LIPSCHITZ REGULARITY FOR DEGENERATE ELLIPTIC INTEGRALS WITH p, q-GROWTH

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Abstract. We establish the local Lipschitz continuity and the higher differentiability of vector-valued local minimizers of a class of energy integrals of the Calculus of Variations. The main novelty is that we deal with possibly degenerate energy densities with respect to the $x-$variable.

1. Introduction

The paper deals with the regularity of minimizers of integral functionals of the Calculus of Variations of the form

$$F(u) = \int_{\Omega} f(x, Du)\, dx \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open set, $u : \Omega \to \mathbb{R}^N$, $N \geq 1$, is a Sobolev map. The main feature of (1.1) is the possible degeneracy of the lagrangian $f(x, \xi)$ with respect to the $x-$variable. We assume that the Carathéodory function $f = f(x, \xi)$ is convex and of class C^2 with respect to $\xi \in \mathbb{R}^{N\times n}$, with $f_{\xi\xi}(x, \xi)$, $f_{\xi x}(x, \xi)$ also Carathéodory functions and $f(\cdot, 0) \in L^1(\Omega)$. We emphasize that the $N \times n$ matrix of the second derivatives $f_{\xi\xi}(x, \xi)$ not necessarily is uniformly elliptic and it may degenerate at some $x \in \Omega$.

In the vector-valued case $N > 1$ minimizers of functionals with general structure may lack regularity, see [17],[48],[42], and it is natural to assume a modulus-gradient dependence for the energy density; i.e. that there exists $g = g(x, t) : \Omega \times [0, +\infty) \to [0, +\infty)$ such that

$$f(x, \xi) = g(x, |\xi|). \tag{1.2}$$

Without loss of generality we can assume $g(x, 0) = 0$; indeed the minimizers of F are minimizers of $u \mapsto \int_{\Omega} (f(x, Du) - f(x, 0))\, dx$ too. Moreover, by (1.2) and the convexity of f, $g(x, t)$ is a non-negative, convex and increasing function of $t \in [0, +\infty)$.

As far as the growth and the ellipticity assumptions are concerned, we assume that there exist exponents p, q, nonnegative measurable functions $a(x), k(x)$ and a constant $L > 0$ such that

$$\begin{cases} a(x)\, (1 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \leq \langle f_{\xi\xi}(x, \xi)\lambda, \lambda\rangle \leq L\, (1 + |\xi|^2)^{\frac{q-2}{2}} |\lambda|^2, \quad 2 \leq p \leq q, \\ |f_{\xi x}(x, \xi)| \leq k(x)\, (1 + |\xi|^2)^{\frac{q-1}{2}} \end{cases} \tag{1.3}$$

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for a.e. $x \in \Omega$ and for every $\xi, \lambda \in \mathbb{R}^{N\times n}$. We allow the coefficient $a(x)$ to be zero, so that $(1.3)_1$ is a not uniform ellipticity condition. As proved in Lemma 2.2, $(1.3)_1$ implies the following possibly degenerate $p, q-$growth conditions for f, for some constant $c > 0$,

$$c\, a(x)\,(1+|\xi|^2)^{\frac{p-2}{2}}|\xi|^2 \le f(x,\xi) \le L(1+|\xi|^2)^{\frac{q}{2}}, \quad \text{a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^{N\times n}. \tag{1.4}$$

Our main result concerns the local Lipschitz regularity and the higher differentiability of the local minimizers of F.

Theorem 1.1. *Let the functional F in* (1.1) *satisfy* (1.2) *and* (1.3)*. Assume moreover that*

$$\frac{1}{a} \in L^s_{\mathrm{loc}}(\Omega), \qquad k \in L^r_{\mathrm{loc}}(\Omega), \tag{1.5}$$

with $r, s > n$ and

$$\frac{q}{p} < \frac{s}{s+1}\left(1+\frac{1}{n}-\frac{1}{r}\right). \tag{1.6}$$

If $u \in W^{1,1}_{loc}(\Omega)$ is a local minimizer of F, then for every ball $B_{R_0} \Subset \Omega$ the following estimates

$$\|Du\|_{L^\infty(B_{R_0/2})} \le C\mathcal{K}^{\vartheta}_{R_0}\left(\int_{B_{R_0}} (1+f(x,Du))\,dx\right)^{\vartheta} \tag{1.7}$$

$$\int_{B_{R_0/2}} a(1+|Du|^2)^{\frac{p-2}{2}}\left|D^2u\right|^2\,dx \le C\mathcal{K}^{\vartheta}_{R_0}\left(\int_{B_{R_0}} (1+f(x,Du))\,dx\right)^{\vartheta}, \tag{1.8}$$

hold with the exponent ϑ depending on the data, the constant C also depending on R_0 and where $\mathcal{K}_{R_0} = 1 + \|a^{-1}\|_{L^s(B_{R_0})}\|k\|^2_{L^r(B_{R_0})}$.

It is well known that to get regularity under $p, q-$growth the exponents q and p cannot be too far apart; usually, the gap between p and q is described by a condition relating p, q and the dimension n. In our case we take into account the possible degeneracy of $a(x)$ and the condition $(1.3)_2$ on the mixed derivatives $f_{\xi x}$ in terms of a possibly unbounded coefficient $k(x)$; then we deduce that the gap depends on s, the summability exponent of a^{-1} that "measures" how much a is degenerate, and the exponent r that tell us how far $k(x)$ is from being bounded. If $s = r = \infty$ then (1.6) reduces to $\frac{q}{p} < 1 + \frac{1}{n}$ that is what one expects, see [12] and for instance [39]. Moreover, if $s = \infty$ and $n < r \le +\infty$, then (1.6) reduces to $\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}$ and we recover the result of [22].

Motivated by applications to the theory of elasticity, recently Colombo and Mingione [8],[9] (see also [2],[24],[18],[19]) studied the so-called *double phase integrals*

$$\int_\Omega |Du|^p + b(x)|Du|^q\,dx, \quad 1 < p < q\,. \tag{1.9}$$

The model case we have in mind here is different: we consider the degenerate functional with non standard growth of the form

$$I(u) = \int_{B_1(0)} a(x)(1+|Du|^2)^{\frac{p}{2}} + b(x)(1+|Du|^2)^{\frac{q}{2}}\,dx \tag{1.10}$$

with $0 \le a(x) \le b(x) \le L$ for some $L > 0$. The integrand of $I(u)$ satisfies $(1.3)_2$ with $k(x) = |Da(x)| + |Db(x)|$. It is worth mentioning that in the literature $a(x)$ is usually assumed positive and bounded away from zero, see e.g. [2],[8], which is not the case here since $a(x)$ may vanish at some point. The counterpart is that we consider the powers of $(1+|Du|^2)^{\frac{1}{2}}$ instead of $|Du|$. We notice that the regularity result of Theorem 1.1 is new also when $p = q \geq 2$, for example for the energy integral

$$F_1(u) = \int_{B_1(0)} a(x)(1+|Du|^2)^{\frac{p}{2}}\, dx \tag{1.11}$$

with $a(x) \geq 0$, $\frac{1}{a} \in L^s(\Omega)$ and $|Da| \in L^r$ with $\frac{1}{s} + \frac{1}{r} < \frac{1}{n}$. As far as we know, the results proposed here are the first approach to the study of the Lipschitz continuity of the local minimizers in the setting of degenerate elliptic integrals under $p, q-$growth.

As well known, weak solutions to the elliptic equation in divergence form of the type

$$-\text{div}\,(A(x, Du)) = 0 \text{ in } \Omega.$$

are locally Lipschitz continuous provided the vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable with respect to ξ and satisfies the uniformly elliptic conditions

$$\Lambda_1(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle A_\xi(x,\xi)\lambda, \lambda\rangle \leq \Lambda_2(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2.$$

Trudinger [49] started the study of the interior regularity of solutions to linear elliptic equation of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_j}(x)\right) = 0\,, \qquad x \in \Omega \subseteq \mathbb{R}^n, \tag{1.12}$$

where the measurable coefficients a_{ij} satisfy the non-uniform condition

$$\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq n^2\mu(x)|\xi|^2 \tag{1.13}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. Here $\lambda(x)$ is the minimum eigenvalue of the symmetric matrix $A(x) = (a_{ij}(x))$ and $\mu(x) := \sup_{ij}|a_{ij}|$. Trudinger proved that any weak solution of (1.12) is locally bounded in Ω, under the following integrability assumptions on λ and μ

$$\lambda^{-1} \in L^r_{\text{loc}}(\Omega) \quad \text{and} \quad \mu_1 = \lambda^{-1}\mu^2 \in L^\sigma_{\text{loc}}(\Omega) \quad \text{with } \frac{1}{r} + \frac{1}{\sigma} < \frac{2}{n}. \tag{1.14}$$

The equation (1.12) is usually called *degenerate* when $\lambda^{-1} \notin L^\infty(\Omega)$, whereas it is called *singular* when $\mu \notin L^\infty(\Omega)$. These names in this case refer to the *degenerate* and the *singular* cases with respect to the $x-$variable, but in the mathematical literature these names are often referred to the gradient variable; this happens for instance with the $p-$*Laplacian operator* $-\text{div}\left(|Du|^{p-2}\,Du\right)$. We do not study in this paper the degenerate case with respect to the gradient variable, but we refer for instance to the analysis made by Duzaar and Mingione [21], who studied an $L^\infty-$gradient bound for solutions to non-homogeneous $p-$Laplacian type systems and equations; see also Cianchi and Maz'ya [7] and the references therein for the rich literature on the subject.

The result by Trudinger was extended in many settings and directions: firstly, by Trudinger himself in [50] and later by Fabes, Kenig and Serapioni in [28]; Pingen in [47] dealt with systems. More recently for the regularity of solutions and minimizers we refer to [3],[5],[10][15], [16],[31]. For the higher integrability of the gradient we refer to [32] (see also [6]). Very recently Calderon-Zygmund's estimates for the $p-$Laplace operator with degenerate weights have been established in [1]. The literature concerning non-uniformly elliptic problems is extensive and we refer the interested reader to the references therein.

The study of the Lipschitz regularity in the $p, q-$growth context started with the papers by Marcellini [34],[35] and, since then, many and various contributions to the subject have been provided, see the references in [41],[39]. The vectorial homogeneous framework was considered in [36],[40] and by Esposito, Leonetti and Mingione [26],[27]. The condition $(1.3)_2$ for general non autonomous integrands $f = f(x, Du)$ has been first introduced in [22],[23],[24]. It is worth to highlight that, due to the $x-$dependence, the study of regularity is significantly harder and the techniques more complex. The research on this subject is intense, as confirmed by the many articles recently published, see e.g. [11],[13],[20],[29],[37],[38],[39],[44],[45],[46].

Let us briefly sketch the tools to get our regularity result. First, for Lipschitz and higher differentiable minimizers, we prove a weighted summability result for the second order derivatives of minimizers of functionals with possibly degenerate energy densities, see Proposition 3.2. Next in Theorem 3.3 we get an *a-priori estimate* for the L^∞-norm of the gradient. To establish the *a-priori estimate* we use the Moser's iteration method [43] for the gradient and the ideas of Trudinger [49]. An approximation procedure allows us to conclude. Actually, if u is a local minimizer of (1.1), we construct a sequence of suitable variational problems in a ball $B_R \subset\subset \Omega$ with boundary value data u. In order to apply the a-priori estimate to the minimizers of the approximating functionals we prove a higher differentiability result (Theorem 4.1) for minimizers of the class of functionals with $p, q-$growth studied in [22], where only the Lipschitz continuity was proved. By applying the previous a-priori estimate to the sequence of the solutions we obtain a uniform control in L^∞ of the gradient which allows to transfer the local Lipschitz continuity property to the original minimizer u.

Another difficulty due to the $x-$dependence of the energy density is that the *Lavrentiev phenomenon* may occur. A local minimizer of F is a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that $f(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega f(x, Du)\, dx \le \int_\Omega f(x, Du + D\varphi)\, dx$$

for every $\varphi \in C^1_0(\Omega)$. If u is a local minimizer of the functional F, by virtue of (1.4) we have that $a(x)|Du|^p \in L^1_{\mathrm{loc}}(\Omega)$ and, by (1.5), $u \in W^{1,\frac{ps}{s+1}}_{\mathrm{loc}}(\Omega)$ since

$$\int_{B_R} |Du|^{\frac{ps}{s+1}}\, dx \le \left(\int_{B_R} a|Du|^p\, dx\right)^{\frac{s}{s+1}} \left(\int_{B_R} \frac{1}{a^s}\, dx\right)^{\frac{1}{s+1}} < +\infty \tag{1.15}$$

for every ball $B_R \subset \Omega$. Therefore in our context a-priori the presence of the Lavrentiev phenomenon cannot be excluded. Indeed, due to the growth assumptions on the energy density, the integral in (1.1) is well defined if $u \in W^{1,q\frac{r}{r-1}}$, but a-priori this is not the case if $u \in W^{1,\frac{ps}{s+1}}(\Omega) \setminus W^{1,q\frac{r}{r-1}}_{\mathrm{loc}}(\Omega)$. However, as a consequence of Theorem 1.1, under the stated assumptions (1.2),(1.3),(1.5),(1.6) the Lavrentiev phenomenon for the integral functional F in (1.1) cannot occur. For the gap in the Lavrentiev phenomenon we refer to [51],[4],[27],[25].

We conclude this introduction by observing that even in the one-dimensional case the Lipschitz continuity of minimizers for non-uniformly elliptic integrals is not obvious. Indeed, if we consider a minimizer u to the one-dimensional integral

$$F(u)=\int_{-1}^{1} a(x)\,|u'(x)|^{p}\,dx\,,\qquad p>1, \tag{1.16}$$

then the Euler's first variation takes the form

$$\int_{-1}^{1} a(x)\,p\,|u'(x)|^{p-2}\,u'(x)\,\varphi'(x)\,dx=0,\quad \forall\,\varphi\in C_0^1(-1,1)\,.$$

This implies that the quantity $a(x)\,|u'(x)|^{p-2}\,u'(x)$ is constant in $(-1,1)$; it is a nonzero constant, unless $u(x)$ itself is constant in $(-1,1)$, a trivial case that we do not consider here. In particular the sign of $u'(x)$ is constant and we get

$$|u'(x)|^{p-1}=\frac{c}{a(x)}\,,\quad \text{a.e. } x\in(-1,1)\,.$$

Therefore if $a(x)$ vanishes somewhere in $(-1,1)$ then $|u'(x)|$ is unbounded (and viceversa), independently of the exponent $p>1$. Thus for $n=1$ the local Lipschitz regularity of the minimizers does not hold in general if the coefficient $a(x)$ vanishes somewhere.

We can compare this one-dimensional fact with the general conditions considered in the Theorem 1.1. In the case $a(x)=|x|^{\alpha}$ for some $\alpha\in(0,1)$ then, taking into account the assumptions in (1.5), for the integral in (1.16) we have $k(x)=a'(x)=\alpha\,|x|^{\alpha-2}\,x$ and

$$\begin{cases} \frac{1}{a}\in L^{s}_{\mathrm{loc}}(-1,1) & \Leftrightarrow \quad 1-\alpha s>0 \quad \Leftrightarrow \quad \alpha<\frac{1}{s} \\ k(x)=a'\in L^{r}_{\mathrm{loc}}(-1,1) & \Leftrightarrow \quad r(\alpha-1)>-1 \quad \Leftrightarrow \quad \alpha>1-\frac{1}{r}. \end{cases}$$

These conditions are compatible if and only if $1-\frac{1}{r}<\frac{1}{s}$. Therefore, also in the one-dimensional case we have a counterexample to the $L^{\infty}-$gradient bound in (1.7) if

$$\frac{1}{r}+\frac{1}{s}>1\,. \tag{1.17}$$

This is a condition that can be easily compared with the assumption (1.6) for the validity of $L^{\infty}-$gradient bound (1.7) in the general $n-$dimensional case. In fact, being $1\le\frac{q}{p}$, (1.6) implies

$$1<\frac{s}{s+1}\left(1+\frac{1}{n}-\frac{1}{r}\right)\quad\Leftrightarrow\quad \frac{1}{r}+\frac{1}{s}<\frac{1}{n}\,,$$

which essentially is the complementary condition to (1.17) when $n=1$.

The plan of the paper is the following. In Section 2 we list some definitions and preliminary results. In Section 3 we prove an a-priori estimates of the L^{∞}-norm of the gradient of local minimizers and an higher differentiability result, see Theorem 3.3. In Section 4 we prove an estimate for the second order derivatives of a minimizer of an auxiliary uniformly elliptic functional, see Theorem 4.1. In the last section we complete the proof of Theorem 1.1.

2. Preliminary results

We shall denote by C or c a general positive constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies will be suitably emphasized using parentheses or subscripts. In what follows, $B(x,r)=B_r(x)=\{y\in\mathbb{R}^n:\ |y-x|<r\}$ will denote the ball centered at x of radius r. We shall omit the dependence on the center and on the radius when no confusion arises.

To prove our higher differentiability result (see Theorem 4.1 below) we use the finite difference operator. For a function $u : \Omega \to \mathbb{R}^k$, Ω open subset of $\mathbb{R}^n$, given $s \in \{1, \dots, n\}$, we define

$$\tau_{s,h}u(x) := u(x + he_s) - u(x), \qquad x \in \Omega_{|h|}, \tag{2.1}$$

where e_s is the unit vector in the x_s direction, $h \in \mathbb{R}$ and

$$\Omega_{|h|} := \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) < |h|\}.$$

We now list the main properties of this operator.

(i) if $u \in W^{1,t}(\Omega)$, $1 \le t \le \infty$, then $\tau_{s,h}u \in W^{1,t}(\Omega_{|h|})$ and

$$D_i(\tau_{s,h}u) = \tau_{s,h}(D_i u),$$

(ii) if f or g has support in $\Omega_{|h|}$, then

$$\int_\Omega f\tau_{s,h}g\, dx = \int_\Omega g\tau_{s,-h}f\, dx,$$

(iii) if $u, u_{x_s} \in L^t(B_R)$, $1 \le t < \infty$, and $0 < \rho < R$, then for every h, $|h| \le R - \rho$,

$$\int_{B_\rho} |\tau_{s,h}u(x)|^t\, dx \le |h|^t \int_{B_R} |u_{x_s}(x)|^t\, dx,$$

(iv) if $u \in L^t(B_R)$, $1 < t < \infty$, and for $0 < \rho < R$ there exists $K > 0$ such that for every h, $|h| < R - \rho$,

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h}u(x)|^t\, dx \le K|h|^t, \tag{2.2}$$

then letting h go to 0, $Du \in L^t(B_\rho)$ and $\|u_{x_s}\|_{L^t(B_\rho)} \le K$ for every $s \in \{1, \dots, n\}$.

We recall the following estimate for the auxiliary function

$$V_p(\xi) := \Big(1 + |\xi|^2\Big)^{\frac{p-2}{4}} \xi, \tag{2.3}$$

which is a convex function since $p \ge 2$ (see the Step 2 in [33] and the proof of [30, Lemma 8.3]).

Lemma 2.1. *Let $1 < p < \infty$. There exists a constant $c = c(n, p) > 0$ such that*

$$c^{-1}\Big(1 + |\xi|^2 + |\eta|^2\Big)^{\frac{p-2}{2}} \le \frac{|V_p(\xi) - V_p(\eta)|^2}{|\xi - \eta|^2} \le c\Big(1 + |\xi|^2 + |\eta|^2\Big)^{\frac{p-2}{2}}$$

for any ξ, $\eta \in \mathbb{R}^n$.

In the next lemma we prove that $(1.3)_1$ implies the, possibly degenerate, $p, q-$growth condition stated in (1.4).

Lemma 2.2. *Let $f = f(x, \xi)$ be convex and of class C^2 with respect to the $\xi-$variable. Assume (1.2) and*

$$a(x)\,(1 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \le \langle D_{\xi\xi}f(x, \xi)\lambda, \lambda\rangle \le b(x)\,(1 + |\xi|^2)^{\frac{q-2}{2}}|\lambda|^2 \tag{2.4}$$

for some exponents $2 \le p \le q$ and nonnegative functions a, b. Then there exists a constant c such that

$$c\, a(x)(1 + |\xi|^2)^{\frac{p-2}{2}}|\xi|^2 \le f(x, \xi) \le b(x)(1 + |\xi|^2)^{\frac{q}{2}} + f(x, 0).$$

Proof. For $x \in \Omega$ and $s \in \mathbb{R}$, let us set $\varphi(s) = g(x, st)$, where we recall that g is linked to f by (1.2). The assumptions on f imply that $\varphi \in C^2(\mathbb{R})$ and that g_t is increasing in the gradient variable $t \in [0; +\infty)$ with $g_t(x, 0) = 0$. Since

$$\varphi'(s) = g_t(x, st) \cdot t\,, \qquad \varphi''(s) = g_{tt}(x, st) \cdot t^2,$$

Taylor expansion formula yields that there exists $\vartheta \in (0, 1)$ such that

$$\varphi(1) = \varphi(0) + \varphi'(0) + \frac{1}{2}\varphi''(\vartheta)\,.$$

Recalling the definition of φ, we get

$$g(x, t) = g(x, 0) + g_t(x, 0) \cdot t + \frac{1}{2} g_{tt}(x, \vartheta t) \cdot t^2 = g(x, 0) + \frac{1}{2} g_{tt}(x, \vartheta t) \cdot t^2. \tag{2.5}$$

Assumption (2.4) translates into

$$a(x)\,(1 + t^2)^{\frac{p-2}{2}} \le g_{tt}(x, t) \le b(x)\,(1 + t^2)^{\frac{q-2}{2}}. \tag{2.6}$$

Inserting (2.6) in (2.5), we obtain

$$\frac{a(x)}{2}\,(1 + (\vartheta t)^2)^{\frac{p-2}{2}} t^2 + g(x, 0) \le g(x, t) \le g(x, 0) + \frac{b(x)}{2}\,[1 + (\vartheta t)^2]^{\frac{q-2}{2}} t^2. \tag{2.7}$$

Note that, since $\vartheta < 1$ and $q > 2$, the right hand side of (2.7) can be controlled with

$$g(x, t) \le g(x, 0) + b(x)\,[1 + (\vartheta t)^2]^{\frac{q-2}{2}} t^2 \le g(x, 0) + b(x)\,[1 + t^2]^{\frac{q-2}{2}} t^2$$

Moreover since $g(x, 0) \ge 0$ and $p \ge 2$, the left hand side of (2.7) can be controlled from below as follows

$$\begin{aligned} g(x, t) &\ge \frac{a(x)}{2}\,(1 + (\vartheta t)^2)^{\frac{p-2}{2}} t^2 + g(x, 0) \ge \frac{a(x)}{2}\,(1 + (\vartheta t)^2)^{\frac{p-2}{2}} t^2 \\ &\ge \frac{a(x)}{2}\,(\vartheta^2 + (\vartheta t)^2)^{\frac{p-2}{2}} t^2 = \vartheta^{p-2} \frac{a(x)}{2} (1 + t^2)^{\frac{p-2}{2}} t^2. \end{aligned}$$

Combining the last two estimates and recalling that $f(x, \xi) = g(x, |\xi|)$, we conclude that there exists a constant $c = c(\vartheta)$ such that

$$c(\vartheta) a(x) (1 + |\xi|^2)^{\frac{p-2}{2}} |\xi|^2 \le f(x, \xi) \le b(x)\,(1 + |\xi|^2)^{\frac{q-2}{2}} |\xi|^2 + f(x, 0)$$

and the conclusion follows. □

We end this preliminary section with a well known property. The following lemma has important applications in the so called *hole-filling method*. Its proof can be found for example in [30, Lemma 6.1] .

Lemma 2.3. *Let $h : [r, R_0] \to \mathbb{R}$ be a nonnegative bounded function and $0 < \vartheta < 1$, $A, B \ge 0$ and $\beta > 0$. Assume that*

$$h(s) \le \vartheta h(t) + \frac{A}{(t - s)^\beta} + B,$$

for all $r \le s < t \le R_0$. Then

$$h(r) \le \frac{cA}{(R_0 - r)^\beta} + cB,$$

where $c = c(\vartheta, \beta) > 0$.

3. The a-priori estimate

The main result in this section is an a-priori estimate of the L^∞-norm of the gradient of local minimizers of the functional F in (1.1) satisfying weaker assumptions than those in Theorem 3.3. Precisely, in this section we consider the following growth conditions

$$\begin{cases} a(x)\,(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \le \langle f_{\xi\xi}(x,\xi)\lambda,\lambda\rangle \le b(x)\,(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2 \\ |f_{\xi x}(x,\xi)| \le k(x)(1+|\xi|^2)^{\frac{q-1}{2}}, \end{cases} \tag{3.1}$$

for a.e. $x\in\Omega$ and for every $\xi,\lambda\in\mathbb{R}^{N\times n}$. Here, a,b,k are non-negative measurable functions. We do not require $a,b\in L^\infty$, but, in the main result of this section, see Theorem 3.3, we assume the following summability properties:

$$\frac{1}{a}\in L^s_{\rm loc}(\Omega),\qquad a\in L^{\frac{rs}{2s+r}}_{\rm loc}(\Omega),\qquad b,\,k\in L^r_{\rm loc}(\Omega),\qquad \text{with } r>n. \tag{3.2}$$

Moreover, we assume (1.6). We use the following weighted Sobolev type inequality, whose proof relies on the Hölder's inequality, see e.g. [16].

Lemma 3.1. *Let $p\ge 2$, $s\ge 1$ and $w\in W_0^{1,\frac{ps}{s+1}}(\Omega;\mathbb{R}^N)$ ($w\in W_0^{1,p}(\Omega;\mathbb{R}^N)$ if $s=\infty$). Let $\lambda:\Omega\to[0,+\infty)$ be a measurable function such that $\lambda^{-1}\in L^s(\Omega)$. There exists a constant $c=c(n)$ such that*

$$\left(\int_\Omega |w|^{\sigma^*}\,dx\right)^{\frac{p}{\sigma^*}} \le c(n)\|\lambda^{-1}\|_{L^s(\Omega)}\int_\Omega \lambda|Dw|^p\,dx, \tag{3.3}$$

where $\sigma=\frac{ps}{s+1}$ ($\sigma=p$ if $s=+\infty$).

In establishing the a-priori estimate, we need to deal with quantities that involve the L^2-norm of the second derivatives of the minimizer weighted with the function $a(x)$. Next result tells that a $W^{2,\frac{2s}{s+1}}$ assumption on the second derivatives implies that they belong to the weighted space $L^2(a(x)dx)$. More precisely, we have

Proposition 3.2. *Consider the functional F in (1.1) satisfying the assumption (3.1) with*

$$a,b\in L^1_{\rm loc}(\Omega),\ k\in L^{\frac{2s}{s-1}}_{loc}(\Omega), \tag{3.4}$$

for some $s\ge 1$. If $u\in W^{1,\infty}_{loc}(\Omega)\cap W^{2,\frac{2s}{s+1}}_{loc}(\Omega)$ is a local minimizer of F then

$$a(x)|D^2u|^2\in L^1_{\rm loc}(\Omega).$$

Proof. Since u is a local minimizer of the functional F, then u satisfies the Euler's system

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(x,Du)\varphi^\alpha_{x_i}(x)\,dx=0\qquad \forall\varphi\in C_0^\infty(\Omega;\mathbb{R}^N),$$

and, using the second variation, for every $s=1,\dots,n$ it holds

$$\int_\Omega\left\{\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)\varphi^\alpha_{x_i}u^\beta_{x_sx_j}+\sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\varphi^\alpha_{x_i}\right\}dx=0\qquad \forall\varphi\in C_0^\infty(\Omega;\mathbb{R}^N). \tag{3.5}$$

Fix $s=1,\dots,n$, a cut off function $\eta\in C_0^\infty(\Omega)$ and define for any $\gamma\ge 0$ the function

$$\varphi^\alpha := \eta^4 u^\alpha_{x_s} \quad \alpha=1,\dots,N.$$

Thanks to our assumptions on the minimizer u, through a standard density argument, we can use φ as test function in the equation (3.5), thus getting

$$\begin{aligned} 0 =& \int_\Omega 4\eta^3 \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)\eta_{x_i}u^\alpha_{x_s}u^\beta_{x_s x_j}\,dx \\ &+\int_\Omega \eta^4 \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\alpha_{x_s x_i}u^\beta_{x_s x_j}\,dx \\ &+\int_\Omega 4\eta^3 \sum_{i,s,\alpha} f_{\xi^\alpha_i x_s}(x,Du)\eta_{x_i}u^\alpha_{x_s}\,dx \\ &+\int_\Omega \eta^4 \sum_{i,s,\alpha} f_{\xi^\alpha_i x_s}(x,Du)u^\alpha_{x_s x_i}\,dx \\ =:& J_1+J_2+J_3+J_4. \end{aligned}$$

By the use of Cauchy-Schwartz and Young's inequalities and by virtue of the second inequality of (3.1), we can estimate the integral I_1 as follows

$$\begin{aligned} |J_1| \le\ & 4\int_\Omega \left\{\eta^2 \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)\eta_{x_i}u^\alpha_{x_s}\eta_{x_j}u^\beta_{x_s}\right\}^{\frac12}\left\{\eta^4 \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\alpha_{x_s x_i}u^\beta_{x_s x_j}\right\}^{\frac12} \\ \le\ & C\int_\Omega \eta^2|D\eta|^2 b(x)(1+|Du|^2)^{\frac q2}\,dx \\ & +\frac12\int_\Omega \eta^4 \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\alpha_{x_s x_i}u^\beta_{x_s x_j}\,dx. \end{aligned}$$

Moreover, by the last inequality in (3.1) we obtain

$$\begin{aligned} |J_3| \le\ & 4\int_\Omega \eta^3 k(x)(1+|Du|^2)^{\frac{q-1}{2}} \sum_{i,s,\alpha} |\eta_{x_i}u^\alpha_{x_s}|\,dx \\ \le\ & 4\int_\Omega \eta^3|D\eta| k(x)(1+|Du|^2)^{\frac q2}\,dx. \end{aligned}$$

and also

$$|J_4| \le \int_\Omega \eta^4 k(x)(1+|Du|^2)^{\frac{q-1}{2}}|D^2u|\,dx.$$

Therefore we get

$$\begin{aligned} &\int_\Omega \eta^4 \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\alpha_{x_s x_i}u^\beta_{x_s x_j}\,dx \le \frac12\int_\Omega \eta^4 \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\alpha_{x_s x_i}u^\beta_{x_s x_j}\,dx \\ &+C\int_\Omega \eta^2|D\eta|^2 b(x)(1+|Du|^2)^{\frac q2}\,dx + 4\int_\Omega \eta^3|D\eta|k(x)(1+|Du|^2)^{\frac q2}\,dx \end{aligned}$$

$$+\int_\Omega \eta^4 k(x)(1+|Du|^2)^{\frac{q-1}{2}}|D^2u|\,dx.$$

Reabsorbing the first integral in the right hand side by the left hand side we obtain

$$\begin{aligned}&\int_\Omega \eta^4 \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)u^\alpha_{x_s x_i}u^\beta_{x_s x_j}\,dx\\ \leq\ & C\int_\Omega \eta^2|D\eta|^2 b(x)(1+|Du|^2)^{\frac{q}{2}}\,dx+4\int_\Omega \eta^3|D\eta|k(x)(1+|Du|^2)^{\frac{q}{2}}\,dx\\ &+\int_\Omega \eta^4 k(x)(1+|Du|^2)^{\frac{q-1}{2}}|D^2u|\,dx.\end{aligned}$$

By the ellipticity assumption in (3.1) and since $u\in W^{1,\infty}_{\rm loc}(\Omega)$ we get

$$\begin{aligned}&\int_\Omega \eta^4 a(x)(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx\\ \leq\ & C\|1+|Du|\|^q_{L^\infty(\mathrm{supp}\eta)}\int_\Omega \eta^2|D\eta|^2 b(x)\,dx+C\|1+|Du|\|^q_{L^\infty(\mathrm{supp}\eta)}\int_\Omega \eta^3|D\eta|k(x)\\ &+C\|1+|Du|\|^{q-1}_{L^\infty(\mathrm{supp}\eta)}\int_\Omega \eta^4 k(x)|D^2u|\,dx.\end{aligned}$$

Hölder's inequality yields

$$\begin{aligned}&\int_\Omega \eta^4 a(x)(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx\\ \leq\ & C\|1+|Du|\|^q_{L^\infty(\mathrm{supp}\eta)}\int_\Omega \eta^2|D\eta|^2 b(x)\,dx+C\|1+|Du|\|^q_{L^\infty(\mathrm{supp}\eta)}\int_\Omega \eta^3|D\eta|k(x)\\ &+C\|1+|Du|\|^{q-1}_{L^\infty(\mathrm{supp}\eta)}\left(\int_\Omega \eta^4 k^{\frac{2s}{s-1}}\,dx\right)^{\frac{s-1}{2s}}\left(\int_\Omega \eta^4|D^2u|^{\frac{2s}{s+1}}\,dx\right)^{\frac{s+1}{2s}}.\end{aligned}\tag{3.6}$$

Since $k\in L^{\frac{2s}{s-1}}_{\rm loc}(\Omega)$ and $u\in W^{2,\frac{2s}{s+1}}_{loc}(\Omega)$ then estimate (3.6) implies that

$$a(x)|D^2u|^2\in L^1_{\rm loc}(\Omega).$$

□

We are now ready to establish the main result of this section.

Theorem 3.3. *Consider the functional F in* (1.1) *satisfying the assumptions* (3.1), (3.2), (1.2) *and* (1.6). *If $u\in W^{1,\infty}_{loc}(\Omega)\cap W^{2,\frac{2s}{s+1}}_{loc}(\Omega)$ is a local minimizer of F then for every ball $B_{R_0}\Subset\Omega$*

$$\|Du\|_{L^\infty(B_{R_0/2})}\leq C\mathcal{K}^\vartheta_{R_0}\left(\int_{B_{R_0}}(1+f(x,Du))\,dx\right)^\vartheta \tag{3.7}$$

$$\int_{B_\rho} a(1+|Du|^2)^{\frac{p-2}{2}}\left|D^2u\right|^2\,dx\leq c\left(\int_{B_{R_0}}(1+f(x,Du)\,dx\right)^\vartheta, \tag{3.8}$$

hold, for any $\rho < \frac{R}{2}$*. Here*

$$\mathcal{K}_{R_0} = 1 + \|a^{-1}\|_{L^s(B_{R_0})} \|k+b\|^2_{L^r(B_{R_0})} + \|a\|_{L^{\frac{rs}{2s+r}}(B_{R_0})},$$

$\vartheta > 0$ *is depending on the data,* C *is depending also on* R_0 *and* c *is depending also on* ρ *and* $\mathcal{K}_{R_0}$*.*

Proof. Since u is a local minimizer of the functional F, then u satisfies the Euler's system

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(x, Du) \varphi^\alpha_{x_i}(x)\, dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega; \mathbb{R}^N),$$

and, using the second variation, for every $s = 1, \dots, n$ it holds

$$\int_\Omega \left\{ \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) \varphi^\alpha_{x_i} u^\beta_{x_s x_j} + \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x, Du) \varphi^\alpha_{x_i} \right\} dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega; \mathbb{R}^N). \quad (3.9)$$

Fix $s = 1, \dots, n$, a cut off function $\eta \in C_0^\infty(\Omega)$ and define for any $\gamma \geq 0$ the function

$$\varphi^\alpha := \eta^4 u^\alpha_{x_s} (1 + |Du|^2)^{\frac{\gamma}{2}} \quad \alpha = 1, \dots, N.$$

One can easily check that

$$\varphi^\alpha_{x_i} = 4\eta^3 \eta_{x_i} u^\alpha_{x_s} (1 + |Du|^2)^{\frac{\gamma}{2}} + \eta^4 u^\alpha_{x_s x_i} (1 + |Du|^2)^{\frac{\gamma}{2}} + \gamma \eta^4 u^\alpha_{x_s} (1 + |Du|^2)^{\frac{\gamma-2}{2}} |Du| \big(|Du|\big)_{x_i}.$$

Thanks to our assumptions on the minimizer u, through a standard density argument, we can use φ as test function in the equation (3.9), thus getting

$$\begin{aligned}
0 = &\int_\Omega 4\eta^3 (1 + |Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) \eta_{x_i} u^\alpha_{x_s} u^\beta_{x_s x_j}\, dx \\
&+ \int_\Omega \eta^4 (1 + |Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, dx \\
&+ \gamma \int_\Omega \eta^4 (1 + |Du|^2)^{\frac{\gamma-2}{2}} |Du| \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u^\alpha_{x_s} u^\beta_{x_s x_j} (|Du|)_{x_i}\, dx \\
&+ \int_\Omega 4\eta^3 (1 + |Du|^2)^{\frac{\gamma}{2}} \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x, Du) \eta_{x_i} u^\alpha_{x_s}\, dx \\
&+ \int_\Omega \eta^4 (1 + |Du|^2)^{\frac{\gamma}{2}} \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x, Du) u^\alpha_{x_s x_i}\, dx \\
&+ \gamma \int_\Omega \eta^4 (1 + |Du|^2)^{\frac{\gamma-2}{2}} |Du| \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x, Du) u^\alpha_{x_s} \big(|Du|\big)_{x_i}\, dx \\
=: &I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \qquad (3.10)
\end{aligned}$$

Estimate of I_1

By the use of Cauchy-Schwartz and Young's inequalities and by virtue of the second inequality in (3.1), we can estimate the integral I_1 as follows

$$
\begin{aligned}
|I_1| \;&\le\; 4\int_\Omega (1+|Du|^2)^{\frac{\gamma}{2}}\Big\{\eta^2\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)\eta_{x_i}u^\alpha_{x_s}\eta_{x_j}u^\beta_{x_s}\Big\}^{\frac12}\Big\{\eta^4\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)u^\alpha_{x_sx_i}u^\beta_{x_sx_j}\Big\}^{\frac12} \\
&\le\; C(\varepsilon)\int_\Omega \eta^2|D\eta|^2 b(x)(1+|Du|^2)^{\frac{q+\gamma}{2}}\,dx \\
&\quad +\varepsilon\int_\Omega \eta^4(1+|Du|^2)^{\frac{\gamma}{2}}\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)u^\alpha_{x_sx_i}u^\beta_{x_sx_j}\,dx, \qquad (3.11)
\end{aligned}
$$

where $\varepsilon>0$ will be chosen later.

Estimate of I_3

Since

$$
f_{\xi_i^\alpha\xi_j^\beta}(x,\xi)=\left(\frac{g_{tt}(x,|\xi|)}{|\xi|^2}-\frac{g_t(x,|\xi|)}{|\xi|^3}\right)\xi_i^\alpha\xi_j^\beta+\frac{g_t(x,|\xi|)}{|\xi|}\delta_{\xi_i^\alpha\xi_j^\beta}
$$

and

$$
(|Du|)_{x_i}=\frac{1}{|Du|}\sum_{\alpha,s}u^\alpha_{x_ix_s}u^\alpha_{x_s} \qquad (3.12)
$$

then

$$
\begin{aligned}
&\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)u^\alpha_{x_s}u^\beta_{x_sx_j}(|Du|)_{x_i} \\
&=\left(\frac{g_{tt}(x,|Du|)}{|Du|^2}-\frac{g_t(x,|Du|)}{|Du|^3}\right)\sum_{i,j,s,\alpha,\beta}u^\alpha_{x_s}u^\beta_{x_sx_j}u^\alpha_{x_i}u^\beta_{x_j}(|Du|)_{x_i} \\
&\quad+\frac{g_t(x,|Du|)}{|Du|}\sum_{i,s,\alpha}u^\alpha_{x_s}u^\alpha_{x_sx_i}(|Du|)_{x_i} \\
&=\left(\frac{g_{tt}(x,|Du|)}{|Du|}-\frac{g_t(x,|Du|)}{|Du|^2}\right)\sum_\alpha\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2 \\
&\quad+g_t(x,|Du|)|D(|Du|)|^2. \qquad (3.13)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_3=\gamma\int_\Omega \eta^4(1+|Du|^2)^{\frac{\gamma-2}{2}}|Du|\Big\{\Big(\frac{g_{tt}(x,|Du|)}{|Du|}-\frac{g_t(x,|Du|)}{|Du|^2}\Big)\sum_\alpha\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2 \\
+g_t(x,|Du|)|D(|Du|)|^2\Big\}\,dx.
\end{aligned}
$$

Using the Cauchy-Schwartz inequality, i.e.

$$
\sum_\alpha\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2\le |Du|^2|D(|Du|)|^2
$$

and observing that

$$
g_t(x,|Du|)\ge 0,
$$

we conclude

$$I_3 \geq \gamma \int_\Omega \eta^4 (1+|Du|^2)^{\frac{\gamma-2}{2}} |Du| \frac{g_{tt}(x,|Du|)}{|Du|} \sum_\alpha \left(\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right)^2 dx \geq 0. \tag{3.14}$$

Estimate of I_4

By using the last inequality in (3.1) we obtain

$$\begin{aligned} |I_4| &\leq 4 \int_\Omega \eta^3 k(x) (1+|Du|^2)^{\frac{q-1+\gamma}{2}} \sum_{i,s,\alpha} |\eta_{x_i} u^\alpha_{x_s}|\, dx \\ &\leq 4 \int_\Omega \eta^3 |D\eta| k(x) (1+|Du|^2)^{\frac{q+\gamma}{2}}\, dx. \end{aligned} \tag{3.15}$$

Estimate of I_5

Using the last inequality in (3.1) and Young's inequality we have that

$$\begin{aligned} |I_5| &\leq \int_\Omega \eta^4 k(x) (1+|Du|^2)^{\frac{q-1+\gamma}{2}} |D^2u|\, dx \\ &\leq \sigma \int_\Omega \eta^4 a(x) (1+|Du|^2)^{\frac{p-2+\gamma}{2}} |D^2u|^2\, dx \\ &\quad + C_\sigma \int_\Omega \eta^4 \frac{k^2(x)}{a(x)} (1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\, dx, \end{aligned} \tag{3.16}$$

where $\sigma \in (0,1)$ will be chosen later and a is the function appearing in (3.1).

Estimate of I_6

Using the last inequality in (3.1) and (3.12), we get

$$\begin{aligned} |I_6| &\leq \gamma \int_\Omega \eta^4 k(x) (1+|Du|^2)^{\frac{q-1+\gamma}{2}} |D(|Du|)|\, dx \\ &\leq \gamma \int_\Omega \eta^4 (1+|Du|^2)^{\frac{q-1+\gamma}{2}} k(x) |D^2u|\, dx \\ &\leq \sigma \int_\Omega \eta^4 a(x) (1+|Du|^2)^{\frac{p-2+\gamma}{2}} |D^2u|^2\, dx \\ &\quad + C_\sigma \gamma^2 \int_\Omega \eta^4 \frac{k^2(x)}{a(x)} (1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\, dx, \end{aligned} \tag{3.17}$$

where we used Young's inequality again.

Since the equality (3.10) can be written as follows

$$I_2 + I_3 = -I_1 - I_4 - I_5 - I_6\,,$$

by virtue of (3.14), we get

$$I_2 \leq |I_1| + |I_4| + |I_5| + |I_6|$$

and therefore, recalling the estimates (3.11), (3.15), (3.16) and (3.17), we obtain

$$\int_\Omega \eta^4 (1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x,Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, dx$$

$$
\begin{aligned}
\leq\ & \varepsilon \int_\Omega \eta^4 (1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, dx \\
& +4 \int_\Omega \eta^3 |D\eta| k(x) (1+|Du|^2)^{\frac{q+\gamma}{2}}\, dx \\
& +2\sigma \int_\Omega \eta^4 a(x) (1+|Du|^2)^{\frac{p-2+\gamma}{2}} |D^2u|^2\, dx \\
& +C_\sigma (1+\gamma^2) \int_\Omega \eta^4 \frac{k^2(x)}{a(x)} (1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\, dx \\
& +C_\varepsilon \int_\Omega \eta^2 |D\eta|^2 b(x) (1+|Du|^2)^{\frac{q+\gamma}{2}}\, dx. \qquad (3.18)
\end{aligned}
$$

Choosing $\varepsilon = \frac{1}{2}$, we can reabsorb the first integral in the right hand side by the left hand side thus getting

$$
\begin{aligned}
& \int_\Omega \eta^4 (1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, dx \\
\leq\ & 4\sigma \int_\Omega \eta^4 a(x) (1+|Du|^2)^{\frac{p-2+\gamma}{2}} |D^2u|^2\, dx \\
& +C \int_\Omega \eta^3 |D\eta| k(x) (1+|Du|^2)^{\frac{q+\gamma}{2}}\, dx + C_\sigma (1+\gamma^2) \int_\Omega \eta^4 \frac{k^2(x)}{a(x)} (1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\, dx \\
& +C \int_\Omega \eta^2 |D\eta|^2 b(x) (1+|Du|^2)^{\frac{q+\gamma}{2}}\, dx. \qquad (3.19)
\end{aligned}
$$

Now, using the ellipticity condition in (3.1) to estimate the left hand side of (3.19), we get

$$
\begin{aligned}
& c_2 \int_\Omega \eta^4 a(x) (1+|Du|^2)^{\frac{p-2+\gamma}{2}} |D^2u|^2\, dx \\
\leq\ & 4\sigma \int_\Omega \eta^4 a(x) (1+|Du|^2)^{\frac{p-2+\gamma}{2}} |D^2u|^2\, dx \\
& +C \int_\Omega \eta^3 |D\eta| k(x) (1+|Du|^2)^{\frac{q+\gamma}{2}}\, dx + C_\sigma (1+\gamma^2) \int_\Omega \eta^4 \frac{k^2(x)}{a(x)} (1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\, dx \\
& +C \int_\Omega \eta^2 |D\eta|^2 b(x) (1+|Du|^2)^{\frac{q+\gamma}{2}}\, dx.
\end{aligned}
$$

We claim that $k \in L^{\frac{2s}{s-1}}_{\mathrm{loc}}(\Omega)$. Since by assumption (3.2), $k \in L^r_{\mathrm{loc}}(\Omega)$, we need to prove that $\frac{2s}{s-1} \leq r$ that is equivalent to $\frac{2}{r} + \frac{1}{s} \leq 1$. This holds true, because by (1.6) and $q \geq p$ we get

$$
\frac{s}{s+1}\left(1+\frac{1}{n}-\frac{1}{r}\right) > 1 \Leftrightarrow \frac{n}{r}+\frac{n}{s} < 1
$$

and we conclude, because $n \geq 2$. Therefore, since by the assumption $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,\frac{2s}{s+1}}_{\mathrm{loc}}(\Omega)$, we can use Proposition 3.2, that implies that the first integral in the right hand side of previous estimate is finite. By choosing $\sigma = \frac{c_2}{8}$, we can reabsorb the first integral in the right hand side by the left hand side thus getting

$$\begin{aligned}
&\int_\Omega \eta^4 a(x)(1+|Du|^2)^{\frac{p-2+\gamma}{2}}|D^2u|^2\,dx\\
\le\ & C\int_\Omega \eta^3|D\eta|k(x)(1+|Du|^2)^{\frac{q+\gamma}{2}}\,dx + C(1+\gamma^2)\int_\Omega \eta^4\frac{k^2(x)}{a(x)}(1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\,dx\\
&+C\int_\Omega \eta^2|D\eta|^2 b(x)(1+|Du|^2)^{\frac{q+\gamma}{2}}\,dx\\
\le\ & C\int_\Omega \left(\eta^2|D\eta|^2+|D\eta|^4\right)a(x)(1+|Du|^2)^{\frac{p+\gamma}{2}}\,dx\\
&+C(\gamma+1)^2\int_\Omega \eta^4\frac{k^2(x)+b^2(x)}{a(x)}(1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\,dx, \qquad (3.20)
\end{aligned}$$

where we used Young's inequality again. Now, we note that

$$\eta^4 a(x)\left|D\big((1+|Du|^2)^{\frac{p+\gamma}{4}}\big)\right|^2 \le c(p+\gamma)^2 a(x)\eta^4(1+|Du|^2)^{\frac{p-2+\gamma}{2}}|D^2u|^2$$

and so fixing $\frac{R_0}{2}\le\rho<t'<t<R<R_0$ with R_0 such that $B_{R_0}\Subset\Omega$, and choosing $\eta\in C_0^\infty(B_t)$ a cut off function between $B_{t'}$ and B_t, by the assumption $a^{-1}\in L^s_{\mathrm{loc}}(\Omega)$ we can use Sobolev type inequality of Lemma 3.1 with $w=\eta^2(1+|Du|^2)^{\frac{p+\gamma}{4}}$, $\lambda=a$ and $p=2$, thus obtaining

$$\begin{aligned}
\left(\int_{B_t}\left(\eta^2(1+|Du|^2)^{\frac{p+\gamma}{4}}\right)^{\left(\frac{2s}{s+1}\right)^*}dx\right)^{\frac{2}{\left(\frac{2s}{s+1}\right)^*}} &\le \frac{c(n)}{(t-t')^2}\int_{B_t} a(x)(1+|Du|^2)^{\frac{p+\gamma}{2}}\,dx\\
&+c(n)(p+\gamma)^2\int_{B_t}\eta^4 a\,(1+|Du|^2)^{\frac{p-2+\gamma}{2}}|D^2u|^2\,dx,
\end{aligned}$$

with a constant $c(n)$ depending only on n.

Using (3.20) to estimate the last integral in previous inequality, we obtain

$$\begin{aligned}
&\left(\int_{B_t}\eta^{\frac{4ns}{n(s+1)-2s}}(1+|Du|^2)^{\frac{(p+\gamma)ns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{n(s+1)-2s}{ns}}\\
\le\ & c\left(\frac{(p+\gamma)^2}{(t-t')^2}+\frac{(p+\gamma)^2}{(t-t')^4}\right)\int_{B_t} a(x)(1+|Du|^2)^{\frac{p+\gamma}{2}}\,dx\\
&+c(p+\gamma)^4\int_{B_t}\frac{k^2(x)+b^2(x)}{a(x)}(1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\,dx\\
\le\ & c\left(\frac{(p+\gamma)^2}{(t-t')^2}+\frac{(p+\gamma)^2}{(t-t')^4}\right)\int_{B_t} a(x)(1+|Du|^2)^{\frac{p+\gamma}{2}}\,dx\\
&+c(p+\gamma)^4\left(\int_{B_t}\frac{1}{a^s}\,dx\right)^{\frac{1}{s}}\left(\int_{B_t}(k^r+b^r)\,dx\right)^{\frac{2}{r}}\\
&\times\left(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p+\gamma)rs}{2(rs-2s-r)}}\,dx\right)^{\frac{rs-2s-r}{rs}},
\end{aligned}$$

where we used assumptions (3.2) and Hölder's inequality with exponents s, $\frac{r}{2}$ and $\frac{rs}{rs-2s-r}$.

Using the properties of η we obtain

$$
\begin{aligned}
&\left(\int_{B_{t'}} (1+|Du|^2)^{\frac{(p+\gamma)ns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{n(s+1)-2s}{ns}}\\
\leq\ & c(p+\gamma)^4\|a^{-1}\|_{L^s(B_{R_0})}\|k+b\|^2_{L^r(B_{R_0})}\left(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p+\gamma)rs}{2(rs-2s-r)}}dx\right)^{\frac{rs-2s-r}{rs}}\\
&+c\left(\frac{(p+\gamma)^2}{(t-t')^2}+\frac{(p+\gamma)^2}{(t-t')^4}\right)\int_{B_t} a(x)(1+|Du|^2)^{\frac{p+\gamma}{2}}\,dx\\
\leq\ & c(p+\gamma)^4\|a^{-1}\|_{L^s(B_{R_0})}\|k+b\|^2_{L^r(B_{R_0})}\left(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p+\gamma)rs}{2(rs-2s-r)}}dx\right)^{\frac{rs-2s-r}{rs}}\\
&+c\left(\frac{(p+\gamma)^2}{(t-t')^2}+\frac{(p+\gamma)^2}{(t-t')^4}\right)\|a\|_{L^{\frac{rs}{2s+r}}(B_{R_0})}\left(\int_{B_t}(1+|Du|^2)^{\frac{(p+\gamma)rs}{2(rs-2s-r)}}\,dx\right)^{\frac{rs-2s-r}{rs}},
\end{aligned}
$$

where we used the assumption $a \in L^{\frac{rs}{2s+r}}_{\mathrm{loc}}(\Omega)$. Setting

$$
\mathcal{K}_{R_0} = 1+\|a^{-1}\|_{L^s(B_{R_0})}\|k+b\|^2_{L^r(B_{R_0})}+\|a\|_{L^{\frac{rs}{2s+r}}}(B_{R_0}) \tag{3.21}
$$

and assuming without loss of generality that $t-t'<1$, we can write the previous estimate as follows

$$
\begin{aligned}
&\left(\int_{B_{t'}} (1+|Du|^2)^{\frac{(p+\gamma)ns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{n(s+1)-2s}{ns}}\\
&\leq c(p+\gamma)^4\mathcal{K}_{R_0}\left(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p+\gamma)rs}{2(rs-2s-r)}}dx\right)^{\frac{rs-2s-r}{rs}}\\
&\qquad + c(p+\gamma)^2\frac{\mathcal{K}_{R_0}}{(t-t')^4}\left(\int_{B_t}(1+|Du|^2)^{\frac{(p+\gamma)rs}{2(rs-2s-r)}}\,dx\right)^{\frac{rs-2s-r}{rs}},
\end{aligned}
$$

and, using the a-priori assumption $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, we get

$$
\begin{aligned}
&\left(\int_{B_{t'}} (1+|Du|^2)^{\frac{(p+\gamma)ns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{n(s+1)-2s}{ns}}\\
&\leq c(p+\gamma)^4\mathcal{K}_{R_0}\left(\|Du\|^{2(q-p)}_{L^\infty(B_R)}+\frac{1}{(t-t')^4}\right)\left(\int_{B_t}(1+|Du|^2)^{\frac{(p+\gamma)rs}{2(rs-2s-r)}}dx\right)^{\frac{rs-2s-r}{rs}}.
\end{aligned} \tag{3.22}
$$

Setting now

$$
m=\frac{rs}{rs-2s-r}
$$

and noting that

$$
\frac{ns}{n(s+1)-2s}=\frac{1}{2}\left(\frac{2s}{s+1}\right)^{*}=:\frac{2^*_s}{2}
$$

we can write (3.22) as follows

$$\begin{aligned}&\left(\int_{B_{t'}}\left((1+|Du|^2)^{\frac{(p+\gamma)m}{2}}\right)^{\frac{2_s^*}{2m}}\,dx\right)^{\frac{2m}{2_s^*}}\\ \leq\ & c(p+\gamma)^{4m}\mathcal{K}_{R_0}^m\frac{\|Du\|_{L^\infty(B_R)}^{2(q-p)m}}{(t-t')^{4m}}\int_{B_t}(1+|Du|^2)^{\frac{(p+\gamma)m}{2}}dx,\end{aligned}\tag{3.23}$$

where, without loss of generality, we supposed $\|Du\|_{L^\infty(B_R)}^{2(q-p)m}\geq 1$. Define now the decreasing sequence of radii by setting

$$\rho_i=\rho+\frac{R-\rho}{2^i}$$

and the increasing sequence of exponents

$$p_0=pm\qquad\qquad p_i=p_{i-1}\left(\frac{2_s^*}{2m}\right)=p_0\left(\frac{2_s^*}{2m}\right)^i$$

As we will prove (see (3.32) below) the right hand side of (3.23) is finite for $\gamma=0$. Then for every $\rho<\rho_{i+1}<\rho_i<R$, we may iterate it on the concentric balls B_{ρ_i} with exponents p_i, thus obtaining

$$\begin{aligned}&\left(\int_{B_{\rho_{i+1}}}(1+|Du|^2)^{\frac{p_{i+1}}{2}}\,dx\right)^{\frac{1}{p_{i+1}}}\\ \leq\ &\prod_{j=0}^{i}\left(C^m\mathcal{K}_{R_0}^m\frac{p_j^{4m}\|Du\|_{L^\infty(B_R)}^{2m(q-p)}}{(\rho_j-\rho_{j+1})^{4m}}\right)^{\frac{1}{p_j}}\left(\int_{B_R}(1+|Du|^2)^{\frac{p_0}{2}}\,dx\right)^{\frac{1}{p_0}}\\ =\ &\prod_{j=0}^{i}\left(C^m\mathcal{K}_{R_0}^m\frac{4^{jm}p_j^{4m}\|Du\|_{L^\infty(B_R)}^{2(q-p)m}}{(R-\rho)^{4m}}\right)^{\frac{1}{p_j}}\left(\int_{B_R}(1+|Du|^2)^{\frac{p_0}{2}}\,dx\right)^{\frac{1}{p_0}}\\ =\ &\prod_{j=0}^{i}\left(4^{jm}p_j^{4m}\right)^{\frac{1}{p_j}}\prod_{j=0}^{i}\left(\frac{C^m\mathcal{K}_{R_0}^m\|Du\|_{L^\infty(B_R)}^{2(q-p)m}}{(R-\rho)^{4m}}\right)^{\frac{1}{p_j}}\left(\int_{B_R}(1+|Du|^2)^{\frac{p_0}{2}}\,dx\right)^{\frac{1}{p_0}}.\end{aligned}\tag{3.24}$$

Since

$$\prod_{j=0}^{i}\left(4^{jm}p_j^{4m}\right)^{\frac{1}{p_j}}=\exp\left(\sum_{j=0}^{i}\frac{1}{p_j}\log(4^{jm}p_j^{4m})\right)\leq\exp\left(\sum_{j=0}^{+\infty}\frac{1}{p_j}\log(4^{jm}p_j^{4m})\right)\leq c(n,r)$$

and

$$\begin{aligned}&\prod_{j=0}^{i}\left(\frac{C^m\mathcal{K}_{R_0}^m\|Du\|_{L^\infty(B_t)}^{2(q-p)m}}{(R-\rho)^{4m}}\right)^{\frac{1}{p_j}}=\left(\frac{C^m\mathcal{K}_{R_0}^m\|Du\|_{L^\infty(B_R)}^{2(q-p)m}}{(R-\rho)^{4m}}\right)^{\sum_{j=0}^{i}\frac{1}{p_j}}\\ \leq\ &\left(\frac{C^m\mathcal{K}_{R_0}^m\|Du\|_{L^\infty(B_R)}^{2(q-p)m}}{(R-\rho)^{4m}}\right)^{\sum_{j=0}^{+\infty}\frac{1}{p_j}}=\left(\frac{C\mathcal{K}_{R_0}\|Du\|_{L^\infty(B_R)}^{2(q-p)}}{(R-\rho)^{4}}\right)^{\frac{2_s^*}{p(2_s^*-2m)}}\end{aligned}$$

we can let $i \to \infty$ in (3.24) thus getting

$$\|Du\|_{L^\infty(B_\rho)} \le C(n,r,p)\left(\frac{\mathcal{K}_{R_0}}{(R-\rho)^4}\right)^{\frac{2_s^*}{p(2_s^*-2m)}} \|Du\|_{L^\infty(B_R)}^{\frac{2(q-p)2_s^*}{p(2_s^*-2m)}} \left(\int_{B_R}(1+|Du|^2)^{\frac{pm}{2}}\,dx\right)^{\frac{1}{pm}},$$

where we used that $p_0 = pm$. Since assumption (1.6) implies that

$$\frac{2(q-p)2_s^*}{p(2_s^*-2m)} < 1, \tag{3.25}$$

we can use Young's inequality with exponents

$$\frac{p(2_s^*-2m)}{2(q-p)2_s^*} > 1 \qquad \text{and} \qquad \frac{p(2_s^*-2m)}{p(2_s^*-2m)-2(q-p)2_s^*}$$

to deduce that

$$\|Du\|_{L^\infty(B_\rho)} \le \frac{1}{2}\|Du\|_{L^\infty(B_R)} + C(n,r,p,s)\left(\frac{\mathcal{K}_{R_0}}{(R-\rho)^4}\right)^{\vartheta}\left(\int_{B_R}(1+|Du|^2)^{\frac{pm}{2}}\,dx\right)^{\varsigma}, \tag{3.26}$$

with $\vartheta = \vartheta(p,q,n,s)$ and $\varsigma = \varsigma(p,q,n,s)$.

We now estimate the last integral. By definition of m and by the assumption on s, i.e., $s > \frac{nr}{r-n}$, we get

$$m < \frac{ns}{n(s+1)-2s}.$$

Thus, by Hölder's inequality,

$$\int_{B_R}(1+|Du|^2)^{\frac{pm}{2}}\,dx \le c(R_0,n,r,s)\left(\int_{B_R}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{r(n(s+1)-2s)}{n(rs-2s-r)}}. \tag{3.27}$$

This last integral can be estimated by using (3.22) with $\gamma = 0$. Indeed, let us re-define t', t and η as follows: consider $R \le t' < t \le 2R-\rho \le R_0$ and η a cut off function, $\eta \equiv 1$ on $B_{t'}$ and $\operatorname{supp}\eta \subset B_t$. By (3.22) with $\gamma = 0$,

$$\begin{aligned} &\left(\int_{B_{t'}}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{n(s+1)-2s}{ns}} \\ \le\ & c\left(\frac{p^2}{(t-t')^2}+\frac{p^2}{(t-t')^4}\right)\int_{B_{R_0}} a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx \\ &+cp^4\mathcal{K}_{B_{R_0}}\left(\int_{B_{R_0}}(k^r+b^r)\,dx\right)^{\frac{2}{r}} \\ &\times\left(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p)rs}{2(rs-2s-r)}}\,dx\right)^{\frac{rs-2s-r}{rs}}. \end{aligned} \tag{3.28}$$

If we denote

$$\tau := \frac{(2q-p)rs}{rs-2s-r}, \quad \tau_1 := \frac{nps}{n(s+1)-2s}, \quad \tau_2 := \frac{ps}{s+1},$$

by (1.6) and $s > \frac{rn}{r-n}$, we get

$$\frac{\tau}{\tau_1} < 1 < \frac{\tau}{\tau_2}.$$

Therefore there exists $\theta \in (0,1)$ such that

$$1 = \theta \frac{\tau}{\tau_1} + (1-\theta)\frac{\tau}{\tau_2}.$$

The precise value of θ is

$$\theta = \frac{ns(qr - pr + p) + qrn}{rs(2q-p)}. \tag{3.29}$$

By Hölder's inequality with exponents $\frac{\tau_1}{\theta\tau}$ and $\frac{\tau_2}{(1-\theta)\tau}$ we get

$$\left(\int_{B_t} (1+|Du|^2)^{\frac{(2q-p)rs}{2(rs-2s-r)}}\,dx\right)^{\frac{rs-2s-r}{rs}} = \left(\int_{B_t} (1+|Du|^2)^{\theta\frac{\tau}{2}+(1-\theta)\frac{\tau}{2}}\,dx\right)^{\frac{2q-p}{\tau}}$$
$$\le \left(\int_{B_t} (1+|Du|^2)^{\frac{\tau_1}{2}}\,dx\right)^{\frac{(2q-p)\theta}{\tau_1}} \left(\int_{B_t} (1+|Du|^2)^{\frac{\tau_2}{2}}\,dx\right)^{\frac{(2q-p)(1-\theta)}{\tau_2}}.$$

Hence, we can use the inequality above to estimate the last integral of (3.28) to deduce that

$$\begin{aligned}
&\left(\int_{B_{t'}} (1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{n(s+1)-2s}{ns}}\\
\le\ & c\left(\frac{p^2}{(t-t')^2} + \frac{p^2}{(t-t')^4}\right)\int_{B_{R_0}} a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx\\
&+C\mathcal{K}_{B_{R_0}}\left(\int_{B_t} (1+|Du|^2)^{\frac{ps}{2(s+1)}}\,dx\right)^{\frac{(1-\theta)(2q-p)(s+1)}{ps}}\\
&\times\left(\int_{B_t} (1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{\theta(ns+n-2s)(2q-p)}{nps}}.
\end{aligned} \tag{3.30}$$

Note that, again by (1.6) and (3.29), we have

$$\frac{\theta(2q-p)}{p} < 1.$$

We can use Young's inequality in the last term of (3.30) with exponents $\frac{p}{p-\theta(2q-p)}$ and $\frac{p}{\theta(2q-p)}$ to obtain that for every $\sigma < 1$

$$\begin{aligned}
&\left(\int_{B_{t'}} (1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{n(s+1)-2s}{ns}}\\
\le\ & C\left(\frac{p^2}{(t-t')^2} + \frac{p^2}{(t-t')^4}\right)\int_{B_{R_0}} a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx\\
&+ C_\sigma\mathcal{K}_{B_{R_0}}^{\frac{p}{p-\theta(2q-p)}}\left(\int_{B_t} (1+|Du|^2)^{\frac{ps}{2(s+1)}}\,dx\right)^{\frac{(1-\theta)(2q-p)}{p-\theta(2q-p)}\frac{s+1}{s}}
\end{aligned}$$

$$+\sigma\left(\int_{B_t}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{n(s+1)-2s}{ns}}.$$

By applying Lemma 2.3, and noting that $2R-\rho-R=R-\rho$, we conclude that

$$\begin{aligned}&\left(\int_{B_R}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\right)^{\frac{n(s+1)-2s}{ns}}\\&\le C\left(\frac{p^2}{(R-\rho)^2}+\frac{p^2}{(R-\rho)^4}\right)\int_{B_{R_0}}a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx\\&+C\mathcal{K}_{B_{R_0}}^{\frac{p}{p-\theta(2q-p)}}\left(\int_{B_{R_0}}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\,dx\right)^{\frac{(1-\theta)(2q-p)}{p-\theta(2q-p)}\frac{s+1}{s}}.\end{aligned}\tag{3.31}$$

Collecting (3.27) and (3.31) we obtain

$$\begin{aligned}&\int_{B_R}(1+|Du|^2)^{\frac{pm}{2}}\,dx\\\le\quad&C\left(\frac{p^2}{(R-\rho)^2}+\frac{p^2}{(R-\rho)^4}\right)\int_{B_{R_0}}a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx\\&+C\mathcal{K}_{B_{R_0}}^{\frac{p}{p-\theta(2q-p)}}\left(\int_{B_{R_0}}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\,dx\right)^{\frac{(1-\theta)(2q-p)}{p-\theta(2q-p)}\frac{s+1}{s}}.\end{aligned}\tag{3.32}$$

Notice that the right hand side is finite, because u is a local minimizer and (1.4) and (1.15) hold. This inequality, together with (3.26), implies

$$\begin{aligned}\|Du\|_{L^\infty(B_\rho)}\le&\frac{1}{2}\|Du\|_{L^\infty(B_R)}+C\left(\frac{\mathcal{K}_{B_{R_0}}}{(R-\rho)^8}\right)^{\theta}\left(\int_{B_{R_0}}a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx\right)^{\tilde\theta}\\&+C\left(\frac{\mathcal{K}_{B_{R_0}}}{(R-\rho)^8}\right)^{\theta}\left(\int_{B_{R_0}}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\,dx\right)^{\tilde\varsigma}\end{aligned}$$

with the constant C depending on the data. Applying Lemma 2.3 we conclude the proof of estimate (3.7). Now, we write the estimate (3.20) for $\gamma=0$ and for a cut off function $\eta\in C_0^\infty(B_{\frac{R}{2}})$, $\eta=1$ on B_ρ for some $\rho<\frac{R}{2}$. This yields

$$\begin{aligned}&\int_{B_\rho}a(x)(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx\\\le\;&C(R)\int_{B_{\frac{R}{2}}}a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx+C\int_{B_{\frac{R}{2}}}\frac{k^2(x)+b^2(x)}{a(x)}(1+|Du|^2)^{\frac{2q-p}{2}}\,dx\\\le\;&C(R)\int_{B_{\frac{R}{2}}}f(x,Du)\,dx+C\|1+|Du|\|^{2q-p}_{L^\infty(B_{\frac{R}{2}})}\int_{B_{\frac{R}{2}}}\frac{k^2(x)+b^2(x)}{a(x)}\,dx\end{aligned}$$

$$\leq \; C(R)\int_{B_{\frac{R}{2}}} f(x,Du)\,dx$$

$$+C(R)\|1+|Du|\|^{2q-p}_{L^\infty(B_{\frac{R}{2}})}\left(\int_{B_{\frac{R}{2}}}(k^r(x)+b^r(x))\,dx\right)^{\frac{2}{r}}\left(\int_{B_{\frac{R}{2}}}\frac{1}{a^s(x)}\,dx\right)^{\frac{1}{s}},$$

where we used Hölder's inequality, since $\frac{1}{s}+\frac{2}{r}<1$ by assumptions. Using (3.7) to estimate the L^∞ norm of $|Du|$ and recalling the definition of $\mathcal{K}_{R_0}$ at (3.21), we get

$$\int_{B_\rho} a(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx \leq c\left(\int_{B_R} 1+f(x,Du)\,dx\right)^{\tilde{\varrho}},$$

i.e. (3.8), with c depending on $p,r,s,n,\rho,R,\ \mathcal{K}_{R_0}$. □

4. An auxiliary functional: higher differentiability estimate

Consider the functional

$$H(v)=\int_\Omega h(x,Dv)\,dx$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 2$, is a Sobolev map and $h:\Omega\times\mathbb{R}^{N\times n}\to[0,+\infty)$ is a Carathéodory function, convex and of class C^2 with respect to the second variable. We assume that there exists $\tilde{h}:\Omega\times[0,+\infty)\to[0,+\infty)$, increasing in the last variable such that

$$h(x,\xi)=\tilde{h}(x,|\xi|). \tag{4.1}$$

Moreover assume that there exist p,q, $1<p<q$, and constant ℓ,ν,L_1,L_2 such that

$$\ell(1+|\xi|^2)^{\frac{p}{2}}\leq h(x,\xi)\leq L_1\,(1+|\xi|^2)^{\frac{q}{2}} \tag{4.2}$$

for a.e. $x\in\Omega$ and for every $\xi\in\mathbb{R}^{N\times n}$. We assume that h is of class C^2 with respect to the $\xi-$variable, and that the following conditions hold

$$\nu\,(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2\leq\langle h_{\xi\xi}(x,\xi)\lambda,\lambda\rangle\leq L_2\,(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2 \tag{4.3}$$

for a.e. $x\in\Omega$ and for every $\xi,\lambda\in\mathbb{R}^{N\times n}$. Moreover, we assume that there exists a non-negative function $k\in L^r_{\rm loc}(\Omega)$ such that

$$|D_{\xi x}h(x,\xi)|\leq k(x)(1+|\xi|^2)^{\frac{q-1}{2}} \tag{4.4}$$

for a.e. $x\in\Omega$ and for every $\xi\in\mathbb{R}^{N\times n}$.

The following is a higher differentiability result for minimizers of H, that, by the result in [22], are locally Lipschitz continuos.

Theorem 4.1. *Let $v\in W^{1,\infty}_{\rm loc}(\Omega)$ be a local minimizer of the functional H. Assume* (4.1)–(4.4) *for a couple of exponents p,q such that*

$$\frac{q}{p}<1+\frac{1}{n}-\frac{1}{r}. \tag{4.5}$$

Then $u \in W^{2,2}_{\rm loc}(\Omega)$ *and the following estimate holds*

$$\begin{aligned}&\int_{B_\rho} |DV_p(Du)|^2\,dx\\ \le\ & \frac{c}{(R-\rho)^2}\|1+|Du|\|_\infty^{2q-p}\int_{B_{2R}}|Du|^2\,dx + c\|1+|Du|\|_\infty^{2q-p}\int_{B_R}k^2(x)\,dx\\ &+c\|1+|Du|\|_\infty^{q-1}\left(\int_{B_R}k^{\frac{p}{p-1}}(x)\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_{2R}}|Du|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

for every ball $B_\rho \subset B_R \subset B_{2R} \Subset \Omega$.

Proof. Since v is a local minimizer of the functional H, then v satisfies the Euler's system

$$\int_\Omega \sum_{i,\alpha} h_{\xi_i^\alpha}(x,Du)\varphi^\alpha_{x_i}(x)\,dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega;\mathbb{R}^N).$$

Let $B_{2R} \Subset \Omega$ and let $\eta \in C_0^\infty(B_R)$ be a cut off function between B_ρ and B_R for some $\rho < R$. Fixed $1 \le s \le n$, and denoted e_s is the unit vector in the x_s direction, consider the finite differential operator $\tau_{s,h}$, see (2.1), from now on simply denoted τ_h. Choosing $\varphi = \tau_{-h}(\eta^2\tau_h u)$ as test function in the Euler's system, we get, by properties (i) and (ii) of τ_h,

$$\int_\Omega \sum_{i,\alpha} \tau_h(h_{\xi_i^\alpha}(x,Du))D_{x_i}(\eta^2\tau_h u^\alpha)\,dx = 0$$

and so

$$\int_\Omega \sum_{i,\alpha} \tau_h(h_{\xi_i^\alpha}(x,Du))(2\eta\eta_{x_i}\tau_h u^\alpha + \eta^2\tau_h u^\alpha_{x_i})\,dx = 0.$$

Exploiting the definition of τ_h, we get

$$\int_\Omega \big(h_{\xi_i^\alpha}(x+he_s, Du(x+he_s)) - h_{\xi_i^\alpha}(x,Du(x))\big)(2\eta\eta_{x_i}\tau_h u^\alpha + \eta^2\tau_h u^\alpha_{x_i})\,dx = 0$$

i.e.

$$\begin{aligned}&\int_\Omega \eta^2[h_{\xi_i^\alpha}(x+he_s,Du(x+he_s)) - h_{\xi_i^\alpha}(x+he_s,Du(x))]\tau_h u^\alpha_{x_i}\,dx\\ =&\int_\Omega \eta^2[h_{\xi_i^\alpha}(x,Du(x)) - h_{\xi_i^\alpha}(x+he_s,Du(x))]\tau_h u^\alpha_{x_i}\,dx\\ &-2\int_\Omega \eta[h_{\xi_i^\alpha}(x+he_s,Du(x+he_s)) - h_{\xi_i^\alpha}(x+he_s,Du(x))]\eta_{x_i}\tau_h u^\alpha\\ &+2\int_\Omega \eta[h_{\xi_i^\alpha}(x,Du(x)) - h_{\xi_i^\alpha}(x+he_s,Du(x))]\eta_{x_i}\tau_h u^\alpha.\end{aligned}$$

We can write previous equality as follows

$$\begin{aligned}I_0 =:&\int_\Omega \eta^2\int_0^1 \sum_{i,j,\alpha,\beta} h_{\xi_i^\alpha\xi_j^\beta}(x+he_s, Du(x)+\sigma\tau_h Du(x))d\sigma\tau_h u^\alpha_{x_i}\tau_h u^\beta_{x_j}\,dx\\ =&h\int_\Omega \eta^2\int_0^1\sum_{i,\alpha} h_{x_s\xi_i^\alpha}(x+\sigma he_s, Du(x))d\sigma\tau_h u^\alpha_{x_i}\,dx\end{aligned}$$

$$
\begin{aligned}
&- 2\int_\Omega \eta \int_0^1 \sum_{i,j,\alpha,\beta} h_{\xi_i^\alpha \xi_j^\beta}(x+he_s, Du(x)+\sigma\tau_h Du(x))d\sigma \eta_{x_i}\tau_h u^\alpha \tau_h u^\beta_{x_j} \\
&+ 2h\int_\Omega \eta \int_0^1 \sum_{i,\alpha} h_{x_s\xi_i^\alpha}(x+\sigma he_s, Du(x))d\sigma \eta_{x_i}\tau_h u^\alpha \\
=:& I_1+I_2+I_3,
\end{aligned}
$$

that implies

$$
I_0 \le |I_1|+|I_2|+|I_3| \tag{4.6}
$$

The ellipticity assumption (4.3) yields

$$
I_0 \ge \nu \int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx \tag{4.7}
$$

By assumption (4.4) we get

$$
\begin{aligned}
|I_1| &\le |h|\int_\Omega \eta^2 \int_0^1 k(x+\sigma he_s)d\sigma(1+|Du(x)|^2)^{\frac{q-1}{2}}|\tau_h Du|\,dx \\
&\le |h|\|1+|Du|\|_\infty^{\frac{2q-p}{2}}\int_\Omega \eta^2\int_0^1 k(x+\sigma he_s)d\sigma(1+|Du(x)|^2)^{\frac{p-2}{4}}|\tau_h Du|\,dx \\
&\le \frac{\nu}{4}\int_\Omega \eta^2(1+|Du(x)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx \\
&\quad + c_\nu |h|^2\|1+|Du|\|_\infty^{2q-p}\int_\Omega \eta^2\left(\int_0^1 k(x+\sigma he_s)d\sigma\right)^2 dx,
\end{aligned} \tag{4.8}
$$

where in the last line we used Young's inequality. The right inequality in (4.3) yields

$$
\begin{aligned}
|I_2| &\le c(L_2)\int_\Omega \eta|D\eta|(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{q-2}{2}}|\tau_h Du||\tau_h u|\,dx \\
&\le c(L_2)\|1+|Du|\|_\infty^{\frac{2q-p}{2}}\int_\Omega \eta|D\eta|(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{4}}|\tau_h Du||\tau_h u|\,dx \\
&\le \frac{\nu}{4}\int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2\,dx \\
&\quad + c_{\nu,L_2}\|1+|Du|\|_\infty^{2q-p}\int_\Omega |D\eta|^2|\tau_h u|^2\,dx.
\end{aligned} \tag{4.9}
$$

Finally, using again assumption (4.4) and Hölder's inequality, we obtain

$$
\begin{aligned}
|I_3| &\le 2|h|\int_\Omega \eta|D\eta|\int_0^1 k(x+\sigma he_s)d\sigma(1+|Du(x)|^2)^{\frac{q-1}{2}}|\tau_h u|\,dx \\
&\le |h|\|1+|Du|\|_\infty^{q-1}\int_\Omega \eta|D\eta|\int_0^1 k(x+\sigma he_s)d\sigma|\tau_h u|\,dx \\
&\le |h|\|1+|Du|\|_\infty^{q-1}\left(\int_\Omega \eta|D\eta|\left(\int_0^1 k(x+\sigma he_s)d\sigma\right)^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}
\end{aligned}
$$

$$\times \left(\int_\Omega \eta |D\eta| |\tau_h u|^p \, dx \right)^{\frac{1}{p}} . \tag{4.10}$$

Inserting (4.7), (4.8), (4.9) and (4.10) in (4.6), we get

$$\begin{aligned}
&\nu \int_\Omega \eta^2 (1 + |Du(x)|^2 + |Du(x + he_s)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 \, dx \\
\le &\frac{\nu}{2} \int_\Omega \eta^2 (1 + |Du(x)|^2 + |Du(x + he_s)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 \, dx \\
&+ c_{\nu, L_2} \|1 + |Du|\|_\infty^{2q-p} \int_\Omega |D\eta|^2 |\tau_h u|^2 \, dx \\
&+ c_\nu |h|^2 \|1 + |Du|\|_\infty^{2q-p} \int_\Omega \eta^2 \left(\int_0^1 k(x + \sigma h e_s) d\sigma \right)^2 dx \\
&+ |h| \|1 + |Du|\|_\infty^{q-1} \left(\int_\Omega \eta |D\eta| \left(\int_0^1 k(x + \sigma h e_s) d\sigma \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\
&\quad \times \left(\int_\Omega \eta |D\eta| |\tau_h u|^p \, dx \right)^{\frac{1}{p}} .
\end{aligned}$$

Reabsorbing the first integral in the right hand side by the left hand side and recalling the properties of the cut off function η, we obtain

$$\begin{aligned}
&\int_{B_\rho} (1 + |Du(x)|^2 + |Du(x + he_s)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 \, dx \\
\le \; &\frac{c}{(R - \rho)^2} \|1 + |Du|\|_\infty^{2q-p} \int_{B_R} |\tau_h u|^2 \, dx \\
&+ c |h|^2 \|1 + |Du|\|_\infty^{2q-p} \int_{B_R} \left(\int_0^1 k(x + \sigma h e_s) d\sigma \right)^2 dx \\
&+ c |h| \|1 + |Du|\|_\infty^{q-1} \left(\int_{B_R} \left(\int_0^1 k(x + \sigma h e_s) d\sigma \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\
&\quad \times \left(\int_{B_R} |\tau_h u|^p \, dx \right)^{\frac{1}{p}} ,
\end{aligned}$$

where $c = c(\nu, L_2, n, N)$. By property (iii) of the finite difference operator we deduce that

$$\begin{aligned}
&\int_{B_\rho} (1 + |Du(x)|^2 + |Du(x + he_s)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 \, dx \\
\le \; &\frac{c}{(R - \rho)^2} |h|^2 \|1 + |Du|\|_\infty^{2q-p} \int_{B_{2R}} |Du|^2 \, dx \\
&+ c |h|^2 \|1 + |Du|\|_\infty^{2q-p} \int_{B_R} \left(\int_0^1 k(x + \sigma h e_s) d\sigma \right)^2 dx
\end{aligned}$$

$$+c|h|^2\|1+|Du|\|_\infty^{q-1}\left(\int_{B_R}\left(\int_0^1 k(x+\sigma h e_s)d\sigma\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}$$
$$\times\left(\int_{B_{2R}}|Du|^p\,dx\right)^{\frac{1}{p}}. \tag{4.11}$$

Dividing (4.11) by $|h|^2$ and using Lemma 2.1 in the left hand side, we get

$$\begin{aligned}
&\int_{B_\rho}\frac{|\tau_h V_p(Du)|^2}{|h|^2}\,dx\\
\leq\ & \frac{c}{(R-\rho)^2}\|1+|Du|\|_\infty^{2q-p}\int_{B_{2R}}|Du|^2\,dx\\
&+c\|1+|Du|\|_\infty^{2q-p}\int_{B_R}\left(\int_0^1 k(x+\sigma h e_s)d\sigma\right)^2 dx\\
&+c\|1+|Du|\|_\infty^{q-1}\left(\int_{B_R}\left(\int_0^1 k(x+\sigma h e_s)d\sigma\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}\\
&\times\left(\int_{B_{2R}}|Du|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Letting h go to 0, by property (iv) of the finite difference operator, we conclude

$$\begin{aligned}
&\int_{B_\rho}|DV_p(Du)|^2\,dx\\
\leq\ & \frac{c}{(R-\rho)^2}\|1+|Du|\|_\infty^{2q-p}\int_{B_{2R}}|Du|^2\,dx+c\|1+|Du|\|_\infty^{2q-p}\int_{B_R}k^2(x)\,dx\\
&+c\|1+|Du|\|_\infty^{q-1}\left(\int_{B_R}k^{\frac{p}{p-1}}(x)\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_{2R}}|Du|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.12}$$

Since $k\in L^r$, with $r>n\geq 2\geq\frac{p}{p-1}$, estimate (4.12) implies the conclusion. □

5. Proof of Theorem 1.1

Using the previous results and an approximation procedure, we can prove of our main result.

Proof of Theorem 1.1. For $f(x,\xi)$ satisfying the assumptions (1.3)–(1.6), let us introduce the sequence

$$f_h(x,\xi)=f(x,\xi)+\frac{1}{h}(1+|\xi|^2)^{\frac{ps}{2(s+1)}}. \tag{5.1}$$

Note that $f_h(x,\xi)$ satisfies the following set of conditions

$$\frac{1}{h}(1+|\xi|^2)^{\frac{ps}{2(s+1)}}\leq f_h(x,\xi)\leq(1+L)\,(1+|\xi|^2)^{\frac{q}{2}}, \tag{5.2}$$

$$\frac{c_1}{h}\,(1+|\xi|^2)^{\frac{ps}{2(s+1)}-2}|\lambda|^2\leq\langle D_{\xi\xi}f_h(x,\xi)\lambda,\lambda\rangle, \tag{5.3}$$

$$|D_{\xi\xi} f_h(x,\xi)| \le c_2(1+L)\,(1+|\xi|^2)^{\frac{q-2}{2}}, \tag{5.4}$$

$$|D_{\xi x} f_h(x,\xi)| \le k(x)(1+|\xi|^2)^{\frac{q-1}{2}} \tag{5.5}$$

for some constants $c_1, c_2 > 0$, for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{N\times n}$.

Now, fix a ball $B_R \Subset \Omega$, and let $v_h \in W^{1,\frac{ps}{s+1}}(B_R,\mathbb{R}^N)$ be the unique solution to the problem

$$\min\left\{\int_{B_R} f_h(x,Dv)\,dx:\; v_h \in u + W_0^{1,\frac{ps}{s+1}}(B_R,\mathbb{R}^N)\right\}. \tag{5.6}$$

Since $f_h(x,\xi)$ satisfies (5.2), (5.3), (5.4) (5.5) with $k \in L^r$, $r > n$, and (1.6) holds, then by the result in [22] we have that $v_h \in W^{1,\infty}_{\mathrm{loc}}(B_R)$ and by Theorem 4.1, used with p replaced by $p\frac{s}{s+1}$, we also have $v_h \in W^{2,2}_{\mathrm{loc}}(B_R)$.

Since $f_h(x,\xi)$ satisfies (5.2), by the minimality of v_h we get

$$\begin{aligned}
&\int_{B_R} |Dv_h|^{\frac{ps}{s+1}}\,dx \le c_s\int_{B_R} a(x)|Dv_h|^p + c_s\int_{B_R}\frac{1}{a^s(x)}\,dx\\
\le\;& c_s\int_{B_R} f_h(x,Dv_h)\,dx + c_s\int_{B_R}\frac{1}{a^s(x)}\,dx\\
\le\;& c_s\int_{B_R} f_h(x,Du)\,dx + c_s\int_{B_R}\frac{1}{a^s(x)}\,dx\\
=\;& c_s\int_{B_R} f(x,Du)\,dx + \frac{c_s}{h}\int_{B_R}(1+|Du|)^{\frac{ps}{s+1}}\,dx + c_s\int_{B_R}\frac{1}{a^s(x)}\,dx\\
\le\;& c_s\int_{B_R} f(x,Du)\,dx + c_s\int_{B_R}(1+|Du|)^{\frac{ps}{s+1}}\,dx + c_s\int_{B_R}\frac{1}{a^s(x)}\,dx.
\end{aligned}$$

Therefore the sequence v_h is bounded in $W^{1,\frac{ps}{s+1}}(B_R)$, so there exists $v \in u + W_0^{1,\frac{ps}{s+1}}(B_R)$ such that, up to subsequences,

$$v_h \rightharpoonup v \qquad \text{weakly in } W^{\frac{ps}{s+1}}(B_R). \tag{5.7}$$

On the other hand, we can apply Theorem 3.3 to $f_h(x,\xi)$ since the assumptions are satisfied, with b replaced by $1+L$. Thus, we are legitimate to apply estimates (3.7) and (3.8) to the solutions v_h to obtain

$$\begin{aligned}
&\|Dv_h\|_{L^\infty(B_\rho)} \le C\mathcal{K}_R^{\tilde\vartheta}\left(\int_{B_R}(1+f_h(x,Dv_h))\,dx\right)^{\tilde\varsigma}\\
\le\;& C\mathcal{K}_R^{\tilde\vartheta}\left(\int_{B_R}(1+f_h(x,Du))\,dx\right)^{\tilde\varsigma}\\
=\;& C\mathcal{K}_R^{\tilde\vartheta}\left(\int_{B_R}(1+f(x,Du)+\frac{1}{h}(1+|Du|^2)^{\frac{ps}{2(s+1)}})\,dx\right)^{\tilde\varsigma}\\
\le\;& C\mathcal{K}_R^{\tilde\vartheta}\left(\int_{B_R}(1+f(x,Du)+(1+|Du|^2)^{\frac{ps}{2(s+1)}})\,dx\right)^{\tilde\varsigma},
\end{aligned} \tag{5.8}$$

with $C, \tilde\vartheta, \tilde\varsigma$ independent of h and $0<\rho<R$. Therefore, up to subsequences,

$$v_h \rightharpoonup v \qquad \text{weakly* in } W^{1,\infty}(B_\rho). \tag{5.9}$$

Our next aim is to show that $v = u$. The lower semicontinuity of $u \mapsto \int_{B_R} f(x, Du)$ and the minimality of v_h imply

$$\begin{aligned}
&\int_{B_R} f(x,Dv)\,dx \le \liminf_h \int_{B_R} f(x,Dv_h)\,dx \le \liminf_h \int_{B_R} f_h(x,Dv_h)\,dx \\
\le\;& \liminf_h \int_{B_R} f_h(x,Du)\,dx \\
=\;& \liminf_h \int_{B_R} \big(f(x,Du) + \frac{1}{h}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\big)\,dx \\
=\;& \int_{B_R} f(x,Du)\,dx.
\end{aligned}$$

The strict convexity of f yields that $u = v$. Therefore passing to the limit as $h \to \infty$ in (5.8) we get

$$\|Du\|_{L^\infty(B_\rho)} \le C\mathcal{K}_R^{\tilde\vartheta}\left(\int_{B_R} (1 + f(x,Du) + (1+|Du|^2)^{\frac{ps}{2(s+1)}})\,dx\right)^{\tilde\varsigma},$$

i.e. (1.7). Moreover, we are legitimate to apply estimate (3.8) to each v_h thus getting

$$\begin{aligned}
\int_{B_\rho} a(x)(1+|Dv_h|^2)^{\frac{p-2}{2}}|D^2v_h|^2\,dx \;&\le\; c\left(\int_{B_R}(1+f_h(x,Dv_h))\,dx\right)^{\tilde\varrho} \\
&\le\; c\left(\int_{B_R}(1+f_h(x,Du))\,dx\right)^{\tilde\varrho} \\
&=\; c\left(\int_{B_R}\left(1+f(x,Du)+\frac{1}{h}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\right)\,dx\right)^{\tilde\varrho},
\end{aligned}$$

where we used the minimality of v_h and the definition of $f_h(x,\xi)$. Since $v_h \to u$ a.e. up to a subsequence, we conclude that

$$\begin{aligned}
&\int_{B_\rho} a(x)(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx \le \liminf_h \int_{B_\rho} a(x)(1+|Dv_h|^2)^{\frac{p-2}{2}}|D^2v_h|^2\,dx \\
&\le c\liminf_h \left(\int_{B_R}\left(1+f(x,Du)+\frac{1}{h}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\right)\,dx\right)^{\tilde\varrho} \\
&= \left(\int_{B_R}\left(1+f(x,Du)\right)\,dx\right)^{\tilde\varrho},
\end{aligned}$$

i.e. (1.8). □

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