# Determining the anisotropic traction state in a membrane by boundary measurements

G. Alessandrini *     E. Cabib †

## 1 Introduction

Consider an elastic thin membrane which occupies a planar region represented by a simply connected bounded open set $\Omega \subset \mathbb{R}^2$. In the same plane a vector force field $T$ is applied on its boundary $\partial\Omega$, so that the membrane is subject to a distributed *pretraction state*, expressed by a positive and symmetric tensor $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, which satisfies the plane equilibrium equations

$$\operatorname{div} \sigma = 0 \ , \text{ in } \Omega \ , \tag{1.1}$$
$$\sigma\nu = T \ , \text{ on } \partial\Omega \ , \tag{1.2}$$

where $\nu$ is the outer unit normal to $\partial\Omega$. The transverse displacement $u(x)$ of the membrane will be governed by the equation

$$-\operatorname{div}(\sigma\nabla u) = f \ , \text{ in } \Omega \ , \tag{1.3}$$
$$u = \varphi \ , \text{ on } \partial\Omega \ , \tag{1.4}$$

where $f(x)$ represents the distributed transverse load applied to it, and $\varphi$ represents the prescribed transverse displacement at the boundary.

In this note we wish to investigate the inverse problem of determining the plane traction state tensor $\sigma$ from boundary measurements on displacements and on the corresponding forces. As an initial attempt, since the external load $f$ has no influence on the pretraction state $\sigma$, and in analogy with many other well-known inverse boundary problems, see for instance [2, 8, 15], it seems natural to treat the case when $f = 0$ in (1.3), and consider as available data an arbitrary transverse displacement $\varphi$ on the boundary and the corresponding transverse load on the boundary, namely the reaction of the boundary constraints

$$\Lambda_\sigma\varphi = \sigma\nabla u \cdot \nu \ . \tag{1.5}$$

Hence, fixing any $\varphi \in H^{1/2}(\partial\Omega)$, if we denote by $u \in H^1(\Omega)$ the weak solution to the Dirichlet problem

$$\operatorname{div}(\sigma\nabla u) = 0 \ , \text{in } \Omega \ , \tag{1.6}$$
$$u = \varphi \ , \text{on } \partial\Omega \ , \tag{1.7}$$

we introduce the Dirichlet-to-Neumann (D-N) map as the bounded linear operator

$$\Lambda_\sigma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \ , \tag{1.8}$$

defined, in weak terms, by the formula

$$< \Lambda_\sigma\varphi, v_{|\partial\Omega} >= \int_\Omega \sigma\nabla u \cdot \nabla v \ , \quad \text{for every } v \in H^1(\Omega) \ , \tag{1.9}$$

*Dipartimento di Matematica e Informatica, Via Valerio, 12 - 34127 Trieste, Italy, email: alessang@univ.trieste.it. Work supported in part by MIUR, PRIN n. 2004011204 and by GNAMPA, INdAM Progetto *Problemi al contorno inversi* 2006 .

†Dipartimento di Ingegneria Civile, Via delle Scienze, 208 - 33100 Udine, Italy, email: cabib@uniud.it

where $v_{|\partial\Omega}$ denotes the trace on $\partial\Omega$ of $v \in H^1(\Omega)$.

Thus we examine here the problem of determining $\sigma$ from the knowledge of $\Lambda_\sigma$. The peculiarity of this problem is that, by its own nature, the tensor $\sigma$ is anisotropic, and since Tartar's example, as reported in [11], it is well-known that a general anisotropic tensor $\sigma$ cannot be uniquely determined by the D-N map $\Lambda_\sigma$. In this case however, we shall see, in the next Section 2 that we can take advantage of the null divergence condition (1.1) and obtain the uniqueness in such restricted class of genuinely anisotropic tensors, see Theorem 2.1.

In Section 3 we also consider the question of stability, that is of the continuous dependence upon the data. And although the present results are very preliminary, they show up interesting phenomena, which are markedly different from those available for the well-known *inverse conductivity problem.* In fact, on one hand, we prove, Theorem 3.2, a qualitative form of stability, when a very weak topology is assigned on the class of tensors, namely the topology of *G-convegence.* And, on the other hand, we show that a Lipschitz stability bound holds for the mean value of $\sigma$.

# 2 Uniqueness

For any given $K \geq 1$ we consider the class of tensors

$$M_K = \{\sigma \in L^\infty(\Omega, M^{2\times 2}) \mid K^{-1}|\xi|^2 \leq \sigma\xi \cdot \xi \leq K|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^2 \ \} \ , \tag{2.1}$$

here $M^{2\times 2}$ denotes the set of $2 \times 2$ symmetric matrices. Let us also introduce

$$\Sigma_K = \{\sigma \in M_K \mid \operatorname{div} \sigma = 0\} \ , \tag{2.2}$$

where the null divergence condition $\operatorname{div} \sigma = 0$ is meant in the weak sense

$$\int_\Omega \sigma \nabla v = 0 \qquad \text{for every } v \in H_0^1(\Omega) \ . \tag{2.3}$$

Let us also introduce

$$M = \cup_{K\geq 1} M_K \ , \ \Sigma = \cup_{K \geq 1} \Sigma_K \ . \tag{2.4}$$

Given a $W^{1,2}$ mapping $\Phi : \Omega \to D \subset \mathbb{R}^2$ we denote its Jacobian matrix as follows

$$D\Phi(x) = \left\{ \frac{\partial \Phi_i(x)}{\partial x_j} \right\} \qquad i,j = 1,2 \ . \tag{2.5}$$

We recall that $\Phi$ is said to be *quasiconformal* if, for some $Q \geq 1$ it satisfies

$$||D\Phi||^2 \leq Q \det D\Phi \text{ a.e. in } \Omega \ , \tag{2.6}$$

and it is invertible, see for instance Ahlfors [1]. Here, for any matrix $A$, we denote $||A||^2 = \operatorname{tr} AA^T$ and the suffix $T$ denotes transpose.

For any tensor $\sigma \in M_K$ and any quasiconformal mapping $\Phi$, we introduce

$$T_\Phi \sigma(y) = \frac{D\Phi \sigma D\Phi^T}{\det D\Phi}(\Phi^{-1}(y)) \ , \qquad \text{for every } y \in D \ . \tag{2.7}$$

This new tensor defined in $D$, called the *push-forward* of $\sigma$ by $\Phi$, is again symmetric and satisfies the ellipticity condition with some possibly new $K \geq 1$. Moreover, one can verify that such operation preserves the bilinear Dirichlet form associated to $\sigma$, that is

$$\int_\Omega \sigma \nabla u \cdot \nabla v = \int_D T_\Phi \sigma \nabla(u \circ \Phi^{-1}) \cdot \nabla(v \circ \Phi^{-1}) \qquad \text{for every } u,v \in H^1(\Omega) \ . \tag{2.8}$$

**Theorem 2.1.** *$\Lambda_\sigma$ uniquely determines $\sigma$ among all tensors in $\Sigma$.*

The proof will be a consequence of the following two Lemmas.

**Lemma 2.2.** *Let $\sigma \in \Sigma$. Then the tensor*

$$T_\Phi \sigma (D\Phi^{-1})^T \tag{2.9}$$

*is divergence free in $D$.*

*Proof.* Condition (2.3) can be rewritten as

$$\int_\Omega \sigma \nabla x_j \cdot \nabla v = 0 \qquad \text{for every } v \in H_0^1(\Omega)\ ,\ j = 1, 2\ . \tag{2.10}$$

By (2.8) one computes

$$\int_D T_\Phi \sigma \nabla((\Phi^{-1})_j) \cdot \nabla(v \circ \Phi^{-1}) = 0 \ \text{ for every } v \in H_0^1(\Omega)\ ,\ j = 1, 2\ , \tag{2.11}$$

and the thesis follows. □

**Lemma 2.3.** *Let $\sigma \in M$ and suppose that for a given mapping $\Phi$ we have $\operatorname{div} T_\Phi \sigma = 0$ in $D$. Then*

$$\operatorname{div}(\sigma D\Phi^T) = 0\ . \tag{2.12}$$

*Proof.* The proof follows immediately from Lemma 2.2, just by reversing the roles of $\Phi$ , $\Phi^{-1}$ and of $\sigma$, $T_\Phi\sigma$, respectively. □

We are now in a position to prove our main result.

*Proof of Theorem 2.1.* By the results of Astala, Päivärinta and Lassas [4, Theorem 1], which have extended to the $L^\infty$ setting those of Sylvester [18] and Nachman [14], we have that $\Lambda_\sigma$ determines uniquely the class

$$E_\sigma = \{\sigma' \in M | \sigma' = T_\Phi \sigma,\ \text{with } \Phi : \Omega \to \Omega \text{ quasiconformal and such that } \Phi_{|\partial\Omega} = I\}\ . \tag{2.13}$$

The class $E_\sigma$ contains at most one divergence free element. In fact, if $\Phi$ is a quasiconformal mappings which fixes the boundary, and such that $\operatorname{div} T_\Phi \sigma = 0$ then, by Lemma 2.3, we have

$$\begin{cases} \operatorname{div}(\sigma D\Phi^T) = 0 & \text{in } \Omega\ , \\ \Phi = I & \text{on } \partial\Omega\ . \end{cases} \tag{2.14}$$

Note that this system, is formed by two uncoupled Dirichlet problems for the two components of the mapping $\Phi$. On the other hand we observe that, if $\sigma$ is divergence free, then the identity mapping $I$ is itself a solution to (2.14) and by uniqueness for the Dirichlet problem, we obtain $\Phi = I$ on $\Omega$. □

**Remark 2.4.** *It is worth mentioning, that the crucial fact used in this proof is that all linear functions are solutions to the elliptic equation* (1.3)*, this is a condition on $\sigma$ which is in fact equivalent to the null divergence condition* (2.3)*. Indeed this property has been used already in a study on optimization of tension structures in the different context of variational problems and G-convergence, see [6], [7] and also Section 3 below.*

*A further application of this property of linear functions is the possibility to identify the traction $T$ applied on the boundary. For every $\xi \in \mathbb{R}^2$ let $\varphi_\xi(x) = \xi \cdot x$ be the Dirichlet data. Since the corresponding solution is $u_\xi(x) = \xi \cdot x$ all over $\Omega$, we have*

$$\Lambda_\sigma \varphi_\xi = \sigma \nabla u_\xi \cdot \nu = \sigma \xi \cdot \nu = \sigma \nu \cdot \xi = T \cdot \xi\ , \tag{2.15}$$

*whose knowledge for every $\xi \in \mathbb{R}^n$ is equivalent to the knowledge of $T$.*

*If the same argument is applied to the particular case of a square network $Q = [0, 1] \times [0, 1]$, that is, a portion of fabrics made by two families of parallel elastic strings which cross orthogonally,* *the identification of $\sigma$ is immediate. The particular situation leads to define as admissible all the tensors of the form*

$$\sigma(x) = \begin{pmatrix} \sigma_1(x_2) & 0 \\ 0 & \sigma_2(x_1) \end{pmatrix} .$$

*with $K^{-1} \leq \sigma_1, \sigma_2 \leq K$. Consider the Dirichlet data $\varphi(x) = x_1$ on $\partial Q$, so that also $u(x) = x_1$ on $Q$, and the corresponding $\psi(x_2) = \Lambda_\sigma \varphi$ on the edge $x_1 = 1$ where $\nu = (1, 0)$. Then we have*

$$\sigma_1(x_2) = \sigma(1, x_2)\nabla u \cdot \nu = \psi(x_2) ,$$

*likewise, $\sigma_2(x_1)$ can be identified as well.*

# 3 Stability

This section is devoted to the continuity properties of the inverse of the map

$$\Sigma_K \ni \sigma \to \Lambda_\sigma \in \mathscr{L}(H^{1/2}(\Omega), H^{-1/2}(\Omega))$$

when we assign to $\Sigma_K$ the topology of G-convergence. Let us recall here the basic notions and some important properties of the G-convergence. A wide literature is available on this subject, we refer for example to the classical papers [10, 13, 16, 17] and to the book by Dal Maso [9] where G-convegence is cast in the more general theory of $\Gamma$-convegence.

**Definition 3.1.** *A sequence $\{\sigma_h\} \subset M_K$ is said to G-converge to $\sigma \in M_K$, and we write $\sigma_h \overset{G}{\to} \sigma$, if for every $f \in H^{-1}(\Omega)$ the corresponding sequence $\{u_h\} \subset H^1_0(\Omega)$ of solutions to the* inhomogeneous *problems*

$$-\operatorname{div}(\sigma_h \nabla u_h) = f \quad \text{in } \Omega \ , \ u_h = 0 \quad \text{on } \partial\Omega \ , \tag{3.1}$$

*converges weakly in $H^1_0(\Omega)$ to the solution $u \in H^1_0(\Omega)$ of the problem*

$$-\operatorname{div}(\sigma \nabla u) = f \quad \text{in } \Omega \ , \ u = 0 \quad \text{on } \partial\Omega \ . \tag{3.2}$$

It is well known that G-convergence is induced by a compact metrizable topology on $M_K$, [17, Remark 4].

It is also worth recalling that the $L^1_{\text{loc}}$-strong convergence implies the G-convergence, [16, Proposition 5], [17, Remark 11].

**Theorem 3.2.** *Given $K \geq 1$, the mapping $\Sigma_K \ni \sigma \to \Lambda_\sigma \in \mathscr{L}(H^{1/2}(\Omega), H^{-1/2}(\Omega))$ has a continuous inverse when $\Sigma_K$ is endowed with the topology of G-convergence.*

*Proof.* First we recall that [7], by the characterization of $\Sigma_K$ as the subclass of those $\sigma \in M_K$ for which all linear functions are solutions to (1.6), implies that $\Sigma_K$ is a closed set in the G-topology and hence it is compact. Let $\{\sigma_h\} \subset \Sigma_K$ and $\sigma \in \Sigma_K$ be such that $\|\Lambda_{\sigma_h} - \Lambda_\sigma\| \to 0$. By the above mentioned compactness , there exists a subsequence $\{\sigma_{r_h}\}$ of $\{\sigma_h\}$ such that $\sigma_{r_h} \overset{G}{\to} \sigma' \in \Sigma_K$ and we prove $\sigma' = \sigma$. For any $\varphi \in H^{1/2}(\partial\Omega)$, let $u_h$, $u'$ the solutions to (1.6) when $\sigma$ is replaced with with $\sigma_h$, $\sigma'$, respectively. Then, by convergence of the energies [17], we have

$$< \Lambda_{\sigma_{r_h}} \varphi, \varphi >= \int_\Omega \sigma_{r_h} \nabla u_{r_h} \cdot \nabla u_{r_h} \to \int_\Omega \sigma' \nabla u' \cdot \nabla u' =< \Lambda_{\sigma'} \varphi, \varphi > , \tag{3.3}$$

on the other hand, $\Lambda_{\sigma_h} \to \Lambda_\sigma$, therefore $< \Lambda_{\sigma'} \varphi, \varphi >=< \Lambda_\sigma \varphi, \varphi >$ for every $\varphi \in H^{1/2}(\partial\Omega)$. From the uniqueness Theorem 2.1, we get $\sigma' = \sigma$. The above argument applies to any subsequence of $\{\sigma_h\}$. Thus we have obtained that for any subsequence of $\{\sigma_h\}$ there is a sub-subsequence which G-converges to $\sigma$, and hence, the full sequence $\{\sigma_h\}$ must G-converge to $\sigma$. □

**Remark 3.3.** *It is worthwhile to compare this result with the case of the inverse conductivity problem, that is when the unknown $\sigma \in M_K$ is a-priori known to be isotropic, that is $\sigma = \gamma I$ where $\gamma \in L^\infty(\Omega)$ is a scalar function satisfying $K^{-1} \leq \gamma \leq K$ and $I$ denotes the identity matrix. Indeed for the inverse conductivity problem, stability with respect to G-convergence fails, see [12] for related arguments. In fact, as is well-known, Marino and Spagnolo [13] proved that there exist a constant $c > 1$, depending only on the space dimension $n$ (in our case $n = 2$), such that any tensor in $M_{K/c}$ can be approximated in the sense of G-convegence by isotropic tensors of $M_K$. Hence if we had stability with respect to G-convergence for isotropic tensors, that would imply the uniqueness in the class $M_{K/c}$ of* anisotropic *tensors, which, by the above mentioned example of Tartar cannot hold true. Hence the stability result above, Theorem 3.2, is crucially based on the property of our set of admissible matrices, $\Sigma_K$, of being G-closed.*

We recall that in [6], and in [7] in a more general context, it was proved that on $\Sigma_K$ the G-convergence is equivalent to the $L^\infty(\Omega)$-weak* convergence. Therefore, as a consequence of Theorem 3.2, we also obtain that for every $\psi \in L^1(\Omega)$ and for every $i, j = 1, 2$ the functional $F$ defined by

$$F(\sigma) = \int_\Omega \psi \sigma_{i,j} \ , \ \sigma \in \Sigma_K \ , \tag{3.4}$$

depends continuously on $\Lambda_\sigma$.

In the very special case when, in (3.4), we choose $\psi \equiv \frac{1}{|\Omega|}$ a concrete stability estimate can be obtained. In fact, in the next Proposition we show that the average of $\sigma$, a quantity which can be interpreted as a global measure of the pretraction field, depends in a Lipschitz continuous fashion on the Dirichlet-to-Neumann map.

**Proposition 3.4.** *For any $\sigma, \sigma' \in \Sigma$ we have*

$$\left\| \frac{1}{|\Omega|} \int_\Omega (\sigma - \sigma') \right\| \leq (1 + (\operatorname{diam} \Omega)^2) \|\Lambda_\sigma - \Lambda_{\sigma'}\| \ . \tag{3.5}$$

*Proof.* Being linear functions solutions, we can use (1.9), with $u = x_i \ , v = x_j \ , i, j = 1, 2$, both for $\sigma$ and $\sigma'$. We obtain

$$\int_\Omega (\sigma - \sigma')_{ij} = < (\Lambda_\sigma - \Lambda_{\sigma'}) x_i, x_j > \ , \tag{3.6}$$

and the thesis follows by straightforward computations. □

**Remark 3.5.** *Also in this case it may be interesting to make a comparison with the inverse conductivity problem. In fact it is an open problem whether, for the average $\frac{1}{|\Omega|} \int_\Omega \gamma$ of an isotropic tensor $\sigma = \gamma I$, the Lipschitz stability in terms of the corresponding the Dirichlet-to-Neumann map holds true, see [3].*

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