

A note about hardening-free viscoelastic models in Maxwellian-type rheologies at large strains

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Abstract. Maxwellian-type rheological models of inelastic effects of creep type at large strains are revisited in relation to inelastic-strain gradient theories. In particular, we observe that a dependence of the stored-energy density on inelastic-strain gradients may lead to spurious hardening effects, preventing the model from accommodating large inelastic slips. The main result of this paper is an alternative inelastic model of creep type, where higher-order energy-contribution is provided by the gradients of the elastic strain and of the plastic strain rate, thus preventing the onset of spurious hardening under large slips. The combination of Kelvin-Voigt damping and Maxwellian creep results in a Jeffreys-type rheological model. Existence of weak solutions is proved via a Faedo-Galerkin approximation.

Keywords: creep at large strains, spurious hardening, gradient of the elastic strain, weak solutions.

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1 Introduction

Inelasticity at large strain has been the focus of an intense research activity for decades, first from the engineering community, see, e.g., the monographs [10, 18, 25], and subsequently also from the mathematical point of view (see, e.g., the recent contributions [9, 20, 26] on large-strain rate-independent processes, incomplete damage, and finite plasticity, respectively as well as the monographs [21, 33] and the references therein).

Within the mathematical purview, there is a general agreement that the rigorous analysis of large-strain inelastic time-evolving phenomena requires higher-order regularizations of the inelastic strains [8, 13, 24, 29, 32–34, 44]. Existence theories without gradient regularization are available only in one space dimension [27], at the incremental level [30, 31, 47], or under stringent modeling restrictions [20, 30]. In the engineering literature, on the other hand, gradient theories at large strains are seldom considered, see [3, 10, 37], [18, Ch. 25], [25, Ch. 8], and existence of solutions not in focus.

Gradient theories for the inelastic strain introduce an internal length-scale in the problem related to the characteristic width of inelastic slip-bands arising during creep, damage, or plastification processes. The occurrence of such scale is however not expected to cause additional hardening. Although sometimes strain or time hardening are to be considered [3, 37], in many applications, inelastic models are ultimately desired not to exhibit any hardening effect during long-lasting slip deformations. In metals, for example, very large irreversible plastification can occur within the phenomenon sometimes referred to as *superplasticity*. Large slips with no hardening are particularly common in rock, soil, or ice mechanics. Typically, the slip on tectonic faults can easily accommodate kilometers during millions of years. Glaciers flow kilometers, with hardening only occurring at temperatures below -70°C [46]. In a very different context, large deformations without hardening can be observed in polymers as well.

As a result, one is interested in identifying inelastic strain-gradient modelizations guaranteeing, on the one hand, that the existence of time-evolution of inelastic phenomena is mathematically well-posed, and on the other hand, that no spurious hardening effects are generated. The focus of this paper is hence on introducing a novel hardening-free inelastic model of creep-type allowing for existence of solutions. In order to accomplish this, the energy of the medium is assumed to contain a term depending on the gradient of the *elastic* strain. This contrasts with usual approaches based on *total* strain-gradient or *inelastic* strain-gradient regularization. Indeed, we present an example in Subsection 2.3 below showing the possible effect of such usual strain-gradient regularizations on the onset of spurious hardening.

Our new model is introduced in Section 2. In addition to elastic-strain hardening, we assume the viscous dissipation to be quadratic and to depend on the gradient of the inelastic-strain rate. This last gradient term does not affect the hardening-free nature of the model.

Eventually, Section 3 focuses on the existence of weak solutions to the model. The proof relies on a Faedo-Galerkin approximation, as well as on compactness, and lower semicontinuity arguments.

2 A hardening-free viscoelastic model

We devote this section to introducing and commenting our modeling choices.

Following the classical mathematical theory of inelasticity at large strains [15, 17, 23], we assume that the elastic behavior of our specimen $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is independent from preexistent inelastic distortions. This can be rephrased as the assumption that the deformation gradient $F := \nabla y$ associated to any deformation $y : \Omega \rightarrow \mathbb{R}^d$ of the body decomposes into an elastic strain and an inelastic one. For linearized theories, this decomposition would have an additive nature; in the setting of large-strain inelasticity, instead, this behavior is traditionally modeled via a multiplicative decomposition. In the mathematical literature different constitutive models have been taken into account, see, e.g., [8, 12, 13, 36] in the framework of finite plasticity. We focus here on the classical multiplicative decomposition

ansatz [19, 22], recently justified in the setting of dislocation systems and crystal plasticity in [6, 7]), in which deformations $y \in H^1(\Omega; \mathbb{R}^d)$ fulfill

$$F = F_{\text{el}}\Pi, \quad (2.1)$$

where F_{el} and Π denote the elastic and inelastic strains, respectively.

2.1 Tensorial notation

In the following, we use capital letters to indicate tensors and tensor-valued functions, independently from their dimensions. For $A, \widehat{A}, \widetilde{A} \in \mathbb{R}^{d \times d}$, $B, \widehat{B} \in \mathbb{R}^{d \times d \times d}$, and $C, \widehat{C} \in \mathbb{R}^{d \times d \times d \times d}$ we use the standard notation for contractions on two, three, and four indices, namely,

$$A:\widehat{A} = A_{ij}\widehat{A}_{ij}, \quad B:\widehat{B} = B_{ijk}\widehat{B}_{ijk}, \quad (C:A)_{ij} = C_{ijkl}A_{kl}, \quad (B:A)_i = B_{ijk}A_{jk}, \quad C::\widehat{C} = C_{ijkl}\widehat{C}_{ijkl}$$

(summation convention over repeated indices). On the other hand, contraction on one index will be marked by \cdot only in case of vectors. In particular, $(CA)_{ijkl} = C_{ijkm}A_{ml}$, $(BA)_{ijk} = B_{ijm}A_{mk}$, etc. The symbol \top indicate transposition of two-tensors, namely $A_{ij}^\top = A_{ji}$, whereas we denote by the superscript t the partial transposition of a four-tensor with respect to the first two indices, namely $C_{ijkl}^t = C_{jikl}$. For $A \in \mathbb{R}^{d \times d}$ we indicate its symmetric part by $\text{sym } A = (A + A^\top)/2$ and, if A is invertible, use the shorthand notation $A^{-\top} = (A^{-1})^\top$. We will use the algebra $A\widehat{A}:\widehat{A} = A:\widehat{A}\widehat{A}^\top$ and $A:\widehat{A}\widehat{A} = \widehat{A}^\top A:\widehat{A}$.

Let us recall that, for a differentiable function $F : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d}$ and $A, \widehat{A} \in \mathbb{R}^{d \times d}$ we have that $DF(A) \in \mathbb{R}^{d \times d \times d \times d}$ and $DF(A):\widehat{A} = (d/d\alpha)F(A + \alpha\widehat{A})|_{\alpha=0}$. In particular, one has that $D(A^{-1}):\widehat{A} = -A^{-1}\widehat{A}A^{-1}$. Moreover, one easily checks that $D(F^\top) = (DF)^\top$, so that one has that $D(A^{-\top}):\widehat{A} = -A^{-\top}\widehat{A}A^{-\top}$. Given two other differentiable functions $\widehat{F} : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d}$ and $f : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$ one has that $D(f \circ F)(A):\widehat{A} = Df(F(A)):DF(A):\widehat{A}$ and $D(\widehat{F} \circ F)(A):\widehat{A} = DF(\widehat{F}(A)):D\widehat{F}(A):\widehat{A}$.

Let the reference domain $\Omega \subset \mathbb{R}^d$ be open and with Lipschitz boundary Γ , and let n be the outward-pointing unit normal vector at the boundary. For a m -tensor valued function $x \in \Omega \mapsto A(x) \in (\mathbb{R}^d)^m$ with $m \geq 1$ we define the gradient $\nabla A(x) \in (\mathbb{R}^d)^{m+1}$ and the divergence $\text{div} A(x) \in (\mathbb{R}^d)^{m-1}$ componentwise as

$$\nabla A(x)_{i_1 \dots i_m j} = \frac{\partial}{\partial x_j} A_{i_1 \dots i_m}(x), \quad (\text{div} A(x))_{i_1 \dots i_{m-1}} = \sum_{j=1}^d \frac{\partial}{\partial x_j} A(x)_{i_1 \dots i_{m-1} j}.$$

For all $x \in \Omega \mapsto A(x) \in \mathbb{R}^{d \times d}$ and $x \in \Omega \mapsto \widehat{A}(x) \in \mathbb{R}^{d \times d}$ we have that $\nabla(A\widehat{A}) = (\widehat{A}^\top \nabla A^\top)^t + A \nabla \widehat{A}$. Let now $x \in \Omega \mapsto v(x) \in \mathbb{R}^d$, $x \in \Omega \mapsto A(x) \in \mathbb{R}^{d \times d}$, and $x \in \Omega \mapsto B(x) \in \mathbb{R}^{d \times d \times d}$ be given. Under suitable regularity assumptions the following Green formulas can be checked

$$\int_{\Omega} A:\nabla v \, dx = - \int_{\Omega} \text{div} A \cdot v \, dx + \int_{\Gamma} (An) \cdot v \, dx, \quad (2.2a)$$

$$\int_{\Omega} B:\nabla A \, dx = - \int_{\Omega} A:\text{div} B \, dx + \int_{\Gamma} (A:B) \cdot n \, dx. \quad (2.2b)$$

Eventually, let div_s denote the $(d-1)$ -dimensional surface divergence on Γ . For vector-valued functions $x \mapsto v(x) \in \mathbb{R}^d$ this is defined as

$$\text{div}_s v = \text{tr} \nabla_s v \quad \text{for} \quad \nabla_s v := \nabla v - \frac{\partial v}{\partial n} \otimes n,$$

where tr stands for the trace. The same definition will be used row-wise for tensor-valued functions. We will use the [11, Formula (34)]

$$\int_{\Gamma} A:\nabla_s v \, dS = - \int_{\Gamma} (\text{div}_s A \cdot v + 2\mathfrak{h}An \cdot v) \, dS, \quad (2.3)$$

where \mathfrak{h} stands for the mean curvature of Γ . Arguing row-wise, an analogous relation can be checked to hold for tensors-valued functions as well.

2.2 Stored energy

Our aim is that of introducing a hardening-free inelastic model. In absence of hardening, the mathematical analysis of inelastic evolution is notoriously challenging. In order to make the existence of weak solutions amenable, we include in the model higher-order (gradient) effects. More specifically, we define

$$\Phi(y, II) = \int_{\Omega} \varphi_E(\nabla y II^{-1}) + \varphi_H(II) + \varphi_G(\nabla(\nabla y II^{-1})) \, dx. \quad (2.4)$$

Here, $\varphi_E : \mathbb{R}^{d \times d} \rightarrow [0, \infty)$ corresponds to the elastic energy density of the medium and will be assumed to be coercive and to control the sign of $\det F_{\text{el}}$, see (3.1a) below. On the other hand, $\varphi_H : \mathbb{R}^{d \times d} \rightarrow [0, \infty]$ plays the role of a constraint on $\det II$. In particular, we are interested in choices of φ_H enforcing the usual isochoric constraint $\det II = 1$ in an approximate sense and keeping $\det II$ away from negative values, see (3.1b) below. An explicit example for such a term is

$$\varphi_H(II) := \begin{cases} \frac{\delta}{\max(1, \det II)^r} + \frac{(\det II - 1)^2}{2\delta} & \text{if } \det II > 0, \\ +\infty & \text{if } \det II \leq 0 \end{cases} \quad (2.5)$$

with $\delta > 0$ small and r big enough; cf. [44, Remark 2.6], [21, Formula (9.4.36)], or [40].

Eventually, $\varphi_G : \mathbb{R}^{d \times d} \rightarrow [0, \infty)$ controls the elastic strain gradient and relates to the length scale of higher-order effects. Specific assumptions are given in (3.1c) below. In particular, the stored energy features a regularizing term depending on the gradient of the elastic strain $F_{\text{el}} = \nabla y II^{-1}$. Note however that no gradient of II appears in the energy, for this might give rise to hardening, as explained in Subsection 2.3 below.

2.3 Spurious hardening from gradients in the stored energy

As already mentioned, the analysis of inelastic evolution models calls for considering inelastic gradient theories. Usual choices in this direction are terms of the form

$$\frac{1}{2}\kappa|\nabla II|^2 \quad (\text{standard choice}), \quad (2.6a)$$

$$\frac{1}{2}\kappa|F^{-\top} \nabla II|^2 \quad (\text{push forward}), \quad (2.6b)$$

$$\frac{1}{2}\kappa|\nabla(II^\top II)|^2 \quad (\text{inelastic metric tensor}). \quad (2.6c)$$

For the *standard choice* in (2.6a), we refer to [14, 21, 24, 34] in the context of plasticity, see also [32] for a more general dependence on ∇II covering also creep models, as well as [1] for

an additional scalar-valued internal variable acting as an effective inelastic strain. The *push-forward* term in (2.6b) has been used in [21, Remark 9.4.12] and [42, Remark 5], whereas the inelastic *metric* tensor in (2.6c) has been analyzed in [13], cf. [39] for a throughout discussion and comparison.

All models (2.6) however exhibit a drawback: the influence of the inelastic gradient terms amplifies when inelastic slips evolve and accommodate large inelastic strains. This, in turn, might result in a spurious hardening effect.

To demonstrate the presence of a non-autonomous spurious hardening effect, we consider $d = 2$ and resort to a stratified situation where F and Π are constant in the x_1 direction, cf. [44] or also [21, Example 9.4.11] for similar examples. We consider a pure *horizontal* shift of the stripe $\Omega = \mathbb{R} \times [-\ell, \ell]$ driven by time-dependent Dirichlet boundary conditions for the displacement on the sides $\mathbb{R} \times \{\pm\ell\}$ and evolving in a steady-state mode. In particular, we assume by symmetry that the deformation has the *stratified* form

$$y(x_1, x_2) = (x_1 + f(t, x_2), x_2)$$

where the slip via the (unspecified) smooth function $f : [0, +\infty) \times [-\ell, \ell] \rightarrow \mathbb{R}$ fulfills the given Dirichlet boundary conditions, say

$$f(t, \pm\ell) = \pm t. \quad (2.7)$$

We specify elastic response by assuming the material to be rigid. In particular, the elastic strain F_{el} is assumed to be the identity matrix. In the setting of plasticity, this would be called a plastic-rigid model. The corresponding inelastic strain reads then

$$\Pi = F = \nabla y = \begin{pmatrix} 1 & \partial_{x_2} f(t, x_2) \\ 0 & 1 \end{pmatrix}. \quad (2.8)$$

Let us note that $\det \Pi = 1$, so that $\varphi_{\text{H}}(\Pi) = 0$ when φ_{H} is defined as in (2.5). The arguments in the κ -term in (2.6) read (see Section 2 for details on the tensorial notation) then as

$$(\nabla \Pi)_{ijk} = \begin{cases} \partial_{x_2}^2 f(t, x_2) & \text{for } i = 1, j = 2, k = 2, \\ 0 & \text{otherwise,} \end{cases}, \quad (2.9a)$$

$$(F^{-\top} \nabla \Pi)_{ijk} = \begin{cases} \partial_{x_2}^2 f(t, x_2) & \text{for } i = 1, j = 2, k = 2, \\ -\partial_{x_2} f(t, x_2) \partial_{x_2}^2 f(t, x_2) & \text{for } i = j = k = 2, \\ 0 & \text{otherwise,} \end{cases}, \quad (2.9b)$$

$$(\nabla(\Pi^\top \Pi))_{ijk} = \begin{cases} \partial_{x_2}^2 f(t, x_2) & \text{for } i = 1, j = 2, k = 2, \\ \partial_{x_2}^2 f(t, x_2) & \text{for } i = 2, j = 1, k = 2, \\ 2\partial_{x_2} f(t, x_2) \partial_{x_2}^2 f(t, x_2) & \text{for } i = j = k = 2, \\ 0 & \text{otherwise,} \end{cases}. \quad (2.9c)$$

Note that $\partial_{x_2} f(t, x_2)$ necessarily depends on time. Indeed, if this were not the case one would have that

$$\dot{f}(t, \ell) - \dot{f}(t, -\ell) = \int_{-\ell}^{\ell} \partial_{x_2} \dot{f}(t, x_2) dx_2 = 0,$$

contradicting the fact that $\dot{f}(t, \pm\ell) = \pm 1$ from (2.7). Hence, in all cases, the argument of the quadratic terms in (2.6) is genuinely time dependent. More precisely, by taking the mean

across the stripe we have that

$$\frac{1}{2\ell} \int_{-\ell}^{\ell} \partial_{x_2} f(t, x_2) dx_2 = \frac{1}{2\ell} (f(t, \ell) - f(t, -\ell)) \stackrel{(2.7)}{=} \frac{t}{\ell}$$

so that the terms in (2.6) would actually be unbounded in time. This shows, that no matter how small the coefficient κ is, the regularizing terms in (2.6) grow indefinitely under large slips, preventing the energy from being bounded and eventually corrupting the modelization. To compensate for these spurious hardening-like effects, one could assume κ to be time dependent, which would however lead to an artificially non-autonomous model, which is also not desirable.

In order to avoid this spurious hardening effect while still retaining regularization, our choice (2.4) for Φ above departs from the classical inelastic-gradient regularization (2.6) by including the gradient of the elastic strain F_{el} instead. Note that in the above example the term ∇F_{el} vanishes, hence allowing for indefinitely large inelastic slips under bounded energy.

Before closing this discussion, let us mention the possibility of considering the alternative inelastic-gradient terms

$$\frac{1}{2}\kappa|\text{curl}\Pi|^2 \quad \text{or} \quad \frac{1}{2}\kappa|\Pi^{-\top}\text{curl}\Pi|^2 \quad (2.10)$$

in the energy Φ . Here, the curl of the tensor Π is taken row-wise in three dimension and is defined as $\text{curl}\Pi = (\partial_1\Pi_{12} - \partial_2\Pi_{11}, \partial_1\Pi_{22} - \partial_2\Pi_{21})$ in two dimensions. These terms correspond to the so-called *dislocation-density* tensor [4] and have been considered in [31, 40, 45] from the viewpoint of existence of solutions of the incremental problems. In case of (2.8), the plastic strain is curl-free and both terms in (2.10) vanish. Therefore these terms exhibit a capability to accumulate large inelastic slips at bounded energy, for they vanish for Π given by (2.8). In particular, at least in elastically “well rigid” materials, they would not generate the spurious hardening effect mentioned above. However, the options (2.8) do not seem to contribute sufficient compactness in order to devise an existence theory at the time-continuous level. Of course, combination of some option from (2.10) with some option from (2.4) in the stored energy is possible and yields analytically good compactifying effects but again the spurious hardening would be involved in the model.

2.4 Dissipation

In order to incorporate inertial effects, a Kelvin-Voigt-type viscosity needs to be included in the model. We consider a purely linear viscous model by assuming the dissipation potential to be quadratic in terms of rates, namely,

$$\mathcal{R}(y, \Pi; \nabla \dot{y}, \dot{\Pi}) = \int_{\Omega} \frac{\nu_m}{2} |\dot{\Pi}|^2 + \frac{\nu_h}{2} |\nabla^2 \dot{\Pi}|^2 + \frac{\nu_{kv}}{2} |\dot{C}_{\text{el}}|^2 dx$$

with $C_{\text{el}} = F_{\text{el}}^{\top} F_{\text{el}} = \Pi^{-\top} \nabla y^{\top} \nabla y \Pi^{-1}$,

(2.11)

where ν_m , ν_h , and ν_{kv} are positive viscous coefficients and C_{el} is the elastic Cauchy-Green tensor. In particular, the Kelvin-Voigt-type viscosity term depends on \dot{C}_{el} in order to ensure frame-indifference [2].

The occurrence of the $\nabla^2 \dot{\Pi}$ term above is motivated by the need of controlling the rate of Π uniformly in space while still avoiding hardening. In other words, differently from gradient

terms acting directly on Π (see Section 2), this term provides a regularization not giving rise to spurious hardening effects, a phenomenon which we want to avoid. This uniform bound in space in turn will allow the control of the nonlinear terms in (2.12) as well as of the inverse Π^{-1} , which is paramount for devising an existence theory. Henceforth, following a suggestion by A. Mielke [28], we augment our dissipation potential by a regularization provided by the gradient of the creep rate.

The only higher-order terms involving the inelastic strain hence occur in the dissipation and are given by the gradient of the inelastic strain rate, i.e. of $\dot{\Pi}$. With reference to the discussion of Subsection 2.3, let us point out that such terms may again be time dependent. Still, they can be expected to show some boundedness with respect to time. In the case of (2.8) one indeed obtains that the mean across the strip

$$\frac{1}{2\ell} \int_{-\ell}^{\ell} \dot{\Pi}(t, x_1, x_2) dx_2 = \frac{1}{2\ell} \int_{-\ell}^{\ell} \begin{pmatrix} 0 & \partial_{x_2} \dot{f}(t, x_2) \\ 0 & 0 \end{pmatrix} dx_2 = \begin{pmatrix} 0 & \frac{\dot{f}(t, \ell) - \dot{f}(t, -\ell)}{2\ell} \\ 0 & 0 \end{pmatrix} \stackrel{(2.7)}{=} \begin{pmatrix} 0 & 1/\ell \\ 0 & 0 \end{pmatrix}$$

is time-independent. A regularization in term of $\nabla \dot{\Pi}$ is hence not expected to generate spurious hardening-like effects.

2.5 Constitutive equations

Following the classical *Coleman-Noll procedure* [5], we identify variations of Φ with respect to y and Π as driving forces in the momentum equation and in the inelastic flow-rule, respectively. More precisely, we have

$$\delta_y \Phi(y, \Pi) = -\operatorname{div} (D\varphi_E(\nabla y \Pi^{-1}) \Pi^{-\top} - \operatorname{div} (D\varphi_G(\nabla(\nabla y \Pi^{-1}))) \Pi^{-\top}) , \quad (2.12a)$$

$$\begin{aligned} \delta_{\Pi} \Phi(y, \Pi) &= \nabla y^{\top} D\varphi_E(\nabla y \Pi^{-1}) : D(\Pi^{-1}) \\ &\quad + D\varphi_H(\Pi) - \operatorname{div} (D\varphi_G(\nabla(\nabla y \Pi^{-1}))) : \nabla y D(\Pi^{-1}) . \end{aligned} \quad (2.12b)$$

In order to consider variations of the dissipation \mathcal{R} , we start by explicitly computing

$$\begin{aligned} \dot{C}_{\text{el}} &= \Pi^{-\top} (\nabla \dot{y}^{\top} \nabla y + \nabla y^{\top} \nabla \dot{y}) \Pi^{-1} + (D(\Pi^{-\top}) : \dot{\Pi}) \nabla y^{\top} \nabla y \Pi^{-1} + \Pi^{-\top} \nabla y^{\top} \nabla y D(\Pi^{-1}) : \dot{\Pi} \\ &= \Pi^{-\top} (\nabla \dot{y}^{\top} \nabla y + \nabla y^{\top} \nabla \dot{y}) \Pi^{-1} - \Pi^{-\top} \dot{\Pi}^{\top} \Pi^{-\top} \nabla y^{\top} \nabla y \Pi^{-1} - \Pi^{-\top} \nabla y^{\top} \nabla y \Pi^{-1} \dot{\Pi} \Pi^{-1} \\ &= \Pi^{-\top} (\nabla \dot{y}^{\top} \nabla y + \nabla y^{\top} \nabla \dot{y}) \Pi^{-1} - 2 \operatorname{sym} (\Pi^{-\top} \nabla y^{\top} \nabla y \Pi^{-1} \dot{\Pi} \Pi^{-1}) . \end{aligned}$$

This Kelvin-Voigt-type viscosity features then both $\nabla \dot{y}$ and $\dot{\Pi}$ terms. It hence contributes to both the momentum equation and to the inelastic flow rule. In particular, setting for brevity $\Sigma := \nu_{\text{kv}} \dot{C}_{\text{el}}$, the contribution of the Kelvin-Voigt-type viscosity to the stress is given by

$$\delta_y \dot{C}_{\text{el}} : \Sigma = -\operatorname{div} (2 \operatorname{sym} (\Pi^{-\top} \nabla y^{\top} \Sigma \Pi^{-1})) .$$

On the other hand, by computing

$$D_{\dot{\Pi}} \dot{C}_{\text{el}} = (\Pi^{-\top} \nabla y^{\top} \nabla y D(\Pi^{-1}))^{\dagger} + \Pi^{-\top} \nabla y^{\top} \nabla y D(\Pi^{-1}) ,$$

we have that the Kelvin-Voigt-type viscous contribution to the inelastic driving force is

$$D_{\dot{\Pi}} \dot{C}_{\text{el}} : \Sigma = -2 \operatorname{sym} (\Pi^{-\top} \nabla y^{\top} \nabla y \Pi^{-1} \Sigma \Pi^{-1}) .$$

2.6 Evolution system

The evolution of the medium is governed by the system of momentum equation and the inelastic flow rule. Let us denote by $\mathcal{K}(\dot{y}) = \frac{1}{2} \int_{\Omega} \varrho |\dot{y}|^2 dx$ the kinetic energy and by $\mathcal{F}(t)$ the external load

$$\langle \mathcal{F}(t), y \rangle = \int_{\Omega} f(t) \cdot y \, dx + \int_{\Gamma} g(t) \cdot y \, dS$$

where f and g denote a given body force density and surface traction density, respectively. The system reads then in abstract form

$$(\delta_{\dot{y}} \mathcal{K}(\dot{y})) \cdot + \delta_{\dot{y}} \mathcal{R}(y, \Pi; \nabla \dot{y}, \dot{\Pi}) + \delta_y \Phi(y, \Pi) = \mathcal{F}(t), \quad (2.13a)$$

$$\delta_{\dot{\Pi}} \mathcal{R}(y, \Pi; \nabla \dot{y}, \dot{\Pi}) + \delta_{\Pi} \Phi(y, \Pi) = 0. \quad (2.13b)$$

Here, we have formally indicated variations with δ . In the following, these relations will be made precise in the weak sense, see (3.3). For the sake of clarity, we present here the strong form of the system, assuming suitable regularity of the ingredients. Owing to our choices (2.4) and (2.11) for energy and dissipation, the latter corresponds to the nonlinear PDE system

$$\begin{aligned} \varrho \ddot{y} - \operatorname{div} \left(\mathbb{D}\varphi_{\mathbb{E}}(\nabla y \Pi^{-1}) \Pi^{-\top} + 2 \operatorname{sym} (\Pi^{-\top} \nabla y^{\top} \Sigma \Pi^{-1}) \right) \\ + \operatorname{div} (\operatorname{div} (\mathbb{D}\varphi_{\mathbb{G}}(\nabla(\nabla y \Pi^{-1}))) \Pi^{-\top}) = f, \end{aligned} \quad (2.14a)$$

$$\begin{aligned} \nu_{\text{m}} \dot{\Pi} + \operatorname{div}^2 (\nu_{\text{h}} \nabla^2 \dot{\Pi}) + \nabla y^{\top} \mathbb{D}\varphi_{\mathbb{E}}(\nabla y \Pi^{-1}) : \mathbb{D}(\Pi^{-1}) \\ - 2 \operatorname{sym} (\Pi^{-\top} \nabla y^{\top} \nabla y \Pi^{-1} \Sigma \Pi^{-1}) + \mathbb{D}\varphi_{\mathbb{H}}(\Pi) \\ - \operatorname{div} (\mathbb{D}\varphi_{\mathbb{G}}(\nabla(\nabla y \Pi^{-1}))) : \nabla y \mathbb{D}(\Pi^{-1}) = 0, \end{aligned} \quad (2.14b)$$

where we have again used the notation

$$\Sigma = \nu_{\text{kv}} \dot{C}_{\text{el}} \quad \text{and} \quad C_{\text{el}} = \Pi^{-\top} \nabla y^{\top} \nabla y \Pi^{-1}. \quad (2.15)$$

Taking into account the formulas (2.2)–(2.3), system (2.14) is intended to be completed by the following boundary conditions

$$\begin{aligned} \mathbb{D}\varphi_{\mathbb{E}}(\nabla y \Pi^{-1}) \Pi^{-\top} n - \operatorname{div} (\mathbb{D}\varphi_{\mathbb{G}}(\nabla(\nabla y \Pi^{-1}))) \Pi^{-\top} n \\ - \operatorname{div}_{\text{s}} (\mathbb{D}\varphi_{\mathbb{G}}(\nabla(\nabla y \Pi^{-1}))) n \Pi^{-\top} - 2 \mathfrak{h} (\mathbb{D}\varphi_{\mathbb{G}}(\nabla(\nabla y \Pi^{-1}))) n \Pi^{-\top} n \\ + 2 \operatorname{sym} (\Pi^{-\top} \nabla y^{\top} \Sigma \Pi^{-1}) n = g, \end{aligned} \quad (2.16a)$$

$$(\mathbb{D}\varphi_{\mathbb{G}}(\nabla(\nabla y \Pi^{-1}))) : (n \otimes (n \Pi^{-1})) = 0, \quad (2.16b)$$

$$\begin{aligned} \mathbb{D}\varphi_{\mathbb{G}}(\nabla(\nabla y \Pi^{-1}))) n : \nabla y \mathbb{D}(\Pi^{-1}) - \operatorname{div} \nu_{\text{h}} \nabla^2 \dot{\Pi} n \\ - \operatorname{div}_{\text{s}} (\nu_{\text{h}} \nabla^2 \dot{\Pi} n) - 2 \nu_{\text{h}} \mathfrak{h} (\nabla^2 \dot{\Pi} n) n = 0, \end{aligned} \quad (2.16c)$$

$$\nu_{\text{h}} \nabla^2 \dot{\Pi} : (n \otimes n) = 0. \quad (2.16d)$$

The energetics of the model can be obtained by formally testing (2.14a) with \dot{y} under (2.16a)–(2.16b) and (2.14b) with $\dot{\Pi}$ under (2.16c)–(2.16d). By considering the initial conditions

$$y(0) = y_0, \quad \dot{y}_0 = v_0, \quad \Pi(0) = \Pi_0, \quad (2.17)$$

the resulting energy balance on the time interval $[0, t]$ is

$$\begin{aligned}
& \int_{\Omega} \frac{\rho}{2} |\dot{y}(t)|^2 + \varphi_{\text{E}}(\nabla y(t) \Pi^{-1}(t)) + \varphi_{\text{G}}(\nabla(\nabla y(t) \Pi^{-1}(t))) + \varphi_{\text{H}}(\Pi(t)) \, dx \\
& + \int_0^t \int_{\Omega} \nu_{\text{kv}} |\dot{C}_{\text{el}}|^2 + \nu_{\text{m}} |\dot{\Pi}|^2 + \nu_{\text{h}} |\nabla^2 \dot{\Pi}|^2 \, dx \, d\tau = \int_0^t \int_{\Omega} f \cdot \dot{y} \, dx \, d\tau + \int_0^t \int_{\Gamma} g \cdot \dot{y} \, dS \, d\tau \\
& + \int_{\Omega} \frac{\rho}{2} |\dot{y}_0|^2 + \varphi_{\text{E}}(\nabla y_0 \Pi_0^{-1}) + \varphi_{\text{G}}(\nabla(\nabla y_0 \Pi_0^{-1})) + \varphi_{\text{H}}(\Pi_0) \, dx. \tag{2.18}
\end{aligned}$$

In particular, the sum of total energy at time t and dissipated energy on $[0, t]$ equals the sum of initial total energy and work of external forces.

Remark 2.1 (*Nonlinear or activated creep*). We assume here the dissipation potential to be quadratic, which makes the occurrence of $\dot{\Pi}$ in (2.14b) linear. In order to generalize this to the nonlinear (or even activated) case, the analysis of the problem would require to check strong compactness for the approximations of Σ . This seems presently out of reach in our setting, where only a weak convergence for such approximants can be guaranteed, cf. (3.13) below.

Remark 2.2 (*Jeffreys rheology*). The combination of two viscous damping mechanisms and one elastic energy-storing mechanism is often referred to as *Jeffreys rheology* [21] (sometimes also called *anti-Zener rheology*). This combination may arise from two different arrangements of rheological elements: one can arrange a Stokes viscous element either in parallel with a Maxwell rheological element or in series with a Kelvin-Voigt rheological one. Recall that a Maxwell (resp. Kelvin-Voigt) rheological element is an arrangement of an elastic and a viscous element in series (resp. in parallel). At small strains, the two possible arrangements giving a Jeffreys rheology are equivalent, cf. [21, Formula (6.6.34)]. On the contrary, equivalence does not hold at large strains. In our model we follow the second variant: the viscous Stokes element is in series with a Kelvin-Voigt rheological element. The reader is referred to [21, Remark 9.4.4], for a model following the first variant instead, which allows for a simpler analysis in spite of a somehow lesser physical relevance.

3 Analysis of the model

In the following we use the standard notation $C(\cdot)$ for the space of continuous functions, L^p for Lebesgue spaces, and $W^{k,p}$ for Sobolev spaces whose k -th distributional derivatives are in L^p . Moreover, we use the abbreviation $H^k = W^{k,2}$ and, for all $p \geq 1$, we let the conjugate exponent $p' = p/(p-1)$ (with $p' = \infty$ if $p = 1$), and use the notation p^* for the Sobolev exponent $p^* = pd/(d-p)$ for $p < d$, $p^* < \infty$ for $p = d$, and $p^* = \infty$ for $p > d$. Thus, $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ or $L^{p^*}(\Omega) \subset (W^{1,p}(\Omega))^*$ = the dual to $W^{1,p}(\Omega)$.

Given the fixed time interval $I = [0, T]$, we denote by $L^p(I; X)$ the standard Bochner space of Bochner-measurable mappings $u : I \rightarrow X$, where X is a Banach space. Moreover, $W^{k,p}(I; X)$ denotes the Banach space of mappings in $L^p(I; X)$ whose k -th distributional derivative in time is also in $L^p(I; X)$.

Let us list here the assumptions on the data which are used in the following:

$\varphi_E : \mathbb{R}^{d \times d} \rightarrow [0, +\infty]$ continuously differentiable on $GL^+(d)$, $\exists \epsilon > 0$, $p_G \in (d, 2^*)$, $r > p_G d / (p_G - d)$,

$$\varphi_E(F_{el}) \geq \begin{cases} \epsilon / (\det F_{el})^r & \text{if } \det F_{el} > 0, \\ +\infty & \text{if } \det F_{el} \leq 0, \end{cases}, \quad (3.1a)$$

$\varphi_H : \mathbb{R}^{d \times d} \rightarrow [0, +\infty]$ continuously differentiable on $GL^+(d)$, $\exists \epsilon > 0$, $s > 2^* d / (2^* - d)$,

$$\varphi_H(\Pi) \geq \begin{cases} \epsilon / (\det \Pi)^s & \text{if } \det \Pi > 0, \\ +\infty & \text{if } \det \Pi \leq 0, \end{cases}, \quad (3.1b)$$

$\varphi_G : \mathbb{R}^{d \times d \times d} \rightarrow [0, +\infty)$ convex, continuously differentiable, $\exists \epsilon > 0$,

$$\forall G, \tilde{G} \in \mathbb{R}^{d \times d \times d} : \begin{aligned} & (D\varphi_G(G) - D\varphi_G(\tilde{G})) : (G - \tilde{G}) \geq \epsilon |G - \tilde{G}|^{p_G} \\ & \varphi_G(G) \geq \epsilon |G|^{p_G}, \quad |D\varphi_G(G)| \leq (1 + |G|^{p_G - 1}) / \epsilon, \end{aligned} \quad (3.1c)$$

$$\varrho > 0, \quad \nu_m, \nu_{kv}, \nu_h > 0, \quad (3.1d)$$

$y_0 \in W^{2, p_G}(\Omega)^d$, $v_0 \in L^2(\Omega)^d$, $\Pi_0 \in H^2(\Omega)^{d \times d}$,

$$\varphi_E(\nabla y_0 \Pi_0^{-1}) \in L^1(\Omega), \quad \varphi_H(\Pi_0) \in L^1(\Omega), \quad (3.1e)$$

$$f \in L^1(I; L^2(\Omega)^d) + L^2(I; L^1(\Omega)^d), \quad g \in L^2(I; L^1(\Gamma)^d). \quad (3.1f)$$

A prototypical choice for φ_G satisfying (3.1c) is $\varphi_G(\cdot) = |\cdot|^{p_G}$. The restriction $p_G < 2^*$ will be instrumental for estimates (3.11) and (3.16) below.

The definition of weak solutions follows directly from system (2.13). It can be recovered by formally testing both equations in (2.14) by smooth functions and use Green formulas (2.2) together with the surface Green formula (2.3), the boundary conditions (2.16), and multiple by-part integration in time, keeping into account the initial conditions (2.17). Altogether, we arrive at the following definition.

Definition 3.1 (Weak formulation of (2.14) with (2.16)-(2.17)). *The pair (y, Π) satisfying*

$$y \in L^\infty(I; W^{2, p_G}(\Omega)^d) \cap H^1(I; L^2(\Omega)^d) \quad \text{with} \quad \nabla y^\top \nabla y \in H^1(I; L^2(\Omega)^{d \times d}),$$

$$\Sigma \in L^2(I \times \Omega)^{d \times d}, \quad \det \nabla y > 0, \quad \text{and} \quad \frac{1}{\det \nabla y} \in L^\infty(I \times \Omega), \quad \text{and} \quad (3.2a)$$

$$\Pi \in H^1(I; H^2(\Omega)^{d \times d}) \quad \text{with} \quad \det \Pi > 0 \quad \text{and} \quad \frac{1}{\det \Pi} \in L^\infty(I \times \Omega) \quad (3.2b)$$

is called a weak solution to the initial-boundary-value problem (2.14), (2.16)-(2.17) if the following two identities hold with Σ from (2.15):

- (i) The weak formulation of the momentum balance (2.14a) with the boundary conditions (2.16a)-(2.16b) and first two initial conditions in (2.17)

$$\begin{aligned} & \int_0^T \int_\Omega \left(D\varphi_E(\nabla y \Pi^{-1}) : (\nabla \tilde{y} \Pi^{-1}) + \varrho y \cdot \ddot{\tilde{y}} + 2 \operatorname{sym}(\Pi^{-\top} \nabla y^\top \Sigma \Pi^{-1}) : \nabla \tilde{y} \right. \\ & \quad \left. + D\varphi_G(\nabla(\nabla y \Pi^{-1})) : \nabla(\nabla \tilde{y} \Pi^{-1}) \right) dx dt = \int_0^T \int_\Omega f \cdot \tilde{y} dx dt \\ & \quad + \int_0^T \int_\Gamma g \cdot \tilde{y} dS dt + \int_\Omega \varrho v_0 \cdot \tilde{y}(0) - \varrho y_0 \cdot \dot{\tilde{y}}(0) dx \end{aligned} \quad (3.3a)$$

holds for any \tilde{y} smooth with $\tilde{y}(T) = \dot{\tilde{y}}(T) = 0$.

(ii) *The weak formulation of the creep flow rule (2.14b) with the boundary conditions (2.16c)–(2.16d) and the last initial condition in (2.17)*

$$\begin{aligned}
& \int_0^T \int_{\Omega} \left(\nabla y^\top \mathrm{D}\varphi_{\mathrm{E}}(\nabla y \Pi^{-1}) : \mathrm{D}(\Pi^{-1}) + \mathrm{D}\varphi_{\mathrm{H}}(\Pi) - 2 \operatorname{sym}(\Pi^{-\top} \nabla y^\top \nabla y \Pi^{-1} \Sigma \Pi^{-1}) \right) : \tilde{\Pi} \\
& \quad - \nu_{\mathrm{m}} \Pi : \dot{\tilde{\Pi}} + \mathrm{D}\varphi_{\mathrm{G}}(\nabla(\nabla y \Pi^{-1})) : \nabla(\nabla y \mathrm{D}(\Pi^{-1}) : \tilde{\Pi}) - \nu_{\mathrm{h}} \nabla^2 \Pi : \nabla^2 \dot{\tilde{\Pi}} \, dx \, dt \\
& \quad = \int_{\Omega} \nu_{\mathrm{m}} \Pi_0 : \tilde{\Pi}(0) + \nu_{\mathrm{h}} \nabla^2 \Pi_0 : \nabla^2 \tilde{\Pi}(0) \, dx
\end{aligned} \tag{3.3b}$$

holds for any $\tilde{\Pi}$ smooth with $\tilde{\Pi}(T) = 0$.

Let us note that, due to (3.2b), we have also $\Pi^{-1} = \operatorname{Cof} \Pi^\top / \det \Pi \in L^\infty(I \times \Omega)^{d \times d}$, as well as $\mathrm{D}\varphi_{\mathrm{G}}(\nabla(\nabla y \Pi^{-1})) \in L^\infty(I; L^{p_{\mathrm{G}}}(\Omega)^{d \times d \times d})$ so that all integrands in (3.3) are well-defined as L^1 -functions.

Our main analytical result is an existence theorem for weak solutions. This is to be seen as a mathematical consistency property of the proposed model. It reads as follows.

Theorem 3.2 (Existence of weak solutions). *Let the assumptions (3.1) hold. Then, there exists a weak solution (y, Π) in the sense of Definition 3.1.*

Proof. As we are working in reference (Lagrangian) coordinates and aim at testing by partial derivatives in time, we can advantageously use the Galerkin discretisation method in space. Let us fix a nested sequence of finite-dimensional subspaces $V_k \subset W^{2,\infty}(\Omega)$, $k \in \mathbb{N}$ whose union is dense in $W^{2,\infty}(\Omega)$. We will use this sequence for all components of deformations y and inelastic strains Π .

Without loss of generality, we may consider an approximation of the initial conditions $y_{0,k} \in V_k^d$, $v_{0,k} \in V_k^d$, and $\Pi_{0,k} \in V_k^{d \times d}$ such that

$$y_{0,k} \rightarrow y_0 \quad \text{strongly in } W^{2,p_{\mathrm{G}}}(\Omega)^d, \tag{3.4a}$$

$$v_{0,k} \rightarrow v_0 \quad \text{strongly in } L^2(\Omega)^d, \tag{3.4b}$$

$$\Pi_{0,k} \rightarrow \Pi_0 \quad \text{strongly in } H^2(\Omega)^{d \times d}. \tag{3.4c}$$

Existence of a finite-dimensional approximate solution $(y_k, \Pi_k) \in W^{2,1}(I; V_k^d) \times C^1(I; V_k^{d \times d})$ of the initial-value problem for the system of nonlinear ordinary differential equations arising from the Galerkin approximation is standard, also using successive prolongation based on uniform L^∞ estimates. Such estimates can be obtained by testing the discrete-in-space equations by \dot{y}_k and $\dot{\Pi}_k$. This leads to the energy balance (2.18) for the Galerkin approximations (y_k, Π_k) . Starting from the energy balance, by using the Gronwall and Hölder inequalities, we obtain a-priori estimates independently of k , namely,

$$\{y_k\}_{k \in \mathbb{N}} \quad \text{is bounded in } W^{1,\infty}(I; L^2(\Omega)^d), \tag{3.5a}$$

$$\{\Pi_k\}_{k \in \mathbb{N}} \quad \text{is bounded in } H^1(I; H^2(\Omega)^{d \times d}) \subset L^\infty(I \times \Omega)^{d \times d}, \tag{3.5b}$$

$$\{F_{\mathrm{el},k}\}_{k \in \mathbb{N}} = \{\nabla y_k \Pi_k^{-1}\}_{k \in \mathbb{N}} \quad \text{is bounded in } L^\infty(I; W^{1,p_{\mathrm{G}}}(\Omega)^{d \times d}), \tag{3.5c}$$

$$\{C_{\mathrm{el},k}\}_{k \in \mathbb{N}} = \{F_{\mathrm{el},k}^\top F_{\mathrm{el},k}\}_{k \in \mathbb{N}} \quad \text{is bounded in } H^1(I; L^2(\Omega)^{d \times d}). \tag{3.5d}$$

Next, we use the classical Healey-Krömer [16] argument, here applied to the plastic strain instead of the deformation gradient, as already exploited in [44]. This is based on the L^∞ -bound of Π_k and on the sufficiently fast blow-up of φ_H , as assumed in (3.1b). It is important that the argument in [16] holds even for the discrete level (as realized already in [21, 35]) and ensures that $\det \Pi_k \geq \delta$ for all time instants and for some $\delta > 0$ independent of k . In particular, we also have that

$$\{\Pi_k^{-1}\}_{k \in \mathbb{N}} \text{ is bounded in } L^\infty(I \times \Omega)^{d \times d}. \quad (3.5e)$$

From (3.5b)–(3.5c) we get that $\{\nabla y_k\}_{k \in \mathbb{N}} = \{F_{\text{el},k} \Pi_k\}_{k \in \mathbb{N}}$ is bounded in $L^\infty(I \times \Omega)^{d \times d \times d}$. From (3.5b) we find that $\nabla(\nabla y_k \Pi_k^{-1}) = (\Pi_k^{-\top} \nabla(\nabla y_k)^\top)^\dagger + \nabla y_k \text{D}(\Pi_k^{-1}) \nabla \Pi_k$ is bounded in $L^\infty(I; L^{p_G}(\Omega)^{d \times d \times d})$. This in particular implies that

$$\begin{aligned} \{\nabla(\nabla y_k)^\top\}_{k \in \mathbb{N}} &= \left\{ \Pi_k^\top \left(\nabla(\nabla y_k \Pi_k^{-1}) - \nabla y_k \text{D}(\Pi_k^{-1}) \nabla \Pi_k \right)^\dagger \right\}_{k \in \mathbb{N}} \\ &\text{is bounded in } L^\infty(I; L^{p_G}(\Omega)^{d \times d \times d}). \end{aligned} \quad (3.5f)$$

From (3.5a), we know that $\{y_k\}_{k \in \mathbb{N}}$ is bounded in $L^\infty(I; L^2(\Omega)^d)$, so that (3.5f) yields a bound in $L^\infty(I; W^{2,p_G}(\Omega)^d)$. We proceed by showing that (3.5d), yields the estimate

$$\{\nabla \dot{y}_k\}_{k \in \mathbb{N}} \text{ is bounded in } L^2(I \times \Omega)^{d \times d}. \quad (3.5g)$$

To prove (3.5g) we argue as in [21, Sect. 9.4.3]. We preliminary observe that by the growth conditions from below on φ_E in (3.1a), as well as by the super-quadratic growth on φ_G in (3.1c), the Healey-Krömer argument yields the existence of $\delta_{\text{el}} > 0$ such that

$$\det F_{\text{el},k} \geq \delta_{\text{el}} \quad \text{in } I \times \Omega$$

for every $k \in \mathbb{N}$. By combining the Cauchy-Binet formula with the bound in (3.5e), we find that

$$\frac{1}{\det \nabla y_k} = \frac{1}{\det(\nabla y_k \Pi_k^{-1} \Pi_k)} = \frac{1}{\det(\nabla y_k \Pi_k^{-1})} \frac{1}{\det \Pi_k}$$

is uniformly bounded in $L^\infty(I \times \Omega)$. Property (3.5g) follows now by applying the generalized Korn inequality by Neff [38] and Pompe [41] as exploited for the Kelvin-Voigt rheology in [35, Thm. 3.3].

For all $k \in \mathbb{N}$ the pair (y_k, Π_k) fulfills the weak formulation (3.3) with initial conditions approximated as (3.4), provided that the test-functions take value in the finite-dimensional space. In particular, we have

$$\begin{aligned} &\int_0^T \int_\Omega \left(\text{D}\varphi_E(\nabla y_k \Pi_k^{-1}) : (\nabla \tilde{y}_k \Pi_k^{-1}) + \varrho y_k \cdot \ddot{\tilde{y}}_k + 2 \text{sym}(\Pi_k^{-\top} \nabla y_k^\top \Sigma_k \Pi_k^{-1}) : \nabla \tilde{y}_k \right. \\ &\quad \left. + \text{D}\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})) : \nabla(\nabla \tilde{y}_k \Pi_k^{-1}) \right) dx dt = \int_0^T \int_\Omega f \cdot \tilde{y}_k dx dt \\ &\quad + \int_0^T \int_\Gamma g \cdot \tilde{y}_k dS dt + \int_\Omega \varrho v_0 \cdot \tilde{y}_k(0) - \varrho y_0 \cdot \dot{\tilde{y}}_k(0) dx \end{aligned} \quad (3.6a)$$

$$\begin{aligned} &\int_0^T \int_\Omega \left(\nabla y_k^\top \text{D}\varphi_E(\nabla y_k \Pi_k^{-1}) : \text{D}(\Pi_k^{-1}) + \text{D}\varphi_H(\Pi_k) - 2 \text{sym}(\Pi_k^{-\top} \nabla y_k^\top \nabla y_k \Pi_k^{-1} \Sigma_k \Pi_k^{-1}) \right) : \tilde{\Pi}_k \\ &\quad - \nu_m \Pi_k : \dot{\tilde{\Pi}}_k + \text{D}\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})) : \nabla(\nabla y_k \text{D}(\Pi_k^{-1}) : \tilde{\Pi}_k) - \nu_h \nabla^2 \Pi_k : \nabla^2 \dot{\tilde{\Pi}}_k dx dt \\ &= \int_\Omega \nu_m \Pi_0 : \tilde{\Pi}_k(0) + \nu_h \nabla^2 \Pi_0 : \nabla^2 \tilde{\Pi}_k(0) dx \end{aligned} \quad (3.6b)$$

for all $\tilde{y}_k \in C^2(I; V_k^d)$ and $\tilde{\Pi}_k \in C^1(I; V_k^{d \times d})$ with $\tilde{y}_k(T) = \dot{\tilde{y}}_k(T) = 0$ and $\tilde{\Pi}_k(T) = 0$.

We are hence ready to address the convergence $\{(y_k, \Pi_k)\}_{k \in \mathbb{N}}$ as $k \rightarrow \infty$. By the Banach selection principle and the Aubin-Lions compact-embedding theorem, we select a not relabeled subsequence converging with respect to the weak* topologies indicated in (3.5). In particular, we have that

$$y_k \rightarrow y \quad \text{weakly* in } W^{1,\infty}(I; L^2(\Omega)^d) \cap L^\infty(I; W^{2,p_G}(\Omega)^d) \\ \text{and strongly in } C(I \times \bar{\Omega})^d, \quad (3.7a)$$

$$\Pi_k \rightarrow \Pi \quad \text{weakly in } H^1(I; H^2(\Omega)^{d \times d}) \text{ and strongly in } L^\infty(I \times \Omega)^{d \times d}, \quad (3.7b)$$

$$\Pi_k^{-1} \rightarrow \Pi^{-1} \quad \text{strongly in } L^\infty(I \times \Omega)^{d \times d}, \quad (3.7c)$$

$$F_{\text{el},k} = \nabla y_k \Pi_k^{-1} \rightarrow F_{\text{el}} = \nabla y \Pi^{-1} \quad \text{weakly* in } L^\infty(I; W^{1,p_G}(\Omega)^{d \times d}), \quad (3.7d)$$

$$C_{\text{el},k} = F_{\text{el},k}^\top F_{\text{el},k} \rightarrow C_{\text{el}} = F_{\text{el}}^\top F_{\text{el}} \quad \text{weakly in } H^1(I; L^2(\Omega)^{d \times d}). \quad (3.7e)$$

In fact, using the Aubin-Lions theorem in the context of Galerkin method when the time derivatives are estimated only in some locally convex space (or alternatively only their Hahn-Banach extension is estimated in a Banach space) requires some attention, as commented in [43, Sect.8.4]. The convergence of Π_k^{-1} is obtained by exploiting the formula $\Pi_k^{-1} = \text{Cof} \Pi_k^\top / \det \Pi_k$, as well as the uniform lower bound $\det \Pi_k \geq \delta$, and the fact that the determinant is a locally Lipschitz function. By recalling that $D(\Pi^{-1}):A = -\Pi^{-1}A\Pi^{-1}$ for all $A \in \mathbb{R}^{d \times d}$ one readily checks that

$$D(\Pi_k^{-1}) \rightarrow D(\Pi^{-1}) \quad \text{strongly in } L^\infty(I \times \Omega)^{d \times d \times d \times d}, \quad (3.8)$$

$$\nabla(\Pi_k^{-1}) = D(\Pi_k^{-1}):\nabla \Pi_k \rightarrow D(\Pi^{-1}):\nabla \Pi = \nabla(\Pi^{-1}) \\ \text{strongly in } L^q(I \times \Omega)^{d \times d \times d} \quad \forall q < 2^*. \quad (3.9)$$

We further proceed by proving that

$$\nabla(\nabla y_k \Pi_k^{-1}) \rightarrow \nabla(\nabla y \Pi^{-1}) \quad \text{strongly in } L^{p_G}(I \times \Omega)^{d \times d \times d}. \quad (3.10)$$

By the uniform monotonicity of $D\varphi_G$, we find:

$$\begin{aligned} & \epsilon \left\| \nabla(\nabla y_k \Pi_k^{-1}) - \nabla(\nabla y \Pi^{-1}) \right\|_{L^{p_G}(I \times \Omega)^{d \times d \times d}}^{p_G} \\ & \leq \int_0^T \int_\Omega (D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})) - D\varphi_G(\nabla(\nabla y \Pi^{-1}))) : (\nabla(\nabla y_k \Pi_k^{-1}) - \nabla(\nabla y \Pi^{-1})) \, dx \, dt \\ & = \int_0^T \int_\Omega D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})) : \nabla(\nabla(y_k - y) \Pi_k^{-1}) \, dx \, dt \\ & \quad + \int_0^T \int_\Omega D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})) : \nabla(\nabla y (\Pi_k^{-1} - \Pi^{-1})) \, dx \, dt \\ & \quad - \int_0^T \int_\Omega D\varphi_G(\nabla(\nabla y \Pi^{-1})) : (\nabla(\nabla y_k \Pi_k^{-1}) - \nabla(\nabla y \Pi^{-1})) \, dx \, dt =: I_{1,k} + I_{2,k} + I_{3,k}. \end{aligned}$$

where $\epsilon > 0$ is from (3.1c). We have $I_{3,k} \rightarrow 0$ owing to (3.7d) and to $D\varphi_G(\nabla(\nabla y \Pi^{-1})) \in L^\infty(I; L^{p'_G}(\Omega)^{d \times d \times d})$ because of the growth assumption in (3.1c). Also $I_{2,k} \rightarrow 0$ since

$$\nabla(\nabla y (\Pi_k^{-1} - \Pi^{-1})) = ((\Pi_k^{-\top} - \Pi^{-\top}) \nabla(\nabla y)^\top)^\dagger + \nabla y (\nabla \Pi_k^{-1} - \nabla \Pi^{-1}) \rightarrow 0 \quad (3.11)$$

strongly in $L^{p_G}(I \times \Omega)^{d \times d \times d}$ by (3.7c); here we also used the convergence $\nabla \Pi_k^{-1} \rightarrow \nabla \Pi^{-1}$ strongly in $L^{p_G}(I \times \Omega)^{d \times d \times d}$ owing to (3.7b). To prove that also the term $I_{1,k}$ converges to

0, we test the momentum equation for the Galerkin approximants by $y_k - \tilde{y}_k$ where \tilde{y}_k is an approximation of the limit y which takes values in the finite-dimensional subspaces V_k^d and which converges to y in $L^2(I; W^{2,p_G}(\Omega)^d) \cap H^1(I; L^2(\Omega)^d)$. We further assume $\tilde{y}_k(0) = y_{0,k}$. Note that here $y_k - \tilde{y}_k$ is not $C^2(I; H^2(\Omega)^d)$ but rather $W^{1,\infty}(I; H^2(\Omega)^d)$. Nevertheless, this regularity is enough for arguing differently from (3.6a) and integrating by-part in time only once. Note also that $y_k(T) - \tilde{y}_k(T) \neq 0$. For this reason, a further term at time T appears in the equation below. Altogether,

$$\begin{aligned} I_{1,k} = & \int_0^T \int_{\Omega} D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})) : (\nabla(\nabla(\tilde{y}_k - y) \Pi_k^{-1})) + \varrho \dot{y}_k \cdot (\dot{y}_k - \dot{\tilde{y}}_k) + f \cdot (y_k - \tilde{y}_k) \\ & - D\varphi_E(\nabla y_k \Pi_k^{-1}) : \nabla(y_k - \tilde{y}_k) \Pi_k^{-1} - 2 \text{sym}(\Pi_k^{-\top} \nabla y_k^{\top} \Sigma_k \Pi_k^{-1}) : \nabla(y_k - \tilde{y}_k) \, dx \, dt \\ & - \int_{\Omega} \dot{y}_k(T) \cdot (y_k(T) - \tilde{y}_k(T)) \, dx + \int_0^T \int_{\Gamma} g \cdot (y_k - \tilde{y}_k) \, dS \, dt \rightarrow 0. \end{aligned}$$

Then, from (3.5g), using (the above mentioned generalization of) the Aubin-Lions theorem, exploiting an information about \ddot{y}_k obtained via a comparison argument in the discrete variant of (2.14a) for the Galerkin approximants, we infer that

$$\dot{y}_k \rightarrow \dot{y} \quad \text{strongly in } L^2(I \times \Omega)^d,$$

and

$$\dot{y}_k(T) \rightarrow \dot{y}(T) \quad \text{weakly in } L^2(\Omega)^d.$$

By (3.5b), (3.7c), (3.7d), and (3.7e) we conclude that $I_{1,k} \rightarrow 0$ and obtain (3.10).

What it is left to prove is that (y, Π) is a weak solution in the sense of Definition 3.1. Let \tilde{y} and $\tilde{\Pi}$ be smooth with $\tilde{y}(T) = \dot{\tilde{y}}(T) = 0$ and $\tilde{\Pi}(T) = 0$, and approximate them via sequences \tilde{y}_k and $\tilde{\Pi}_k$ as in (3.6), so that $\tilde{y}_k \rightarrow \tilde{y}$ strongly in $H^2(I; W^{2,p_G}(\Omega)^d)$ and $\tilde{\Pi}_k \rightarrow \tilde{\Pi}$ strongly in $H^1(I; H^2(\Omega)^d)$. One needs to check that convergences (3.7) are sufficient to pass to the limit in all terms in (3.3). Let us start by the momentum balance (3.6a). By the continuity of the superposition operator we have that

$$D\varphi_E(\nabla y_k \Pi_k^{-1}) \Pi_k^{-\top} \rightarrow D\varphi_E(\nabla y \Pi^{-1}) \Pi^{-\top} \quad \text{strongly in } L^\infty(I \times \Omega)^{d \times d}, \quad (3.12)$$

cf. the growth condition (3.1a). Estimate (3.5d) ensures that

$$\Sigma_k = \nu_{\text{kv}} \dot{C}_{\text{el},k} \rightarrow \Sigma \quad \text{weakly in } L^2(I \times \Omega)^{d \times d}. \quad (3.13)$$

The limit Σ can be identified as $\Sigma = \nu_{\text{kv}} \dot{C}_{\text{el}}$ since we have convergence (3.7e). Owing to (3.13), (3.5f), and (3.7c) we deduce that

$$\Pi_k^{-\top} \nabla y_k^{\top} \Sigma_k \Pi_k^{-1} \rightarrow \Pi^{-\top} \nabla y^{\top} \Sigma \Pi^{-1} \quad \text{weakly in } L^2(I \times \Omega)^{d \times d}. \quad (3.14)$$

Let us now compute

$$\begin{aligned} D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})) : \nabla(\nabla \tilde{y}_k \Pi_k^{-1}) &= D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})) : (\Pi_k^{-\top} \nabla(\nabla \tilde{y}_k)^{\top})^t \\ &\quad + D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})) : (\nabla \tilde{y}_k D(\Pi_k^{-1}) : \nabla \Pi_k) \end{aligned}$$

Convergences (3.7) suffice to pass to the weak limit in both terms in the right-hand side. In fact, taking into account (3.8) and (3.10), we have the following strong convergences (even

though weak ones would be enough for our existence proof):

$$\begin{aligned} D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})):(\Pi_k^{-\top} \nabla(\nabla \tilde{y}_k)^\top)^\dagger &\rightarrow D\varphi_G(\nabla(\nabla y \Pi^{-1})):(\Pi^{-\top} \nabla(\nabla \tilde{y})^\top)^\dagger \\ &\text{in } L^p(I; L^{p'_G}(\Omega)^{d \times d}) \quad \forall p < +\infty, \text{ and} \end{aligned} \quad (3.15)$$

$$\begin{aligned} D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})):(\nabla \tilde{y}_k D(\Pi_k^{-1}):\nabla \Pi_k) &\rightarrow D\varphi_G(\nabla(\nabla y \Pi^{-1})):(\nabla \tilde{y} D(\Pi^{-1}):\nabla \Pi) \\ &\text{in } L^p(I; L^q(\Omega)^{d \times d}) \quad \forall p < +\infty, q < \frac{2^* p'_G}{2^* + p'_G}. \end{aligned} \quad (3.16)$$

Since all the remaining terms in the momentum balance (3.6a) are linear, convergences (3.12)–(3.16) allow to pass to the limit and obtain (3.3a).

Let us now move to the flow rule (3.6b). Arguing as above, by (3.5f) we have that

$$\begin{aligned} \nabla y_k^\top D\varphi_E(\nabla y_k \Pi_k^{-1}):D(\Pi_k^{-1}) &\rightarrow \nabla y^\top D\varphi_E(\nabla y \Pi^{-1}):D(\Pi^{-1}) \\ &\text{strongly in } L^\infty(I \times \Omega)^{d \times d}. \end{aligned} \quad (3.17)$$

By using again convergence (3.13) we also get that

$$\Pi_k^{-\top} \nabla y_k^\top \nabla y_k \Pi_k^{-1} \Sigma_k \Pi_k^{-1} \rightarrow \Pi^{-\top} \nabla y^\top \nabla y \Pi^{-1} \Sigma \Pi^{-1} \quad \text{weakly in } L^2(I \times \Omega)^{d \times d}. \quad (3.18)$$

Eventually, we use convergences (3.7a), (3.9), and (3.10) in order to check that

$$\begin{aligned} D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})):\nabla(\nabla y_k D(\Pi_k^{-1}):\tilde{\Pi}_k) &= -D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})):\nabla(\nabla y_k \Pi_k^{-1} \tilde{\Pi}_k \Pi_k^{-1}) \\ &= -D\varphi_G(\nabla(\nabla y_k \Pi_k^{-1})):[(\tilde{\Pi}_k \Pi_k^{-1})^\top \nabla(\nabla y_k \Pi_k^{-1})^\dagger]^\dagger + \nabla y_k \Pi_k^{-1} \nabla(\tilde{\Pi}_k \Pi_k^{-1}) \\ &\rightarrow D\varphi_G(\nabla(\nabla y \Pi^{-1})):\nabla(\nabla y D(\Pi^{-1}):\tilde{\Pi}) \quad \text{strongly in } L^1(I \times \Omega)^{d \times d \times d}. \end{aligned}$$

All remaining terms in the flow rule (3.6b) are linear and convergences (3.17)–(3.18) suffice to pass to the limit and obtain (3.3b). \square

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