positive parameter $\varepsilon > 0$.

Proof of Theorem II. We argue by contradiction and assume that there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \in (\mathcal{M}_T(\Omega)\backslash\{f^*\})^{\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{\mathcal{J}_T^\varepsilon(f^*) - \mathcal{J}_T^\varepsilon(f_k)}{\int_0^T \left(-\frac{\partial p_\varepsilon^*}{\partial r}(t, r^*)\right) \|f_k(t,\cdot) - f^*\|^2_{L^1(\Omega)}} = 0, \tag{4.56}$$

where we recall that p_ε^* is the solution of

$$\begin{cases} \frac{\partial p_\varepsilon^*}{\partial t} + \Delta p_\varepsilon^* = -u^* \text{ in } (0;T)\times\Omega\,, \\ p_\varepsilon^*(T,\cdot) = \varepsilon u^*(T,\cdot)\,, \\ p_\varepsilon^*(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega. \end{cases} \tag{4.57}$$

In the same way, if $f \in \mathcal{M}_T(\Omega)$, $p_{\varepsilon,f}$ stands for the solution of (4.57) with u^* replaced by u_f. By Proposition 8, the derivative of $\mathcal{J}_T^\varepsilon$ at f in a direction h is given by

$$\dot{\mathcal{J}}_T^\varepsilon(f)[h] = \iint_{(0;T)\times\Omega} h p_{\varepsilon,f}. \tag{4.58}$$

Let us then begin with the following Claim:

Claim 17. *For any $T, \varepsilon > 0$ and any $y_0 \in (0; r^*)$, there exists $\alpha(y_0; \varepsilon, T)$ such that*

$$\inf_{(0;T)\times(y_0;R)} \left(-\frac{\partial p_\varepsilon^*}{\partial r}(t,r)\right) \geqslant \alpha(y_0;\varepsilon,T) > 0. \tag{4.59}$$

Proof of Claim 17. We define $q_\varepsilon^*(t,\cdot) := p_\varepsilon^*(T-t,\cdot)$. Since u^* is radially symmetric, q_ε is radially symmetric as well and satisfies, in radial coordinates,

$$\begin{cases} \frac{\partial q_\varepsilon^*}{\partial t} - \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial q_\varepsilon^*}{\partial r}\right) = u^*(T-t,\cdot) \text{ in } (0;T)\times(0;R)\,, \\ q_\varepsilon^*(0,\cdot) = \varepsilon u^*(T,\cdot)\,, \\ q_\varepsilon^*(t,R) = \frac{\partial q_\varepsilon^*}{\partial r}(t,0) = 0. \end{cases} \tag{4.60}$$

It follows from standard Schauder estimates [27, Theorem 4.9, p.59] and Proposition 6 that $q_\varepsilon^* \in \mathscr{C}^{1,\alpha}((0;T)\times\Omega)$. Besides, since $u^* \geqslant 0$, we also have $q_\varepsilon \geqslant 0$ and, by the strong parabolic maximum principle, $q_\varepsilon^* > 0 \in (0;T)\times\Omega$.

As a consequence of the Hopf Lemma and of the fact that $\frac{\partial u^*}{\partial r}(T,R)<0$, defining $\Phi_\varepsilon^* := \frac{\partial q_\varepsilon^*}{\partial r}$, we obtain

$$\forall t \in [0;T]\,, \Phi_\varepsilon^*(t,R)<0. \tag{4.61}$$

From Proposition 9, $\Phi_\varepsilon^*(0,\cdot)\leqslant 0\,, <0$ in $(0;R]$. Differentiating (4.60), Φ_ε^* thus solves

$$\begin{cases} \frac{\partial \Phi_\varepsilon^*}{\partial t}-\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial \Phi_\varepsilon^*}{\partial r}\right)=\frac{\partial u^*(T-t,\cdot)}{\partial r}-\frac{(n-1)\Phi_\varepsilon^*}{r^2} \text{ in } (0;T)\times(0;R)\,,\\ \Phi_\varepsilon^*(0,r)<0 \text{ if } r>0\,, \Phi_\varepsilon^*(t,0)=0\,,\\ \Phi_\varepsilon^*(t,R)<0.\end{cases} \tag{4.62}$$

Since (Proposition 9) $\frac{\partial u^*}{\partial r}<0$ for almost every $t,r>0$, Φ_ε^* solves, in $(0;T)\times\Omega$, the differential inequality

$$\frac{\partial \Phi_\varepsilon^*}{\partial t}-\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial \Phi_\varepsilon^*}{\partial r}\right)<-\frac{(n-1)\Phi_\varepsilon^*}{r^2} \text{ in } (0;T)\times(0;R). \tag{4.63}$$

We can then apply the maximum principle, as was done in Proposition 9, to ensure that for any $t\in(0;T]$ and any $r>y_0$,

$$\Phi_\varepsilon^*(t,r)<0. \tag{4.64}$$

As $\Phi_\varepsilon^*(0,r^*)=\varepsilon\frac{\partial u_\varepsilon^*}{\partial r}(T,r^*)<0$ it follows that

$$\forall t\in[0;T]\,, \Phi_\varepsilon^*(t,r^*)<0. \tag{4.65}$$

Since Φ_ε is continuous in time, we can define

$$\alpha(y_0;\varepsilon,T):=\inf_{t\in[0;T],r\in(y_0;R)}\left(-\Phi_\varepsilon^*(t,r)\right)>0 \tag{4.66}$$

and the conclusion follows. □

Using this Claim we can come back to the sequence $\{f_k\}_{k\in\mathbb{N}}\in(\mathcal{M}_T(\Omega)\backslash\{f^*\})^{\mathbb{N}}$ satisfying (4.56). Since f^* is the unique maximiser of $\mathcal{J}_T^\varepsilon$, we must have

$$\int_0^T\left(-\frac{\partial p_\varepsilon^*}{\partial r}(t,r^*)\right)\|f_k-f^*\|^2_{L^1(\Omega)}\underset{k\to\infty}{\rightarrow}0. \tag{4.67}$$

If this were not the case, it would follow that the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges weakly in $\mathcal{M}_T(\Omega)$ to some $f_\infty\neq f^*$. As a consequence, the sequence $\{u_{f_k}\}_{k\in\mathbb{N}}$ would converge in $\mathscr{C}^0((0;T)\times\Omega)$ (using the uniform Hölder bounds from Proposition 6) to u_{f_∞}, and so

$$\mathcal{J}_T^\varepsilon(f_k)\underset{k\to\infty}{\rightarrow}\mathcal{J}_T^\varepsilon(f_\infty)>\mathcal{J}_T^\varepsilon(f^*). \tag{4.68}$$

This would yield

$$\lim_{k\to\infty}\frac{\mathcal{J}_T^\varepsilon(f^*)-\mathcal{J}_T^\varepsilon(f_k)}{\int_0^T\left(-\frac{\partial p_\varepsilon^*}{\partial r}(t,r^*)\right)\|f_k(t,\cdot)-f^*\|^2_{L^1(\Omega)}}=\frac{\mathcal{J}_T^\varepsilon(f^*)-\mathcal{J}_T^\varepsilon(f_\infty)}{\int_0^T\left(-\frac{\partial p_\varepsilon^*}{\partial r}(t,r^*)\right)\|f_\infty(t,\cdot)-f^*\|^2_{L^1(\Omega)}}>0, \tag{4.69}$$

a contradiction.

Hence we work under the assumption that (4.67) holds. From Claim 17 this implies

$$\int_0^T\|f_k(t,\cdot)-f^*\|^2_{L^1(\Omega)}\underset{k\to\infty}{\rightarrow}0. \tag{4.70}$$

Hence, by Jensen's inequality,

$$\|f_k - f^*\|_{L^1((0;T)\times\Omega)} = \int_0^T \|f_k(t,\cdot) - f^*\|_{L^1(\Omega)} dt \underset{k\to\infty}{\to} 0. \tag{4.71}$$

As a consequence of standard parabolic estimates (Proposition 6) we have, for any $\alpha \in (0;1)$,

$$u_{f_k} \underset{k\to\infty}{\to} u_{f^*} = u^* \text{ in } \mathscr{C}^{0,\alpha}(\Omega) \tag{4.72}$$

Defining, for any $k \in \mathbb{N}$, $p_\varepsilon^{f_k}$ as the solution of

$$\begin{cases} \frac{\partial p_\varepsilon^{f_k}}{\partial t} + \Delta p_\varepsilon^{f_k} = -u_{f_k} \text{ in } (0;T)\times\Omega\,, \\ p_\varepsilon^{f_k}(T,\cdot) = \varepsilon u_{f_k} \text{ in } \Omega\,, \\ p_\varepsilon^{f_k}(t,\cdot) = 0 \text{ in } (0;T)\times\partial\Omega\,, \end{cases} \tag{4.73}$$

This in in turn implies, by Schauder's estimates [35, Theorem 48.2]

$$p_\varepsilon^{f_k} \underset{k\to\infty}{\to} p_\varepsilon^* \text{ in } \mathscr{C}^{1,\alpha}((0;T)\times\Omega). \tag{4.74}$$

Hence, for any $y_0 \in (0;r^*)$, there exists $k(y_0) > 0$ such that for any $k \geqslant k(y_0)$, by Claim 17, there holds,

$$\forall (t,r) \in (0;T)\times(y_0;R)\,, \left(-\frac{\partial p_\varepsilon^{f_k}}{\partial r}(t,r)\right) \geqslant -\frac{1}{2}\frac{\partial p_\varepsilon^*}{\partial r}(t,r) > 0, \tag{4.75}$$

and, for any $k > 0$ large enough, $\mathbb{B}(0;r^*)$ is a uniquely defined level set of $p_\varepsilon^{f_k}$: there exists c_k such that

$$\mathbb{B}(0;r^*) = \{p_\varepsilon^{f_k} > c_k\}. \tag{4.76}$$

As a consequence, choosing y_0 small enough, we can ensure that all the assumptions of Proposition 14 are satisfied.

Finally, let us note that, by the same argument, these property also hold for any $p_\varepsilon^{f^*+\tau(f_k-f^*)}$ for any $\tau \in (0;1)$ and any k large enough. In all the reasoning above, it suffices to add τ as another parameter in the family.

This allows us to apply Proposition 14: there exists a constant $\underline{\omega} > 0$ such that

$$\begin{aligned} &\forall f \in \mathcal{M}(\Omega)\,, \forall k \text{ large enough, } \forall t \in (0;T)\,, \forall \tau \in (0;1)\,, \\ &\int_\Omega p_\varepsilon^{f^*+\tau(f_k-f^*)}(t,\cdot)(f^* - f_k(t,\cdot)) \geqslant \underline{\omega}\left(-\frac{\partial p_\varepsilon^*}{\partial r}(t,r^*)\right)\|f_k(t,\cdot) - f^*\|_{L^1(\Omega)}^2. \end{aligned} \tag{4.77}$$

Let us now apply, for any k large enough, the mean value theorem to the map

$$T_k = [0;1] \ni \tau \mapsto \mathcal{J}_T^\varepsilon(f^* + \tau(f_k - f^*)). \tag{4.78}$$

There exists $\overline{\tau} \in (0;1)$ such that

$$\mathcal{J}_T^\varepsilon(f_k - f^*) = \iint_{(0;T)\times\Omega} p_\varepsilon^{f^*+\overline{\tau}(f_k-f^*)}\,(f_k - f^*)\,. \tag{4.79}$$

Using (4.77) we get

$$\mathcal{J}_T^\varepsilon(f_k - f^*) = \iint_{(0;T)\times\Omega} p_\varepsilon^{f^*+\overline{\tau}(f_k-f^*)}\,(f_k - f^*) \geqslant \underline{\omega}\int_0^T \left(-\frac{\partial p_\varepsilon^*}{\partial r}(t,r^*)\right)\|f_k(t,\cdot) - f^*\|_{L^1(\Omega)}^2. \tag{4.80}$$

This is a contradiction, and the Theorem follows.

□

5 Proof of Theorem III: quantitative inequalities via shape derivatives and bathtub principle

5.1 Presentation and plan of the proof

The proof relies on the use of shape derivatives and on the study of an auxiliary problem. The structure of the proof is inspired by a previous work of the author [31] and we will refer to this paper when needed. The main point is here to show an example of how shape derivatives may be used for parabolic problems.

Let us define, for any $\delta > 0$, the class

$$\overline{\mathcal{M}}(\Omega, \delta) := \left\{ f \in \overline{\mathcal{M}}(\Omega) \, , \| f - f^* \|_{L^1(\Omega)} = \delta \right\} . \qquad (\overline{\mathbf{Adm}}(\delta))$$

We first consider the auxiliary variational problem

$$\inf_{f \in \overline{\mathcal{M}}(\Omega, \delta)} \mathcal{J}_T(f) \qquad (\mathbf{P}_\delta)$$

and prove that it admits a solution f_δ (Lemma 19 below). Once this is done, we prove (Lemma 20 below) that Theorem III is equivalent to proving that

$$\liminf_{\delta \to 0} \left(\frac{\mathcal{J}_T(f^*) - \mathcal{J}_T(f_\delta)}{\delta^2} \right) > 0. \qquad (5.1)$$

Remark 18. *At this stage, one may argue to explicitly characterize f_δ as a radially symmetric solution, and thus bypass the part about shape derivatives. However, as our goal is also to provide a full analysis of shape hessians for time-dependent problems, and to present, in the Conclusion, possible generalisations to other settings where the explicit characterisation of optimisers of such a penalised problem are no longer available, we choose to not take advantage of that fact here.*

We then recall that $f^* = \mathbb{1}_{\mathbb{B}^*}$. We consider, for smooth enough vector fields Φ, the deformed set $\mathbb{B}^*_\Phi := (Id + \Phi)(\mathbb{B}^*)$ and, with a slight abuse of notation, we write

$$\mathcal{J}_T(\mathbb{B}^*_\Phi) := \mathcal{J}_T(\mathbb{1}_{\mathbb{B}^*_\Phi}).$$

We will prove (Proposition 21) that whenever Φ is "small" enough (in a sense made precise in the section devoted to shape derivatives) there holds

$$\mathcal{J}_T(\mathbb{B}^*) - \mathcal{J}_T(\mathbb{B}^*_\Phi) \geqslant C \, \mathrm{Vol}(\mathbb{B}^*_\Phi \Delta \mathbb{B}^*)^2 \qquad (5.2)$$

for some constant $C > 0$.

We also prove a quantitative bathtub principle (Proposition 29), and finally conclude as in [31] by comparing any competitor with one of the level sets of the switch function, and then this level set with the set $\mathbb{B}^*$. The key to conclude here is the convexity of the cost functional $\mathcal{J}_T$.

To proceed, we need some basic informations about the optimality conditions for Problem ($\mathbf{I}_1$).

Optimality conditions for ($\mathbf{I}_1$) We recall, from Proposition 8 that for any admissible perturbation $h \in L^\infty(\Omega)$ (that is, such that, for any $\varepsilon > 0$ small enough, $f^* + \varepsilon h \in \overline{\mathcal{M}}(\Omega)$) the Gâteaux-derivative of u_f in the direction h, thereafter noted $\dot{u}$ solves

$$\begin{cases} \frac{\partial \dot{u}}{\partial t} - \Delta \dot{u} = h \text{ in } (0;T) \times \Omega \, , \\ \dot{u} = 0 \text{ on } (\Omega; T) \times \partial\Omega \, , \\ \dot{u}(0, \cdot) \equiv 0 \end{cases} \qquad (5.3)$$

and that, introducing the solution p_f of

$$\begin{cases} \frac{\partial p_f}{\partial t} + \Delta p_f = -u_f \text{ in } (0;T)\times\Omega\,, \\ p_f = 0 \text{ on } (0;T)\times\partial\Omega\,, \\ p_f(T,\cdot)\equiv 0. \end{cases} \tag{5.4}$$

we get the following expression for the Gâteaux-derivative of $\mathcal{J}_T$:

$$\dot{\mathcal{J}}_T(h) = \iint_{(0;T)\times\Omega} hp_f = \int_\Omega h(x)\left(\int_0^T p_f(t,x)dt\right)dx. \tag{5.5}$$

Let us define

$$\Psi(x) := \int_0^T p_f(t,x)dt. \tag{5.6}$$

Hence it follows that

$$\dot{\mathcal{J}}_T(h) = \int_\Omega h\Psi. \tag{5.7}$$

5.2 Reduction to an auxiliary problem

We now justify the study of the auxiliary problem

$$\inf_{f\in\overline{\mathcal{M}}(\Omega,\delta)} \mathcal{J}_T(f) \tag{$\mathbf{P}_\delta$}$$

where $\overline{\mathcal{M}}(\Omega,\delta)$ was defined in ($\overline{\mathbf{Adm}}(\delta)$).

Lemma 19. *For any $\delta>0$, the variational problem ($\mathbf{P}_\delta$) has a solution f_δ.*

This Lemma is an adaptation of [31, Proposition 22]; for the sake of readability, its proof is only given in Appendix A. Throughout the rest of the proof of Theorem III we adopt the following notation:

$$\text{For any } \delta>0,\ f_\delta \text{ is a solution of } (\mathbf{P}_\delta). \tag{5.8}$$

We now explain why we will focus on the study of f_δ as a competitor; it is the subject of the following Lemma:

Lemma 20. *Theorem III is equivalent to proving that*

$$\liminf_{\delta\to0}\frac{\mathcal{J}_T(f^*)-\mathcal{J}_T(f_\delta)}{\delta^2} \geqslant C_0 > 0 \tag{5.9}$$

for some positive constant C_0.

The proof of this result is an adaptation of [31, Lemma 23] and mostly relies on the uniqueness of maximisers. We postpone the proof to Appendix A. The rest of the proof of Theorem III is going to be devoted to the proof of Estimate (5.9), see Proposition 5.5 below. To prove it, we need a local inequality for deformations of the optimal set $\mathbb{B}^*$ and a quantitative bathtub principle which will be used in combination with the convexity of the functional.

5.3 Quantitative inequalities for deformations of $\mathbb{B}^*$: using shape derivatives

Let us consider a $\mathscr{C}^1$ set E of volume V_0 such that $E\cap\partial\Omega=\emptyset$ and a smooth, compactly supported in Ω, vector field Φ. We define

$$E_\Phi := (Id+\Phi)(E). \tag{5.10}$$

We recall that we see $\mathcal{J}_T$ as a shape functional by defining, with a slight abuse of notations,

$$\mathcal{J}_T(E) := \mathcal{J}_T(\mathbb{1}_E). \tag{5.11}$$

Our goal is the following proposition:

Proposition 21. *There exist a constant $C>0$, a parameter $\eta>0$ and $p\in(1;+\infty)$ such that, for any compactly supported vector field Φ satisfying $\|\Phi\|_{W^{2,p}}$ there holds*

$$\mathcal{J}_T(\mathbb{B}^*)-\mathcal{J}_T(\mathbb{B}^*_\Phi)\geqslant C\,\mathrm{Vol}\left(\mathbb{B}^*_\Phi\Delta\mathbb{B}^*\right)^2. \tag{5.12}$$

The proof of this Proposition follows the synthetic presentation of quantitative inequalities for deformations of optimal sets presented in [16]; their proof holds for shape optimisation of the domain Ω, and we have presented in [31] how to adapt their method to the optimisation of a subdomain $E\subset\Omega$. Let us present the main steps of the proof of Proposition 21:

1. The first one is to prove that $\mathbb{B}^*$ is a critical shape in the following sense: computing, for any compactly supported vector field $\Phi\in W^{2,p}$ the first order shape derivative $\mathcal{J}_T'(\mathbb{B}^*)[\Phi]$ we need to prove that, if Φ additionally satisfies the linearised constraint

$$\int_{\partial\mathbb{B}^*}(\Phi\cdot\nu)=0 \tag{5.13}$$

   then there holds

$$\mathcal{J}_T'(E^*)[\Phi]=0. \tag{5.14}$$

   This allows to consider, for the computation and analysis of second-order shape derivatives, vector fields Φ that are normal to $\partial\mathbb{B}^*$, and also to define a Lagrangian associated with a Lagrange multiplier

$$\mathcal{L}_T(E):=\mathcal{J}_T(E)+\mu\,\mathrm{Vol}(E), \tag{5.15}$$

   which satisfies, for any compactly supported vector field $\Phi\in W^{2,p}$ not necessarily satisfying (5.13)

$$\mathcal{L}_T'(\mathbb{B}^*)[\Phi]=0. \tag{5.16}$$

2. As a second step, we compute the second order shape derivative of the Lagrangian $\mathcal{L}_T''(\mathbb{B}^*)[\Phi,\Phi]$ and prove an L^2 coercivity estimate, i.e that there exists a constant $c_0>0$ such that

$$\forall\Phi\in W^{2,p}(\Omega;\mathbb{R}^2)\,,\int_{\partial\mathbb{B}^*}\phi\cdot\nu=0\Rightarrow\mathcal{L}_T''(\mathbb{B}^*)[\Phi,\Phi]\leqslant -c_0\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)}. \tag{5.17}$$

   This is done using a comparison principle previously used for elliptic equations [30, 29], and our contribution here is to show how it extends to the case of parabolic equations.

3. We then define for a compactly supported vector field $\Phi\in W^{2,p}$ the map

$$j_\Phi:[0;1]\ni t\mapsto\mathcal{L}_T(\mathbb{B}^*_{t\Phi})+C(\mathrm{Vol}(\mathbb{B}^*_{t\Phi})-V_0)^2 \tag{5.18}$$

for some C large enough such that

$$j''_\Phi(0) \leqslant -\tilde{c}_0 \|\Phi \cdot \nu\|^2_{L^2(\partial \mathbb{B}^*)} \tag{5.19}$$

and prove that there exists a modulus of continuity η, that is, a continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(0) = 0$, such that

$$|j''_\Phi(t) - j''_\Phi(0)| \leqslant \eta\left(\|\Phi\|_{W^{2,p}}\right) \|\Phi \cdot \nu\|^2_{L^2(\partial \mathbb{B}^*)}, \tag{5.20}$$

and conclude using the Taylor-Lagrange formula

$$j_\Phi(1) - j_\Phi(0) = \int_0^1 j''_\Phi(s)ds \leqslant \left(-\tilde{c}_0 + \omega\left(\|\Phi\|_{W^{2,p}}\right)\right) \|\Phi \cdot \nu\|^2_{L^2(\partial \mathbb{B}^*)}. \tag{5.21}$$

All these steps rely on fine properties of first and second order shape derivatives. We begin with the computations of the shape derivatives of the Lagrange multiplier associated with the volume constraint and of the diagonalisation of the associated shape hessian at E^*.

5.3.1 Computation of first and second order shape derivatives, computation of the Lagrange multiplier and diagonalisation of the shape Hessian

Computation and analysis of the first order shape derivative Let us define, for any subdomain E of Ω the function u_E as the solution of (1.9) associated with $f = \mathbb{1}_E$. It should be noted that the shape differentiability of first and second order of the shape functional $\mathcal{J}_T : E \mapsto \mathcal{J}_T(E)$ follows from the same arguments as in [11], and so does the computation of the first order shape derivative. The computations are a straightforward adaptation of [11] and we only give here a heuristic approach. Let us, then, consider a $\mathscr{C}^1$ shape, and a $W^{2,p}$ compactly supported vector field E. The shape derivative of $E \mapsto u_E$ in the direction Φ is denoted by u' for the sake of notational simplicity. The differentiation of the main equation of (1.9) gives, in a weak form, that, for any test function v,

$$-\iint_{(0;T)\times\Omega} \frac{\partial v}{\partial t} u' + \iint_{(0;T)\times\Omega} \langle \nabla u', \nabla v \rangle = \iint_{(0;T)\times\partial E} v\left(\Phi \cdot \nu\right). \tag{5.22}$$

Remark 22. *Alternatively, at a formal level: the differentiation of the initial condition yields*

$$u'(0, \cdot) \equiv 0. \tag{5.23}$$

The differentiation of the main equation gives

$$\frac{\partial u'}{\partial t} - \Delta u' = 0. \tag{5.24}$$

Finally, the structural condition given by the weak formulation of (1.9) *is that there is no jump of the normal derivative on* $\partial \mathbb{B}^*$ *or, mathematically, that*

$$\left[\!\left[\frac{\partial u_E}{\partial \nu} \right]\!\right]\Big|_{\partial E} = 0. \tag{5.25}$$

We refer to Subsection 1.5 for the definition of the jump. Differentiating (5.25) *yields*

$$\left[\!\left[\frac{\partial u'}{\partial \nu} \right]\!\right]\Big|_{\partial E} = -\left(\Phi \cdot \nu\right). \tag{5.26}$$

In conclusion, u' satisfies

$$\begin{cases} \frac{\partial u'}{\partial t} - \Delta u' = 0 \text{ in } (0;T) \times \Omega\,, \\ u'(t,\cdot) = 0 \text{ on } (0;T) \times \partial\Omega\,, \\ \left[\!\left[\frac{\partial u'}{\partial \nu} \right]\!\right]\Big|_{\partial E} = -\left(\Phi \cdot \nu\right), \\ u'(0,\cdot) \equiv 0. \end{cases} \tag{5.27}$$

Furthermore, if we consider the adjoint state $p_{\mathbb{1}_E}$, which we abbreviate as p_E for notational simplicity, given by Equation (5.4) we obtain

$$\begin{aligned} \mathcal{J}_T'(E)[\Phi] &= \iint_{(0;T)\times\Omega} u_E u' \\ &= -\iint_{(0;T)\times\Omega} \left(\frac{\partial p_E}{\partial t} + \Delta p_E \right) u' \\ &= -\iint_{(0;T)\times\partial E} \left[\!\left[\frac{\partial u'}{\partial \nu} \right]\!\right] p_E \\ &= \iint_{(0;T)\times\partial E} \left(\Phi \cdot \nu\right) p_E. \end{aligned}$$

Let us single out this last identity:

$$\mathcal{J}_T'(E)[\Phi] = \int_{\partial E} \left(\Phi \cdot \nu\right) \left(\int_0^T p_E \right). \tag{5.28}$$

This allows us to obtain the following result:

Lemma 23. *$\mathbb{B}^*$ is a critical shape in the following sense: for any compactly supported vector field $\Phi \in W^{2,p}(\Omega;\mathbb{R}^2)$*

$$\int_{\partial\mathbb{B}^*} \left(\Phi \cdot \nu\right) = 0 \Rightarrow \mathcal{J}_T(\mathbb{B}^*)[\Phi] = 0. \tag{5.29}$$

Proof of Lemma 23. From Proposition 9, u^* is a radially symmetric function. Hence, the associated adjoint state $p^* = p_{\mathbb{B}^*}$ is also radially symmetric, so that the map

$$\Psi : \mathbb{B}(0;R) \ni x \mapsto \int_0^T p^*(t,x)dt \tag{5.30}$$

is radially symmetric. Letting $\overline{\Psi}_{\partial\mathbb{B}^*} := \Psi|_{\partial\mathbb{B}^*}$ we obtain

$$\mathcal{J}_T'(\mathbb{B}^*)[\Phi] = \overline{\Psi}_{\partial\mathbb{B}^*} \int_{\partial\mathbb{B}^*} \left(\Phi \cdot \nu\right) = 0. \tag{5.31}$$

□

It follows that the Lagrange multiplier associated with the volume constraint is $\mu = -\overline{\Psi}_{\partial\mathbb{B}^*}$ and we can hence define the Lagrangian

$$\mathcal{L}_{\mathbb{B}^*}(E) := \mathcal{J}_T(E) - \overline{\Psi}_{\partial\mathbb{B}^*}\,\mathrm{Vol}(E) \tag{5.32}$$

and observe that, since $\mathrm{Vol}'(E)[\Phi] = \int_{\partial E} \left(\Phi \cdot \nu\right)$ we have, for any compactly supported vector field $\Phi \in W^{2,p}(\Omega;\mathbb{R}^2)$

$$\mathcal{L}_{\mathbb{B}^*}'(\mathbb{B}^*)[\Phi] = 0. \tag{5.33}$$

As a consequence of [22, Theorem 5.9.2 and the remark below], the second-order shape derivative in a direction Φ only depends on the normal trace of Φ and we hence work under the Assumption:

$$\Phi \text{ is normal to } \partial\mathbb{B}^*. \tag{$\mathbf{A}_\nu$}$$

Computation of the shape hessian and diagonalisation of the shape hessian at the ball We can now turn to the computation of the second order shape derivative. We once again choose a $\mathscr{C}^2$ shape E and a compactly supported vector field $\Phi \in W^{2,p}(\Omega;\mathbb{R}^2)$. It is well-known [22, Proposition 5.4.18] that

$$\mathrm{Vol}''(E)[\Phi,\Phi] = \int_{\partial E} \mathscr{H}\,(\Phi\cdot\nu)^2\,, \tag{5.34}$$

where $\mathscr{H}$ is the mean curvature of ∂E. Furthermore, differentiating (5.28) and using once again [22, Proposition 5.4.18] we obtain

$$\mathcal{J}_T''(E)[\Phi,\Phi] = \int_{\partial E}(\Phi\cdot\nu)\left(\int_{(0;T)} p'\right) + \int_{\partial E}(\Phi\cdot\nu)^2\left(\mathscr{H}\int_0^T p_E + \int_0^T \frac{\partial p_E}{\partial\nu}\right) \tag{5.35}$$

where p' satisfies

$$\begin{cases} \frac{\partial p'}{\partial t} + \Delta p' = -u' \text{ in } (0;T)\times\Omega, \\ p' = 0 \text{ on } (0;T)\times\Omega\,, \\ p'(T,\cdot) \equiv 0 \text{ in } \Omega. \end{cases} \tag{5.36}$$

In particular, the shape hessian of the Lagrangian at the ball is given by

$$\mathcal{L}_{\mathbb{B}^*}''(\mathbb{B}^*)[\Phi,\Phi] = \int_{\partial\mathbb{B}^*}(\Phi\cdot\nu)\left(\int_{(0;T)} p'\right) + \int_{\partial\mathbb{B}^*}(\Phi\cdot\nu)^2\left(\mathscr{H}\,\overline{\Psi}\big|_{\partial\mathbb{B}^*} + \int_0^T \frac{\partial p^*}{\partial\nu}\right) - \overline{\Psi}\big|_{\partial\mathbb{B}^*}\int_{\partial\mathbb{B}^*}\mathscr{H}\,(\Phi\cdot\nu)^2$$

so that simplifying the terms involving the mean curvature we are left with

$$\mathcal{L}_{\mathbb{B}^*}''(\mathbb{B}^*)[\Phi,\Phi] = \int_{\partial\mathbb{B}^*}(\Phi\cdot\nu)\left(\int_{(0;T)} p'\right) + \int_{\partial\mathbb{B}^*}(\Phi\cdot\nu)^2\int_0^T \frac{\partial p}{\partial\nu}. \tag{5.37}$$

Let us now diagonalise it. Since Φ is a vector field that is normal to $\mathbb{S}^* := \partial\mathbb{B}^*$ from Assumption ($\mathbf{A}_\nu$) it follows that we can decompose it, in angular coordinates, as

$$\Phi\cdot\nu = \sum_{k=1}^{\infty}\alpha_k\cos(k\cdot) + \beta_k\sin(k\cdot). \tag{5.38}$$

Remark 24. *The fact that the sum involving the cosines starts at $k=1$ is a consequence of the fact that to compute the optimality condition for second order shape derivative we need to work in the space satisfying the linearised constraint or, in this case, to assume that*

$$\int_{\partial\mathbb{B}^*}\Phi\cdot\nu = 0. \tag{5.39}$$

Let us first define u_k' (resp. v_k') as the solution of (5.27) associated with $\Phi\cdot\nu = \cos(k\cdot)$ (resp. $\sin(k\cdot)$). It is straightforward to see that these two functions write

$$u_k'(r,\theta) = y_k(r)\cos(k\theta)\,, v_k'(r,\theta) = y_k(r)\sin(k\theta) \tag{5.40}$$

where y_k solves, for any $k\in\mathbb{N}^*$,

$$\begin{cases} \frac{\partial y_k}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial y_k}{\partial r}) = -\frac{k^2}{r^2}y_k \text{ in } (0;R)\,, \\ [\![y_k']\!]\,(r^*) = -1\,, \\ y_k(R,\cdot) = 0\,, \\ y_k'(0) = 0. \end{cases} \tag{5.41}$$

Let us also introduce g'_k (resp. w'_k) the solution of (5.36) associated with $\Phi \cdot \nu = \cos(k\cdot)$ (resp. $\sin(k\cdot)$). It is straightforward to see that these two functions write

$$g'_k(r,\theta) = z_k(r)\cos(k\theta)\,, w'_k(r,\theta) = z_k(r)\sin(k\theta) \tag{5.42}$$

where z_k solves, for any $k \in \mathbb{N}^*$,

$$\begin{cases} \frac{\partial z_k}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial z_k}{\partial r}) = \frac{k^2}{r^2} z_k - y_k \text{ in } (0;R)\,, \\ z_k(R,\cdot) = 0\,, \\ z_k(T) = 0. \end{cases} \tag{5.43}$$

Furthermore, since p^* is a radially symmetric function let us introduce the function $\overline{p}$ such that

$$p^*(t,r,\theta) = \overline{p}(t,r). \tag{5.44}$$

This allows to recast the second order shape derivative (5.37) through the following Lemma:

Lemma 25. *If $\Phi \cdot \nu$ is of the form (5.38) then there holds*

$$\mathcal{L}''_{\mathbb{B}^*}(\mathbb{B}^*)[\Phi,\Phi] = \frac{r^*}{2}\sum_{k=1}^{\infty} \omega_k \left\{\alpha_k^2 + \beta_k^2\right\} \tag{5.45}$$

where for every $k \in \mathbb{N}^*$ we have defined

$$\omega_k := \int_0^T z_k(t,r^*)dt + \int_0^T \frac{\partial \overline{p}}{\partial r}(t,r^*)dt. \tag{5.46}$$

Proof of Lemma 25. We can write (5.37) as

$$\begin{aligned} \mathcal{L}''_{\mathbb{B}^*}(\mathbb{B}^*)[\Phi,\Phi] &= \int_0^{2\pi} (\Phi\cdot\nu)\left(\int_{(0;T)} p'\right) + \int_0^{2\pi} (\Phi\cdot\nu)^2 \int_0^T \frac{\partial p^*}{\partial \nu} \\ &= \sum_{k,k'=1}^{\infty} \int_0^{2\pi}\int_0^T (\alpha_k\alpha_{k'}\cos(k\theta)\cos(k'\theta) + \beta_k\beta_{k'}\sin(k\theta)\sin(k'\theta) \\ &+ +\alpha_k\beta_{k'}\cos(k\theta)\sin(k'\theta))\, z_k(t,r^*)dtd\theta \\ &+ \sum_{k=1}^{\infty}\int_0^{2\pi}\int_0^T \frac{1}{2}\left(\alpha_k^2+\beta_k^2\right)\frac{\partial \overline{p}}{\partial r}(t,r^*)dt. \end{aligned}$$

All the crossed terms disappear for $k \neq k'$, and the conclusion follows by integrating in polar coordinates. ∎

We may now state the main result of this subsection:

Proposition 26. *There exists a constant $c_0 > 0$ such that for any $\Phi \in W^{2,p}$ satisfying ($\mathbf{A}_\nu$) there holds*

$$\mathcal{L}''_{\mathbb{B}^*}(\mathbb{B}^*)[\Phi,\Phi] \leqslant -c_0 \int_{\partial\mathbb{B}^*} (\Phi\cdot\nu)^2\,. \tag{5.47}$$

Proof of Proposition 26. Given Lemma 25 it suffices to prove that there exists a constant $c_0 > 0$ such that

$$\forall k \in \mathbb{N}^*\,, \omega_k \leqslant -c_0. \tag{5.48}$$

Equation (5.48) is obviously provided the following Claim holds:

Claim 27. *The sequence* $\{\omega_k\}_{k\in\mathbb{N}^*}$ *is decreasing. Furthermore,* $\omega_1 < 0$.

Indeed, it then suffices to take $c_0 = -\frac{2}{r^*}\omega_1$ and we can then bound

$$\mathcal{L}''_{\mathbb{B}^*}(\mathbb{B}^*)[\Phi,\Phi] = \frac{r^*}{2}\sum_{k=1}^{\infty}\omega_k(\alpha_k^2+\beta_k^2) \leqslant -c_0\sum_{k=1}^{\infty}(\alpha_k^2+\beta_k^2) = -c_0\int_{\partial\mathbb{B}^*}(\Phi\cdot\nu)^2\,. \tag{5.49}$$

We now focus on the proof of this last Claim.

Proof of Claim 27. Let us note that from Lemma 25 we have

$$\forall k\in\mathbb{N}^*\,,\omega_k-\omega_1 = \int_0^T (z_k-z_1)(t,r^*)dt. \tag{5.50}$$

The fact that $\{\omega_k\}_{k\in\mathbb{N}}$ is decreasing is thus guaranteed provided the following estimate holds:

$$\forall k\in\mathbb{N}^*\,, z_k\leqslant z_1. \tag{5.51}$$

(5.51) will be proved using a comparison principle. If we want to compare z_k and z_1, we need to compare, for any $k\in\mathbb{N}^*$, y_k and y_1. The first thing to observe is that

$$y_1\geqslant 0. \tag{5.52}$$

Proof of (5.52)*.* We already know that y_1 satisfies

$$\begin{cases}\frac{\partial y_1}{\partial t}-\frac{1}{r}\frac{\partial}{dr}(r\frac{\partial y_1}{dr}) = -\frac{1}{r^2}y_1 \text{ in } (0;R)\,,\\ [\![y_1']\!](r^*) = -1\,,\\ y_1(R,\cdot)=0\,,\\ y_1'(0,\cdot)=0.\end{cases} \tag{5.53}$$

We consider the negative part y_1^- of y_1. We have

$$[\![(y_1^-)']\!](t,r^*)\begin{cases}=0 \text{ if } y_1(t,r^*)>0,\\ =1 \text{ if } y_1(t,r^*)<0,\\ >0 \text{ if } y_1(t,\cdot) \text{ locally changes sign at } r^*.\end{cases}$$

In any case, we obtain

$$[\![(y_1^-)']\!]\geqslant 0. \tag{5.54}$$

Multiplying the equation by y_1^- and integrating by parts in space and time as in the proof of Proposition 9 gives

$$\frac{1}{2}\int_\Omega (y_1^-)^2(T,\cdot)+\iint_{(0;T)\times\Omega}|\nabla y_1^-|^2+\iint_{(0;T)\times\partial\mathbb{B}^*}y_1^-\,[\![(y_1^-)']\!]+\iint_{(0;T)\times\Omega}\frac{1}{r}(y_1^-)^2=0. \tag{5.55}$$

As a conclusion, $y_1^-\equiv 0$, which concludes the proof.

□

Using this information, we can now prove:

$$\forall k\in\mathbb{N}^*\,, y_k\leqslant y_1. \tag{5.56}$$

Proof of (5.56). Let us define, for any $k \in \mathbb{N}^*$,

$$\Psi_k := y_k - y_1. \tag{5.57}$$

Then, in $(0;T) \times \Omega$, Ψ_k solves

$$\frac{\partial \Psi_k}{\partial t} - \Delta \Psi_k = -\frac{k^2}{r^2} y_k + \frac{1}{r^2} y_1 \leqslant -\frac{k^2}{r^2}(y_k - y_1) = -\frac{k^2}{r^2} \Psi_k, \tag{5.58}$$

where the last inequality comes from the fact that y_1 is non-negative. Furthermore,

$$[\![\Psi_k']\!](t, r^*) = 0, \tag{5.59}$$

so that, following exactly the main line of reasoning, we obtain

$$\Psi_k \leqslant 0, \tag{5.60}$$

which concludes the proof.

□

We now pass to the next step:

$$z_1 \geqslant 0 \text{ in } (0;T) \times \Omega. \tag{5.61}$$

Proof of (5.61). The function z_1 satisfies

$$\begin{cases} \frac{\partial z_1}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial z_1}{\partial r}) = \frac{1}{r^2} z_1 - y_1 \text{ in } (0;T) \times (0;R)\,, \\ z_1(R, \cdot) = 0\,, \\ z_1(T, 0) = \partial_r z_1(t, 0) = 0. \end{cases} \tag{5.62}$$

Since $y_1 \geqslant 0$ from (5.52) z_1 solves, in particular,

$$\begin{cases} \frac{\partial z_1}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial z_1}{\partial r}) \leqslant \frac{1}{r^2} z_1 \text{ in } (0;T) \times (0;R)\,, \\ z_1(R, \cdot) = 0\,, \\ z_1(T, 0) = \partial_r z_1(t, 0) = 0. \end{cases} \tag{5.63}$$

Let us now define $\overline{z}_1 = z_1(T - t, \cdot)$. Straightforward computations show that $\overline{z}_1$ solves

$$\partial_t \overline{z}_1 - \frac{1}{r}\partial_r(r \partial_r \overline{z}_1) \geqslant -\frac{1}{r^2}\overline{z}_1. \tag{5.64}$$

Multiplying this identity by $\overline{z}_1^-$ and integrating by parts, we obtain in the same way

$$z_1 \geqslant 0, \tag{5.65}$$

as claimed. □

We are now in a position to prove (5.51):

Proof of (5.51). We define, for any $k \in \mathbb{N}$, $\mathscr{Z}_k := z_k - z_1$. It is clear that $\mathscr{Z}_k$ solves

$$\frac{\partial \mathscr{Z}_k}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathscr{Z}_k}{\partial r}\right) = \frac{k^2}{r^2} z_k - \frac{1}{r^2} z_1 + y_1 - y_k. \text{ in } (0;T) \times (0;R) \tag{5.66}$$

From Estimate (5.56) there holds

$$\frac{\partial \mathscr{Z}_k}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathscr{Z}_k}{\partial r}\right) \geqslant \frac{k^2}{r^2} z_k - \frac{1}{r^2} z_1 \text{ in } (0;T) \times (0;R) \tag{5.67}$$

and so, from Estimate (5.61) we get

$$\frac{\partial \mathscr{Z}_k}{\partial t}+\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathscr{Z}_k}{dr}\right)\geqslant \frac{k^2}{r^2}z_k-\frac{k^2}{r^2}z_1=\frac{k^2}{r^2}\mathscr{Z}_k \text{ in } (0;T)\times(0;R). \tag{5.68}$$

From the same reasoning, we obtain

$$\mathscr{Z}_k\leqslant 0 \tag{5.69}$$

and so

$$z_k\leqslant z_1 \text{ in } (0;T)\times(0;R). \tag{5.70}$$

□

The proof of the first part of Claim 27 is thus finished, and it hence remains to prove that

$$\omega_1<0. \tag{5.71}$$

Proof of (5.71). We recall that

$$\omega_1=\int_0^T z_1(t,r^*)dt+\int_0^T\frac{\partial\overline{p}}{\partial r}(t,r^*)dt.$$

First of all, is is easy to see that p is non-negative.

Let us define $\overline{\varphi}:=\frac{\partial\overline{p}}{\partial r}$. Straightforward computations show that $\overline{\varphi}$ solves

$$\begin{cases}\frac{\partial\overline{\varphi}}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\overline{\varphi}}{\partial r}\right)=-\frac{\partial\overline{u}}{\partial r}+\frac{1}{r^2}\overline{\varphi} \text{ in } (0;T)\times(0;R)\,,\\ \overline{\varphi}(t,R)\leqslant 0\,,\\ \overline{\varphi}(T,\cdot)\equiv 0.\end{cases} \tag{5.72}$$

If we define $\overline{\Phi}:=\overline{\varphi}+z_1$ we thus have

$$\frac{\partial\overline{\Phi}}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\overline{\Phi}}{\partial r}\right)=\frac{1}{r^2}\overline{\Phi}-\frac{\partial\overline{u}}{\partial r}\geqslant\frac{1}{r^2}\overline{\Phi}. \tag{5.73}$$

The last inequality comes from Proposition 9. Furthermore we have $\overline{\Phi}(t,R)\leqslant 0$. As a consequence, we have

$$\overline{\Phi}\leqslant 0 \text{ in } (0;T)\times\Omega. \tag{5.74}$$

Furthermore, we necessarily have $\overline{\Phi}(t,r^*)<0$ in a subset of positive measure of $(0;T)$, for otherwise we have $\frac{\partial\overline{u}}{\partial r}(t,r^*)=0$ on this subset, which is absurd given Proposition 9. As a conclusion, we obtain

$$\omega_1=\int_0^T\overline{\Phi}(t,r^*)dr<0, \tag{5.75}$$

as claimed. □

□

□

With this Proposition available, we are in a position to prove Proposition 21. Let us recall that, for a normal deformation $\Phi \in W^{2,p}$ we have defined

$$j_\Phi(\xi) := \mathcal{L}_{\mathbb{B}^*}(\mathbb{B}^*_{t\Phi}) + C(\mathrm{Vol}(\mathbb{B}^*_{t\Phi}) - V_0)^2. \tag{5.76}$$

Since $\mathbb{B}^*$ is a critical shape we obtain

$$j'_\Phi(0) = 0 \tag{5.77}$$

so that the Taylor-Lagrange formula with integral remainder writes, in the case where $\mathrm{Vol}(\mathbb{B}^*_\Phi) = V_0$,

$$\mathcal{L}_{\mathbb{B}^*}(\mathbb{B}^*_\Phi) - \mathcal{L}_{\mathbb{B}^*}(\mathbb{B}^*) = \int_0^1 j'' \leqslant j''_\Phi(0) + \int_0^1 |j''_\Phi(\xi) - j''_\Phi(0)|\, d\xi. \tag{5.78}$$

The key is now to prove the following Lemma:

Lemma 28. *There exists a modulus of continuity, that is, a continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(0) = 0$, such that*

$$|j''_\Phi(t) - j''_\Phi(0)| \leqslant \eta\left(\|\Phi\|_{W^{2,p}}\right) \|\Phi \cdot \nu\|^2_{L^2(\partial\mathbb{B}^*)}. \tag{5.79}$$

Indeed, Lemma 28 implies Proposition 21 in the following way: assuming it holds then

$$\begin{aligned}
\mathcal{L}_{\mathbb{B}^*}(\mathbb{B}^*_\Phi) - \mathcal{L}_{\mathbb{B}^*}(\mathbb{B}^*) &= \int_0^1 j'' \leqslant j''_\Phi(0) + \int_0^1 |j''_\Phi(\xi) - j''_\Phi(0)|\, d\xi && (5.80)\\
&\leqslant -c_0\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)} + \eta\left(\|\Phi\|_{W^{2,p}}\right)\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)} && (5.81)\\
&\leqslant -\frac{c_0}{2}\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)} \text{ for } \|\Phi\|_{W^{2,p}} \text{ small enough} && (5.82)\\
&\leqslant -\frac{c_0}{2}\frac{1}{\mathrm{Per}(\partial\mathbb{B}^*)}\left(\int_{\partial\mathbb{B}^*}|\Phi\cdot\nu|\right)^2 \text{ by the Cauchy-Schwarz inequality} && (5.83)\\
&\leqslant -\tilde{c}_0\,\mathrm{Vol}\left(\mathbb{B}^*_\Phi \Delta \mathbb{B}^*\right)^2. && (5.84)
\end{aligned}$$

The proof of Lemma 28 is extremely similar to the proof of [31, Proposition 23] and is mostly a technical adaptation of [16]. For this reason, we postpone it to Appendix A.3 and briefly sketch here why this L^2 norm of $\Phi \cdot \nu$ is, in contrast to the $H^{\frac{1}{2}}$ usually required in shape optimisation [16], the optimal norm here. If we consider, for instance, at at given shape E the second order shape derivative of the Lagrangian, we have

$$\mathcal{L}''_{\mathbb{B}^*}(E)[\Phi,\Phi] = \underbrace{\iint_{(0;T)\times\partial E} p'\,(\Phi\cdot\nu)}_{=I_1} + \underbrace{\int_{\partial E}(\Phi\cdot\nu)^2\left(\mathscr{H}\int_0^T p_E + \int_0^T \frac{\partial p_E}{\partial\nu}\right) - \overline{\Psi}_{\partial\mathbb{B}^*}\int_{\partial E}\mathscr{H}\,(\Phi\cdot\nu)^2}_{=:I_2} \tag{5.85}$$

where:

1. $\mathscr{H}$ is the mean curvature of ∂E,
2. p_E solves

$$\begin{cases} \frac{\partial p_E}{\partial t} + \Delta p_E = -u_E \text{ in } (0;T)\times\Omega\,, \\ p_E(T,\cdot) = 0\,, \\ p_E(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega, \end{cases} \tag{5.86}$$

3. u' solves

$$\begin{cases}\frac{\partial u'}{\partial t}-\Delta u'=0 \text{ in } (0;T)\times\Omega\,,\\ u'(0,\cdot)=0\,,\\ [\![\partial_\nu u']\!]=-1 \text{ on } (0;T)\times\partial E,\\ u'(t,\cdot)=0 \text{ on } (0;T)\times\partial\Omega,\end{cases} \tag{5.87}$$

4. p' solves

$$\begin{cases}\frac{\partial p'}{\partial t}+\Delta p'=-u' \text{ in } (0;T)\times\Omega\,,\\ p'(T,\cdot)=0\,,\\ p'(t,\cdot)=0 \text{ on } (0;T)\times\partial\Omega,\end{cases} \tag{5.88}$$

5. and

$$\overline{\Psi}_{\partial\mathbb{B}^*}=\int_0^T p^*(t,\cdot)\Big|_{\partial\mathbb{B}^*}$$

   is the Lagrange multiplier associated with the volume constraint.

Now, by the regularity estimates of Proposition 6 and by standard Schauder estimates, it is natural to expect that

$$\|I_2\|\leqslant M\,\|\Phi\cdot\nu\|^2_{L^2(\partial E)}\,. \tag{5.89}$$

To prove that the same estimate holds for I_1, it suffices, by continuity of the trace, to obtain

$$\iint_{(0;T)\times\Omega}|\nabla p'|^2\leqslant M\int_{\partial E^*}(\Phi\cdot\nu)^2. \tag{5.90}$$

However, this just follows from standard parabolic estimates, provided we can prove that

$$\iint_{(0;T)\times\Omega}(u')^2\leqslant M\int_{\partial E^*}(\Phi\cdot\nu)^2. \tag{5.91}$$

To prove (5.91), we use u' as a test function in the weak equation on u' and obtain, by the Cauchy-Schwarz inequality and the continuity of the trace,

$$\frac{\partial}{\partial t}\int_\Omega(u')^2(t,\cdot)+\int_\Omega|\nabla u'|^2=\int_{\partial E}(\Phi\cdot\nu)u' \tag{5.92}$$

$$\leqslant M\|\Phi\cdot\nu\|_{L^2(\partial E)}\|\nabla u'\|_{L^2(\Omega)}. \tag{5.93}$$

Integrating this inequality in time yields the required result and we hence obtain

$$|\mathcal{L}''_{\mathbb{B}^*}(E)[\Phi,\Phi]|\leqslant M\|\Phi\cdot\nu\|^2_{L^2(\partial E)}. \tag{5.94}$$

As a consequence, the L^2 norm should be the optimal coercivity norm.

5.4 Quantitative bathtub principle: using the convexity of the functional

In this section, we will fully exploit the convexity of the functional. We first heuristically explain how we are going to make use of it.

Heuristics Let us assume that we are working with a competitor $f \in \overline{\mathcal{M}}(\Omega)$, and let us define p_f as the adjoint state associated to f (solution of (5.4)). Hence, for an admissible perturbation h at f (i.e, such that $f + th \in \overline{\mathcal{M}}(\Omega)$ for any $t \geqslant 0$ small enough), the derivative of $\mathcal{J}_T$ at f in the direction h is given by (Proposition 8)

$$\dot{\mathcal{J}}_T(f)[h] = \int_\Omega h(x)\left(\int_0^T p_f(t,x)dt\right)dx. \tag{5.95}$$

Since $\mathcal{J}_T$ is convex (Proposition 7), we have

$$\mathcal{J}_T(f+h) - \mathcal{J}_T(f) \geqslant \dot{\mathcal{J}}_T(f)[h]. \tag{5.96}$$

As a consequence, let us assume that $f = \mathbb{1}_E$. In order to maximise the right hand side of (5.96), we need to choose h such that, defining $\overline{\Psi} := \int_0^T p_f(t,\cdot)dt$, and choosing $\overline{c} > 0$ such that $\operatorname{Vol}\left(\{\overline{\Psi} > \overline{c}\}\right) = V_0$ (assuming this set is uniquely defined and regular),

$$f + h = \mathbb{1}_{\{\overline{\Psi} > \overline{c}\}} = \mathbb{1}_{\overline{E}} \tag{5.97}$$

and so we obtain the lower bound

$$\mathcal{J}_T(\mathbb{1}_{\overline{E}}) - \mathcal{J}_T(\mathbb{1}_E) \geqslant \int_{\overline{E}} \overline{\Psi} - \int_E \overline{\Psi}. \tag{5.98}$$

Now, as we will see, when f is close enough to f^*, $\overline{E}$ should be a normal deformation of $\mathbb{B}^*$, and the only thing left is thus to quantify

$$\int_{\overline{E}} \overline{\Psi} - \int_E \overline{\Psi}. \tag{5.99}$$

Indeed, using (5.98) we obtain

$$\mathcal{J}_T(\mathbb{1}_{\mathbb{B}^*}) - \mathcal{J}_T(\mathbb{1}_E) \geqslant J_T(\mathbb{1}_{\mathbb{B}^*}) - \mathcal{J}_T(\mathbb{1}_{\overline{E}}) + \mathcal{J}_T(\mathbb{1}_{\overline{E}}) - \mathcal{J}_T(\mathbb{1}_E) \geqslant C \operatorname{Vol}(E^* \Delta \overline{E})^2 + \mathcal{J}_T(\mathbb{1}_{\overline{E}}) - \mathcal{J}_T(\mathbb{1}_E). \tag{5.100}$$

Here, $C > 0$ is given by Proposition 21.

Since $\overline{f} := \mathbb{1}_{\overline{E}}$ is a maximiser of $T_{\overline{\Psi}} : f \mapsto \int_\Omega f\overline{\Psi}$ in $\overline{\mathcal{M}}(\Omega)$, it turns out that estimating (5.99) amounts to providing a quantitative estimate for the linear optimisation problem

$$\sup_{f \in \overline{\mathcal{M}}(\Omega)} T_{\overline{\Psi}}(f) \tag{5.101}$$

which is exactly the quantitative version of the bathtub principle.

The goal of the present paragraph is to give a uniform bathtub principle that was presented in a slightly different form in the section devoted to Theorem II, see Proposition 14 above.

Proposition 29. *Let $\beta > 0$ and let $\{\psi_i\}_{i\in I} \subset \mathscr{C}^{1,\beta}(\Omega;\mathbb{R}_+)^I$ be a closed subset of $\mathscr{C}^{1,\beta'}(\Omega)$ for some $\beta' < \beta$. We assume that:*

1. *For every $i \in I$ there exists a unique c_i such that, up to a set of measure 0,*

$$\Omega_i := \{\psi_i > c_i\} = \{\psi_i \geqslant c_i\} \tag{5.102}$$

and

$$\forall i \in I\,, \operatorname{Vol}(\Omega_i) = V_0\,, \overline{L} = \sup_{i \in I} \operatorname{Per}(\Omega_i) < +\infty. \tag{5.103}$$

We define, for any $i \in I$,

$$\overline{f}_i := \mathbb{1}_{\Omega_i}.$$

2. There exists $M > 0$ such that

$$\sup_{i\in I} \|\psi_i\|_{\mathscr{C}^{1,\beta}} \leqslant M_I. \tag{5.104}$$

3. There exists $\underline{\mu} > 0$ such that

$$\inf_{i\in I} \inf_{\partial\Omega_i} \left\{ -\frac{\partial\psi_i}{\partial\nu} \right\} \geqslant \underline{\mu}. \tag{5.105}$$

Then there exists a constant $\overline{\omega} > 0$ such that

$$\forall i \in I\,, \forall f \in \overline{\mathcal{M}}(\Omega)\,, \int_\Omega (\overline{f}_i - f)\psi_i \geqslant \overline{\omega}\|\overline{f}_i - f\|^2_{L^1(\Omega)}. \tag{5.106}$$

The proof of this Proposition is very similar to that of Proposition 14.

Proof of Proposition 29. We just need to prove that, thanks to our assumption, we can bring ourselves back to the proof of Proposition 14. This is done using the Schwarz rearrangement, as was done in [31].

From the bathtub principle we have, for any $i \in I$, that $f_i^* := \mathbb{1}_{\Omega_i}$ is the unique solution of

$$\sup_{f\in\mathcal{M}(\Omega)} \int_\Omega f\psi_i. \tag{5.107}$$

By the uniform Hölder continuity of $\{\nabla\psi\}_{i\in I}$ there exists $\overline{\varepsilon} > 0$ that only depends on M_I and $\underline{\mu}$ such that

$$\forall x \in \Omega\,, \forall i \in I\,, \psi_i(x) \in \Big(c_i - \overline{\varepsilon}; c_i + \overline{\varepsilon}\Big) \Rightarrow |\nabla\psi_i|\,(x) \geqslant \frac{\underline{\mu}}{2}\,, \sup_{\varepsilon\in(-\overline{\varepsilon};\overline{\varepsilon})} \sup_{i\in I} \mathrm{Per}(\{\psi_i = c_i + \varepsilon\}) < +\infty. \tag{5.108}$$

Let us fix such an $\overline{\varepsilon}$.

We now reduce ourselves to the case of radially symmetric function:

Reduction to radially symmetric functions For any $i \in I$, let us consider the distribution function $\mathscr{L}_i$ of ψ_i. From (5.108), $\mathscr{L}_i$ is $\mathscr{C}^1$ in $(c_i - \overline{\varepsilon}; c_i + \overline{\varepsilon})$ and so, letting $\psi_i^\#$ be the Schwarz rearrangement of ψ_i, we have

$$\int_{\partial\{\psi_i > c_i\}} \frac{1}{\left|\frac{\partial\psi_i}{\partial\nu}\right|} = -\mathscr{L}_i'(c_i) = \int_{\partial\mathbb{B}(0;r^*)} \frac{1}{\left|\frac{\partial\psi_i^\#}{\partial\nu}\right|}.$$

Given the uniform perimeter bound (5.108) on the level sets close to $\{\psi_i = c_i\}$, it thus follows that there exists a constant $C > 0$ and $\underline{\varepsilon} > 0$ such that $\{\psi_i^\dagger\}_{i\in I}$ satisfies, in a $(r^* - \underline{\varepsilon}; r^* + \underline{\varepsilon})$,

$$\forall i \in I\,, \left|\frac{d\psi_i^\dagger}{dr}\right| \geqslant C\underline{\mu}. \tag{5.109}$$

We can then observe the following thing: by equimeasurability of the Schwarz rearrangement, we have, for every $f \in \mathcal{M}(\Omega)$, the following property: if $\|f - \overline{f}_i\|_{L^1(\Omega)} = \delta$ then, defining $\mathbb{A}_\delta$ as the unique annulus such that $\mathrm{Vol}(\mathbb{A}_\delta) = V_0\,, \mathrm{Vol}(\mathbb{A}_\delta \Delta \mathbb{B}^*) = \delta$, the Haryd-Littlewood inequality and the equimeasurability of the rearrangement ensure that

$$\int_\Omega (\overline{f}_i - f)\psi_i \geqslant \int_\Omega (f^* - \mathbb{1}_{\mathbb{A}_\delta})\psi_i^\#. \tag{5.110}$$

Hence, it suffices to prove that

$$\int_{\Omega}(f^*-\mathbb{1}_{\mathbb{A}_\delta})\psi_i^{\#}\geqslant \underline{\omega}\delta^2 \tag{5.111}$$

where $\underline{\omega}$ does not depend on i. Thanks to (5.109), the rest of the proof follows along the same exact lines as Proposition 14. □

5.5 Combining the bathtub principle and shape derivatives

To conclude the proof of Theorem III, it thus only remains to prove the following proposition:

Proposition 30. *Estimate (5.9) holds.*

Proof. We argue by contradiction and assume that Estimate (5.9) does not hold. Let us then consider a sequence $\{f_k\}_{k\in\mathbb{N}}$ such that

$$\frac{\mathcal{J}_T(f^*)-\mathcal{J}_T(f_k)}{\|f_k-f^*\|^2_{L^1(\Omega)}}\underset{k\to\infty}{\to}0. \tag{5.112}$$

As in the proof of Theorem II, the only closure point of $\{f_k\}_{k\in\mathbb{N}}$ (in a weak $L^\infty-*$ sense) is f^*. We introduce, for any $k\in\mathbb{N}$,

$$\delta_k:=\|f_k-f^*\|_{L^1(\Omega)}. \tag{5.113}$$

Up to replacing f_k with f_{δ_k}, we can assume that $f_k=f_{\delta_k}$.

Let us define, for any $k\in\mathbb{N}$, p_k as the adjoint state (solution of (5.4)) with $f=f_k$. From standard parabolic regularity and Proposition 6, for any $\beta\in(0;1)$ there exists M_β such that for any $t\in[0;T]$

$$\|p_k\|_{\mathscr{C}^{2,\beta}(\Omega)}\leqslant M_\beta, \tag{5.114}$$

and hence, since $f_k\underset{k\to\infty}{\to}f^*$, we obtain

$$p_k\overset{\mathscr{C}^{2,\beta}((0;T)\times\Omega)}{\underset{k\to\infty}{\to}}p^* \tag{5.115}$$

where p^* is the adjoint state associated with $f=f^*$.

Let, for any $k\in\mathbb{N}$,

$$\Psi_k:=\int_0^T p_k. \tag{5.116}$$

From the same arguments,

$$\Psi_k\overset{\mathscr{C}^{2,\beta}(\Omega)}{\underset{k\to\infty}{\to}}\Psi^*:=\int_0^T p^*. \tag{5.117}$$

Ψ^* is radially symmetric, it is decreasing and its only level set of volume V_0 is $\mathbb{B}^*$. Furthermore, from the same arguments as in Claim 17, we also have

$$\forall\eta>0\,,\inf_{r>\eta}\left|\frac{\partial\Psi^*}{\partial r}\right|(r^*)=\ell(\eta)>0. \tag{5.118}$$

Let, for any $k\in\mathbb{N}$, c_k be such that $\operatorname{Vol}(\{\Psi_k\geqslant c_k\})\geqslant V_0\,,\operatorname{Vol}(\{\Psi_k>c_k\})\leqslant V_0$.

Since $f_k\underset{k\to\infty}{\to}f^*$ and since $\mathcal{J}_T(f_k)\geqslant\mathcal{J}_T(\mathbb{1}_{\{\Psi_k>c_k\}})$ by convexity of the functional, it follows that $\left\{\mathbb{1}_{\{\Psi_k>c_k\}}\right\}_{k\in\mathbb{N}}$ converges weakly to f^*. Since f^* is an extreme point of $\overline{\mathcal{M}}(\Omega)$, this convergence occurs in L^1. We choose $\eta>0$ small enough so that, for any $k\in\mathbb{N}$ large enough,

$\{\psi_k = c_k\} \cap \{0 < r < \eta\} = \emptyset$. This is possible because Ψ^* is radially symmetric and decreasing: indeed, argue by contradiction and assume that there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ converging to 0 such that for any $k \in \mathbb{N}$, $\Psi_k(x_k) = c_k$. Let c be the limit of the sequence $\{c_k\}$. Since $\operatorname{Vol}(\{\Psi_k > c_k\}) = V_0$, there exists $\eta' > 0$ such that for any $k \in \mathbb{N}$ there exists y_k , $\|y_k\| > \eta'$ such that $\Psi_k(y_k) > c_k$. Passing to the limit, there exists $y' > 0$ such that $\Psi^*(y') \geqslant c = \lim_{k\to\infty} \Psi_k(x_k) = \Psi^*(0)$ and so Ψ^* can not be decreasing. Hence such an $\eta > 0$ exists.

As a consequence, for such an η we have, for any k large enough,

$$\inf_{x\,,\|x\|>\eta} |\nabla \Psi_k|(x) \geqslant \frac{\ell(\eta)}{2}. \tag{5.119}$$

Thus, the level set $\{\Psi_k = c_k\}$ is a $\mathscr{C}^1$ curve and

$$\inf_{\{\Psi_k = c_k\}} |\nabla \Psi_k| \geqslant \frac{\ell(\eta)}{2}. \tag{5.120}$$

Since $\{\Psi_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $\mathscr{C}^2(\Omega)$, these sets have uniformly Lipschitz boundaries. It follows that the sequence of sets $\{\{\Psi_k > c_k\}\}_{k\in\mathbb{N}}$ converges in Hausdorff distance to $\{\Psi^* > c^*\} = \mathbb{B}^*$ where $c^* = \Psi^*(r^*)$.

Finally, for any $k \in \mathbb{N}$ large enough, $E_k := \{\Psi_k = c_k\}$ is a normal deformation of $\mathbb{B}^*$. Indeed, assuming that it is not, there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \in (\partial\mathbb{B}^*)^{\mathbb{N}}$ and two sequences $\{t_{i,k}\}_{i=1,2,k\in\mathbb{N}}$ converging to 0 such that $\Psi_k(x_k + t_{i,k}\nu(x_k)) = c_k$, $i = 1, 2$. This gives the existence of a t_k, converging to 0 as $k \to \infty$, such that $\langle \nabla\Psi_k(x_k + t_k\nu(x_k)), \nu(x_k)\rangle = 0$, which yields a contradiction when passing to the limit. Thus, ∂E_k converges $W^{2,p}$ to $\partial\mathbb{B}^*$ for all $p > 1$, and in $\mathscr{C}^2$, $\beta \in (0; 1)$, and the sequence $\{\operatorname{Per}(\{\Psi_k = c_k\})\}_{k\in\mathbb{N}}$ is bounded.

We can now prove Estimate (5.9): from the convexity of the functional and the fact that, for any $k \in \mathbb{N}$, f_k solves (P_{δ_k}), there exists a subset F_k of Ω such that $f_k = \mathbb{1}_{F_k}$. Let $E_k = \{\Psi_k > c_k\}$ be the unique level-set of Ψ_k of measure V_0. By convexity of the functional,

$$\mathscr{J}_T(\mathbb{1}_{E_k}) - \mathscr{J}_T(f_k) \geqslant \int_\Omega (\mathbb{1}_{E_k} - \mathbb{1}_{F_k})\Psi_k. \tag{5.121}$$

We are now in a position to apply Proposition 29: with the $\overline{\varpi}$ given in Proposition 29, we thus have

$$\mathscr{J}_T(\mathbb{1}_{E_k}) - \mathscr{J}_T(f_k) \geqslant \overline{\varpi}\operatorname{Vol}(F_k \Delta E_k)^2. \tag{5.122}$$

Then, as E_k is a normal deformation of $\mathbb{B}^*$ we can apply Proposition 21 and obtain, for $C > 0$ given by Proposition 21,

$$\mathscr{J}_T(\mathbb{1}_{\mathbb{B}^*}) - \mathscr{J}_T(\mathbb{1}_{E_k}) \geqslant C\operatorname{Vol}(E_k \Delta \mathbb{B}^*)^2. \tag{5.123}$$

We obtain the existence a $C' > 0$ such that

$$\mathscr{J}_T(\mathbb{1}_{\mathbb{B}^*}) - \mathscr{J}_T(\mathbb{1}_{E_k}) \geqslant C'\left(\operatorname{Vol}(E_k\Delta\mathbb{B}^*)^2 + \operatorname{Vol}(F_k\Delta E_k)^2\right). \tag{5.124}$$

However, by the triangle inequality in L^1 and the arithmetic-geometric inequality,

$$\operatorname{Vol}(\mathbb{B}^*\Delta F_k)^2 \leqslant \frac{1}{2}\left(\operatorname{Vol}(F_k\Delta E_k)^2 + \operatorname{Vol}(E_k\Delta\mathbb{B}^*)^2\right). \tag{5.125}$$

The conclusion follows.

□

6 Conclusion

6.1 Structure of the problem, structure of the proof

In this paper, we have investigated two possible approaches to quantitative inequalities for time-evolving optimal control problems. While Theorem II, dealing with time-dependent controls, is more powerful than Theorem III, it is likely that its proof does not generalise easily to other domains. Indeed, the first step of the proof is to identify, explicitly, the maximisers of an auxiliary optimisation problem, which can not be done in general, non-spherical domains.

On the other hand, the proof of Theorem III is susceptible of applying to other cases. Let us specify what we mean: considering a controlled heat equation

$$\frac{\partial u_f}{\partial t} - \Delta u_f = f \tag{6.1}$$

with Dirichlet boundary conditions, and where $f \in \overline{\mathcal{M}}(\Omega)$, let f^* be a solution of ($\mathbf{I}_1$). The convexity of the functional $\mathcal{J}_T$ (Proposition 7) holds independently of the geometry of the domain an so any maximiser f^* writes $\mathbb{1}_{E^*}$ for some subset E^* of Ω. In order to carry out the proof of Theorem III in this new domain, several things are in order:

1. The regularity of optimal sets: each set E^* such that $f^* = \mathbb{1}_{E^*}$ is a solution of ($\mathbf{I}_1$) needs to be smooth enough that shape derivatives of the criterion may be computed. It is unclear at this stage whether or not the classical regularity works valid in the stationary case may be applied to obtain such regularity.

2. The coercivity of shape Lagrangians: defining $I^* := \{E^* \subset \Omega\,, \mathbb{1}_{E^*} \text{ solves } (\mathbf{I}_1)\}$ and assuming that each $E^* \in I^*$ is smooth enough to compute first and second order shape derivatives, one needs to check that, defining the Lagrangian L_{E^*} associated with the volume constraint, there exists a constant $\alpha > 0$ such that, for any $E^* \in I^*$ and any admissible vector field Φ at E^*, there holds
$$L_{E^*}(E^*)[\Phi, \Phi] \geqslant \alpha \|\Phi \cdot \nu\|^2_{L^2(\partial E^*)}. \tag{6.2}$$
This kind of estimate seems to be extremely challenging to obtain in general, as indicated by the fact that, in this paper, such a coercivity was obtained by explicit diagonalisation of the shape hessian. Such diagonalisation may not be available in general.

If these two assumptions are satisfied, then we believe that the method of proof of Theorem III may adapt.

6.2 The optimal coercivity norm for other types of constraints

As mentioned in the Introduction, an interesting question is that of knowing whether or not the coercivity norm obtained in Theorem II remains unchanged when considering other types of L^1 constraints. Indeed, let us consider the following variation: defining

$$\tilde{\mathcal{M}}(\Omega) := \left\{ f \in L^\infty((0;T) \times \Omega)\,, 0 \leqslant f \leqslant 1 \text{ a.e.}, \iint_{(0;T)\times\Omega} f = V_0 \right\} \tag{6.3}$$

we investigate the optimisation problem

$$\sup_{f \in \tilde{\mathcal{M}}(\Omega)} J_T(f). \tag{6.4}$$

Here, the convexity of the functional $\mathcal{J}_T$ is still valid, so that a solution of this new problem writes $f^* = \mathbb{1}_{E^*}$, with E^* a measurable subset of E^*. Then, if one were to compare f^* with a competitor $f = \mathbb{1}_E$, it would be more natural to expect the “classical ”discrepancy norm

$$\mathrm{Vol}(E^*\Delta E)^2 = \left(\iint_{(0;T)\times\Omega} |f - f^*|\right)^2 \tag{6.5}$$

to be optimal. We do not believe this to be true, however, and we believe that the correct discrepancy norm remains

$$\int_0^T \mathrm{Vol}(E^*(t)\Delta E(t))^2 dt. \tag{6.6}$$

To give some explanation as to why we believe this is to be expected, we can once again consider the case of the ball $\Omega = \mathbb{B}(0;R)$. Once again, the rearrangement arguments used throughout the paper remain valid, and, for almost every $t \in (0;T)$, $E^*(t)$ is a centred ball of radius $r(t) \geqslant 0$. We expect several difficulties in treating this problem (most notably, we expect the (non)-degeneracy of r, or, in other terms, the control of the set $\{r = 0\}$, to be very hard to obtain) but the methods of Theorem II should once again provide a quadratic estimate at each time t, yielding the aforementioned stronger estimate. We underline once again that, at the present moment, it is unclear to us how one may fully analyse this type of global constraint.

6.3 Theorem II: on the Assumption $\varepsilon > 0$

One may also argue that the assumption $\varepsilon > 0$ is artificial. At this stage, and since we use in a crucial manner the uniform non-degeneracy of the switch function (Claim 17), we are not yet in a position to give a proof that would bypass this assumption. However, it should be noted that our proof makes use of very strong regularity properties in order to derive the uniform bathtub principle. It would be interesting to see if, using the general quantitative Hardy-Littlewood inequality [15] one could bypass the strength required in the present proof to obtain the case $\varepsilon = 0$ (and, in general, it would be extremely interesting to use [15] to see if Theorem II could be obtained in more general domain).

6.4 Using the quantitative isoperimetric inequality to obtain our results

We touch on another way which it would be interesting to investigate, that of using the quantitative isoperimetric inequality in order to obtain Theorem II. It would amount, in the approach of [33] (see Section 3), to supplementing the isoperimetric inequality in (3.19). We expect that this would lead to a control of the isoperimetric deficit $\mathcal{A}(t,\tau)$ of the level sets $\{u_f(t,\cdot) > \tau\}$ in the sense that we could give a lower bound of the form $\int_0^T \int_0^{\|u_f\|_{L^\infty}} \mathcal{A}(t,\tau)^2 d\tau dt$, but is unclear how this would then translate to a control of the isoperimetric deficit of f.

6.5 Minimisation problems

We believe the proof for minimisation problems works in exactly the same way, as we also have an explicit description of minimisers using rearrangement techniques.

6.6 Technical obstructions and possible generalisations for bilinear control problems

Finally, we touch upon bilinear control problems. Let us assume that we are working with the state equation

$$\begin{cases} \frac{\partial u_f}{\partial t} - \Delta u_f = f u_f \text{ in } (0;T) \times \Omega\,, \\ u_f(t=0) = u_0 \geqslant 0\,, u_0 \neq 0\,, \\ u_f(t,\cdot) = 0 \text{ on } (0;T) \times \partial\Omega. \end{cases} \tag{6.7}$$

The maximisation problem reads the same:

$$\sup_{f \in \mathcal{M}_T(\Omega)} \frac{1}{2} \iint_{(0;T)\times\Omega} u_f^2. \tag{6.8}$$

Here we con once again explicitly characterise the maximisers using rearrangement techniques. However: the convexity of the functional is no longer obvious, and it can be checked that the switch function is here given

$$\Psi = p_f u_f \tag{6.9}$$

where p_f solves

$$\begin{cases} \frac{\partial p_f}{\partial t} + \Delta p_f = -f p_f - u_f\,, \\ p_f(T,\cdot) = 0\,, \\ p_f(t,\cdot) = 0 \text{ on } (0;T) \times \partial\Omega. \end{cases} \tag{6.10}$$

Here we see our first difference with our approach, which is that the switch function can merely be expected to be $\mathscr{C}^{0,\alpha}$, which is in contrast with the $\mathscr{C}^{2,\alpha}$ regularity we obtained in our paper. Maybe it is possible to bypass this problem using the tools of [15].

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A Proof of technical lemmas

A.1 Proof of Lemma 19

Proof of Lemma 19. This Lemma relies on two elements: the first one is the weak continuity of $\mathcal{J}_T$, given in the following claim

Claim 31. *Assume $\{f_k\}_{k\in\mathbb{N}} \in \overline{\mathcal{M}}(\Omega)^{\mathbb{N}}$ converges weakly $L^\infty - *$ to f_∞. Then*

$$\mathcal{J}_T(f_k) \underset{k\to\infty}{\to} \mathcal{J}_T(f_\infty). \tag{A.1}$$

Proof of Claim 31. This proof relies on standard parabolic estimates. □

The second element is the following property of the class $\overline{\mathcal{M}}(\Omega,\delta)$:

Claim 32. *The class $\overline{\mathcal{M}}(\Omega,\delta)$ is weakly $L^\infty - *$ compact.*

Proof of Claim 32. For any $f \in \overline{\mathcal{M}}(\Omega,\delta)$, let us define

$$h := f - f^*.$$

The condition that $\int_\Omega = \int_\Omega f^*$ rewrites

$$\int_\Omega h = 0. \tag{A.2}$$

Since $f^* = \mathbb{1}_{\mathbb{B}^*}$ is the characteristic function of a set, we also have

$$h \leqslant 0 \text{ in } \mathbb{B}^*\,, h \geqslant 0 \text{ in } (\mathbb{B}^*)^c. \tag{A.3}$$

This, and Equation (A.2), allows to rewrite the condition $\|f - f^*\|_{L^1(\Omega)}$ as

$$\int_{\mathbb{B}^*} |h| = -\int_{\mathbb{B}^*} h = \int_{(\mathbb{B}^*)^c} h = \int_{(\mathbb{B}^*)} |h| = \frac{\delta}{2}. \tag{A.4}$$

Finally, $-1 \leqslant h \leqslant 1$ as a consequence of its definition.

Let us then consider a sequence $\{f_k\}_{k\in\mathbb{N}} \in \overline{\mathcal{M}}(\Omega,\delta)^{\mathbb{N}}$ and define, for any $k \in \mathbb{N}$,

$$h_k := f_k - f^*. \tag{A.5}$$

Since $\overline{\mathcal{M}}(\Omega)$ is compact for the weak $L^\infty - *$ convergence, let us assume that there exists $f_\infty \in \overline{\mathcal{M}}(\Omega)$ such that

$$f_k \underset{k\to\infty}{\rightharpoonup} f_\infty \tag{A.6}$$

and define $h_\infty = f_\infty - f^*$. It is clear that

$$h_k \underset{k\to\infty}{\rightharpoonup} h_\infty. \tag{A.7}$$

Since (A.3)-(A.4) are satisfied by h_k for every $k \in \mathbb{N}$, it follows that they are satisfied by h_∞. As a consequence,

$$\int_{\mathbb{B}^*} |h_\infty| = -\int_{\mathbb{B}^*} h_\infty = \int_{(\mathbb{B}^*)^c} h_\infty = \int_{(\mathbb{B}^*)} |h_\infty\,| = \frac{\delta}{2} \tag{A.8}$$

and so

$$f_\infty \in \overline{\mathcal{M}}(\Omega,\delta), \tag{A.9}$$

so that the Claim follows.

□

Thus, to conclude the proof of the Lemma, it suffices to consider a minimising sequence $\{f_k\}_{k\in\mathbb{N}} \in \overline{\mathcal{M}}(\Omega,\delta)^{\mathbb{N}}$ for the variational problem ($\mathbf{P}_\delta$). One can extract a wek $L^\infty - *$ converging subsequence that converges to $f_\infty \in \overline{\mathcal{M}}(\Omega,\delta)$, and the Claim 31 enables one to pass to the limit. As a conclusion we obtain

$$\mathcal{J}_T(f_\infty) = \min_{f\in\overline{\mathcal{M}}(\Omega,\delta)} \mathcal{J}_T(f). \tag{A.10}$$

□

A.2 Proof of Lemma 20

Proof of Lemma 20. First of all, the fact that Theorem III implies the conclusion of Lemma 20 is trivial. Conversely, assume Lemma 20 holds. Let us define the functional

$$\mathcal{G}_T : \overline{\mathcal{M}}(\Omega)\backslash\{f^*\} \ni f \mapsto \frac{\mathcal{J}_T(f^*) - \mathcal{J}_T(f)}{\|f-f^*\|^2_{L^1(\Omega)}}. \tag{A.11}$$

Proving Theorem III is equivalent to proving

$$\inf_{f\in\overline{\mathcal{M}}(\Omega)} \mathcal{G}_T(f) > 0. \tag{A.12}$$

We consider a minimising sequence $\{f_k\}_{k\in\mathbb{N}^*}$ for $\mathcal{G}_T$. Let us consider a closure point f_∞ of this sequence. If $f_\infty \neq f^*$ then from Claim 31 we have

$$\lim_{k\to\infty} \mathcal{J}_T(f^*) - \mathcal{J}_T(f_k) = \mathcal{J}_T(f^*) - \mathcal{J}_T(f^\infty) = A > 0$$

where the last inequality is strict because f^* is the unique maximiser of $\mathcal{J}_T$ (Theorem I) and so, using the trivial bound $\|f-f^*\|_{L^1(\Omega)} \leqslant 2\,\mathrm{Vol}(\Omega)$ we obtain

$$\lim_{k\to\infty} \mathcal{G}_T(f_k) \geqslant \frac{\mathcal{J}_T(f^*) - \mathcal{J}_T(f_\infty)}{2\,\mathrm{Vol}(\Omega)} > 0. \tag{A.13}$$

If on the other hand we have $f_\infty = f^*$ then, f^* being an extreme point of the convex set $\overline{\mathcal{M}}(\Omega)$, the convergence $f_k \underset{k\to\infty}{\rightharpoonup} f^*$ is strong in $L^1(\Omega)$ ([22, Proposition 2.2.1]). As a consequence we can define the sequence

$$\forall k \in \mathbb{N}\,, \delta_k := \|f_k - f^*\|_{L^1(\Omega)}. \tag{A.14}$$

It then follows that there holds

$$\liminf_{k\to\infty} \mathcal{G}_T(f_k) \geqslant \liminf_{k\to\infty} \frac{\mathcal{J}_T(f^*) - \mathcal{J}_T(f_{\delta_k})}{\delta_k^2} \tag{A.15}$$

$$> 0 \tag{A.16}$$

if Estimate (5.9) holds, and this concludes the proof of the equivalence between the two results. □

A.3 Proof of the coercivity estimate-Lemma 28

Proof of Lemma 28. To alleviate the proof, we first note that such a continuity is standard to prove for the term $C(\mathrm{Vol}(\mathbb{B}^*_{t\Phi}) - V_0)^2$ and we hence omit it. Let us then define the function $\overline{j}_\Phi := \mathcal{L}_{\mathbb{B}^*}$ and prove this estimate for this term. First of all, standard computations show that $\overline{j}_\Phi$ is twice differentiable in the sense of shapes. Furthermore, the second order shape derivatives at a given shape E such that $E \cap \partial\Omega = \emptyset$, in the direction Φ, where $\Phi \in W^{2,p}(\Omega;\mathbb{R}^2)$ is compactly supported in Ω, is given by

$$\mathcal{L}''_{\mathbb{B}^*}(E)[\Phi,\Phi] = \iint_{(0;T)\times\partial E} p'\left(\Phi\cdot\nu\right) + \int_{\partial E}\left(\Phi\cdot\nu\right)^2\left(\mathscr{H}\int_0^T p_E + \int_0^T \frac{\partial p_E}{\partial\nu}\right) - \overline{\Psi}_{\partial\mathbb{B}^*}\int_{\partial E}\mathscr{H}\left(\Phi\cdot\nu\right)^2 \tag{A.17}$$

where:

1. $\mathscr{H}$ is the mean curvature of ∂E,
2. p_E solves
$$\begin{cases} \frac{\partial p_E}{\partial t} + \Delta p_E = -u_E \text{ in } (0;T)\times\Omega\,,\\ p_E(T,\cdot) = 0\,,\\ p_E(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega, \end{cases} \tag{A.18}$$
3. u' solves
$$\begin{cases} \frac{\partial u'}{\partial t} - \Delta u' = 0 \text{ in } (0;T)\times\Omega\,,\\ u'(0,\cdot) = 0\,,\\ [\![\partial_\nu u']\!] = -1 \text{ on } (0;T)\times\partial E,\\ u'(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega, \end{cases} \tag{A.19}$$
4. and p' solves
$$\begin{cases} \frac{\partial p'}{\partial t} + \Delta p' = -u' \text{ in } (0;T)\times\Omega\,,\\ p'(T,\cdot) = 0\,,\\ p'(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega, \end{cases} \tag{A.20}$$
5. and
$$\overline{\Psi}_{\partial\mathbb{B}^*} = \int_0^T p^*(t,\cdot)\Big|_{\partial\mathbb{B}^*}$$
is the Lagrange multiplier associated with the volume constraint.

Let us now assume that $E = \mathbb{B}^*_{\tau\Phi}$ for a fixed compactly supported vector field $\Phi \in W^{2,p}$ normal to $\partial\mathbb{B}^*$, and for some $\tau \in (0;1)$. We first use a change of variables: let us define

$$T_\tau := Id + \tau\Phi\,, J_{\Sigma,\tau}(\Phi) := \det(\nabla T_\tau)\left|({}^T\nabla T_\tau^{-1})\nu\right|\,,$$
$$J_{\Omega,\tau} := \det(\nabla\tau\Phi)\,, A_\tau := J_{\Omega,\tau}(\Phi)(Id+\tau\nabla\Phi)^{-1}(Id+\tau{}^T\nabla\Phi)^{-1},$$

$u_\tau := u_{\mathbb{B}^*_{\tau\Phi}}$ and $\hat{u}_\tau := u_{\mathbb{B}^*_{\tau\Phi}} \circ T_\tau$. By a change of variable, we see that $\hat{u}_\tau$ satisfies

$$\begin{cases} J_{\Omega,\tau}\frac{\partial\hat{u}_\tau}{\partial\tau} - \nabla\cdot(A_\tau\nabla\hat{u}_\tau) = J_{\Omega,\tau}f^* \text{ in } (0;T)\times\Omega\,,\\ \hat{u}_\tau(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega\,,\\ \hat{u}_\tau(0,\cdot) = 0, \end{cases} \tag{A.21}$$

while the function $\hat{u}'_{\mathbb{B}^*_{\tau\Phi},\Phi} := u'_{E,\tau\Phi}\circ T_\tau$ which we abbreviate as $\hat{u}'_\tau$, satisfies

$$\begin{cases} J_{\Omega,\tau}\frac{\partial \hat{u}'_\tau}{\partial t} - \nabla\cdot(A_\tau \nabla \hat{u}'_\tau) = 0 \text{ in } (0;T)\times\Omega\,, \\ [\![A_\tau \frac{\partial \hat{u}'_\tau}{\partial \nu}]\!]_{\partial\mathbb{B}^*} = -J_{\Sigma,\tau}\left(\Phi\cdot\nu\right)\,, \\ \hat{u}'_\tau = 0 \text{ on } \partial\Omega. \end{cases} \tag{A.22}$$

With the same notation, $\hat{p}_\tau\,,\hat{p}'_\tau$ satisfy

$$\begin{cases} J_{\Omega,\tau}\frac{\partial \hat{p}_\tau}{\partial \tau} + \nabla\cdot(A_\tau \nabla \hat{p}_\tau) = -J_{\Omega,\tau}\hat{u}_\tau \text{ in } (0;T)\times\Omega\,, \\ \hat{p}_\tau(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega\,, \\ \hat{p}_\tau(T,\cdot) = 0, \end{cases} \tag{A.23}$$

and

$$\begin{cases} J_{\Omega,\tau}\frac{\partial \hat{p}'_\tau}{\partial t} + \nabla\cdot(A_\tau \nabla \hat{p}'_\tau) = -J_{\Omega,\tau}\hat{u}'_\tau \text{ in } (0;T)\times\Omega \\ \hat{p}'_\tau = 0 \text{ on } \partial\Omega\,, \\ \hat{p}'_\tau(T,\cdot) = 0. \end{cases} \tag{A.24}$$

Finally, we set $\hat{\mathscr{H}_\tau} := \mathscr{H}_\tau\circ T_\tau$, $\mathscr{H}_\tau$ being the mean curvature of $\mathbb{B}^*_{\tau\Phi}$.

Using these notations, the difference which has to be controlled is hence

$$\mathcal{L}''_{\mathbb{B}^*}(\mathbb{B}^*_{\tau\Phi})[\Phi,\Phi] - \mathcal{L}''_{\mathbb{B}^*}(\mathbb{B}^*)[\Phi,\Phi] = \iint_{(0;T)\times\partial\mathbb{B}^*}(\Phi\cdot\nu)\left\{J_{\Sigma,\tau}\hat{p}'_\tau - p'_{\mathbb{B}^*}\right\} \tag{$\mathbf{R_1}(\tau,\Phi)$}$$

$$+ \iint_{(0;T)\times\partial\mathbb{B}^*} J_{\Sigma,\tau}\hat{\mathscr{H}_\tau}\left(\hat{p}_\tau - p_{\mathbb{B}^*}\right)(\Phi\cdot\nu)^2 \tag{$\mathbf{R_2}(\tau,\Phi)$}$$

$$+ \iint_{(0;T)\times\partial\mathbb{B}^*}\left(J_{\Sigma,\tau}\frac{\partial\hat{p}_\tau}{\partial\nu} - \frac{\partial p_{\mathbb{B}^*}}{\partial\nu}\right)(\Phi\cdot\nu)^2\,. \tag{$\mathbf{R_3}(\tau,\Phi)$}$$

We will control each of these three terms separately and we first recall several geometric estimates from [16].

Proposition 33 (Geometric estimates, [16, Lemma 4.8])**.** *For any $p\in(1;+\infty)$, for any $\Phi\in\mathcal{X}_1(\mathbb{B}^*)\cap W^{2,p}(\Omega;\mathbb{R}^2)\cap W^{1,\infty}(\Omega;\mathbb{R}^2)$ such that $\|\Phi\|_{W^{1,\infty}}\leqslant M_0$ fixed, there exists a constant M_p such that, for any $\tau\in(0;1)$:*

- $$\|\hat{J}_{\Sigma,\tau}-1\|_{L^\infty(\partial\mathbb{B}^*)}\leqslant M_p\|\Phi\cdot\nu\|_{W^{1,\infty}(\partial\mathbb{B}^*)}. \tag{A.25}$$
- $$\|\hat{\mathscr{H}_\tau}-\mathscr{H}_{\mathbb{B}^*}\|_{L^p(\partial\mathbb{B}^*)}\leqslant M_p\|\Phi\cdot\nu\|_{W^{2,p}(\partial\mathbb{B}^*)}. \tag{A.26}$$
- $$\|A_\tau - Id\|_{L^\infty}+\|A_\tau - Id\|_{W^{1,p}(\Omega)}\leqslant M_P\|\Phi\|_{W^{2,p}(\mathbb{B}^*)} \tag{A.27}$$

Control of ($\mathbf{R_2}(\tau,\Phi)$)**-**($\mathbf{R_3}(\tau,\Phi)$) Our goal is to obtain the existence of a constant $M>0$ and of a modulus of continuity η such that

$$|\mathbf{R_2}(\tau,\Phi)|+|\mathbf{R_3}(\tau,\Phi)|\leqslant M\eta(\|\Phi\|_{W^{2,p}(\Omega)})\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)}. \tag{A.28}$$

From Proposition 33 and standard Schauder estimates, such an estimate follows if there exists a modulus of continuity η such that, for some $\beta\in(0;1)$,

$$\|u_\tau - u^*\|_{\mathscr{C}^{0,\beta}((0;T)\times\Omega)}\leqslant M\eta(\|\Phi\|_{W^{2,p}}). \tag{A.29}$$

In turn, using Proposition 33 and the Hölder continuity of u_τ (Proposition 6), (A.29) is implied by the following: there exist a constant $M > 0$ and a modulus of continuity η such that

$$\|\hat{u}_\tau - u^*\|_{\mathscr{C}^{0,\beta}((0;T)\times\Omega)} \leqslant M\eta\left(\|\Phi\|_{W^{2,p}}\right). \tag{A.30}$$

Proof of (A.30). Straightforward computations show that $z_\tau := \hat{u}_\tau - u^*$ solves

$$\begin{cases} J_{\Omega,\tau}\frac{\partial z_\tau}{\partial \tau} - \nabla\cdot(A_\tau\nabla z_\tau) = f^*(J_{\Omega,\tau}-1) + \nabla\cdot((A_\tau - 1)\nabla u^*) + (J_{\Omega,\tau}-1)\frac{\partial u_{\mathbb{B}^*}}{\partial t} \text{ in } (0;T)\times\Omega\,, \\ z_\tau(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega\,, \\ z_\tau(0,\cdot) = 0. \end{cases} \tag{A.31}$$

Standard L^p estimates imply that for any $p\in(1;+\infty)$ there exists a constant M_p such that

$$\|z_\tau\|_{W^{1,p}((0;T)\times\Omega)} \leqslant M_p\left(\|\Phi\|_{W^{2,p}} + \|\Phi\|_{W^{1,\infty}}\right) \tag{A.32}$$

so that Sobolev embeddings conclude the proof. □

Control of ($\mathbf{R_1}(\tau,\Phi)$) To control this term it suffices to show that there exists a constant M_p such that

$$\left\|\int_0^T \hat{p}'_\tau - p'_{\mathbb{B}^*}\right\|_{L^2(\partial\mathbb{B}^*)} \leqslant M_p\|\Phi\|_{W^{2,p}}\|\Phi\cdot\nu\|_{L^2(\partial\mathbb{B}^*)}. \tag{A.33}$$

By the continuity of the trace it follows that it suffices to prove that

$$\int_0^T \|\hat{p}'_\tau - p'_{\mathbb{B}^*}\|^2_{W^{1,2}(\Omega)} \leqslant M_p\|\Phi\|^2_{W^{2,p}}\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)}. \tag{A.34}$$

Let us define $z'_\tau := \hat{p}'_\tau - p'_{\mathbb{B}^*}$. Straightforward computations show that

$$\begin{cases} J_{\Omega,\tau}\frac{\partial z'_\tau}{\partial t} + \nabla\cdot(A_\tau\nabla z'_\tau) = -\hat{u}'_\tau + u'_{\mathbb{B}^*} - \nabla\cdot((A_\tau-1)\nabla p'_{\mathbb{B}^*}) + (J_{\Omega,\tau}-1)\frac{\partial p'_{\mathbb{B}^*}}{\partial t}\,, \\ z'_\tau(T,\cdot) = 0\,, \\ z'_\tau(t,\cdot) = 0 \text{ on } (0;T)\times\Omega \end{cases} \tag{A.35}$$

and so (A.34) follows from standard $W^{1,2}$ estimates if we can prove that

$$\int_0^T \left\|\frac{\partial p'_{\mathbb{B}^*}}{\partial t}\right\|^2_{L^2(\Omega)} + ||\nabla p'_{\mathbb{B}^*}||^2_{L^2(\Omega)} \leqslant M_p\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)} \tag{A.36}$$

and that

$$\int_0^T \|u'_{\mathbb{B}^*} - \hat{u}'_\tau\|^2_{L^2(\Omega)} \leqslant M_p\eta(\|\Phi\|_{W^{2,p}})^2\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)} \tag{A.37}$$

for some constant M_p. To prove these two inequalities, we begin with a first estimate:

$$\iint_{(0;T)\times\Omega}\left(\frac{\partial u'_{\mathbb{B}^*}}{\partial t}\right)^2 + \iint_{(0;T)\times\Omega}(u'_{\mathbb{B}^*})^2 \leqslant M\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)}. \tag{A.38}$$

Proof of (A.38). For the sake of readability, we abbreviate $u'_{\mathbb{B}^*}$ as u' here. Multiplying the equation on u' by u' and integrating by parts, we obtain

$$\int_\Omega (u')^2(T,\cdot) + \iint_{(0;T)\times\Omega}|\nabla u'|^2 = \iint_{(0;T)\times\partial\mathbb{B}^*}(\Phi\cdot\nu)\,u' \leqslant \|\Phi\cdot\nu\|_{L^2(\partial\mathbb{B}^*)}\|u'(t,\cdot)\|_{L^2((0.T)\times\partial\mathbb{B}^*)}. \tag{A.39}$$

By continuity of the trace and by the Poincaré inequality, we obtain

$$\iint_{(0;T)\times\Omega} |\nabla u'|^2 \leqslant \int_\Omega (u')^2(T,\cdot) + \iint_{(0;T)\times\Omega} |\nabla u'|^2 \leqslant M\|\Phi\cdot\nu\|_{L^2(\partial\mathbb{B}^*)}\|\nabla u'\|_{L^2((0;T)\times\Omega)}. \quad \text{(A.40)}$$

This first gives

$$\|\nabla u'\|_{L^2((0;T)\times\Omega)} \leqslant M\|\Phi\cdot\nu\|_{L^2(\partial\mathbb{B}^*)} \quad \text{(A.41)}$$

which in turn implies

$$\iint_{(0;T)\times\Omega} (u')^2 \leqslant MT\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)}. \quad \text{(A.42)}$$

To obtain the estimates on $\iint_{(0;T)\times\Omega}\left(\frac{\partial u'_{\mathbb{B}^*}}{\partial t}\right)^2$ we proceed as follows: using $\frac{\partial u'_{\mathbb{B}^*}}{\partial t}$ as a test function we obtain

$$\iint_{(0;T)\times\Omega}\left(\frac{\partial u'_{\mathbb{B}^*}}{\partial t}\right)^2 + \int_\Omega |\nabla u'_{\mathbb{B}^*}|^2(T,\cdot) = \iint_{(0;T)\times\partial\mathbb{B}^*} (\Phi\cdot\nu)\,\frac{\partial u'_{\mathbb{B}^*}}{\partial t} \quad \text{(A.43)}$$

$$= \int_{\partial\mathbb{B}^*} (\Phi\cdot\nu)\,(u'_{\mathbb{B}})(T,\cdot) \quad \text{(A.44)}$$

$$\leqslant M\|\Phi\cdot\nu\|_{L^2(\partial\mathbb{B}^*)}\|\nabla u'_{\mathbb{B}^*}\|_{L^2(\Omega)}(T,\cdot), \quad \text{(A.45)}$$

which gives the conclusion: indeed, we apply Young's inequality $2ab \leqslant \varepsilon a^2 + \frac{b^2}{\varepsilon}$ to the right hand side and conclude.

□

Let us then turn to (A.36)

Proof of (A.36). Let us define $q'(t,\cdot) := p'_{\mathbb{B}^*}(T-t,\cdot)$. Then q' satisfies, with $u' = u'_{\mathbb{B}^*}$,

$$\begin{cases} \frac{\partial q'}{\partial t} - \Delta q' = u'(T-t,\cdot) \text{ in } (0;T)\times\Omega\,, \\ q'(0,\cdot) = 0\,, \\ q'(t,\cdot) = 0 \text{ on } (0;T)\times\partial\Omega, \end{cases} \quad \text{(A.46)}$$

so that by standard parabolic estimates for the heat equation we obtain, for some constant M

$$\iint_{(0;T)\times\Omega}\left(\frac{\partial p'_{\mathbb{B}^*}}{\partial t}\right)^2 = \iint_{(0;T)\times\Omega}\left(\frac{\partial q'}{\partial t}\right)^2 \leqslant MT\iint_{(0;T)\times\Omega}(u')^2 \leqslant MT\|\Phi\cdot\nu\|^2_{L^2(\partial\mathbb{B}^*)}. \quad \text{(A.47)}$$

□

Finally, let us prove (A.37).

Proof of (A.37). Let us recall that the weak formulation of the equations on $\hat{u}'_\tau$ and on $u' := u'_{\mathbb{B}^*}$ are: for any test function v,

$$\iint_{(0;T)\times\Omega} J_{\Omega,\tau}\frac{\partial \hat{u}'_\tau}{\partial t}v + \iint_{(0;T)\times\Omega}\langle A_\tau\nabla\hat{u}'_\tau, \nabla v\rangle = \iint_{(0;T)\times\partial\mathbb{B}^*} J_{\Sigma,\tau}\,(\Phi\cdot\nu)\,v \quad \text{(A.48)}$$

and

$$\iint_{(0;T)\times\Omega}\frac{\partial u'}{\partial t}v + \iint_{(0;T)\times\Omega}\langle\nabla u', \nabla v\rangle = \iint_{(0;T)\times\partial\mathbb{B}^*}(\Phi\cdot\nu)\,v. \quad \text{(A.49)}$$

Substracting these two weak formulations and setting $w'_\tau := \hat{u}'_\tau - u'_{\mathbb{B}^*}$ we obtain, on w'_τ, the weak formulation

$$\iint_{(0;T)\times\Omega} J_{\Omega,\tau}\frac{\partial w'_\tau}{\partial t}v + \iint_{(0;T)\times\Omega}\langle A_\tau\nabla w'_\tau, \nabla v\rangle = \iint_{(0;T)\times\partial\mathbb{B}^*}(J_{\Sigma,\tau}-1)v + \iint_{(0;T)\times\Omega}(J_{\Omega,\tau}-1)\frac{\partial u'}{\partial t}v + \iint_{(0;T)\times\Omega}\langle (A_\tau - Id)\nabla u', \nabla v\rangle. \quad (A.50)$$

Using $v = w'_\tau$ as a test function we obtain in the same way, using Poincaré inequality and the continuity of the trace for any t, up to a multiplicative constant that does not depend on Φ,

$$\int_\Omega (w'_\tau)^2(t,\cdot) + \iint_{(0;t)\times\Omega}|\nabla w'_\tau|^2 \leqslant \|J_{\Sigma,\tau}-1\|_{L^2(\partial\mathbb{B})}\|\nabla w'_\tau\|_{L^2((0;t)\times\Omega)}(t) \quad (A.51)$$

$$+ \|J_{\Omega,\tau}-1\|_{L^\infty}\left\|\frac{\partial u'}{\partial t}\right\|_{L^2((0;t)\times\Omega)}\|\nabla w'_\tau\|_{L^2((0;t)\times\Omega)} \quad (A.52)$$

$$+ \|A_\tau - 1\|_{L^\infty}\|\nabla u'\|_{L^2((0;t)\times\Omega)}\|\nabla w'_\tau\|_{L^2((0;t)\times\Omega)}. \quad (A.53)$$

The conclusion then follows. □

□

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