Transport equations with nonlocal diffusion and applications to Hamilton-Jacobi equations

Alessandro Goffi

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Abstract

We investigate regularity and a priori estimates for Fokker-Planck and Hamilton-Jacobi equations with unbounded ingredients driven by the fractional Laplacian of order $s \in (1/2, 1)$. As for Fokker-Planck equations, we establish integrability estimates under a fractional version of the Aronson-Serrin interpolated condition on the velocity field and Bessel regularity when the drift has low Lebesgue integrability with respect to the solution itself. Using these estimates, through the Evans' nonlinear adjoint method we prove new integral, sup-norm and Hölder estimates for weak and strong solutions to fractional Hamilton-Jacobi equations with unbounded right-hand side and polynomial growth in the gradient. Finally, by means of these latter results, exploiting Calderón-Zygmund-type regularity for linear nonlocal PDEs and Gagliardo-Nirenberg inequalities, we deduce maximal L^q-regularity for fractional Hamilton-Jacobi equations having first order terms below the natural growth.

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Keywords. Fractional Fokker-Planck equations, Fractional Hamilton-Jacobi equations with unbounded data, Maximal regularity, Hölder regularity, Adjoint method.

Contents

1 Introduction

In this paper, we analyze regularity properties of transport equations of Fokker-Planck-type and Hamilton-Jacobi equations with fractional diffusion driven by a fractional power of the Laplacian, $(-\Delta)^s$, with subcritical order $s \in (\frac{1}{2}, 1)$. In particular, we address well-posedness, parabolic Bessel regularity and integrability estimates for solutions to (backward) fractional Fokker-Planck equations of the form

$$\begin{cases} -\partial_t \rho(x,t) + (-\Delta)^s \rho(x,t) + \mathrm{div}(b(x,t)\,\rho(x,t)) = 0 & \text{in } Q_\tau := \mathbb{T}^d \times (0,\tau)\,, \\ \rho(x,\tau) = \rho_\tau(x) & \text{in } \mathbb{T}^d\,, \end{cases} \tag{1}$$

where the nonlocal diffusion operator is defined on the flat torus $\mathbb{T}^d \equiv \mathbb{R}^d \backslash \mathbb{Z}^d$ [81], under minimal integrability conditions on the drift, mainly when either $b \in L_t^{\mathcal{Q}}(L_x^{\mathcal{P}})$ (cf (3) below) or $b \in L^k(\rho\, dxdt)$, $k > 1$, without requiring a control on its divergence.
We finally apply our results for the above transport-like equation to obtain a priori gradient estimates for strong solutions and regularization effects for weak solutions of fractional Hamilton-Jacobi equations with subcritical diffusion of the form

$$\begin{cases} \partial_t u(x,t) + (-\Delta)^s u(x,t) + H(x, Du(x,t)) = f(x,t) & \text{in } Q_T\,, \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}^d, \end{cases} \tag{2}$$

where $f \in L^q(Q_T)$ for some $q > 1$ and $H(x, Du) \sim |Du|^\gamma$, $\gamma > 1$, i.e. H has superlinear gradient growth. Following the approach in [35, 36], we derive new integral, sup-norm and Hölder bounds by means of the nonlinear adjoint method introduced by L.C. Evans [44, 43], that in turn requires to exploit the regularity properties of the adjoint of the linearization of (2), which is of the form (1). Then, we combine these latter estimates with Gagliardo-Nirenberg interpolation inequalities and maximal regularity in Lebesgue spaces for fractional heat equations to deduce gradient bounds for (2). This approach to deduce a priori estimates for nonlinear problems is inspired by [6] (see also [16, 15]), where semi-linear equations with quadratic growth are studied. These interpolation methods have been also employed in e.g. [74] (see also the references therein) and recently revived in [48, 47, 36] in the context of Mean-Field Games [63, 62].

As announced, in order to study the regularity properties of (1) we need to extend well-known results for linear viscous equations with discontinuous coefficients to the fractional framework. In the viscous case, the first works date back to [59, 8, 9] for linear and quasi-linear problems, see also [19] for the case of measurable ingredients. Within this framework, well-posedness and integrability estimates are well-established when $b \in L_t^{\mathcal{Q}}(L_x^{\mathcal{P}})$ with $\mathcal{Q}, \mathcal{P}$ satisfying

$$\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} \leq \frac{1}{2}\,, \mathcal{P} \in [d, \infty]\,, \mathcal{Q} \in [2, \infty]\,.$$

We emphasize that this assumption on the drift is sharp to get integrability estimates in Lebesgue classes, at least in the viscous case, cf [18].
As far as we know, nonlocal equations with unbounded coefficients are treated in [65] for fractional heat equations, and [2, 3, 53] for gradient perturbations of the fractional Laplacian. Most of the regularity results we present here are new when the velocity field satisfies a fractional version of the above Aronson-Serrin condition, i.e. $b \in L_t^{\mathcal{Q}}(L_x^{\mathcal{P}})$ with $\mathcal{Q}, \mathcal{P}$ satisfying

$$\frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} < \frac{2s-1}{2s}\,. \tag{3}$$

The second step in the analysis of (1) concerns fractional Bessel regularity estimates of solutions to (1) in terms of the crossed quantity

$$\iint_{Q_\tau} |b|^k \rho \, dxdt \; , k > 1 \; , \tag{4}$$

which is widely analyzed for classical viscous Fokker-Planck equations in [75, 79]. In particular, we prove that for some $k > 1$ one has the estimate

$$\|(-\Delta)^{s-\frac{1}{2}}\rho\|_{\sigma'} \lesssim \iint_{Q_\tau} |b|^k \rho + \|\rho_\tau\|_{p'}$$

for σ' in some range determined in terms of the regularity p' of the terminal data. This bound is obtained by duality, following [75, 35, 36], via a maximal regularity estimate for nonlocal equations with divergence-type terms of the form

$$\|(-\Delta)^{s-\frac{1}{2}}\rho\|_{\sigma'} \lesssim \|b\rho\|_{\sigma'} + \|\rho_\tau\|_{p'} \; .$$

These estimates are crucial to study regularity properties for PDEs arising in Mean Field Games, cf [39, 35, 79, 80], see also [22] for the time-fractional framework. We remark that when $b = -D_pH(x, Du)$, (1) becomes the adjoint equation to (2) and bounds on the quantity (4) are natural for the mean field equations by duality [79, 34, 35].

Owing to these results for (1), we deduce sup-norm, integral and Hölder estimates for solutions to (2) with $f \in L^q$. These bounds are accomplished with the nonlinear adjoint method [44] and exploit the aforementioned Bessel regularity properties of solutions to Fokker-Planck equations. In particular, our Hölder estimates for weak solutions to Hamilton-Jacobi equations hold for $2s \le \gamma < \frac{1}{2-2s}$. To our knowledge, these are the first Hölder regularity results for Hamilton-Jacobi equations with unbounded ingredients and fractional diffusion in the literature. Furthermore, these also show a Hölder's regularization effect similar to [84] (where $\gamma = 2s$), although here solutions are meant in a different sense and the equation presents unbounded right-hand side.

Finally, we are able to partially prove optimal L^q-regularity as addressed in [37, 36] for elliptic and evolutive equations respectively, driven by the Laplacian. This states, within our context, a control on $\partial_t u, (-\Delta)^s u, |Du|^\gamma \in L^q$ in terms of $f \in L^q$ for an appropriate range of the integrability exponent $q > \bar{q}$, see Section 5.5.1 for details on the threshold $\bar{q}$. As a result, we find that (2) behaves in terms of regularity as the fractional heat equation for appropriate values of q despite the nonlinear character of the coercive term $H = H(x, p) \sim |p|^\gamma$, at least when $\gamma < 2s$. In the classical viscous case, the proof has been given refining the integral Bernstein method, which, however, does not seem the right path to treat both fractional and time-dependent problems like (2).
We recall that maximal L^q-regularity properties of Calderón-Zygmund-type are well-known for general abstract linear evolution equations, see e.g. [50, 61, 71], and are recalled in Lemma 5.15 below. Combining integral estimates for solutions to Hamilton-Jacobi equations, maximal regularity for linear problems and generalized Gagliardo-Nirenberg inequalities involving integral norms of the form

$$\|Du\|^{\gamma}_{L^{\gamma q}_{x,t}} \lesssim \|u\|^{\gamma\theta}_{L^q(H^{2s}_q)} \|u\|^{\gamma(1-\theta)}_{L^\infty_t(L^r_x)}$$

for some $r \in (1, \infty)$, $\theta \in (0, 1)$ such that $\theta\gamma < 1$, we show that maximal L^q-regularity for (2) holds for strong solutions when $f \in L^q(Q_T)$, $q > \frac{d+2s}{(2s-1)\gamma'}$, $\gamma < 2s$, i.e. we have

$$\|\partial_t u\|_{L^q(Q_T)} + \|u\|_{L^q(0,T;H^{2s}_q(\mathbb{T}^d))} + \||Du|^\gamma\|_{L^q(Q_T)} \le C(\|f\|_{L^q(Q_T)}, \|u(0)\|_{W^{2s-2s/q,q}(\mathbb{T}^d)}, q, d, T, s, H) \; .$$

Recall that $\|u\|_{L^q(0,T;H_q^{2s}(\mathbb{T}^d))} \sim \|u\|_{L^q(Q_T)} + \|(-\Delta)^s u\|_{L^q(Q_T)}$. We remark that, in the classical viscous case, the regime $\gamma = 2$ is usually easier to address using the Hopf-Cole transform, cf [70, 37, 36]. Here, however, when $\gamma = 2s$ it is not known whether there exists a fractional analogue of that transformation reducing (2) into a simpler fractional PDE. Thus, even the natural growth case becomes not trivial to analyze. We emphasize in passing that our results are at this stage conditional to the well-posedness of the adjoint equation to (1), see assumption (I) and Remarks 4.5 and 5.5. We conclude by saying that the range $s \in (0, \frac{1}{2}]$ is out of the methods proposed in the present paper.

We now recall some related results for (1) and (2). As for fractional Fokker-Planck equations, when $b \in L^\infty$ or some control on the divergence is assumed, we refer to [30] for stationary problems and to [34] for the evolutive case. Instead, the viscous case is widely analyzed, even under weaker assumptions on the velocity field [59, 75, 19, 18, 39, 35, 79, 36].
As for Hamilton-Jacobi equations, Hölder's regularity results have been largely investigated for parabolic problems in the borderline cases $s = 0$ and $s = 1$. For first order and second order degenerate problems, we refer to [25, 23, 31], while to [27] for Sobolev regularity estimates. In the uniformly parabolic case, the first results we are aware of go back to [60]. Recently, second order degenerate problems with unbounded right-hand side have been analyzed in [29], see also [91] for PDEs driven by the Laplacian via De Giorgi's techniques. Hölder, integral and sup-norm estimates for the parabolic problem have been addressed in the paper [36] for the viscous case $s = 1$, and we recover those results by letting $s \to 1$. As for integrability estimates, we refer to [26] for the degenerate case and [36] for the viscous problem.
Hölder's regularity of nonlocal Hamilton-Jacobi equations with supernatural growth has been first treated in [28]. Hölder's regularization effect of solutions to fractional Hamilton-Jacobi equations in the subcritical growth case $\gamma = 2s$ starting from a bounded initial data has been observed by L. Silvestre in [84]. There, the author has also obtained Hölder bounds in the fractional supercritical regime $2s < \gamma < 2s+\varepsilon$ imposing some smallness conditions on the data. More recently, the regularization effect in Besov spaces when $s = 1/2$ has been investigated in [52] under a smallness condition on the initial data.
Instead, the literature on Lipschitz regularity is huge. The conservation of Lipschitz regularity (i.e. with $u(0) \in W^{1,\infty}$) for every $s \in (0, 1)$ goes back to [42, Theorem 5] (see also [51, 54]). Besides, Lipschitz regularity has been investigated in the case of critical diffusion $s = 1/2$ by L. Silvestre in [85]. Gradient regularity for viscosity solutions of coercive fractional Hamilton-Jacobi equations has been widely analyzed using viscosity solutions' techniques. In [12] the authors have analyzed Lipschitz regularity of solutions via Ishii-Lions method when f is bounded (which requires the restriction $\gamma < 2s$, as for the classical viscous case $s = 1$) and via a weak version of the Bernstein method in the periodic setting [13], where $f \in W^{1,\infty}$ in the space variable and $\gamma > 1$, even for more general integro-differential operators than fractional powers of the Laplacian. We finally mention that fractional Hamilton-Jacobi-type PDEs and regularity issues have been recently investigated in the framework of periodic homogenization problems [10].
As for the stationary counterpart of (2) with unbounded terms in Lebesgue scales we mention the works [2, 1]. Related results for Hamilton-Jacobi equations can be found in [37, 68, 11, 14] and the references therein. Other regularity estimates for space-fractional Fokker-Planck equations, also combined with Hamilton-Jacobi equations in the context of Mean Field Games, can be found in [34, 30], while for advection equations with fractional diffusion we refer the reader to [86, 87].

Outline. Section 2 presents a list of the main results and the assumptions used throughout the paper. Section 3 is devoted to introduce the main functional spaces and related embedding properties. Section 4 concerns the analysis of the well-posedness, Bessel regularity and integrability estimates for fractional Fokker-Planck equations, while Section 5 comprises the applications to regularity issues for equations of Hamilton-Jacobi type with nonlocal diffusion.

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2 Assumptions and main results

Throughout all the manuscript we will assume $s \in (1/2, 1)$ unless otherwise stated. Our first main results concern the fractional Fokker-Planck equation (1): in the first one, b is assumed to belong to mixed Lebesgue classes in the fractional Aronson-Serrin zone, while in the second one parabolic Bessel regularity is studied in terms of the crossed quantity (4). More precisely, in this section for $\mu \in \mathbb{R}$ we deal with anisotropic spaces of the form

$$\mathcal{H}_p^\mu(Q_T) := \{u \in L^p(0,T; H_p^\mu(\mathbb{T}^d)); \partial_t u \in L^p(0,T; H_p^{\mu-2s}(\mathbb{T}^d))\} ,$$

where $H_p^\mu(\mathbb{T}^d)$ is the space of Bessel potentials on the torus. We refer to Section 3.2 for additional properties of these spaces.
We will assume the following additional assumption, and we refer to Remarks 4.5 and 5.5 for additional comments on this hypothesis.

(I) There exists a unique weak solution to the dual problem of (1)

$$\begin{cases} \partial_t v + (-\Delta)^s v - b \cdot Dv = 0 & \text{in } Q_\omega := \mathbb{T}^d \times (\omega, \tau) , \\ v(x, \omega) = v_\omega(x) & \text{in } \mathbb{T}^d . \end{cases} \tag{5}$$

with $\omega \in [0, \tau)$, where b satisfies (3). In addition, if $v(\omega) \geq 0$, then $v \geq 0$ a.e. on Q_ω.

Theorem 2.1. *Let $b \in L^{\mathcal{Q}}(0, \tau; L^{\mathcal{P}}(\mathbb{T}^d))$ with $\mathcal{P} \in (d/(2s-1), \infty)$ and $\mathcal{Q} \in (2s/(2s-1), \infty]$ satisfying*

$$\frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} < \frac{2s-1}{2s} ,$$

and $\rho_\tau \in H^{s-1}(\mathbb{T}^d)$. Then, there exists a weak solution $\rho \in \mathcal{H}_2^{2s-1}(Q_\tau)$ to (1)*. If, in addition, $\rho_\tau \in L^p(\mathbb{T}^d)$, $p \in (1, \infty]$, then $\rho \in L^\infty(0, \tau; L^p(\mathbb{T}^d))$. Finally, if (I) holds and $\rho_\tau \geq 0$, the solution is unique and $\rho \geq 0$ a.e. on Q_τ.*

Theorem 2.2. *Let ρ be a (non-negative) weak solution to* (1) *and $1 < \sigma' < \frac{d+2s}{2s-1}$.*

(i) There exists $C > 0$, depending on T, σ', d, s such that

$$\|\rho\|_{\mathcal{H}_{\sigma'}^{2s-1}(Q_\tau)} \leq C \left(\iint_{Q_\tau} |b(x,t)|^{m'} \rho(x,t)\, dxdt + \|\rho_\tau\|_{L^1(\mathbb{T}^d)} \right) ,$$

where

$$1 < \sigma' < \frac{d+2s}{d+2s-1}$$

and

$$m' = 1 + \frac{d+2s}{\sigma(2s-1)} .$$

(ii) There exists $C > 0$, depending on T, σ', d, s such that

$$\|\rho\|_{\mathcal{H}_{\sigma'}^{2s-1}(Q_\tau)} \leq C \left(\iint_{Q_\tau} |b(x,t)|^{m'} \rho(x,t)\, dxdt + \|\rho_\tau\|_{L^{p'}(\mathbb{T}^d)} \right) ,$$

where either

$$\sigma' = \frac{d+2s}{d+2s-1} \text{ with any } p > 1 \,,$$

or

$$\sigma' > \frac{d+2s}{d+2s-1} \text{ with } p = \frac{d\sigma}{d+2s-\sigma}$$

and

$$m' = 1 + \frac{d+2s}{\sigma(2s-1)}.$$

As for (2), we suppose that $H(x,p)$ is $C^1(\mathbb{T}^d \times \mathbb{R}^d)$, convex in p and has polynomial growth in the gradient entry, i.e.

$$\begin{aligned} &\text{there exist constants } \gamma > 1 \text{ and } C_H > 0 \text{ such that} \\ &C_H^{-1}|p|^\gamma - C_H \le H(x,p) \le C_H(|p|^\gamma + 1) \,, \\ &D_pH(x,p)\cdot p - H(x,p) \ge C_H^{-1}|p|^\gamma - C_H \,, \\ &C_H^{-1}|p|^{\gamma-1} - C_H \le |D_pH(x,p)| \le C_H|p|^{\gamma-1} + C_H \,, \end{aligned} \tag{H}$$

for every $x \in \mathbb{T}^d$, $p \in \mathbb{R}^d$. Moreover, we suppose without loss of generality that $H \ge 0$. Recall that the Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, $L(x,\nu) := \sup_p\{p\cdot\nu - H(x,p)\}$, namely the Legendre transform of H in the p-variable, is well defined by the superlinear behavior of $H(x,\cdot)$. Moreover, by convexity of $H(x,\cdot)$,

$$H(x,p) = \sup_{\nu\in\mathbb{R}^d}\{\nu\cdot p - L(x,\nu)\},$$

and

$$H(x,p) = \nu\cdot p - L(x,\nu) \quad \text{if and only if} \quad \nu = D_pH(x,p). \tag{6}$$

The following properties of L are standard (see, e.g. [24]): for some $C_L > 0$,

$$C_L^{-1}|\nu|^{\gamma'} - C_L \le L(x,\nu) \le C_L|\nu|^{\gamma'} \tag{L1}$$

(7)

for all $\nu \in \mathbb{R}^d$.

Concerning the case $\gamma \ge 2$ and when dealing with Hölder regularity, we will further assume some additional regularity in the x-variable, i.e for $\alpha \in (0,1)$ to be determined,

$$H(x,p) - H(x+\xi,p) \le C_H|\xi|^\alpha\left(|D_pH(x,p)|^{\gamma'} + 1\right) \tag{H_α}$$

for all $x,\xi \in \mathbb{T}^d$ and $p \in \mathbb{R}^d$. A prototype example of H satisfying (H) is

$$H(x,p) = h(x)|p|^\gamma + b(x)\cdot p, \qquad 0 < h_0 \le h(x), \quad h,b \in C(\mathbb{T}^d).$$

Note that whenever $h \in C^\alpha(\mathbb{T}^d)$, this model Hamiltonian satisfies also (H_α). Unless otherwise stated, in the next results we will always assume that (I) holds to have the full well-posedness of the adjoint problem. Then, our first results for (2) concern sup-norm and integral estimate for strong solutions as in Definition 5.2 when $f \in L^q$, obtained via the nonlinear adjoint method via the strategy already implemented in [35, 36].

Theorem 2.3. *Let* (H) *be in force and* $\gamma > 1$, $q > \frac{d+2s}{2s}$. *Then, there exists a constant* $C > 0$ *depending solely on* $T, d, s, \|f\|_{L^q(Q_\tau)}, \|u_0\|_{C(\mathbb{T}^d)}$ *such that any global weak solution to* (2) *satisfies*

$$\|u(\cdot,\tau)\|_{C(\mathbb{T}^d)} \le C \text{ for all } \tau \in [0,\tau] \,.$$

The estimate holds even for strong solutions to (2).

Theorem 2.4. *Let* (H) *be in force and* $\gamma \in (1, 2s)$*. Let also* $u \in \mathcal{H}_q^{2s}(Q_T)$ *be a strong solution to* (2) *with* $q > \frac{d+2s}{(2s-1)\gamma'}$*. The following assertions hold*

(A) There exists a constant $C_1 > 0$*, depending on* $T, d, s, C_H, \|f\|_{L^q(Q_\tau)}, \|u_0\|_{W^{2s-2s/q,q}(\mathbb{T}^d)}$, $q = \frac{d+2s}{2s}$*, such that any strong solution to* (2) *satisfies*

$$\|u\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))} \leq C_1$$

for $1 \leq p < \infty$.

(B) There exists constant $C_2 > 0$*, depending on* $T, d, s, C_H \|f\|_{L^q(Q_\tau)}, \|u_0\|_{W^{2s-2s/q,q}(\mathbb{T}^d)}$*, such that any strong solution to* (2) *satisfies*

$$\|u\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))} \leq C_2$$

with $p = \frac{dq}{d+2s-2sq}$ *if* $q < \frac{d+2s}{2s}$.

Owing to a similar approach and following the scheme of [36], spatial Hölder's regularity estimates for weak energy solutions to fractional Hamilton-Jacobi equations with unbounded right-hand side are provided.

Theorem 2.5. *Assume* (H), *(H_α) and* $2s \leq \gamma < \frac{1}{2-2s}$.

(C) Let $f \in L^q(Q_T)$ *with* $q > \frac{d+2s}{(2s-1)\gamma'}$*. Let u be a local weak solution to* (2)*. Then, there exists* $C_1 > 0$ *depending on* $t_1, C_H, \|u\|_{C(\overline{Q}_T)}, \|f\|_{L^q(Q_T)}, q, d, T, s$ *such that*

$$\sup_{t\in(t_1,T)} [u(\cdot, t)]_{C^\alpha(\mathbb{T}^d)} \leq C_1$$

with $\alpha \leq \gamma'(2s-1) - \frac{d+2s}{q}$ *if* $q < \frac{d+2s}{\gamma'(2s-1)-1}$ *and* $\alpha \in (0, 1)$ *if* $q \geq \frac{d+2s}{\gamma'(2s-1)-1}$.

(D) If u is a strong solution in $\mathcal{H}_q^{2s}(Q_T)$ *with* $q > \frac{d+2s}{(2s-1)\gamma'}$ *to* (2)*, then* $u \in L^\infty(0, T; C^\alpha(\mathbb{T}^d))$*. In particular, there exists a positive constant* C_2 *depending on* $C_H, \|u_0\|_{C^\alpha(\mathbb{T}^d)}, \|f\|_{L^q(Q_T)}, q, d, T, s$ *such that*

$$\sup_{t\in(0,T)} [u(\cdot, t)]_{C^\alpha(\mathbb{T}^d)} \leq C_2 \ .$$

with $\alpha \leq \gamma'(2s-1) - \frac{d+2s}{q}$ *if* $q < \frac{d+2s}{\gamma'(2s-1)-1}$ *and* $\alpha \in (0, 1)$ *if* $q \geq \frac{d+2s}{\gamma'(2s-1)-1}$.

In the last part of the paper, we address optimal L^q-regularity in the form of a priori estimates for strong solutions, that is stated in the next

Theorem 2.6. *Assume* (H)*. Let* $u \in \mathcal{H}_q^{2s}(Q_T)$ *be a strong solution to* (2) *with* $\gamma \in (\frac{d+6s-2}{d+2s}, 2s)$ *and* $q > \frac{d+2s}{(2s-1)\gamma'}$*, and assume that there exists* $\tilde{K}_1 > 0$ *such that*

$$\|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2s-\frac{2s}{q},q}(\mathbb{T}^d)} \leq \tilde{K}_1 \ .$$

Then, there exists a constant $C_1 > 0$ *depending on* $\tilde{K}_1, q, d, T, s, C_H$ *such that*

$$\|\partial_t u\|_{L^q(Q_T)} + \|u\|_{L^q(0,T;H_q^{2s}(\mathbb{T}^d))} + \||Du|^\gamma\|_{L^q(Q_T))} \leq C_1 \ .$$

We finally point out that our results, and in particular the Hölder bounds, apply to the so-called fractional Kardar-Parisi-Zhang equations, see [54], of the form

$$\partial_t z + (-\Delta)^s z = G(x, Dz(x, t)) - f(x, t) \text{ in } Q_T$$

where G satisfies (H). In other terms, the sign in front of H is not important since $u = -z$ solves (2) with $H(x, p) = G(x, -p)$.

3 Functional spaces

3.1 Stationary spaces: definitions and useful results

We denote by $L^p(\mathbb{T}^d)$ the space of all measurable and periodic functions belonging to $L^p_{loc}(\mathbb{R}^d)$ endowed with the norm $\|\cdot\|_p = \|\cdot\|_{L^p((0,1)^d)}$. Let k be a nonnegative integer. We denote by $W^{k,p}(\mathbb{T}^d)$ the space of $L^p(\mathbb{T}^d)$ functions with distributional derivatives in $L^p(\mathbb{T}^d)$ up to order k. For $\mu \in \mathbb{R}$ and $p \in (1,\infty)$, the space of Bessel potentials $H^\mu_p(\mathbb{T}^d)$ comprises those distributions verifying the integrability condition $(I-\Delta)^{\frac{\mu}{2}}u \in L^p(\mathbb{T}^d)$, where $(I-\Delta)^{\frac{\mu}{2}}$ is the operator defined in terms of Fourier series as

$$(I-\Delta)^{\frac{\mu}{2}}u = \sum_{k\in\mathbb{Z}^d}(1+4\pi^2k^2)^{\frac{\mu}{2}}\hat{u}(k)e^{2\pi i k\cdot x} ,$$

with

$$\hat{u}(k) = \int_{\mathbb{T}^d} u(x)e^{-2\pi i k\cdot x} .$$

We denote the norm in $H^\mu_p(\mathbb{T}^d)$ as

$$\|u\|_{\mu,p} := \|(I-\Delta)^{\frac{\mu}{2}}u\|_p \simeq \|u\|_p + \|(-\Delta)^{\frac{\mu}{2}}u\|_p .$$

The proof of the latter equivalence is given in [34, Remark 2.3]. Let us also remark that when $\mu = k$ is a nonnegative integer, $W^{k,p}$ is isomorphic to H^k_p, see e.g. [34, Remark 2.3]. Moreover, it can be seen that the operator $(I-\Delta)^{\frac{\mu}{2}}$ maps isometrically $H^{\eta+\mu}_p$ in H^η_p for any $\eta,\mu \in \mathbb{R}$, see again [34, Remark 2.3] for the proof. As a final remark, we recall that H^μ_p can be defined by complex interpolation among L^p and integer-order Sobolev spaces [71].

Let now $\mu \in (0,1)$ and $1 \le p,q \le \infty$. The Besov space $B^\mu_{pq}(\mathbb{T}^d)$ consists of all functions $u \in L^p(\mathbb{T}^d)$ such that the norm

$$\|u\|_{B^\mu_{pq}(\mathbb{T}^d)} := \|u\|_{L^p(\mathbb{T}^d)} + \left(\int_{\mathbb{T}^d}\frac{\|f(x+h)-f(x)\|^q_{L^p(\mathbb{T}^d)}}{|h|^{d+\mu q}}\,dh\right)^{\frac{1}{q}}$$

is finite. When $p = q = \infty$ and $\mu \in (0,1)$ we have $B^\mu_{\infty\infty}(\mathbb{T}^d) \simeq C^\mu(\mathbb{T}^d)$ (cf [82, Section 3.5.4 p. 168-169]), i.e. the classical Hölder space, which is endowed with the equivalent norm

$$\|u\|_{C^\alpha(\mathbb{T}^d)} := \|u\|_{C(\mathbb{T}^d)} + \sup_{x\neq y\in\mathbb{T}^d}\frac{|u(x)-u(y)|}{\mathrm{dist}(x,y)^\alpha} ,$$

where $\mathrm{dist}(x,y)$ is the geodesic distance among $x,y \in \mathbb{T}^d$. When $p = q$ and μ is not an integer, it is immediate to recognize that $B^\mu_{pp}(\mathbb{T}^d) \simeq W^{\mu,p}(\mathbb{T}^d)$, where $W^{\mu,p}(\mathbb{T}^d)$ is the classical Sobolev-Slobodeckij scale in the periodic setting. When $q = \infty$ and p is finite, the space $B^\mu_{p\infty}(\mathbb{T}^d) \simeq N^{\mu,p}(\mathbb{T}^d)$ is known as Nikol'skii space [76] and the above norm is interpreted in the usual sense via

$$\|u\|_{N^{\mu,p}(\mathbb{T}^d)} := \|u\|_{L^p(\mathbb{T}^d)} + \sup_{|h|>0}|h|^{-\mu}\|u(x+h)-u(x)\|_{L^p(\mathbb{T}^d)} ,$$

see [64, Chapter 17] for the whole space case and [92, p. 460], [82, Section 3.5.4] for the definition in the periodic case.

We now recall another characterization of Besov classes. We say that two normed spaces $(X_0,\|\cdot\|_{X_0})$ and $(X_1,\|\cdot\|_{X_1})$ are an admissible pair if they are embedded onto a Hausdorff topological vector space. For $\theta \in (0,1)$ and $q \in [1,\infty]$, we define the real interpolation space

$$(X_0,X_1)_{\theta,q} := \{x \in X_0 + X_1 : \|x\|_{\theta,q} < \infty\} ,$$

where for $1 \leq q < \infty$

$$\|x\|_{\theta,q} := \left(\int_0^\infty (K(x,t))^q \frac{dt}{t^{1+\theta q}} \right)^{\frac{1}{q}} ,$$

while for $q = \infty$

$$\|x\|_{\theta,\infty} := \sup_{t>0} t^{-\theta} K(x,t) ,$$

where for $t > 0$, $K(x,t)$ is the map defined by $x \in X_0 + X_1 \longmapsto K(x,t) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1}\}$, the so-called K-functional in interpolation theory.
For $m \in \mathbb{N}$, $p, q \in [1,\infty]$ and $\theta \in (0,1)$ we have

$$(L^p(\mathbb{T}^d), W^{m,p}(\mathbb{T}^d))_{\theta,q} \simeq B^{\theta m}_{pq}(\mathbb{T}^d) ,$$

with equivalence of the respective norms, see e.g. [71], [64, Theorem 17.24]. We recall some standard embeddings we will use in the sequel among the aforementioned spaces.

Lemma 3.1. *(i) Let $\nu, \mu \in \mathbb{R}$ with $\nu \leq \mu$, then $H^\mu_p(\mathbb{T}^d) \hookrightarrow H^\nu_p(\mathbb{T}^d)$.*

(ii) If $p\mu > d$ and $\mu - d/p$ is not an integer, then $H^\mu_p(\mathbb{T}^d) \hookrightarrow C^{\mu-d/p}(\mathbb{T}^d)$.

(iii) Let $\nu, \mu \in \mathbb{R}$ with $\nu \leq \mu$, $p, q \in (1,\infty)$ and

$$\mu - \frac{d}{p} = \nu - \frac{d}{q} ,$$

then $H^\mu_p(\mathbb{T}^d) \hookrightarrow H^\nu_q(\mathbb{T}^d)$.

(iv) In particular, for $\mu > 0$ such that $\mu p < d$ and $1 < q < \frac{dp}{d-\mu p}$ the embedding of $H^\mu_p(\mathbb{T}^d)$ onto $L^q(\mathbb{T}^d)$ is compact.

(v) If $p\mu = d$, then $H^\mu_p(\mathbb{T}^d) \hookrightarrow L^q(\mathbb{T}^d)$ for all $1 \leq q < \infty$.

Proof. For (i)-(iii) see [34, Lemma 2.5] and the references therein, while for (v) see [4]. To prove (iv) one can argue by interpolation. We restrict without loss of generality to the case $\mu \in (0,1)$. One observes then that $H^\mu_p(\mathbb{T}^d) = [L^p(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d)]_\theta$, where $[X,Y]_\theta$ denotes the complex interpolation among X, Y, cf [71]. It is well-known that $W^{1,p}(\mathbb{T}^d)$ is compactly embedded onto $L^r(\mathbb{T}^d)$ for all r such that $1 < r < \frac{dp}{d-p}$ by Rellich-Kondrachov Theorem, and hence the identity map $T : W^{1,p}(\mathbb{T}^d) \to L^p(\mathbb{T}^d)$, $T(u) = u$ is compact. Moreover, T is also continuous from $L^p(\mathbb{T}^d)$ onto itself. Therefore, by classical compactness results in interpolation theory (see e.g. [67, Chapter V]), we have the compact embedding of $H^\mu_p(\mathbb{T}^d)$ onto $L^p(\mathbb{T}^d)$. We now take a bounded sequence u_n in $H^\mu_p(\mathbb{T}^d)$. Therefore, one can extract a subsequence u_{n_k} converging strongly in $L^p(\mathbb{T}^d)$. By interpolation, for every $p < q < \frac{dp}{d-\mu p}$, there exists $\theta \in (0,1)$ such that

$$\|u_{n_k} - u_{n_j}\|_q \leq \|u_{n_k} - u_{n_j}\|_p^{1-\theta} \|u_{n_k} - u_{n_j}\|^\theta_{\frac{dp}{d-\mu p}} \to 0$$

as $j, k \to \infty$ since u_{n_k} is bounded in $H^\mu_p(\mathbb{T}^d)$, which is in turn continuously embedded onto $L^{\frac{dp}{d-\mu p}}(\mathbb{T}^d)$ by (iii). Then we have the strong convergence in L^q with q as above, as desired.

□

Lemma 3.2. *(i) Let $\nu, \mu \in \mathbb{R}$ with $\nu \leq \mu$, then $W^{\mu,p}(\mathbb{T}^d) \subset W^{\nu,p}(\mathbb{T}^d)$.*

(ii) If $p\mu > d$ and $\mu - d/p$ is not an integer, then $W^{\mu,p}(\mathbb{T}^d) \subset C^{\mu-d/p}(\mathbb{T}^d)$.

(iii) Let $\nu, \mu \in \mathbb{R}$ with $\nu \leq \mu$, $p, q \in (1, \infty)$ and

$$\mu - \frac{d}{p} = \nu - \frac{d}{q} \ ,$$

then $W^{\mu,p}(\mathbb{T}^d) \subset W^{\nu,q}(\mathbb{T}^d)$.

(iv) If $p\mu = d$, then $W^{\mu,p}(\mathbb{T}^d) \subset L^q(\mathbb{T}^d)$ for all $1 \leq q < \infty$.

Proof. For (i)-(iii) we refer to [82, Section 3.5.5], while for (iv) see [41, Theorem 6.9]. □

Lemma 3.3. *We have the following inclusions for $\mu \in \mathbb{R}$.*

(i) $B^{\mu}_{pp}(\mathbb{T}^d) \subseteq H^{\mu}_p(\mathbb{T}^d) \subseteq B^{\mu}_{p,2}(\mathbb{T}^d)$ for $1 < p \leq 2$.

(ii) $B^{\mu}_{p,2}(\mathbb{T}^d) \subseteq H^{\mu}_p(\mathbb{T}^d) \subseteq B^{\mu}_{pp}(\mathbb{T}^d)$ for $2 \leq p < \infty$.

Proof. The result on $\mathbb{R}^d$ is proven in [93, Section 2.3.3] (see also [17, Theorem 6.4.4]. Recalling that H^{μ}_p is isomorphic to a Triebel-Lizorkin scale (see [82, Theorem 3.5.4-(v)] and the same chapter for the definition of this space), one uses [82, Remark 3.5.1.4-(20)] to show (i) and (ii). □

We first recall the following Gagliardo-Nirenberg inequality

Lemma 3.4. *Let $1 < q, r < \infty$, $1 < z < \infty$ and $\mu \in \mathbb{R}$. Let $u(\cdot, t) \in H^{\mu}_p(\mathbb{T}^d) \cap L^z(\mathbb{T}^d)$ for a.e. $t \in (0, T)$. Then, for $0 \leq j < \mu$ there exists a constant C depending on d, q, z, j, μ and $\theta \in (0, 1)$ such that*

$$\|D^j u(\cdot, t)\|_{L^r(\mathbb{T}^d)} \leq C \|u(\cdot, t)\|^{\theta}_{H^{\mu}_q(\mathbb{T}^d)} \|u(\cdot, t)\|^{1-\theta}_{L^z(\mathbb{T}^d)} \quad \text{for a.e. } t \in (0, T) \ . \tag{8}$$

where

$$\frac{1}{r} = \frac{j}{d} + \theta \left(\frac{1}{q} - \frac{\mu}{d} \right) + \frac{1-\theta}{z} \ , j \leq \mu\theta \ .$$

Proof. The full inequality (8) has been obtained in [49, Corollary 1.5] and [65, Theorem 6] (for $j = 0$ and $\mu = 2s$) on the whole space. We sketch the proof of the inequality for $r, z \in (1, \infty)$ on the torus via complex interpolation methods following [73]. We write via [17, Theorem 6.4.5-(7)]

$$H^{\mu\theta}_h(\mathbb{T}^d) \simeq [L^z(\mathbb{T}^d), H^{\mu}_r(\mathbb{T}^d)]_{\theta} \ ,$$

where h satisfies

$$\frac{1}{h} = \frac{1-\theta}{z} + \frac{\theta}{r} \ . \tag{9}$$

for $\theta \in (0, 1)$. This in particular gives

$$\|u(\cdot, t)\|_{H^{\mu\theta}_h(\mathbb{T}^d)} \leq C \|u(\cdot, t)\|^{1-\theta}_{L^z(\mathbb{T}^d)} \|u(\cdot, t)\|^{\theta}_{H^{\mu}_r(\mathbb{T}^d)} \ .$$

We then take $0 \leq j < \mu$ satisfying

$$\frac{1}{r} = \frac{j}{d} + \theta \left(\frac{1}{r} - \frac{\mu}{d} \right) + \frac{1-\theta}{z} \tag{10}$$

and observe that (9) and (10) yield

$$j - \frac{d}{r} = \mu\theta - \frac{d}{h} \ .$$

This allows to apply the embedding $H^{\mu\theta}_h(\mathbb{T}^d) \hookrightarrow H^j_r(\mathbb{T}^d)$ stated in Lemma 3.1-(iii) for $\mu\theta \geq j$. Being j integer, we note that $H^j_r(\mathbb{T}^d) \simeq W^{j,r}(\mathbb{T}^d)$.

□

We now provide a Gagliardo-Nirenberg interpolation inequality involving Hölder and Bessel potential scales. To prove this, we closely follow the approach proposed in [73], while more general inequalities on Besov scales can be found in [49, Theorem 1.2].

Lemma 3.5. *Let $u \in H_r^{m+\theta}(\mathbb{T}^d) \cap C^{k,\alpha}(\mathbb{T}^d)$ with $m \in \mathbb{N}$, $m \geq 1$, $\theta \in (0,1)$, $r \in [2,\infty)$, $k \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,1)$ and $k < m$. For j satisfying*

$$\max\{k+\alpha, m+\theta-\frac{d}{r}\} < j < m+\theta\ ,$$

we have $u \in W^{j,q}(\mathbb{T}^d)$, and there exists a constant $c > 0$ depending on $d, m, \theta, r, k, \alpha, \beta$ such that

$$\|u\|_{W^{j,p}(\mathbb{T}^d)} \leq c\|u\|^{\beta}_{H_r^{m+\theta}(\mathbb{T}^d)}\|u\|^{1-\beta}_{C^{k,\alpha}(\mathbb{T}^d)}$$

with

$$\frac{1}{p} = \frac{j}{d} + \beta\left(\frac{1}{r} - \frac{m+\theta}{d}\right) - (1-\beta)\frac{k+\alpha}{d}\ ,$$

and

$$\beta \in \left[\frac{j-(k+\alpha)}{m+\theta-(k+\alpha)}, 1\right)$$

with $(1-\beta)(k+\alpha)+\beta(m+\theta)$ noninteger.

Proof. Recall that $F^{m+\theta}_{p_0 2}(\mathbb{T}^d) \simeq H^{m+\theta}_{p_0}(\mathbb{T}^d)$ from e.g. [82, 21], where $F^{m+\theta}_{p_0 2}(\mathbb{T}^d)$ is the periodic Triebel-Lizorkin scale [82], and $B^{k+\alpha}_{\infty\infty}(\mathbb{T}^d) \simeq C^{k,\alpha}(\mathbb{T}^d)$. Note that for $p_0 = r \geq 2$ we have the inclusion

$$F^{m+\theta}_{r2}(\mathbb{T}^d) \subseteq B^{m+\theta}_{rr}(\mathbb{T}^d)\ .$$

Moreover, it holds

$$B^{m+\theta}_{rr}(\mathbb{T}^d) \cap B^{k+\alpha}_{\infty\infty}(\mathbb{T}^d) \subseteq (B^{k+\alpha}_{\infty\infty}(\mathbb{T}^d), B^{m+\theta}_{rr}(\mathbb{T}^d))_{\beta,q}\ ,$$

where $(X,Y)_{\beta,q}$ stands for the real interpolation couple among X, Y. Note now that by [17, Theorem 6.4.5-(3)] we have

$$(B^{k+\alpha}_{\infty\infty}(\mathbb{T}^d), B^{m+\theta}_{rr}(\mathbb{T}^d))_{\beta,q} \simeq B^h_{qq}(\mathbb{T}^d)$$

for $h = (1-\beta)(k+\alpha)+\beta(m+\theta)$ non-integer and $\frac{1}{q} = \frac{\beta}{r}$. Moreover, since $r \geq 2$ by Lemma 3.3

$$\|u\|_{B^h_{qq}(\mathbb{T}^d)} \leq c_1\|u\|^{\beta}_{B^{m+\theta}_{rr}(\mathbb{T}^d)}\|u\|^{1-\beta}_{B^{k+\alpha}_{\infty\infty}(\mathbb{T}^d)} \leq c_2\|u\|^{\beta}_{H^{m+\theta}_r(\mathbb{T}^d)}\|u\|^{1-\beta}_{B^{k+\alpha}_{\infty\infty}(\mathbb{T}^d)}\ .$$

Set

$$\max\{k+\alpha, m+\theta-\frac{d}{r}\} < j < m+\theta\ ,$$

$$\beta \in \left[\frac{j-(k+\alpha)}{m+\theta-(k+\alpha)}, 1\right)$$

with h noninteger as before. For every β as above, we take

$$\frac{1}{p} = \frac{j}{d} + \beta\left(\frac{1}{r} - \frac{m+\theta}{d}\right) - (1-\beta)\frac{k+\alpha}{d}$$

so that it results $1 < q \leq p < \infty$, $0 < j \leq h < \infty$ and

$$h - \frac{d}{q} = j - \frac{d}{p}$$

and hence

$$B^h_{qq}(\mathbb{T}^d) \simeq W^{h,q}(\mathbb{T}^d) \subseteq W^{j,p}(\mathbb{T}^d)$$

by the embeddings in Lemma 3.2-(iii), and this concludes the proof. □

We conclude this section with a fractional Poincaré-Wirtinger inequality.

Lemma 3.6. *There exists $C = C(d, p, \mu)$ such that for $\mu \in (0, 1]$ and $p \in (1, \infty)$*

$$\|u - u_M\|_{L^p(\mathbb{T}^d)} \le C[u]_{W^{\mu,p}(\mathbb{T}^d)} ,$$

where $u_M = \fint_{\mathbb{T}^d} u\, dx$ and $[\cdot]_{W^{\mu,p}(\mathbb{T}^d)}$ stands for the Gagliardo seminorm. As a consequence, for $s \in (1/2, 1)$ we have

$$\|u - u_M\|_{L^p(\mathbb{T}^d)} \le C\|(-\Delta)^{s-\frac{1}{2}}u\|_{L^p(\mathbb{T}^d)} .$$

Proof. The first inequality can be found in [7, Proposition B.11], [64, Chapter 17]. The second one follows from the first and the inclusions among Bessel and Sobolev-Slobodeckii spaces $H_p^{\mu+\epsilon}(\mathbb{T}^d) \subseteq W^{\mu,p}(\mathbb{T}^d) \subseteq H_p^{\mu-\epsilon}(\mathbb{T}^d)$, $\epsilon > 0$, $\mu \in \mathbb{R}$, cf [34, Lemma 2.14]. □

3.2 Parabolic spaces: definitions and embeddings

In this section we introduce some functional spaces involving time and space weak derivatives. Let again $\mu \in \mathbb{R}$ and $p \in (1, \infty)$. We denote by $\mathbb{H}_p^\mu(Q) := L^p(0, T; H_p^\mu(\mathbb{T}^d))$, $Q := \mathbb{T}^d \times I$, the space of measurable functions $u : (0, T) \to H_p^\mu(\mathbb{T}^d)$ endowed with the norm

$$\|u\|_{\mathbb{H}_p^\mu(Q)} := \left(\int_0^T \|u(\cdot, t)\|^p_{H_p^\mu(\mathbb{T}^d)}\, dt\right)^{\frac{1}{p}} .$$

We define the space $\mathcal{H}_p^\mu(Q)$ as the space of functions $u \in \mathbb{H}_p^\mu(Q)$ with $\partial_t u \in (\mathbb{H}_{p'}^{2s-\mu}(Q))'$ equipped with the norm

$$\|u\|_{\mathcal{H}_p^\mu(Q)} := \|u\|_{\mathbb{H}_p^\mu(Q)} + \|\partial_t u\|_{(\mathbb{H}_{p'}^{2s-\mu}(Q))'} .$$

We refer the reader to [34, 32]. These are natural spaces in the standard parabolic setting $s = 1$: when $s = 1$ and $\mu = 2$ we have $\mathcal{H}_p^2 \simeq W_p^{2,1}$, cf [59], see also [58, 56], [39], [20, Chapter 6] for further properties in the case $s = 1$. Note that $(\mathbb{H}_{p'}^{2s-\mu}(Q))'$ coincides with $\mathbb{H}_p^{\mu-2s}(Q)$. In the sequel we denote by $Q_\tau = \mathbb{T}^d \times (0, \tau)$ and $Q_{\omega,\tau} = \mathbb{T}^d \times (\omega, \tau)$.
We now recall the following trace result of functions in $\mathcal{H}_p^\mu$ on the hyperplane $t = 0$, which extends [59, Lemma II.3.4] for classical spaces associated to heat PDEs to the fractional framework.

Lemma 3.7. *If $u \in \mathcal{H}_p^\mu(Q_T)$, $\mu \in \mathbb{R}$ and $p > 1$ satisfying $\mu - 2s/p > 0$, then $u(0) \in W^{\mu-2s/p,p}(\mathbb{T}^d)$. In addition, the space $\mathcal{H}_p^\mu$ is continuously embedded onto $C([0, T]; W^{\mu-\frac{2s}{p}}(\mathbb{T}^d))$.*

Proof. The first statement is a consequence of [71, Corollary 1.14], the embedding properties for the domain of the fractional Laplacian $D(-(-\Delta)^s)$ and the Reiteration Theorem in interpolation theory. The second fact can be deduced again by [71, Corollary 1.14], see also [5, Theorem III.4.10.2] and [36] for the case $s = 1$. □

We now recall some fractional parabolic embedding theorems partially proved in [34] and in [46].

Lemma 3.8. *(i) If $1 < p < \frac{d+2s}{\mu}$, then $\mathcal{H}_p^\mu(Q_T)$ is continuously embedded onto $L^q(Q_T)$ for $1 \le q \le \frac{(d+2s)p}{d+2s-\mu p}$.*

(ii) For $p > 1$ the parabolic space $\mathcal{H}_p^{2s-1}(Q_T)$ is continuously embedded into $L^\delta(0, T; W^{\alpha,\delta}(\mathbb{T}^d))$, where $\delta > p$ and

$$\alpha = 2s - 1 + \frac{d + 2s}{\delta} - \frac{d + 2s}{p} .$$

(iii) If $p > \frac{d+2s}{\mu}$, then $\mathcal{H}_p^{2s-1}(Q_T)$ is continuously embedded onto $C^{\alpha,\alpha/2s}$ for some $\alpha \in (0,1)$. Moreover, the space $\mathcal{H}_p^{2s}(Q_T)$ is continuously embedded onto $C([0,T];C^{2s-\frac{d+2s}{p}}(\mathbb{T}^d))$.

Proof. (i) is proven in [34, Proposition 2.11] for $1 \le q < \frac{(d+2s)p}{d+2s-\mu p}$. A slight modification of that proof allows even to prove the endpoint case $q = \frac{(d+2s)p}{d+2s-\mu p}$, which we provide below. Here, we distinguish the cases $1 < p \le 2$ and $2 < p < \infty$ in view of the inclusions stated in Lemma 3.3. To prove the first case $1 < p \le 2$, we note that for any $\theta \in (0,1)$, if $\nu = \nu(\theta) = (\mu - 2s/p)(1-\theta) + \mu\theta$, then H_p^ν can be obtained by complex interpolation between H_p^μ and $H_p^{\mu-2s/p}$ (see, e.g., [17, Theorem 6.4.5]). Moreover, H_p^ν is continuously embedded in $H_q^{\nu+d/q-d/p}$ in view of Lemma 3.1. Hence, for a.e. t,

$$c(d,p,s,q)\,\|u(t)\|_{\nu-\frac{d}{p}+\frac{d}{q},q} \le \|u(t)\|_{\nu,p} \le \|u(t)\|_{\mu-2s/p,p}^{1-\theta}\,\|u(t)\|_{\mu,p}^{\theta}\,.$$

Therefore, for all $\eta \le \nu - \frac{d}{p} + \frac{d}{q} = \mu + \frac{d}{q} - \frac{d+2s(1-\theta)}{p}$,

$$\begin{aligned}\left(\int_0^T \|u(t)\|_{\eta,q}^{\frac{p}{\theta}}\,dt\right)^\theta &\le C_1\left(\int_0^T \|u(t)\|_{\mu-2s/p,p}^{(1-\theta)\frac{p}{\theta}}\,\|u(t)\|_{\mu,p}^{p}\,dt\right)^\theta\\ &\le C_2\left(\int_0^T \|u(t)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}^{(1-\theta)\frac{p}{\theta}}\,\|u(t)\|_{\mu,p}^{p}\,dt\right)^\theta\,,\end{aligned}$$

where we used that for $1 < p \le 2$, $W^{\mu-2s/p,p}$ is embedded onto $H_p^{\mu-2s/p}$ (cf Lemma 3.3-(i)). Then, the last inequality is less than or equal to

$$\begin{aligned}&C\sup_{t\le T}\|u(t)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}^{(1-\theta)p}\left(\int_0^T \|u(t)\|_{\mu,p}^{p}\,dt\right)^\theta\\ &\qquad\le C(\|u\|_{\mathcal{H}_p^\mu(Q_T)} + \|u(0)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)})^{(1-\theta)p}\,\|u(t)\|_{\mathbb{H}_p^\mu(Q_T)}^{\theta p}\\ &\qquad\le C(\|u\|_{\mathcal{H}_p^\mu(Q_T)} + \|u(0)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)})^{p}\end{aligned}$$

where, in the second inequality we used the embedding in Lemma 3.7

$$\mathcal{H}_p^\mu(Q_T) \hookrightarrow C([0,T];W^{\mu-2s/p,p}(\mathbb{T}^d))$$

while, in the last one, Young's inequality.
As for the case $2 < p < \infty$, we interpolate in the Sobolev-Slobodeckii scale. In particular, one uses that $W^{\nu,p}$ can be obtained by real interpolation among $W^{\mu,p}$ and $W^{\mu-2s/p,p}$. Moreover, $W^{\nu,p}$ is continuously embedded in $W^{\nu+d/q-d/p,q}$ in view of Lemma 3.2-(iii). Hence, for a.e. t,

$$c(d,p,s,q)\,\|u(t)\|_{W^{\nu-\frac{d}{p}+\frac{d}{q},q}(\mathbb{T}^d)} \le \|u(t)\|_{W^{\nu,p}(\mathbb{T}^d)} \le \|u(t)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}^{1-\theta}\,\|u(t)\|_{W^{\mu,p}(\mathbb{T}^d)}^{\theta}\,.$$

Then, for all η verifying $\eta \le \nu - \frac{d}{p} + \frac{d}{q} \le \mu + \frac{d}{q} - \frac{d+2s(1-\theta)}{p}$ we have

$$\begin{aligned}\left(\int_0^T \|u(t)\|_{W^{\eta,q}(\mathbb{T}^d)}^{\frac{p}{\theta}}\,dt\right)^\theta &\le C_1\left(\int_0^T \|u(t)\|_{W^{\nu-\frac{d}{p}+\frac{d}{q},q}(\mathbb{T}^d)}^{\frac{p}{\theta}}\,dt\right)^\theta\\ &\le C_2\left(\int_0^T \|u(t)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}^{(1-\theta)p}\,\|u(t)\|_{W^{\mu,p}(\mathbb{T}^d)}^{p}\,dt\right)^\theta\\ &\le C_3\sup_{t\le T}\|u(t)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}^{(1-\theta)p}\left(\int_0^T \|u(t)\|_{\mu,p}^{p}\,dt\right)^\theta\end{aligned}$$

where we used that H^μ_p is embedded onto $W^{\mu,p}$ when $p > 2$ (see Lemma 3.3). At this stage, one has to use the maximal regularity embedding in Lemma 3.7 to get

$$\mathcal{H}^\mu_p(Q_T) \hookrightarrow C([0,T]; W^{\mu-2s/p,p}(\mathbb{T}^d))$$

and finally conclude the assertion setting $\eta = 0$ to get

$$\left(\int_0^T \|u(t)\|_q^q \, dt\right)^{\frac{p}{q}} \le C(\|u\|_{\mathcal{H}^\mu_p(Q_T)} + \|u(0)\|^p_{W^{\mu-2s/p,p}(\mathbb{T}^d)}) \, .$$

Item (ii) is then a consequence of the above computations setting $\eta = \alpha$, $q = \delta$, $\mu = 2s-1$, $\theta = p/q = p/\delta$, while (iii) is a consequence of [34, Theorem 2.6], while last assertion is a byproduct of the embedding in Lemma 3.7 and Lemma 3.1, since

$$\mathcal{H}^{2s}_p(Q_T) \hookrightarrow C([0,T]; W^{2s-\frac{2s}{p},p}(\mathbb{T}^d)) \hookrightarrow C([0,T]; C^{2s-\frac{d+2s}{p}}(\mathbb{T}^d))$$

□

Lemma 3.9. *Let $1 < p < \frac{d+2s}{\mu}$, $\mu \in \mathbb{R}$, $\mu > 0$. Then, the space $\mathcal{H}^\mu_p(Q_T)$ is compactly embedded onto $L^q(Q_T)$ for $1 \le q < \frac{(d+2s)p}{d+2s-\mu p}$.*

Proof. To show the compactness, we restrict to consider the case $\mu \in (0, 2s]$, the general case being consequence of the isometry property of the operator $(I - \Delta)^{\frac{\mu}{2}}$ on spaces of Bessel potentials. The idea is to exploit the so-called Aubin-Lions-Simon Lemma. Let $\mu \in \mathbb{R}$ and $0 < \mu \le 2s$ with p satisfying $1 < p < \frac{d+2s}{\mu}$. Note first that $H^\mu_{p'}(\mathbb{T}^d)$ is reflexive and separable. Therefore the space $L^p(0,T; (H^\mu_{p'}(\mathbb{T}^d))')$ is isomorphic to $(L^{p'}(0,T; H^\mu_{p'}(\mathbb{T}^d)))' \equiv (\mathbb{H}^\mu_{p'}(Q_T))'$. One can easily see that, by definition, $\mathcal{H}^\mu_p(Q_T)$ is isomorphic to

$$E := \{u \in L^p(0,T; H^\mu_p(\mathbb{T}^d)), \partial_t u \in L^p(0,T; (H^{2s-\mu}_{p'}(\mathbb{T}^d))'\} \, .$$

Note also that $H^\mu_p(\mathbb{T}^d)$ is compactly embedded into $L^p(\mathbb{T}^d)$ by Lemma 3.1-(iv) and $L^p(\mathbb{T}^d)$ is continuously embedded in $(H^{2s-\mu}_{p'}(\mathbb{T}^d))'$ since $\mu \le 2s$. Then, Aubin-Lions-Simon Lemma (see [88] and [83, Proposition III.1.3]) implies that E is compactly embedded into $L^p(Q_T)$. Hence $\mathcal{H}^\mu_p(Q_T)$ is compactly embedded in $L^q(Q_T)$ for any $1 \le q \le p$. Let u_n be a bounded sequence in $\mathcal{H}^\mu_p(Q_T)$. By the previous discussion we may extract a subsequence u_{n_k} converging to u strongly in $L^p(Q_T)$. For any $p < q < \frac{(d+2s)p}{d+2s-\mu p}$, arguing by interpolation, we may assert the existence of $0 < \theta < 1$ such that

$$\left\|u_{n_k} - u_{n_j}\right\|_{L^q(Q_T)} \le \left\|u_{n_k} - u_{n_j}\right\|^\theta_{L^p(Q_T)} \left\|u_{n_k} - u_{n_j}\right\|^{1-\theta}_{L^{\frac{(d+2s)p}{d+2s-\mu p}}} \to 0$$

as $j, k \to +\infty$, since u_{n_k} belongs to $\mathcal{H}^\mu_r(Q_T)$, which is in turn continuously embedded onto $L^{\frac{(d+2s)p}{d+2s-\mu p}}$ in view of Lemma 3.8, so u_{n_k} converges strongly also in $L^q(Q_T)$. □

We now recall a maximal regularity theorem for fractional heat equations. Consider the problem

$$\begin{cases} \partial_t u + (-\Delta)^s u = f(x,t) & \text{in } Q_T \, , \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}^d \, . \end{cases} \tag{11}$$

We have the following result for strong solutions to (11), i.e. $u \in \mathcal{H}^{2s}_q$, the equation is solved a.e. and $u(0)$ is meant in the sense of traces.

Theorem 3.10. *Let $p > 1$. Suppose that $u \in \mathcal{H}_p^\mu(Q_T)$ solves* (11)*. Then, every strong solution to* (11) *verifies*

$$\|u\|_{\mathcal{H}_p^\mu(Q_T)} \leq C(\|f\|_{\mathbb{H}_p^{\mu-2s}(Q_T)} + \|u_0\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}).$$

where $C > 0$ depends on d, T, p, s (but remains bounded for bounded values of T).

Proof. The proof is a consequence of well-known results for abstract evolution equations when $\mu = 2s$, see e.g. [61, 50]. The general case can be handled using the isometry of the Bessel operator as in [57], and it is proved in [32]. In particular, in [32] the proof is provided for stochastic PDEs, which makes necessary the restriction $p > 2$. However, for standard PDEs one simply requires $p > 1$, as it can be seen in [32, Lemma 3.2 and Lemma 3.4]. □

The last part of the section is devoted to present a Sobolev embedding theorem for the parabolic Bessel potential class $\mathcal{H}_p^{2s-1}$ with traces on. the hyperplane $t = 0$ in L^1. This can be regarded as a nonlocal counterpart of [35, Proposition A.2]. We prove the result via interpolation theory arguments, although a different proof can be done as in [35] via duality.

Lemma 3.11. *Let $s \in (\frac{1}{2}, 1)$. If $1 < \sigma' < (d + 2s)/(d + 2s - 1)$, then $\mathcal{H}_{\sigma'}^{2s-1}(Q_T)$ is continuously embedded onto $L^p(Q_T)$ for*

$$\frac{1}{p} = \frac{1}{\sigma'} - \frac{2s-1}{d+2s}.$$

Moreover, if $u \in \mathcal{H}_{\sigma'}^{2s-1}(Q_T)$ and $u(\cdot, 0) \in L^1(\mathbb{T}^d)$, we have

$$\|u\|_{L^p(Q_T)} \leq C\left(\|u\|_{\mathcal{H}_{\sigma'}^{2s-1}(Q_T)} + \|u(0)\|_{L^1(\mathbb{T}^d)} \right), \tag{12}$$

where the constant C depends on d, p, σ', T, but remains bounded for bounded values of T.

Proof. The result is a consequence of Lemma 3.8-(i) with $p = \sigma'$, $\mu = 2s - 1$ and the fact that

$$\|u(0)\|_{W^{2s-1-2s/\sigma',\sigma'}(\mathbb{T}^d)} \leq \tilde{C}\|u(0)\|_{L^1(\mathbb{T}^d)}$$

for some positive constant $\tilde{C} > 0$ provided that $\sigma > d + 2s$, i.e. $\sigma' < \frac{d+2s}{d+2s-1} (< 2)$. □

The next result is proven in [92],[90], see also [36, Lemma A.3] for the periodic setting.

Lemma 3.12. *Let $\mu > 0$ and $1 \leq p < \infty$. Then $W^{\mu,p}(\mathbb{T}^d) \subseteq N^{\mu,p}(\mathbb{T}^d)$ with continuous embedding, and there exists a constant $C = C(d, \mu, p)$ such that*

$$\|u\|_{N^{\mu,p}(\mathbb{T}^d)} \leq C\|u\|_{W^{\mu,p}(\mathbb{T}^d)}$$

In particular, the space $L^p(I; W^{\mu,p}(\mathbb{T}^d)) \subseteq L^p(I; N^{\mu,p}(\mathbb{T}^d))$ with continuous inclusion, where $I \subset \mathbb{R}$.

4 Fractional Fokker-Planck equations

4.1 Weak solutions for the fractional Fokker-Planck equation

This part is devoted to study the following Fokker-Planck equation with fractional diffusion

$$\begin{cases} -\partial_t \rho(x,t) + (-\Delta)^s \rho(x,t) + \mathrm{div}(b(x,t)\,\rho(x,t)) = 0 & \text{in } Q_\tau\,, \\ \rho(x,\tau) = \rho_\tau(x) & \text{in } \mathbb{T}^d\,. \end{cases} \tag{13}$$

Note that when the vector field $b(x,t) = -D_pH(x, Du(x,t))$, then (13) becomes the adjoint equation of the linearization of (2). Here, $\tau \in (0, T]$ and $Q_\tau := \mathbb{T}^d \times (0, \tau)$. From now on, unless otherwise specified, we will focus on $d > 2$. We will consider the following notion of weak solution

Definition 4.1. Let $b \in L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))$ with $\mathcal{P} \in (d/(2s-1),\infty)$ and $\mathcal{Q} \in (2s/(2s-1),\infty]$ be such that

$$\frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} < \frac{2s-1}{2s}\ , \tag{14}$$

and $\rho_\tau \in H^{s-1}(\mathbb{T}^d)$. A (weak) solution ρ to (13) belongs to $\mathcal{H}_2^{2s-1}(Q_\tau)$ and satisfies

$$\int_0^\tau \int_{\mathbb{T}^d} \partial_t\rho\varphi\, dxdt + \iint_{Q_\tau} (-\Delta)^{s-\frac{1}{2}}\rho\, (-\Delta)^{\frac{1}{2}}\varphi - b\rho\cdot D\varphi\, dxdt = \int_{\mathbb{T}^d} \rho_\tau(x)\varphi(x,\tau)\, dx \tag{15}$$

for all $\varphi \in C^\infty(\mathbb{T}^d \times (0,\tau])$.

In particular, the above formulation holds even when test functions are chosen to belong to the class $\mathcal{H}_2^1(Q_\tau) := \{\varphi \in L^2(0,\tau;H^1(\mathbb{T}^d))\ , \partial_t\varphi \in L^2(0,\tau;H^{-2s+1}(\mathbb{T}^d))\}$. We stress out that when $s=1$ the above setting falls within the classical matter described in [19, 59, 18] . We remark in passing that $\rho \in \mathcal{H}_2^{2s-1}(Q_\tau) \hookrightarrow C([0,T];(H^{2s-1}(\mathbb{T}^d),H^{-1}(\mathbb{T}^d))_{1/2,2}) \simeq C([0,T];H^{s-1}(\mathbb{T}^d))$ in view of the classical abstract trace result [40, Section XVIII.3 eq. (1.61)].

Remark 4.2. We point out that time-integration by parts

$$\iint_{Q_\tau} \varphi\partial_t\rho + \iint_{Q_\tau} \partial_t\varphi\rho\, dxdt = \int_{\mathbb{T}^d} \varphi(x,\tau)\rho(x,\tau)\, dx - \int_{\mathbb{T}^d} \varphi(x,\omega)\rho(x,\omega)\, dx \tag{16}$$

holds, where duality pairings are hidden here. To prove this fact, one represents $\mathcal{H}_2^{2s-1}(Q_\tau)$ as

$$\mathcal{H}_2^{2s-1}(Q_\tau) = \{u \in L^2(0,\tau;H^{2s-1}(\mathbb{T}^d))\ , \partial_t u \in L^2(0,\tau;H^{-1}(\mathbb{T}^d))\}$$

which coincides with the space $W(0,\tau,H^{2s-1}(\mathbb{T}^d),H^{-1}(\mathbb{T}^d))$ defined in [40, Chapter XVIII, Section 3]. Then, one uses that $C_0^\infty([0,\tau];H^{2s-1}(\mathbb{T}^d))$ is dense in $\mathcal{H}_2^{2s-1}(\mathbb{T}^d)$, the embedding $\mathcal{H}_2^{2s-1}(Q_\tau) \hookrightarrow C([0,T];(H^{2s-1}(\mathbb{T}^d),H^{-1}(\mathbb{T}^d))_{1/2,2}) \simeq C([0,T];H^{s-1}(\mathbb{T}^d))$ to give sense to the traces and the fact that (16) is true for $\varphi,\rho \in C_0^\infty([0,\tau];H^{2s-1}(\mathbb{T}^d))$ by the theory of integration and derivation in Banach spaces. In this setting, it is sufficient to have $H^{2s-1}(\mathbb{T}^d) \hookrightarrow H^{-1}(\mathbb{T}^d)$, with $H^{2s-1}(\mathbb{T}^d)$ dense in $H^{-1}(\mathbb{T}^d)$, cf [66, Proposition 3.3], as described in [40].

Throughout this section we will assume that

$$\rho_\tau \in H^{s-1}(\mathbb{T}^d), \quad \rho_\tau \geq 0, \quad \text{and} \quad \int_{\mathbb{T}^d} \rho_\tau(x)\, dx = 1\ . \tag{17}$$

We further observe that since $s > 1/2$ we have $\rho \in \mathcal{H}_2^{2s-1}$ and $\mathcal{H}_2^{2s-1} \hookrightarrow L^{\frac{2(d+2s)}{d+2-2s}} \hookrightarrow L^2 \hookrightarrow L^1$ and hence $\rho(t) \in L^1(\mathbb{T}^d)$ for a.e. t. Therefore, by using $\varphi \equiv 1$ as a test function one obtains $\int_{\mathbb{T}^d} \rho(t) = 1$ for $t \in (0,T)$.

Remark 4.3. Note that on $\mathbb{R}^d \times (0,T)$ it is easy to check that the equation

$$\partial_t\rho + (-\Delta)^s\rho + \operatorname{div}(b(x,t)\rho) = 0$$

is invariant under the scalings

$$\rho_\lambda(x,t) := \rho(\lambda x, \lambda^{2s}t) \text{ and } b_\lambda(x,t) := \lambda^{2s-1}b(\lambda x,\lambda^{2s}t)\ .$$

Therefore, when looking at small scales, for $s \in (1/2,1)$ one has to check the effect of the scaling on the Lebesgue norm of the velocity filed. In such case, the subcritical space turns out to be the mixed space $L^{\mathcal{Q}}(L^{\mathcal{P}})$ when the exponents $\mathcal{P} \geq d/(2s-1)$ and $\mathcal{Q} \geq 2s/(2s-1)$ fulfill the condition

$$\frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} \leq \frac{2s-1}{2s}\ ,$$

which can be seen as the fractional counterpart of the classical Aronson-Serrin interpolated condition for viscous problems with unbounded coefficients [59, 18, 19] mentioned in the introduction. This condition allows to give a distributional sense to the transport term. Indeed, for $\varphi \in \mathcal{H}_2^1$, $\rho \in \mathcal{H}_2^{2s-1}$ and $\mathcal{P} = \mathcal{Q}$, we have by Hölder's inequality

$$\iint \operatorname{div}(b(x,t)\rho)\varphi \simeq \iint b(x,t)\rho \cdot D\varphi \le \|b\|_{L^{\frac{d+2s}{2s-1}}} \|\rho\|_{L^{\frac{2(d+2s)}{d+2-2s}}} \|D\varphi\|_{L^2} \lesssim \|b\|_{L^{\frac{d+2s}{2s-1}}} \|\rho\|_{\mathcal{H}_2^{2s-1}} \|D\varphi\|_{L^2} .$$

Classical Fokker-Planck equations with low regularity on the drift have been studied in [79, 75, 20, 35] and references therein.

4.2 Existence and integrability estimates for fractional Fokker-Planck equations

We now present the main result of this section and comment upon the assumption (I) at the end of the result.

Proof of Theorem 2.1. Step 1. Existence in the energy space $\mathcal{H}_2^{2s-1}(Q_\tau)$. We apply Leray-Schauder fixed point theorem for the existence (see [45, Theorem 11.6]) on the space $\mathcal{H}_2^{2s-1}(Q_\tau)$. Consider the map $\mathcal{M} : \mathcal{H}_2^{2s-1} \times [0,1] \to \mathcal{H}_2^{2s-1}(Q_\tau)$ defined by $m \longmapsto \rho = \mathcal{M}[m;\sigma]$ given by solving the following parametrized PDE

$$-\partial_t \rho + (-\Delta)^s \rho = \sigma \operatorname{div}(b(x,t)m) \text{ in } Q_\tau \ , \rho(x,\tau) = \sigma\rho_\tau(x) \text{ in } \mathbb{T}^d \ .$$

Note that $\mathcal{M}[m;0] = 0$ by standard results for fractional heat equations. We first show that G is well-defined. We start with the case $\mathcal{P} = \mathcal{Q}$ (whence condition (14) becomes $\mathcal{P} > \frac{d+2s}{2s-1}$). By parabolic Calderón-Zygmund regularity theory (cf Theorem 3.10) we have

$$\begin{aligned} \|\rho\|_{\mathcal{H}_2^{2s-1}(Q_\tau)} &\le C(\sigma\|bm\|_{L^2(Q_\tau)} + \sigma\|\rho_\tau\|_{H^{s-1}(\mathbb{T}^d)}) \\ &\le C(\|b\|_{L^{\mathcal{P}}(Q_\tau)}\|m\|_{L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(Q_\tau)} + \|\rho_\tau\|_{H^{s-1}(\mathbb{T}^d)}) \ . \quad (18) \end{aligned}$$

Now, note that

$$1 < \frac{2\mathcal{P}}{\mathcal{P}-2} < \frac{2(d+2s)}{d+2-2s} \ .$$

We then argue by interpolation, exploit the embedding of $\mathcal{H}_2^{2s-1}(Q_\tau) \hookrightarrow L^{\frac{2(d+2s)}{d+2-2s}}(Q_\tau)$ in Lemma 3.8 and the fact that $m \in L^1(Q_\tau)$ to show applying also Young's inequality

$$\|m\|_{L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(Q_\tau)} \ \le \ C_1\|m\|^\theta_{L^1(Q_\tau)}\|m\|^{1-\theta}_{L^{\frac{2(d+2s)}{d+2-2s}}(Q_\tau)} \ = \ C_1\tau^\theta\|m\|^{1-\theta}_{L^{\frac{2(d+2s)}{d+2-2s}}(Q_\tau)} \ \le \ C_2 + \varepsilon\|m\|_{\mathcal{H}_2^{2s-1}(Q_\tau)}$$

for some $\theta \in (0,1)$, $\varepsilon > 0$. Then, for $\varepsilon = 1/2$ we have

$$\|\rho\|_{\mathcal{H}_2^{2s-1}(Q_\tau)} \le C_2(\|b\|_{L^{\mathcal{P}}(Q_\tau)} + \|\rho_\tau\|_{H^{s-1}(\mathbb{T}^d)}) + \frac{1}{2}\|m\|_{\mathcal{H}_2^{2s-1}(Q_\tau)} \ .$$

This shows that $\mathcal{M}$ is well-defined from $\mathcal{H}_2^{2s-1}(Q_\tau)$ into itself, since $m \in \mathcal{H}_2^{2s-1}(Q_\tau)$. Moreover, if $\rho \in \mathcal{H}_2^{2s-1}(Q_\tau)$ and $\sigma \in [0,1]$ is a fixed point of the map $\rho = \mathcal{M}[\rho;\sigma]$ we have that $\rho \in \mathcal{H}_2^{2s-1}(Q_\tau)$ is a solution of (13) and the a priori estimate (18) carry through uniformly on $\sigma \in [0,1]$. Thus, we obtain the existence of a constant $M > 0$ depending only on the data (namely $\|b\|_{L^{\mathcal{P}}(Q_\tau)}$, ρ_τ, T, s) such that

$$\|\rho\|_{\mathcal{H}_2^{2s-1}(Q_\tau)} \le M \ .$$

We finally show that the map T is compact. Let m_n be a bounded sequence in $\mathcal{H}_2^{2s-1}(Q_\tau)$ and let $\rho_n = \mathcal{M}[m_n;\sigma]$. Since $|b|m_n \in L^2(Q_\tau)$ we have that $\mathrm{div}(bm_n) \in \mathbb{H}_2^{-1}(Q_\tau)$ and hence by Theorem 3.10 we deduce $\rho_n \in \mathcal{H}_2^{2s-1}(Q_\tau)$. By the compactness of $\mathcal{H}_2^{2s-1}$ onto $L^2(Q_\tau)$ (cf Lemma 3.9), which is ensured by the restriction $s > 1/2$, we have that, along a subsequence, ρ_n converges strongly in $L^2(Q_\tau)$ to ρ and $(-\Delta)^{s-1/2}\rho_n$ converges weakly to $(-\Delta)^{s-1/2}\rho$ in $L^2(Q_\tau)$. We use $(-\Delta)^{s-1}(\rho_n-\rho) \in \mathcal{H}_2^1(Q_\tau)$ as admissible test function in the weak formulation of the equation satisfied by ρ_n to conclude

$$\begin{aligned}\iint_{Q_\tau} &|(-\Delta)^{s-\frac{1}{2}}(\rho_n-\rho)|^2\,dxdt \\ &\le C\iint_{Q_\tau}|b|m_n||(-\Delta)^{s-\frac{1}{2}}(\rho_n-\rho)|\,dxdt - \iint_{Q_\tau}(-\Delta)^s\rho(-\Delta)^{s-1}(\rho_n-\rho)\,dxdt \\ &\qquad + \iint_{Q_\tau}\partial_t\rho(-\Delta)^{s-1}(\rho_n-\rho)\,dxdt\end{aligned}$$

Since $|b|m_n \in L^2(Q_\tau)$ and $(-\Delta)^{s-1/2}\rho_n$ converges weakly to $(-\Delta)^{s-1/2}\rho$ in $L^2(Q_\tau)$ the first term on the right-hand side of the above inequality converges to 0. Similarly, since $\partial_t\rho \in \mathbb{H}_2^{-1}(Q_\tau)$ and exploiting again the weak convergence of $(-\Delta)^{s-1/2}\rho_n$ in $L^2(Q_\tau)$ the third term goes to 0. Similar motivations provide the convergence of the second term. This shows that $(-\Delta)^{s-1/2}\rho_n$ converges strongly to $(-\Delta)^{s-1/2}\rho$ in $L^2(Q_\tau)$. Finally, to show the strong convergence of $\partial_t\rho_n$ to $\partial_t\rho$ in $\mathbb{H}_2^{-1}(Q_\tau)$ we argue by duality. For every $\varphi \in \mathbb{H}_2^1(Q_\tau)$ we have

$$\begin{aligned}\left|\iint_{Q_\tau}\partial_t(\rho_n-\rho)\varphi\,dxdt\right| &\le \left|\iint_{Q_\tau}(-\Delta)^s(\rho_n-\rho)\varphi\,dxdt\right| + \left|\iint_{Q_\tau}\mathrm{div}(b(\rho_n-\rho))\varphi\,dxdt\right| \\ &\le C\iint_{Q_\tau}|(-\Delta)^{s-\frac{1}{2}}(\rho_n-\rho)||D\varphi|\,dxdt + \iint_{Q_\tau}|\rho_n-\rho||b||D\varphi|\,dxdt\,,\end{aligned}$$

which yields the strong convergence of $\partial_t\rho_n$ to $\partial_t\rho$ in $\mathbb{H}_2^{-1}(Q_\tau)$ in view of the previous claims. The general case $\mathcal{P} \neq \mathcal{Q}$ can be dealt with similarly. Indeed, in the borderline case $\mathcal{Q} = \infty$, we observe that

$$\|bm\|_{L^2(Q_\tau)} \le c\||b|\|_{L^\infty(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))}\|m\|_{L^2(0,\tau;L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(\mathbb{T}^d))}\,.$$

We then observe that

$$1 < \frac{2\mathcal{P}}{\mathcal{P}-2} < \frac{2d}{d-2(2s-1)}\,,$$

which yields by interpolation for $\mathcal{P} > \frac{d}{2s-1}$ the inequality

$$\|m\|_{L^2(0,\tau;L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(\mathbb{T}^d))} \le C\|m\|^{1-\theta}_{L^2(0,\tau;L^{\frac{2d}{d-2(2s-1)}}(\mathbb{T}^d))}$$

for a.e. $t \in (0,\tau)$. Using the Sobolev embedding $H_2^{2s-1}(\mathbb{T}^d) \hookrightarrow L^{\frac{2d}{d-2(2s-1)}}(\mathbb{T}^d))$ we conclude

$$\|m\|_{L^2(0,\tau;L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(\mathbb{T}^d))} \le C\|m\|^{\theta}_{\mathcal{H}_2^{2s-1}(Q_\tau)}$$

and then proceed as above. When $\mathcal{P}, \mathcal{Q}$ are finite, we have

$$\|bm\|_{L^2(Q_\tau)} \le \||b|\|_{L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))}\|m\|_{L^{\frac{2\mathcal{Q}}{\mathcal{Q}-2}}(0,\tau;L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(\mathbb{T}^d))}\,.$$

We now use interpolation with η, δ, ζ (cf [9, Lemma 1]) satisfying

$$\frac{\mathcal{Q}-2}{2\mathcal{Q}} = \frac{1-\theta}{\zeta} + \frac{\theta(\eta-2)}{2\eta}\,,$$

$$\frac{\mathcal{P}-2}{2\mathcal{P}} = 1-\theta+\frac{\theta(\delta-2)}{2\delta} \ .$$

for $\theta \in (0,1)$. This gives, using that $m \in L^1(\mathbb{T}^d)$,

$$\|m\|_{L^{\frac{2\mathcal{Q}}{\mathcal{Q}-2}}(0,\tau;L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(\mathbb{T}^d))} \leq C\|m\|^{\theta}_{L^{\frac{2\eta}{\eta-2}}(0,\tau;L^{\frac{2\delta}{\delta-2}}(\mathbb{T}^d))}$$

We now exploit the mixed-norm embedding in [34, Proposition 2.11] (applied with $p=2$, $q=\frac{2\delta}{\delta-2}$, $\theta=\frac{\eta-2}{\eta}$, $\mu=2s-1$) to conclude

$$\|m\|^{\theta}_{L^{\frac{2\eta}{\eta-2}}(0,\tau;L^{\frac{2\delta}{\delta-2}}(\mathbb{T}^d))} \leq C\|m\|^{\theta}_{\mathcal{H}_2^{2s-1}(Q_\tau)}$$

provided that

$$\frac{d}{2s\delta}+\frac{1}{\eta} < \frac{2s-1}{2s} \ ,$$

i.e. when (14) holds for $\mathcal{P}, \mathcal{Q}$. The uniqueness of solutions can be obtained by duality, see Step 3 below for a similar argument.

Step 2. A priori estimates via Duhamel's formula. The proof we are going to present is not formal, and can be made rigorous by regularization (cf [80, Lemma 2.3]), using Duhamel's formula for the regularized PDE and then passing to the limit. A similar regularization for nonlocal transport equations is given in [94]. The approach is inspired by [18] and has been also recently implemented in [33] to get embeddings in mixed Lebesge scales.
We claim that there exists $t^* \in (0,\tau]$ independently of $\rho_\tau \in L^p(\mathbb{T}^d)$ such that

$$\|\rho(\cdot,t)\|_{L^p(\mathbb{T}^d)} \leq C_2\|\rho_\tau\|_{L^p(\mathbb{T}^d)} \text{ for all } t \in [t^*,\tau]$$

for some $C_2>0$. Set $\tilde{\rho}(\cdot,t) := \rho(\cdot,\tau-t)$ for all $t\in[0,\tau]$. We use Duhamel's formula to represent the solution as

$$\tilde{\rho}(t) = \mathcal{T}_t\rho_\tau - \int_0^t \mathcal{T}_{t-\omega}\mathrm{div}(b\tilde{\rho})(\cdot,\omega)d\omega \ .$$

where $\mathcal{T}_t = e^{-t(-\Delta)^s}$. We have

$$\begin{aligned}\|\tilde{\rho}(t)\|_{L^p(\mathbb{T}^d)} &\leq \|\mathcal{T}_t\rho_\tau\|_{L^p(\mathbb{T}^d)} + \left\|\int_0^t \mathcal{T}_{t-\omega}\mathrm{div}(b\tilde{\rho})(\cdot,\omega)d\omega\right\|_{L^p(\mathbb{T}^d)} \\ &\leq \|\rho_\tau\|_{L^p(\mathbb{T}^d)} + \int_0^t (t-\omega)^{-\frac{d}{2s}(\frac{1}{b}-\frac{1}{p})-\frac{1}{2s}}\|\mathrm{div}(b\tilde{\rho})(\cdot,\omega)\|_{H_b^{-1}(\mathbb{T}^d)}d\omega \\ &\leq \|\rho_\tau\|_{L^p(\mathbb{T}^d)} + \int_0^t (t-\omega)^{-\frac{d}{2s}(\frac{1}{b}-\frac{1}{p})-\frac{1}{2s}}\|b\tilde{\rho}(\cdot,\omega)\|_{L^b(\mathbb{T}^d)}d\omega \ ,\end{aligned}$$

where we applied the decay estimates of the fractional heat semigroup among spaces of Bessel potentials

$$\|\mathcal{T}_t u\|_{L^p(\mathbb{T}^d)} \leq Ct^{-\frac{d}{2s}(\frac{1}{b}-\frac{1}{p})-\frac{1}{2s}}\|u\|_{H_b^{-1}(\mathbb{T}^d)}$$

(cf [34]). We then use Hölder's inequality to bound the right-hand side of the last inequality with

$$\begin{aligned}&\|\tilde{\rho}\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))}\int_0^\tau (t-\omega)^{-\frac{d}{2s}(\frac{1}{b}-\frac{1}{p})-\frac{1}{2s}}\|b(\cdot,\omega)\|_{L^{\mathcal{P}}(\mathbb{T}^d)}d\omega \\ &\qquad\leq \left(\int_0^t (t-\omega)^{[-\frac{d}{2s}(\frac{1}{b}-\frac{1}{p})-\frac{1}{2s}]\mathcal{Q}'}\right)^{\frac{1}{\mathcal{Q}'}} \|b\|_{L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))}\|\tilde{\rho}\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))}\end{aligned}$$

where

$$\frac{1}{b} = \frac{1}{\mathcal{P}} + \frac{1}{p} \ .$$

In particular, the above integral term is well-posed provided that

$$\alpha := \left(-\frac{d}{2s\mathcal{P}} - \frac{1}{2s}\right) Q' > -1 \ ,$$

which is indeed satisfied precisely when

$$\frac{d}{2s\mathcal{P}} + \frac{1}{Q} < \frac{2s-1}{2s} \ .$$

Hence

$$\|\tilde{\rho}\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))} \le \|\rho_\tau\|_{L^p(\mathbb{T}^d)} + C\|b\|_{L^{Q}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))} t^{\frac{\alpha+1}{Q'}} \|\tilde{\rho}\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))} \ ,$$

which gives

$$\|\tilde{\rho}\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))} \le 2\|\rho_\tau\|_{L^p(\mathbb{T}^d)}$$

by taking

$$t \ge t^* := \left(\frac{1}{2C\|b\|_{L^{Q}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))}}\right)^{\frac{Q'}{\alpha+1}}$$

and hence the validity of the estimate on $[0, t^*]$. Note that t^* does not depend on $\|\rho_\tau\|_{L^p(\mathbb{T}^d)}$ and hence one can iterate the argument to get the estimate in $[0, \tau]$ as in [18].

Step 3. Positivity and uniqueness. The result can be proven e.g. by duality exploiting the results for the adjoint equation via assumption (I). We take $\varphi = v$ as a test function in the weak formulation of (13), where v solves (5) in $Q_\omega = \mathbb{T}^d \times (\omega, \tau)$ with $v(\omega) = v_\omega \ge 0$ and ρ as a test function to (5). By summing the expressions one obtains

$$\int_{\mathbb{T}^d} v(\omega)\rho(\omega)\,dx = \int_{\mathbb{T}^d} \rho(\tau)v(\tau)\,dx$$

and since the right-hand side is nonnegative, the left-hand side is so since $v_\omega \ge 0$. The uniqueness follows in a similar manner. □

Remark 4.4. In Step 1 we can actually reach the threshold

$$\frac{d}{2s\mathcal{P}} + \frac{1}{Q} = \frac{2s-1}{2s}$$

by assuming a smallness condition on $\|b\|_{L^{Q}(L^{\mathcal{P}})}$, since interpolation inequalities are no longer available (cf [89] for the elliptic viscous case).

Remark 4.5. As for assumption (I), we emphasize that this is a standard well-posedness result in the viscous case, see e.g. [19, 59]. Uniqueness and positivity of weak solutions to (5) has been discussed in e.g. [3, Theorem 6.1] under a slightly stronger condition on the velocity field. In this sense, uniqueness and positivity results for (13) are at this stage conditional to those for (5), see also Remark 5.5.

4.3 Parabolic Bessel regularity for fractional Fokker-Planck equations

We finally describe further regularity results that rely on the information $b \in L^k(\rho\,dxdt)$ for some $k > 1$, that will be used in the forthcoming section. We start with the following maximal L^q-regularity result for PDEs with divergence-type terms and terminal data in L^1.

Proposition 4.6. *Let ρ be a (non-negative) weak solution to* (13) *and*

$$1 < \sigma' < \frac{d+2s}{d+2s-1} \ . \tag{19}$$

Then, there exists $C > 0$, depending on σ', d, T, s such that

$$\|\rho\|_{\mathcal{H}^{2s-1}_{\sigma'}(Q_\tau)} \le C(\|b\rho\|_{L^{\sigma'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}). \tag{20}$$

Note that C here does not depend on $\tau \in (0, T]$.

Proof. Let ρ be smooth, the general case follows by an approximation argument. Let φ be a smooth test function vanishing at the initial time $\varphi(\cdot, 0) = 0$. The strategy follows the proof of [35, Proposition 2.4] and it is based on duality arguments. A different proof giving the same result, which uses maximal L^q-regularity tools, is provided in the next Remark 4.7, see also Theorem 4.8. Using the weak formulation of (31), we write for φ as above

$$\iint_{Q_\tau} \rho(\partial_t\varphi + (-\Delta)^s\varphi - b\cdot D\varphi)\,dxdt = \int_{\mathbb{T}^d} \rho_\tau(x)\varphi(x,\tau)\,dx \ .$$

Let $\delta > 0$ and $\psi = \psi_\delta$ be the solution to the forward fractional heat equation

$$\begin{cases} \partial_t\psi + (-\Delta)^s\psi = (|(-\Delta)^{s-\frac12}\rho|^2 + \delta)^{\frac{\sigma'-2}{2}}(-\Delta)^{s-\frac12}\rho & \text{in } Q_\tau \ , \\ \psi(x,0) = 0 & \text{in } \mathbb{T}^d \ . \end{cases}$$

By maximal L^σ-regularity, we get

$$\begin{aligned}\|\psi\|_{\mathcal{H}^{2s}_\sigma(Q_\tau)} &\le C\|(|(-\Delta)^{s-\frac12}\rho|^2 + \delta)^{\frac{\sigma'-2}{2}}(-\Delta)^{s-\frac12}\rho\|_{L^\sigma(Q_\tau)} \\ &\le C\||(-\Delta)^{s-\frac12}\rho|^{\sigma'-1}\|_{L^\sigma(Q_\tau)} = C\|(-\Delta)^{s-\frac12}\rho\|^{\sigma'-1}_{L^{\sigma'}(Q_\tau)} \ .\end{aligned}$$

We take $\varphi = (-\Delta)^{s-\frac12}\psi$ in the weak formulation to see that after integrating by parts

$$\iint_{Q_\tau} (|(-\Delta)^{s-\frac12}\rho|^2 + \delta)^{\frac{\sigma'-2}{2}}|(-\Delta)^{s-\frac12}\rho|^2\,dxdt \le \|b\rho\|_{L^{\sigma'}(Q_\tau)}\|D\varphi\|_{L^\sigma(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}\|\varphi(\cdot,\tau)\|_\infty \ . \tag{21}$$

Now, observe that

$$\begin{aligned}\|\psi\|_{\mathcal{H}^{2s}_\sigma(Q_T)} &\ge C\|\psi\|_{C(W^{2s-2s/\sigma,\sigma}(\mathbb{T}^d))} = C\|(I-\Delta)^{s-\frac12}\psi\|_{C(W^{1-2s/\sigma,\sigma}(\mathbb{T}^d))} \\ &\ge C\|\varphi(\cdot,\tau)\|_{W^{1-2s/\sigma,\sigma}(\mathbb{T}^d)} \ge C\|\varphi(\tau)\|_\infty\end{aligned}$$

when $\sigma > d+2s$ for possibly different positive constants always denoted by C. Here, we used the isometry properties of the Bessel potential operator on Sobolev-Slobodeckii scales [17, Theorem 6.2.7] and the embeddings in Lemma 3.2. We then get

$$\begin{aligned}\iint_{Q_\tau} (|(-\Delta)^{s-\frac12}\rho|^2 + \delta)^{\frac{\sigma'-2}{2}}|(-\Delta)^{s-\frac12}\rho|^2\,dxdt &\le C_1(\|b\rho\|_{L^{\sigma'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)})\|\psi\|_{\mathcal{H}^{2s}_\sigma(Q_\tau)} \\ &\le C_2(\|b\rho\|_{L^{\sigma'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)})\||(-\Delta)^{s-\frac12}\rho|\|^{\sigma'-1}_{L^{\sigma'}(Q_\tau)}\end{aligned}$$

and let $\delta \to 0$ to conclude

$$\|(-\Delta)^{s-\frac12}\rho\|_{L^{\sigma'}(Q_\tau)} \le C_2(\|b\rho\|_{L^{\sigma'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}) \ .$$

The estimate on $\rho \in L^{\sigma'}(Q_\tau)$ follows by using that of $\|(-\Delta)^{s-\frac{1}{2}}\rho\|_{L^{\sigma'}(Q_\tau)}$ and the fractional Poincaré-Wirtinger inequality in Lemma 3.6. The estimate on the time derivative can be obtained by duality. Indeed, for any $\varphi \in L^\sigma(0,\tau;H^1_\sigma(\mathbb{T}^d))$ we have

$$\left|\iint_{Q_\tau} \partial_t\rho\varphi\, dxdt\right| = \left|\iint_{Q_\tau} (-\Delta)^{s-\frac{1}{2}}\rho(-\Delta)^{\frac{1}{2}}\varphi\, dxdt\right| + \|b\rho\|_{L^{\sigma'}(Q_\tau)}\|D\varphi\|_{L^\sigma(Q_\tau)}$$
$$\leq C(\|(-\Delta)^{s-\frac{1}{2}}\rho\|_{L^{\sigma'}(Q_\tau)} + \|b\rho\|_{L^{\sigma'}(Q_\tau)})\|(-\Delta)^{\frac{1}{2}}\varphi\|_{L^\sigma(Q_\tau)}$$

where we used that $W^{1,\sigma} \simeq H^1_\sigma$ and Hölder's inequality. □

Remark 4.7. A slightly different proof of the above result can be obtained as follows. We observe that $\rho \in \mathcal{H}^{2s-1}_2$ readily implies $\rho \in \mathcal{H}^{2s-1}_{\sigma'}$ for every $1 < \sigma' < 2$. Let us rewrite equation (13) as a perturbation of the fractional heat equation

$$-\partial_t\rho + (-\Delta)^s\rho = \operatorname{div}(b(x,t)\rho) \text{ on } Q_\tau$$

with terminal data $\rho(x,\tau) := \rho_\tau(x)$ on $\mathbb{T}^d$. By parabolic regularity theory (see Theorem 3.10) ρ enjoys the estimate

$$\|\rho\|_{\mathcal{H}^{2s-1}_{\sigma'}(Q_\tau)} \leq C(\|b\rho\|_{L^{\sigma'}(Q_\tau)} + \|\rho_\tau\|_{W^{2s-1-2s/\sigma',\sigma'}(\mathbb{T}^d)}) .$$

By exploiting Sobolev embedding for fractional Sobolev spaces in Lemma 3.2, one immediately obtains that

$$\|\rho_\tau\|_{W^{2s-1-2s/\sigma',\sigma'}(\mathbb{T}^d)} \leq C\|\rho_\tau\|_{L^1(\mathbb{T}^d)}$$

whenever $1 < \sigma' < \frac{d+2s}{d+2s-1}$. Indeed, by definition, we have

$$\|\rho_\tau\|_{W^{2s-1-2s/\sigma',\sigma'}(\mathbb{T}^d)} = \sup_{\varphi\in W^{2s/\sigma'-2s+1,q}(\mathbb{T}^d)\ ,\|\varphi\|_{W^{2s/\sigma'-2s+1,\sigma}(\mathbb{T}^d)}=1} \left|\int_{\mathbb{T}^d}\rho_\tau\varphi\, dx\right|$$
$$\leq \|\varphi\|_\infty\|\rho_\tau\|_{L^1(\mathbb{T}^d)} \leq C\|\varphi\|_{W^{2s/\sigma'-2s+1,\sigma}(\mathbb{T}^d)}\|\rho_\tau\|_{L^1(\mathbb{T}^d)} \leq C\|\rho_\tau\|_{L^1(\mathbb{T}^d)} ,$$

where the last inequality is a consequence of the embedding $W^{2s/\sigma'-2s+1,\sigma}(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$ (cf Lemma 3.2-(ii)) when

$$(2s/\sigma' - 2s + 1)\sigma > d ,$$

that is $\sigma > d+2s$ or, in other words, when σ' satisfies (19). This highlights that the range of σ' is imposed by the heat part of the equation.

The next results asserts fractional Sobolev regularity of the fractional Fokker-Planck equation when the trace ρ_τ belongs to some suitable Lebesgue class. A different proof has been proposed in [36, Proposition 2.2].

Proposition 4.8. *Let ρ be a (non-negative) weak solution to* (13)*, $\rho_\tau \in L^{p'}(\mathbb{T}^d)$ and either*

$$\sigma' = \frac{d+2s}{d+2s-1} \text{ with any } p > 1 ,$$

or

$$\sigma' > \frac{d+2s}{d+2s-1} \text{ with } p' = \frac{d\sigma}{(d+1)\sigma - (d+2s)} .$$

Then, there exists $C > 0$, depending on σ', d, T, s such that

$$\|\rho\|_{\mathcal{H}^{2s-1}_{\sigma'}(Q_\tau)} \leq C(\|b\rho\|_{L^{\sigma'}(Q_\tau)} + \|\rho_\tau\|_{L^{p'}(\mathbb{T}^d)}). \tag{22}$$

Proof. We can proceed as above, except for the treatment of the term involving ρ_τ. We modify (21) as

$$\begin{aligned}\|\psi\|_{\mathcal{H}^{2s}_{\sigma}(Q_T)} \geq C\|\psi\|_{C(W^{2s-2s/\sigma,\sigma}(\mathbb{T}^d))} &= C\|(I-\Delta)^{s-\frac{1}{2}}\psi\|_{C(W^{1-2s/\sigma,\sigma}(\mathbb{T}^d))} \\ &\geq C\|\varphi(\cdot,\tau)\|_{W^{1-2s/\sigma,\sigma}(\mathbb{T}^d)} \geq C\|\varphi(\tau)\|_{L^{\frac{d\sigma}{d+2s-\sigma}}(\mathbb{T}^d)}\end{aligned}$$

for $\sigma < d+2s$, and

$$\begin{aligned}\|\psi\|_{\mathcal{H}^{2s}_{q}(Q_T)} \geq C\|\psi\|_{C(W^{2s-2s/\sigma,\sigma}(\mathbb{T}^d))} &= C\|(I-\Delta)^{s-\frac{1}{2}}\psi\|_{C(W^{1-2s/\sigma,\sigma}(\mathbb{T}^d))} \\ &\geq C\|\varphi(\cdot,\tau)\|_{W^{1-2s/\sigma,\sigma}(\mathbb{T}^d)} \geq C\|\varphi(\tau)\|_{L^{p}(\mathbb{T}^d)} .\end{aligned}$$

for any $p \in (1,\infty)$ when $\sigma = d+2s$. □

As a consequence, the above results yield the proof of Theorem 2.2.

Proof of Theorem 2.2. We use Proposition 4.6 and Proposition 4.8, depending on the range of σ', and the generalized Hölder's inequality to conclude

$$\|\rho\|_{\mathcal{H}^{2s-1}_{\sigma'}(Q_\tau)} \leq C(\|b\rho^{1/m'}\rho^{1/m}\|_{L^{\sigma'}(Q_\tau)}+1) \leq C\left(\left(\iint_{Q_\tau}|b|^{m'}\rho\,dxdt\right)^{1/m'}\|\rho\|^{1/m}_{L^{\zeta}(Q_\tau)}+1\right), \tag{23}$$

for $p > \sigma'$ satisfying

$$\frac{1}{\sigma'} = \frac{1}{m'} + \frac{1}{m\zeta}. \tag{24}$$

Then, by Young's inequality, for all $\varepsilon > 0$

$$\|\rho\|_{\mathcal{H}^{2s-1}_{\sigma'}(Q_\tau)} \leq C\left(\frac{1}{\varepsilon}\iint_{Q_\tau}|b|^{m'}\rho\,dxdt + \varepsilon\|\rho\|_{L^{\zeta}(Q_\tau)}+1\right). \tag{25}$$

One can verify that the identity $m' = 1+\frac{d+2s}{\sigma(2s-1)}$ and (24) yield

$$\frac{1}{\zeta} = \frac{1}{\sigma'} - \frac{2s-1}{d+2s}.$$

Indeed, (24) gives

$$\frac{1}{\zeta} = \frac{m}{\sigma'} - \frac{m}{m'} = \frac{1}{\sigma'} - \frac{m-1}{\sigma}$$

and then the definition of m' in (2.2) yields the conclusion. The continuous embedding of $\mathcal{H}^{2s-1}_{\sigma'}(Q_\tau)$ in $L^p(Q_\tau)$ stated in Lemma 3.11 then implies

$$\|\rho\|_{L^{\zeta}(Q_\tau)} \leq C_1\left(\|\rho\|_{\mathcal{H}^{2s-1}_{\sigma'}(Q_\tau)}+\tau\right),$$

finally giving

$$\|\rho\|_{L^{\zeta}(Q_\tau)} \leq CC_1\left(\frac{1}{\varepsilon}\iint_{Q_\tau}|b|^{m'}\rho\,dxdt + \varepsilon\|\rho\|_{L^{\zeta}(Q_\tau)}+1\right), \tag{26}$$

Hence, the term $\varepsilon\|\rho\|_{L^{\zeta}(Q_\tau)}$ can be absorbed by the left hand side of (26) by choosing $\varepsilon = (2CC_1)^{-1}$, thus providing the assertion. □

Corollary 4.9. *Let ρ be a nonnegative weak solution to* (13)*. Then, there exists $C > 0$ depending on d, q', T, s such that*

$$\sup_{t\in[0,\tau]} \|\rho(t)\|_{W^{\frac{d(2s-1)}{d+2s}-\frac{2s}{q'},\frac{d+2s}{d+2s+(2s-1)q'}}(\mathbb{T}^d)} + \|\rho\|_{L^{q'}(Q_\tau)} \le C\left(\iint_{Q_\tau} |b|^{\frac{d+2s}{(2s-1)q}}\rho\, dxdt + \|\rho_\tau\|_{L^{p'}(\mathbb{T}^d)}\right)$$

with $p = \frac{dq}{d+2s-2sq}$ if $q < \frac{d+2s}{2s}$, $p > 1$ arbitrarily large when $q = \frac{d+2s}{2s}$ and $p = \infty$ if $q > \frac{d+2s}{2s}$.

Proof. The result follows from Theorem 2.2 applied with $\frac{1}{\sigma'} = \frac{1}{q'} + \frac{2s-1}{d+2s}$ and the continuous embedding of $\mathcal{H}^{2s-1}_{\sigma'}$ onto $L^{q'}(Q_T)$ and $C([0,\tau]; W^{\frac{d(2s-1)}{d+2s}-\frac{2s}{q'},\frac{d+2s}{d+2s+(2s-1)q'}}(\mathbb{T}^d))$ □

5 On fractional Hamilton-Jacobi equations

5.1 On the notions of solutions

We first provide the following notion of weak solution to (2) we will need to discuss Hölder's regularization effects for (2). See [35] for a similar definition in the viscous case $s = 1$.

Definition 5.1. We say that

i) u is a *local weak* solution to (2) if for all $0 < \omega < T$

$$u \in \mathcal{H}^1_2(\mathbb{T}^d \times (\omega, T)) \cap C(\overline{Q}_T), \quad H(\cdot, Du) \in L^1(Q_{\omega,T}), \tag{27}$$

$$\text{and } D_pH(\cdot, Du) \in L^{\mathcal{Q}}(\omega, T; L^{\mathcal{P}}(\mathbb{T}^d)) \tag{28}$$

$$\text{for some } \frac{d}{2s-1} < \mathcal{P} < \infty, \text{ and } \frac{2s}{2s-1} < \mathcal{Q} < \infty \text{ such that } \frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} < \frac{2s-1}{2s}, \tag{29}$$

and for all $0 < \omega < \tau \le T$, $\varphi \in \mathcal{H}^{2s-1}_2(\mathbb{T}^d \times (\omega, \tau)) \cap L^\infty(Q_{\omega,\tau})$

$$\int_\omega^\tau \int_T \partial_t u\varphi\, dxdt + \iint_{\mathbb{T}^d\times(\omega,\tau)} (-\Delta)^{\frac{1}{2}} u\, (-\Delta)^{s-\frac{1}{2}}\varphi + H(x, Du)\varphi\, dxdt = \iint_{\mathbb{T}^d\times(\omega,\tau)} f\varphi\, dxdt\,. \tag{30}$$

ii) u is a *global weak* solution if (27)-(28)-(29) hold for all $0 \le \omega < T$, that is, on all Q_T (and therefore, (30) is also satisfied up to $\omega = 0$).

Definition 5.2. We say that $u \in \mathcal{H}^{2s}_q(Q_T)$, $q > 1$, is a strong solution to (2) the equation is solved for a.e. $(x, t) \in Q_T$ and the initial condition holds in trace sense, i.e. $u(0) \in W^{2s-2s/q,q}(\mathbb{T}^d)$.

Remark 5.3. It is immediate to verify by using Sobolev embeddings that whenever $q > \frac{d+2s}{2s}$ and $u \in \mathcal{H}^{2s}_q(Q_T)$, then $u \in \mathcal{H}^1_2(Q_T)$, $u \in C([0,T]; W^{2s-2s/q,q}(\mathbb{T}^d))$ and hence u is a global weak solution. This means that (30) is satisfied up to $\omega = 0$.
We note that under the restriction $q > \frac{d+2s}{(2s-1)\gamma'}$ the results in [34, Proposition 2.11] (applied with $p = q$, $\theta = 1/\gamma$ and replacing q with γq) gives the embedding

$$\mathcal{H}^{2s}_q(Q_T) \hookrightarrow L^{\gamma q}(0, T; W^{\gamma q}(\mathbb{T}^d))\,.$$

Furthermore, we restrict to consider

$$\gamma > \frac{d + 6s - 2}{d + 2s}\ (> 1)$$

so that $u \in L^2(0,T;W^{1,2}(\mathbb{T}^d))$ (and in particular $u \in \mathcal{H}_2^1(Q_T)$) by parabolic Sobolev embedding and the weak formulation can be safely used. Indeed, embeddings from [34] yields that $\mathcal{H}_q^{2s}(Q_T)$ whenever

$$1 < 2s + \frac{d}{2} - \frac{d + 2s(1 - q/2)}{q} \implies q > \frac{2(d+2s)}{d+6s-2} .$$

In particular, we have

$$\frac{d+2s}{(2s-1)\gamma'} > \frac{2(d+2s)}{d+6s-2}$$

whence $\gamma > \frac{d+6s-2}{d+2s}$. This is consistent with [36], where the condition $\gamma > \frac{d+4}{d+2}$ is needed when $s = 1$. We believe that this restriction can be relaxed to

$$\gamma > \frac{d+2s}{d+1}$$

so that $q > \frac{d+2s}{(2s-1)\gamma'} > 1$ using techniques from renormalized solutions, cf [72].

Remark 5.4. By classical embedding properties for Sobolev-Slobodeckii spaces we have the following inclusions

$$W^{2s-\frac{2s}{q},q}(\mathbb{T}^d) \hookrightarrow \begin{cases} C^{2s-\frac{d+2s}{q}}(\mathbb{T}^d) & \text{for } q > \frac{d+2s}{2s} , \\ L^p(\mathbb{T}^d) & \text{for } p \in [1,\infty) \text{ and } q = \frac{d+2s}{2s} , \\ L^p(\mathbb{T}^d) & \text{for } p \in [1, \frac{dq}{d+2s-2sq}] \text{ and } q < \frac{d+2s}{2s} . \end{cases}$$

Remark 5.5. As far as the assumption (I) is concerned, this hypothesis is needed when using ρ as a test function in the variational formulation to (2) to have uniqueness and positivity of the adjoint variable. When $u \in \mathcal{H}_q^{2s}$ with $q > \frac{d+2s}{2s-1}$, Sobolev embeddings yield $Du \in L^\infty(Q_T)$ and (I) is satisfied being D_pH in the Aronson-Serrin zone, due to [3, Theorem 6.1]. In the other cases, when q lies below the threshold $q = \frac{d+2s}{2s-1}$, our uniqueness and positivity results for (13) are conditional to those for (5). Nonetheless, one can strengthen a little the a priori integrability assumptions in Definition 5.1 and impose that D_pH falls in the class of [3, Theorem 6.1] so that (I) is satisfied. However, we remark that uniqueness and positivity of (5) is classical when $s = 1$ under the Aronson-Serrin condition.

5.2 Further estimates for the adjoint variable via duality

First, we prove a simple representation formula for (2) by duality with the adjoint problem

$$\begin{cases} -\partial_t \rho + (-\Delta)^s \rho - \operatorname{div}(D_pH(x, Du(x,t)\rho(x,t)) = 0 & \text{in } Q_\tau , \\ \rho(x,\tau) = \rho_\tau & \text{in } \mathbb{T}^d \end{cases} \tag{31}$$

where $\rho_\tau \in C^\infty(\mathbb{T}^d)$, $\rho_\tau \geq 0$.

Lemma 5.6. *Let u be a local weak solution to* (2)*. Assume that ρ is a weak solution to* (31)*. Then, for all $\omega \in (0,\tau)$ we have*

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)\,dx = \int_{\mathbb{T}^d} u(x,\omega)\rho(x,\omega)\,dx + \iint_{Q_{\omega,\tau}} L(x, D_pH(x,Du))\rho\,dxdt + \iint_{Q_{\omega,\tau}} f\rho\,dxdt \tag{32}$$

Moreover, if u is either a strong solution in $\mathcal{H}_q^{2s}(Q_T)$ or a global weak solution, the previous identity holds up to $\omega = 0$.

Proof. Using $-\rho \in \mathcal{H}_2^{2s-1}(Q_{\omega,\tau}) \cap L^\infty(Q_{\omega,\tau})$ as a test function in the weak formulation of (2) and $u \in \mathcal{H}_2^1(Q_{\omega,\tau})$ as a test function in the corresponding adjoint equation, after summing both expressions we obtain

$$-\int_\omega^\tau \langle \partial_t u(t), \rho(t)\rangle - \int_\omega^\tau \langle \partial_t \rho(t), u(t)\rangle + \iint_{Q_{\omega,\tau}} (D_pH(x,Du)\cdot Du - H(x,Du))\rho\, dxdt \\ + \iint_{Q_{\omega,\tau}} f\rho\, dxdt = 0$$

□

We are now ready to prove the crossed integrability bound on D_pH with respect to the density ρ.

Proposition 5.7. *Let u be a local weak solution to* (2) *and ρ be a weak solution to* (31) *with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$. Then, there exist a positive constant C (depending on C_H, $\|u\|_{C(Q_T)}$, $\|f\|_{L^q(Q_T)}$, q, d, T, s) such that*

$$\iint_{Q_\tau} |D_pH(x, Du(x,t))|^{\gamma'}\rho(x,t)\, dxdt \le C\ , \tag{33}$$

Remark 5.8. An immediate consequence of (33) is the bound

$$\iint_{Q_\tau} |Du(x,t)|^\beta \rho(x,t)\, dxdt \le C_\beta \qquad \text{for all } 1 \le \beta \le \gamma. \tag{34}$$

Indeed, by (H) and $\int_{\mathbb{T}^d} \rho(t) = 1$ for a.e. t, $\iint_{Q_\tau} |Du(x,t)|^\gamma \rho(x,t)\, dxdt \le C$, which yields (34) for $\beta = \gamma$. For $\beta < \gamma$ it is sufficient to use Young's inequality and $\|\rho(t)\|_{L^1(\mathbb{T}^d)} = 1$.

Proof. Rearrange the representation formula (32) to get, for $0 < \tau_1 < \tau < T$,

$$\iint_{Q_{\tau_1,\tau}} L(x, D_pH(x,Du))\rho\, dxdt = \int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx - \int_{\mathbb{T}^d} u(x,\tau_1)\rho(x,\tau_1)\, dx \\ - \iint_{Q_{\tau_1,\tau}} f\rho\, dxdt. \tag{35}$$

Fix η to be determined such that

$$\eta > \frac{d+2s}{(2s-1)\gamma'}\ .$$

We use the bounds on the Lagrangian and Hölder's inequality to get

$$C_L^{-1}\iint_{Q_\tau} |D_pH(x,Du)|^{\gamma'}\rho \le \iint_{Q_\tau} L(x, D_pH(x,Du)\rho\, dxdt \le 2\|u\|_{C(\overline{Q}_\tau)} + \|f\|_{L^\eta(Q_\tau)}\|\rho\|_{L^{\eta'}(Q_\tau)}$$

We let σ be such that

$$\frac{1}{\eta'} = \frac{1}{\sigma'} - \frac{2s-1}{d+2s}\ .$$

By Lemma 3.8 we have

$$\mathcal{H}_{\sigma'}^{2s-1}(Q_\tau) \hookrightarrow L^{\eta'}(Q_\tau)$$

Moreover, the choice $\eta > \frac{d+2s}{2s}$ assures that $\sigma' < \frac{d+2s}{d+2s-1}$. Then, in view of Theorem 2.2 we have

$$\|\rho\|_{L^{\eta'}(Q_\tau)} \le C(\|\rho\|_{\mathcal{H}_{\sigma'}^{2s-1}(Q_\tau)} + 1) \le C_1\left(\iint_{Q_\tau} |D_pH(x,Du)|^{m'}\rho\, dxdt + 1\right)$$

for

$$m' = 1 + \frac{d+2s}{(2s-1)\sigma} \ .$$

Thus, we get

$$C_L^{-1} \iint_{Q_\tau} |D_pH(x,Du)|^{\gamma'}\rho \le 2\|u\|_{C(\overline{Q}_\tau)} + C_1\|f\|_{L^\eta(Q_\tau)} \left(\iint_{Q_\tau} |D_pH(x,Du)|^{m'}\rho\, dxdt + 1 \right) \ .$$

Finally, the right-hand side can be absorbed in the left-hand side when $r' < \gamma'$, i.e.

$$m' = 1 + \frac{d+2s}{(2s-1)\sigma} = \frac{d+2s}{(2s-1)\eta} < \gamma' \ .$$

One then obtain (33) by letting $\tau_1 \to 0$ (note that here constants remain bounded for $\tau_1 \in (0,\tau)$). □

The crossed integrability of D_pH against the adjoint variable ρ finally provides the $L^{q'}$ regularity of $(-\Delta)^{s-1/2}\rho$. This extends [35, Corollary 3.4] to the fractional framework.

Corollary 5.9. *Let u be a weak solution to* (2) *and ρ be a local weak solution to* (13)*. Let $\bar\sigma$ be such that*

$$\bar\sigma > d+2s \quad \textit{and} \quad \bar\sigma \ge \frac{d+2s}{(\gamma'-1)(2s-1)}.$$

Then, there exists a positive constant C such that

$$\|\rho\|_{\mathcal{H}^{2s-1}_{\bar\sigma'}(Q_\tau)} \le C \ ,$$

where C depends in particular on C_H, $\|f\|_{L^{\bar\sigma}(Q_T)}$, $\|u\|_{C(\overline{Q}_T)}$, η, d, T, s (but not on τ, ρ_τ), $\bar\sigma > \frac{d+2s}{2s}$.

Remark 5.10. Note that condition on $\bar\sigma$ can be rewritten as

$$\bar\sigma > \begin{cases} d+2s & \text{if } \gamma \le 2s \\ \frac{d+2s}{(\gamma'-1)(2s-1)} & \text{if } \gamma > 2s \ . \end{cases}$$

Proof. Since $\bar\sigma' < \frac{d+2s}{d+2s-1}$, (2.2) applies (with $\sigma = \bar\sigma$), yielding

$$\|\rho\|_{\mathcal{H}^{2s-1}_{\bar\sigma'}(Q_\tau)} \le C\left(\iint_{Q_\tau} |D_pH(x,Du(x,t))|^{r'}\rho(x,t)\, dxdt + 1 \right),$$

with

$$m' = 1 + \frac{d+2s}{\bar\sigma(2s-1)} \le \gamma'.$$

If $m' = \gamma'$, use Proposition 5.7 to conclude. Otherwise, if $m' < \gamma'$ use Young's inequality first to control $\iint |D_pH(x,Du(x,t))|^{r'}\rho\, dxdt$ with $\iint |D_pH(x,Du)|^{\gamma'}\, dxdt + \tau$. □

5.3 Sup-norm and integral estimates

We are now ready to prove the sup-norm estimate for global weak solutions to fractional Hamilton-Jacobi equations in terms of $\|u_0\|_{C(\mathbb{T}^d)}$.

Proof of Theorem 2.3. We argue as in [35, Proposition 3.7]. First, we prove a bound from above for u

$$u(x,\tau) \leq \|u_0\|_{C(\mathbb{T}^d)} + C\|f\|_{L^q(\mathbb{T}^d)}$$

for all $\tau \in (0,T)$, $x \in \mathbb{T}^d$ and $q > \frac{d+2s}{2s}$. Consider indeed the strong solution to the backward problem

$$\begin{cases} -\partial_t \mu + (-\Delta)^s \mu = 0 & \text{in } \mathbb{T}^d \times (0,\tau) \\ \mu(x,\tau) = \mu_\tau(x) & \text{in } \mathbb{T}^d \end{cases}$$

with $\mu_\tau \in C^\infty(\mathbb{T}^d)$, $\mu_\tau \geq 0$ and $\|\mu_\tau\|_{L^1(\mathbb{T}^d)} = 1$. We use μ as a test function in the weak formulation of the fractional Hamilton-Jacobi equation to deduce

$$\int_{\mathbb{T}^d} u(x,\tau)\mu_\tau(x)\,dx = \int_{\mathbb{T}^d} u(x,0)\mu(x,0)\,dx + \iint_{Q_\tau} f\mu\,dxdt - \iint_{Q_\tau} H(x,Du)\mu\,dxdt\,. \tag{36}$$

Using Theorem 4.6 with $b = 0$ we find $\|\mu\|_{\mathcal{H}^{2s-1}_{\sigma'}(Q_\tau)} \leq C$ and by Lemma 3.8-(i) we conclude

$$\|\mu\|_{L^{q'}(Q_\tau)} \leq C\,.$$

with $q' < \frac{d+2s}{d}$. Then, using the above estimate, Hölder's inequality to the second term of the right-hand side of the above inequality and exploiting that $\|\mu_\tau\|_{L^1(\mathbb{T}^d)} = 1$ one has, choosing $q > \frac{d+2s}{2s}$,

$$\int_{\mathbb{T}^d} u(x,0)\mu(x,0)\,dx + \iint_{Q_\tau} f\mu\,dxdt \leq \|u_0\|_{C(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)}\,.$$

By the assumption $H \geq 0$ one concludes

$$\int_{\mathbb{T}^d} u(x,\tau)\mu_\tau(x)\,dx \leq \|u_0\|_{C(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)}$$

and the claimed estimate from above by duality after passing to the supremum over $\mu_\tau \in L^1(\mathbb{T}^d)$.
To prove the bound from below, we argue using (32) with $\omega = 0$. Fix some q such that

$$q > \frac{d+2s}{(2s-1)\gamma'}\,.$$

We use the bounds on the Lagrangian, the upper bound obtained in Step 1, and Hölder's inequality to get

$$\begin{aligned} C_L^{-1}\iint_{Q_\tau} |D_pH(x,Du)|^{\gamma'}\rho \leq \iint_{Q_\tau} L(x,D_pH(x,Du)\rho\,dxdt &\leq 2\|u\|_{C(\overline{Q}_\tau)} + \|f\|_{L^q(Q_\tau)}\|\rho\|_{L^{q'}(Q_\tau)} \\ &\leq 2\|u_0\|_{C(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)} + \|f\|_{L^q(Q_\tau)}\|\rho\|_{L^{q'}(Q_\tau)} \end{aligned}$$

We let $\bar{q}$ be such that

$$\frac{1}{q'} = \frac{1}{\sigma'} - \frac{2s-1}{d+2s}\,.$$

By Lemma 3.8 we have

$$\mathcal{H}^{2s-1}_{\sigma'}(Q_\tau) \hookrightarrow L^{q'}(Q_\tau)$$

Choose $q > \frac{d+2s}{2s}$ so that $\sigma' < \frac{d+2s}{d+2s-1}$. Then, owing to Theorem 2.2, we write

$$\|\rho\|_{L^{q'}(Q_\tau)} \leq C(\|\rho\|_{\mathcal{H}^{2s-1}_{\sigma'}(Q_\tau)} + 1) \leq C_1\left(\iint_{Q_\tau} |D_pH(x,Du)|^{m'}\rho\,dxdt + 1\right)$$

for

$$m' = 1 + \frac{d+2s}{(2s-1)\sigma} \ .$$

We then have

$$C_L^{-1} \iint_{Q_\tau} |D_pH(x,Du)|^{\gamma'}\rho \le \|u_0\|_{C(\mathbb{T}^d)} + C_1\|f\|_{L^q(Q_\tau)} \left(\iint_{Q_\tau} |D_pH(x,Du)|^{m'}\rho \, dxdt + 1 \right)$$

Then, the right-hand side can be absorbed in the left-hand side when $m' < \gamma'$, i.e.

$$m' = 1 + \frac{d+2s}{(2s-1)\sigma} = \frac{d+2s}{(2s-1)q} < \gamma' \ .$$

This provides a bound on $\iint_{Q_\tau} |D_pH(x,Du)|^{\gamma'}\rho$ and thus on $\|\rho\|_{L^{q'}(Q_\tau)}$, depending on $\|u_0\|_{C(\mathbb{T}^d)}$. Going back to (36) we have

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau \, dx \ge \int_{\mathbb{T}^d} u(x,0)\rho(x,0) - C_L \iint_{Q_\tau} \rho \, dxdt + \iint_{Q_\tau} f\rho \, dxdt$$

Since $\iint f\rho$ can be bounded from below by Hölder's inequality we get

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x) \, dx \ge -\|u(\cdot,0)\|_{C(\mathbb{T}^d)} - C_L\tau - C$$

Since ρ_τ can be arbitrarily chosen so that $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$, we conclude the desired result. □

To deduce integral estimates we use a L^p version of the adjoint method. We recall that integral estimates for parabolic viscous Hamilton-Jacobi equations are obtained in [36] using similar methods, [48], see also [26, Theorem 3.1] for degenerate problems.

Remark 5.11. Results in Theorem 2.4 can be regarded as an a priori estimates since

$$\mathcal{H}_q^{2s} \hookrightarrow C([0,T]; W^{2s-2s/q,q}(\mathbb{T}^d)) \hookrightarrow C([0,T]; L^p(\mathbb{T}^d))$$

with $p = \frac{dq}{d+2s-2sq}$

Proof of Theorem 2.4. We denote by $T_k(\omega) = \min(k,\omega)$ the truncation at level $k > 0$, $u^+ = \max\{u, 0\}$ and $u^- = (-u)^+$.
Step 1. We first prove that

$$\|u^+(\cdot,\tau)\|_{L^p(\mathbb{T}^d)} \le C(\|u_0^+\|_{L^p(\mathbb{T}^d)} + \|f\|_{L^q(Q_\tau)}) \tag{37}$$

for all $\tau \in (0,T)$ and $x \in \mathbb{T}^d$, $p \in [1,\infty)$. For $k > 0$ consider the weak nonnegative solution to the following backward problem for the fractional heat equation

$$\begin{cases} -\partial_t\mu(x,t) + (-\Delta)^s\mu(x,t) = 0 & \text{in } Q_\tau \ , \\ \mu(x,\tau) = \mu_\tau(x) & \text{in } \mathbb{T}^d \ , \end{cases}$$

with $\mu_\tau(x) = \frac{[T_k(u^+(x,\tau)]^{p-1}}{\|u^+(\tau)\|_p^{p-1}}$. First, observe that $\|\mu_\tau\|_{p'} \le 1$. By Corollary 4.9 with $b \equiv 0$ we deduce

$$\|\mu\|_{L^{q'}(Q_\tau)} \le C \ ,$$

for $\frac{d+2s}{(2s-1)\gamma'} < q < \frac{d+2s}{2s}$ and $p = \frac{dq}{d+2s-2sq}$ or $q = \frac{d+2s}{2s}$ and any $p > 1$, with C not depending on k. By parabolic Kato's inequality, cf [65, Theorem 34], u^+ is a subsolution to

$$\partial_t u^+(x,t) + (-\Delta)^s u^+(x,t) \le [f(x,t) - H(x, Du(x,t))]\chi_{\{u>0\}} \text{ in } Q_\tau \ .$$

Using μ as a test function in the weak formulation of the above equation we show that

$$\int_{\mathbb{T}^d} u^+(x,\tau)\mu_\tau(x)\, dx \le \int_{\mathbb{T}^d} u_0^+(x)\mu(x,0)\, dx + \iint_{Q_\tau\cap\{u>0\}} f\mu\, dxdt - \iint_{Q_\tau\cap\{u>0\}} H(x,Du)\mu\, dxdt \ .$$

We apply Hölder's inequality to the second term of the right-hand side of the above equality, the fact that $H \ge 0$ and $\|\mu(t)\|_{L^{p'}(\mathbb{T}^d)} \le C$ for all $t \in [0,\tau]$ to get, after sending $k \to \infty$,

$$\|u^+(\tau)\|_{L^p(\mathbb{T}^d)} \le C(\|u_0^+\|_{L^p(\mathbb{T}^d)} + \|f^+\|_{L^q(Q_\tau)}) \ .$$

Step 2. To prove the bound on the negative part, consider for $k > 0$ the solution $\rho = \rho_k$ to the adjoint problem

$$\begin{cases} -\partial_t\rho(x,t) + (-\Delta)^s\rho(x,t) + \mathrm{div}(D_pH(x,Du)\chi_{\{u>0\}}\rho) & \text{in } Q_\tau \ , \\ \rho(x,\tau) = \mu_\tau(x) & \text{in } \mathbb{T}^d \ . \end{cases}$$

First, by Corollary 4.9 we have

$$\|\rho(0)\|_{W^{\frac{d(2s-1)}{d+2s}-\frac{2s}{q'},\frac{d+2s}{d+2s+(2s-1)q'}}(\mathbb{T}^d)} + \|\rho\|_{L^{q'}(Q_\tau)} \le C\left(\iint_{Q_\tau} |D_pH(x,Du)|^{\frac{d+2s}{(2s-1)q}}\chi_{\{u<0\}}\rho\, dxdt + \|\rho_\tau\|_{L^{p'}(\mathbb{T}^d)}\right) \tag{38}$$

with C not depending on k. We note again that in view of the parabolic Kato's inequality u^- is a weak subsolution to

$$\partial_t u^-(x,t) + (-\Delta)^s u^-(x,t) \le [-f(x,t) + H(x, Du(x,t))]\chi_{\{u<0\}} \text{ in } Q_\tau \ .$$

Owing to the representation formula we get

$$\int_{\mathbb{T}^d} u^-(\tau)\rho(\tau)\, dx + \iint_{Q_\tau} [-D_pH(x,Du)\cdot Du^- - H(x,Du)]\chi_{\{u<0\}}\rho\, dxdt \le \int_{\mathbb{T}^d} u_0^-\rho(0)\, dx - \iint_{Q_\tau\cap\{u<0\}} f\rho \ .$$

By the assumption on the Lagrangian we get

$$[-D_pH(x,Du)\cdot Du^- - H(x,Du)]\chi_{\{u<0\}} = L(x, D_pH(x,-Du^-))\chi_{\{u<0\}} \ge [C_L^{-1}|D_pH(x,Du)|^{\gamma'} - C_L]\chi_{\{u<0\}} \ .$$

Then, we have

$$\begin{aligned}\int_{\mathbb{T}^d} u^-(\tau)\rho(\tau)\, dx &+ C_L^{-1}\iint_{Q_\tau} |D_pH(x,Du)|^{\gamma'}\rho\, dxdt - C_L\iint_{Q_\tau}\chi_{\{u<0\}}\rho\, dxdt \\ &\le \|u_0^-\|_{W^{\frac{2s}{q'}-\frac{d(2s-1)}{d+2s},\frac{(d+2s)q'}{q'(d+1)-(d+2s)}}(\mathbb{T}^d)}\|\rho(0)\|_{W^{\frac{d(2s-1)}{d+2s}-\frac{2s}{q'},\frac{d+2s}{d+2s+(2s-1)q'}}(\mathbb{T}^d)} + \|\rho\|_{L^{q'}(Q_\tau)}\|f^-\|_{L^q(Q_\tau)} \ .\end{aligned}$$

By Lemma 3.2

$$\|u_0^-\|_{W^{\frac{2s}{q'}-\frac{d(2s-1)}{d+2s},\frac{(d+2s)q'}{\eta'(d+1)-(d+2s)}}(\mathbb{T}^d)} \le C\|u_0^-\|_{W^{2s-2s/q,q}(\mathbb{T}^d)}$$

since

$$2s - \frac{2s}{q} - \frac{d}{q} = \frac{2s}{q'} - \frac{d(2s-1)}{d+2s} - d\frac{(d+1)q' - (d+2s)}{(d+2s)q'} \ .$$

Then, we get

$$\int_{\mathbb{T}^d} u^-(\tau)\rho(\tau)\,dx + C_L^{-1}\iint_{Q_\tau} |D_pH(x,Du)|^{\gamma'}\rho\,dxdt - C_L\iint_{Q_\tau}\chi_{\{u<0\}}\rho\,dxdt$$
$$\leq C\|u_0^-\|_{W^{2s-2s/q,q}(\mathbb{T}^d)}\|\rho(0)\|_{W^{\frac{d(2s-1)}{d+2s}-\frac{2s}{q'},\frac{d+2s}{d+2s+(2s-1)q'}}(\mathbb{T}^d)} + \|\rho\|_{L^{q'}(Q_\tau)}\|f^-\|_{L^q(Q_\tau)}\,. \quad (39)$$

The term $\|\rho(0)\|_{W^{\frac{d(2s-1)}{d+2s}-\frac{2s}{q'},\frac{d+2s}{d+2s+(2s-1)q'}}(\mathbb{T}^d)}$ can be controlled owing to Corollary 4.9, so using (38) we conclude

$$\int_{\mathbb{T}^d} u^-(\tau)\rho(\tau)\,dx + C_L^{-1}\iint_{Q_\tau} |D_pH(x,Du)|^{\gamma'}\rho\,dxdt$$
$$\leq C(\|u_0^-\|_{W^{2s-2s/q,q}(\mathbb{T}^d)}+\|f^-\|_{L^q(Q_\tau)})\left(\iint_{Q_\tau}|D_pH(x,Du)|^{\frac{d+2s}{(2s-1)q}}\chi_{\{u<0\}}\rho\,dxdt+1\right)+C_L\iint_{Q_\tau}\chi_{\{u<0\}}\rho\,dxdt\,.$$

Since $q > \frac{d+2s}{(2s-1)\gamma'}$ one can use Young's inequality and that $\int \rho(t)\,dx \leq \|\mu_\tau\|_{p'} \leq 1$ to conclude, after letting $k \to \infty$,

$$\int_{\mathbb{T}^d} u^-(\tau)\rho(\tau)\,dx \leq C(\|u_0^-\|_{W^{2s-2s/q,q}(\mathbb{T}^d)} + \|f^-\|_{L^q(Q_\tau)})$$
$$+ CC_L\tau((\|u_0^-\|_{W^{2s-2s/q,q}(\mathbb{T}^d)} + \|f^-\|_{L^q(Q_\tau)})^{\frac{\gamma' q(2s-1)}{\gamma' q(2s-1)-(d+2s)}} + C_L\tau\,. \quad (40)$$

Combining Step 1 and Step 2 we conclude

$$\|u\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))} \leq C$$

for p as above. □

5.4 Hölder regularity results

We are now in position to prove Hölder bounds for solutions to the fractional Hamilton-Jacobi equation (2) as in Theorem 2.5.

Proof of Theorem 2.5. Since H is convex and superlinear, we write for a.e. $(x,t) \in Q_T$

$$H(x, Du(x,t)) = \sup_{\nu\in\mathbb{R}^d}\{\nu\cdot Du(x,t) - L(x,\nu)\}\,.$$

Let $0 < \omega < \tau < T$. By the weak formulation of (2) we obtain

$$\int_\omega^\tau \langle\partial_t u(t),\varphi(t)\rangle dt + \iint_{Q_{\omega,\tau}} (-\Delta)^{\frac{1}{2}}u(x,t)\,(-\Delta)^{s-\frac{1}{2}}\varphi(x,t) + [\Theta(x,t)\cdot Du(x,t) - L(x,\Theta(x,t))]\varphi\,dxdt$$
$$\leq \iint_{Q_{\omega,\tau}} f(x,t)\varphi(x,t)\,dxdt \quad (41)$$

for all test functions $\varphi \in \mathcal{H}_2^{2s-1}(Q_{\omega,\tau}) \cap L^\infty(Q_{\omega,\tau})$ and measurable $\Theta : Q_{\omega,\tau} \to \mathbb{R}^d$ such that $L(\cdot,\Theta(\cdot,\cdot)) \in L^1(Q_{\omega,\tau})$ and $\Theta\cdot Du \in L^1(Q_{\omega,\tau})$. Note that the previous inequality becomes an equality if $\Theta(x,t) = D_pH(x,Du(x,t))$ in $Q_{\omega,\tau}$.

We fix $\rho_\tau \in C^\infty(\mathbb{T}^d)$, $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$ and $\rho_\tau \geq 0$. Set

$$w(x,t) = \eta(t)u(x,t).$$

Use now (41) with $\Xi(x,t) = D_pH(x, Du(x,t))$ and $\varphi = \eta\rho \in \mathcal{H}_2^{2s-1}(Q_{\omega,\tau}) \cap L^\infty(Q_{\omega,\tau})$, where ρ is the adjoint variable (i.e. the weak solution to (31)) to find

$$\int_\omega^\tau \langle \partial_t w(t), \rho(t)\rangle dt + \iint_{Q_{\omega,\tau}} (-\Delta)^{\frac{1}{2}} u(x,t)\, (-\Delta)^{s-\frac{1}{2}} \varphi(x,t) + D_pH(x,Du)\cdot Dw\rho - L(x, D_pH(x,Du))\eta\rho\, dxdt$$
$$= \iint_{Q_{\omega,\tau}} f\eta\rho\, dxdt + \iint_{Q_{\omega,\tau}} u\eta'\rho\, dxdt. \quad (42)$$

Then, use $w \in \mathcal{H}_2^1(Q_T)$ as a test function in the weak formulation of the equation satisfied by ρ to get

$$-\int_\omega^\tau \langle \partial_t \rho(t), w(t)\rangle dt + \iint_{Q_{\omega,\tau}} (-\Delta)^{s-\frac{1}{2}}\rho(-\Delta)^{\frac{1}{2}} w + D_pH(x,Du)\rho \cdot Dw\, dxdt = 0. \quad (43)$$

We now fix ω small so that $\eta(\omega) = 0$. We then obtain, subtracting the previous equality to (42), and integrating by parts in time

$$\int_{\mathbb{T}^d} w(x,\tau)\rho_\tau(x)dx = \iint_{Q_{s,\tau}} \eta(t) f(x,t)\rho(x,t)dxdt$$
$$+ \iint_{Q_{s,\tau}} \eta(t) L\big(x, D_pH(x, Du(x,t))\big)\rho(x,t)dxdt + \iint_{Q_{s,\tau}} \eta'(t)u(x,t)\rho(x,t)dxdt. \quad (44)$$

For $h > 0$ and $\xi \in \mathbb{R}^d$, $|\xi| = 1$, define $\hat\rho(x,t) := \rho(x - h\xi, t)$. After a change of variables in (31), it can be seen that $\hat\rho$ satisfies, using w as a test function,

$$-\int_\omega^\tau \langle \partial_t \hat\rho(t), w(t)\rangle dt + \iint_{Q_{\omega,\tau}} (-\Delta)^{s-\frac{1}{2}}\hat\rho(-\Delta)^{\frac{1}{2}} w\, dxdt$$
$$+ \iint_{Q_{\omega,\tau}} D_pH(x - h\xi, Du(x - h\xi, t))\hat\rho(x,t) \cdot Dw(x,t)\, dxdt = 0. \quad (45)$$

As before, plugging $\Theta(x,t) = D_pH(x - h\xi, Du(x - h\xi, t))$ and $\varphi = \eta\hat\rho$ in (41) yields

$$\int_\omega^\tau \langle \partial_t w(t), \hat\rho(t)\rangle dt +$$
$$\iint_{Q_{\omega,\tau}} (-\Delta)^{s-\frac{1}{2}}\hat\rho(-\Delta)^{\frac{1}{2}} w + D_pH(x-h\xi, Du(x-h\xi,t))\cdot Dw\hat\rho - L(x, D_pH(x-h\xi, Du(x-h\xi,t)))\eta\hat\rho\, dxdt$$
$$\le \iint_{Q_{\omega,\tau}} f\eta\hat\rho\, dxdt + \iint_{Q_{\omega,\tau}} u\eta'\hat\rho\, dxdt.$$

Hence, subtracting (45) to the previous inequality,

$$\int_{\mathbb{T}^d} w(x,\tau)\hat\rho_\tau(x)dx \le \iint_{Q_{\omega,\tau}} L(x, D_pH(x - h\xi, Du(x - h\xi, t)))\eta\hat\rho\, dxdt + \iint_{Q_{\omega,\tau}} f\eta\hat\rho\, dxdt$$
$$+ \iint_{Q_{\omega,\tau}} u\eta'\hat\rho\, dxdt,$$

which, after the change of variables $x \mapsto x + h\xi$, becomes

$$\int_{\mathbb{T}^d} w(x + h\xi, \tau)\rho_\tau(x)dx \le \iint_{Q_{\omega,\tau}} \eta(t)L(x + h\xi, D_pH(x, Du(x,t)))\rho(x,t)\, dxdt + \iint_{Q_{\omega,\tau}} f\eta\hat\rho\, dxdt$$
$$+ \iint_{Q_{\omega,\tau}} u\eta'\hat\rho\, dxdt, \quad (46)$$

Taking the difference between (46) and (44) we obtain

$$
\begin{aligned}
\int_{\mathbb{T}^d} (w(x+h\xi,\tau) - w(x,\tau))\rho_\tau(x)dx \\
\leq \iint_{Q_{\omega,\tau}} \eta(t)\Big(L(x+h\xi, D_pH(x,Du(x,t))) - L(x,D_pH(x,Du(x,t)))\Big)\rho(x,t)\,dxdt \\
+ \iint_{Q_{\omega,\tau}} \eta(t)f(x,t)\big(\rho(x-h\xi,t)-\rho(x,t)\big)\,dxdt \\
+ \iint_{Q_{\omega,\tau}} \eta'(t)u(x,t)\big(\rho(x-h\xi,t)-\rho(x,t)\big)\,dxdt = (I)+(II)+(III) \quad (47)
\end{aligned}
$$

Step 2. We now estimate all the right hand side terms of (47). We stress that constants $C, C_1, \ldots$ are not going to depend on τ, ρ_τ, h, ξ.
First, we observe that in view of the representation formula, we have

$$
\begin{aligned}
\iint_{Q_{\omega,\tau}} L(x, D_pH(x,Du(x,t))\rho(x,t)\,dxdt = \int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)\,dx - \int_{\mathbb{T}^d} u(x,\omega)\rho(x,\omega)\,dx \\
- \iint_{Q_{\omega,\tau}} f(x,t)\rho(x,t)\,dxdt\ .
\end{aligned}
$$

Then, one obtains

$$
C_L^{-1}\iint_{Q_{\omega,\tau}} |D_pH(x,Du(x,t))|^{\gamma'}\rho(x,t)\,dxdt \leq C + \|f\|_{L^q(Q_\tau)}\|\rho\|_{L^{q'}(Q_{\omega,\tau})}\ .
$$

We use Corollary 4.9 to get

$$
\begin{aligned}
C_L^{-1}\iint_{Q_{\omega,\tau}} |D_pH(x,Du(x,t))|^{\gamma'}\rho(x,t)\,dxdt \\
\leq C + C_1\|f\|_{L^q(Q_\tau)}\left(\iint_{Q_{\omega,\tau}} |D_pH(x,Du)|^{\frac{d+2s}{(2s-1)q}}\rho(x,t)\,dxdt + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}\right)\ .
\end{aligned}
$$

A straightforward application of Young's inequality yields a control on $\iint_{Q_{\omega,\tau}} |D_pH(x,Du(x,t))|^{\gamma'}\rho(x,t)\,dxdt$ and hence we derive by Theorem 2.2

$$
\iint_{Q_{\omega,\tau}} |D_pH(x,Du(x,t))|^{\gamma'}\rho(x,t)\,dxdt + \|\rho\|_{\mathcal{H}^{2s-1}_{\frac{d+2s}{d+4s-1-\gamma'(2s-1)}}(Q_{\omega,\tau})} \leq C_2\ .
$$

Let $\alpha \in (0,1)$ to be determined. For $\nu = D_pH(x,p)$ we have $L(x,\nu) = \nu\cdot p - H(x,p)$ and thus

$$
L(x+h\xi, D_pH(x,Du(x,t))) - L(x,D_pH(x,Du(x,t)) \leq H(x,Du(x,t)) - H(x+\xi,Du(x,t))\ .
$$

Next, using (H_α) and the above inequality, we get

$$
\begin{aligned}
\left|\iint_{Q_{\omega,\tau}} \eta(t)\Big(L(x+h\xi, D_pH(x,Du(x,t))) - L(x,D_pH(x,Du(x,t)))\Big)\rho(x,t)\,dxdt\right| \\
\leq C_2C_L|h|^\alpha \iint_{Q_{\omega,\tau}} \big(|D_pH(x,Du(x,t))|^{\gamma'}+1\big)\rho(x,t)\,dxdt \leq C|h|^\alpha.
\end{aligned}
$$

We can apply the Sobolev embedding in Lemma 3.8-(ii) with $\delta = q'$, $p = \frac{d+2s}{d+4s-1-\gamma'(2s-1)}$, giving $\alpha = \gamma'(2s-1) - \frac{d+2s}{q}$, and Lemma 3.12, to show

$$\left|\iint_{Q_{\omega,\tau}} \eta(t) f(x,t)\big(\rho(x-h\xi,t)-\rho(x,t)\big)\,dxdt\right|$$
$$\leq |h|^\alpha \iint_{Q_{\omega,\tau}} |f(x,t)| \frac{|(\rho(x-h\xi,t)-\rho(x,t))|}{|h|^\alpha}\,dxdt \leq |h|^\alpha \|f\|_{L^q(Q_{\omega,\tau})}\|\rho\|_{L^{q'}(\omega,\tau;N^{\alpha,q'}(\mathbb{T}^d))}$$
$$\leq C_1|h|^\alpha \|f\|_{L^q(Q_{\omega,\tau})}\|\rho\|_{L^{q'}(\omega,\tau;W^{\alpha,q'}(\mathbb{T}^d))} \leq C_2|h|^\alpha \|f\|_{L^q(Q_{\omega,\tau})}\|\rho\|_{\mathcal{H}^{2s-1}_{\frac{d+2s}{d+4s-1-\gamma'(2s-1)}}(Q_{\omega,\tau})} \leq C_3|h|^\alpha \|f\|_{L^q(Q_{\omega,\tau})}.$$

Finally, as above, we conclude

$$\left|\iint_{Q_{\omega,\tau}} \eta'(t) u(x,t)\big(\rho(x-h\xi,t)-\rho(x,t)\big)\,dxdt\right| \leq |h|^\alpha \Big(\sup_{(0,T)}|\eta'(t)|\Big)\|u\|_{L^{q'}(Q_{\omega,\tau})}\|\rho\|_{L^{q'}(\omega,\tau;N^{\alpha,q'}(\mathbb{T}^d))}$$
$$\leq C_1|h|^\alpha \Big(\sup_{(0,T)}|\eta'(t)|\Big)\|u\|_{C(\overline{Q}_T)}\|\rho\|_{L^{q'}(\omega,\tau;W^{\alpha,q'}(\mathbb{T}^d))} \leq C_2|h|^\alpha \sup_{(0,T)}|\eta'(t)|.$$

Plugging all the estimates in (47) we obtain

$$\int_{\mathbb{T}^d} (w(x+h\xi,\tau)-w(x,\tau))\rho_\tau(x)dx \leq C|h|^\alpha\Big(\sup_{(0,T)}|\eta'(t)|+1\Big). \tag{48}$$

when $q > \frac{d+2s}{(2s-1)\gamma'}$.

Step 3. Since (48) holds for all smooth $\rho_\tau \geq 0$ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$, we get

$$\eta(\tau)[u(x+h\xi,\tau)-u(x,\tau)] \leq C|h|^\alpha\Big(\sup_{(0,T)}|\eta'(t)|+1\Big)$$

for all $x \in \mathbb{T}^d$, $\xi \in \mathbb{R}^d$, $h > 0$. Thus, $u(\cdot,\tau)$ is Hölder continuous, and

$$\eta(\tau)[u(\cdot,\tau)]_{C^\alpha(\mathbb{T}^d)} \leq C\Big(\sup_{(0,T)}|\eta'(t)|+1\Big).$$

Since C does not depend on $\tau \in (0,T)$, we take $t_1 \in (0,T)$, $\eta = \eta(t)$ nonnegative smooth function on $[0,T]$ such that $\eta(t) \leq 1$, $\eta(t) = 1$ on $[t_1,T]$ and vanishing on $[0,t_1/2]$. This proves Theorem 2.5-(C).

To prove the global-in-time bound in (D) one may observe that if $u \in \mathcal{H}^{2s}_q(Q_T)$ is a strong solution with $q > \frac{d+2s}{(2s-1)\gamma'}$, $\gamma \geq 2s$ (or a global weak solution), then the solution is global in time and one can take $\omega = 0$ throughout the proof setting also $\eta \equiv 1$ on $[0,T]$. Being the solution global, norms $\|u\|_{C(\overline{Q_T})}$ can be replaced by $\|u_0\|_{C(\mathbb{T}^d)}$ by Theorem 2.3, which are in turn bounded by $\|u_0\|_{C^\alpha(\mathbb{T}^d)}$. Now, an additional term of the form

$$\int_{\mathbb{T}^d} \frac{u(x+h,0)-u(x,0)}{|h|^\alpha}\rho(x,0)\,dx$$

arises, which can be immediately bounded by $[u_0]_{C^\alpha(\mathbb{T}^d)}$ since $\int_{\mathbb{T}^d}\rho(0) = 1$.

□

Remark 5.12. Using the same scheme of Theorem 2.5, we believe one can even handle the case $\gamma < 2s$ by considering appropriate weak solutions (not continuous on the whole cylinder Q_T) to (2).

Remark 5.13. An approach similar to that for the Hölder bounds and the one in [35, Theorem 1.1], which exploits the regularity properties in Corollary 5.9, yields a Lipschitz regularization effect for (2) whenever $f \in L^q(0,T;H^{2-2s}_q(\mathbb{T}^d))$ for $\sigma = q$ as in Corollary 5.9. This requires to impose that $|D_xH(x,p)| \leq C_H(|p|^\gamma + 1)$ instead of (H_α), see [46, Chapter 7].

5.5 Maximal L^q-regularity for time-dependent fractional Hamilton-Jacobi equations

5.5.1 An overview of the results in the viscous case

Let us first consider the following viscous problem

$$-\Delta u + |Du|^\gamma = f(x) \text{ in } \Omega\,, \gamma > 1\,, \tag{49}$$

where f is an unbounded source term belonging to a suitable Lebesgue space L^q. In [69, 70] P.-L. Lions proposed the following conjecture:

Conjecture 5.14. Let $f \in L^q(\Omega)$, $q > 1$, for some

$$q > \frac{d}{\gamma'} = \frac{d(\gamma-1)}{\gamma}\,. \tag{50}$$

and $\gamma > 1$. Then, every solution to (49) satisfies the a priori estimate

$$\|D^2u\|_{L^q} + \||Du|^\gamma\|_{L^q} \le C(\|f\|_{L^q}, d, q, \gamma)\,.$$

Moreover, the estimate is false when $q \le d/\gamma'$.

A byproduct of this statement is a maximal L^q-regularity for solutions to (49) and says that (49) behaves in terms of regularity as the Poisson equation (cf [45, Theorem 9.9]) under the regime (50) of the integrability exponent q of the right-hand side f. Maximal regularity at the endpoints $q = 1$ and $q = \infty$ fails to be true even for the Poisson equation, see [78], and one needs to consider Besov scales [77]. Conjecture 5.14 completes the results in [68] for the subcritical range of the integrability of the forcing term, where it is shown a Lipschitz regularity result when $f \in L^q$, $q > d$, and every $\gamma > 1$, obtained via an integral Bernstein method. A proof of Conjecture 5.14 has been proposed in [37] appropriately modifying the Bernstein method, while an extension to the parabolic viscous framework has been already provided in [36]. More precisely, in [36] it is proved that maximal regularity for viscous (2) occurs for strong solutions when $f \in L^q(Q_T)$ with

$$q > \begin{cases} (d+2)\frac{\gamma-1}{\gamma} & \text{if } 1 + \frac{2}{d+2} < \gamma < 2 \\ (d+2)\frac{\gamma-1}{2} & \text{if } \gamma \ge 2. \end{cases}$$

Note that the threshold $q = (d+2)(\gamma-1)/\gamma$ can be regarded as a parabolic analogue to the one in (50). We also refer to [38] for more recent maximal regularity results for viscous ($s = 1$) quadratic problems with right-hand sides in mixed Lebesgue scales and to [55] (and the references therein) for some maximal regularity properties for fully nonlinear uniformly parabolic problems in the context of L^p viscosity solutions.

5.5.2 Maximal regularity below the natural growth

We first recall the following Calderón-Zygmund regularity result for the fractional heat equation with unbounded potential.

Lemma 5.15. *Let $u \in \mathcal{H}_q^{2s}(Q_T)$, $q > 1$, be a strong solution to*

$$\begin{cases} \partial_t u + (-\Delta)^s u = V(x,t) & \text{in } Q_T \\ u(x,0) = u_0 & \text{in } \mathbb{T}^d\,. \end{cases}$$

with $V \in L^q(Q_T)$ and $u_0 \in (L^q(\mathbb{T}^d), H_q^{2s}(\mathbb{T}^d))_{1-1/q,q} \simeq W^{2s-2s/q,q}(\mathbb{T}^d)$. Then, there exists a constant C that remains bounded for bounded values of T, such that

$$\|u\|_{\mathcal{H}_q^{2s}(Q_T)} = \|\partial_t u\|_{L^q(Q_T)} + \|u\|_{L^q(0,T;H_q^{2s}(\mathbb{T}^d))} \le C\left(\|V\|_{L^q(Q_T)} + \|u_0\|_{W^{2s-2s/q,q}(\mathbb{T}^d)}\right) .$$

As a consequence, every strong solution to (2) *with $u_0 \in W^{2s-2s/q,q}(\mathbb{T}^d)$ satisfies*

$$\|u\|_{\mathcal{H}_q^{2s}(Q_T)} \le C(\|f\|_{L^q(Q_T)} + \|H(x,Du)\|_{L^q(Q_T)} + \|u_0\|_{W^{2s-2s/q,q}(\mathbb{T}^d)}) .$$

By gathering all the previous results and the estimates in Theorems 2.3 and Theorem 2.4 we have the following maximal L^q-regularity result for (2) with (fractional) sub-natural growth.

Proof of Theorem 2.6. We exploit the Gagliardo-Nirenberg inequality in Lemma 3.4 to get for $\gamma \in (1, 2s)$

$$\||Du(t)\|_{L^{\gamma q}(Q_T)} \le C_1 \|u(t)\|^{\theta}_{H_q^{2s}(\mathbb{T}^d)} \|u(t)\|^{1-\theta}_{L^z(\mathbb{T}^d)} \tag{51}$$

for $z \in (1,\infty)$ and $\theta \in \left[\frac{1}{2s}, 1\right)$ with

$$\frac{1}{\gamma q} = \frac{1}{d} + \theta\left(\frac{1}{q} - \frac{2s}{d}\right) + \frac{1-\theta}{z} .$$

By Theorems 2.3 and 2.4 we have

$$\sup_t \|u(t)\|_{L^z(\mathbb{T}^d)} < \infty$$

for any $z \le p = \frac{dq}{d+2s-2sq}$ if $q < \frac{d+2s}{2s}$, $z \in [1,\infty)$ when $q = \frac{d+2s}{2s}$, $z = \infty$ for $q > \frac{d+2s}{2s}$. Since $q > \frac{d+2s}{(2s-1)\gamma'}$, we conclude $p > \frac{d(\gamma-1)}{2s-\gamma}$. Then, we choose z close to $\frac{d(\gamma-1)}{2s-\gamma}$ so that

$$\theta\gamma = \frac{\frac{1}{z} + \frac{1}{d} - \frac{1}{\gamma q}}{\frac{1}{z} + \frac{2s}{d} - \frac{1}{q}}\gamma < 1$$

and $\theta \in [1/2s, 1/\gamma)$. Raising (51) to γq and integrating in time we have

$$\|Du\|_{L^{\gamma q}(Q_T} \le C_2 \|u\|^{\gamma\theta}_{L^q(0,T;H_q^{2s}(\mathbb{T}^d))} \|u\|^{\gamma(1-\theta)}_{L^\infty(0,T;L^z(\mathbb{T}^d))} .$$

Then, by (H) we deduce for positive constants $C_3, C_4 > 0$

$$\|H(x,Du)\|_{L^q(Q_T)} \le C_3(1 + \||Du|\|^{\gamma}_{L^{\gamma q}(Q_T)}) \le C_4(\|u\|^{\gamma\theta}_{L^q(0,T;H_q^{2s}(\mathbb{T}^d))} \|u\|^{\gamma(1-\theta)}_{L^\infty(0,T;L^z(\mathbb{T}^d))} + 1) .$$

Using Lemma 5.15 and Young's inequality we have

$$\begin{aligned}
\|u\|_{\mathcal{H}_q^{2s}(Q_T)} &\le C_5(\|H(x,Du)\|_{L^q(Q_T)} + \|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}) \\
&\le C_6(\|u\|^{\gamma\theta}_{L^q(0,T;H_q^{2s}(\mathbb{T}^d))} \|u\|^{\gamma(1-\theta)}_{L^\infty(0,T;L^z(\mathbb{T}^d))} + \|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2s-\frac{2s}{q},q}(\mathbb{T}^d)}) \\
&\le \frac{1}{2}\|u\|_{L^q(0,T;H_q^{2s}(\mathbb{T}^d))} + C_7 \|u\|^{\frac{\gamma(1-\theta)}{1-\gamma\theta}}_{L^\infty(0,T;L^z(\mathbb{T}^d))} + C_6(\|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2s-\frac{2s}{q},q}(\mathbb{T}^d)}) .
\end{aligned}$$

We then absorb the term $\frac{1}{2}\|u\|_{\mathcal{H}_q^{2s}(Q_T)}$ on the left-hand side and use the integral estimate in Theorem 2.4 in $L^\infty(L^z)$ to conclude the assertion. Note that here we used Remark 5.4 to make sure that $\|u_0\|_{L^p(\mathbb{T}^d)}$ for all finite p is controlled by $\|u_0\|_{W^{2s-2s/q,q}(\mathbb{T}^d)}$. Then, we have

$$\||Du|\|_{L^{\gamma q}(Q_T)} \le C(\|f\|_{L^q(Q_T)}, \|u_0\|_{W^{2s-\frac{2s}{q},q}(\mathbb{T}^d)})$$

for a possibly different constant $C > 0$. □

Remark 5.16. The case $\gamma \ge 2s$ can be handled using Gagliardo-Nirenberg inequalities as in Lemma 3.5 provided that $(1-\beta)\alpha + \beta 2s$ with $\beta = \frac{1-\alpha}{2s-\alpha}$ can be integer, as in [36].

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alessandro.goffi@math.unipd.it
Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova
via Trieste 63, 35121 Padova (Italy)