Variational convergences for functionals and differential operators depending on vector fields

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INTRODUCTION

The results presented in this Ph.D. thesis concern variational convergences for functionals and differential operators depending on Lipschitz continuous vector fields. This setting has been introduced by Folland and Stein in [59] and it has found numerous applications in recent years, see e.g. [2, 16, 27, 40, 58, 76, 95, 113].

The two convergences taken into account, namely Γ-*convergence* and H-*convergence*, were developed during the '60s by two different mathematical schools: the school of Pisa of Ennio De Giorgi, which studied abstract results and applications of Γ-convergence, and the school of Paris of Jacques-Louis Lions, which formulated the theory of H-convergence.

The notion of Γ-convergence, introduced by Ennio De Giorgi and Tullio Franzoni in [50, 51], occupies a prominent place in the world of variational convergences by its applications in material sciences. Moreover, the vastness of results concerning Γ-compactness of integral functionals and the fact that almost all the other notions of convergences can be expressed in its language enhance its importance. The precursors of this theory are *Epi-convergence*, originally called *infimal convergence* by Robert Arthur Wijsman in [124], which consists in the Hausdorff convergence of the epigraphs, *Mosco convergence*, introduced by Umberto Mosco in [97], which deals with sequences of functions (and convex sets) in infinite dimensional Banach spaces, and G-*convergence*, developed by Sergio Spagnolo in [116] to study asymptotic behaviours of sequences of elliptic operators in divergence form. We also remind that Dirichlet forms, whose theory was initiated by Arne Beurling and Jacques Deny in [18] and that allows to study Laplace and heat equations on spaces that are not manifolds, have applications related to Γ-convergence. We refer the interested reader to [12, 29, 30, 31, 33, 34, 35, 36, 37, 38, 47] for a general overview on Γ-convergence, to [24, 25, 31, 54, 61, 108, 109, 110, 111, 122] for applications of Γ-convergence and to [5, 10, 11, 19, 23, 43, 52, 74, 85, 91, 92, 96, 98, 112, 115] for what concerns Epi-convergence, Mosco convergence, G-convergence and Dirichlet forms.

The results presented in the first part of the thesis, contained in the papers [86, 87, 88, 90], deal with a class of integral functionals that can be represented by

$$F(u) = \int_\Omega f(x, Xu(x))\, dx \tag{1}$$

where Ω is an open subset of $\mathbb{R}^n$, u is sufficiently smooth on Ω and $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Borel measurable function, which is convex in the second variable and satisfies the growth condition

$$c_0\,|\eta|^p - a_0(x) \leq\; f(x,\eta) \leq\; c_1\,|\eta|^p + a_1(x)$$

a.e. $x \in \Omega$ for each $\eta \in \mathbb{R}^m$, with $p > 1$, $c_0 \leq c_1$ positive constants and $a_0, a_1 \in L^1(\Omega)$ nonnegative functions.

The X-*gradient* $X = (X_1, \ldots, X_m)$ that appears in (1), represents a family of m vector fields with locally Lipschitz coefficients on Ω satisfying a *linear independence condition*, (LIC) condition in short, which consists in requiring the existence of negligible closed subsets of Ω outside of which $X_1(x), \ldots, X_m(x)$ are linearly independent as vectors of $\mathbb{R}^n$. Vector fields of this form embraces many relevant families already present in literature: for instance, if $m = n$, then $X = D := (\partial_1, \ldots, \partial_n)$ trivially satisfies the (LIC) condition (other examples can be found in Example 1.1.2).

The main result of the thesis is Theorem 2.3.12, that shows the Γ-convergence, up to subsequences, of sequences of functionals as in (1) in the strong topology of $L^p(\Omega)$, for $p > 1$. If the convexity properties and the growth conditions of the sequence of integrands are also uniform, Theorem 2.3.12 ensures that the Γ-limits are also represented by integral functionals of the same form. In Section 2.3.4, we show that the same conclusions still remain true in two interesting subcases: from one side, the class of integrands that are quadratic forms with respect to the second variable (this result will be crucial in Chapter 3) and, from the other side, the subclass of integrands that do not depend anymore on the point, but just on the X-gradient. This last result, combined with Theorem 2.2.12, will provide a Γ-compactness theorem for sequences of *left-invariant* functionals on Carnot groups, namely Theorem 2.3.33.

A problem strictly related to the Γ-convergence of integral functionals as in (1) is the asymptotic behaviour of solutions of elliptic partial differential equations whose coefficients are subject to strong perturbations. This kind of problems are object of study of the theory of the H-convergence, which was initiated by François Murat and Luc Tartar in the '70s. In our framework, H-convergence studies simultaneously the convergence of solutions and momenta of sequences of differential operators appearing in problems of the form

$$\mu u + \mathrm{div}_X(a(x)X)(u) = g \quad \text{in } \Omega \tag{2}$$

where, called $W^{1,p}_X(\Omega)$ the Sobolev space of L^p-functions whose derivatives with respect to the vector fields X_j still belongs to $L^p(\Omega)$ for any $j = 1,\dots,m$ and $1 \leq p < \infty$, u belongs to the closure of $\mathbf{C}^1_c(\Omega) \cap W^{1,2}_X(\Omega)$ in $W^{1,2}_X(\Omega)$, namely $H^1_{X,0}(\Omega)$, $g \in L^2(\Omega)$, $\mu \geq 0$ and $a = [a_{ij}]$ is a $m \times m$ symmetric matrix such that $a_{ij} \in L^\infty(\Omega)$ for any $i, j = 1,\dots,m$ and satisfying the standard ellipticity and continuity conditions

$$c_0|\eta|^2 \leq \langle a(x)\eta, \eta\rangle_{\mathbb{R}^m} \leq c_1|\eta|^2 \quad \text{a.e. } x \in \Omega \ \forall \eta \in \mathbb{R}^m,\ c_0, c_1 > 0.$$

In Chapter 3, we study H-compactness results for two families of operators depending on vector fields with two different approaches. In the first case, we show that the class of linear differential operators in X-divergence form, whose domain $D(\mathcal{L})$ is the set of functions $u \in W^{1,2}_X(\Omega)$ such that the distribution defined by the right hand side of

$$\mathcal{L} := \mathrm{div}_X(a(x)X) := \sum_{i,j=1}^{m} X_j^T(a_{ij}(x)X_i)$$

belongs to $L^2(\Omega)$, is closed in the topology of the H-convergence. The variational technique adopted here, which makes a comparison between Γ-convergence and H-convergence, was developed by Nadia Ansini, Gianni Dal Maso and Caterina Ida Zeppieri in [7, 8, 9]. We remind that a comparison between Spagnolo's G-convergence and Γ-convergence already existed in literature (see e.g. [5, 49]). Moreover, as for Γ-convergence, also H-convergence for subelliptic PDEs have been also widely studied, always assuming the X-gradient satisfying the Hörmander condition (see, for instance, [20, 21, 22, 64, 73]).

The results contained in the second part of Chapter 3, which are studied in [86] and are set in the sub-Riemannian framework of Carnot groups, are motivated by the recent works of Annalisa Baldi, Bruno Franchi, Nicoletta Anna Tchou and Maria Carla Tesi [13, 14, 72], where the authors studied linear counterparts of Tartar's H-compactness theorem for monotone operators ([122, Chapter 11]) in Carnot groups. The willing of adapting the original techniques in [13, 14, 72] needed a generalization of the Murat and Tartar' *Div-curl Lemma* [122, Lemma 7.2], which is a classical tool in the theory of the H-convergence. The monotone operators taken into account in [13, 14, 72] are of the form

$$\mathcal{A}(u) := -\mathrm{div}_{\mathbb{G}}(A(x)\nabla_{\mathbb{G}} u)$$

where A is a $(m \times m)$-matrix-valued measurable function, $m \leq n$ is the dimension of the first layer of the Lie algebra associated with the Carnot group $\mathbb{G}$ and $\nabla_{\mathbb{G}}$ and $\mathrm{div}_{\mathbb{G}}$ are, respectively, the intrinsic gradient and the intrinsic divergence (see Section 1.3 for details).

Differently from the works of Baldi, Franchi, Tchou and Tesi, we deal with a class of *nonlinear* monotone operators of the form

$$\mathcal{A}(u) := -\mathrm{div}_{\mathbb{G}}(A(x, \nabla_{\mathbb{G}} u))$$

with Ω open subset of $\mathbb{G}$ and for a given Carathéodory function A such that $A(x, 0) = 0$ and satisfying the following ellipticity and continuity conditions

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta \rangle_{\mathbb{R}^m} \geq \alpha |\xi - \eta|^p$$
$$|A(x,\xi) - A(x,\eta)| \leq \beta \left[1 + |\xi|^p + |\eta|^p\right]^{\frac{p-2}{p}} |\xi - \eta|$$

for every $\xi, \eta \in \mathbb{R}^m$ a.e. $x \in \Omega$, for $\alpha \leq \beta$ positive constants and $p \geq 2$. In Theorem 3.2.9, we show that our class of monotone operators is still closed in the topology of the H-convergence by using standard techniques and adopting the version of the Div-curl Lemma proved in [13]. We remind the interested reader to [17, 99, 100, 103, 118, 119, 120, 121, 122, 123] for a general overview on H-convergence.

A characterization of the *fractional* Sobolev spaces depending on Lipschitz continuous vector fields form the basis of the fourth chapter of the thesis. In the classical theory of (Euclidean) fractional Sobolev spaces, a well-known defect of the fractional Gagliardo seminorms is that they do not converge when the fractional parameter s tends to 0 or 1 since, by definition

$$\|u\|_{W_0^{s,p}(\mathbb{R}^n)} := \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy$$

where, for any $s\in(0,1)$ and $1\le p<\infty$, called $W^{s,p}(\mathbb{R}^n)$ the fractional Sobolev space of L^p-functions such that $\frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+s}}\in L^p(\mathbb{R}^n\times\mathbb{R}^n)$, the space $W_0^{s,p}(\mathbb{R}^n)$ is defined as the closure of $\mathbf{C}_c^1(\Omega)\cap W^{s,p}(\mathbb{R}^n)$ in $W^{s,p}(\mathbb{R}^n)$.

At the beginning of this century, many mathematicians tried to answer a natural question: "*There exist nontrivial correctors $f=f(s)$ and $g=g(s)$, depending only on s, such that the following limits converge and recover, up to constants, their local counterparts, that is*

$$\lim_{s\uparrow 1} f(s)\|u\|^p_{W_0^{s,p}(\mathbb{R}^n)} = c_1\,\|u\|^p_{W_0^{1,p}(\mathbb{R}^n)}$$

$$\lim_{s\downarrow 0} g(s)\|u\|^p_{W_0^{s,p}(\mathbb{R}^n)} = c_2\,\|u\|^p_{L^p(\mathbb{R}^n)}$$

with c_1, c_2 positive constants, independent of s, for suitable functions u?"

In 2001, Jean Bourgain, Haim Brezis and Petru Mironescu showed in [28] that, in any smooth bounded domain Ω of $\mathbb{R}^n$, the fractional Gagliardo seminorm recovers its local counterpart as s goes to 1 by choosing the corrector $f(s)=(1-s)$, in the sense that

$$\lim_{s\uparrow 1}(1-s)\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = K_1(n,p)\int_\Omega |\nabla u(x)|^p\,dx$$

for any $u\in L^p(\Omega)$, $1\le p<\infty$, where the constant K_1 depends only on n and p, with the convention that $\|u\|_{W^{1,p}(\Omega)}=\infty$ if $u\notin W^{1,p}(\Omega)$.

The complementary question was instead answered, one year later, by Vladimir Maz'ya and Tatyana Shaposhnikova in [93], as a consequence of the following *Hardy-type inequality*

$$\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}dx \le K \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|}{|x-y|^{n+sp}}dx\,dy$$

with K positive constant independent of s. In [93], the authors showed that

$$\lim_{s\downarrow 0} s \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = K_2(n,p)\int_{\mathbb{R}^n} |u|^p\,dx$$

for any $u \in \bigcup_{s\in(0,1)} W_0^{s,p}(\mathbb{R}^n)$. The constant K_2 still depends only on n and p.

The last chapter of the thesis is devoted to some generalizations of the previous formulas for a particular class of anisotropic fractional Sobolev spaces depending on vector fields and on Orlicz functions (or Nice Young functions) in the setting of Carnot groups. The fractional seminorms taken into account are, respectively

$$\iint_{\mathbb{G}\times\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}$$

where $\mathbb{G}$ is a step k Carnot group of homogeneous dimension Q and φ is an Orlicz function, for a generalization of the Bourgain-Brezis-Mironescu formula, namely (BBM) formula, and

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \varphi\left(\left|\frac{u(x)-e^{\mathrm{i}(x-y)A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^s}\right|\right)\frac{dx\,dy}{|x-y|^n}$$

for a generalization of the Maz'ya-Shaposhnikova formula, (MS) formula, in fractional *magnetic* Orlicz-Sobolev spaces. We refer to Section 1.4 for details.

The generalization of the (BBM) formula was obtained in [41] in collaboration with Marco Capolli, Ariel Martin Salort and Eugenio Vecchi, by adapting the technique introduced by Fernández Bonder and Salort in [57] to families of homogeneous norms $\|\cdot\|_{\mathbb{G}}$ that are invariant under horizontal rotations and that satisfy the triangular inequality (see Remark 1.3.2 for details). Instead, the generalization of the (MS) formula in the framework of fractional magnetic Orlicz-Sobolev spaces on $\mathbb{R}^n$ was obtained in collaboration with Salort and Vecchi in [89], as a consequence of a Hardy-type inequality, proved recently in [3, Theorem 5.1] by Angela Alberico, Andrea Cianchi, Luboš Pick and Lenka Slavíková. We finally remind that [89] complements the paper [26], where the case $s\uparrow 1$ was studied.

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Chapter One

Function spaces depending on vector fields

Notation

Throughout the following chapters, $\Omega \subset \mathbb{R}^n$ is a fixed open set and $\overline{\mathbb{R}} = [-\infty, \infty]$. If $v, w \in \mathbb{R}^n$, we denote by $|v|$ and $\langle v, w \rangle$ the Euclidean norm and the scalar product, respectively. If Ω and Ω' are subsets of $\mathbb{R}^n$ then $\Omega' \Subset \Omega$ means that Ω' is compactly contained in Ω. Moreover, $B(x, r)$ is the open Euclidean ball of radius r centered at x. If $A \subset \mathbb{R}^n$, then $|A|$ is its n-dimensional Lebesgue measure $\mathcal{L}^n$ and, by notation *a.e.* $x \in A$, we will simply mean $\mathcal{L}^n$-a.e. $x \in A$. $\mathbb{1}_A$ and χ_A denote, respectively, the indicator and the characteristic function of A, that is,

$$\mathbb{1}_A(x) := \begin{cases} 0 & \text{if } x \in A \\ \infty & \text{otherwise} \end{cases} \quad \text{and} \quad \chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

Finally, we denote by $\mathbf{C}^k(\Omega)$ the space of $\mathbb{R}$-valued functions k times continuously differentiable, by $\mathbf{C}^k_c(\Omega)$ the subspace of $\mathbf{C}^k(\Omega)$ whose functions have support compactly contained in Ω, by $\mathcal{D}(\Omega) := \mathbf{C}^\infty_c(\Omega)$ and by $\mathcal{D}'(\Omega)$ its dual space.

1.1 Framework and examples

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We define *X-gradient* a family of first order linear differential operators with Lipschitz coefficients $X(x) := (X_1(x), \dots, X_m(x))$, that is,

$$X_j(x) = \sum_{i=1}^{n} c_{ji}(x)\partial_i$$

with $c_{ji}(x) \in Lip(\Omega)$ for $j = 1, \dots, m$ and $i = 1, \dots, n$.

In the sequel, we will identify each X_j with the vector field $(c_{j1}(x), \dots, c_{jn}(x)) \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$. Moreover, we define

$$C(x) = [c_{ji}(x)]_{\substack{j=1,\dots,m \\ i=1,\dots,n}}, \tag{1.1}$$

the *coefficient matrix of the X-gradient.*

The following structural assumption on the X-gradient turns out to be a key point.

Definition 1.1.1. We say that the X-gradient satisfies the *linear independence condition*, (LIC) condition in short, on an open set $\Omega \subset \mathbb{R}^n$, if there exists a set $\mathcal{N}_X \subset \Omega$, closed in the topology of Ω, such that $|\mathcal{N}_X| = 0$ and, for each $x \in \Omega_X := \Omega \setminus \mathcal{N}_X$, $X_1(x), \dots, X_m(x)$ are linearly independent as vectors of $\mathbb{R}^n$.

Many relevant families of vector fields embraces the (LIC) condition, as shown by the following example:

Example 1.1.2.

(i) (Euclidean gradient) Let $X = D := (\partial_1, \dots, \partial_n)$. In this case, the coefficients matrix $C(x)$ of X is the $n \times n$ matrix

$$C(x) = \mathrm{I_n} \quad \forall\, x \in \mathbb{R}^n\,,$$

denoting I_n the identity matrix of order n.

(ii) (Grushin vector fields) Let $X = (X_1, X_2)$ be the family of vector fields on $\mathbb{R}^2$, defined as

$$X_1(x) :=\ \partial_{_1}, \quad X_2(x) :=\ x_1\, \partial_{_2} \text{ if } x = (x_1, x_2) \in \mathbb{R}^2\,.$$

Here, $C(x)$ is the 2×2 matrix

$$C(x) := \begin{bmatrix} 1 & 0 \\ 0 & x_1 \end{bmatrix} \quad \forall\, x \in \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \neq 0\}.$$

(iii) (Heisenberg vector fields) Let $X = (X_1, X_2)$ be the family of vector fields on $\mathbb{R}^3$, defined as

$$X_1(x) := \partial_1 - \frac{x_2}{2}\partial_3,\ X_2(x) := \partial_2 + \frac{x_1}{2}\partial_3 \text{ if } x = (x_1, x_2, x_3) \in \mathbb{R}^3 .$$

In this case, $C(x)$ is the 2×3 matrix

$$C(x) := \begin{bmatrix} 1 & 0 & -x_2/2 \\ 0 & 1 & x_1/2 \end{bmatrix} \quad \forall\, x \in \mathbb{R}^3.$$

(iv) (Vector Fields not satisfying the *Hörmander condition*) Let $X = (X_1, X_2)$ be the family of vector fields on $\mathbb{R}^3$, defined as

$$X_1(x) := \partial_1, \quad X_2(x) := \partial_2 \text{ if } x = (x_1, x_2, x_3) \in \mathbb{R}^3 .$$

Now, $C(x)$ is the 2×3 matrix

$$C(x) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \forall\, x \in \mathbb{R}^3.$$

Let us notice that, if $X = (X_1, \ldots, X_m)$ satisfies (LIC), then $m \leq n$. Moreover, by the well-known extension result for Lipschitz functions, without loss of generality, we can assume that vector fields' coefficients $c_{ji} \in Lip_{\text{loc}}(\mathbb{R}^n)$ for each $j = 1, \ldots, m$, $i = 1, \ldots, n$.

1.2 Sobolev spaces depending on vector fields

Definition 1.2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. For $1 \leq p \leq \infty$ we define

$$W_X^{1,p}(\Omega) := \{u \in L^p(\Omega) : X_j u \in L^p(\Omega) \text{ for } j = 1, \dots, m\}$$

$$W_{X;loc}^{1,p}(\Omega) := \left\{u : u|_{\Omega'} \in W_X^{1,p}(\Omega') \text{ for every open set } \Omega' \Subset \Omega\right\}.$$

Moreover, we set

$$H_X^{1,p}(\Omega) := \text{closure of } \mathbf{C}^1(\Omega) \cap W_X^{1,p}(\Omega) \text{ in } W_X^{1,p}(\Omega).$$

The following Proposition is proved in [59]:

Proposition 1.2.2. $W_X^{1,p}(\Omega)$ *endowed with the norm*

$$\|u\|_{W_X^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{i=1}^{m} \|X_j u\|_{L^p(\Omega)}$$

is a Banach space, reflexive if $1 < p < \infty$.

Proposition 1.2.3. *The following properties hold for functions in* $W_{X;\text{loc}}^{1,p}(\Omega)$*:*

(i) let $u \in L^p(\Omega)$ *and assume the existence of an open set* $A \subset \Omega$ *such that* $u|_A \in W_{X;\text{loc}}^{1,p}(A)$*. Then, for every open set* $A' \Subset A$*, there exists*

$$w \in W_X^{1,p}(\Omega) \text{ such that } u|_{A'} = w|_{A'}\,. \tag{1.2}$$

(ii) Let $A \subset \Omega$ *be an open subset and let* $u \in L^p(A)$*. Suppose that there exists* $M > 0$ *such that* $\|u\|_{W_X^{1,p}(A')} \leq M$ *for any* $A' \Subset A$*. Then* $u \in W_X^{1,p}(A)$*.*

(iii) Let $\{A_1, \dots, A_N\}$ *be a finite family of open subsets of* Ω *and let* $u \in L^p(\Omega)$*. If* $u_{|A_i} \in W_X^{1,p}(A_i)$ *for all* $i = 1, \dots, N$ *then* $u \in W_X^{1,p}\left(\bigcup_{i=1}^N A_i\right)$*.*

(iv) Let $A \subset \Omega$ *be an open subset and let* $u \in W_X^{1,p}(A)$*. Then* $u_{|B} \in W_X^{1,p}(B)$ *for any open set* $B \subseteq A$*.*

Proof. (i) Let $\varphi \in \mathbf{C}^1_c(A)$ be a cut-off function such that $\varphi \equiv 1$ in A'. If

$$w(x) := u(x)\,\varphi(x) \text{ if } x \in \Omega\,,$$

then it is easy to see that w satisfies (1.2).

(ii) Let us consider a sequence of open subsets of A, $\{A_i\}_{i\in\mathbb{N}}$ with $A_i \Subset A_{i+1}$ and $A \subseteq \bigcup_{i=1}^{\infty} A_i$. Then

$$\int_A |Xu|^p\,dx \leq \int_{\bigcup_{i=1}^{\infty} A_i} |Xu|^p\,dx = \lim_{i\to\infty}\int_{A_i} |Xu|^p\,dx \leq M$$

and the conclusion follows.

(iii) For every $A' \Subset \bigcup_{i=1}^{N} A_i$ there exists a partition of unity subordinate to the covering $\{A_1,\dots,A_N\}$, i.e., nonnegative functions $\{\eta_1,\dots,\eta_N\} \subset C^\infty_c\left(\bigcup_{i=1}^{N} A_i\right)$ such that each η_j has support in some A_i and $\sum_{j=1}^{N}\eta_j(x) = 1$ for all $x \in A'$. Set $u_j := u\eta_j$. Since the support of η_j is contained in some A_i, it is clear that $u_j \in W^{1,p}_X\left(\bigcup_{i=1}^{N} A_i\right)$ and $u \in W^{1,p}_X(A')$. Let us estimate

$$\|u\|_{W^{1,p}_X(A')} \leq \sum_{j=1}^{N} \|u\eta_j\|_{W^{1,p}_X(A')} \leq C\|u\|_{W^{1,p}_X(A_i)}$$

where $C > 0$ is independent of A'. The conclusion then follows using (ii).

(iv) The thesis follows easily observing that $C^\infty_c(B) \subseteq C^\infty_c(A)$. □

Remark 1.2.4. Since vector fields X_j have locally Lipschitz continuous coefficients, then $\partial_i c_{j,i} \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ for each $j = 1,\dots,m$, $i = 1,\dots,n$. Thus, by definition, it is immediate that, for each open bounded set $\Omega \subset \mathbb{R}^n$,

$$W^{1,p}(\Omega) \subset W^{1,p}_X(\Omega) \quad \forall\, p \in [1,\infty] \tag{1.3}$$

and, for any $u \in W^{1,p}(\Omega)$,

$$Xu(x) = C(x)\,Du(x) \qquad \text{for a.e. } x \in \Omega\,, \tag{1.4}$$

where $W^{1,p}(\Omega)$ denotes the classical Sobolev space, or, equivalently, the space $W^{1,p}_X(\Omega)$ associated to $X = D := (\partial_1,\dots,\partial_n)$ (see Example 1.1.2 (i)). Moreover, it is easy to see that inclusion (1.3) can be strict and turns out to be continuous, as well as the inclusion

$$W^{1,p}_{\text{loc}}(\Omega) \subset W^{1,p}_{X;\text{loc}}(\Omega) \quad \forall\, p \in [1,\infty]\,.$$

1.2.1 Approximation by regular functions

Let us recall in this section some results concerning approximation by regular functions in these anisotropic Sobolev spaces. First, we recall the following version of the Jensen's Inequality in Banach spaces proved, for instance, in [47].

Lemma 1.2.5. *[47, Lemma 23.2] Let X be a Banach space and let $F : X \to [0,\infty]$ be a lower semicontinuous convex function. Let (E,ϵ,μ) be a measure space with $\mu \geq 0$ and $\mu(E) = 1$. Then,*

$$F\left(\int_E u(s)\,d\mu(s)\right) \leq \int_E F(u(s))\,d\mu(s) \tag{1.5}$$

for every μ-integrable function $u : E \to X$.

Definition 1.2.6. Let $\{\rho_\varepsilon\}_\varepsilon$ be a family of mollifiers, i.e., $\rho_\varepsilon \in \mathbf{C}_c^\infty(\mathbb{R}^n)$, $\rho_\varepsilon \geq 0$ on $\mathbb{R}^n$, $\int_{\mathbb{R}^n} \rho_\varepsilon\,dx = 1$ and $\operatorname{supp}(\rho_\varepsilon) \subset B(0,\varepsilon)$. For any $u \in L^p_{\text{loc}}(\mathbb{R}^n)$, the convolution $\rho_\varepsilon * u$ is defined by

$$(\rho_\varepsilon * u)(x) := \int_{B(x,\varepsilon)} \rho_\varepsilon(x-y)u(y)\,dy = \int_{B(0,\varepsilon)} \rho_\varepsilon(y)u(x-y)\,dy. \tag{1.6}$$

Here and in the sequel, if $u : \Omega \to \overline{\mathbb{R}}$, we will denote by $\bar{u} : \mathbb{R}^n \to \overline{\mathbb{R}}$ its extension to the whole $\mathbb{R}^n$ being 0 outside Ω.

Proposition 1.2.7. *[68, Proposition 1.2.2] Assume $u \in W^{1,p}_X(\Omega)$ for $1 \leq p < \infty$. Then, if $\Omega' \Subset \Omega$*

$$\lim_{\varepsilon\to 0} \|\rho_\varepsilon * \bar{u} - u\|_{W^{1,p}_X(\Omega')} = 0\,,$$

where $\rho \in C_c^\infty(B(0,1))$ is a smooth compactly supported function and $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(\varepsilon^{-1}|x|)$ is a mollifier supported in $B(0,\varepsilon)$.

The following theorem, proved independently in [68] and [75], is the analogous of the celebrated Meyers-Serrin Theorem [94, Theorem]. Analogous results in the weighted cases and in metric measure spaces are proved in [67] and [6], respectively.

Theorem 1.2.8. *[68, Theorem 1.2.3] Let Ω be an open subset of $\mathbb{R}^n$ and $1 \le p < \infty$. Then*

$$H^{1,p}_X(\Omega) = W^{1,p}_X(\Omega).$$

The following proposition will be useful in the sequel.

Proposition 1.2.9. *(i) Let $\{u_h\}_h$ and u be in $L^p_{loc}(\mathbb{R}^n)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that*

$$u_h \to u \textit{ in } L^1_{loc}(\Omega) \textit{ as } h \to \infty\,.$$

Then, for each open set $\Omega' \Subset \Omega$, for given $0 < \varepsilon < \operatorname{dist}(\Omega', \mathbb{R}^n \setminus \Omega)$,

$$\rho_\varepsilon * u_h \to \rho_\varepsilon * u \textit{ uniformly on } \Omega',\textit{ as } h \to \infty\,. \tag{1.7}$$

(ii) Let $f : \mathbb{R}^m \to [0,\infty)$ be a convex function and let $w \in L^1_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)$. Then, for each bounded open sets Ω' and Ω with $\Omega' \Subset \Omega$, for each $0 < \varepsilon < \operatorname{dist}(\Omega', \mathbb{R}^n \setminus \Omega)$,

$$\int_{\Omega'} f(\rho_\varepsilon * w)\,dx \le \int_\Omega f(w)\,dx.$$

Proof. (i) Let $\Omega' \Subset \Omega$ and let $\varepsilon \in (0, \operatorname{dist}(\Omega', \mathbb{R}^n \setminus \Omega))$. For any $x \in \Omega'$, it holds that

$$\begin{aligned} |(\rho_\varepsilon * u_h)(x) - (\rho_\varepsilon * u)(x)| &= \left| \int_{B(x,\varepsilon)} [u_h(y) - u(y)]\rho_\varepsilon(x-y)\,dy \right| \\ &\le \int_{B(x,\varepsilon)} |u_h(y) - u(y)|\rho_\varepsilon(x-y)\,dy \\ &\le \|\rho_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \int_{B(x,\varepsilon)} |u_h(y) - u(y)|\,dy. \end{aligned}$$

Therefore, passing to the supremum in Ω', and taking the limit as $h \to \infty$, we get (1.7).

(ii) The result trivially follows by the Jensen's inequality. In fact, by the Jensen's inequality and a change of variables, it holds that

$$\begin{aligned} \int_{\Omega'} f(\rho_\varepsilon * w)\,dx &= \int_{\Omega'} f\left(\int_{B(0,\varepsilon)} w(x-y)[\rho_\varepsilon(y)\,dy] \right) dx \\ &\le \int_{\Omega'} \int_{B(0,\varepsilon)} (f \circ w)(x-y)\rho_\varepsilon(y)\,dx\,dy \\ &\le \int_\Omega (f \circ w)(z)\,dz \int_{B(0,\varepsilon)} \rho_\varepsilon(y)\,dy = \int_\Omega f(w)\,dz. \end{aligned}$$

□

1.2.2 Approximation by piecewise affine functions

It is well known (see, for instance, [55, Chap. X, Proposition 2.9]) that the class of piecewise affine functions is dense in the classical Sobolev space $W^{1,p}(\Omega)$, provided that Ω is a bounded open set with Lipschitz boundary. This result is crucial in the proof of the classical integral representation theorem with respect to the Euclidean gradient (see, for instance, [47, Theorem 20.1]). The aim of this section is to prove that no results of this kind are available for a general family $X = (X_1, \dots, X_m)$ in $\mathbb{R}^n$ by extending, in a natural way, the notion to be affine with respect to the X-gradient.

Definition 1.2.10. We say that $u \in \mathbf{C}^\infty(\mathbb{R}^n)$ is X-*affine* if there exists $c \in \mathbb{R}^n$ such that

$$Xu(x) = c \quad \text{for any } x \in \mathbb{R}^n.$$

Let $\Omega \subset \mathbb{R}^n$ be open. We say that $u : \Omega \to \mathbb{R}$ is X-affine if it is the restriction to Ω of a X-affine function over $\mathbb{R}^n$. Moreover, we say that $u : \mathbb{R}^n \to \mathbb{R}$ is X-*piecewise affine* if it is continuous and there is a partition of $\mathbb{R}^n$ into a negligible set and a finite number of open sets on which u is X-affine.

Let us show that, for Heisenberg and Grushin vector fields, the approximation of functions in $W^{1,p}_X(\Omega)$ by X-piecewise affine functions may fail.

Example 1.2.11. (**a**) *Let $X = (X_1, X_2)$ be the Heisenberg vector field on $\mathbb{R}^3$ (see Example 1.1.2 (iii)). Let us notice that any function $u \in \mathbf{C}^\infty(\mathbb{R}^3)$ is X-affine if and only if*

$$u(x) = c_1x_1 + c_2x_2 + c_3 \text{ for each } x = (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{1.8}$$

for suitable constants $c_i \in \mathbb{R}$ $i = 1, 2, 3$. Indeed, it is trivial that a function u in (1.8) is X-affine. Conversely, if $X_1u = c_1$ and $X_2u = c_2$ on $\mathbb{R}^3$, for some $u \in \mathbf{C}^\infty(\mathbb{R}^3)$, then the commutator

$$[X_1, X_2]\, u := (X_1X_2 - X_2X_1)\, u = \partial_3 u = 0 \quad \text{on } \mathbb{R}^3,$$

which gives $u(x) = c_1x_1 + c_2x_2 + c_3$ for each $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, for some $c_3 \in \mathbb{R}$.

Let $u(x) = x_3$. Then $u \in W^{1,p}_X(\Omega)$, whenever $|\Omega| < \infty$. Since any X-piecewise affine function does not depend on x_3, there cannot be any sequence of X-piecewise affine functions $\{u_h\}_h$ such that $u_h(x_1, x_2, x_3) \to u(x_1, x_2, x_3)$ for a.e. $(x_1, x_2, x_3) \in \Omega$.

(**b**) *Let $X = (X_1, X_2)$ be the Grushin vector fields on $\mathbb{R}^2$ (see Example 1.1.2 (ii)). Let $u \in \mathbf{C}^\infty(\mathbb{R}^2)$ be such that $X_1 u = c_1$ and $X_2 u = c_2$ on $\mathbb{R}^2$. Then it is easy to prove, arguing as before, that $u(x) = c_1 x_1 + c_3$ for each $x = (x_1, x_2) \in \mathbb{R}^2$, for some $c_3 \in \mathbb{R}$. The conclusion follows as in the previous case taking $u(x_1, x_2) = x_2$, which belongs to $W^{1,p}_X(\Omega)$ for any $p \geq 1$ and for any bounded open set $\Omega \subset \mathbb{R}^2$.*

1.2.3 Poincaré inequality

Definition 1.2.12. For $1 \leq p \leq \infty$ we set

$$W^{1,p}_{X,0}(\Omega) := \text{closure of } \mathbf{C}^1_c(\Omega) \cap W^{1,p}_X(\Omega) \text{ in } W^{1,p}_X(\Omega)$$

and, given $\varphi \in W^{1,p}_X(\Omega)$, we define the affine subspace $W^{1,p}_{X,\varphi}(\Omega)$ of $W^{1,p}_X(\Omega)$ as

$$W^{1,p}_{X,\varphi}(\Omega) := \{u \in W^{1,p}_X(\Omega) \mid u - \varphi \in W^{1,p}_{X,0}(\Omega)\}.$$

It is proven in [59] that, for any $1 \leq p \leq \infty$, the normed spaces $(H^{1,p}_X(\Omega), \| \cdot \|_{W^{1,p}_X(\Omega)})$ and $(W^{1,p}_{X,0}(\Omega), \| \cdot \|_{W^{1,p}_X(\Omega)})$ are Banach spaces. We conclude this section proving a Poincaré-type inequality, which ensures that, when Ω is bounded and the family X satisfies suitable properties, then

$$\|u\|_{W^{1,p}_{X,0}} := \left(\int_\Omega |Xu|^p \, dx \right)^{\frac{1}{p}} \tag{1.9}$$

defines an equivalent norm on $W^{1,p}_{X,0}(\Omega)$. Moreover, we show that $W^{1,p}_X(\Omega)$ can be compactly embedded in $L^p_{loc}(\Omega)$ for any $1 \leq p < \infty$.

To this aim, we should ask for stronger hypotheses on the family X.

Definition 1.2.13. Let Ω be a bounded open set and $X = (X_1, \ldots, X_m)$ be a family of Lipschitz continuous vector fields in a neighbourhood Ω_0 of $\overline{\Omega}$ satisfying (LIC) on Ω. Let us now define the following three conditions:

(H1) Let $d : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ be the so-called Carnot-Carathéodory distance function induced by X (see, for instance, [67, Section 2]). We assume $d(x,y) < \infty$ for any $x, y \in \Omega_0$, so that d is a standard distance in Ω_0. Moreover, we assume that d is continuous with respect to the usual topology of $\mathbb{R}^n$.

(H2) For any compact set $K \subset \Omega_0$ and for any $r < r_K$ and any $x \in K$ there exists a constant $C_K > 0$ such that

$$|B_d(x,2r)| \leq C_K |B_d(x,r)|\,,$$

where $B_d(x,r) := \{y \in \Omega_0 \mid d(x,y) < r\}$ is the (open) metric ball with respect to d.

(H3) There exist geometric constants $c, C > 0$ such that for any $B = B_d(\overline{x}, r)$, $\overline{x} \in \Omega_0$, if $cB := B_d(\overline{x}, cr) \subseteq \Omega_0$, then

$$\left| f(x) - \frac{1}{|B|}\int_B f(y)dy \right| \leq C \int_{cB} |Xf(y)| \frac{d(x,y)}{|B_d(x, d(x,y))|} dy$$

for any $f \in \mathrm{Lip}(c\overline{B})$ and $x \in B$.

Remark 1.2.14. Assumptions (H1), (H2) and (H3) are satisfied by several important families of vector fields. Let us point out two classes of vector fields:

1) If the vector fields are smooth and the rank of the Lie algebra generated by $X_1, \ldots, X_m$ equals n at any point of Ω_0 (the so-called Hörmander condition), then (H1), (H2) and (H3) hold (see, [101] for (H1) and (H2), and [66] for (H3)).

2) If the vector fields are as in [62], [63] and [65], then conditions (H1), (H2) and (H3) still hold (see, [62, 63, 65] for (H1) and (H2), and [67, Remark 2.8] for (H3)).

Before proving the main results of this section, that is, Proposition 1.2.17 and Proposition 1.2.18, we need to recall the following theorems, proved in [67].

Theorem 1.2.15. *[67, Theorem 2.11] Let Ω and Ω_0 be, respectively, a bounded open set and an open set with $\bar{\Omega} \subset \Omega_0$. Let $1 \leq p < \infty$ and let $X = (X_1, \ldots, X_m)$ be a family of Lipschitz* *continuous vector fields defined on Ω_0. If X satisfies conditions (H1), (H2) and (H3), then, for each metric ball $B = B_d(x,r) \subset \Omega$ and for every $u \in W^{1,p}_X(\Omega)$, there exist constants $c(u,B) \in \mathbb{R}$ and $C \in \mathbb{R}$, independent of u, such that*

$$\int_B |u(x) - c(u,B)|^p \, dx \le C\, r^p \int_B |Xu|^p \, dx\,.$$

Theorem 1.2.16. *[67, Theorem 3.4] Let $\Omega \Subset \Omega_0$ be a bounded open set, $1 \le p < \infty$ and $X = (X_1, \dots, X_m)$ be a family of Lipschitz continuous vector fields defined on Ω_0. If X satisfies conditions (H1), (H2) and (H3), then $W^{1,p}_{X,0}(\Omega)$ is compactly embedded in $L^p(\Omega)$.*

A first consequence of Theorem 1.2.16 is the following result.

Proposition 1.2.17. *Under the assumptions of Theorem 1.2.16, $W^{1,p}_X(\Omega)$ can be compactly embedded in $L^p_{loc}(\Omega)$.*

Proof. Since Ω is open, there exist open subsets $\{\Omega_i\}_{i\in\mathbb{N}}$ of Ω such that $\overline{\Omega}_i$ is compact for every $i \in \mathbb{N}$, $\emptyset \neq \Omega_1 \subseteq \overline{\Omega}_1 \subseteq \Omega_2 \subseteq \dots$ and $\bigcup_{i=1}^{\infty} \overline{\Omega}_i = \Omega$. Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}_X(\Omega)$ be a bounded sequence and let $\rho \in \mathbf{C}^\infty_c(\Omega)$, with $0 \le \rho(x) \le 1$ for every $x \in \Omega$ and $\rho \equiv 1$ on Ω_1. Then, the sequence $\{v^{(1)}_n\}_{n\in\mathbb{N}}$, whose general term is $v^{(1)}_n := \rho\, u_n|_{\Omega_1}$, is a bounded sequence in $W^{1,p}_{X,0}(\Omega)$. Therefore, by Theorem 1.2.16, there exist a subsequence $\{v^{(1)}_{n_k}\}_{k\in\mathbb{N}}$ of $\{v^{(1)}_n\}_{n\in\mathbb{N}}$ and $u^{(1)} \in L^p(\Omega)$ such that

$$v^{(1)}_{n_k} \to u^{(1)} \quad \text{in } L^p(\Omega) \quad \text{and} \quad u^{(1)}_n \to u^{(1)} \quad \text{in } L^p(\Omega_1),$$

where $\{u^{(1)}_n\}_{n\in\mathbb{N}}$ is the subsequence of $\{u_n\}_{n\in\mathbb{N}}$ such that $v^{(1)}_{n_k} = \rho u^{(1)}_n|_{\Omega_1}$.

Starting now from the sequence $\{u^{(1)}_n\}_{n\in\mathbb{N}}$, let us repeat the same procedure described above, finding the existence of $u^{(2)} \in L^p(\Omega)$ and $\{u^{(2)}_n\}_{n\in\mathbb{N}}$, a subsequence of $\{u^{(1)}_n\}_{n\in\mathbb{N}}$, such that

$$u^{(2)}_n \to u^{(2)} \quad \text{in } L^p(\Omega_2) \quad \text{and} \quad u^{(1)} = u^{(2)} \quad \text{a.e. in } \Omega_1.$$

Let us iterate the same procedure for any $i \in \mathbb{N}$ and let us define $\{v_n\}_{n\in\mathbb{N}}$, whose general term is $v_n := u^{(n)}_n$, the $n-$th element of the $n-$th subsequence of $\{u_n\}_{n\in\mathbb{N}}$, and $\overline{u}(x) := u^{(i)}(x)$, if $x \in \Omega_i$. By construction, $\overline{u}$ is well-defined and $\overline{u} \in L^p(\Omega)$. To conclude the proof, let us show that

$$v_n \to \overline{u} \quad \text{in } L^p(\tilde{\Omega}) \quad \text{for any open set } \tilde{\Omega} \Subset \Omega.$$

This conclusion trivially follows since, for any open set $\tilde{\Omega} \Subset \Omega$, there exists $i \in \mathbb{N}$ such that $\tilde{\Omega} \Subset \Omega_i$ and since

$$\int_{\tilde{\Omega}} |v_n - \overline{u}|^p \, dx \le \int_{\Omega_i} |v_n - \overline{u}|^p \, dx = \int_{\Omega_i} |u_n^{(n)} - u^{(i)}|^p \, dx \to 0.$$

□

As a consequence of Theorem 1.2.15 and Theorem 1.2.16, a global Poincaré inequality holds in $W^{1,p}_{X,0}(\Omega)$.

Proposition 1.2.18. *Under the assumptions of Theorem 1.2.16, and also assuming that Ω is connected, then there exists a positive constant $c_{p,\Omega} > 0$ such that*

$$\int_{\Omega} |u|^p \, dx \le \; c_{p,\Omega} \int_{\Omega} |Xu|^p \, dx \text{ for each } u \in W^{1,p}_{X,0}(\Omega)\,. \tag{1.10}$$

Proof. By contradiction, assume the existence of a sequence $\{u_h\}_h \subset W^{1,p}_{X,0}(\Omega)$ such that

$$\int_{\Omega} |u_h|^p \, dx > \; h \int_{\Omega} |Xu_h|^p \, dx \text{ for each } h \in \mathbb{N}\,. \tag{1.11}$$

By definition of $W^{1,p}_{X,0}(\Omega)$, there exists a sequence $\{v_h\}_h \subset \mathbf{C}^\infty_c(\Omega)$ such that

$$\|u_h - v_h\|_{W^{1,p}_X(\Omega)} < \frac{1}{h^{\frac{1}{p}}} \text{ for each } h \in \mathbb{N}\,. \tag{1.12}$$

Moreover, by (1.11) and (1.12), it follows that

$$\int_{\Omega} |v_h|^p \, dx + \frac{1}{h} > \; h \left(\int_{\Omega} |Xv_h|^p \, dx - \frac{1}{h} \right) \text{ for each } h \in \mathbb{N} \tag{1.13}$$

and, by homogeneity and (1.13), we can assume

$$\int_{\Omega} |v_h|^p \, dx = 1 - \frac{1}{h} \text{ for each } h \ge \; 2\,, \tag{1.14}$$

and

$$\int_{\Omega} |Xv_h|^p \, dx < \frac{2}{h-1} \text{ for each } h \geq 2 \,. \tag{1.15}$$

Let Ω_1 be a bounded connected open set such that $\Omega \Subset \Omega_1 \Subset \Omega_0$ and let $\{\bar{v}_h\}_h \subset \mathbf{C}_c^\infty(\Omega_1)$, where $\bar{v}_h : \Omega_1 \to \mathbb{R}$ is the extension of v_h defined as $\bar{v}_h \equiv 0$ in $\Omega_1 \setminus \Omega$. By (1.14),(1.15) and Theorem 1.2.16, up to subsequences, there exists $v \in W^{1,p}_{X,0}(\Omega_1)$ such that

$$\bar{v}_h \to v \text{ in } L^p(\Omega_1) \text{ and a.e. in } \Omega_1 \,, \tag{1.16}$$

$$\int_{\Omega_1} |v|^p \, dx = 1 \tag{1.17}$$

and

$$Xv = (0, \ldots, 0) \text{ a.e. in } \Omega_1 \,. \tag{1.18}$$

By Theorem 1.2.15 and (1.18), it follows that v is locally constant on the connected set Ω_1. Then v is constant, that is, there exists $k \in \mathbb{R}$ such that

$$v(x) = k \text{ for a.e. } x \in \Omega_1 \,.$$

By (1.17), $k \neq 0$ and this yields a contradiction since, by (1.16), $\bar{v}_h \equiv 0$ in $\Omega_1 \setminus \bar{\Omega}$ for each h and, therefore, $v = k = 0$ a.e. in $\Omega_1 \setminus \bar{\Omega}$. □

Corollary 1.2.19. *Let p, Ω and X as in Proposition 1.2.18. Then, the function $\| \cdot \|_{W^{1,p}_{X,0}}$, defined in* (1.9)*, is a norm in $W^{1,p}_{X,0}(\Omega)$ equivalent to $\| \cdot \|_{W^{1,p}_X(\Omega)}$.*

1.3 Sobolev spaces on Carnot groups

Sometimes, in the sequel, we will work in frameworks requiring a stronger structure than before, the *Carnot groups*. Here, we present few definitions and properties concerning Carnot groups and we refer the interested readers to [27], for a complete reading on this topic.

A Carnot group $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is a connected, simply connected and nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification, namely there exist linear subspaces, usually called *layers*, such that

$$\mathfrak{g} = V_1 \oplus .. \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k,$$

where k is usually called the *step* of the group $(\mathbb{G}, \cdot)$ and

$$[V_i, V_j] := \operatorname{span}\{[X, Y] : \, X \in V_i, Y \in V_j\}.$$

The explicit expression of the group law $\cdot$ can be deduced from the Hausdorff-Campbell formula, see e.g. [27], and the group law can be used to define a diffeomorphism, usually called *left-translation* $\gamma_y : \mathbb{G} \to \mathbb{G}$ for every $y \in \mathbb{G}$, defined as

$$\gamma_y(x) := y \cdot x \quad \text{for every } x \in \mathbb{G}.$$

A Carnot group $\mathbb{G}$ is also endowed with a family of automorphisms of the group $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, $\lambda \in \mathbb{R}^+$, called *dilations*, given by

$$\delta_\lambda(x_1, \dots, x_n) := (\lambda^{d_1} x_1, .., \lambda^{d_n} x_n),$$

where $(x_1, \dots, x_n)$ are the *exponential coordinates* of $x \in \mathbb{G}$, $d_j \in \mathbb{N}$ for every $j = 1, \dots, n$ and $1 = d_1 = \ldots = d_m < d_{m+1} \leq \ldots \leq d_n$ for $m := \dim(V_1)$. Here the group $\mathbb{G}$ and the algebra $\mathfrak{g}$ are identified through the exponential mapping.

It is customary to denote with $Q := \sum_{i=1}^k i \, \dim(V_i)$ the homogeneous dimension of $\mathbb{G}$, which corresponds to the Hausdorff dimension of $\mathbb{G}$ (w.r.t. an appropriate sub-Riemannian distance, see below). Q is generally greater than or equal to the topological dimension of

$\mathbb{G}$ and it coincides with it only when $\mathbb{G}$ is the Euclidean group $(\mathbb{R}^n, +)$, which is the only Abelian Carnot group.

Carnot groups are also naturally endowed with sub-Riemannian distances which make them interesting examples of metric spaces. A first well-known example of such metrics is provided by the Carnot-Carathéodory distance d_{cc}, see e.g. [27, Definition 5.2.2], which is a path-metric resembling the classical Riemannian distance. In our case, we will work with metrics induced by homogeneous norms.

Definition 1.3.1. A homogeneous norm $|\cdot|_{\mathbb{G}} : \mathbb{G} \to \mathbb{R}_0^+$ is a continuous function with the following properties:

(*i*) $|x|_{\mathbb{G}} = 0$ if and only if $x = 0$ for every $x \in \mathbb{G}$;

(*ii*) $|x^{-1}|_{\mathbb{G}} = |x|_{\mathbb{G}}$ for every $x \in \mathbb{G}$;

(*iii*) $|\delta_\lambda x|_{\mathbb{G}} = \lambda |x|_{\mathbb{G}}$ for every $\lambda \in \mathbb{R}^+$ and for every $x \in \mathbb{G}$.

A homogeneous norm induces a left-invariant homogeneous distance by

$$d(x, y) := |y^{-1} \cdot x|_{\mathbb{G}} \quad \text{for every } x, y \in \mathbb{G}.$$

We remind that a generic distance d is left-invariant if and only if $d(z \cdot x, z \cdot y) = d(x, y)$ for every $x, y, z \in \mathbb{G}$. A concrete example of such kind of homogeneous distance is given by the Korányi distance, see e.g. [45].

For our purposes, we are also interested in introducing the right-invariant distance

$$d^{\mathcal{R}}(x, y) := |x \cdot y^{-1}|_{\mathbb{G}} \quad \text{for every } x, y \in \mathbb{G}.$$

As before, $d^{\mathcal{R}}$ is right-invariant if and only if $d^{\mathcal{R}}(x \cdot z, y \cdot z) = d^{\mathcal{R}}(x, y)$ for every $x, y, z \in \mathbb{G}$.

From now on we will write $B(x, \varepsilon)$ and $B^{\mathcal{R}}(x, \varepsilon)$ to denote the balls of center $x \in \mathbb{G}$ and radius $\varepsilon > 0$ w.r.t the distances d and $d^{\mathcal{R}}$ respectively. We notice that for any $\varepsilon > 0$

$$B(0, \varepsilon) = B^{\mathcal{R}}(0, \varepsilon).$$

Remark 1.3.2. In chapter 4, we will ask for the following stronger hypotheses on the homogeneous norm $|\cdot|_{\mathbb{G}}$:

(*iv*) invariance under horizontal rotations;

(*v*) the validity of the classical triangular inequality

$$\big||y|_{\mathbb{G}} - |x|_{\mathbb{G}}\big| \le |y^{-1}\cdot x|_{\mathbb{G}} \le |x|_{\mathbb{G}} + |y|_{\mathbb{G}}.$$

An example of such kind of norm, whose induced distance is equivalent to the Carnot-Carathéodory distance, is given in [69, 71].

We also define two *left-translation operators*, one acting on functions and the other one acting on sets, which will be relevant in the upcoming sections.

Definition 1.3.3. Let $y \in \mathbb{G}$. We define $\tau_y : L^p_{\text{loc}}(\mathbb{R}^n) \to L^p_{\text{loc}}(\mathbb{R}^n)$ as

$$\tau_y u(x) := u(y^{-1}\cdot x) \quad \text{for every } x \in \mathbb{G}.$$

With an abuse of notation, we also define $\tau_y : \mathcal{A}_0 \to \mathcal{A}_0$ as

$$\tau_y A := y \cdot A = \{x \in \mathbb{G} : y^{-1}\cdot x \in A\},$$

where $\mathcal{A}_0$ denotes the family of all bounded open sets of $\mathbb{G}$.

Let $u : \mathbb{G} \to \mathbb{R}$ be a sufficiently smooth function and let $(X_1, .., X_m)$ be a basis of the horizontal layer V_1, made of left-invariant vector fields, i.e.,

$$X_j(\tau_y u) = \tau_y(X_j u) \quad \text{for any } j = 1, \dots, m, \text{ for any } y \in \mathbb{G}.$$

The sub-bundle of the tangent bundle $T\mathbb{G}$, which is spanned by the vector fields $X_1, .., X_m$, is called the *horizontal bundle* and it is denoted by $H\mathbb{G}$. The sections of $H\mathbb{G}$ are called *horizontal sections* and are identified with canonical coordinates with respect to this moving frame, that is, a section Φ will be identified with a function $\Phi = (\Phi_1, .., \Phi_m) : \mathbb{G} \to \mathbb{R}^m$.

Definition 1.3.4. Let $(X_1, .., X_m)$ be a basis of V_1, let $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, for which the partial derivatives $X_i u$ exist in the sense of distributions, and let $\Phi = (\Phi_1, .., \Phi_m)$ a horizontal section such that $X_i \Phi_i \in L^1_{\text{loc}}(\mathbb{R}^n)$ for $i = 1, .., m$. The *intrinsic gradient* of u and the *intrinsic divergence* of Φ are defined, respectively, as

$$\nabla_{\mathbb{G}} u := \sum_{j=1}^{m} (X_j u) X_j = (X_1 u, .., X_m u);$$
$$\operatorname{div}_{\mathbb{G}}(\Phi) := \sum_{i=1}^{m} X_i \Phi_i.$$

A definition of *intrinsic curl* in the setting of Carnot groups, $\operatorname{curl}_{\mathbb{G}}$, can be found in [13, Section 5].

Let us also recall the notion of intrinsic differentiability, due to Pansu [102], or inspired by his ideas. We refer the interested reader to [70, 71] for more details.

Definition 1.3.5. A map $L : \mathbb{G} \to \mathbb{R}$ is $\mathbb{G}$-linear if it is a homomorphism from $\mathbb{G}$ to $(\mathbb{R}, +)$ and if it is positively homogeneous of degree 1 with respect to the dilations of $\mathbb{G}$, that is, if

$$L(\delta_\lambda(x)) = \lambda L(x) \quad \text{for every } \lambda > 0,\ x \in \mathbb{G}.$$

If $(X_1, .., X_m)$ is a basis of V_1, then L is $\mathbb{G}$-linear if and only if there exists $a \in \mathbb{R}^m$ such that

$$L(x) = \sum_{j=1}^{m} a_j x_j \quad \text{for each } x = (x_1, \dots, x_m, x_{m+1}, \dots, x_n) \in \mathbb{G}.$$

Definition 1.3.6. A function $f : \mathbb{G} \to \mathbb{R}$ is said to be *Pansu differentiable* (or $\mathbb{G}$-differentiable) at $x \in \mathbb{G}$ if there exists a $\mathbb{G}$-linear map $L^f_x : \mathbb{G} \to \mathbb{R}$, called Pansu differential, such that

$$\lim_{|h|_{\mathbb{G}} \to 0} \frac{f(x \cdot h) - f(x) - L^f_x(h')}{|h|_{\mathbb{G}}} = 0\,,$$

where $h = (h', h'') \in \mathbb{G}$, with $h' = (h_1, \dots, h_m)$ and $h'' = (h_{m+1}, \dots, h_n)$. We will say that f is Pansu differentiable in $\mathbb{G}$ if it is Pansu differentiable at any $x \in \mathbb{G}$.

Notice that, if f is Pansu differentiable at $x \in \mathbb{G}$, then $X_j f(x)$ exist for $j = 1, \dots, m$ and

$$L^f_x(h') = \langle \nabla_{\mathbb{G}} f, h' \rangle_{\mathbb{R}^m} = \sum_{j=1}^{m} X_j f(x) h_j.$$

A notion of smoothness for functions defined on Carnot groups is that of being $\mathbf{C}^1$ functions with respect to the horizontal vector fields $(X_1, \ldots, X_m)$.

Definition 1.3.7. $f : \mathbb{G} \to \mathbb{R}$ is said to be in $\mathbf{C}^1_{\mathbb{G}}(\mathbb{R}^n)$ if $X_j f : \mathbb{G} \to \mathbb{R}$ exist and are continuous for $j = 1, \ldots, m$. Moreover, we define $\mathbf{C}^1_{\mathbb{G}}(\mathbb{R}^n, H\mathbb{G})$ the set of all sections Φ of $H\mathbb{G}$, whose canonical coordinates $\Phi_j \in \mathbf{C}^1_{\mathbb{G}}(\mathbb{R}^n)$ for $j = 1, \ldots, m$.

The previous definition is stronger that being Pansu-differentiable, but requires less regularity than being $\mathbf{C}^1$ with respect to the Euclidean vector fields, since

$$\mathbf{C}^1(\mathbb{R}^n) \subset \mathbf{C}^1_{\mathbb{G}}(\mathbb{R}^n).$$

We also remind that the inclusion can be strict (see, for instance, [71, Remark 5.9]).

Theorem 1.3.8. *[71, Theorem 5.10] If $f \in \mathbf{C}^1_{\mathbb{G}}(\mathbb{R}^n)$, then f is Pansu-differentiable at any point $x \in \mathbb{G}$.*

The n-dimensional Lebesgue measure $\mathcal{L}^n$ of $\mathbb{R}^n$ provides the Haar measure on $\mathbb{G}$, see e.g. [27, Proposition 1.3.21]. The following proposition provides its construction on $\mathbb{G}$.

Proposition 1.3.9. *[15, Proposition 2.10] The Haar measure on $\mathbb{G}$ is given by the image through exponential mapping of the Lebesgue measure on the Lie algebra $\mathfrak{g}$ associated to $\mathbb{G}$, that is, given $f : \mathbb{G} \to \mathbb{R}$, the integration on $\mathbb{G}$ can be expressed as the following integral in canonical coordinates on $\mathbb{R}^n$*

$$\int_{\mathbb{G}} f(x)\, d\mu(x) := \int_{\mathbb{R}^n} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n,$$

where $(x_1, \ldots, x_n)$ are the exponential coordinates of $x \in \mathbb{G}$.

Remark 1.3.10. From now on, we will keep the same notation for integrals on $\mathbb{G}$ and $\mathbb{R}^n$.

Proposition 1.3.11. *Let $f \in L^1(\mathbb{R}^n)$. Then, the Haar measure on $\mathbb{G}$*

(i) is invariant under left and right translations, i.e.

$$\int_{\mathbb{G}} f(x)\, dx = \int_{\mathbb{G}} f(x \cdot y)\, dx = \int_{\mathbb{G}} f(y \cdot x)\, dx \quad \forall y \in \mathbb{G};$$

(ii) scales under group dilations by the homogeneous dimension of $\mathbb{G}$, *that is*

$$\int_{\mathbb{G}} f(\delta_\lambda x)\,dx = \lambda^Q \int_{\mathbb{G}} f(x)\,dx \quad \forall \lambda > 0.$$

Remark 1.3.12. It trivially follows that $|B(x,r)| = r^Q|B| = r^Q C_b$ for all $x \in \mathbb{G}$ and $r > 0$, where $B = B(0,1)$ and C_b denotes its Lebesgue measure.

The following three propositions will be very useful in the sequel.

Proposition 1.3.13. *[59, Proposition 1.13] Let* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ *be an homogeneous function of degree* $-Q$, *i.e.*, $f(\delta_\lambda x) = \lambda^{-Q} f(x)$. *Then, there exists a constant* M_f, *mean value of* f, *such that*

$$\int_{\mathbb{G}} f(x) g(|x|_{\mathbb{G}})\,dx = M_f \int_0^{+\infty} g(r)\frac{dr}{r}$$

for any $g \in L^1(\mathbb{R}^+, \frac{dr}{r})$.

As a consequence of the previous result, we are able to compute explicitly integrals on balls of radial functions, in terms of integrals on the real line.

Proposition 1.3.14. *Let* $f \in L^1(\mathbb{R}^+)$ *and* $R > 0$. *Then*

$$\int_{B(y,R)} f(|y^{-1}\cdot x|_{\mathbb{G}})\,dx = \int_{B(0,R)} f(|x|_{\mathbb{G}})\,dx = QC_b \int_0^R r^{Q-1} f(r)\,dr$$

and

$$\int_{\mathbb{G}\setminus B(y,R)} f(|y^{-1}\cdot x|_{\mathbb{G}})\,dx = \int_{\mathbb{G}\setminus B(0,R)} f(|x|_{\mathbb{G}})\,dx = QC_b \int_R^{+\infty} r^{Q-1} f(r)\,dr.$$

Proof. At first, let us compute the constant M_f for the function $f(x) = |x|_{\mathbb{G}}^{-Q}$.
By Proposition 1.3.13, taking $g(|x|_{\mathbb{G}}) = |x|_{\mathbb{G}}^Q \chi_{[0,1]}(|x|_{\mathbb{G}})$, we get

$$C_b = \int_B dx = \int_{\mathbb{G}} |x|_{\mathbb{G}}^{-Q}|x|_{\mathbb{G}}^{Q}\chi_{[0,1]}(|x|_{\mathbb{G}})\,dx = M_{|x|_{\mathbb{G}}^{-Q}} \int_0^1 r^{Q-1}\,dr = \frac{M_{|x|_{\mathbb{G}}^{-Q}}}{Q},$$

i.e., $M_{|x|_{\mathbb{G}}^{-Q}} = QC_b$.

Therefore, still by Proposition 1.3.13, we have

$$\begin{aligned}\int_{B(0,R)} f(|x|_{\mathbb{G}})\,dx &= \int_{\mathbb{G}} |x|_{\mathbb{G}}^{-Q}|x|_{\mathbb{G}}^{Q}\chi_{[0,R]}(|x|_{\mathbb{G}}) f(|x|_{\mathbb{G}})\,dx \\ &= M_{|x|_{\mathbb{G}}^{-Q}} \int_0^R r^{Q-1} f(r)\,dr = QC_b \int_0^R r^{Q-1} f(r)\,dr\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{G}\setminus B(0,R)} f(|x|_{\mathbb{G}})\,dx &= \int_{\mathbb{G}} |x|_{\mathbb{G}}^{-Q}|x|_{\mathbb{G}}^{Q}\chi_{[R,+\infty[}(|x|_{\mathbb{G}})f(|x|_{\mathbb{G}})\,dx\\ &= QC_b\int_R^{+\infty} r^{Q-1}f(r)\,dr.\end{aligned}$$

□

Proposition 1.3.15. *[59, Proposition 1.15] There exists a unique Radon measure σ on S such that for all $u \in L^1(\mathbb{R}^n)$*

$$\int_{\mathbb{G}} u(x)dx = \int_0^{+\infty}\left(\int_S u(\delta_r z)r^{Q-1}\,d\sigma(z)\right)dr$$

where S is the unit sphere in $\mathbb{G}$.

1.3.1 Approximation by regular functions

The following section is entirely devoted to the introduction and a brief recap of the main properties of global and local convolution on Carnot groups, since this tool is far more delicate in this sub-Riemannian framework. We refer the interested reader to [27, 44] for more details.

First, we need to recall the notion of smooth mollifiers. Given a standard mollifier ρ, that is, $\rho \in \mathbf{C}_c^\infty(\mathbb{R}^n)$, $\mathrm{supp}(\rho) \subset B(0,1)$ and $\int_{\mathbb{G}} \rho(x)\,dx = 1$, for $\varepsilon > 0$ we define $\rho_\varepsilon : \mathbb{G} \to \mathbb{R}$ as

$$\rho_\varepsilon(x) := \tfrac{1}{\varepsilon^Q}\rho\left(\delta_{\varepsilon^{-1}}x\right).$$

The sequence $\{\rho_\varepsilon\}_\varepsilon$ is still a family of mollifiers, in the sense of Definition 1.2.6.

Definition 1.3.16. Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and let $x \in \mathbb{G}$. We define the global convolution on the Carnot group $\mathbb{G}$ as

$$u_\varepsilon(x) := (\rho_\varepsilon * u)(x) := \int_{\mathbb{G}} \rho_\varepsilon(x \cdot y^{-1})u(y)\,dy \quad \text{for any } \varepsilon > 0.$$

Following [44], we move to a proper definition of *local* convolution on Carnot groups. From now on, let $\Omega \subset \mathbb{G}$ be an open and, for every $\varepsilon > 0$, define the open set

$$\Omega_\varepsilon^{\mathcal{R}} := \left\{ x \in \mathbb{G} : \mathrm{dist}^{\mathcal{R}}(x, \mathbb{G} \setminus \Omega) > \varepsilon \right\},$$

where

$$\mathrm{dist}^{\mathcal{R}}(x, \mathbb{G} \setminus \Omega) := \inf \left\{ d^{\mathcal{R}}(x, y) : y \in \mathbb{G} \setminus \Omega \right\}.$$

Definition 1.3.17. For any $u \in L^1_{\mathrm{loc}}(\Omega)$ and $x \in \mathbb{G}$, we define the local convolution as

$$u_\varepsilon(x) := (\rho_\varepsilon * u)(x) := \int_\Omega \rho_\varepsilon(x \cdot y^{-1}) u(y)\, dy\,.$$

If we restrict the domain of definition by considering $x \in \Omega_\varepsilon^{\mathcal{R}}$, we can write

$$\begin{aligned}(\rho_\varepsilon * u)(x) &= \int_{B^{\mathcal{R}}(x,\varepsilon)} \rho_\varepsilon(x \cdot y^{-1}) u(y)\, dy = \int_{B(0,\varepsilon)} \rho_\varepsilon(y) u(y^{-1} \cdot x)\, dy \\ &= \int_{B(0,1)} \rho(z) u\left((\delta_\varepsilon z)^{-1} \cdot x \right)\, dz\end{aligned} \tag{1.19}$$

since, for every $\varepsilon > 0$, $B^{\mathcal{R}}(0, \varepsilon) = B(0, \varepsilon)$.

We are finally ready to state the natural counterparts of the classical results holding for the Euclidean convolution, see e.g. [56].

Proposition 1.3.18. *Let $1 \le p < \infty$, $u \in L^p_{\mathrm{loc}}(\Omega)$ and let $\{\rho_\varepsilon\}_\varepsilon$ a family of mollifiers. Then*

$$\rho_\varepsilon * u \longrightarrow u \quad \textit{(strongly) in } L^p_{\mathrm{loc}}(\Omega). \tag{1.20}$$

Moreover, if $u \in W^{1,p}_{\mathbb{G};\mathrm{loc}}(\Omega)$, then

$$\rho_\varepsilon * u \longrightarrow u \quad \textit{(strongly) in } W^{1,p}_{\mathbb{G};\mathrm{loc}}(\Omega). \tag{1.21}$$

Proof. Let $u \in L^p_{\mathrm{loc}}(\Omega)$ and let $x \in V \Subset W \Subset \Omega$, with V, W being open sets. Since for $\varepsilon > 0$ small enough $V \subset \Omega_\varepsilon^{\mathcal{R}}$, we can exploit (1.19). We first prove an auxiliary estimate which holds true for $p \in (1, \infty)$. In this case, let us set p' to be the conjugate exponent of p, namely

$\frac{1}{p} + \frac{1}{p'} = 1$. We find

$$\begin{aligned} |u_\varepsilon(x)| &\le \int_{B(0,1)} \rho(z) \left| u\left((\delta_\varepsilon z)^{-1} \cdot x \right) \right| \, dz = \int_{B(0,1)} \rho(z)^{\frac{1}{p}} \rho(z)^{\frac{1}{p'}} \left| u\left((\delta_\varepsilon z)^{-1} \cdot x \right) \right| \, dz \\ &\le \left(\int_{B(0,1)} \rho(z) \, dx \right)^{\frac{1}{p'}} \left(\int_{B(0,1)} \rho(z) \left| u\left((\delta_\varepsilon z)^{-1} \cdot x \right) \right|^p \, dz \right)^{\frac{1}{p}} \\ &= \left(\int_{B(0,1)} \rho(z) \left| u\left((\delta_\varepsilon z)^{-1} \cdot x \right) \right|^p \, dz \right)^{\frac{1}{p}}. \end{aligned}$$

Hence, and now for every $p \in [1, \infty)$, we obtain that

$$\begin{aligned} \|u_\varepsilon\|^p_{L^p(V)} &\le \int_{B(0,1)} \rho(z) \left(\int_V \left| u\left((\delta_\varepsilon z)^{-1} \cdot x \right) \right|^p \, dx \right) dz \\ &\le \int_W |u(y)|^p \, dy = \|u\|^p_{L^p(W)} \end{aligned}$$

for $\varepsilon > 0$ sufficiently small.

Let us now fix $\delta > 0$. Since $u \in L^p(W)$, there exists $v \in \mathbf{C}(\overline{W})$ such that $\|u - v\|_{L^p(W)} \le \delta$. Moreover, by the last estimate, $\|u_\varepsilon - v_\varepsilon\|_{L^p(V)} \le \|u - v\|_{L^p(W)} \le \delta$. Thus

$$\|u_\varepsilon - u\|_{L^p(V)} \le \|u_\varepsilon - v_\varepsilon\|_{L^p(V)} + \|v_\varepsilon - v\|_{L^p(V)} + \|v - u\|_{L^p(V)} \le 3\delta$$

since v_ε converges uniformly to v in any compact subset of W.

Let us now move to the proof of (1.21). Thanks to (1.20), it is enough to prove that

$$X_j u_\varepsilon = \rho_\varepsilon * X_j u \quad \text{in } \Omega^{\mathcal{R}}_\varepsilon$$

for every $j = 1, \dots, m$. Let us fix $x \in \Omega^{\mathcal{R}}_\varepsilon$. By the left-invariance of each vector field X_j, we get

$$\begin{aligned} X_j u_\varepsilon(x) &= X_j \left(\int_{B(0,\varepsilon)} \rho_\varepsilon(y) u(y^{-1} \cdot z) \, dy \right) \Big|_{z=x} = \int_{B(0,\varepsilon)} X_j(\rho_\varepsilon(y) u(y^{-1} \cdot x)) \, dy \\ &= \int_{B(0,\varepsilon)} \rho_\varepsilon(y) (X_j u)(y^{-1} \cdot x) \, dy = (\rho_\varepsilon * X_j u)(x) \,, \end{aligned}$$

as desired. □

1.4 Fractional Orlicz-Sobolev spaces depending on vector fields

Definition 1.4.1. Let $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a real valued function such that:

(i) $\phi(0) = 0$, and $\phi(t) > 0$ for any $t > 0$;

(ii) ϕ is nondecreasing on $\mathbb{R}_0^+$;

(iii) ϕ is right-continuous in $\mathbb{R}_0^+$ and $\lim_{t\to\infty} \phi(t) = \infty$.

Then, the real valued function defined on $\mathbb{R}_0^+$ by

$$\varphi(t) = \int_0^t \phi(s)\, ds$$

is called Orlicz function (or Nice Young function).

It is easy to show that hypotheses $(i) - (iii)$ imply that φ is continuous, locally Lipschitz continuous, strictly increasing and convex on $\mathbb{R}_0^+$. Moreover, $\varphi(0) = 0$ and φ is superlinear at zero and at infinity, i.e.,

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0 \qquad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

Up to normalization, we can assume $\varphi(1) = 1$. Hypotheses $(i) - (iii)$ also guarantee the existence of $\varphi^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, which is continuous, concave and strictly increasing, with $\varphi^{-1}(0) = 0$ and $\varphi^{-1}(1) = 1$.

From now on, the following growth condition will be required on φ:

$$p^- \le \frac{t\,\phi(t)}{\varphi(t)} \le p^+ \quad \forall t > 0, \tag{L}$$

where $p^- \le p^+$ are positive constants grater than 1. It holds that

$$s^{\overline{p}}\varphi(t) \le \varphi(st) \le s^{\tilde{p}}\varphi(t), \tag{φ_1}$$

$$\varphi(s+t) \le \frac{2^{p^+}}{2}(\varphi(s) + \varphi(t)). \tag{φ_2}$$

for any $s, t \in \mathbb{R}_0^+$, where $s^{\tilde{p}} := \max\{s^{p^-}, s^{p^+}\}$ and $s^{\overline{p}} := \min\{s^{p^-}, s^{p^+}\}$.

Let us notice that $p^- = p^+$ if and only if $\phi(t) = t^p$, being $\varphi(1) = 1$.

We remind that the conjugate function of φ, defined as its Legendre's transform, is

$$\varphi^*(s) := \sup_{t>0}\{st - \varphi(t)\}.$$

Definition 1.4.2. The smallest $C \in \mathbb{R}^+$ such that the following Δ_2-condition holds

$$\varphi(2t) \leq C\varphi(t) \quad \forall t \in \mathbb{R}_0^+,$$

is called the Δ_2-constant and it is denoted by $\mathbf{C}$. By (φ_2), we have that

$$2 < \mathbf{C} \leq 2^{p^+}. \tag{1.22}$$

It is not difficult to show that (L) is equivalent to require the Δ_2-condition both on φ and φ^* (see for instance [104, Chapter 4]).

The following Lemma can be seen as an improvement of (φ_2).

Lemma 1.4.3. *[57, Lemma 2.6] Let φ be an Orlicz function and let $s, t \in \mathbb{R}_0^+$. Then, for any $\delta > 0$, there exists a positive constant C_δ such that*

$$\varphi(s+t) \leq C_\delta \varphi(s) + (1+\delta)^{p^+}\varphi(t).$$

We conclude this section recalling a fundamental definition which is the natural counterpart of [57, Remark 2.15] in the context of Carnot groups. From now on, when necessary, a generic $z \in \mathbb{G}$ will be denoted as $z = (z', z'')$ where $z' = (z_1, .., z_m)$ is the horizontal part and $z'' = (z_{m+1}, .., z_n)$ is the vertical one.

Definition 1.4.4. For an Orlicz function φ and $t \in \mathbb{R}^+$, we define the bounded function

$$\tilde{\varphi}^+(t) := \limsup_{s\uparrow 1}(1-s)\int_0^1 \left(\int_S \varphi(t|z'|_{\mathbb{R}^m} r^{1-s})\, d\sigma(z)\right)\frac{dr}{r}.$$

A similar definition with lim inf instead of lim sup is used to define $\tilde{\varphi}^-$. When they coincide, we will define

$$\tilde{\varphi}(t) := \lim_{s\uparrow 1}(1-s)\int_0^1 \left(\int_S \varphi(t|z'|_{\mathbb{R}^m} r^{1-s})\, d\sigma(z)\right)\frac{dr}{r}. \tag{1.23}$$

Proposition 1.4.5. *The functions* $\tilde{\varphi}^{\pm}$ *are still Orlicz functions, both of them equivalent to* φ*, i.e., there exist* $c_1, c_2 > 0$ *such that*

$$c_1\varphi(t) \leq \tilde{\varphi}^{\pm}(t) \leq c_2\varphi(t)$$

for any $t \in \mathbb{R}^+$*.*

Proof. $\tilde{\varphi}^{\pm}$ are Orlicz functions by similar arguments of [57, Proposition 2.16]. Moreover, by (φ_1), we can notice that

$$\begin{aligned}\int_0^1 \left(\int_S \varphi(t|z'|_{\mathbb{R}^m} r^{1-s})\, d\sigma(z)\right)\frac{dr}{r} &\leq \int_S |z'|_{\mathbb{R}^m}^{p^-} d\sigma(z) \int_0^1 \varphi(tr^{1-s})\frac{dr}{r} \\ &\leq QC_b\varphi(t)\int_0^1 r^{(1-s)p^- -1}dr = \frac{QC_b}{(1-s)p^-}\varphi(t)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \left(\int_S \varphi(t|z'|_{\mathbb{R}^m} r^{1-s})\, d\sigma(z)\right)\frac{dr}{r} &\geq \int_S |z'|_{\mathbb{R}^m}^{p^+} d\sigma(z) \int_0^1 \varphi(tr^{1-s})\frac{dr}{r} \\ &\geq QC_b\varphi(t)\int_0^1 r^{(1-s)p^+ -1}dr = \frac{QC_b}{(1-s)p^+}\varphi(t).\end{aligned}$$

Thus, taking $c_1 := \frac{QC_b}{p^+}$ and $c_2 := \frac{QC_b}{p^-}$, we get the thesis. □

Remark 1.4.6. Let us notice that $c_1 = c_2 = \frac{QC_b}{p}$ if and only if $\varphi(t) = t^p$. We also remind that explicit examples of $\tilde{\varphi}$, in the Euclidean case, are given in [57, Example 2.17].

We conclude this section defining the Orlicz function $\overline{\varphi}\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$, still naturally associated to φ, and which is defined as

$$\overline{\varphi}(t) := \int_0^t \varphi(\tau)\frac{d\tau}{\tau}, \quad t \in \mathbb{R}_0^+. \tag{1.24}$$

Following [4], φ and $\overline{\varphi}$ are still equivalent Orlicz function and, in particular, the following estimate holds true

$$\varphi\left(\frac{t}{2}\right) \leq \overline{\varphi}(t) \leq \varphi(t) \quad \text{for every } t \in \mathbb{R}_0^+. \tag{1.25}$$

1.4.1 Fractional Orlicz-Sobolev spaces on Carnot groups

Definition 1.4.7. Let $\mathbb{G}$ be a Carnot group, let φ be an Orlicz function and let $0 < s \le 1$. We define, with a little abuse of notation, the Orlicz-Lebesgue space and the fractional Orlicz-Sobolev spaces, respectively, as

$$
\begin{aligned}
L^{\varphi}(\mathbb{R}^n) &\coloneqq \{u : \mathbb{G} \to \mathbb{R} \text{ measurables such that } \Phi_{\varphi}(u) < \infty\} \\
W^{s,\varphi}_{\mathbb{G}}(\mathbb{R}^n) &\coloneqq \{u \in L^{\varphi}(\mathbb{R}^n) \text{ such that } \Phi_{s,\varphi}(u) < \infty\},
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_{\varphi}(u) &\coloneqq \int_{\mathbb{G}} \varphi(|u(x)|)\, dx, \\
\Phi_{s,\varphi}(u) &\coloneqq \begin{cases} \displaystyle\iint_{\mathbb{G}\times\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|^s_{\mathbb{G}}}\right) \frac{dx\,dy}{|y^{-1}\cdot x|^Q_{\mathbb{G}}} & \text{if } 0 < s < 1 \\ \Phi_{\varphi}(|\nabla_{\mathbb{G}} u|_{\mathbb{R}^m}) & \text{if } s = 1 \end{cases}.
\end{aligned}
$$

These spaces are usually endowed with the so-called Luxemburg norms, studied by Luxemburg in [84], and defined as

$$
\begin{aligned}
\|u\|_{\varphi} &\coloneqq \inf\{\lambda > 0 : \Phi_{\varphi}\left(\frac{u}{\lambda}\right) \le 1\} \\
\|u\|_{s,\varphi} &\coloneqq \|u\|_{\varphi} + [u]_{s,\varphi}
\end{aligned}
$$

where

$$
[u]_{s,\varphi} \coloneqq \inf\{\lambda > 0 : \Phi_{s,\varphi}\left(\frac{u}{\lambda}\right) \le 1\}
$$

is the (s,φ)-Gagliardo seminorm. Moreover, the space $W^{s,\varphi}_{\mathbb{G},0}(\mathbb{R}^n)$ in defined as the counterpart of the one introduced in Definition 1.2.12, in the framework of Carnot groups and depending on Orlicz functions.

By well-known results given in [53, 77] for the Euclidean case, it is easy to characterize these spaces as follows.

Theorem 1.4.8. *Let φ be an Orlicz function. Then, $L^{\varphi}(\mathbb{R}^n)$ and $W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$ are separable Banach spaces. Moreover, if both φ and φ^* satisfy the Δ_2-condition, then the spaces $L^{\varphi}(\mathbb{R}^n)$*

and $W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$ *are also reflexive and the dual space of* $L^{\varphi}(\mathbb{R}^n)$ *can be identified with* $L^{\varphi^*}(\mathbb{R}^n)$*. Finally,* $\mathbf{C}^{\infty}_c(\mathbb{R}^n)$ *is dense in both* $L^{\varphi}(\mathbb{R}^n)$ *and* $W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$*.*

The proof of Theorem 1.4.8 trivially follows from the Euclidean case. The reader can see for instance [53, Theorem 2.3.13, Theorem 2.5.10] and [77, Theorem 5.3, Theorem 5.5, Corollary 3.7, Corollary 3.9], where a more general theory is treated.

Following the same technique of [57, Proposition 2.11], we can also state the following theorem.

Theorem 1.4.9. *Let us assume the same hypotheses of Theorem 1.4.8. Then, for each* $s \in (0,1)$*, the space* $W^{s,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$ *is a reflexive and separable Banach space. Moreover,* $\mathbf{C}^{\infty}_c(\mathbb{R}^n)$ *is dense in* $W^{s,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$*.*

As in the Euclidean case, the immersion of the space $W^{s,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$ can be compactly embedded into $L^{\varphi}(\mathbb{R}^n)$.

Theorem 1.4.10. *Let* $0 < s < 1$ *and let* φ *be an Orlicz function. Then, from every bounded sequence* $\{u_n\}_n \subset W^{s,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$*, there exist* $u \in W^{s,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$ *and* $\{u_{n_k}\}_k \subset \{u_n\}_n$ *such that*

$$u_{n_k} \to u \quad \textit{in } L^{\varphi}(\mathbb{R}^n).$$

The proof of Theorem 1.4.10 is a consequence of the following theorem.

Theorem 1.4.11. *[80, Theorem 11.4] Any sequence of functions* $\{v_k\}_k \subset L^{\varphi}(\mathbb{R}^n)$ *is compact if and only if the following two conditions are satisfied:*

(i) $\Phi_{\varphi}(v_k)$ *is bounded;*

(ii) for any $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\Phi_{\varphi}(\tau_h v_k - v_k) < \varepsilon$ *for any* $h \in \mathbb{G}$ *such that* $|h|_{\mathbb{G}} < \delta$*, where* $\tau_h u(x) := u(x \cdot h)$ *for any* $x \in \mathbb{G}$*.*

Proof of Theorem 1.4.10. Let us fix $u \in W^{s,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$. In order to apply Theorem 1.4.11, we want to show the existence of a constant $M > 0$ such that

$$\Phi_{\varphi}(\tau_h u - u) \leq M |h|^{sp^-}_{\mathbb{G}} \Phi_{s,\varphi}(u) \tag{1.26}$$

for every $h \in \mathbb{G}$ such that $|h|_{\mathbb{G}} < \frac{1}{2}$. For any $y \in B(x, |h|_{\mathbb{G}})$, by the monotonicity of φ, the Δ_2-condition and since $|B(x,r)| = r^Q |B| = r^Q C_b$, by Remark 1.3.12, then we have

$$\begin{aligned}
\Phi_\varphi(\tau_h u - u) &= \int_{\mathbb{G}} \varphi(|u(x\cdot h) - u(y) + u(y) - u(x)|)\,dx \\
&\le \frac{\mathbf{C}}{2}\left[\int_{\mathbb{G}} \varphi(|u(x\cdot h) - u(y)|)\,dx + \int_{\mathbb{G}} \varphi(|u(y) - u(x)|)\,dx\right] \\
&= \frac{\mathbf{C}}{2C_b|h|_{\mathbb{G}}^Q} \int_{B(x,|h|_{\mathbb{G}})} \left(\int_{\mathbb{G}} \varphi(|u(x\cdot h) - u(y)|)\,dx\right) dy \\
&\quad + \frac{\mathbf{C}}{2C_b|h|_{\mathbb{G}}^Q} \int_{B(x,|h|_{\mathbb{G}})} \left(\int_{\mathbb{G}} \varphi(|u(y) - u(x)|)\,dx\right) dy = \frac{\mathbf{C}}{2C_b|h|_{\mathbb{G}}^Q}(I_1 + I_2).
\end{aligned} \tag{1.27}$$

Let us notice that, by the triangular inequality,

$$|y^{-1}\cdot x\cdot h|_{\mathbb{G}} \le |y^{-1}\cdot x|_{\mathbb{G}} + |h|_{\mathbb{G}} \le 2|h|_{\mathbb{G}}.$$

Therefore, by (φ_1), the monotonicity of φ and a change of variables, we have

$$\begin{aligned}
I_1 &= \int_{B(x,|h|_{\mathbb{G}})} \left(\int_{\mathbb{G}} \varphi\left(\frac{|u(x\cdot h) - u(y)|}{|y^{-1}\cdot x\cdot h|_{\mathbb{G}}^s}|y^{-1}\cdot x\cdot h|_{\mathbb{G}}^s\right) \frac{|y^{-1}\cdot x\cdot h|_{\mathbb{G}}^Q}{|y^{-1}\cdot x\cdot h|_{\mathbb{G}}^Q}\,dx\right) dy \\
&\le 2^Q|h|_{\mathbb{G}}^Q \int_{B(x,|h|_{\mathbb{G}})} \left(\int_{\mathbb{G}} \varphi\left(\frac{|u(x\cdot h) - u(y)|}{|y^{-1}\cdot x\cdot h|_{\mathbb{G}}^s}(2|h|_{\mathbb{G}})^s\right) \frac{dx}{|y^{-1}\cdot x\cdot h|_{\mathbb{G}}^Q}\right) dy \\
&\le 2^{sp^- + Q}|h|_{\mathbb{G}}^{sp^- + Q} \int_{B(x,|h|_{\mathbb{G}})} \left(\int_{\mathbb{G}} \varphi\left(\frac{|u(z) - u(y)|}{|y^{-1}\cdot z|_{\mathbb{G}}^s}\right) \frac{dz}{|y^{-1}\cdot z|_{\mathbb{G}}^Q}\right) dy \\
&\le 2^{sp^- + Q}|h|_{\mathbb{G}}^{sp^- + Q}\Phi_{s,\varphi}(u).
\end{aligned}$$

Similarly

$$I_2 \le |h|_{\mathbb{G}}^{sp^- + Q}\Phi_{s,\varphi}(u).$$

Thus, by (1.27), we finally have

$$\Phi_\varphi(\tau_h u - u) \le \frac{\mathbf{C}}{2C_b}(2^{sp^- + Q} + 1)|h|_{\mathbb{G}}^{sp^-}\Phi_{s,\varphi}(u) := M|h|_{\mathbb{G}}^{sp^-}\Phi_{s,\varphi}(u).$$

Let now $\{u_n\}_n \subset W^{s,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$ be a bounded sequence in $W^{s,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$. In particular, $\{u_n\}_n$ is bounded in $L^\varphi(\mathbb{R}^n)$. Therefore, by (1.26)

$$\sup_{n\in\mathbb{N}} \Phi_\varphi(\tau_h u_n - u_n) \le \sup_{n\in\mathbb{N}}(\Phi_{s,\varphi}(u_n) + \Phi_\varphi(u_n))M|h|_{\mathbb{G}}^{sp^-}.$$

Thus, by Theorem 1.4.11, there exist $u \in L^{\varphi}(\mathbb{R}^n)$ and $\{u_{n_k}\}_k \subset \{u_n\}_n$ such that $u_{n_k} \to u$ in $L^{\varphi}(\mathbb{R}^n)$. In order to conclude the proof, we show that $u \in W^{s,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$.

By Fatou's Lemma and the continuity of φ, we have

$$\begin{aligned}\Phi_{s,\varphi}(u) &= \iint_{\mathbb{G}\times\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|^s_{\mathbb{G}}}\right)\frac{dx\,dy}{|y^{-1}\cdot x|^Q_{\mathbb{G}}}\\ &\leq \liminf_{k\to\infty}\iint_{\mathbb{G}\times\mathbb{G}} \varphi\left(\frac{|u_{n_k}(x)-u_{n_k}(y)|}{|y^{-1}\cdot x|^s_{\mathbb{G}}}\right)\frac{dx\,dy}{|y^{-1}\cdot x|^Q_{\mathbb{G}}}\\ &\leq \sup_{n\in\mathbb{N}}\Phi_{s,\varphi}(u_{n_k})<\infty.\end{aligned}$$

□

The two following Lemmas will be useful in Chapter 4.

Lemma 1.4.12. *Let $u \in L^{\varphi}(\mathbb{R}^n)$ and let $\{u_\varepsilon\}_\varepsilon$ be a sequence of regularized functions of u, in the sense of Definition 1.3.16. Then*

$$\Phi_{s,\varphi}(u_\varepsilon) \leq \Phi_{s,\varphi}(u)$$

for any $\varepsilon > 0$ and $0 < s < 1$.

Proof. Let $x, y \in \mathbb{G}$ and let $h := y^{-1}\cdot x$. Then, by the Jensen's inequality and the monotonicity of φ, we have

$$\begin{aligned}\varphi\left(\frac{|u_\varepsilon(x\cdot h)-u_\varepsilon(x)|}{|h|^s_{\mathbb{G}}}\right) &\leq \varphi\left(\int_{B(0,\varepsilon)}\frac{|u(y^{-1}\cdot x\cdot h)-u(y^{-1}\cdot x)|}{|h|^s_{\mathbb{G}}}\rho_\varepsilon(y)\,dy\right)\\ &\leq \int_{B(0,\varepsilon)}\varphi\left(\frac{|u(y^{-1}\cdot x\cdot h)-u(y^{-1}\cdot x)|}{|h|^s_{\mathbb{G}}}\right)\rho_\varepsilon(y)\,dy.\end{aligned}$$

Therefore, by the invariance of the norm under translations, we have

$$\begin{aligned}\int_{\mathbb{G}}\varphi\left(\frac{|u_\varepsilon(x\cdot h)-u_\varepsilon(x)|}{|h|^s_{\mathbb{G}}}\right)\frac{dx}{|h|^Q_{\mathbb{G}}} &\leq \int_{\mathbb{G}}\left(\int_{B(0,\varepsilon)}\varphi\left(\frac{|u(y^{-1}\cdot x\cdot h)-u(y^{-1}\cdot x)|}{|h|^s_{\mathbb{G}}}\right)\rho_\varepsilon(y)\,dy\right)\frac{dx}{|h|^Q_{\mathbb{G}}}\\ &= \int_{\mathbb{G}}\left(\int_{\mathbb{G}}\varphi\left(\frac{|u(y^{-1}\cdot x\cdot h)-u(y^{-1}\cdot x)|}{|h|^s_{\mathbb{G}}}\right)\frac{dx}{|h|^Q_{\mathbb{G}}}\right)\rho_\varepsilon(y)\,dy\\ &= \int_{\mathbb{G}}\varphi\left(\frac{|u(x\cdot h)-u(x)|}{|h|^s_{\mathbb{G}}}\right)\frac{dx}{|h|^Q_{\mathbb{G}}}.\end{aligned}$$

The thesis follows by integrating in $\mathbb{G}$ with respect to h. □

Definition 1.4.13. Given $\eta \in \mathbf{C}_c^\infty(\mathbb{R}^n)$ such that $0 \le \eta \le 1$, $\mathrm{supp}(\eta) \subset B(0,2)$, $\eta = 1$ in $B(0,1)$ and $|\nabla_{\mathbb{G}}\eta|_{\mathbb{R}^m} \le 2$, we define the family of cut-off functions $\{\eta_k\}_k$ as

$$\eta_k(x) := \eta(\delta_{k^{-1}}x) \quad \text{for any } k \in \mathbb{N},$$

that is, $0 \le \eta_k \le 1$, $\eta_k = 1$ in $B(0,k)$, $\mathrm{supp}(\eta_k) \subset B(0,2k)$ and $|\nabla_{\mathbb{G}}\eta_k|_{\mathbb{R}^m} \le \frac{2}{k}$. For any $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the family of truncated functions $\{u_k\}_k$ as

$$u_k := \eta_k u \quad \text{for any } k \in \mathbb{N}.$$

Lemma 1.4.14. *Let $u \in L^\varphi(\mathbb{R}^n)$ and let $\{u_k\}_k$ be the sequence of truncated functions of u. Then*

$$\Phi_{s,\varphi}(u_k) \le \frac{\mathbf{C}}{2}\left(\Phi_{s,\varphi}(u) + \left(\frac{2}{k}\right)^{p^-} \frac{QC_b}{(1-s)p^+}\Phi_\varphi(u) + 2^{p^+}\frac{QC_b}{sp^-}\Phi_\varphi(u)\right)$$

for any $k \in \mathbb{N}$ and $0 < s < 1$.

Proof. Let us fix $x, y \in \mathbb{G}$. Then, by the Δ_2-condition and the monotonicity of φ, we have

$$\begin{aligned}\varphi\left(\frac{|u_k(x)-u_k(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right) &= \varphi\left(\frac{|\eta_k(x)u(x)-\eta_k(y)u(x)+\eta_k(y)u(x)-\eta_k(y)u(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right)\\ &\le \frac{\mathbf{C}}{2}\varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right) + \frac{\mathbf{C}}{2}\varphi\left(\frac{|\eta_k(y)||u(x)-u(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right).\end{aligned}$$

Hence, being $\eta_k \le 1$ for any $k \in \mathbb{N}$, we get

$$\begin{aligned}\Phi_{s,\varphi}(u_k) &= \iint_{\mathbb{G}\times\mathbb{G}} \varphi\left(\frac{|u_k(x)-u_k(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{|y^{-1}\cdot x|_{\mathbb{G}}^Q}\\ &\le \frac{\mathbf{C}}{2}\Phi_{s,\varphi}(u) + \frac{\mathbf{C}}{2}\iint_{\mathbb{G}\times\mathbb{G}}\varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{|y^{-1}\cdot x|_{\mathbb{G}}^Q}\\ &= \frac{\mathbf{C}}{2}\Phi_{s,\varphi}(u) + \frac{\mathbf{C}}{2}\left[\int_{\mathbb{G}}\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}<1\}}\varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{|y^{-1}\cdot x|_{\mathbb{G}}^Q}\right.\\ &\quad\left.+\int_{\mathbb{G}}\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\ge 1\}}\varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{|y^{-1}\cdot x|_{\mathbb{G}}^Q}\right].\end{aligned}$$

Since $|\nabla\eta_k|_{\mathbb{R}^m} \le \frac{2}{k}$, then, by (φ_1), assuming without loss of generality $k > 2$, and by

Proposition 1.3.14, we have

$$\begin{aligned}
&\int_{\mathbb{G}}\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}<1\}} \varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{|y^{-1}\cdot x|_{\mathbb{G}}^Q}\\
&\qquad \le \int_{\mathbb{G}}\left(\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\le 1\}} \varphi\left(\frac{2}{k}\frac{|u(x)|}{|y^{-1}\cdot x|_{\mathbb{G}}^{s-1}}\right)\frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^Q}\right)dx\\
&\qquad \le \left(\frac{2}{k}\right)^{p^-}\int_{\mathbb{G}}\left(\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\le 1\}} \frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^{(s-1)p^++Q}}\right)\varphi(|u(x)|)\,dx\\
&\qquad = \Phi_\varphi(u)\left(\frac{2}{k}\right)^{p^-} QC_b\int_0^1 r^{(1-s)p^+-1}\,dr = \left(\frac{2}{k}\right)^{p^-}\frac{QC_b}{(1-s)p^+}\Phi_\varphi(u).
\end{aligned}$$

Moreover

$$\begin{aligned}
&\int_{\mathbb{G}}\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\ge 1\}} \varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{|y^{-1}\cdot x|_{\mathbb{G}}^Q}\\
&\qquad \le \int_{\mathbb{G}}\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\ge 1\}} \varphi\left(\frac{2|u(x)|}{|y^{-1}\cdot x|_{\mathbb{G}}^{s}}\right)\frac{dx\,dy}{|y^{-1}\cdot x|_{\mathbb{G}}^Q}\\
&\qquad \le 2^{p^+}\int_{\mathbb{G}}\left(\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\ge 1\}} \frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^{sp^-+Q}}\right)\varphi(|u(x)|)\,dx\\
&\qquad = \Phi_\varphi(u)2^{p^+}QC_b\int_1^\infty r^{-sp^--1}\,dr = 2^{p^+}\frac{QC_b}{sp^-}\Phi_\varphi(u).
\end{aligned}$$

□

1.4.2 Fractional magnetic Orlicz-Sobolev spaces on $\mathbb{R}^n$

We conclude the first chapter of the thesis by introducing the function spaces appearing in Theorem 4.3.1. To simplify the readability, we will use the following compact notation: let $s \in (0,1)$, we denote the magnetic Hölder quotient of order s as

$$D_s^A u(x,y) := \frac{u(x) - e^{\mathrm{i}(x-y)A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^s}. \tag{1.28}$$

We will also denote

$$d\mu(x,y) := \frac{dx\,dy}{|x-y|^n}.$$

We notice that when $A \equiv 0$, $D_s^0 u = D_s u := \dfrac{u(x)-u(y)}{|x-y|^s}$ is the usual s-Hölder quotient appearing in the definition of fractional Sobolev spaces.

Definition 1.4.15. Let φ be an Orlicz function, $s \in (0,1)$ be a fractional parameter and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth enough vector potential. Similarly as section 1.4.1, we define the spaces $L^\varphi(\mathbb{R}^n;\mathbb{C})$ and $W_A^{s,\varphi}(\mathbb{R}^n;\mathbb{C})$ as follows:

$$L^\varphi(\mathbb{R}^n;\mathbb{C}) := \{u \colon \mathbb{R}^n \to \mathbb{C} \text{ measurable} \colon \Phi_\varphi(u) < \infty\},$$
$$W_A^{s,\varphi}(\mathbb{R}^n;\mathbb{C}) := \left\{u \in L^\varphi(\mathbb{R}^n;\mathbb{C}) \colon \Phi_{s,\varphi}^A(u) < \infty\right\},$$

where Φ_φ and $\Phi_{s,\varphi}^A$ are defined as

$$\Phi_\varphi(u) := \int_{\mathbb{R}^n} \varphi(|u(x)|)\, dx$$

and

$$\Phi_{s,\varphi}^A(u) := \iint_{\mathbb{R}^{2n}} \varphi\left(|D_s^A u(x,y)|\right)\, d\mu. \tag{1.29}$$

These spaces become Banach spaces when endowed with the so-called Luxemburg norms defined through $\Phi_{s,\varphi}^A$, namely

$$\|u\|_{s,\varphi}^A = \|u\|_\varphi + |u|_{s,\varphi}^A,$$

where

$$\|u\|_\varphi := \inf\{\lambda > 0 \colon \Phi_\varphi(\tfrac{u}{\lambda}) \le 1\}$$

is the usual (Luxemburg) norm on $L^\varphi(\mathbb{R}^n;\mathbb{C})$ and

$$|u|_{s,\varphi}^A := \inf\{\lambda > 0 \colon \Phi_{s,\varphi}^A(\tfrac{u}{\lambda}) \le 1\}.$$

Finally, we define the fractional Magnetic Orlicz-Sobolev space $W_{A,0}^{s,\varphi}(\mathbb{R}^n;\mathbb{C})$ as the closure of $\mathbf{C}_c^1(\mathbb{R}^n;\mathbb{C})$ with respect to the magnetic fractional Orlicz seminorm $|u|_{s,\varphi}^A$.

Remark 1.4.16. We note that, when $\varphi(t) = t^p$, we recover the magnetic fractional Sobolev spaces defined in [105, 106, 117]. At the same time, if we assume $A \equiv 0$, the above definitions lead to the fractional Orlicz-Sobolev spaces considered in [3, 4]. Combining the last observations, it is also obvious that for $\varphi(t) = t^p$ and $A \equiv 0$ we recover the classical fractional Sobolev spaces.

Remark 1.4.17. In the definition of the Hölder quotient of order s, and hence in the definition of the fractional Orlicz-Sobolev spaces, we actually chose the so-called midpoint prescription

$$(x,y) \mapsto A\left(\frac{x+y}{2}\right),$$

which is closely related to the magnetic fractional Laplacian

$$(-\Delta)^s_A u(x) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^{n+2s}} dy, \qquad x \in \mathbb{R}^n. \tag{1.30}$$

It is noteworthy to mention that actually, for $s = \frac{1}{2}$, the definition of the fractional operator $(-\Delta)^s_A$ dates back to the '80s, and it is closely related to the proper definition of a quantized operator corresponding to the symbol of the classical relativistic Hamiltonian, namely

$$\sqrt{(\xi - A(x))^2 + m^2} + V(x), \quad (\xi, x) \in \mathbb{R}^n \times \mathbb{R}^n.$$

In particular, it is related to the *kinetic part* of the above symbol. In [78], it is explained that there are at least three definitions for such a quantized operator appearing in the literature: two of them are given in terms of pseudo-differential operators, while the third one as the square root of a suitable non-negative operator. In [78], Ichinose showed that these three nonlocal operators are in general not the same, but they do coincide whenever dealing with a linear vector potential A. A well studied example of a linear potential A is the so-called Ahronov-Bohm potential. We finally notice that, in the physically relevant case of $\mathbb{R}^3$, the linearity of A is actually equivalent to require a constant magnetic field.

Remark 1.4.18. As mentioned before, one may then replace midpoint prescription with other prescriptions, e.g. the averaged one

$$(x,y) \mapsto \int_0^1 A\left((1-\vartheta)x + \vartheta y\right) d\vartheta =: A_\sharp(x,y).$$

From the physical point of view, the latter has the advantage that the magnetic fractional Laplacian associated to it, i.e. $(-\Delta)^s_{A_\sharp}$, turns out to be Gauge covariant (see e.g. [78, Proposition 2.8]), namely

$$(-\Delta)^s_{(A+\nabla\phi)_\sharp} = e^{\mathrm{i}\phi}(-\Delta)^s_{A_\sharp} e^{-\mathrm{i}\phi}.$$

Chapter Two

Γ-convergence

Notation

In this chapter, we identify the space of real matrices of order $m \times n$ with $\mathbb{R}^{mn}$ or $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, where $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ denotes the class of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ endowed with its operator norm. Given a matrix $A = [a_{ij}]$ of order $m \times n$, its *operator norm* is defined as

$$\|A\| := \sup_{|z|=1} |Az|$$

and its *Hilbert-Schmidt norm* as

$$\|A\|_{\mathbb{R}^{mn}} := \sqrt{\sum_{i,j} a_{ij}^2}$$

(see [82, Chap. 7]). Since the norms are equivalent, we can also identify the spaces

$$\mathbf{C}^0(\Omega_X, \mathbb{R}^{mn}) \equiv \mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)),$$

where we recall that $\Omega_X = \Omega \setminus \mathcal{N}_X$ (see Definition 1.1). For each $x \in \Omega$, let $L_x : \mathbb{R}^n \to \mathbb{R}^m$ be the linear map

$$L_x(v) := C(x)v \text{ if } v \in \mathbb{R}^n \tag{2.1}$$

where $C(x)$ denotes the matrix in (1.1). Let N_x and V_x respectively denote the subspaces of $\mathbb{R}^n$ defined as

$$N_x := \ker(L_x), \quad V_x := \{C(x)^T z \ : \ z \in \mathbb{R}^m\}. \tag{2.2}$$

It is well-known that N_x and V_x are orthogonal complements in $\mathbb{R}^n$, that is

$$\mathbb{R}^n = N_x \oplus V_x \,.$$

Therefore, for each $x \in \Omega$ and $\xi \in \mathbb{R}^n$, there exist $\xi_{N_x} \in N_x$ and $\xi_{V_x} \in V_x$, unique vectors of $\mathbb{R}^n$, such that

$$\xi = \xi_{N_x} + \xi_{V_x}. \tag{2.3}$$

Let $\Pi_x : \mathbb{R}^n \to V_x \subset \mathbb{R}^n$ be the projection

$$\Pi_x(\xi) := \xi_{V_x}. \tag{2.4}$$

Moreover, by using the definition of V_x it is easy to see that

$$V_x = \operatorname{span}_{\mathbb{R}} \{X_1(x), \ldots, X_m(x)\} \,,$$

i.e., the so-called *horizontal bundle*, denoted also by H_x.

2.1 Characterizations of local functionals depending on vector fields

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 \le p < \infty$ and let X be an X-gradient. For a given Borel function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$, we consider $F, F_1 : L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$ defined by

$$F(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in \mathbf{C}^1(\Omega) \\ \infty & \text{otherwise} \end{cases} \tag{2.5}$$

and

$$F_1(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in W^{1,1}_{\text{loc}}(\Omega) \\ \infty & \text{otherwise} \end{cases}, \tag{2.6}$$

where the integrands taken into account satisfy the following standard structural conditions:

(I_1) for every $\eta \in \mathbb{R}^m$, $f(\cdot, \eta) : \Omega \to \mathbb{R}$ is Borel measurable on Ω;

(I_2) for a.e. $x \in \Omega$, $f(x, \cdot) : \mathbb{R}^m \to \mathbb{R}$ is convex;

(I_3) there exist two positive constants $c_0 \le c_1$ and two nonnegative functions $a_0,\ a_1 \in L^1(\Omega)$ such that

$$c_0|\eta|^p - a_0(x) \le f(x, \eta) \le c_1|\eta|^p + a_1(x) \tag{2.7}$$

for a.e. $x \in \Omega$ and for each $\eta \in \mathbb{R}^m$.

Definition 2.1.1. We denote by $I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ the class of functions satisfying hypotheses $(I_1) - (I_3)$. When $a_0 = a_1 \equiv 0$, the class $I_{m,p}(\Omega, c_0, c_1, 0, 0)$ will be simply denoted by $I_{m,p}(\Omega, c_0, c_1)$.

Notice that both functionals (2.5) and (2.6) always admit an integral representation with respect to the *Euclidean gradient*, that is, taking for instance in mind functional (2.5), it can be represented by

$$F(u) = \int_\Omega f_e(x, Du(x))\, dx \quad \text{for each } u \in \mathbf{C}^1(\Omega)$$

where $f_e : \Omega \times \mathbb{R}^n \to \mathbb{R}$ denotes the *Euclidean integrand*, defined as

$$f_e(x,\xi) := f(x, C(x)\xi) \quad \text{for a.e. } x \in \Omega \text{ for each } \xi \in \mathbb{R}^n, \tag{2.8}$$

where $C(x)$ is the coefficient matrix of the X-gradient.

Now, set $\mathcal{A}$ the class of open sets contained in Ω, we are going to study whether a local functional $F : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$, defined by

$$F(u,A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A} \text{ and } u \in \mathbf{C}^1(A) \\ \infty & \text{otherwise} \end{cases}, \tag{2.9}$$

can be equivalently represented both with respect to the X-gradient and the Euclidean gradient.

In virtue of (2.8), it is clear that, for each $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$, the following characterization of (2.9) can be given:

$$F(u,A) = \int_A f(x, Xu(x))\, dx = \int_A f(x, C(x)Du(x))\, dx = \int_A f_e(x, Du(x))\, dx\,. \tag{2.10}$$

Question: Given an X-gradient, an integrand $f_e : \Omega \times \mathbb{R}^n \to \mathbb{R}$, convex in the second variable for a.e. $x \in \Omega$, and a functional $F : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$, defined by

$$F(u,A) = \int_A f_e(x, Du(x))\, dx \quad u \in \mathbf{C}^1(A)\,, \tag{2.11}$$

there exists a function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that

$$F(u,A) = \int_A f(x, Xu(x))\, dx \quad \text{for every } u \in \mathbf{C}^1(A)\,?$$

The following counterexample shows that, in general, the answer is negative, without further additional assumptions on f_e.

Counterexample 2.1.2. Let X be the family of Heisenberg vector fields (see Example 1.1.2 *(iii)*), let $\Omega \subset \mathbb{R}^3$ be a bounded open set containing the origin and let $F : L^2(\Omega) \times \mathcal{A} \to [0,\infty]$ be the local functional defined as

$$F(u,A) := \begin{cases} \int_A |Du(x)|^2\, dx & \text{if } u \in \mathbf{C}^1(A) \\ \infty & \text{otherwise} \end{cases}.$$

Let us show that there not exist any function $f : \Omega \times \mathbb{R}^2 \to [0, \infty)$ such that

$$F(u, A) = \int_A f(x, Xu(x))\, dx \quad \text{for every } u \in \mathbf{C}^1(A)\,.$$

By contradiction, if such integrand f exists, then, by (2.26), we get

$$|\xi|^2 = f_e(x, \xi) = f(x, C(x)\xi) = f_e(x, \Pi_x(\xi)) = |\Pi_x(\xi)|^2$$

for a.e. $x \in \Omega$, for any $\xi \in \mathbb{R}^3$, where Π_x is defined in (2.4). Since, by Lemma 2.1.3 (iii), function $\Omega \ni x \mapsto \Pi_x(\xi)$ is continuous, the previous identity must hold for any $x \in \Omega$ and $\xi \in \mathbb{R}^3$.

Let $x = 0$, then $\Pi_0(\xi) = (\xi_1, \xi_2, 0)$ for each $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Thus, if we choose for instance $\xi = (0, 0, 1)$, then the previous identity is not satisfied and then we have a contradiction.

In Lemma 2.1.9, in according with Theorem 2.1.5, we will show that, if there exist a nonnegative function $a \in L^1_{\text{loc}}(\Omega)$ and a positive constant b such that for a.e. $x \in \Omega$

$$f_e(x, \xi) \le a(x) + b|C(x)\xi|^p \quad \forall\, \xi \in \mathbb{R}^n\,,$$

then

$$F(u, A) = \int_A f_e(x, Du(x))\, dx = \int_A f(x, Xu(x))\, dx \quad \forall A \in \mathcal{A} \quad \forall u \in \mathbf{C}^1(A)\,,$$

that is, the answer to the previous question is positive.

Before proving Lemma 2.1.9, let us study some algebraic properties of our framework.

Lemma 2.1.3. *Assume that the family X satisfies (LIC) on Ω. Let $C(x)$ be the matrix in* (1.1) *and L_x be the map in* (2.1). *Then*

(i) $\dim V_x = m$ *for each $x \in \Omega_X$ and* $L_x(V_x) = \text{range}(L_x) = \mathbb{R}^m$, *where* $\text{range}(L_x)$ *denotes the range of L_x, that is,* $\text{range}(L_x) := \{L_x(v) : v \in \mathbb{R}^n\}$. *In particular $L_x : V_x \to \mathbb{R}^m$ is an isomorphism.*

(ii) Let

$$B(x) := C(x)\,C^T(x) \quad x \in \Omega\,. \tag{2.12}$$

Then, for each $x \in \Omega_X$, $B(x)$ is a symmetric invertible matrix of order m. Moreover the map $B^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, defined as

$$B^{-1}(x)(z) := B(x)^{-1}z \quad \text{if } z \in \mathbb{R}^m\,, \tag{2.13}$$

is continuous.

(iii) For each $x \in \Omega_X$, the projection Π_x can be represented as

$$\Pi_x(\xi) = \xi_{V_x} = C(x)^T B(x)^{-1} C(x)\,\xi, \quad \forall\,\xi \in \mathbb{R}^n\,.$$

If $m = n$, then $\Pi_x = \mathrm{Id}_n : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map in $\mathbb{R}^n$.

Proof. **(i)** The claim is a well-known result of basic linear algebra.

(ii) It is straightforward that $B(x)$ a symmetric matrix of order m for each $x \in \Omega$. We have only to show that it is invertible for each $x \in \Omega_X$ or, equivalently, that

$$\text{if } B(x)z = 0 \text{ for some } z \in \mathbb{R}^m, \text{ then } z = 0. \tag{2.14}$$

Let z^T denote the transpose of a column vector $z \in \mathbb{R}^m$. If $B(x)z = 0$, then

$$0 = z^T B(x) z = z^T C(x) C^T(x) z = \left|C^T(x) z\right|^2_{\mathbb{R}^m} \quad \Longleftrightarrow \quad C^T(x) z = 0\,. \tag{2.15}$$

By (LIC) and (2.15) and, since

$$\operatorname{rank} C(x) = \operatorname{rank} C^T(x) = m \quad \forall x \in \Omega_X\,,$$

then we get that $z = 0$ and (2.14) follows. Let us now prove that the map (2.13) is continuous. Let us recall that, given a matrix $A \in \mathbf{C}^0(\Omega_X, \mathbb{R}^{m^2})$, by the definition of determinant (see, for instance, [82, Chap.3, Theorem 6]), the determinant map

$$\det A : \Omega_X \to \mathbb{R}, \quad (\det A)(x) := \det(A(x))$$

is continuous. Moreover

$$A(x) \text{ is invertible } \iff \det A(x) \neq 0\,.$$

By Cramer's rule (see, for instance, [82, Chap.3, Theorem 7]), if $B(x)^{-1} = [b^*_{ij}(x)]$, then

$$b^*_{ij}(x) = (-1)^{i+j}\frac{\det B_{ij}(x)}{\det B(x)} \quad x \in \Omega_X,\ i, j = 1, \dots, m\,,$$

where B_{ij} is the $(m-1) \times (m-1)$ matrix obtained by striking out the ith row and jth column of B, i.e., the (ij)th minor of B.

This implies that $B^{-1} \in \mathbf{C}^0(\Omega_X, \mathbb{R}^{m^2}) \equiv \mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m))$.

(iii) We have

$$\Pi_x(\xi) = \xi_{V_x} = C(x)^T w \tag{2.16}$$

for a suitable (unique) $w = w(x, \xi) \in \mathbb{R}^m$ depending on x and ξ. On the other hand, by (2.16),

$$C(x)\xi = L_x(\xi) = L_x(\xi_{N_x}) + L_x(\xi_{V_x}) = C(x)\xi_{V_x} = C(x)C(x)^T w = B(x)w\,. \tag{2.17}$$

Since $B(x)$ is invertible, then, by (2.16) and (2.17), we get the desired conclusion. □

Corollary 2.1.4. *Assume that the family X satisfies (LIC) condition on Ω. Then, the map $L_x : V_x \to \mathbb{R}^m$ is invertible and the map $L^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, defined as*

$$L^{-1}(x) := L_x^{-1} \text{ if } x \in \Omega_X, \tag{2.18}$$

belongs to $\mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$.

Proof. The fact that the map $L_x : V_x \to \mathbb{R}^m$ is invertible follows from Lemma 2.1.3 (i). Let us now prove that

$$L_x^{-1}(z) = C^T(x)B(x)^{-1}z \quad \forall\, z \in \mathbb{R}^m\,, \tag{2.19}$$

where $B(x)$ is the matrix in (2.12). Let us fix $z \in \mathbb{R}^m$ and let $v = L_x^{-1}(z) \in V_x$. By Lemma 2.1.3 (iii), there exists $w \in \mathbb{R}^m$ such that $v = C^T(x)w$. Hence

$$z = L_x(v) = C(x)C^T(x)w = B(x)w\,.$$

By Lemma 2.1.3 (*ii*), it holds $w = B(x)^{-1}z$. Therefore, we get

$$L_x^{-1}(z) = v = C^T(x)B(x)^{-1}z \tag{2.20}$$

and (2.19) follows. Let us define

$$A(x) := C^T(x)B(x)^{-1} \quad \text{if } x \in \Omega_X \,.$$

Then, from Lemma 2.1.3 (*ii*), $A \in \mathbf{C}^0(\Omega_X, \mathbb{R}^{mn}) \equiv \mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$ and, by (2.20), we get the desired conclusion. □

The following theorem provides a necessary and sufficient condition for integral functionals depending on the Euclidean gradient to be represented with respect to a given X-gradient.

Theorem 2.1.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let X be an X-gradient satisfying (LIC) condition on Ω. Moreover, let $F : \mathbf{C}^1(\Omega) \times \mathcal{A} \to \mathbb{R}$ be defined as*

$$F(u, A) = \int_A f_e(x, Du(x))\, dx \quad \text{if } A \in \mathcal{A}, u \in \mathbf{C}^1(A)\,,$$

where $f_e : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Borel measurable function satisfying

$$f_e(\cdot, \xi) \in L^1_{\text{loc}}(\Omega) \text{ for each } \xi \in \mathbb{R}^n \tag{2.21}$$

and

$$f_e(x, \cdot) : \mathbb{R}^n \to \mathbb{R} \text{ convex for a.e. } x \in \Omega\,. \tag{2.22}$$

Finally, define $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ as

$$f(x, \eta) := \begin{cases} f_e(x, L^{-1}(x)(\eta)) & \text{if } (x, \eta) \in \Omega_X \times \mathbb{R}^m \\ 0 & \text{otherwise} \end{cases}, \tag{2.23}$$

where $L^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is the map in (2.18). Then, f is a Borel measurable function satisfying

$$f(x, \cdot) : \mathbb{R}^m \to \mathbb{R} \text{ convex for a.e. } x \in \Omega\,. \tag{2.24}$$

Moreover, for any $A \in \mathcal{A}$ *and* $u \in \mathbf{C}^1(A)$

$$F(u,A) = \int_A f_e(x, Du(x))\, dx = \int_A f(x, Xu(x))\, dx \tag{2.25}$$

if and only if

$$f_e(x,\xi) = f_e(x, \Pi_x(\xi)) \quad \textit{for a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^n\,. \tag{2.26}$$

In addition, function f *(for which* (2.25) *holds true) is unique, that is, if there exists another Borel measurable function* $f^* : \Omega \times \mathbb{R}^m \to \mathbb{R}$ *satisfying* $f^*(x,\cdot) : \mathbb{R}^m \to \mathbb{R}$ *convex a.e.* $x \in \Omega$ *and satisfying* (2.25)*, then* $f(x,\eta) = f^*(x,\eta)$ *for a.e.* $x \in \Omega$ *and for each* $\eta \in \mathbb{R}^m$.

Remark 2.1.6. If $m = n$, providing X satisfies (LIC) condition on Ω, then condition (2.26) always holds true, since $\Pi_x \equiv \mathrm{I}_n$, in virtue of Lemma 2.1.3 (*iii*).

Proof. **1st step.** Let us prove that f is Borel measurable. Let $\Psi : \Omega_X \times \mathbb{R}^m \to \Omega_X \times \mathbb{R}^n$ denote the map

$$\Psi(x,\eta) := (x, L^{-1}(x)(\eta)) \quad \text{if } (x,\eta) \in \Omega_X \times \mathbb{R}^m.$$

By Corollary 2.1.4, Ψ is continuous and, therefore, it is also Borel measurable. Since f_e is Borel measurable, the composition $f = f_e \circ \Psi : \Omega_X \times \mathbb{R}^m \to \mathbb{R}$ is still Borel measurable.

Moreover, since $f_e(x,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is convex for a.e. $x \in \Omega$ and $L^{-1}(x) : \mathbb{R}^m \to \mathbb{R}^n$ is linear for $x \in \Omega_X$, then (2.24) follows by

$$f(x,\cdot) = f_e(x,\cdot) \circ L^{-1}(x) \quad \forall\, x \in \Omega_X.$$

2nd step. Let us prove the uniqueness of representation in (2.25). Assume that

$$\int_A f(x, Xu(x))\, dx = \int_A f^*(x, Xu(x))\, dx = \int_A f_e(x, Du(x))\, dx \quad \forall\, A \in \mathcal{A}, u \in \mathbf{C}^1(A) \tag{2.27}$$

for given Borel measurable functions $f,\ f^* : \Omega \times \mathbb{R}^m \to \mathbb{R}$, convex in the second variable a.e. $x \in \Omega$. Let

$$u(x) = u_\xi(x) := \langle \xi, x\rangle \quad x \in \mathbb{R}^n,\ \text{for fixed } \xi \in \mathbb{Q}^n\,. \tag{2.28}$$

By (2.21) and (2.27), it follows that the functions

$$\Omega \ni y \mapsto f(y, C(y)\xi) \text{ and } \Omega \ni y \mapsto f^*(y, C(y)\xi) \text{ are in } L^1_{\text{loc}}(\Omega)\,.$$

Let $A = B(x,r)$, $r > 0$. Then, by (2.27) and by Lebesgue's differentiation theorem, we get the existence of a negligible set $\mathcal{N}_\xi \subset \Omega$ such that for any $x \in \Omega \setminus \mathcal{N}_\xi$

$$f(x, L_x(\xi)) = f(x, C(x)\xi) = f^*(x, C(x)\xi) = f^*(x, L_x(\xi))\,. \tag{2.29}$$

If $\mathcal{N} := \cup_{\xi\in\mathbb{Q}^n}\mathcal{N}_\xi$, then (2.29) holds for any $x \in \Omega \setminus \mathcal{N}$ and $\xi \in \mathbb{Q}^n$. Moreover, since $f(x,\cdot), f^*(x,\cdot) : \mathbb{R}^m \to \mathbb{R}$ are continuous for each $x \in \Omega \setminus \mathcal{N}$, then (2.29) holds for any $x \in \Omega \setminus \mathcal{N}$ and $\xi \in \mathbb{R}^n$ and, since the map $L_x : \mathbb{R}^n \to \mathbb{R}^m$ is onto, then we get the desired conclusion.

3rd step. Let us assume (2.26). To prove (2.25) it is sufficient to show that for each $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$

$$f(x, Xu(x)) = f_e(x, Du(x)) \quad \text{a.e. } x \in \Omega\,. \tag{2.30}$$

Given $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$, let us recall that

$$Xu(x) = C(x)Du(x) \quad \forall\, x \in A\,.$$

Thus, by (2.26), Lemma 2.1.3 (*iii*) and the definition of V_x, if $v_x := Du(x)$

$$\begin{aligned} f(x, Xu(x)) &= f(x, C(x)v_x) = f(x, L_x(\Pi_x(v_x))) = f_e(x, L_x^{-1}(L_x(\Pi_x(v_x))) \\ &= f_e(x, \Pi_x(v_x)) = f_e(x, v_x) = f_e(x, Du(x)) \end{aligned} \tag{2.31}$$

a.e. $x \in \Omega$ and (2.30) follows. On the other hand, let us assume that for every $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$

$$\int_A f_e(x, Du(x))\, dx = \int_A f(x, Xu(x))\, dx$$

where f is the function in (2.23). By (2.31), for every $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$

$$f(x, Xu(x)) = f_e(x, \Pi_x(Du(x))) \quad \forall\, x \in A\,,$$

which implies

$$\int_A f(x, Xu(x))\,dx = \int_A f_e(x, \Pi_x(Du(x)))\,dx\,.$$

Thus, for every $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$

$$\int_A f_e(x, \Pi_x(Du(x)))\,dx = \int_A f_e(x, Du(x))\,dx$$

and the conclusion now follows by proceeding as in the second step of the proof. □

Remark 2.1.7. Observe that (2.30) actually holds for each $u \in W^{1,p}(A)$. As a consequence, (2.25) holds for each $A \in \mathcal{A}$ and $u \in W^{1,p}(A)$.

Counterexample 2.1.8. *If the X-gradient does not satisfy (LIC) condition, the uniqueness of representation* (2.25) *may trivially fail. For instance, let $X = (X_1, X_2) := (\partial_1, 0)$ be the family of vector fields on $\Omega = \mathbb{R}^2$ and let $f, f^* : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(\eta) := \eta_1^2 + g(\eta_2)$ and $f^*(\eta) := \eta_1^2 + g^*(\eta_2)$ for each $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, where g, $g^* : \mathbb{R} \to \mathbb{R}$ are convex functions satisfying*

$$g(0) = g^*(0) = 0\,, \quad \text{but } g \neq g^*.$$

Then, it is clear that f and f^* are integrands of the same functional F defined in (2.25)*, even though $f \neq f^*$.*

The following lemma, that will turn out to be a key result through the thesis, provides a sufficient condition to represent an integral functional both with respect to the Euclidean gradient and the X-gradient.

Lemma 2.1.9. *Let $f_e : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function such that*

(i) for a.e. $x \in \Omega$, $f_e(x, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is convex;

(ii) there exist a nonnegative function $a \in L^1_{\text{loc}}(\Omega)$ and a positive constant b such that

$$f_e(x, \xi) \le a(x) + b|C(x)\xi|^p \quad \text{for a.e. } x \in \Omega, \text{ for each } \xi \in \mathbb{R}^n\,.$$

Then, f_e satisfies (2.26).

Proof. Let us prove that, for a.e. $x \in \Omega$,

$$f_e(x, \xi_{N_x} + \zeta) = f_e(x, \zeta) \quad \forall \xi, \zeta \in \mathbb{R}^n, \tag{2.32}$$

which is equivalent to (2.26). Since $t\xi_{N_x} \in N_x$ for every $t \in \mathbb{R}$, then, by (ii),

$$f_e(x, t\xi_{N_x} + \zeta) \leq a(x) + b|C(x)\zeta|^p \quad \text{for a.e. } x \in \Omega, \text{ for every } t \in \mathbb{R}.$$

Therefore, for a.e. $x \in \Omega$, the function $\mathbb{R} \ni t \to f_e(x, t\xi_{N_x} + \zeta)$ is bounded from above and also convex, in virtue of (i). Hence, it is constant and considering its values at $t = 1$ and $t = 0$, then we obtain (2.32). □

2.2 Representation theorems

Let us recall some notation about set functions on $\mathcal{A}$, the class of open sets contained in Ω, and local functionals defined on $L^p(\Omega) \times \mathcal{A}$, in according with [47].

Definition 2.2.1. Let $\alpha : \mathcal{A} \to [0, \infty]$ be a set function. We say that:

(i) α is *increasing* if $\alpha(A) \leq \alpha(B)$, for each A, $B \in \mathcal{A}$ with $A \subseteq B$;

(ii) α is *inner regular* if $\alpha(A) = \sup\{\alpha(B) : B \in \mathcal{A}, B \Subset A\}$ for each $A \in \mathcal{A}$;

(iii) α is *subadditive* if $\alpha(A) \leq \alpha(A_1) + \alpha(A_2)$ for every A, A_1, $A_2 \in \mathcal{A}$ with $A \subset A_1 \cup A_2$;

(iv) α is *superadditive* if $\alpha(A) \geq \alpha(A_1) + \alpha(A_2)$ for every A, A_1, $A_2 \in \mathcal{A}$ with $A_1 \cup A_2 \subseteq A$ and $A_1 \cap A_2 = \emptyset$;

(v) α is a *measure* if there exists a Borel measure $\mu : \mathcal{B}(\Omega) \to [0, \infty]$ such that $\alpha(A) = \mu(A)$ for every $A \in \mathcal{A}$.

Remark 2.2.2. Let us recall that, if $\alpha : \mathcal{A} \to [0, \infty]$ is an increasing set function, then it is a measure if and only if it is subadditive, superadditive and inner regular (see [47, Theorem 14.23]).

Definition 2.2.3. Let $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$. We say that:

(*i*) F is *increasing* if, for every $u \in L^p(\Omega)$, $F(u, \cdot) : \mathcal{A} \to [0, \infty]$ is increasing as set function;

(*ii*) F is *inner regular* (on $\mathcal{A}$) if it is increasing and, for each $u \in L^p(\Omega)$, $F(u, \cdot) : \mathcal{A} \to [0, \infty]$ is inner regular as set function;

(*iii*) F is a *measure* if, for every $u \in L^p(\Omega)$, $F(u, \cdot) : \mathcal{A} \to [0, \infty]$ is a measure as set function;

(*iv*) F is *local* if

$$F(u, A) = F(v, A)$$

for each $A \in \mathcal{A}$ and for each $u, v \in L^p(\Omega)$ such that $u = v$ a.e. on A;

(*v*) F is *lower semicontinuous* if, for every $A \in \mathcal{A}$, $F(\cdot, A) : L^p(\Omega) \to [0, \infty]$ is lower semicontinuous;

(*vi*) F is *convex* if, for every $A \in \mathcal{A}$, $F(\cdot, A) : L^p(\Omega) \to [0, \infty]$ is convex.

Remark 2.2.4. Let $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ be a non-negative increasing functional such that $F(u, \emptyset) = 0$ for every $u \in L^p(\Omega)$. Then, by [47, Theorem 14.23], F is a measure if and only if F is subadditive, superadditive and inner regular.

2.2.1 Functionals depending on vector fields

The following representation theorem is the main result of this section.

Theorem 2.2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 \le p < \infty$ and let X be an X-gradient satisfying (LIC) condition on Ω. Moreover, let $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ be an increasing functional satisfying the following properties:*

(a) *F is local;*

(b) *F is a measure;*

(c) *F is lower semicontinuous;*

(d) *$F(u + c, A) = F(u, A)$ for each $u \in L^p(\Omega)$, $A \in \mathcal{A}$ and $c \in \mathbb{R}$;*

(e) *there exist a nonnegative function $a \in L^1_{\text{loc}}(\Omega)$ and a positive constant b such that*

$$0 \le F(u, A) \le \int_A (a(x) + b\,|Xu(x)|^p)\; dx$$

for each $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$.

Then, there exists a Borel function $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ such that:

(i) *for each $u \in L^p(\Omega)$ and for each $A \in \mathcal{A}$ with $u|_A \in W^{1,p}_{X;\text{loc}}(A)$*

$$F(u, A) = \int_A f(x, Xu(x))\, dx\,;$$

(ii) *for a.e. $x \in \Omega$, $f(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ is convex;*

(iii) *for a.e. $x \in \Omega$,*

$$0 \le f(x, \eta) \le a(x) + b\,|\eta|^p \quad \forall\, \eta \in \mathbb{R}^m\,.$$

Remark 2.2.6. Let us first observe that inequality in assumption (e) can be extended to each $u \in W^{1,p}_X(A)$, $A \in \mathcal{A}$.

Let $A \in \mathcal{A}$, let $u \in W^{1,p}_X(A) \cap L^p(\Omega)$ and let $\bar{u}$ be the extension of u to $\mathbb{R}^n$ which vanishes outside Ω. Moreover, let $\{\rho_\varepsilon\}_\varepsilon$ be a family of mollifiers, and let

$$u_\varepsilon = \bar{u} * \rho_\varepsilon(x) \quad x \in \mathbb{R}^n.$$

By Proposition 1.2.7, for each $A' \in \mathcal{A}$ with $A' \Subset A$, we have

$$u_\varepsilon \to u \text{ in } L^p(\Omega)\,; \tag{2.33}$$

$$u_\varepsilon|_{A'} \in W^{1,p}_X(A') \text{ and } u_\varepsilon \to u \text{ in } W^{1,p}_X(A')\,. \tag{2.34}$$

Therefore, by assumptions (c) and (e), and by (2.33) and (2.34), it follows that

$$\begin{aligned} F(u, A') &\le \liminf_{\varepsilon\to 0} F(u_\varepsilon, A') \le \lim_{\varepsilon\to 0}\left(\int_{A'} \left(a(x) + b\,|Xu_\varepsilon(x)|^p\right)\,dx\right) \\ &= \int_{A'} \left(a(x) + b\,|Xu(x)|^p\right)\,dx. \end{aligned}$$

Since $F(u,\cdot)$ is a measure, it is also inner regular (see Remark 2.2.4). Thus, taking the supremum on all $A' \in \mathcal{A}$ with $A' \Subset A$, we get the desired conclusion.

Proof of Theorem 2.2.5. We will divide the proof into three steps.

1st step. Let us first prove that there exists an integral representation of F with respect to a Euclidean integrand, that is, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0,\infty)$ and a positive constant b_2 such that

$$F(u, A) = \int_A f_e(x, Du(x))\,dx \tag{2.35}$$

for each $u \in L^p(\Omega)$ and $A \in \mathcal{A}$ with $u|_A \in W^{1,p}_{\mathrm{loc}}(A)$;

$$\text{for a.e. } x \in \Omega,\ f_e(x,\cdot) : \mathbb{R}^n \to [0,\infty) \text{ is convex;} \tag{2.36}$$

$$\text{for a.e. } x \in \Omega,\ 0 \le f_e(x,\xi) \le a(x) + b_2\,|\xi|^p \quad \forall\, \xi \in \mathbb{R}^n; \tag{2.37}$$

$$\text{(2.26) holds, that is, for a.e. } x \in \Omega,\ f_e(x,\xi) = f_e(x, \Pi_x(\xi)) \quad \forall\, \xi \in \mathbb{R}^n. \tag{2.38}$$

By (1.4), if $u \in W^{1,p}(\Omega)$, then, for a.e. $x \in \Omega$, there exists $\bar{b} < \infty$ such that

$$|Xu(x)|^p \le \sup_{x\in\Omega} \|C(x)\|^p\,|Du(x)|^p = \bar{b}\,|Du(x)|^p\,, \tag{2.39}$$

since the coefficients of the X-gradient are Lipschitz on Ω. By (2.39) and assumption (e), it follows that

$$0 \le F(u, A) \le \int_A \left(a(x) + b_2|Du(x)|^p\right)\,dx \tag{2.40}$$

for every $u \in W^{1,p}(\Omega)$ and for every $A \in \mathcal{A}$, with $b_2 := b\,\bar{b}$. Therefore by $(a), (b), (c), (d)$ and (2.40) and in virtue of [47, Theorem 20.1], there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$ satisfying (2.35), (2.36) and (2.37).

Observe now that, by (2.35) and assumption (e), for $u = u_\xi$ (see (2.28)) we get

$$\int_A f_e(x, \xi)\, dx \leq \int_A (a(x) + b|C(x)\xi|^p)\, dx \quad \forall\, A \in \mathcal{A} \quad \forall\, \xi \in \mathbb{R}^n\,.$$

From this integral inequality, we can infer the pointwise inequality, that is, there exists a negligible set $\mathcal{N} \subset \Omega$, such that, for each $x \in \Omega \setminus \mathcal{N}$

$$f_e(x, \xi) \leq\, a(x) + b|C(x)\xi|^p \quad \forall\, \xi \in \mathbb{R}^n\,. \tag{2.41}$$

From (2.36), (2.41) and Lemma 2.1.9, (2.38) holds.

2nd step. Let us prove the existence of a Borel function $f : \Omega \times \mathbb{R}^m \to [0, \infty)$ such that

$$F(u, A) = \int_A f(x, Xu(x))\, dx \tag{2.42}$$

for each $A \in \mathcal{A}$, $u \in \mathbf{C}^1(A)$ satisfying claims (ii) and (iii).

By (2.36), (2.37) and (2.38), (2.42) follows at once with $f : \Omega \times \mathbb{R}^m \to [0, \infty)$ defined as in (2.23), which also satisfies claim (ii), in virtue of Theorem 2.1.5.

Moreover, by assumption (e) and (2.42), assuming $u = u_\xi$, it follows that

$$0 \leq \int_A f(y, C(y)\xi)\, dy \leq \int_A (a(y) + b\,|C(y)\xi|^p)\ dy \quad \forall\, A \in \mathcal{A} \quad \forall\, \xi \in \mathbb{R}^n\,.$$

Taking $A = B(x, r)$, applying Lebesgue's differentiation theorem and arguing as before, from the previous inequality we get the following pointwise estimate: for a.e. $x \in \Omega$, it holds that

$$0 \leq\, f(x, C(x)\xi) \leq\, a(x) +\, b\,|C(x)\xi|^p \quad \forall \xi \in \mathbb{R}^n\,.$$

Observe now that, by (LIC), for a.e. $x \in \Omega$ the map $L_x : \mathbb{R}^n \to \mathbb{R}^m$, $L_x(\xi) := C(x)\xi$, is surjective. Then, claim (iii) also follows.

3rd step. Let us prove that the integral representation in (2.42) can be extended to functions $u \in W^{1,p}_{X;\mathrm{loc}}(A)$. Therefore, claim (i) will follow.

Let us begin to observe that, given $A \in \mathcal{A}$, the functional

$$W^{1,p}_X(A) \ni u \mapsto \int_A f(x, Xu(x))\, dx \text{ is (strongly) continuous.} \tag{2.43}$$

Indeed, since for a.e. $x \in \Omega$, $f(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ is continuous and claim (iii) holds, we can apply the Carathéodory's continuity theorem (see, for instance, [47, Example 1.22]).

Let $u \in W^{1,p}_X(\Omega)$ and let $A,\, A' \in \mathcal{A}$ with $A' \Subset A$. Since $F(\cdot, A') : L^p(\Omega) \to [0, \infty]$, then by (2.33), (2.34), assumptions (c) and the dominated convergence theorem, it follows that

$$F(u, A') \leq \liminf_{\varepsilon \to 0^+} F(u_\varepsilon, A') = \lim_{\varepsilon \to 0^+} \int_{A'} f(x, Xu_\varepsilon(x))\, dx = \int_{A'} f(x, Xu(x))\, dx\,.$$

Since F is a measure and, therefore, inner regular by Remark 2.2.4, we get

$$F(u, A) \leq \int_A f(x, Xu(x))\, dx \tag{2.44}$$

for every $u \in W^{1,p}_X(\Omega)$, for each $A \in \mathcal{A}$.

Let us fix $w \in W^{1,p}_X(\Omega)$ and let us consider the functional $G : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$

$$G(u, A) := F(u + w, A)\,.$$

It is easy to check that G still satisfies assumptions $(a) - (e)$.

Thus, by the second step, there exists a Borel function $g : \Omega \times \mathbb{R}^m \to [0, \infty)$ satisfying claims (ii) and (iii) with $f \equiv g$, for suitable $a \in L^1_{\text{loc}}(\Omega)$ and $b > 0$, such that

$$G(u, A) = \int_A g(x, Xu(x))\, dx \tag{2.45}$$

for each $A \in \mathcal{A}$, $u \in \mathbf{C}^1(A)$ and

$$G(u, A) \leq \int_A g(x, Xu(x))\, dx \tag{2.46}$$

for every $u \in W^{1,p}_X(\Omega)$, for each $A \in \mathcal{A}$. Moreover, arguing as in (2.43), one can prove that, for each $A \in \mathcal{A}$, the functional

$$W^{1,p}_X(A) \ni u \mapsto \int_A g(x, Xu(x))\, dx \text{ is (strongly) continuous.} \tag{2.47}$$

Let $\bar{w}$ be the extension of w to $\mathbb{R}^n$, which vanishes outside Ω, and let

$$w_\varepsilon := \bar{w} * \rho_\varepsilon : \mathbb{R}^n \to \mathbb{R}$$

and fix $A \in \mathcal{A}$. Thus, by Proposition 1.2.7, for every $A' \in \mathcal{A}$ with $A' \Subset A$, as $\varepsilon \to 0^+$,

$$w_\varepsilon \to w \text{ in } L^p(\Omega) \text{ and } w_\varepsilon \to w \text{ in } W^{1,p}_X(A')\,.$$

Moreover, by (2.43), (2.44), (2.45) and (2.47), we obtain

$$\begin{aligned}\int_{A'} g(x,0)\,dx = G(0,A') = F(w,A') \le \int_{A'} f(x,Xw(x))\,dx = \lim_{\varepsilon\to 0^+}\int_{A'} f(x,Xw_\varepsilon(x))\,dx \\ = \lim_{\varepsilon\to 0^+} F(w_\varepsilon,A') = \lim_{\varepsilon\to 0^+} G(w_\varepsilon - w,A') = \lim_{\varepsilon\to 0^+}\int_{A'} g(x,Xw_\varepsilon(x) - Xw(x))\,dx \\ = \int_{A'} g(x,0)\,dx\,.\end{aligned}$$

This implies that

$$F(w,A') = \int_{A'} f(x,Xw(x))\,dx \quad \text{for each } A' \in \mathcal{A} \text{ with } A' \Subset A\,.$$

Taking the supremum for $A' \Subset A$ in the previous identity, we get that

$$F(w,A) = \int_A f(x,Xw(x))\,dx \text{ for each } w \in W^{1,p}_X(\Omega) \text{ and } A \in \mathcal{A}\,. \tag{2.48}$$

If $u \in L^p(\Omega)$, $A \in \mathcal{A}$ and $u|_A \in W^{1,p}_{X;\mathrm{loc}}(A)$ then, for every $A' \in \mathcal{A}$ with $A' \Subset A$, by Proposition 1.2.3 (i), there exists $w \in W^{1,p}_X(\Omega)$ such that

$$u|_{A'} = w|_{A'}\,.$$

Since F is local, by (2.48), we obtain

$$\int_{A'} f(x,Xu(x))\,dx = F(u,A') = F(w,A') = \int_{A'} f(x,Xw(x))\,dx$$

and, by taking the supremum for $A' \Subset A$, we finally get

$$F(u,A) = \int_A f(x,Xu(x))\,dx\,,$$

which concludes the proof. □

2.2.2 Left-invariant functionals on Carnot groups

In this section, $\mathbb{G} = (\mathbb{R}^n, \cdot)$ denotes a step k Carnot group and $(X_1, .., X_m)$ is a basis of the horizontal layer V_1 of $\mathbb{G}$, made of left-invariant vector fields. Moreover, taking in mind Definition 1.3.3, if $\mathcal{A}_0$ denotes the class of all bounded open subsets of $\mathbb{G}$, we define the class of *left-invariant* functionals on $\mathbb{G}$ as follows.

Definition 2.2.7. A functional $F : L^p_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A}_0 \to \overline{\mathbb{R}}$ is left-invariant if, for every $y \in \mathbb{G}$,

$$F(\tau_y u, \tau_y A) = F(u, A) \tag{2.49}$$

for every $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and for every $A \in \mathcal{A}_0$.

We stress that, whenever $\mathbb{G} = (\mathbb{R}^n, +)$, the above definition boils down to the one considered in [47, Chapter 23] and it is therefore possible to provide many examples of left-invariant functionals. A less trivial example, directly adapted to the Carnot group setting, is provided by the functional

$$F(u, A) := \begin{cases} \int_A f(\nabla_{\mathbb{G}} u(x))\, dx & \text{if } u \in W^{1,1}_{\mathbb{G};\mathrm{loc}}(A) \\ \infty & \text{otherwise} \end{cases}, \tag{2.50}$$

where f is a non-negative Borel function. We remind that the functional F, defined above, is increasing, subadditive, superadditive and inner regular on $\mathcal{A}_0$ and, therefore, it is a measure on $\mathcal{A}_0$ by Remark 2.2.4 (for details, see e.g. [47, Example 15.4]).

Proposition 2.2.8. *Let F be as in* (2.50)*. Then, F is left-invariant.*

Proof. First, we notice that, due to the left-invariance of the vector fields $X_1, \ldots, X_m$, for any $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $A \in \mathcal{A}_0$

$$\tau_y u \in W^{1,1}_{\mathbb{G};\mathrm{loc}}(\tau_y A) \quad \text{if and only if} \quad u \in W^{1,1}_{\mathbb{G};\mathrm{loc}}(A).$$

Therefore, it is sufficient to prove the result for functions $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that the restriction $u|_A \in W^{1,1}_{\mathbb{G};\mathrm{loc}}(A)$.

Let us fix $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $A \in \mathcal{A}_0$ such that $u|_A \in W^{1,1}_{\mathbb{G};\mathrm{loc}}(A)$. By a change of variables, it follows that

$$F(\tau_y u, \tau_y A) = \int_{\tau_y A} f(\nabla_{\mathbb{G}} \tau_y u(x))\, dx = \int_A f(\nabla_{\mathbb{G}} u(z))\, dz = F(u, A)\,,$$

as desired. □

Remark 2.2.9. The previous result trivially holds true by replacing $W^{1,1}_{\mathbb{G};\mathrm{loc}}(A)$ with $W^{1,p}_{\mathbb{G};\mathrm{loc}}(A)$, for any $p > 1$.

Before stating the main theorem of this section, we need two preliminary results.

Theorem 2.2.10. *Let $F : L^p_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A}_0 \to \overline{\mathbb{R}}$ be a left-invariant, increasing, convex and lower semicontinuous functional and let $\{\rho_h\}_h$, $h \in \mathbb{N}$, be a sequence of mollifiers. Then*

$$F(u, A') \leq \liminf_{h\to\infty} F(\rho_h * u, A') \leq \limsup_{h\to\infty} F(\rho_h * u, A') \leq F(u, A)$$

for every $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and for every $A, A' \in \mathcal{A}_0$ with $A' \Subset A$.

Proof. The first inequality follows from the lower semicontinuity of F, while the second one is always trivially satisfied. It remains to prove that

$$\limsup_{h\to\infty} F(u_h, A') \leq F(u, A)\,, \tag{2.51}$$

where $u_h := \rho_h * u$ for any $h \in \mathbb{N}$.

Let $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and let $A, A' \in \mathcal{A}_0$ be such that $A' \Subset A$. Moreover, let $h \in \mathbb{N}$ be such that $\frac{1}{h} < \mathrm{dist}^{\mathcal{R}}(A', \mathbb{G} \setminus A)$ and let us define $B_h := B(0, \frac{1}{h})$. We can notice that, for every $x \in A'$

$$u_h(x) = \int_{B_h} u(y^{-1} \cdot x)\rho_h(y)\, dy = \int_{B_h} \tau_y u(x) \rho_h(y)\, dy\,. \tag{2.52}$$

By (2.52), Lemma 1.2.5 and by the left-invariance of F, we get

$$\begin{aligned} F(u_h, A') &= F\left(\int_{B_h} \tau_y u\ \rho_h(y)\, dy, A'\right) \leq \int_{B_h} F(\tau_y u, A')\rho_h(y)\, dy \\ &= \int_{B_h} F(u, \tau_{y^{-1}} A')\rho_h(y)\, dy \leq \int_{B_h} F(u, A)\rho_h(y)\, dy = F(u, A) \end{aligned}$$

where the last inequality follows observing that $\tau_{y^{-1}}A' \subset A$ for each $y \in B_h$. Indeed, for any $x \in \tau_{y^{-1}}A'$, which means $y \cdot x \in A'$ in according with Definition 1.3.3, if $x \in \mathbb{G} \setminus A$ we would have

$$d^{\mathcal{R}}(x, y \cdot x) = |y|_{\mathbb{G}} < \frac{1}{h} < d^{\mathcal{R}}(A', \mathbb{G} \setminus A)\,,$$

which is impossible. Thus, taking the lim sup as $h \to \infty$, we get (2.51). □

The next result yields the lower semicontinuity of integral functionals of the form (2.50), under appropriate assumptions on the integrand. See [114] for the Euclidean case.

Theorem 2.2.11. *Let $f : \mathbb{R}^m \to [0, \infty]$ be a convex and lower semicontinuous function and let A be an open subset of $\mathbb{G}$. Then, the functional $F : W^{1,1}_{\mathbb{G};\mathrm{loc}}(A) \to [0, \infty]$, defined as*

$$F(u) := \int_A f(\nabla_{\mathbb{G}} u(x))\, dx\,,$$

is lower semicontinuous on $W^{1,1}_{\mathbb{G};\mathrm{loc}}(A)$ with respect to the topology induced by $L^1_{\mathrm{loc}}(A)$.

Proof. Let us fix A open subset of $\mathbb{G}$ and $u_h, u \in W^{1,1}_{\mathbb{G};\mathrm{loc}}(A)$ such that $u_h \to u$ in $L^1_{\mathrm{loc}}(A)$. We need to show that

$$F(u) \le \liminf_{h\to\infty} \int_A f(\nabla_{\mathbb{G}} u_h(x))\, dx$$

and, since F is inner regular, it is sufficient to show that

$$\int_{A'} f(\nabla_{\mathbb{G}} u(x))\, dx \le \liminf_{h\to\infty} \int_A f(\nabla_{\mathbb{G}} u_h(x))\, dx \quad \text{for any } A' \Subset A\,. \tag{2.53}$$

To this aim, let us fix $A' \Subset A$, $k \in \mathbb{N}$ such that $\frac{1}{k} < \mathrm{dist}^{\mathcal{R}}(A', \mathbb{G} \setminus A)$ and let us consider a sequence of mollifiers $\{\rho_k\}_k$. Moreover, let us denote $B_k := B\left(0, \frac{1}{k}\right)$. Then, by Lemma 1.2.5 and Proposition 2.2.8, and using similar arguments used in the proof of Theorem 2.2.10, we have

$$\begin{aligned}
\int_{A'} f(\nabla_{\mathbb{G}}(\rho_k * u_h)(x))\,dx &= \int_{A'} f((\rho_k * \nabla_{\mathbb{G}} u_h)(x))\,dx \\
&= \int_{A'} f\left(\int_{B_k} \nabla_{\mathbb{G}} u_h(y^{-1}\cdot x)\rho_k(y)\,dy\right)dx \\
&\le \int_{A'} \left(\int_{B_k} f(\nabla_{\mathbb{G}} u_h(y^{-1}\cdot x))\rho_k(y)\,dy\right)\,dx \\
&= \int_{B_k} \left(\int_{A'} f(\nabla_{\mathbb{G}} \tau_y u_h(x))\,dx\right)\rho_k(y)\,dy \\
&= \int_{B_k} \left(\int_{\tau_{y^{-1}}A'} f(\nabla_{\mathbb{G}} u_h(x))\,dx\right)\rho_k(y)\,dy \\
&\le \int_{B_k} \left(\int_{A} f(\nabla_{\mathbb{G}} u_h(x))\,dx\right)\rho_k(y)\,dy = \int_A f(\nabla_{\mathbb{G}} u_h(x))\,dx.
\end{aligned} \tag{2.54}$$

Let us now show that

$$\rho_k * u_h \to \rho_k * u \text{ in } C^\infty(\overline{A'}). \tag{2.55}$$

Recalling that $u_h \to u$ in $L^1_{\mathrm{loc}}(A)$ as $h\to\infty$, then, for each $\alpha, h \in \mathbb{N}$ and for every $x \in \overline{A'}$ and $j = 1,..,m$, it holds that

$$\begin{aligned}
\left|(X_j^\alpha(\rho_k * u_h) - X_j^\alpha(\rho_k * u))(x)\right| &= \left|X_j^\alpha(\rho_k * u_h - \rho_k * u)(x)\right| \\
&= \left|X_j^\alpha\left(\int_{B^{\mathcal{R}}\left(x,\frac{1}{k}\right)} (u_h(y)-u(y))\rho_k(x\cdot y^{-1})\,dy\right)\right| \\
&= \left|\int_{B^{\mathcal{R}}\left(x,\frac{1}{k}\right)} (u_h(y)-u(y))X_j^\alpha\rho_k(x\cdot y^{-1})\,dy\right| \\
&\le \int_{B^{\mathcal{R}}\left(x,\frac{1}{k}\right)} |u_h(y)-u(y)|\left|X_j^\alpha\rho_k(x\cdot y^{-1})\right|\,dy \\
&\le \|X_j^\alpha\rho_k\|_{L^\infty(A)}\int_{A'}|u_h(y)-u(y)|\,dy\,.
\end{aligned}$$

Passing to the supremum in $\overline{A'}$ and taking the limit as $h\to\infty$, we get (2.55). As a consequence, the sequence $\{\nabla_{\mathbb{G}}(\rho_k * u_h)\}_h$ uniformly converges to $\nabla_{\mathbb{G}}(\rho_k * u)$ in A', as $h\to\infty$. We can also notice that, by the lower semicontinuity of f, by (2.54) and applying Fatou's lemma, then

$$\int_{A'} f(\nabla_{\mathbb{G}}(\rho_k * u)(x))\,dx \le \liminf_{h\to\infty}\int_{A'} f(\nabla_{\mathbb{G}}(\rho_k * u_h)(x))\,dx \le \liminf_{h\to\infty}\int_A f(\nabla_{\mathbb{G}} u_h(x))\,dx\,. \tag{2.56}$$

Moreover, since $\nabla_{\mathbb{G}}(\rho_k * u)$ converges to $\nabla_{\mathbb{G}} u$ in $L^1(A')$, in according with Proposition 1.3.18, then, by the lower semicontinuity of f, Fatou's lemma and by (2.56) we finally get

$$\int_{A'} f(\nabla_{\mathbb{G}} u(x))\,dx \le \liminf_{k\to\infty} \int_{A'} f(\nabla_{\mathbb{G}}(\rho_k * u)(x))\,dx \le \liminf_{h\to\infty} \int_A f(\nabla_{\mathbb{G}} u_h(x))\,dx$$

and (2.53) holds. □

The main result of this section is the following representation theorem.

Theorem 2.2.12. *Let $1 \le p < \infty$ and let $\mathcal{A}_0$ the class of all bounded open subsets of $\mathbb{G}$. Let $F : L^p_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A}_0 \to [0,\infty]$ be an increasing functional satisfying the following properties:*

- (a) *F is local and left-invariant;*
- (b) *F is a measure;*
- (c) *F is convex and lower semicontinuous;*
- (d) *$F(u+c, A) = F(u, A)$ for each $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $A \in \mathcal{A}_0$ and $c \in \mathbb{R}$;*
- (e) *there exist two positive constants a, b such that*

$$0 \le F(u, A) \le \int_A (a + b\,|\nabla_{\mathbb{G}} u(x)|^p)\,dx$$

for each $u \in W^{1,1}_{\mathbb{G},\mathrm{loc}}(A)$, $A \in \mathcal{A}_0$.

Then, there exists a convex function $f : \mathbb{R}^m \to [0,\infty)$ such that:

- (i) *for each $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, for each $A \in \mathcal{A}_0$ with $u|_A \in W^{1,1}_{\mathbb{G},\mathrm{loc}}(A)$*

$$F(u, A) = \int_A f(\nabla_{\mathbb{G}} u(x))\,dx\,;$$

- (ii)

$$0 \le f(\eta) \le a + b\,|\eta|^p \quad \textit{for each } \eta \in \mathbb{R}^m.$$

Proof. We start defining the auxiliary function $u_\xi : \mathbb{G} \to \mathbb{R}$ as

$$u_\xi(x) := \langle \xi, \Pi(x) \rangle_{\mathbb{R}^m} \quad \text{for every } \xi \in \mathbb{R}^m, \tag{2.57}$$

where $\Pi : \mathbb{R}^n \to \mathbb{R}^m$ denotes the projection over the horizontal layer V_1, here identified with $\mathbb{R}^m$. By definition, u_ξ is smooth and

$$\nabla_{\mathbb{G}} u_\xi(x) = \xi \quad \text{for every } x \in \mathbb{G}. \tag{2.58}$$

Moreover

$$\tau_x u_\xi(y) = u_\xi(x^{-1} \cdot y) = \langle \xi, \Pi(x^{-1} \cdot y) \rangle_{\mathbb{R}^m} = \sum_{i=1}^m \xi_i (y_i - x_i) = u_\xi(y) - u_\xi(x) \tag{2.59}$$

for every $x, y \in \mathbb{G}$. Therefore, by the left-invariance of F, (2.59) and assumption (d), we get

$$\begin{aligned} F(u_\xi, B_\rho(0)) &= F(u_\xi(y), B_\rho(0)) = F(\tau_x u_\xi(y), \tau_x B_\rho(0)) \\ &= F(u_\xi(y) - c, B_\rho(x)) = F(u_\xi, B_\rho(x)) \end{aligned} \tag{2.60}$$

for every $x \in \mathbb{G}$. We stress that $c := u_\xi(x)$ is a constant with respect to y.

Now, in virtue of Theorem 2.2.5, there exists a function $f : \mathbb{G} \times \mathbb{R}^m \to [0, \infty)$ such that

$$F(u_\xi, B_\rho(x)) = \int_{B_\rho(x)} f(z, \nabla_{\mathbb{G}} u_\xi(z))\, dz\,.$$

Since, by Remark 1.3.12 $|B_\rho(x)| = |B_\rho(0)|$ for every $x \in \mathbb{G}$, then, by (2.60) and Lebesgue's differentiation theorem, taking the limit as $\rho \to 0^+$ we have

$$f(0, \xi) \leftarrow \frac{1}{|B_\rho(0)|} \int_{B_\rho(0)} f(z, \nabla_{\mathbb{G}} u_\xi(z))\, dz = \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} f(z, \nabla_{\mathbb{G}} u_\xi(z))\, dz \to f(x, \xi)$$

for every $x \in \mathbb{G}$. Therefore, considering the well-defined function $f_0 : \mathbb{R}^m \to [0, \infty)$, given by

$$f_0(\xi) := f(0, \xi) \quad \text{for every } \xi \in \mathbb{R}^m,$$

which inherits all the properties of f proved to hold in Theorem 2.2.5, i.e., f_0 is convex and

$$0 \le f_0(\xi) \le a + b|\xi|^p \quad \text{for every } \xi \in \mathbb{R}^m,$$

then we get

$$F(u, A) = \int_A f_0(\nabla_{\mathbb{G}} u(x))\, dx \tag{2.61}$$

for every $u \in \mathbf{C}^\infty(\mathbb{R}^n)$ and for every $A \in \mathcal{A}_0$.

It remains to show that the same representation (2.61) holds for every $u \in L^p_{\text{loc}}(\mathbb{R}^n)$ with $u|_A \in W^{1,1}_{\mathbb{G};\text{loc}}(A)$ (for every $A \in \mathcal{A}_0$).

Let $A' \in \mathcal{A}_0$ such that $A' \Subset A$ and let $\{\rho_h\}_h$ be a family of smooth mollifiers (here with $h = \frac{1}{\varepsilon}$). Hence, by (1.21), Fatou's lemma, the representation among smooth functions (2.61) and by Theorem 2.2.10, we get

$$\int_{A'} f_0(\nabla_{\mathbb{G}} u(x))\, dx \le \liminf_{h\to\infty} \int_{A'} f_0(\nabla_{\mathbb{G}} u_h(x))\, dx = \liminf_{h\to\infty} F(u_h, A') \le F(u, A)\,,$$

where $u_h := \rho_h * u$ for every $h \in \mathbb{N}$.

Therefore, by taking the supremum for $A' \Subset A$, we get

$$\int_A f_0(\nabla_{\mathbb{G}} u(x))\, dx \le F(u, A)\,. \tag{2.62}$$

We now proceed with the proof of the opposite inequality. Since, by Theorem 2.2.11, functional F is lower semicontinuous in $W^{1,1}_{\mathbb{G};\text{loc}}(A)$, then

$$F(u, A') \le \liminf_{h\to\infty} F(u_h, A')\,. \tag{2.63}$$

Now, as before, we denote by $B_h := B\left(0, \frac{1}{h}\right)$ for every $h \in \mathbb{N}$. Whenever $\frac{1}{h} < \text{dist}^{\mathcal{R}}(A', \mathbb{G}\setminus A)$ then, by Lemma 1.2.5 and the left-invariance of F, it holds that

$$\begin{aligned} F(u_h, A') &= \int_{A'} f_0\left(\int_{B_h} \nabla_{\mathbb{G}} u(y^{-1}\cdot x)\rho_h(y)\, dy\right) dx \\ &\le \int_{A'} \left(\int_{B_h} f_0(\nabla_{\mathbb{G}} u(y^{-1}\cdot x))\rho_h(y)\, dy\right) dx \\ &= \int_{B_h} \left(\int_{A'} f_0(\nabla_{\mathbb{G}} u(y^{-1}\cdot x))\, dx\right) \rho_h(y)\, dy \\ &\le \int_{B_h} \left(\int_{A} f_0(\nabla_{\mathbb{G}} u(x))\, dx\right) \rho_h(y)\, dy = \int_A f_0(\nabla_{\mathbb{G}} u(x))\, dx\,. \end{aligned} \tag{2.64}$$

We stress that the last inequality follows from the same argument used in the proof of Theorem 2.2.10.

Combining (2.63) with (2.64), we get

$$F(u, A') \leq \int_A f_0(\nabla_{\mathbb{G}} u(x))\, dx\,,$$

which in turn gives

$$F(u, A) \leq \int_A f_0(\nabla_{\mathbb{G}} u(x))\, dx\,, \tag{2.65}$$

by passing to the supremum for $A' \Subset A$. The thesis follows by (2.62) and (2.65). □

As for the classical case, we can prove that left-invariant functionals are uniquely determined on $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ by their prescription on a class of regular functions. First, let us recall a definition which will be useful in the sequel. See, for instance, [47, Chapter 15] for details.

Definition 2.2.13. Let X be a topological space and let $F : X \times \mathcal{A}_0 \to \overline{\mathbb{R}}$ be an increasing functional, in according with Definition 2.2.3. We define the *inner regular envelope* of F the increasing functional $F_- : X \times \mathcal{A}_0 \to \overline{\mathbb{R}}$ defined as

$$F_-(x, A) := \sup\{F(x, B) : B \in \mathcal{A}_0, B \Subset A\}$$

for every $x \in X$ and for every $A \in \mathcal{A}_0$.
Moreover, we define the *lower semicontinuous envelope* of F, $sc^-F : X \times \mathcal{A}_0 \to \overline{\mathbb{R}}$, as

$$(sc^-F)(x, A) := \sup_{U \in \mathcal{N}(x)} \inf_{y \in U} F(y, A)$$

for every $x \in X$ and for every $A \in \mathcal{A}_0$, where $\mathcal{N}(x)$ denotes the set of all open neighbourhoods of x in X and, finally, we define the *inner regular envelope of the lower semicontinuous envelope* of F as

$$\overline{F} := (sc^-F)_-\,.$$

Remark 2.2.14. If the functional F is increasing and lower semicontinuous, then F_- is also increasing, lower semicontinuous and inner regular. If F is just increasing, then sc^-F is still increasing and lower semicontinuous, but, in general, it is not inner regular, even if F is inner regular. Finally, $\overline{F}$ is the greatest increasing, inner regular and lower semicontinuous functional less than or equal to F. See e.g. [47, Example 15.11] for details.

Before stating the last result of this section, we need the following preliminary theorem.

Theorem 2.2.15. *Let $F : L^p_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A}_0 \to [0, \infty]$ be an increasing functional satisfying assumptions $(a) - (e)$ of Theorem 2.2.12 and let $f : \mathbb{R}^m \to [0, \infty)$ be as in Theorem 2.2.12. Moreover, let $\mathcal{F} : L^p_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A}_0 \to [0, \infty]$ be defined as*

$$\mathcal{F}(u, A) := \begin{cases} \int_A f(\nabla_{\mathbb{G}} u(x)) dx & \text{if } u \in W^{1,1}_{\mathbb{G};\mathrm{loc}}(A) \\ \infty & \text{otherwise} \end{cases}$$

and let $\overline{\mathcal{F}}$ be the inner regular envelope of the lower semicontinuous envelope of $\mathcal{F}$. Then

$$\overline{\mathcal{F}}(u, A) = \int_A f(\nabla_{\mathbb{G}} u(x))\, dx \tag{2.66}$$

for every $A \in \mathcal{A}_0$ and for every $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u|_A \in W^{1,1}_{\mathbb{G};\mathrm{loc}}(A)$ and also

$$F(u, A) = \overline{\mathcal{F}}(u, A) \tag{2.67}$$

for every $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and for every $A \in \mathcal{A}_0$.

Proof. By Theorem 2.2.11, the functional $\mathcal{F}$ is lower semicontinuous on $W^{1,1}_{\mathbb{G};\mathrm{loc}}(A)$ with respect to the topology induced by $L^1_{\mathrm{loc}}(A)$. Moreover, by Proposition 2.2.8, $\mathcal{F}$ is also left-invariant. Finally, it is easy to check that $\overline{\mathcal{F}}$ satisfies properties $(a) - (e)$ of Theorem 2.2.12 and, therefore, (2.66) directly follows.

Concerning (2.67), we first recall that $\overline{\mathcal{F}}$ is an increasing, inner regular and lower semicontinuous functional, which is also the greatest functional with these properties less than or equal to $\mathcal{F}$. Therefore, since

$$F(u, A) \leq \mathcal{F}(u, A) \quad \forall A \in \mathcal{A}_0,\, \forall u \in L^p_{\mathrm{loc}}(\mathbb{R}^n),$$

then we get

$$F(u, A) \leq \overline{\mathcal{F}}(u, A) \quad \forall A \in \mathcal{A}_0,\, \forall u \in L^p_{\mathrm{loc}}(\mathbb{R}^n).$$

To conclude the proof we need to show that the opposite inequality holds true as well. To this aim, let us consider $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $A, A' \in \mathcal{A}_0$ such that $A' \Subset A$. Moreover, let $\{\rho_h\}_h$ be a sequence of mollifiers.

Since $u_h := \rho_h * u$ is smooth, then, by Theorem 2.2.12 (see in particular (2.61)), we have

$$\mathcal{F}(u_h, A') = F(u_h, A') \quad \text{for every } h \in \mathbb{N}.$$

Now, the lower semicontinuity of $\overline{\mathcal{F}}$, implies that

$$\overline{\mathcal{F}}(u, A') \leq \liminf_{h\to\infty} \mathcal{F}(u_h, A') \leq \limsup_{h\to\infty} F(u_h, A') \leq F(u, A)\,,$$

by Theorem 2.2.10.

Finally, since $\overline{\mathcal{F}}$ is inner regular, then, taking the supremum among sets $A' \Subset A$, we get

$$\overline{\mathcal{F}}(u, A) \leq F(u, A)$$

for every $A \in \mathcal{A}_0$ and for every $u \in L^p_{\rm loc}(\mathbb{R}^n)$. □

As a direct consequence, we can finally prove the following result.

Theorem 2.2.16. *Let $F, G : L^p_{\rm loc}(\mathbb{R}^n) \times \mathcal{A}_0 \to [0, \infty]$ be two increasing functionals satisfying $(a) - (e)$ of Theorem 2.2.12. Let $\emptyset \neq A_0 \in \mathcal{A}_0$ and, for every $\xi \in \mathbb{R}^m$, let $u_\xi : \mathbb{G} \to \mathbb{R}$ be defined as in* (2.57)*. Moreover, assume that*

$$F(u_\xi, A_0) = G(u_\xi, A_0) \quad \textit{for every } \xi \in \mathbb{R}^m. \tag{2.68}$$

Then, $F = G$ on $L^p_{\rm loc}(\mathbb{R}^n) \times \mathcal{A}_0$.

Proof. By Theorem 2.2.12, there exist two convex functions $f, g : \mathbb{R}^m \to [0, \infty)$ such that

$$F(u, A) = \int_A f(\nabla_{\mathbb{G}} u(x))\, dx \quad \text{and} \quad G(u, A) = \int_A g(\nabla_{\mathbb{G}} u(x))\, dx$$

for every $A \in \mathcal{A}_0$ and for every $u \in L^p_{\rm loc}(\mathbb{R}^n)$ such that $u|_A \in W^{1,1}_{\mathbb{G};{\rm loc}}(A)$.

Moreover, by (2.58) and (2.68), we get

$$f(\xi)|A_0| = F(u_\xi, A_0) = G(u_\xi, A_0) = g(\xi)|A_0| \quad \text{for every } \xi \in \mathbb{R}^m.$$

The thesis follows by applying Theorem 2.2.15. □

In the last section of this chapter we will come back to the setting of Carnot group to prove a Γ-compactness result for classes of left-invariant functionals (see Theorem 2.3.33 for details).

2.3 Γ-convergence for functionals depending on vector fields

This is the main section of the thesis and it is devoted to studying Γ-convergence results for sequences of integral functionals depending on vector fields, in the strong topologies of $L^p(\Omega)$ and $W^{1,p}_X(\Omega)$, with $1 < p < \infty$. The main results of this section are Theorem 2.3.12 and Theorem 2.3.26. In the first one, we study Γ-compactness of sequences of integral functionals whose integrands belong to $I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, while the second result is an extension of Theorem 2.3.12 to the subclasses J_1 and J_2 made, respectively, of integrands in $I_{m,2}(\Omega, c_0, c_1)$ that are quadratic forms with respect to the second variable and integrands belonging to $I_{m,p}(\Omega, c_0, c_1)$ that are independent on the point.

In the sequel, functionals F and F_1, defined in (2.5) and (2.6), will be seen as *local functionals*. With a little abuse of notation, in according with [47, Chapter 15] and called $\mathcal{A}$ the class of open sets contained in Ω, let F, $F_1 : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ be defined as

$$F(u, A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A} \text{ and } u \in \mathbf{C}^1(A) \cap L^p(A) \\ \infty & \text{otherwise} \end{cases},$$

$$F_1(u, A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A} \text{ and } u \in W^{1,1}_{\text{loc}}(A) \cap L^p(A) \\ \infty & \text{otherwise} \end{cases}.$$

A first step in the study of the Γ-convergence consists on the characterization of the relaxed functionals of F and F_1. This is the topic of the next paragraph.

2.3.1 Characterization of relaxed functionals

We are going to characterize the *relaxed functionals* of F and F_1 in the strong topology of $L^p(\Omega)$. Following [32, 60], let us recall the definition of relaxation of a functional.

Definition 2.3.1. Let $G : L^p(\Omega) \to [0, \infty]$. The relaxed functional of G is defined as

$$\bar{G}(u) := \inf \left\{ \liminf_{h\to\infty} G(u_h) : \{u_h\}_h \subset L^p(\Omega),\ u_h \to u \text{ in } L^p(\Omega) \right\} \tag{2.69}$$

for any $u \in L^p(\Omega)$. It is well known that $\bar{G}$ is the greatest $L^p(\Omega)$-lower semicontinuous functional smaller than or equal to G and that it coincides with G in $\mathbf{C}^1(\Omega) \cap L^p(\Omega)$. See e.g. [32] for details.

The study of the relaxed functionals $\bar{F}$ and $\bar{F}_1$ dated back to [68] and it is summed up in the following result.

Theorem 2.3.2. *Let Ω be an open subset of $\mathbb{R}^n$, $1 < p < \infty$ and let $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$. Then*

(i) $\operatorname{dom} \bar{F} := \left\{ u \in L^p(\Omega) : \bar{F}(u) < \infty \right\} = W^{1,p}_X(\Omega)$;

(ii) $\bar{F}(u) = \int_\Omega f(x, Xu(x))\, dx$ *for every* $u \in W^{1,p}_X(\Omega)$;

(iii) $\bar{F}(u) = \bar{F}_1(u)$ *for each* $u \in L^p(\Omega)$.

Proof. Claims (i) and (ii) are proved in [68, Theorem 3.1.1] for a smaller class of integrands. We adapt that proof to our framework for the sake of completeness.

(i) Let $u \in \operatorname{dom} \bar{F}$. By (2.69), there exists a sequence $\{u_h\}_h \subset \mathbf{C}^1(\Omega) \cap L^p(\Omega)$ such that $u_h \to u$ in $L^p(\Omega)$. Moreover, by (2.7)

$$\liminf_{h\to\infty} \int_\Omega f(x, Xu_h(x))\, dx \le c_1 \liminf_{h\to\infty} \int_\Omega |Xu_h(x)|^p\, dx + \|a_1\|_{L^1(\Omega)} < \infty$$

and

$$c_0 \int_\Omega |Xu_h(x)|^p\, dx - \|a_0\|_{L^1(\Omega)} \le \int_\Omega f(x, Xu_h(x))\, dx < \infty\,.$$

Therefore, $\{u_h\}_h$ is bounded in $W^{1,p}_X(\Omega)$ and, by the reflexivity of the space, $u \in W^{1,p}_X(\Omega)$.

On the other hand, if $u \in W^{1,p}_X(\Omega)$ then, by Theorem 1.2.8, there exists a sequence $\{u_h\}_h \subset \mathbf{C}^1(\Omega) \cap W^{1,p}_X(\Omega)$ such that $u_h \to u$ in $W^{1,p}_X(\Omega)$. Thus, by the lower semicontinuity of $\bar{F}$ and since $\bar{F} = F$ in $\mathbf{C}^1(\Omega) \cap W^{1,p}_X(\Omega)$, it holds that

$$\bar{F}(u) \le \liminf_{h\to\infty} \bar{F}(u_h) = \liminf_{h\to\infty} F(u_h) = \liminf_{h\to\infty} \int_\Omega f(x, Xu_h(x))\, dx < \infty\,.$$

(ii) Let $F^* : L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$ denote the functional

$$F^*(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases}\,. \tag{2.70}$$

By [32, Theorem 2.3.1], F^* is $L^p(\Omega)$-lower semicontinuous and, therefore, $\bar{F}^* = F^*$. Moreover, since by definition $F^* \le F$ on $L^p(\Omega)$, then $F^* \le \bar{F}$ on $L^p(\Omega)$.

Let now $u \in W^{1,p}_X(\Omega)$. By Theorem 1.2.8, there exists $\{u_h\}_h \subset \mathbf{C}^1(\Omega) \cap W^{1,p}_X(\Omega)$ such that $u_h \to u$ in $W^{1,p}_X(\Omega)$ and, up to subsequences, $u_h \to u$ and $Xu_h \to Xu$ a.e. in Ω. Since, $\{f(\cdot, Xu_h)\}_h$ is bounded by a function $g \in L^p(\Omega)$ by (2.7) then, by dominated convergence theorem

$$\bar{F}(u) \le \lim_{h\to\infty} \bar{F}(u_h) = \lim_{h\to\infty} \int_\Omega f(x, Xu_h(x))\, dx = \int_\Omega f(x, Xu(x))\, dx = F^*(u)\,.$$

(iii) Let $u \in L^p(\Omega)$ and $\{u_h\}_h \subset \mathbf{C}^1(\Omega) \cap L^p(\Omega)$ be such that $u_h \to u$ in $L^p(\Omega)$. In particular, $\{u_h\}_h \subset W^{1,1}_{loc}(\Omega) \cap L^p(\Omega)$. Therefore

$$\bar{F}_1(u) \le \liminf_{h\to\infty} \bar{F}_1(u_h) = \liminf_{h\to\infty} \int_\Omega f(x, Xu_h(x))\, dx = \liminf_{h\to\infty} F(u_h)\,,$$

which implies

$$\bar{F}_1(u) \le \bar{F}(u)\,. \tag{2.71}$$

Let now F^* be as in (2.70) and let $u \in \operatorname{dom}(F_1)$. Then, by (2.7), it holds that

$$c_0 \int_\Omega |Xu(x)|^p\, dx - \|a_0\|_{L^1(\Omega)} \le F_1(u) < \infty\,.$$

Therefore $u \in W^{1,p}_X(\Omega)$, $\operatorname{dom} F_1 \subset W^{1,p}_X(\Omega)$, $F^* \le F_1$ on $L^p(\Omega)$ and, by lower semicontinuity of F^*, $F^* \le \bar{F}_1$ on $L^p(\Omega)$. Thus, by (2.71) and (ii), we have

$$F^* \le \bar{F}_1 \le \bar{F} = F^* \quad \text{on } L^p(\Omega),$$

which completes the proof. □

When $p = 1$, the domain of relaxed functional $\bar{F}$ gives rise to the space of functions of bounded variation function associated to X, $BV_X(\Omega)$. See [68, Theorem 3.2.3] for details.

Remark 2.3.3. From Theorem 2.3.2 we get the non-occurrence of Lavrentiev phenomenon for $\bar{F}$, as well as the existence of minimizers of suitable perturbation of $\bar{F}$ in $W^{1,p}_X(\Omega)$ and in $W^{1,p}_{X,\varphi}(\Omega)$ for any $\varphi \in W^{1,p}_X(\Omega)$ (see Section 2.3.5). We refer the interested reader to [39, 81] for more details about Lavrentiev phenomenon.

2.3.2 A quick overview on Γ-convergence

Let us now recall some notions and results concerning Γ-convergence's theory, which are contained in the fundamental monograph [47] and to which we will refer through this section. We also recommend monograph [29] as exhaustive account on this topic, containing also interesting applications of Γ-convergence.

Definition 2.3.4. Let (X, τ) be a topological space and let $\{F_h\}_h$ be a sequence of functionals from the space (X, τ) to $\bar{\mathbb{R}}$. For every $x \in X$, if $\mathcal{U}(x)$ denotes the family of open neighbourhoods of x, we define the Γ-*lower limit* and Γ-*upper limit* of the sequence $\{F_h\}_h$ in the topology τ as

$$(\Gamma(\tau)\text{-}\liminf_{h\to\infty} F_h)(x) := \sup_{U\in\mathcal{U}(x)} \liminf_{h\to\infty} \inf_{y\in U} F_h(y)\,,$$

$$(\Gamma(\tau)\text{-}\limsup_{h\to\infty} F_h)(x) := \sup_{U\in\mathcal{U}(x)} \limsup_{h\to\infty} \inf_{y\in U} F_h(y)$$

and, if $F : (X, \tau) \to \bar{\mathbb{R}}$, we say that $\{F_h\}_h$ Γ-*converges to* F *in the topology* τ or, more briefly, that $\{F_h\}_h$ $\Gamma(\tau)$-*converges to* F, at $x \in X$, if

$$(\Gamma(\tau)\text{-}\liminf_{h\to\infty} F_h)(x) = (\Gamma(\tau)\text{-}\limsup_{h\to\infty} F_h)(x) = F(x)$$

and we write

$$F(x) = (\Gamma(\tau)\text{-}\lim_{h\to\infty} F_h)(x)\,.$$

The following result will be useful in the sequel.

Theorem 2.3.5. *Let F_h and F be functionals from space (X,τ) to $\bar{\mathbb{R}}$, $h \in \mathbb{N}$.*

(i) ([47, Proposition 6.1]) *If $\{F_h\}_h$ $\Gamma(\tau)$-converges to F, then each of its subsequences still $\Gamma(\tau)$-converges to F.*

(ii) ([47, Proposition 6.3]) *Let τ_1 and τ_2 be two topologies on X such that τ_1 is weaker than τ_2. If $\{F_h\}_h$ $\Gamma(\tau_1)$-converges to F_1 and $\{F_h\}_h$ $\Gamma(\tau_2)$-converges to F_2, then $F_1 \leq F_2$.*

(iii) ([47, Theorem 7.8]) (Fundamental theorem of Γ-convergence) *Assume that $\{F_h\}_h$ is equicoercive (on X), that is, for each $t \in \mathbb{R}$ there exists a closed countably compact set $K_t \subset X$ such that*

$$\{x \in X : F_h(x) \leq t\} \subset K_t \quad \text{for each } h \in \mathbb{N}\,.$$

Let us also assume that $\{F_h\}_h$ $\Gamma(\tau)$-converges to F. Then, F is coercive and

$$\min_{x\in X} F(x) = \lim_{h\to\infty} \inf_{x\in X} F_h(x)\,.$$

(iv) ([47, Proposition 8.1]) *Assume that (X,τ) satisfies the first countability axiom. Then $\{F_h\}_h$ $\Gamma(\tau)$-converges to F if and only if the following two conditions hold:*

(1) (Γ-lim inf inequality) *for any $x \in X$ and for any sequence $\{x_h\}_h$ converging to x in X one has*

$$F(x) \leq \liminf_{h\to\infty} F_h(x_h)\,;$$

(2) (Γ-lim equality) *for any $x \in X$, there exists a sequence $\{\bar{x}_h\}_h$ converging to x in X such that*

$$F(x) = \lim_{h\to\infty} F_h(\bar{x}_h)\,.$$

(v) ([47, Theorem 8.5]) *Assume that (X,τ) satisfies the second countability axiom, that is, there is a countable base for the topology τ. Then, every sequence of functionals from X to $\bar{\mathbb{R}}$ has a $\Gamma(\tau)$-convergent subsequence.*

Remark 2.3.6. It is well-known that inequality in Theorem 2.3.5 (*ii*) can be strict, even in the case of a (infinite dimensional) Banach space. Take e.g. $\tau_1 \equiv$ weak topology of X and $\tau_2 \equiv$ strong topology of X (see [47, Example 6.6]). An instance of such a phenomenon can occur in the case of non-coercive quadratic integral functionals (see [1] for details).

Definition 2.3.7. Let $F_h : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$, $h \in \mathbb{N}$, be a sequence of increasing functionals. We say that $\{F_h\}_h$ $\bar{\Gamma}$-converges to $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$, and we will write $F = \bar{\Gamma}\text{-}\lim_{h\to\infty} F_h$, if F is increasing, inner regular and lower semicontinuous and the following conditions are satisfied:

(1) ($\bar{\Gamma}$-$\liminf$ inequality) for each $u \in L^p(\Omega)$, for every $A \in \mathcal{A}$ and $\{u_h\}_h \subset L^p(\Omega)$ converging to u in $L^p(\Omega)$, it holds

$$F(u, A) \leq \liminf_{h\to\infty} F_h(u_h, A)\,;$$

(2) ($\bar{\Gamma}$-$\limsup$ inequality) for each $u \in L^p(\Omega)$, for each A, $B \in \mathcal{A}$ with $A \Subset B$, there exists $\{\bar{u}_h\}_h \subset L^p(\Omega)$ converging to u in $L^p(\Omega)$ with

$$F(u, B) \geq \limsup_{h\to\infty} F_h(\bar{u}_h, A)\,.$$

Remark 2.3.8. Let us $F_h : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ be a sequence of increasing functionals and assume the existence of a measure functional $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that $\{F_h(\cdot, A)\}_h$ Γ-converges to $F(\cdot, A)$ for each $A \in \mathcal{A}$. Then $\{F_h\}_h$ $\bar{\Gamma}$-converges to F. Indeed, since F is a Γ-limit, it is lower semicontinuous (see [47, Proposition 6.8]) and it is also increasing and inner regular, because it is a measure. Inequalities (1) and (2) immediately follows by the characterization of the Γ-limit, given in Theorem 2.3.5 (*iv*).

Definition 2.3.9. Let $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ be a non-negative functional. We say that F satisfies the *fundamental estimate* if, for every $\varepsilon > 0$ and for every A', A'', $B \in \mathcal{A}$, with $A' \Subset A''$, there exists a constant $M > 0$ with the following property: for every u, $v \in L^p(\Omega)$, there exists a function $\varphi \in \mathbf{C}_c^\infty(A'')$, with $0 \leq \varphi \leq 1$ on A'', $\varphi = 1$ in a neighbourhood of A', such that

$$F\Big(\varphi u + (1-\varphi)v, A' \cup B\Big) \leq (1+\varepsilon)\Big(F(u, A'') + F(v, B)\Big) \\ + \varepsilon\Big(\|u\|^p_{L^p(S)} + \|v\|^p_{L^p(S)} + 1\Big) + M\|u-v\|^p_{L^p(S)},$$

where $S = (A'' \setminus A') \cap B$. Moreover, if $\mathcal{F}$ is a class of non-negative functionals on $L^p(\Omega) \times \mathcal{A}$, we say that the *fundamental estimate holds uniformly in* $\mathcal{F}$ if each element F of $\mathcal{F}$ satisfies the fundamental estimate with M depending only on ε, A', A'' and B, while φ may depend also on F, u, and v.

Remark 2.3.10. Let us recall that, if $F_h : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ are *measures*, then $F = \bar{\Gamma}\text{-}\lim_{h\to\infty} F_h$ need not be a measure (see [47, Examples 16.13 and 16.14]). In fact, if the sequence $\{F_h\}_h$ satisfies the fundamental estimates uniformly with respect to h, then F is a measure (see [47, Theorem 18.5]).

Let us now state a result which assures the coincidence between the $\bar{\Gamma}\text{-}\lim F_h$ and $\Gamma\text{-}\lim F_h$ for a sequence of local functional $F_h : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$, provided that the fundamental estimate holds uniformly for the sequence $\{F_h\}_h$ (see [47, Theorem 18.7] for details).

Theorem 2.3.11. *Let $\{F_h\}_h$ be a sequence of non-negative increasing functionals, defined on $L^p(\Omega) \times \mathcal{A}$, which $\bar{\Gamma}$-converges to a functional F. Assume the existence of two constants $c_1 \geq 1$ and $c_2 \geq 0$, a non-negative increasing functional $G : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ and a non-negative Radon measure $\mu : \mathcal{B}(\Omega) \to [0, \infty]$ such that*

$$G(u, A) \leq F_h(u, A) \leq c_1 G(u, A) + c_2\|u\|^p_{L^p(A)} + \mu(A)$$

for every $u \in L^p(\Omega)$, $A \in \mathcal{A}$ and $h \in \mathbb{N}$. Assume, in addition, that G is a lower semicontinuous measure and that the fundamental estimate holds uniformly for the sequence $\{F_h\}_h$. Then, $\{F_h(\cdot, A)\}_h$ Γ-converges in $L^p(\Omega)$ to $F(\cdot, A)$ for every $A \in \mathcal{A}$ such that $\mu(A) < \infty$.

2.3.3 Γ-compactness in the strong topologies of $L^p(\Omega)$ and $W^{1,p}_X(\Omega)$

The main result of this section is the following Γ-compactness theorem.

Theorem 2.3.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 < p < \infty$ and let X be an X-gradient satisfying (LIC) condition on Ω. Let $\{f_h\}_h \subset I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and, for any $h \in \mathbb{N}$, let $F^*_h : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ be the local functional defined as*

$$F^*_h(u, A) := \begin{cases} \int_A f_h(x, Xu(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}. \tag{2.72}$$

Then, up to a subsequence, there exist a local functional $F : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ and $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ such that:

(i)

$$F(\cdot, A) = \Gamma(L^p(\Omega))\text{-}\lim_{h\to\infty} F^*_h(\cdot, A) \quad \text{for each } A \in \mathcal{A}; \tag{2.73}$$

(ii) F admits the following representation

$$F(u, A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A}, u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}. \tag{2.74}$$

The proof of Theorem 2.3.12 requires several preliminary results. Let us begin to recall a fundamental result about representation of $\bar{\Gamma}$-limits with respect to a Euclidean integrand, namely [47, Theorem 20.3], which applies to a large class of integral functionals, defined as follows.

Definition 2.3.13. Let c_1, c_2, c_3 be positive constants and let $a \in L^1(\Omega)$ be nonnegative. We denote $\mathcal{H} = \mathcal{H}(p, c_1, c_2, c_3, a)$ the class of all local functionals $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ for which there exist two Borel functions f_e, $g : \Omega \times \mathbb{R}^n \to [0, \infty)$ such that:

(a) $$F(u, A) := \begin{cases} \int_A f_e(x, Du(x))dx & \text{if } A \in \mathcal{A}, u \in W^{1,1}_{\text{loc}}(A) \\ \infty & \text{otherwise} \end{cases};$$

(*b*) $g(x,\xi) \leq f_e(x,\xi) \leq c_1 g(x,\xi) + a(x)$;

(*c*) $0 \leq g(x,\xi) \leq c_2\,(|\xi|^p + 1)$;

(*d*) $g(x,\cdot)$ is convex on $\mathbb{R}^n$;

(*e*) $g(x,2\xi) \leq c_3\,(g(x,\xi) + 1)$

for every $u \in L^p(\Omega)$, $x \in \Omega$, $\xi \in \mathbb{R}^n$.

Theorem 2.3.14. *For every sequence $\{F_h\}_h \subset \mathcal{H}$ there exist a subsequence $\{F_{h_k}\}_k$ and an increasing functional $F : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ such that*

$$\{F_{h_k}\}_k \quad \bar{\Gamma}\text{-converges to } F.$$

Moreover, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0,\infty)$ such that

(*i*) *for a.e. $x \in \Omega$, $f_e(x,\cdot)$ is convex on $\mathbb{R}^n$;*

(*ii*) $0 \leq f_e(x,\xi) \leq a(x) + c_1\,c_2|\xi|^p$ *for a.e. $x \in \Omega$, for each $\xi \in \mathbb{R}^n$*

and

$$F(u,A) := \int_A f_e(x, Du(x))\,dx \tag{2.75}$$

for every $A \in \mathcal{A}$, for every $u \in L^p(\Omega)$ such that $u|_A \in W^{1,p}_{\rm loc}(A)$.

We recall an useful criterion for proving that a class of local functionals on $L^p(\Omega) \times \mathcal{A}$ satisfies the fundamental estimate uniformly [47, Theorem 19.4] and a $\bar{\Gamma}$-compactness result in this class [47, Theorem 19.5].

Theorem 2.3.15. *Let c_1, c_2, c_3, c_4 be nonnegative real numbers and let $\sigma : \mathcal{A} \to [0,\infty]$ be a superadditive increasing set function such that $\sigma(A) < \infty$ for each $A \Subset \Omega$. Moreover, let $\mathcal{F}' = \mathcal{F}'(p, c_1, c_2, c_3, c_4)$ be the class of all non-negative increasing local functionals F,* *defined on* $L^p(\Omega) \times \mathcal{A}$*, with the following properties:* F *is a measure and there exists a non-negative increasing local functional* $G: L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ *(depending on* F*) such that* G *is a measure and*

$$G(u, A) \leq F(u, A) \leq c_1 G(u, A) + c_2 \|u\|^p_{L^p(A)} + \sigma(A)\,; \tag{2.76}$$

$$G(\varphi u + (1 - \varphi)v, A) \leq c_4 \left(G(u, A) + G(v, A)\right) + c_3 c_4 \max_{\Omega} |D\varphi|^p \|u - v\|^p_{L^p(A)} + 2c_4 \sigma(A) \tag{2.77}$$

for every u, $v \in L^p(\Omega)$, $A \in \mathcal{A}$, $\varphi \in \mathbf{C}^\infty_c(\Omega)$ *with* $0 \leq \varphi \leq 1$*. Then, the fundamental estimate holds uniformly on* $\mathcal{F}'$*.*

Theorem 2.3.16. *Let* $\mathcal{F}' = \mathcal{F}'(p, c_1, c_2, c_3, c_4)$ *be the class of local functionals defined in Theorem 2.3.15. For every sequence* $\{F_h\}_h \subset \mathcal{F}'$*, there exists a subsequence* $\{F_{h_k}\}_k$ *which* $\bar{\Gamma}$*-converges to a lower semicontinuous functional* $F \in \mathcal{F}'$*.*

Let us now introduce some results concerning functionals depending on vector fields. Let us first prove a Γ-compactness result (see Theorem 2.3.18) for a class of local functionals on $L^p(\Omega) \times \mathcal{A}$ satisfying suitable growth conditions with respect to the local functional $\Psi_p: L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ defined as

$$\Psi_p(u, A) := \begin{cases} \int_A |Xu(x)|^p \, dx & \text{if } A \in \mathcal{A}, u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases} . \tag{2.78}$$

As a consequence, we will get a Γ-compactness result for a class of integral functionals represented with respect to Euclidean integrands, but still with growth condition with respect to Ψ_p (see Theorem 2.3.20). The former is an extension of [47, Theorem 19.6], the latter of [47, Theorem 20.4].

Lemma 2.3.17. *Let* $1 < p < \infty$*. Then* $\Psi_p: L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ *is a measure and lower semicontinuous.*

Proof. Let us start by proving that for any $A \in \mathcal{A}$ the function $u \to \Psi_p(u, A)$ is lower semicontinuous in $L^p(\Omega)$, i.e., for any $A \in \mathcal{A}$ and $\{u_h\}_h \subset L^p(\Omega)$, $u_h \to u$ in $L^p(\Omega)$, it satisfies

$$\Psi_p(u, A) \leq \liminf_{h\to\infty} \Psi_p(u_h, A). \tag{2.79}$$

We can assume $\liminf_{h\to\infty} \Psi_p(u_h, A) < \infty$ and, up to subsequences, we can also assume that $\lim_{h\to\infty} \Psi_p(u_h, A)$ exists. Hence $\{u_h\}_h$ is bounded in $W^{1,p}_X(A)$ and, since $W^{1,p}_X(A)$ is reflexive (recall Proposition 1.2.2 and that $p > 1$), then there exists a subsequence (not relabeled) such that $u_h \rightharpoonup u$ in $W^{1,p}_X(A)$ and, in particular, $Xu_h \rightharpoonup Xu$ in $L^p(A)$. Thus the conclusion follows, recalling the lower semicontinuity of the L^p-norm with respect to the weak convergence.

We now prove that for any $u \in L^p(\Omega)$ the function $\Psi_p(u, \cdot) : \mathcal{A} \to [0, \infty]$ is a measure, i.e., there exists a Borel measure $\mu_u : \mathcal{B}(\Omega) \to [0, \infty]$ such that $\Psi_p(u, A) = \mu_u(A)$ for every $A \in \mathcal{A}$. Since $\Psi_p(u, \cdot)$ is nonnegative, increasing and $\Psi_p(u, \emptyset) = 0$, by Remark 2.2.2, it suffices to prove that $\Psi_p(u, \cdot)$ is subadditive, superadditive and inner regular.

$\Psi_p(u, \cdot)$ is subadditive if for every $A, A_1, A_2 \in \mathcal{A}$, with $A \subseteq A_1 \cup A_2$, it holds

$$\Psi_p(u, A) \leq \Psi_p(u, A_1) + \Psi_p(u, A_2). \tag{2.80}$$

We can assume $u \in W^{1,p}_X(A_1) \cap W^{1,p}_X(A_2)$ and $A_1, A_2 \in \mathcal{A}$, otherwise the conclusion is trivial. Proposition 1.2.3 (*ii*) gives $u \in W^{1,p}_X(A_1 \cup A_2)$, therefore $\Psi_p(u, A_1 \cup A_2) = \int_{A_1\cup A_2} |Xu(x)|^p\, dx$ and (2.80) follows.

$\Psi_p(u, \cdot)$ is superadditive if for every $A, A_1, A_2 \in \mathcal{A}$, with $A_1 \cup A_2 \subseteq A$ and $A_1 \cap A_2 = \emptyset$

$$\Psi_p(u, A) \geq \Psi_p(u, A_1) + \Psi_p(u, A_2). \tag{2.81}$$

We can assume $u \in W^{1,p}_X(A)$ and $A \in \mathcal{A}$, otherwise the conclusion is trivial. Proposition 1.2.3 (*iv*) gives $u \in W^{1,p}_X(B)$ for any open set $B \subseteq A$. Let $A, A_1, A_2 \in \mathcal{A}$, $A_1 \cup A_2 \subseteq A$ and $A_1 \cap A_2 = \emptyset$. Then

$$\Psi_p(u, A_1) + \Psi_p(u, A_2) = \int_{A_1\cup A_2} |Xu(x)|^p\, dx \leq \int_A |Xu(x)|^p\, dx$$

and (2.81) follows.

$\Psi_p(u,\cdot)$ is finally inner regular if for every $A \in \mathcal{A}$

$$\Psi_p(u,A) = \sup\{\Psi_p(u,B) \mid B \in \mathcal{A},\ B \Subset A\}\,. \tag{2.82}$$

Let $M := \sup\{\Psi_p(u,B) \mid B \in \mathcal{A},\ B \Subset A\} \in [0,\infty]$. If $M = +\infty$, there exists $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{A}$, $B_i \Subset A$ such that $\Psi_p(u,B_i) \to \infty$ as $i \to \infty$ and the conclusion follows by observing that for all $i \in \mathbb{N}$, $\Psi_p(u,B_i) \le \Psi_p(u,A)$. If $M \in [0,\infty)$, then $\|u\|_{W^{1,p}_X(B)} \le M$ for any $B \in \mathcal{A}$, $B \Subset A$. Then, Remark (1.2.3) (iii) gives $u \in W^{1,p}_X(A)$ and, by definition, $\Psi_p(u,A) = \int_A |Xu(x)|^p\,dx$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E |Xu(x)|^p\,dx \le \varepsilon$ for any $E \in \mathcal{A}$ with $|E| \le \delta$. Let $B \Subset A$ such that $|A \setminus \overline{B}| \le \delta$, then

$$\int_A |Xu(x)|^p\,dx = \int_B |Xu(x)|\,dx + \int_{A\setminus\overline{B}} |Xu(x)|^p\,dx \le \int_B |Xu(x)|^p\,dx + \varepsilon$$

and the thesis follows. □

Theorem 2.3.18. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, $c_0 \le c_1$ be positive constants and let $a \in L^1(\Omega)$ be a nonnegative function. Denote by $\mathcal{M} = \mathcal{M}(p,c_0,c_1,a)$ the class of local functionals $F : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ such that F is a measure and*

$$c_0\,\Psi_p(u,A) \le\ F(u,A) \le\ c_1\left(\Psi_p(u,A) + \|u\|^p_{L^p(A)}\right) + \|a\|_{L^1(A)} \tag{2.83}$$

for every $u \in L^p(\Omega)$ and for every $A \in \mathcal{A}$. Then, the fundamental estimate holds uniformly in $\mathcal{M}$ and every sequence $\{F_h\}_h \subset \mathcal{M}$ has a subsequence $\{F_{h_k}\}_k$ which $\bar{\Gamma}$-converges to a functional F of the class $\mathcal{M}$. Moreover, $\{F_{h_k}(\cdot,A)\}_k$ Γ-converges to $F(\cdot,A)$ in the strong topology of $L^p(\Omega)$ and

$$\operatorname{dom} F(\cdot,A) := \{u \in L^p(\Omega) :\ F(u,A) <\ \infty\} = W^{1,p}_X(A) \tag{2.84}$$

for every $A \in \mathcal{A}$.

Proof. Let us begin to prove that the fundamental estimate holds uniformly in $\mathcal{M}$. Let

$$g(x,\xi) :=\ c_0\,|C(x)\xi|^p \quad \text{ if } x \in \Omega,\ \xi \in \mathbb{R}^n\,. \tag{2.85}$$

Notice that, since the entries of matrix $C(x)$ are Lipschitz continuous functions, then

$$g(x,\xi) \le c_0 \sup_\Omega \|C(x)\|^p \, |\xi|^p := c_2|\xi|^p \quad \text{if } x\in\Omega,\ \xi\in\mathbb{R}^n\,, \tag{2.86}$$

$$g(x,2\xi) = 2^{p-1}\, 2g(x,\xi) := c_3\, 2g(x,\xi) \quad \text{if } x\in\Omega,\ \xi\in\mathbb{R}^n \tag{2.87}$$

and

$$g(x,\cdot) \text{ is convex on } \mathbb{R}^n\,. \tag{2.88}$$

Thus, from (2.86), (2.87) and (2.88), arguing as in [47, (19.6)], it follows that

$$g(x, t\xi + (1-t)\eta + \zeta) \le c_3\left(g(x,\xi) + g(x,\eta) + c_2\,|\zeta|^p\right) \tag{2.89}$$

for every $x\in\Omega$, $t\in[0,1]$, ξ, $\eta\in\mathbb{R}^n$. In order to apply Theorem 2.3.15, observe that, choosing $G = c_0\,\Psi_p$ then, from (2.83), (2.76) immediately holds true with

$$c_1 \equiv \frac{c_1}{c_0},\ c_2\equiv c_1,\ \sigma(A) = \int_A a(x)\,dx\,.$$

Let us prove (2.77). By (2.89), it follows that

$$\begin{aligned}
G\left(\varphi u + (1-\varphi)v, A\right) &= \int_A g\left(x, \varphi Du + (1-\varphi)Dv + (u-v)D\varphi\right)dx \\
&\le \int_A c_3\left(g(x,Du) + g(x,Dv) + c_2|D\varphi|^p|u-v|^p\right)dx \\
&\le c_3\left(G(u,A) + G(v,A) + c_2\left(\max_\Omega|D\varphi|^p\right)\|u-v\|^p_{L^p(A)}\right)
\end{aligned}$$

for each u, $v\in L^p(\Omega)$, $A\in\mathcal{A}$, $\varphi\in\mathbf{C}^\infty_c(\Omega)$ with $0\le\varphi\le 1$. Thus (2.77) holds true with

$$c_4\equiv c_3 \text{ and } c_3c_4\equiv c_2c_3$$

and we get the desired conclusion.

From Theorem 2.3.16, every sequence $\{F_h\}_h\subset\mathcal{M}$ has a subsequence $\{F_{h_k}\}_k$ $\bar\Gamma$-converging to a functional $F: L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ which is a measure. As each functional F_h satisfies (2.83), the functional F satisfies (2.83), since Ψ_p is lower semicontinuous and inner regular by Lemma 2.3.17 and Remark 2.2.2. By applying Theorem 2.3.11, we get that $\{F_{h_k}(\cdot,A)\}_k$ Γ-converges to $F(\cdot,A)$ in $L^p(\Omega)$ for each $A\in\mathcal{A}$, since Ω is bounded. Finally, by (2.83), (2.84) follows. □

Definition 2.3.19. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, $c_0 \leq c_1$ be positive constants and let $a \in L^1(\Omega)$ be a nonnegative function. We denote by $\mathcal{I} = \mathcal{I}(p, c_0, c_1, a)$ the class of local functionals $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ for which there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$ such that

(i) claim (a) of properties defining $\mathcal{H}$ holds;

(ii) $c_0\, |C(x)\xi|^p \leq\, f_e(x, \xi) \leq\, c_1\, |C(x)\xi|^p + a(x)$ a.e. $x \in \Omega$, for each $\xi \in \mathbb{R}^n$.

Theorem 2.3.20. *For every sequence* $\{F_h\}_h \subset \mathcal{I}$ *there exist a subsequence* $\{F_{h_k}\}_k$ *and a measure functional* $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ *such that* $\{F_{h_k}(\cdot, A)\}_k$ Γ*-converges to* $F(\cdot, A)$ *in the strong topology of* $L^p(\Omega)$ *and* (2.84) *holds for every* $A \in \mathcal{A}$*. Moreover, there exists a Borel function* $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$*, convex in the second variable and satisfying* (ii) *of* $\mathcal{I}$*, for which* (2.75) *holds.*

Proof. By Theorem 2.3.18, for each $\{F_h\}_h \subset \mathcal{I}$ there exist a subsequence $\{F_{h_k}\}_k$ and an inner regular functional $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that $\{F_{h_k}(\cdot, A)\}_k$ Γ-converges to $F(\cdot, A)$ in $L^p(\Omega)$ for every $A \in \mathcal{A}$. Moreover, since Ψ_p is lower semicontinuous and inner regular, then

$$c_0\, \Psi_p(u, A) \leq\, F(u, A) \leq\, c_1 \Psi_p(u, A) + \int_A a(x)\, dx \tag{2.90}$$

for any $u \in L^p(\Omega)$ and $A \in \mathcal{A}$, where Ψ_p is the local functional in (2.78). If $g(x, \xi)$ is as in (2.85), then $\mathcal{I}(p, c_0.c_1, a) \subset \mathcal{H}(p, c_1, c_2, 2c_3, a)$ and, in virtue of Theorem 2.3.14, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$, also convex in the second variable, for which (2.75) holds.

Let us now prove that (ii) of properties defining $\mathcal{I}$ holds. Let $u_\xi(x) := \langle \xi, x \rangle_{\mathbb{R}^n}$ for any $x \in \Omega$. From (2.90), it follows that

$$c_0 \int_A |C(x)\xi|^p\, dx \leq \int_A f_e(x, \xi)\, dx \leq\, c_1 \int_A |C(x)\xi|^p\, dx + \int_A a(x)\, dx$$

for each $\xi \in \mathbb{R}^n$ and $A \in \mathcal{A}$. By means of the usual procedure, we can infer that there exists a negligible set $\mathcal{N} \subset \Omega$ such that, for each $x \in \Omega \setminus \mathcal{N}$,

$$c_0\, |C(x)\xi|^p \leq\, f_e(x, \xi) \leq\, c_1\, |C(x)\xi|^p + a(x) \quad \forall\, \xi \in \mathbb{Q}^n\, .$$

Finally, since $f_e(x,\cdot) : \mathbb{R}^n \to [0,\infty)$ is continuous a.e. $x \in \Omega$, then we can extend the previous inequality to any $\xi \in \mathbb{R}^n$. □

Theorem 2.3.21. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, $\{f_h\}_h \subset I_{m,p}(\Omega, c_0, c_1, 0, a)$ and, for each $h \in \mathbb{N}$, let $F_h^* : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ be the local functional defined in* (2.72)*. Then, there exist a subsequence $\{F_{h_k}^*\}_k$ and a measure functional $F : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ such that $\{F_{h_k}^*(\cdot, A)\}_k$ Γ-converges to $F(\cdot, A)$ in the strong topology of $L^p(\Omega)$ and* (2.84) *holds for every $A \in \mathcal{A}$. Moreover, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0,\infty)$, convex in the second variable, satisfying (ii) of properties defining $\mathcal{I}$, for which* (2.75) *holds.*

Proof. For any $h \in \mathbb{N}$, let $f_{e,h} : \Omega \times \mathbb{R}^n \to [0,\infty)$ denote the Euclidean integrand

$$f_{e,h}(x,\xi) := f_h(x, C(x)\xi) \quad x \in \Omega,\, \xi \in \mathbb{R}^n \tag{2.91}$$

and let $F_h : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ be the local functional defined by

$$F_h(u, A) := \begin{cases} \int_A f_{e,h}(x, Du(x))dx & \text{if } A \in \mathcal{A},\, u \in W^{1,1}_{\text{loc}}(A) \\ \infty & \text{otherwise} \end{cases} . \tag{2.92}$$

Thus, by (2.7) and (2.92), $\{F_h\}_h$ belongs to the class $\mathcal{I}$ and, in virtue of Theorem 2.3.20, there exist a subsequence $\{F_{h_k}\}_k$ and a measure functional $F : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ such that $\{F_{h_k}(\cdot, A)\}_k$ Γ-converges to $F(\cdot, A)$ in the strong topology of $L^p(\Omega)$, for every $A \in \mathcal{A}$. Moreover, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0,\infty)$, convex in the second variable, satisfying (ii) of properties defining $\mathcal{I}$, for which (2.75) holds.

If $\bar{F}_h(\cdot, A) : L^p(\Omega) \to [0,\infty]$ denotes the relaxed functional of $F_h(\cdot, A)$ with respect to the strong topology of $L^p(\Omega)$, then, by Theorem 2.3.2 (iii), it follows that

$$F_h^*(\cdot, A) = \bar{F}_h(\cdot, A) \quad \text{for each } h \in \mathbb{N}, A \in \mathcal{A}$$

and, in virtue of [47, Propostion 6.11], we finally get that $\{F_{h_k}^*(\cdot, A)\}_k$ Γ-converges to $F(\cdot, A)$ in the strong topology of $L^p(\Omega)$, for every $A \in \mathcal{A}$. □

Theorem 2.3.22. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let X be an X-gradient satisfying (LIC) condition on Ω. Let $\{f_h\}_h \subset I_{m,p}(\Omega, c_0, c_1, 0, a)$ and let $\{F_h^*\}_h$ be the sequence of local functionals defined in* (2.72)*. Assume that:*

(i) there exists a measure functional $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that, for each $A \in \mathcal{A}$, $\{F_h^*(\cdot, A)\}_h$ Γ-converges to $F(\cdot, A)$ in the strong topology of $L^p(\Omega)$;

(ii) there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$, convex in the second variable, satisfying (ii) of $\mathcal{I}$ and for which F admits the integral representation (2.75).

(iii) (2.84) *holds for every $A \in \mathcal{A}$.*

Then, there exists $f \in I_{m,p}(\Omega, c_0, c_1, 0, a)$ for which F admits representation (2.74).

Proof. Let us first notice that f_e satisfies the assumptions of Lemma 2.1.9, taking $b = c_1$. Thus, we can assume that f_e satisfies (2.26).

Let $f : \Omega \times \mathbb{R}^m \to [0, \infty)$ be defined as

$$f(x, \eta) := \begin{cases} f_e(x, L^{-1}(x)(\eta)) & \text{if } (x, \eta) \in \Omega_X \times \mathbb{R}^m \\ 0 & \text{otherwise} \end{cases}, \tag{2.93}$$

where $L^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is the map in (2.18). Let us show that $f \in I_{m,p}(\Omega, c_0, c_1, 0, a)$. Properties (I_1) and (I_2) directly follow from Theorem 2.1.5. Moreover, since f_ε satisfies (ii) of properties defining class $\mathcal{I}$, then (I_3) holds.

By Theorem 2.1.5 and Remark 2.1.7, F admits the integral representation (2.74), but only for functions $u \in W^{1,p}(A)$. We are going to extend this representation to all functions $u \in W^{1,p}_X(A)$, by means of Theorem 2.2.5. Since F a Γ-limit, then it is lower semicontinuous (see [47, Proposition 6.8]) and, by [47, Proposition 16.15], it is also local and, by assumptions, it is a measure.

Hence, assumptions (a), (b) and (c) of Theorem 2.2.5 are satisfied. Let us prove assumption (d). For every $h \in \mathbb{N}$, we have $F_h^*(u + c, A) = F_h^*(u, A)$ whenever $u \in L^p(\Omega)$, with $c \in \mathbb{R}$. Then, it is easy to see that this property also holds for the Γ-limit F.

Let us now prove assumption (e). By the integral representation (2.75) and Remark 2.1.7, it follows that, for each $A \in \mathcal{A}$, $u \in W^{1,p}(A)$

$$F(u,A) = \int_A f_e(x, Du(x))\,dx = \int_A f(x, Xu(x))\,dx \le \int_A (a(x) + c_1|Xu(x)|^p)\,dx\,, \quad (2.94)$$

which implies property (e).

Therefore, there exists a Borel function $f^* : \Omega \times \mathbb{R}^m \to [0,\infty)$ satisfying (i) and (ii) of Theorem 2.2.5. In particular, for each $A \in \mathcal{A}$ and $u \in W^{1,p}_X(A)$

$$F(u,A) = \int_A f^*(x, Xu(x))\,dx\,.$$

By (2.94) and Theorem 2.1.5, we get that $f(x,\eta) = f^*(x,\eta)$ for a.e. $x \in \Omega$ and for each $\eta \in \mathbb{R}^m$. This concludes the proof. □

We are now in the position to prove Theorem 2.3.12.

Proof of Theorem 2.3.12. For any $f_h \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, let $\varphi_h : \Omega \times \mathbb{R}^m \to [0,\infty)$ be defined as

$$\varphi_h(x,\eta) := f_h(x,\eta) + a_0(x) \quad \text{a.e. } x \in \Omega, \text{ for each } \eta \in \mathbb{R}^m \text{ and } h \in \mathbb{N}.$$

Then, $\{\varphi_h\}_h \subset I_{m,p}(\Omega, c_0, c_1, 0, a_0 + a_1)$ and if $\Phi^*_h : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ is the local functional defined by

$$\Phi^*_h(u,A) := \begin{cases} \int_A \varphi_h(x, Xu(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases},$$

then, in virtue of Theorems 2.3.21 and 2.3.22, (up to subsequences) there exist a local functional $\Phi : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ and $\varphi \in I_{m,p}(\Omega, c_0, c_1, 0, a_0 + a_1)$ such that

(i)

$$\Phi(\cdot, A) = \Gamma(L^p(\Omega))\text{-}\lim_{h\to\infty} \Phi^*_h(\cdot, A) \quad \text{for each } A \in \mathcal{A}\,;$$

(ii) Φ admits the following representation

$$\Phi(u,A) := \begin{cases} \int_A \varphi(x, Xu(x))dx & \text{if } A \in \mathcal{A}, u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}.$$

Let now $F: L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ be the local functional defined by

$$F(u,A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A}, u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}$$

where the integrand $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$, defined as

$$f(x,\eta) := \varphi(x,\eta) - a_0(x) \quad \text{a.e. } x \in \Omega, \text{ for each } \eta \in \mathbb{R}^m,$$

belongs to the class $I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ by construction. Hence, by (i) and in virtue of [47, Proposition 6.21], we finally get

$$F(\cdot, A) = \Gamma(L^p(\Omega))\text{-}\lim_{h\to\infty} F^*_h(\cdot, A) \quad \text{for each } A \in \mathcal{A}.$$

□

Following [47, Theorem 21.1], we prove the following Γ-compactness result for functionals including boundary conditions.

Theorem 2.3.23. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, let X be an X-gradient satisfying (LIC) condition on Ω, let $f_h, f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and, for each $h \in \mathbb{N}$, let $F_h, F: L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ be the local functionals defined by*

$$F_h(u,A) := \begin{cases} \int_A f_h(x, Xu(x))dx & \textit{if } A \in \mathcal{A}, \ u \in W^{1,p}_X(A) \\ \infty & \textit{otherwise} \end{cases} \tag{2.95}$$

and

$$F(u,A) := \begin{cases} \int_A f(x, Xu(x))dx & \textit{if } A \in \mathcal{A}, \ u \in W^{1,p}_X(A) \\ \infty & \textit{otherwise} \end{cases}.$$

Moreover, fixed $\varphi \in W^{1,p}_X(\Omega)$*, let* $F^\varphi_h, F^\varphi : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ *be the local functionals defined by*

$$F^\varphi_h(u, A) := \begin{cases} \int_A f_h(x, Xu(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_{X,\varphi}(A) \\ \infty & \text{otherwise} \end{cases}$$

and

$$F^\varphi(u, A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_{X,\varphi}(A) \\ \infty & \text{otherwise} \end{cases}. \tag{2.96}$$

Suppose that $\{F_h(\cdot, \Omega)\}_h$ Γ*-converges to* $F(\cdot, \Omega)$ *in the strong topology of* $L^p(\Omega)$*. Then,* $\{F^\varphi_h(\cdot, \Omega)\}_h$ Γ*-converges to* $F^\varphi(\cdot, \Omega)$ *in the strong topology of* $L^p(\Omega)$*.*

Proof. **Case 1.** Let us first show the result for $f_h, f : \Omega \times \mathbb{R}^m \to [0, \infty)$, $h \in \mathbb{N}$, belonging to the class $I_{m,p}(\Omega, c_0, c_1, 0, a)$, for a given nonnegative function $a \in L^1(\Omega)$.

Let $\varphi \in W^{1,p}_X(\Omega)$. By Theorem 2.3.18, there exist a subsequence $\{F_{h_k}\}_k$ of $\{F_h\}_h$ and a functional $G \in \mathcal{M}$ such that $\{F_{h_k}(\cdot, A)\}_k$ Γ-converges to $G(\cdot, A)$ in the strong topology of $L^p(\Omega)$ for every $A \in \mathcal{A}$. To conclude the proof of the first case, it is enough to show that $\{F^\varphi_{h_k}(\cdot, \Omega)\}_k$ Γ-converges to $G^\varphi(\cdot, \Omega)$ in the strong topology of $L^p(\Omega)$, where G^φ is defined by

$$G^\varphi(u, \Omega) := \begin{cases} G(u, \Omega) & \text{if } u \in W^{1,p}_{X,\varphi}(\Omega) \\ \infty & \text{otherwise} \end{cases}.$$

In fact, since $\{F_h(\cdot, \Omega)\}_h$ Γ-converges to $F(\cdot, \Omega)$ in $L^p(\Omega)$, then $F(\cdot, \Omega) = G(\cdot, \Omega)$ and $F^\varphi(\cdot, \Omega) = G^\varphi(\cdot, \Omega)$. Therefore, the Γ-limit of $\{F^\varphi_{h_k}(\cdot, \Omega)\}_h$ does not depend on the choice of the subsequence of $\{F^\varphi_h(\cdot, \Omega)\}_h$ and, in virtue of [47, Proposition 8.3], the whole sequence $\{F^\varphi_h(\cdot, \Omega)\}_h$ Γ-converges to $F^\varphi(\cdot, \Omega)$.

We divide the proof in two steps.

1st step. First, let us prove that

$$G^\varphi(u, \Omega) \geq (\Gamma\text{-}\limsup_{k\to\infty} F^\varphi_{h_k})(u, \Omega) \quad \text{for every } u \in L^p(\Omega). \tag{2.97}$$

Let us assume $u \in W^{1,p}_{X,\varphi}(\Omega)$, otherwise (2.97) is trivial. Since $\{F_{h_k}(\cdot,\Omega)\}_k$ Γ-converges to $G(\cdot,\Omega)$ in $L^p(\Omega)$ then, by Theorem 2.3.5 (*iv*), there exists $\{u_k\}_k \subset L^p(\Omega)$ converging to u in $L^p(\Omega)$ such that

$$G(u,\Omega) = \lim_{k\to\infty} F_{h_k}(u_k,\Omega)\,. \tag{2.98}$$

Therefore, $\{F_{h_k}(u_k,\Omega)\}_k$ is bounded and so $u_k \in W^{1,p}_X(\Omega)$ for any $k \in \mathbb{N}$.

Since $u \in W^{1,p}_X(\Omega)$, then for any $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that

$$\int_{\Omega\setminus K} (|Xu(x)|^p + |u(x)|^p + 1)\,dx < \varepsilon\,. \tag{2.99}$$

Let now $A', A'' \in \mathcal{A}$ be such that $K \subset A' \Subset A'' \Subset \Omega$ and let $B := \Omega \setminus K$. Then, by Theorem 2.3.18, there exist $M \geq 0$ and a sequence of cut-off functions between A' and A'', namely $\{\varrho_k\}_k$, such that for each $k \in \mathbb{N}$

$$\begin{aligned} F_{h_k}\Big(\varrho_k u_k + (1-\varrho_k)u,\Omega\Big) \leq\ &(1+\varepsilon)\Big(F_{h_k}(u_k,A'') + F_{h_k}(u,B)\Big) \\ &+ \varepsilon\Big(\|u_k\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} + 1\Big) + M\|u_k - u\|^p_{L^p(\Omega)}\,. \end{aligned}$$

If $w_k := \varrho_k u_k + (1-\varrho_k)u$, then $w_k \in W^{1,p}_{X,\varphi}(\Omega)$ for any $k \in \mathbb{N}$. Moreover, by the previous inequality and (2.83), it follows that

$$\begin{aligned} F^{\varphi}_{h_k}(w_k,\Omega) \leq (1+\varepsilon)&\Big(F_{h_k}(u_k,\Omega) + \tilde{c}_1 \int_B (|Xu|^p + |u|^p + 1)\,dx\Big) \\ &+ \varepsilon\left(\|u_k\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} + 1\right) + M\|u_k - u\|^p_{L^p(\Omega)}\,, \end{aligned}$$

where $\tilde{c}_1 := \max\{c_1, \|a\|_{L^1(B)}\}$. Thus, by (2.98), (2.99), by [47, Proposition 8.1] (*c*) and since $\{w_k\}_k$ converges to u in $L^p(\Omega)$, then

$$(\Gamma\text{-}\limsup_{k\to\infty} F^{\varphi}_{h_k})(u,\Omega) \leq \limsup_{k\to\infty} F^{\varphi}_{h_k}(w_k,\Omega) \leq (1+\varepsilon)\,(G(u,\Omega) + \tilde{c}_1\varepsilon) + \varepsilon\left(2\|u\|^p_{L^p(\Omega)} + 1\right).$$

Since $G(u,\Omega) = G^{\varphi}(u,\Omega)$, then, as ε goes to zero, we get (2.97).

2nd step. We conclude the proof of the first case by showing that

$$G^{\varphi}(u,\Omega) \leq (\Gamma\text{-}\liminf_{k\to\infty} F^{\varphi}_{h_k})(u,\Omega) \quad \text{for every } u \in L^p(\Omega). \tag{2.100}$$

Let $u \in L^p(\Omega)$ and assume that $(\Gamma\text{-}\liminf_{h\to\infty} F^{\varphi}_{h_k})(u,\Omega) < \infty$, otherwise the conclusion is trivial. Then, by [47, Proposition 8.1] (a), $\liminf_{h\to\infty} F^{\varphi}_{h_k}(u_k,\Omega) < \infty$ for a suitable sequence $\{u_k\}_k$ converging to u in $L^p(\Omega)$. It implies the existence of a subsequence $\{u_{k_j}\}_j$ of $\{u_k\}_k$ such that

$$\sup_{j\in\mathbb{N}} F^{\varphi}_{h_{k_j}}(u_{k_j},\Omega) < \infty$$

that is, $u_{k_j} \in W^{1,p}_{X,\varphi}(\Omega)$ and $\{u_{k_j}\}_j$ is bounded in $W^{1,p}_X(\Omega)$. Thus, by the reflexivity of the space $W^{1,p}_X(\Omega)$, $u \in W^{1,p}_X(\Omega)$ and $\{u_{k_j}\}_j$ converges to u weakly in $W^{1,p}_X(\Omega)$. Moreover, since $W^{1,p}_{X,\varphi}(\Omega)$ is closed in the weak topology of $W^{1,p}_X(\Omega)$ and $u_{k_j} \in W^{1,p}_{X,\varphi}(\Omega)$ for every $j \in \mathbb{N}$, then $u \in W^{1,p}_{X,\varphi}(\Omega)$, $G^{\varphi}(u,\Omega) = G(u,\Omega)$ and, since $F_{h_k}(\cdot,\Omega) \le F^{\varphi}_{h_k}(\cdot,\Omega)$, then

$$G^{\varphi}(u,\Omega) = G(u,\Omega) = (\Gamma\text{-}\lim_{k\to\infty} F_{h_k})(u,\Omega) \le (\Gamma\text{-}\liminf_{k\to\infty} F^{\varphi}_{h_k})(u,\Omega)$$

and (2.100) follows.

Case 2. Let $\bar{f}_h, \bar{f} : \Omega \times \mathbb{R}^m \to [0,\infty)$, $h \in \mathbb{N}$, be defined by

$$\bar{f}_h(x,\eta) := f_h(x,\eta) + a_0(x) \quad \text{and} \quad \bar{f}(x,\eta) := f(x,\eta) + a_0(x)\,.$$

By definition, $\{\bar{f}_h\}_h \subset I_{m,p}(\Omega, c_0, c_1, 0, a_0 + a_1)$ and, set $\mathcal{F}_h, \mathcal{F}^{\varphi}_h : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$, where

$$\mathcal{F}_h(u,A) := \begin{cases} \int_A \bar{f}_h(x, Xu(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}$$

and

$$\mathcal{F}^{\varphi}_h(u,A) := \begin{cases} \mathcal{F}_h(u,A) & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_{X,\varphi}(A) \\ \infty & \text{otherwise} \end{cases},$$

then, in virtue of [47, Proposition 6.21], $\{\mathcal{F}_h(\cdot,\Omega)\}_h$ Γ-converges to $\mathcal{F}(\cdot,\Omega)$, defined by

$$\mathcal{F}(u,A) := \begin{cases} \int_A \bar{f}(x, Xu(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}$$

and, by Case 1, $\{\mathcal{F}_h^\varphi(\cdot,\Omega)\}_h$ Γ-converges to $\mathcal{F}^\varphi(\cdot,\Omega)$ in the strong topology of $L^p(\Omega)$.

Finally, since

$$F^\varphi(u,\Omega) = \mathcal{F}_h^\varphi(\cdot,\Omega) - \|a_0\|_{L^1(\Omega)}\,,$$

then the conclusion follows by [47, Proposition 6.21]. □

We conclude this section dealing with functionals $F : W^{1,p}_X(\Omega) \to \mathbb{R}$ of the form

$$F(u) := \int_\Omega f(x, Xu(x))\,dx\,, \tag{2.101}$$

with Ω bounded open subset of $\mathbb{R}^n$, $p > 1$ and $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, and showing that the pointwise convergence of a sequence of functionals as in (2.101) is equivalent to the Γ-convergence of the sequence in the strong topology of $W^{1,p}_X(\Omega)$.

Proposition 2.3.24. *Let $\{f_h\}_h$ and f be functions in $I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, for any $h \in \mathbb{N}$, and let F_h, $F : W^{1,p}_X(\Omega) \to \mathbb{R}$ be the corresponding integral functionals as in (2.101). Then*

$$F_h \to F \text{ (pointwise) in } W^{1,p}_X(\Omega) \quad \text{it and only if} \quad \{F_h\}_h\ \Gamma\text{-converges to } F$$

in the strong topology of $W^{1,p}_X(\Omega)$, i.e.,

$$F(u) = \ (\Gamma(W^{1,p}_X(\Omega))\text{-}\lim_{h\to\infty} F_h)(u) \quad \forall\, u \in W^{1,p}_X(\Omega)\,.$$

Proof. Let $\{f_h\}_h$, $h \in \mathbb{N}$, be a sequence of functions in $I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and let $\{F_h\}_h$ be the sequence of the corresponding integral functionals. Then, by (I_2), F_h is convex on $W^{1,p}_X(\Omega)$ for any $h \in \mathbb{N}$ and, by (I_3), $\{F_h\}_h$ is equibounded in any neighbourhood of any $u \in W^{1,p}_X(\Omega)$.

Therefore, by [47, Proposition 5.11], $\{F_h\}_h$ is equicontinuous in $W^{1,p}_X(\Omega)$ and, in virtue of [47, Proposition 5.9], we get the thesis. □

2.3.4 Two important subclasses

Definition 2.3.25. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 < p < \infty$ and let $c_0 \le c_1$ be positive constants. We define:

- $J_1 \equiv J_1(\Omega, c_0, c_1)$ the subclass of $I_{m,2}(\Omega, c_0, c_1)$ composed of integrands $f \in I_{m,2}(\Omega, c_0, c_1)$ which are quadratic forms with respect to η, that is,

$$f(x, \eta) = \langle a(x)\eta, \eta\rangle_{\mathbb{R}^m} = \sum_{i,j=1}^{m} a_{ij}(x)\eta_i\eta_j \quad \text{a.e. } x \in \Omega, \forall\, \eta \in \mathbb{R}^m ,$$

  with $a(x) = [a_{ij}(x)]$ $m \times m$ symmetric matrix.

- $J_2 \equiv J_2(\Omega, c_0, c_1)$ the subclass of integrands $f \in I_{m,p}(\Omega, c_0, c_1)$ such that $f = f(\eta)$, that is, f is independent of x.

Theorem 2.3.26. *Let X be an X-gradient satisfying (LIC) condition on Ω and, for $i = 1, 2$, let $\{f_h\}_h \subset J_i(\Omega, c_0, c_1)$ and $\{F_h^*\}_h$ be the sequence of local functionals defined in* (2.72). *Then, up to a subsequence, there exist a local functional $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ and $f \in J_i(\Omega, c_0, c_1)$ $(i = 1, 2)$ such that*

(i) (2.73) *holds;*

(ii) F *admits representation* (2.74).

Proof. **1st case.** Let us first show the conclusion for the subclass J_1.

Let $\{f_h\}_h \subset J_1$. By definition, we can assume that

$$f_h(x, \eta) := \langle a_h(x)\eta, \eta\rangle_{\mathbb{R}^m} \quad x \in \Omega,\ \eta \in \mathbb{R}^m ,$$

where $a_h(x) = [a_{h,ij}(x)]$ is a $m \times m$ symmetric matrix satisfying

$$c_0\, |\eta|^2 \le \langle a_h(x)\eta, \eta\rangle_{\mathbb{R}^m} \le c_1 \left(|\eta|^2 + 1\right) \text{ a.e. } x \in \Omega,\ \forall \eta \in \mathbb{R}^m \tag{2.102}$$

$$a_{h,ij} \in L^\infty(\Omega) \text{ for each } i, j = 1, \dots, m,\ h \in \mathbb{N} . \tag{2.103}$$

By Theorem 2.3.12, there exist $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ and $f \in I_{m,2}(\Omega, c_0, c_1)$ such that, up to subsequences, (2.73) holds and F admits representation (2.74). Let us show that

$$f \in J_1 . \tag{2.104}$$

By previous considerations, we can assume that F admits representation (2.75) with

$$f_e(x,\xi) := f(x, C(x)\xi) \text{ for a.e. } x \in \Omega, \text{ for each } \xi \in \mathbb{R}^n .$$

Moreover, by Theorem 2.1.5 (see (2.19) and (2.23)), it also holds the opposite representation, that is, for each $x \in \Omega_X$,

$$f(x,\eta) = f_e(x, L_x^{-1}(\eta)) \quad \forall\, \eta \in \mathbb{R}^m , \tag{2.105}$$

with

$$L_x^{-1}(\eta) := C(x)^T B(x)^{-1}\eta .$$

Let us now consider the sequence of Euclidean integrands

$$\begin{aligned} f_{e,h}(x,\xi) :&= f_h(x, C(x)\xi) = \langle a_h(x)C(x)\xi, C(x)\xi\rangle_{\mathbb{R}^m} \\ &= \langle C(x)^T a_h(x) C(x)\xi, \xi\rangle_{\mathbb{R}^m} = \langle a_{e,h}(x)\xi, \xi\rangle_{\mathbb{R}^m} \end{aligned}$$

and the related local functionals $F_h : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ defined in (2.92). Since for each $u \in W^{1,1}_{\mathrm{loc}}(A)$ $F_h(u,A) = F_h^*(u,A)$, by using well-known results of Γ-convergence for quadratic functionals (see [47, Theorem 22.1] and Remark 2.3.8), then there exists a $n \times n$ symmetric matrix $a_e = [a_{e,ij}]$, with $a_{e,ij} \in L^\infty(\Omega)$ for each $i, j = 1, \ldots, n$ such that

$$f_e(x,\xi) = \langle a_e(x)\xi, \xi\rangle_{\mathbb{R}^n} \text{ a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^n .$$

By (2.105), for each $x \in \Omega_X$,

$$\begin{aligned} f(x,\eta) :&= f_e(x, L_x^{-1}(\eta)) = \langle a_e(x) C(x)^T B(x)^{-1}\eta, C(x)^T B(x)^{-1}\eta\rangle_{\mathbb{R}^m} \\ &= \langle (B(x)^{-1})^T C(x) a_e(x) C(x)^T B(x)^{-1}\eta, \eta\rangle_{\mathbb{R}^m} = \langle a(x)\eta, \eta\rangle_{\mathbb{R}^m} \end{aligned} \tag{2.106}$$

with

$$a(x) := (B(x)^{-1})^T C(x) a_e(x) C(x)^T B(x)^{-1}$$

$m \times m$ symmetric matrix. Then $f(x,\cdot)$ turns out to be a quadratic form on $\mathbb{R}^m$, induced by the matrix $a(x)$ for a.e. $x \in \Omega$. Thus (2.104) follows.

2nd case. Let $\{f_h\}_h \subset J_2$ and notice that, for each $h \in \mathbb{N}$, $f_h : \mathbb{R}^m \to [0, \infty)$ is a sequence of locally bounded, convex functions. Thus, by a well-known result (see, for instance, [47, Proposition 5.11]), $\{f_h\}_h$ is also locally equilipschitz continuous. From Ascoli-Arzelà's Theorem, we can assume that, up to subsequences, there exists $f \in J_2$ such that

$$f_h \to f \text{ uniformly on bounded sets of } \mathbb{R}^n \text{ as } h \to \infty\,. \tag{2.107}$$

Let us define $\tilde{F} : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ as

$$\tilde{F}(u, A) := \begin{cases} \int_A f(Xu(x))dx & \text{if } A \in \mathcal{A}, u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}$$

and let us prove that, for each $A \in \mathcal{A}$,

$$\lim_{h\to\infty} F^*_h(u, A) = \tilde{F}(u, A) \quad \forall\, u \in W^{1,p}_X(A)\,, \tag{2.108}$$

where F^*_h is defined in (2.72). Let us fix $A \in \mathcal{A}$ and $u \in W^{1,p}_X(A)$. Since $|Xu(x)| < \infty$ for a.e. $x \in A$, by (2.107), it follows that

$$\lim_{h\to\infty} f_h(Xu(x)) = f(Xu(x)) \text{ for a.e. } x \in A\,. \tag{2.109}$$

On the other hand, as

$$0 \le f_h(Xu(x)) \le c_1(1 + |Xu(x)|^p) \quad \text{for a.e. } x \in A, \text{for each } h \in \mathbb{N}\,,$$

by (2.109) and the dominated convergence theorem, (2.108) follows. Let us show that

$$F(u, A) = \tilde{F}(u, A) \quad \forall\, A \in \mathcal{A}, \forall\, u \in L^p(\Omega)\,, \tag{2.110}$$

in order to get our desired conclusion. By (2.84), it is sufficient to prove (2.110) for each $A \in \mathcal{A}$ and for each $u \in W^{1,p}_X(A)$. Inequality

$$F(u, A) \le \tilde{F}(u, A) \quad \forall\, A \in \mathcal{A}, \forall\, u \in W^{1,p}_X(A) \tag{2.111}$$

follows by noticing that, for each $u \in W^{1,p}_X(A)$, Γ- lim inf inequality and (2.108) implies that

$$F(u,A) \le \liminf_{h\to\infty} F^*_h(u,A) = \tilde F(u,A)\,.$$

Let us now prove the opposite inequality

$$F(u,A) \ge \tilde F(u,A) \quad \forall\, A \in \mathcal{A},\, \forall\, u \in W^{1,p}_X(A)\,. \tag{2.112}$$

Let us first recall that, for each $A \in \mathcal{A}$, by (2.108) and Proposition 2.3.24,

$$\tilde F(u,A) = (\Gamma(W^{1,p}_X(A))\text{-}\lim_{h\to\infty} F^*_h)(u) \quad \forall\, u \in W^{1,p}_X(A)\,. \tag{2.113}$$

Fix $A \in \mathcal{A}$ and let $u \in L^p(\Omega)$ with $u|_A \in W^{1,p}_X(A)$. By Γ- lim equality, there exists a sequence $\{u_h\}_h \subset L^p(\Omega)$ such that

$$u_h \to u \text{ in } L^p(\Omega), \text{ as } h \to \infty \tag{2.114}$$

and

$$\lim_{h\to\infty} F^*_h(u_h,A) = F(u,A) < \infty\,. \tag{2.115}$$

By (2.115), we can assume that

$$\{u_h|_A\}_h \subset W^{1,p}_X(A)\,. \tag{2.116}$$

Let $A' \in \mathcal{A}$ with $A' \Subset A$. From Proposition 1.2.9 (ii), if $w := \overline{Xu_h} : \mathbb{R}^n \to \mathbb{R}^m$, that is, $\overline{Xu_h} = Xu_h$ on A and $\overline{Xu_h} = 0$ outside, then, for each $0 < \varepsilon < \operatorname{dist}(A', \mathbb{R}^n \setminus A)$

$$\int_{A'} f_h(\rho_\epsilon * \overline{Xu_h})\,dx \le \int_A f_h(\overline{Xu_h})\,dx \text{ for each } h \in \mathbb{N}\,. \tag{2.117}$$

By (2.114), (2.116) and Proposition 1.2.9 (i), for given $0 < \varepsilon < \operatorname{dist}(A', \mathbb{R}^n \setminus A)$,

$$X(\rho_\epsilon * \bar u_h) \to X(\rho_\epsilon * \bar u) \text{ uniformly on } A' \text{ as } h \to \infty \tag{2.118}$$

and

$$\rho_\epsilon * \overline{Xu_h} \to \rho_\epsilon * \overline{Xu} \text{ uniformly on } A' \text{ as } h \to \infty\,. \tag{2.119}$$

In particular,

$$\rho_\varepsilon * \bar{u}_h \to \rho_\varepsilon * \bar{u} \text{ in } W^{1,p}_X(A') \text{ as } h \to \infty. \tag{2.120}$$

Observe now that, by (2.117), for each $0 < \varepsilon < \operatorname{dist}(A', \mathbb{R}^n \setminus A)$ and for each $h \in \mathbb{N}$,

$$\begin{aligned} F^*_h(\rho_\epsilon * \bar{u}_h, A') &= \int_{A'} f_h(X(\rho_\epsilon * \bar{u}_h)(x))\,dx \\ &= \int_{A'} f_h(\rho_\epsilon * \overline{Xu_h}(x))\,dx + \int_{A'} \big(f_h(X(\rho_\epsilon * \bar{u}_h)(x)) - f_h(\rho_\epsilon * \overline{Xu_h}(x))\big)\,dx \\ &\le \int_{A} f_h(Xu_h(x))\,dx + \int_{A'} \big(f_h(X(\rho_\epsilon * \bar{u}_h)(x)) - f_h(\rho_\epsilon * \overline{Xu_h}(x))\big)\,dx \\ &=: F^*_h(u_h, A) + R_{\epsilon,h}. \end{aligned} \tag{2.121}$$

From (2.107), (2.118) and (2.119), it follows that, for given $0 < \varepsilon < \operatorname{dist}(A', \mathbb{R}^n \setminus A)$

$$\lim_{h\to\infty} R_{\epsilon,h} = R_\epsilon := \int_{A'} \big(f(X(\rho_\epsilon * \bar{u})(x)) - f(\rho_\epsilon * \overline{Xu}(x))\big)\,dx\,. \tag{2.122}$$

For given $0 < \varepsilon < \operatorname{dist}(A', \mathbb{R}^n \setminus A)$, by (2.113), (2.115), (2.120), and (2.122), passing to the limit in (2.121) as $h \to \infty$, it follows that

$$\begin{aligned} \tilde{F}(\rho_\varepsilon * \bar{u}, A') &\le \liminf_{h\to\infty} F^*_h(\rho_\varepsilon * \bar{u}_h, A') \\ &\le \lim_{h\to\infty} F^*_h(u_h, A) + \lim_{h\to\infty} R_{\epsilon,h} = F(u, A) + R_\epsilon. \end{aligned} \tag{2.123}$$

Let us notice that, since f is continuous,

$$X(\rho_\epsilon * \bar{u}) \to Xu \text{ and } \rho_\epsilon * \overline{Xu} \to Xu \text{ in } L^p(A'), \text{ as } \epsilon \to 0^+$$

and

$$f(X(\rho_\epsilon * \bar{u})) \le c_1(1 + |X(\rho_\epsilon * \bar{u})|^p) \text{ and } f(\rho_\epsilon * \overline{Xu}) \le c_1(1 + |\rho_\epsilon * \overline{Xu}|^p) \text{ a.e. in } A'\,.$$

Therefore, in virtue of Vitali's convergence theorem,

$$\lim_{\epsilon\to 0^+} R_\epsilon = 0\,. \tag{2.124}$$

By the semicontinuity of $\tilde{F}$, with respect to the L^p-topology, and by (2.124), we can pass to the limit as $\varepsilon \to 0^+$ in (2.123) and we get

$$\tilde{F}(u, A') \leq \lim_{\varepsilon \to 0^+} \tilde{F}(\rho_\varepsilon * \bar{u}, A') \leq F(u, A) \text{ for each } A' \Subset A\,. \tag{2.125}$$

Finally, taking the supremum in (2.125) on all $A' \in \mathcal{A}$ with $A' \Subset A$, we get (2.112). □

2.3.5 Convergence of minima and minimizers

Let $\{f_h\}_h \subset I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and let $F_h(\cdot, \Omega) : L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$ be the local functionals defined in (2.72). In virtue of Theorem 2.3.12, there exists $F : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ such that, up to subsequences,

$$F(\cdot, A) = \Gamma(L^p(\Omega))\text{-}\lim_{h \to \infty} F_h(\cdot, A)$$

and F admits the representation (2.74). If, in addition, $\{F_h(\cdot, A)\}_h$ is equicoercive in the strong topology of $L^p(\Omega)$ for each $A \in \mathcal{A}$, then, in virtue of Theorem 2.3.5 (*iii*), $F(\cdot, A)$ attains its minimum in $L^p(\Omega)$ and

$$\min_{u \in L^p(\Omega)} F(u, A) = \lim_{h \to \infty} \inf_{u \in L^p(\Omega)} F_h(u, A) = \lim_{h \to \infty} \min_{u \in L^p(\Omega)} F_h(u, A).$$

Let us now study related minimum problems and the convergence of minima and minimizers in the weak topology of $W^{1,p}_X(\Omega)$ and in the strong topology of $L^p(\Omega)$.

Theorem 2.3.27. *Let $\Omega \Subset \Omega_0$ be a bounded open set, $1 < p < \infty$, let $X = (X_1, \ldots, X_m)$ be a family of Lipschitz continuous vector fields defined on Ω_0 and satisfying conditions (H1), (H2) and (H3). Moreover, let $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that there exist two positive constants $d_0 \leq d_1$ and two nonnegative functions $b_0, b_1 \in L^1(\Omega)$ such that*

$$d_0|s|^p - b_0(x) \leq g(x, s) \leq d_1|s|^p + b_1(x) \tag{2.126}$$

for a.e $x \in \Omega$ and for every $s \in \mathbb{R}$. Finally, let $F, G : W^{1,p}_X(\Omega) \to \mathbb{R}$ be the functionals defined, respectively, by

$$F(u) := \int_\Omega f(x, Xu(x))\, dx \quad \text{and} \quad G(u) := \int_\Omega g(x, u(x))\, dx\,,$$

and, fixed $\varphi \in W^{1,p}_X(\Omega)$, let $\mathbb{1}_\varphi : W^{1,p}_X(\Omega) \to \{0; \infty\}$ be the indicator function of $W^{1,p}_{X,\varphi}(\Omega)$ and let $\Xi, \Xi^\varphi : W^{1,p}_X(\Omega) \to \mathbb{R}$ be defined, respectively, by

$$\Xi := F + G \quad \text{and} \quad \Xi^\varphi := F + G + \mathbb{1}_\varphi.$$

Then, the minimum problems

$$\min_{u \in W^{1,p}_X(\Omega)} \Xi(u) \tag{2.127}$$

and

$$\min_{u \in W^{1,p}_{X,\varphi}(\Omega)} \Xi^\varphi(u) \tag{2.128}$$

have at least a solution. Moreover,

$$\min_{u \in W^{1,p}_X(\Omega)} \Xi(u) = \inf_{u \in \mathbf{C}^1(\Omega) \cap W^{1,p}_X(\Omega)} \Xi(u) \tag{2.129}$$

and, if in addition $g(x, \cdot)$ is strictly convex on $\mathbb{R}$ for a.e. $x \in \Omega$, then (2.127) and (2.128) have exactly one solution.

Before giving the proof of Theorem 2.3.27, we need two preliminary results.

Lemma 2.3.28. *Under the hypotheses of Theorem 2.3.27, functional G is sequentially lower semicontinuous in the weak topology of $W^{1,p}_X(\Omega)$.*

Proof. Let $\{u_h\}_h \subset W^{1,p}_X(\Omega)$ and $u \in W^{1,p}_X(\Omega)$ be such that $u_h \to u$ in the weak topology of $W^{1,p}_X(\Omega)$ and, in virtue of Proposition 1.2.17, in the strong topology of $L^p_{\text{loc}}(\Omega)$. The thesis follows from the lower semicontinuity of G in the strong topology of $L^p_{\text{loc}}(\Omega)$ (see e.g. [47, Example 1.21]). □

Lemma 2.3.29. *If $p > 1$, the functional $\|\cdot\|_{W^{1,p}_X(\Omega)} : W^{1,p}_X(\Omega) \to [0,\infty)$, defined by*

$$\|u\|_{W^{1,p}_X(\Omega)} := \int_\Omega |u|^p \, dx + \int_\Omega |Xu|^p \, dx \,,$$

is lower semicontinuous and sequentially coercive in the weak topology of $W^{1,p}_X(\Omega)$.

Proof. The lower semicontinuity of the functional follows since it is convex and continuous in the strong topology of $W^{1,p}_X(\Omega)$.

Let us show that the functional is sequentially coercive in the weak topology of $W^{1,p}_X(\Omega)$. For any $t \in \mathbb{R}^+_0$, the set $\{\|\cdot\|_{W^{1,p}_X(\Omega)} \leq t\}$ is a closed ball in a reflexive Banach space and, therefore, sequentially compact in the weak topology of $W^{1,p}_X(\Omega)$. □

Proof of Theorem 2.3.27. Let $\Psi_p : W^{1,p}_X(\Omega) \to [0,\infty)$ be defined as

$$\Psi_p(u) := \int_\Omega |Xu(x)|^p \, dx \,.$$

By growth conditions (2.7) and (2.126), it holds that

$$\begin{aligned}\Xi(u) &\geq c_0 \Psi_p(u) + d_0 \|u\|^p_{L^p(\Omega)} - \|a_0\|_{L^1(\Omega)} - \|b_0\|_{L^1(\Omega)} \\ &\geq \tilde{c} \|u\|^p_{W^{1,p}_X(\Omega)} - \|a_0\|_{L^1(\Omega)} - \|b_0\|_{L^1(\Omega)}\end{aligned}$$

for every $u \in W^{1,p}_X(\Omega)$, where $\tilde{c} := \min\{c_0, d_0\}$ and, therefore, Ξ is sequentially coercive, in virtue of Lemma 2.3.29. Moreover, since by Theorem 2.3.2 (*ii*) and Lemma 2.3.28 both F and G are sequentially lower semicontinuous in the weak topology of $W^{1,p}_X(\Omega)$, then Ξ satisfies the hypotheses of [47, Theorem 1.15] and the existence of a minimizer of (2.127) immediately follows.

In order to prove (2.129), we start observing that Ξ is continuous with respect to the strong topology of $W^{1,p}_X(\Omega)$.

Let $\{u_h\}_h \subset W^{1,p}_X(\Omega)$ and $u \in W^{1,p}_X(\Omega)$ be such that $u_h \to u$ in $W^{1,p}_X(\Omega)$ and, additionally, let assume, up to subsequences, that $u_h \to u$ and $Xu_h \to Xu$ almost everywhere. Then, from (2.7) and (2.126), we get

$$\begin{aligned}\Xi(u_h) &\le c_1\Psi_p(u_h) + d_1\|u_h\|^p_{L^p(\Omega)} + \|a_1\|_{L^1(\Omega)} + \|b_1\|_{L^1(\Omega)}\\ &\le \bar{c}\|u_h\|^p_{W^{1,p}_X(\Omega)} + \|a_1\|_{L^1(\Omega)} + \|b_1\|_{L^1(\Omega)}\end{aligned}$$

where $\bar{c} := \max\{c_1, d_1\}$ and, via Pratt's Theorem (see, for instance, [107, Theorem A.10]), we can infer that

$$\Xi(u_h) \to \Xi(u) \quad \text{as } h \to \infty.$$

Since this holds for a subsequence of any subsequence of the original sequence $\{u_h\}_h$, then the strong continuity of Ξ follows. Thus, (2.129) now follows readily, since $\mathbf{C}^\infty(\Omega)$ is (strongly) dense in $W^{1,p}_X(\Omega)$.

Let us now define $\Psi^\varphi_p : W^{1,p}_X(\Omega) \to [0,\infty]$ as

$$\Psi^\varphi_p(u) := \begin{cases} \int_\Omega |Xu(x)|^p\,dx\,. & \text{if } u \in W^{1,p}_{X,\varphi}(\Omega)\\ \infty & \text{otherwise}\end{cases}.$$

Then, for every $u \in W^{1,p}_{X,\varphi}(\Omega)$, it holds that

$$\begin{aligned}\Xi^\varphi(u) &\ge c_0\Psi^\varphi_p(u) + d_0\|u\|^p_{L^p(\Omega)} - \|a_0\|_{L^1(\Omega)} - \|b_0\|_{L^1(\Omega)}\\ &\ge \tilde{c}\|u\|^p_{W^{1,p}_X(\Omega)} - \|a_0\|_{L^1(\Omega)} - \|b_0\|_{L^1(\Omega)}\end{aligned}$$

and, as before, we conclude that Ξ^φ is sequentially coercive in the weak topology of $W^{1,p}_X(\Omega)$. Therefore, the existence of a minimum follows since F, G and $\mathbb{1}_\varphi$ are sequentially lower semicontinuous in the weak topology of $W^{1,p}_X(\Omega)$ (see e.g. [47, Example 1.6]).

Finally, if $g(x,\cdot)$ is also strictly convex for a.e. $x \in \Omega$, then both Ξ and Ξ^φ are strictly convex and, therefore, the solutions of (2.127) and (2.128) are unique. □

The following result ensures the existence and convergence of minima and minimizers, in the strong topology of $L^p(\Omega)$, for a class of functionals that will be useful in the sequel.

Theorem 2.3.30. *Let Ω, Ω_0, p, X and g satisfy the hypotheses of Theorem 2.3.27, and also assume that X satisfies (LIC) condition on Ω connected. Let $f_h, f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, $F_h, F : L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$ be the functionals defined, respectively, by*

$$F_h(u) := \begin{cases} \int_\Omega f_h(x, Xu(x))dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases}$$

and

$$F(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases},$$

and let $G : L^p(\Omega) \to \mathbb{R}$ be the functional

$$G(u) := \int_\Omega g(x, u(x))\, dx\,.$$

Finally, for any $h \in \mathbb{N}$, fixed $\varphi \in W^{1,p}_X(\Omega)$, let $\Xi^\varphi_h, \Xi^\varphi : L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$ be, respectively, defined as

$$\Xi^\varphi_h := F_h + G + \mathbb{1}_\varphi \quad \text{and} \quad \Xi^\varphi := F + G + \mathbb{1}_\varphi\,. \tag{2.130}$$

If $\{F_h\}_h$ Γ-converges to F in the strong topology of $L^p(\Omega)$, then

(i) for each $h \in \mathbb{N}$, both Ξ^φ_h and Ξ^φ attain their minima in $L^p(\Omega)$ and

$$\min_{u \in L^p(\Omega)} \Xi^\varphi(u) = \lim_{h \to \infty} \min_{u \in L^p(\Omega)} \Xi^\varphi_h(u)\,; \tag{2.131}$$

(ii) if $\{u_h\}_h \subset L^p(\Omega)$ is a sequence of minimizers of $\{\Xi^\varphi_h\}_h$, i.e.,

$$\Xi^\varphi_h(u_h) = \min_{u \in L^p(\Omega)} \Xi^\varphi_h(u) \quad \text{for any } h \in \mathbb{N}\,,$$

then there exists $\bar{u} \in W^{1,p}_{X,\varphi}(\Omega)$ such that, up to subsequences,

$$u_h \to \bar{u} \text{ weakly in } W^{1,p}_X(\Omega) \text{ and strongly in } L^p(\Omega)$$

and

$$\Xi^\varphi(\bar{u}) = \min_{u \in L^p(\Omega)} \Xi^\varphi(u)\,.$$

Proof. **(i)** By (2.128), for each $h \in \mathbb{N}$, both functionals Ξ_h^φ and Ξ^φ attain their minima in the weak topology of $W^{1,p}_{X,\varphi}(\Omega)$ and, by definition, in the strong topology of $L^p(\Omega)$. Moreover, by Theorem 2.3.23, $\{F_h + \mathbb{1}_\varphi\}_h$ Γ-converges to $F + \mathbb{1}_\varphi$ and, since G is continuous in the strong topology of $L^p(\Omega)$ (it is readily seen, proceeding exactly as in the proof of Theorem 2.3.27), then $\{\Xi_h^\varphi\}_h$ Γ-converges to Ξ^φ in the strong topology of $L^p(\Omega)$, in virtue of [47, Proposition 6.21].

Finally, arguing as in [47, Propositions 2.10 and 2.11], it is readily seen that the sequence of functionals $\{\Xi_h^\varphi\}_h$ is equicoercive in the strong topology of $L^p(\Omega)$ and, in virtue of Theorem 2.3.5 (iii), Ξ^φ is coercive in the strong topology of $L^p(\Omega)$ and (2.131) follows.

(ii) Let $\{u_h\}_h \subset L^p(\Omega)$ be a sequence of minimizers of $\{\Xi_h^\varphi\}_h$. Without loss of generality, in virtue of Theorem 2.3.27, we may assume $\{u_h\}_h \subset W^{1,p}_{X,\varphi}(\Omega)$ and since for any $h \in \mathbb{N}$

$$\infty > \Xi_h^\varphi(u_h) \geq \min\{c_0, c_1\} \|u\|^p_{W^{1,p}_X(\Omega)} - \|a_0\|_{L^1(\Omega)} - \|b_0\|_{L^1(\Omega)}\,,$$

then $\{u_h\}_h$ in bounded in $W^{1,p}_{X,\varphi}(\Omega)$. Hence, the reflexivity of $W^{1,p}_X(\Omega)$ ensures the existence of $\bar{u} \in W^{1,p}_X(\Omega)$ such that, up to subsequences

$$u_h \rightharpoonup \bar{u} \text{ weakly in } W^{1,p}_X(\Omega), \text{ as } h \to \infty$$

and so

$$u_h - \varphi \rightharpoonup \bar{u} - \varphi \text{ weakly in } W^{1,p}_X(\Omega), \text{ as } h \to \infty\,.$$

Moreover, since $u_h - \varphi \in W^{1,p}_{X,0}(\Omega)$, which is closed with respect to the weak convergence of $W^{1,p}_X(\Omega)$, then $\bar{u} \in W^{1,p}_{X,\varphi}(\Omega)$ and, by Theorem 1.2.16, we yield that

$$u_h \to \bar{u} \text{ in } L^p(\Omega), \text{ as } h \to \infty\,. \tag{2.132}$$

Finally, in virtue of [47, Corollary 7.20], it holds that

$$\Xi^\varphi(\bar{u}) = \min_{u \in L^p(\Omega)} \Xi^\varphi(u)\,.$$

□

2.3.6 Convergence of perturbed functionals

In this section we provide the following Γ-convergence result for perturbed functionals depending on vector fields.

Theorem 2.3.31. *Let* $\{f_h\}_h \subset I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ *and let* $\{F_h\}_h$ *be the sequence of functionals defined in* (2.95)*. Assume that, for each* $A \in \mathcal{A}$*, there exists*

$$F(\cdot, A) = \left(\Gamma(L^p(\Omega))\text{-}\lim_{h\to\infty} F_h\right)(\cdot, A) \tag{2.133}$$

and

$$F(u, A) := \begin{cases} \int_A f(x, Xu(x))\, dx & \text{if } u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}, \tag{2.134}$$

with $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$*. Let* $\Phi \in L^p(\Omega)^m$ *and* $G_h^\Phi : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ *be defined by*

$$G_h^\Phi(u, A) := \begin{cases} \int_A f_h(x, Xu(x) + \Phi(x))\, dx & \text{if } u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}.$$

Then, for each $A \in \mathcal{A}$*, there exists*

$$G^\Phi(\cdot, A) = \left(\Gamma(L^p(\Omega))\text{-}\lim_{h\to\infty} G_h^\Phi\right)(\cdot, A)$$

and

$$G^\Phi(u, A) := \begin{cases} \int_A f(x, Xu(x) + \Phi(x))\, dx & \text{if } u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}.$$

Before giving the proof of Theorem 2.3.31, let us recall the following well-known result (see, for instance [46, Proposition 2.32]).

Lemma 2.3.32. *Let* $g \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$*. There exists a positive constant* $c_2 = c_2(p, c_1)$*, depending only on* p *and* c_1*, such that*

$$|g(x, \eta_1) - g(x, \eta_2)| \le c_2\, |\eta_1 - \eta_2| \left(|\eta_1| + |\eta_2| + a_1(x)^{1/p}\right)^{p-1} \tag{2.135}$$

for a.e. $x \in \Omega$*, for each* $\eta_1, \eta_2 \in \mathbb{R}^m$*.*

Proof of Theorem 2.3.31. Let us begin to observe that, by (2.135), it follows that

$$|f_h(x,\eta_1) - f_h(x,\eta_2)| \le c_2\,|\eta_1-\eta_2|\,\left(|\eta_1|+|\eta_2|+a_1(x)^{1/p}\right)^{p-1} \tag{2.136}$$

for each h, for a.e. $x\in\Omega$, for each η_1, $\eta_2\in\mathbb{R}^m$. Define, for every $x\in\Omega$, for each h, for each $\eta\in\mathbb{R}^m$

$$g_h^{\Phi}(x,\eta) := f_h(x,\eta+\Phi(x))\,.$$

Thus, $g_h^{\Phi}\in I_{m,p}(\Omega,c_0,c_3,\tilde a_0,\tilde a_1)$, with $\tilde a_0(x) := a_0(x)-c_0|\Phi|^p$ and $\tilde a_1(x) := a_1(x)+c_3|\Phi|^p$ for a suitable constant $c_3=c_3(c_1,p)>0$, depending only on c_1 and p. Therefore, in virtue of Theorem 2.3.12, up to subsequences and for each $A\in\mathcal{A}$, the sequence of functionals $\{G_h^{\Phi}(\cdot,A)\}_h$ $\Gamma(L^p(\Omega))$-converges to a functional of the form

$$G^{\Phi}(u,A) := \begin{cases}\int_A g^{\Phi}(x,Xu(x))\,dx & \text{if } u\in W^{1,p}_X(A)\\ \infty & \text{otherwise}\end{cases},$$

with

$$g^{\Phi}\in I_{m,p}(\Omega,c_0,c_3,\tilde a_0,\tilde a_1)\,. \tag{2.137}$$

We want to show that, for each $A\in\mathcal{A}$,

$$G^{\Phi}(u,A) = \int_A f(x,Xu(x)+\Phi(x))\,dx \quad \text{if } u\in W^{1,p}_X(A)\,. \tag{2.138}$$

We split the proof of (2.138) in three steps.

1st step. First, let us show the existence of a positive constant $c_4=c_4(c_0,c_1,c_2,a_0,a_1,p)$, depending only on c_0, c_1, c_2, a_0,a_1 and p, such that

$$|G^{\Phi_1}(u,A)-G^{\Phi_2}(u,A)| \le c_4\,\|\Phi_1-\Phi_2\|_{L^p}\,\left(\|Xu\|_{L^p}+\|\Phi_1\|_{L^p}+\|\Phi_2\|_{L^p}+1\right)^{p-1} \tag{2.139}$$

for each Φ_1, $\Phi_2\in L^p(\Omega)^m$, $A\in\mathcal{A}$ and $u\in W^{1,p}_X(A)$. All norms above refer to the set A.

Let Φ_1, $\Phi_2\in L^p(\Omega)^m$, let $A\in\mathcal{A}$ and let $u\in W^{1,p}_X(A)$. By definition of $\Gamma(L^p(\Omega))$-limit, there exists $\{u_h\}_h\subset L^p(\Omega)\cap W^{1,p}_X(A)$ such that

$$u_h\to u \text{ in } L^p(\Omega) \text{ and } G_h^{\Phi_2}(u_h,A)\to G^{\Phi_2}(u,A),\text{ as } h\to\infty\,. \tag{2.140}$$

Applying (2.136) and Hölder's inequality, we get that

$$\begin{aligned}|G_h^{\Phi_1}(u_h,A)-G_h^{\Phi_2}(u_h,A)| &\leq \int_A |f_h(x,Xu_h+\Phi_1)-f_h(x,Xu_h+\Phi_2)|\,dx \\ &\leq \alpha_1 \|\Phi_1-\Phi_2\|_{L^p}\,(\|Xu_h\|_{L^p}+\|\Phi_1\|_{L^p}+\|\Phi_2\|_{L^p}+1)^{p-1} \\ &\leq \alpha_2 \|\Phi_1-\Phi_2\|_{L^p}\,\left(G_h^{\Phi_2}(u_h,A)^{1/p}+\|\Phi_1\|_{L^p}+\|\Phi_2\|_{L^p}+1\right)^{p-1}\end{aligned}$$

for some $\alpha_i > 0$ $(i=1,2)$ depending only c_0, c_1, c_2, a_0, a_1 and p, where all norms above refer to set A. Then, (2.140) and $\Gamma-\liminf$ inequality give

$$|G^{\Phi_1}(u,A)-G^{\Phi_2}(u,A)| \leq \alpha_2 \|\Phi_1-\Phi_2\|_{L^p}\,\left(G^{\Phi_2}(u,A)^{1/p}+\|\Phi_1\|_{L^p}+\|\Phi_2\|_{L^p}+1\right)^{p-1}.$$

Using (2.137), the upper bounds in (I_3) and exchanging the roles of Φ_1 and Φ_2, then we obtain (2.139).

2nd step. Let us prove (2.138) if Φ has the form

$$\Phi(x) = C(x)\,\tilde{\Phi}(x) \text{ a.e. } x \in \Omega\,, \tag{2.141}$$

for some $\tilde{\Phi} \in L^p(\Omega)^n$, where $C(x)$ denotes the coefficient matrix of the X-gradient. Let us divide this step in three cases.

Case 1. Suppose that $\Phi = \Phi_\xi$ with $\Phi_\xi(x) := C(x)\xi$, that is, $\tilde{\Phi}(x) = \xi =$ constant. Let $u_\xi(x) := \langle \xi, x\rangle_{\mathbb{R}^n}$ if $x \in \mathbb{R}^n$. By definition,

$$G_h^{\Phi_\xi}(u,A) = F_h(u+u_\xi,A)$$

hence

$$G^{\Phi_\xi}(u,A) = F(u+u_\xi,A)$$

so that

$$G^{\Phi_\xi}(u,A) = \int_A f(x,Xu(x)+C(x)\xi)\,dx = \int_A f(x,Xu(x)+\Phi_\xi(x))\,dx$$

for every $u \in W_X^{1,p}(A)$, for each $A \in \mathcal{A}$.

Case 2. Suppose that $\Phi(x) = C(x)\,\tilde{\Phi}(x)$ with $\tilde{\Phi}$ *piecewise constant*, that is,

$$\tilde{\Phi}(x) := \sum_{i=1}^{N} \chi_{A_i}(x)\,\xi^i$$

with $\xi^1, \dots, \xi^N \in \mathbb{R}^n$ and $A_1, \dots, A_N$ pairwise disjoint open sets such that $|\Omega \setminus \cup_{i=1}^{N} A_i| = 0$. Since $G^{\Phi}(u, \cdot)$ is a measure, by additivity on pairwise disjoint open sets and by locality, we have

$$G^{\Phi}(u, A) = \sum_{i=1}^{N} G^{\Phi}(u, A \cap A_i) = \sum_{i=1}^{N} G^{\Phi_{\xi_i}}(u, A \cap A_i)\,.$$

Hence, by case 1,

$$\begin{aligned} G^{\Phi}(u, A) &= \sum_{i=1}^{N} \int_{A \cap A_i} f(x, Xu(x) + C(x)\xi^i)\,dx = \sum_{i=1}^{N} \int_{A \cap A_i} f(x, Xu(x) + \Phi^{\xi^i}(x))\,dx \\ &= \int_A f(x, Xu(x) + \Phi(x))\,dx\,. \end{aligned}$$

Case 3. Let Φ be as in (2.141), let $\{\tilde{\Phi}_j\}_j$ be a sequence of piecewise constant functions converging to $\tilde{\Phi}$ strongly in $L^p(\Omega)^n$ and let $\Phi_j := C\,\tilde{\Phi}_j$. Since $C \in L^\infty(\Omega)^{mn}$, it also holds that $\{\Phi_j\}_j$ strongly in $L^p(\Omega)^m$. Moreover, (2.139) implies that

$$G^{\Phi_j}(u, A) \to G^{\Phi}(u, A)$$

for every $A \in \mathcal{A}$, $u \in W^{1,p}_X(A)$. By case 2, (2.135) and (I_3), we obtain that

$$G^{\Phi_j}(u, A) = \int_A f(x, Xu(x) + \Phi_j(x))\,dx \to \int_A f(x, Xu(x) + \Phi(x))\,dx$$

as $j \to \infty$. Therefore, (2.138) holds for each Φ of form (2.141).

3rd step. Let us now prove (2.138) in the general case. Let $\Phi \in L^p(\Omega)^m$ and let us recall that, in virtue of Lemma 2.1.3, for each $x \in \Omega_X$ the exists $\tilde{\Phi}(x) \in \mathbb{R}^n$ such that

$$C(x)\tilde{\Phi}(x) = \Phi(x)\,.$$

Moreover, $\tilde{\Phi}$ can be represented as

$$\tilde{\Phi}(x) = C(x)^T\,B(x)^{-1}\Phi(x) \quad \text{if } x \in \Omega_X\,, \tag{2.142}$$

where $B(x)$ is the $m \times m$ matrix defined by

$$B(x) := C(x)\, C(x)^T ,$$

with $B(x)$ invertible for each $x \in \Omega_X$. Since $B(x)$ is positive semi-definite for each $x \in \Omega$ and it is a positive definite matrix if and only if $x \in \Omega_X$, it holds that

$$|\mathcal{N}_X| = |\Omega \setminus \Omega_X| = 0 \tag{2.143}$$

and

$$\Omega_X = \{x \in \Omega : \det B(x) > 0\}\, , \mathcal{N}_X = \{x \in \Omega : \det B(x) = 0\}\, . \tag{2.144}$$

Let

$$\Omega_\varepsilon := \{x \in \Omega : \det B(x) > \varepsilon\}\, , \text{ if } \varepsilon > 0\, .$$

Observe that, since $B \in L^\infty(\Omega)^{m^2}$ then, by Cramer's rule,

$$B^{-1} \in L^\infty(\Omega_\varepsilon)^{m^2}. \tag{2.145}$$

By (2.142) and (2.145), it follows that $\tilde{\Phi} \in L^p(\Omega_\varepsilon)^n$.

Let $\tilde{\Phi}_\varepsilon : \Omega \to \mathbb{R}^n$ be defined by

$$\tilde{\Phi}_\varepsilon(x) := \begin{cases} C(x)^T\, B(x)^{-1}\Phi(x) & \text{if } x \in \Omega_\varepsilon \\ 0 & \text{if } x \in \Omega \setminus \Omega_\varepsilon \end{cases}$$

and let $\Phi_\varepsilon : \Omega \to \mathbb{R}^m$ be

$$\Phi_\varepsilon(x) := C(x)\tilde{\Phi}_\varepsilon(x) = \begin{cases} C(x)C(x)^T\, B(x)^{-1}\Phi(x) = \Phi(x) & \text{if } x \in \Omega_\varepsilon \\ 0 & \text{if } x \in \Omega \setminus \Omega_\varepsilon \end{cases} .$$

By (2.143) and (2.144), it follows that

$$\Phi_\varepsilon \to \Phi \text{ in } L^p(\Omega)^m, \text{ as } \varepsilon \to 0\, . \tag{2.146}$$

Notice now that, for each $A \in \mathcal{A}$ and $u \in W^{1,p}_X(A)$, by the second step of the proof, for $\varepsilon > 0$

$$\left|G^\Phi(u,A) - \int_A f(x, Xu + \Phi)\, dx\right| \leq \left|G^\Phi(u,A) - G^{\Phi_\varepsilon}(u,A)\right| + \left|\int_A f(x, Xu + \Phi_\varepsilon)\, dx - \int_A f(x, Xu + \Phi)\, dx\right|.$$

Therefore, by the first step of the proof and since f satisfies also (2.135), then, by using Hölder's inequality, we can pass to the limit as $\varepsilon \to 0$ in the previous inequality, and (2.138) follows. □

2.3.7 Γ-convergence for left-invariant functionals on Carnot groups

We conclude this chapter with an application of Theorem 2.2.12 and Theorem 2.3.26 to sequences of left-invariant functionals in the framework of Carnot groups.

Theorem 2.3.33. *Let $1 < p < \infty$, let $\mathcal{A}_0$ be the class of all open bounded subsets of $\mathbb{G}$ and let $\{f_h\}_h \subset J_2(\mathbb{R}^n, c_0, c_1)$. Moreover, for each $h \in \mathbb{N}$, let $F_h : L^p_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A}_0 \to [0,\infty]$ be defined as*

$$F_h(u,A) := \begin{cases} \int_A f_h(\nabla_{\mathbb{G}} u(x))dx & \text{if } A \in \mathcal{A}_0,\ u \in W^{1,p}_{\mathbb{G},\mathrm{loc}}(A) \\ \infty & \text{otherwise} \end{cases}.$$

Then, up to subsequences, there exist $f \in J_2(\mathbb{R}^n, c_0, c_1)$ and a local left-invariant functional $F :\ L^p_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A} \to [0,\infty]$ such that:

(i)

$$F(\cdot, A) = \Gamma\text{-}\lim_{h\to\infty} F_h(\cdot, A) \quad \text{for each } A \in \mathcal{A}_0$$

in the strong topology of $L^p_{\mathrm{loc}}(\mathbb{R}^n)$;

(ii) F admits the following representation

$$F(u,A) := \begin{cases} \int_A f(\nabla_{\mathbb{G}} u(x))dx & \text{if } A \in \mathcal{A}_0, u \in W^{1,p}_{\mathbb{G},\mathrm{loc}}(A) \\ \infty & \text{otherwise} \end{cases}.$$

Proof. By Theorem 2.3.26, there exist a local functional $F : L^p_{\text{loc}}(\mathbb{R}^n) \times \mathcal{A}_0 \to [0, \infty]$ and $f \in J_2(\mathbb{R}^n, c_0, c_1)$ such that (i) and (ii) hold.

Finally, by Proposition 2.2.8 and Remark 2.2.9, we can infer that the Γ-limit F is left-invariant. □

Remark 2.3.34. We remind that Theorem 2.3.33 cannot be expected to hold in general even for $p = 1$, since this is false already in the Euclidean case. We refer to [47, Example 3.14] for more details.

Chapter Three

H-convergence

Notation

Let $X = (X_1, \ldots, X_m)$ be a family of Lipschitz continuous vector fields on an open neighbourhood Ω_0 of Ω, open set of $\mathbb{R}^n$ and let $u \in L^1(\Omega)$. Since for any $j = 1, \ldots, m$

$$X_j(x) = \sum_{i=1}^{n} c_{ji}(x)\partial_i \quad \text{with } c_{ji}(x) \in Lip(\Omega)\,,$$

then, Xu is an element of $\mathcal{D}'(\Omega; \mathbb{R}^m)$ and, for any $\psi = (\psi_1, \ldots, \psi_m) \in \mathcal{D}(\Omega; \mathbb{R}^m)$ and for any $j = 1, \ldots, m$, it holds that

$$\begin{aligned} X_j u(\psi_j) :&= \langle X_j u, \psi_j \rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \int_\Omega X_j u \psi_j \, dx = \int_\Omega \sum_{i=1}^{n} c_{ji}\partial_i u \psi_j \, dx \\ &= \int_\Omega \sum_{i=1}^{n} \partial_i u (c_{ji}\psi_j)\, dx = -\int_\Omega u \sum_{i=1}^{n} \partial_i (c_{ji}\psi_j)\, dx = \int_\Omega u X_j^T \psi_j \, dx\,, \end{aligned}$$

where, once identified each X_j with the vector field $(c_{j1}(x), \ldots, c_{jn}(x)) \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$, then the (formal) adjoint of X_j in $L^2(\Omega)$ is defined as

$$X_j^T \varphi := -\sum_{i=1}^{n} \partial_i (c_{ji}\,\varphi) = -\left(\mathrm{div}(X_j) + X_j\right)\varphi \quad \forall\, \varphi \in \mathbf{C}_c^\infty(\Omega)\,. \tag{3.1}$$

Thus, if we set $X^T\psi := (X_1^T\psi_1, \ldots, X_m^T\psi_m)$, then the aspect of $Xu(\psi)$ becomes even more familiar, i.e.

$$Xu(\psi) := (X_1 u(\psi_1), \ldots, X_m u(\psi_m)) = \int_\Omega u X^T \psi \, dx \qquad \forall \psi \in \mathbf{C}_c^\infty(\Omega; \mathbb{R}^m).$$

3.1 Linear operators depending on vector fields

In this section, we are going to study H-convergence results for linear differential operators depending on vector fields. In according with [47, Chapter 13], the class of operators in *X-divergence form*, we are interested in, is defined as follows.

Definition 3.1.1. Let $a = [a_{ij}(x)]$ be a $m \times m$ symmetric matrix such that

$$\begin{aligned} &a_{ij} \in L^\infty(\Omega) \quad \forall i,j = 1,\dots,m; \\ &c_0|\eta|^2 \le \langle a(x)\eta,\eta\rangle_{\mathbb{R}^m} \le c_1|\eta|^2 \quad \text{a.e. } x\in\Omega,\ \forall \eta \in \mathbb{R}^m. \end{aligned} \tag{3.2}$$

We denote $\mathcal{E}(\Omega; c_0, c_1)$ or, equivalently, $\mathcal{E}(\Omega)$, the class of linear differential operators in X-divergence form, that is

$$\mathcal{L} := \operatorname{div}_X(a(x)X) := \sum_{i,j=1}^m X_j^T(a_{ij}(x)X_i)\,. \tag{3.3}$$

The domain of $\mathcal{L}$ is the set

$$D(\mathcal{L}) = \left\{ u \in W_X^{1,2}(\Omega) : \sum_{i,j=1}^m X_j^T(a_{ij}(x)X_i) \in L^2(\Omega) \right\}.$$

The following H-compactness theorem for operators belonging to $\mathcal{E}(\Omega)$ is the main result of this section. From now on, the space $H^1_{X,0}(\Omega)$ identifies $W^{1,2}_{X,0}(\Omega)$ and $H_X^{-1}(\Omega)$ its dual space.

Theorem 3.1.2. *Let Ω and Ω_0 be, respectively, a bounded open set and an open set, with $\bar\Omega \subset \Omega_0$ and let $X = (X_1,\dots,X_m)$ be a family of Lipschitz continuous vector fields defined on Ω_0 and satisfying conditions (H1), (H2) (H3) and (LIC) on Ω. Moreover, let $\mathcal{L}_h \in \mathcal{E}(\Omega)$, $h \in \mathbb{N}$, and let $a^h(x) = [a^h_{ij}(x)]$ be the associate matrix, in according with Definition 3.1.1. Then, up to subsequences, there exist a symmetric matrix $a^{\mathit{eff}} = [a^{\mathit{eff}}_{ij}(x)]$ satisfying* (3.2) *and an operator $\mathcal{L}_\infty := \operatorname{div}_X(a^{\mathit{eff}}(x)X) \in \mathcal{E}(\Omega)$ such that if, for any $g \in L^2(\Omega)$, $\mu \ge 0$ and $h \in \mathbb{N}$, u_h and u_∞ denote, respectively, the (unique) solutions of*

$$\begin{cases} \mu u + \mathcal{L}_h(u) = g \ \text{in } \Omega \\ u \in H^1_{X,0}(\Omega) \end{cases} \quad \textit{and} \quad \begin{cases} \mu u + \mathcal{L}_\infty(u) = g \ \text{in } \Omega \\ u \in H^1_{X,0}(\Omega) \end{cases},$$

then, as $h \to \infty$, the following convergences hold:

$$u_h \to u_\infty \text{ strongly in } L^2(\Omega) \quad \text{(convergence of solutions)} \tag{3.4}$$

and

$$a^h X u_h \rightharpoonup a^{\mathit{eff}} X u_\infty \text{ weakly in } L^2(\Omega)^m \quad \text{(convergence of momenta).} \tag{3.5}$$

Remark 3.1.3. If, in Theorem 3.1.2, we just assume $g \in H_X^{-1}(\Omega)$, then the conclusion remains true, by replacing the strong convergence of the solutions in $L^2(\Omega)$ with the weak one in $H^1_{X,0}(\Omega)$.

Proof of Theorem 3.1.2. We divide the proof of the theorem in two steps.

1st step. Let us prove that, up to subsequences, there exists a limit operator $\mathcal{L}_\infty \in \mathcal{E}(\Omega)$ for which (3.4) holds true.

Let $\{a^h\}_h$ be the sequence of matrices associated to $\{\mathcal{L}_h\}_h$ and let $F_h : L^2(\Omega) \times \mathcal{A} \to [0, \infty]$ be the quadratic functionals defined by

$$F_h(u, A) := \begin{cases} \frac{1}{2} \int_A \langle a^h(x) X u(x), X u(x) \rangle_{\mathbb{R}^m} \, dx & \text{if } A \in \mathcal{A},\ u \in W_X^{1,2}(A) \\ \infty & \text{otherwise} \end{cases}.$$

Then, by Theorem 2.3.26, there exist a subsequence $\{F_{h_k}\}_k$ of $\{F_h\}_h$, a local functional $F : L^2(\Omega) \times \mathcal{A} \to [0, \infty]$ and a $m \times m$ symmetric matrix $a^{\text{eff}}(x)$ satisfying (3.2), such that

$$\{F_{h_k}(\cdot, \Omega)\}_k \ \Gamma\text{-converges to } F(u, \Omega)$$

in the strong topology of $L^2(\Omega)$ and $F(u, \Omega)$ is represented by

$$F(u, \Omega) = \begin{cases} \frac{1}{2} \int_\Omega \langle a^{\text{eff}}(x) X u(x), X u(x) \rangle_{\mathbb{R}^m} \, dx & \text{if } u \in W_X^{1,2}(\Omega) \\ \infty & \text{otherwise} \end{cases}.$$

Let now $\mathcal{L}_\infty$ be the elliptic operator associated with a^{eff}, that is

$$\mathcal{L}_\infty := \operatorname{div}_X(a^{\text{eff}}(x) X) \,.$$

Taking into account [47, Definition 12.8], it is easy to see that $\mathcal{L}_\infty$ is the operator associated to the functional $F^0 : L^2(\Omega) \to [0,\infty]$

$$F^0(u) = \begin{cases} \frac{1}{2}\int_\Omega \langle a^{\mathrm{eff}}(x)Xu(x), Xu(x)\rangle_{\mathbb{R}^m}\,dx & \text{if } u \in H^1_{X,0}(\Omega) \\ \infty & \text{otherwise} \end{cases}.$$

Let us consider the sequence of functionals $F^0_h : L^2(\Omega) \to [0,\infty]$ defined by

$$F^0_h(u) = \begin{cases} \frac{1}{2}\int_\Omega \langle a^h(x)Xu(x), Xu(x)\rangle_{\mathbb{R}^m}\,dx & \text{if } u \in H^1_{X,0}(\Omega) \\ \infty & \text{otherwise} \end{cases},$$

whose associated operators are $\{\mathcal{L}_h\}_h$. Using Theorem 2.3.23, with $\varphi = 0$ and $A = \Omega$, we get that

$$\left\{F^0_h\right\}_h \ \Gamma\text{-converges to } F^0 \text{ in } L^2(\Omega)\,. \tag{3.6}$$

Let now $\mu \geq 0$ and $g \in L^2(\Omega)$. We denote by $G : L^2(\Omega) \to \mathbb{R}$ the functional

$$G(u) := \int_\Omega \left(\frac{\mu}{2}u^2 - gu\right)dx.$$

Since G is (strongly) continuous in $L^2(\Omega)$, then, by [47, Proposition 6.21] and (3.6), it follows that

$$\left\{F^0_h + G\right\}_h \ \Gamma\text{-converges to } (F^0 + G) \text{ in } L^2(\Omega)\,. \tag{3.7}$$

Let us show that, for any $h \in \mathbb{N}$, the functions u_h and u_∞ are, respectively, the unique elements of the sets

$$\operatorname{argmin}\left\{F^0_h(u) + G(u) \mid u \in H^1_{X,0}(\Omega)\right\}$$
$$\operatorname{argmin}\left\{F^0(u) + G(u) \mid u \in H^1_{X,0}(\Omega)\right\}.$$

Let $\varepsilon > 0$. Then, for any $h \in \mathbb{N}$, by (3.2), Young inequality and in virtue of Proposition 1.2.18, there exists a positive constant $c_{2,\Omega} > 0$ such that

$$\begin{aligned} F^0_h(u) + G(u) &\geq \frac{c_0}{2}\|Xu\|^2_{L^2(\Omega)} + \frac{\mu}{2}\|u\|^2_{L^2(\Omega)} - \left(\frac{\varepsilon}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\|g\|^2_{L^2(\Omega)}\right) \\ &\geq \left(\frac{c_0}{2} - \frac{c_{2,\Omega}\varepsilon}{2}\right)\|u\|^2_{H^1_{X,0}(\Omega)} - \frac{1}{2\varepsilon}\|g\|^2_{L^2(\Omega)} \end{aligned}$$

which, combined with Theorem 1.2.16, gives that $\{F_h^0 + G\}_h$ is equicoercive in $H^1_{X,0}(\Omega)$, upon taking ε sufficiently small ($F^0 + G$ is coercive in virtue of Proposition 2.3.5 (*iii*)).

Therefore, by Theorem 2.3.30 (*ii*), we get (3.4). Moreover

$$\lim_{h\to\infty}\big(F_h^0(u_h) + G(u_h)\big) = \lim_{h\to\infty}\min_{u\in H^1_{X,0}(\Omega)}\big(F_h^0(u) + G(u)\big)$$
$$= \min_{u\in H^1_{X,0}(\Omega)}\big(F^0(u) + G(u)\big) = F^0(u_\infty) + G(u_\infty)\,.$$

□

Remark 3.1.4. Since no definition of *curl* is already given in this general setting, we cannot adapt standard techniques to prove the convergence of the momenta. In the following section, we are going to study a new variational technique to obtain the convergence of momenta, to conclude the previous proof. This technique was introduced by Ansini, Dal Maso and Zeppieri in [7, 8, 9]. We will come back later on the proof of Theorem 3.1.2, after proving few auxiliary results.

3.1.1 Convergence of momenta by variational methods

Remark 3.1.5. Let $1 < p < \infty$ and $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and assume that the function $f(x,\cdot)$ is of class $\mathbf{C}^1(\mathbb{R}^m)$ for a.e. $x \in \Omega$. Moreover, let us denote by $\partial_\eta f(x,\eta)$ the gradient of $f(x,\cdot)$ at $\eta \in \mathbb{R}^m$ for a.e. $x \in \Omega$. Since f satisfies (2.135) in virtue of Lemma 2.3.32, then there exists a positive constant $c_2 = c_2(p, c_1)$, depending only on p and c_1, such that

$$|\partial_\eta f(x,\eta)| \le c_2\,(2\,|\eta| + a_1(x)^{1/p})^{p-1} \text{ for a.e. } x \in \Omega,\ \forall \eta \in \mathbb{R}^m\,.$$

Therefore, the functional $\mathcal{F} : L^p(\Omega)^m \to [0,\infty)$, defined as

$$\mathcal{F}(\Phi) := \int_\Omega f(x,\Phi)\,dx \quad \text{for any } \Phi \in L^p(\Omega)^m,$$

is of class $\mathbf{C}^1$ and its Gateaux derivative, $\partial_\Phi \mathcal{F} : L^p(\Omega)^m \to L^{p'}(\Omega)^m$, is given by

$$\partial_\Phi \mathcal{F}(\Phi) = \partial_\eta f(x,\Phi) \quad \text{for any } \Phi \in L^p(\Omega)^m \text{ a.e. } x \in \Omega.$$

Theorem 3.1.6. *Let $F_h : L^p(\Omega) \to [0,\infty]$ and $\mathcal{F}_h : L^p(\Omega)^m \to [0,\infty]$ be, respectively, defined by*

$$F_h(u) = F_h(u,\Omega) := \begin{cases} \int_\Omega f_h(x, Xu(x))\, dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases},$$

$$\mathcal{F}_h(\Phi) := \int_\Omega f_h(x,\Phi(x))\, dx \quad \text{if } \Phi \in L^p(\Omega)^m$$

with $f_h \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ nonnegative for each $h \in \mathbb{N}$. Moreover, assume that

(i) $f_h(x,\cdot) : \mathbb{R}^m \to [0,\infty)$ belongs to $\mathbf{C}^1(\mathbb{R}^m)$ and, for $0 < \alpha < \min\{1, p-1\}$, there exist a constant $c_2 > 0$ and a non negative function $b \in L^p(\Omega)$, such that

$$|\partial_\eta f_h(x,\eta_1) - \partial_\eta f_h(x,\eta_2)| \leq c_2 |\eta_1 - \eta_2|^\alpha \left(|\eta_1| + |\eta_2| + b(x)\right)^{p-1-\alpha}$$

for a.e. $x \in \Omega$ and for each $h \in \mathbb{N}$;

(ii) there exists $F = \Gamma(L^p(\Omega))\text{-}\lim_{h\to\infty} F_h$, with

$$F(u) = F(u,\Omega) := \begin{cases} \int_\Omega f(x, Xu(x))\, dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases}$$

and $f(x,\cdot) : \mathbb{R}^m \to [0,\infty)$ belongs to $\mathbf{C}^1(\mathbb{R}^m)$ for a.e. $x \in \Omega$;

(iii) there exist a sequence $\{u_h\}_h \subset L^p(\Omega)$ and a function $u \in L^p(\Omega)$ such that

$$u_h \to u \text{ in } L^p(\Omega) \text{ and } \mathcal{F}_h(Xu_h) \to \mathcal{F}(Xu) \text{ as } h \to \infty\,,$$

where $\mathcal{F}(\Phi) := \int_\Omega f(x,\Phi(x))\, dx$ if $\Phi \in L^p(\Omega)^m$.

Then

$$\partial_\Phi \mathcal{F}_h(Xu_h) \to \partial_\Phi \mathcal{F}(Xu) \text{ weakly in } L^{p'}(\Omega)^m \text{ as } h \to \infty\,. \tag{3.8}$$

Remark 3.1.7. We do not know whether assumption (i) of Theorem 3.1.6 actually implies that $f(x,\cdot) \in \mathbf{C}^1(\mathbb{R}^m)$ a.e. $x \in \Omega$, as in in the Euclidean setting (see [8, Theorem 2.8]). On the other hand, the extension of the result from the Euclidean setting to the one of vector fields looks nontrivial. We are now working to solve this issue. However, the application of Theorem 3.1.6 to obtain the convergence of momenta in Theorem 3.1.2 is justified by the fact that the integrands are quadratic forms and, therefore, the $\mathbf{C}^1$-assumption is satisfied.

Proof of Theorem 3.1.6. The proof follows the techniques of [8, Theorem 4.5]. We repeat it here for the sake of completeness. In order to get (3.8) it is sufficient to show that

$$\langle \partial_\Phi \mathcal{F}(Xu), \Psi \rangle_{L^{p'}(\Omega)^m \times L^p(\Omega)^m} \leq \liminf_{h\to\infty} \langle \partial_\Phi \mathcal{F}_h(Xu_h), \Psi \rangle_{L^{p'}(\Omega)^m \times L^p(\Omega)^m} \tag{3.9}$$

for every $\Psi \in L^p(\Omega)^m$.

Let $\{t_j\}_j$ be a sequence of positive numbers converging to 0. By assumptions $(ii), (iii)$ and by Theorem 2.3.31, for every $j \in \mathbb{N}$, it follows that

$$\frac{\mathcal{F}(Xu + t_j\Psi) - \mathcal{F}(Xu)}{t_j} \leq \liminf_{h\to\infty} \frac{\mathcal{F}_h(Xu_h + t_j\Psi) - \mathcal{F}_h(Xu_h)}{t_j}.$$

Therefore, there exists an increasing sequence of integers $\{h_j\}_j$ such that for each $h \geq h_j$

$$\frac{\mathcal{F}(Xu + t_j\Psi) - \mathcal{F}(Xu)}{t_j} - \frac{1}{j} \leq \frac{\mathcal{F}_h(Xu_h + t_j\Psi) - \mathcal{F}_h(Xu_h)}{t_j}. \tag{3.10}$$

Let now $\varepsilon_h := t_j$ for $h_j \leq h < h_{j+1}$, $j \in \mathbb{N}$. Then, from (3.10), we get

$$\liminf_{h\to\infty} \frac{\mathcal{F}(Xu + \varepsilon_h\Psi) - \mathcal{F}(Xu)}{\varepsilon_h} \leq \liminf_{h\to\infty} \frac{\mathcal{F}_h(Xu_h + \varepsilon_h\Psi) - \mathcal{F}_h(Xu_h)}{\varepsilon_h}.$$

Moreover, since both $\mathcal{F}_h$ and $\mathcal{F}$ are of class $\mathbf{C}^1$, then

$$\langle \partial_\Phi \mathcal{F}(Xu), \Psi \rangle_{L^{p'}(\Omega)^m \times L^p(\Omega)^m} = \lim_{h\to\infty} \frac{\mathcal{F}(Xu + \varepsilon_h\Psi) - \mathcal{F}(Xu)}{\varepsilon_h}$$

and, by mean value theorem, for each $h \in \mathbb{N}$, there exists $\tau_h \in (0, \varepsilon_h)$ such that

$$\frac{\mathcal{F}_h(Xu_h + \varepsilon_h\Psi) - \mathcal{F}_h(Xu_h)}{\varepsilon_h} = \langle \partial_\Phi \mathcal{F}_h(Xu_h + \tau_h\Psi), \Psi \rangle_{L^{p'}(\Omega)^m \times L^p(\Omega)^m}.$$

Finally, by assumption (i), we are under the hypotheses of [8, Lemma 4.4], with $H_k = \partial_\eta f_k$, $\Phi_k = Xu_k$ and $\Psi_k = \tau_k\Psi$ and, therefore, we get

$$\liminf_{h\to\infty} \langle \partial_\Phi \mathcal{F}_h(Xu_h + \tau_h\Psi), \Psi \rangle_{L^{p'}(\Omega)^m \times L^p(\Omega)^m} = \liminf_{h\to\infty} \langle \partial_\Phi \mathcal{F}_h(Xu_h), \Psi \rangle_{L^{p'}(\Omega)^m \times L^p(\Omega)^m}.$$

□

Corollary 3.1.8. *Let $\Omega \Subset \Omega_0$ be a bounded and connected open set, let $1 < p < \infty$ and let $X = (X_1, \ldots, X_m)$ be a family of Lipschitz continuous vector fields defined on Ω_0 and satisfying conditions (H1), (H2) and (H3) and (LIC) on Ω. Moreover, let $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that there exist two positive constants $d_0 \leq d_1$ and two non negative functions $b_0, b_1 \in L^1(\Omega)$ such that*

$$d_0|s|^p - b_0(x) \leq g(x,s) \leq d_1|s|^p + b_1(x) \tag{3.11}$$

for a.e $x \in \Omega$ and for every $s \in \mathbb{R}$, and let $G : L^p(\Omega) \to \mathbb{R}$ be defined as

$$G(u) := \int_\Omega g(x, u(x))\, dx\,.$$

Finally, let F_h, f_h, F and f satisfy the hypotheses of Theorem 3.1.6 and (2.73)*, and let $\varphi \in W^{1,p}_X(\Omega)$. We consider the functionals $\Xi_h^\varphi, \Xi^\varphi : L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$ defined by*

$$\Xi_h^\varphi := F_h + G + \mathbb{1}_\varphi \quad \text{and} \quad \Xi^\varphi := F + G + \mathbb{1}_\varphi\,,$$

that is

$$\Xi_h^\varphi(u) = \Xi_h^\varphi(u,\Omega) := \begin{cases} \int_\Omega \left(f_h(x, Xu(x)) + g(x,u(x))\right)\, dx & \text{if } u \in W^{1,p}_{X,\varphi}(\Omega) \\ \infty & \text{otherwise} \end{cases}$$

and

$$\Xi^\varphi(u) = \Xi^\varphi(u,\Omega) := \begin{cases} \int_\Omega \left(f(x, Xu(x)) + g(x,u(x))\right)\, dx & \text{if } u \in W^{1,p}_{X,\varphi}(\Omega) \\ \infty & \text{otherwise} \end{cases}.$$

If $\{u_h\}_h$ is a sequence of minimizers of $\{\Xi_h^\varphi\}_h$ then, up to subsequences, there exists a minimum u of Ξ^φ such that

$$u_h \to u \text{ weakly in } W^{1,p}_X(\Omega) \text{ and strongly in } L^p(\Omega).$$

Moreover, (3.8) *holds.*

Proof. By Theorem 2.3.30 (i) and (ii), up to subsequences, there exists a minimum u of Ξ^φ such that

$$u_h \to u \text{ weakly in } W^{1,p}_X(\Omega) \text{ and strongly in } L^p(\Omega)\,, \tag{3.12}$$

$$\Xi^\varphi_h(u_h) \to \Xi^\varphi(u)\,. \tag{3.13}$$

Since G is continuous, by (3.12), we have that

$$G(u_h) \to G(u)\,. \tag{3.14}$$

Thus, by (3.13) and (3.14), we also have that

$$\mathcal{F}_h(Xu_h) = F_h(u_h) \to F(u) = \mathcal{F}(Xu)\,. \tag{3.15}$$

Therefore, by (3.12) and (3.15), we can apply Theorem 3.1.6 and the proof is accomplished. □

Continue of the proof of Theorem 3.1.2. **2nd step.** Let

$$f_h(x,\eta) := \langle a_h(x)\eta,\eta\rangle_{\mathbb{R}^m} \text{ and } f(x,\eta) := \langle a(x)\eta,\eta\rangle_{\mathbb{R}^m} \quad \text{for any } x\in\Omega \text{ and } \eta\in\mathbb{R}^m\,.$$

Then, it is easy to see that the integrands f_h and f satisfies assumptions (i) and (ii) of Theorem 3.1.6 and that

$$\partial_\Phi \mathcal{F}_h(Xu_h) = a_h X u_h \quad \text{and} \quad \partial_\Phi \mathcal{F}(Xu) = aXu\,.$$

Thus, by the first step of the proof and by Corollary 3.1.8, (3.4) and (3.5) follow. □

3.2 Monotone operators on Carnot groups

The second part of this chapter is devoted to a H-compactness result for monotone operators on Carnot groups, namely, Theorem 3.2.9. As explained later, Theorem 3.2.9 generalizes well-known results of Murat and Tartar [122, Theorem 11.2], for the Euclidean setting, and Baldi, Franchi, Tchou and Tesi [72, Theorem 4.4], [14, Theorem 6.4] and [13, Theorem 5.4], for the setting of Carnot groups. Differently from Theorem 3.1.2, the proof of Theorem 3.2.9 relies on a classical tool introduced by Murat and Tartar [122, Theorem 7.2], the *Div-curl lemma*, whose adaptation to the setting of Carnot groups, Theorem 3.2.12, was studied by Baldi, Franchi, Tchou and Tesi in [13, Theorem 5.1]. Nowadays, this classical technique is still not adaptable in the general case of Sobolev spaces depending on locally-Lipschitz continuous vector fields, object of the first part of the chapter, since no definition of *curl* is already given in that framework.

Let $\Omega \subset \mathbb{G}$ be open, connected and bounded, $2 \leq p < \infty$ and let p, p' be a Hölder conjugate pair. From now on, we denote $V := W^{1,p}_{\mathbb{G},0}(\Omega)$ and $V^* := (W^{1,p}_{\mathbb{G},0}(\Omega))' = W^{-1,p'}_{\mathbb{G}}(\Omega)$. In this section, we are interested in (nonlinear) operators $\mathcal{A} : V \to V^*$ of the form

$$\mathcal{A}(u) := -\mathrm{div}_{\mathbb{G}}(A(x, \nabla_{\mathbb{G}} u)) \tag{3.16}$$

for a given $A \in \mathcal{M}(\alpha, \beta; \Omega)$, where the class $\mathcal{M}(\alpha, \beta; \Omega)$ is defined as follows:

Definition 3.2.1. Let $\Omega \subset \mathbb{G}$ be open, $2 \leq p < \infty$ and let $\alpha \leq \beta$ be positive constants. We define $\mathcal{M}(\alpha, \beta; \Omega)$ the class of Carathéodory functions $A : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ such that

(*i*) $A(x, 0) = 0\,;$

(*ii*) $\langle A(x,\xi) - A(x,\eta), \xi - \eta \rangle_{\mathbb{R}^m} \geq \alpha |\xi - \eta|^p\,;$

(*iii*) $|A(x,\xi) - A(x,\eta)| \leq \beta \left[1 + |\xi|^p + |\eta|^p\right]^{\frac{p-2}{p}} |\xi - \eta|$

for every $\xi, \eta \in \mathbb{R}^m$ a.e. $x \in \Omega$.

Operators as in (3.16) are *monotone* in the sense of the following definition (see e.g. [79, Chapter III] for more details).

Definition 3.2.2. Let V be a reflexive Banach space, V^* its dual space and let $\mathcal{A} : V \to V^*$ be a mapping. We say that

- $\mathcal{A}$ is *monotone*, if

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_{V^* \times V} \geq 0 \quad \text{for all } u, v \in V;$$

- $\mathcal{A}$ is *coercive*, if there exists an element $v \in V$ such that

$$\frac{\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_{V^* \times V}}{\|u - v\|_V} \to \infty \quad \text{as } \|u\|_V \to \infty\,;$$

- $\mathcal{A}$ is *continuous on finite dimensional subspaces* of V if, for any finite dimensional subspace $M \subset V$, $\mathcal{A} : M \to V^*$ is weakly continuous.

The following result shows the existence and uniqueness of (weak) solutions for Dirichlet problems associated to $\mathcal{A}$.

Proposition 3.2.3. *Let $A \in \mathcal{M}(\alpha, \beta; \Omega)$. Then, for every $f \in V^*$, there exists a unique (weak) solution $u \in V$ of*

$$- \operatorname{div}_{\mathbb{G}}(A(x, \nabla_{\mathbb{G}} u)) = f \quad \text{in } \Omega\,, \tag{3.17}$$

i.e.,

$$\int_\Omega \langle A(x, \nabla_{\mathbb{G}} u), \nabla_{\mathbb{G}} \varphi \rangle_{\mathbb{R}^m}\, dx = \int_\Omega f \varphi\, dx \quad \forall \varphi \in \mathbf{C}_c^\infty(\Omega). \tag{3.18}$$

Remark 3.2.4. By Proposition 1.3.18, (3.18) holds for every $\varphi \in V$.

The proof of Proposition 3.2.3, follows from the following well-known result.

Theorem 3.2.5. *[79, Corollary 1.8, Chapter III] Let X be a Banach space, let K be a closed, nonempty and convex subset of X and let $A : K \to X^*$ be monotone, coercive and continuous on finite dimensional subspaces of K. Then, there exists $u \in K$ such that*

$$\langle A(u), v - u \rangle_{X^* \times K} \geq 0 \quad \text{for any } v \in K.$$

Proof of Proposition 3.2.3. Let $f \in V^*$ and let $\Phi : V \to V^*$ be defined as

$$\langle \Phi(u), v\rangle_{V^* \times V} := \int_\Omega [\langle A(x, \nabla_{\mathbb{G}} u), \nabla_{\mathbb{G}} v\rangle_{\mathbb{R}^m} - fv]\, dx \quad \forall u, v \in V.$$

To prove the existence of solutions of (3.17) as a consequence of Theorem 3.2.5, let us show that Φ is monotone, coercive and continuous on finite dimensional subspaces of V.

For any $u, v \in V$, we have

$$\begin{aligned}\langle \Phi(u) - \Phi(v), u - v\rangle_{V^* \times V} &= \int_\Omega \langle A(x, \nabla_{\mathbb{G}} u) - A(x, \nabla_{\mathbb{G}} v), \nabla_{\mathbb{G}}(u - v)\rangle_{\mathbb{R}^m}\, dx \\ &\geq \alpha \int_\Omega |\nabla_{\mathbb{G}} u - \nabla_{\mathbb{G}} v|^p\, dx = \alpha \|u - v\|_V^p \geq 0\end{aligned}$$

and

$$\frac{\langle \Phi(u) - \Phi(v), u - v\rangle_{V^* \times V}}{\|u - v\|_V} \geq \alpha \|u - v\|_V^{p-1},$$

that is, Φ is monotone and coercive. Let now $\{u_n\}_n \subset V$ be convergent to u in V. By the Hölder inequality, it holds that

$$\begin{aligned}\langle \Phi(u_n) - \Phi(u), u_n - u\rangle_{V^* \times V} &= \int_\Omega \langle A(x, \nabla_{\mathbb{G}} u_n) - A(x, \nabla_{\mathbb{G}} u), \nabla_{\mathbb{G}}(u_n - u)\rangle_{\mathbb{R}^m}\, dx \\ &\leq \|A(\cdot, \nabla_{\mathbb{G}} u_n) - A(\cdot, \nabla_{\mathbb{G}} u)\|_{L^{p'}(\Omega, H\mathbb{G})} \|u_n - u\|_V\end{aligned}$$

and since

$$\begin{aligned}\|A(\cdot, \nabla_{\mathbb{G}} u_n) - A(\cdot, \nabla_{\mathbb{G}} u)\|_{L^{p'}(\Omega, H\mathbb{G})}^{p'} &= \int_\Omega |A(x, \nabla_{\mathbb{G}} u_n) - A(x, \nabla_{\mathbb{G}} u)|^{p'}\, dx \\ &\leq \beta^{p'} \int_\Omega [1 + |\nabla_{\mathbb{G}} u_n|^p + |\nabla_{\mathbb{G}} u|^p]^{\frac{p-2}{p-1}} |\nabla_{\mathbb{G}} u_n - \nabla_{\mathbb{G}} u|^{p'}\, dx \\ &\leq \beta^{p'} [|\Omega| + \|u_n\|_V^p + \|u\|_V^p]^{\frac{p-2}{p} p'} \|u_n - u\|_V^{p'},\end{aligned}$$

then Φ is strongly continuous on V, Banach space, and therefore, continuous on finite dimensional subspaces of V.

Thus, in virtue of Theorem 3.2.5, there exists $u \in V$ such that

$$\langle \Phi(u), v - u\rangle_{V^* \times V} \geq 0 \quad \forall v \in V$$

and, by choosing $v_1 := u + \varphi$ and $v_2 := u - \varphi$, we finally get

$$\langle \Phi(u), \varphi \rangle_{V^* \times V} = 0 \quad \forall \varphi \in V.$$

We conclude the proof by showing the uniqueness of the solutions. Let $u, v \in V$ be weak solutions of (3.17). Thus, taking into account Remark 3.2.4

$$\int_\Omega \langle A(x, \nabla_{\mathbb{G}} u) - A(x, \nabla_{\mathbb{G}} v), \nabla_{\mathbb{G}} \varphi \rangle_{\mathbb{R}^m} \, dx = 0 \quad \forall \varphi \in V.$$

Therefore, choosing $\varphi := u - v \in V$, we get the desired conclusion, since

$$0 = \int_\Omega \langle A(x, \nabla_{\mathbb{G}} u) - A(x, \nabla_{\mathbb{G}} v), \nabla_{\mathbb{G}} u - \nabla_{\mathbb{G}} v \rangle_{\mathbb{R}^m} \, dx \geq \alpha \|u - v\|_V^p \geq 0.$$

□

Corollary 3.2.6. *The operator $\mathcal{A}$ is continuous and invertible in V.*

Let us now show three estimates that will be useful in the sequel.

Proposition 3.2.7. *Let $A \in \mathcal{M}(\alpha, \beta; \Omega)$, let $\mathcal{A}$ be defined as in* (3.16) *and let $\mathcal{A}^{-1} : V^* \to V$ be its inverse operator. Then, the following estimates hold:*

(*a*) $\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_{V^* \times V} \geq \alpha \|u - v\|_V^p$;

(*b*) $\|\mathcal{A}^{-1}(f) - \mathcal{A}^{-1}(g)\|_V^p \leq \left(\frac{1}{\alpha}\right)^{p'} \|f - g\|_{V^*}^{p'}$;

(*c*) $\|\mathcal{A}(u) - \mathcal{A}(v)\|_{V^*} \leq \beta \left[|\Omega| + \|u\|_V^p + \|v\|_V^p\right]^{\frac{p-2}{p}} \|u - v\|_V$

for any $u, v \in V$, for any $f, g \in V^*$.

Proof. Let $u, v \in V$ and let $f, g \in V^*$ be such that $\mathcal{A}(u) = f$ and $\mathcal{A}(v) = g$ in Ω. Estimate (*a*) immediately follows from the definition of $\mathcal{M}(\alpha, \beta; \Omega)$, since

$$\begin{aligned}\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_{V^* \times V} &= \int_\Omega \langle A(x, \nabla_{\mathbb{G}} u) - A(x, \nabla_{\mathbb{G}} v), \nabla_{\mathbb{G}} u - \nabla_{\mathbb{G}} v \rangle_{\mathbb{R}^m} \, dx \\ &\geq \alpha \int_\Omega |\nabla_{\mathbb{G}} u - \nabla_{\mathbb{G}} v|^p \, dx = \alpha \|u - v\|_V^p.\end{aligned}$$

Moreover, recalling that

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_{V^* \times V} \leq \|\mathcal{A}(u) - \mathcal{A}(v)\|_{V^*} \|u - v\|_V \quad \forall u, v \in V$$

and applying (a), with $u = \mathcal{A}^{-1}(f)$ and $v = \mathcal{A}^{-1}(g)$, we get

$$\alpha \|\mathcal{A}^{-1}(f) - \mathcal{A}^{-1}(g)\|_V^p \leq \|f - g\|_{V^*} \|\mathcal{A}^{-1}(f) - \mathcal{A}^{-1}(g)\|_V,$$

which implies (b). Finally, estimate (c) holds since

$$\|A(\cdot, \nabla_{\mathbb{G}} u) - A(\cdot, \nabla_{\mathbb{G}} v)\|_{L^{p'}(\Omega, H\mathbb{G})} \leq \beta \left[|\Omega| + \|u\|_V^p + \|v\|_V^p\right]^{\frac{p-2}{p}} \|u - v\|_V$$

and, therefore,

$$\begin{aligned}
\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_{V^* \times V} &= \int_\Omega \langle A(x, \nabla_{\mathbb{G}} u) - A(x, \nabla_{\mathbb{G}} v), \nabla_{\mathbb{G}} u - \nabla_{\mathbb{G}} v \rangle_{\mathbb{R}^m} \, dx \\
&\leq \|A(\cdot, \nabla_{\mathbb{G}} u) - A(\cdot, \nabla_{\mathbb{G}} v)\|_{L^{p'}(\Omega, H\mathbb{G})} \|u - v\|_V \\
&\leq \beta \left[|\Omega| + \|u\|_V^p + \|v\|_V^p\right]^{\frac{p-2}{p}} \|u - v\|_V^2.
\end{aligned}$$

□

Let us now state an adaptation of the notion of *H-convergence* to this framework.

Definition 3.2.8. Let $\{A^n\}_n \subset \mathcal{M}(\alpha, \beta; \Omega)$ and let $A^{\text{eff}} \in \mathcal{M}(\alpha', \beta'; \Omega)$ for some positive constants $\alpha \leq \beta$, $\alpha' \leq \beta'$. Moreover, for any $f \in W_{\mathbb{G}}^{-1,p'}(\Omega)$ and $n \in \mathbb{N}$, let u_n and u_∞ be, respectively, the unique solutions of

$$\begin{cases} -\text{div}_{\mathbb{G}}(A^n(x, \nabla_{\mathbb{G}} u)) = f \text{ in } \Omega \\ u \in W_{\mathbb{G},0}^{1,p}(\Omega) \end{cases} \quad \text{and} \quad \begin{cases} -\text{div}_{\mathbb{G}}(A^{\text{eff}}(x, \nabla_{\mathbb{G}} u)) = f \text{ in } \Omega \\ u \in W_{\mathbb{G},0}^{1,p}(\Omega) \end{cases}.$$

We say that $\{A^n\}_n$ H-converges to A^{eff} if the following convergences hold:

$$u_n \rightharpoonup u_\infty \text{ weakly in } W_{\mathbb{G},0}^{1,p}(\Omega) \quad \text{(convergence of solutions)}$$

and

$$A^n(\cdot, \nabla_{\mathbb{G}} u_n) \rightharpoonup A^{\text{eff}}(\cdot, \nabla_{\mathbb{G}} u_\infty) \text{ weakly in } L^{p'}(\Omega, H\mathbb{G}) \quad \text{(convergence of momenta)},$$

where $L^p(\Omega, H\mathbb{G})$ denotes the set of all measurable sections $\Phi \in L^p(\Omega)^m$.

The main theorem of this section is the following H-compactness result.

Theorem 3.2.9. *Let $\Omega \subset \mathbb{G}$ be open, connected and bounded and let $2 \leq p < \infty$. Moreover, let $\alpha \leq \beta$ be positive constants and let $\{A^n\}_n \subset \mathcal{M}(\alpha, \beta; \Omega)$. Then, up to subsequences, there exists $A^{eff} \in \mathcal{M}(\alpha, \beta; \Omega)$ such that $\{A^n\}_n$ H-converges to A^{eff}.*

The proof of Theorem 3.2.9 consists on a combination of several parts. At first, let us show the convergence of solutions.

Lemma 3.2.10. *Let $\{A^n\}_n \subset \mathcal{M}(\alpha, \beta; \Omega)$ and let $\mathcal{A}_n : W^{1,p}_{\mathbb{G},0}(\Omega) \to W^{-1,p'}_{\mathbb{G}}(\Omega)$ be monotone operators of the form*

$$\mathcal{A}_n(u) := -\mathrm{div}_{\mathbb{G}}(A^n(x, \nabla_{\mathbb{G}} u)) \text{ in } \Omega,\, n \in \mathbb{N}.$$

Then, there exist a continuous and invertible operator $\mathcal{A}_\infty : W^{1,p}_{\mathbb{G},0}(\Omega) \to W^{-1,p'}_{\mathbb{G}}(\Omega)$ and a subsequence $\{\mathcal{A}_m\}_m$ of $\{\mathcal{A}_n\}_n$ such that, for every $f \in W^{-1,p'}_{\mathbb{G}}(\Omega)$

$$\mathcal{A}_m^{-1}(f) \rightharpoonup \mathcal{A}_\infty^{-1}(f) \text{ weakly in } W^{1,p}_{\mathbb{G},0}(\Omega).$$

Proof. For the sake of simplicity, let us still denote by $V = W^{1,p}_{\mathbb{G},0}(\Omega)$ and by V^* its dual space $W^{-1,p'}_{\mathbb{G}}(\Omega)$. We divide the proof of the lemma in three steps.

1st step. Fixed X a countable and dense subspace of V^*, let us show that $\{u_n\}_n$, sequence of unique solutions of

$$\mathcal{A}_n(u) = f \text{ in } \Omega,\, n \in \mathbb{N}, \tag{3.19}$$

weakly converges in V, up to subsequences, for any fixed $f \in X$. Moreover, let us provide an upper-bound for its limit, in terms of f.

Let $f \in X$ and let $\{u_n\}_n \subset V$ be the sequence of solutions of (3.19). By Proposition 3.2.3 and Proposition 3.2.7 (b), it holds that

$$\begin{aligned} u_n &= \mathcal{A}_n^{-1}(f)\,, \\ \|u_n\|_V &\leq \left(\frac{1}{\alpha}\right)^{\frac{1}{p-1}} \|f\|_{V^*}^{\frac{1}{p-1}} \quad \text{for any } n \in \mathbb{N}, \end{aligned}$$

that is, the sequence $\{u_n\}_n$ is bounded in V, reflexive Banach space.

Thus, there exist $u_\infty(f) \in V$, dependent on f, and $\{u_m\}_m$, diagonal subsequence of $\{u_n\}_n$, such that

$$u_m \rightharpoonup u_\infty(f) \text{ weakly in } V.$$

By the lower semicontinuity of the norm $\|\cdot\|_V$, and in virtue of Proposition 3.2.7 (a), it holds that

$$\begin{aligned}\langle f, u_\infty\rangle_{V^*\times V} &= \lim_{m\to\infty}\langle f, u_m\rangle_{V^*\times V} = \lim_{m\to\infty}\langle \mathcal{A}_m(u_m), u_m\rangle_{V^*\times V}\\ &\geq \alpha \lim_{m\to\infty}\|u_m\|_V^p \geq \alpha \liminf_{m\to\infty}\|u_m\|_V^p \geq \alpha\|u_\infty\|_V^p\end{aligned}$$

and, since

$$\langle f, u_\infty\rangle_{V^*\times V} \leq \|f\|_{V^*}\|u_\infty\|_V\,,$$

then

$$\|u_\infty\|_V \leq \left(\frac{1}{\alpha}\right)^{\frac{1}{p-1}} \|f\|_{V^*}^{\frac{1}{p-1}}\,.$$

2nd step. Let $S : X \to V$ be defined by

$$S(f) := \lim_{m\to\infty} \mathcal{A}_m^{-1}(f) \quad \text{for any } f \in X.$$

We show now that S can be extended to the whole space V^*. Since X is countable and dense in V^*, then we just show that S is continuous in the space $(X, \|\cdot\|_{V^*})$.

Let $f, g \in X$. Then, by Proposition 3.2.7 (b), it holds that

$$\|\mathcal{A}_m^{-1}(f) - \mathcal{A}_m^{-1}(g)\|_V \leq \left(\frac{1}{\alpha}\right)^{\frac{1}{p-1}} \|f-g\|_{V^*}^{\frac{1}{p-1}} \quad \text{for any } m \in \mathbb{N}.$$

Therefore, by the lower semicontinuity of the norm $\|\cdot\|_V$, we get

$$\|S(f) - S(g)\|_V \leq \liminf_{m\to\infty}\|\mathcal{A}_m^{-1}(f) - \mathcal{A}_m^{-1}(g)\|_V \leq \left(\frac{1}{\alpha}\right)^{\frac{1}{p-1}} \|f-g\|_{V^*}^{\frac{1}{p-1}}.$$

For the sake of completeness, the extension of S to $V^* \setminus X$ is defined by

$$S(f) := \lim_{n\to\infty} S(f_n)$$

for any $f \in V^*$ and $\{f_n\}_n \subset X$ such that $f_n \to f$ in V^*.

3rd step. We show the invertibility of S in V^*, in terms of Theorem 3.2.5. First, let us show that S is monotone and coercive.

Let $f, g \in V^*$. Then, by Proposition 3.2.7 (a), it holds that

$$\begin{aligned}
\langle S(f) - S(g), f - g\rangle_{V\times V^*} &= \lim_{m\to\infty} \left\langle \mathcal{A}_m^{-1}(f) - \mathcal{A}_m^{-1}(g), f - g\right\rangle_{V\times V^*} \\
&= \lim_{m\to\infty} \langle \mathcal{A}_m(u_m) - \mathcal{A}_m(v_m), u_m - v_m\rangle_{V^*\times V} \\
&\geq \alpha \lim_{m\to\infty} \|u_m - v_m\|_V^p \geq 0.
\end{aligned}$$

Moreover, by Proposition 3.2.7

$$\begin{aligned}
\|\mathcal{A}_m(u_m) - \mathcal{A}_m(v_m)\|_{V^*}^p &\leq \beta^p \left[|\Omega| + \|u_m\|_V^p + \|v_m\|_V^p\right]^{p-2} \|u_m - v_m\|_V^p \\
&\leq \frac{\beta^p}{\alpha} \left[|\Omega| + \|u_m\|_V^p + \|v_m\|_V^p\right]^{p-2} \langle \mathcal{A}_m(u_m) - \mathcal{A}_m(v_m), u_m - v_m\rangle_{V^*\times V} \\
&\leq \frac{\beta^p}{\alpha} \left[|\Omega| + \left(\frac{1}{\alpha}\right)^{p'} \|f\|_{V^*}^{p'} + \left(\frac{1}{\alpha}\right)^{p'} \|g\|_{V^*}^{p'}\right]^{p-2} \langle \mathcal{A}_m^{-1}(f) - \mathcal{A}_m^{-1}(g), f - g\rangle_{V\times V^*}
\end{aligned}$$

and, passing to the limit, we get

$$\|f - g\|_{V^*}^p \leq \frac{\beta^p}{\alpha} \left[|\Omega| + \left(\frac{1}{\alpha}\right)^{p'} \|f\|_{V^*}^{p'} + \left(\frac{1}{\alpha}\right)^{p'} \|g\|_{V^*}^{p'}\right]^{p-2} \langle S(f) - S(g), f - g\rangle_{V\times V^*}.$$

Therefore, in virtue of Theorem 3.2.5, S is invertible in the whole space V^*.

We get the thesis by defining $\mathcal{A}_\infty : V \to V^*$ as

$$\mathcal{A}_\infty(u) := S^{-1}(u) \quad \text{for any } u \in V.$$

□

3.2.1 Convergence of momenta by Div-curl lemma

Differently from the first section of this chapter, the convergence of momenta for monotone operators on Carnot groups is obtained by adapting the well-known technique of *compensated compactness* introduced by Murat and Tartar in the '70s and well explained in [122].

The first result in this direction shows that the sequence of momenta converges to a continuous operator M in the weak topology of $L^{p'}(\Omega, H\mathbb{G})$.

Lemma 3.2.11. *Let $\{\mathcal{A}_n\}_n$ satisfy the hypotheses of Lemma 3.2.10. Then, up to subsequences, there exists a continuous operator $M : W_{\mathbb{G}}^{-1,p'}(\Omega) \to L^{p'}(\Omega, H\mathbb{G})$ such that*

$$A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f)) \rightharpoonup M(f) \text{ weakly in } L^{p'}(\Omega, H\mathbb{G})$$

for every $f \in W_{\mathbb{G}}^{-1,p'}(\Omega)$.

Proof. Let X be a countable and dense subspace of $L^{p'}(\Omega, H\mathbb{G})$ and let $f \in X$. Repeating the same techniques of the previous lemma, let us show the existence of a diagonal subsequence of $\{A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f))\}_n$, weakly convergent in $L^{p'}(\Omega, H\mathbb{G})$.

Since $\{A^n\}_n \subset \mathcal{M}(\alpha, \beta; \Omega)$ then, by the Hölder inequality, it holds that

$$\begin{aligned}
\int_\Omega |A^n(x, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f))|^{p'}\, dx &\leq \beta^{p'} \int_\Omega \left[1 + |\nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f)|^p\right]^{\frac{p-2}{p-1}} |\nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f)|^{p'}\, dx \\
&\leq \beta^{p'} \left(\int_\Omega \left[1 + |\nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f)|^p\right]\, dx\right)^{\frac{p-2}{p-1}} \left(\int_\Omega |\nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f)|^p\, dx\right)^{\frac{p'}{p}} \\
&= \beta^{p'} \left[|\Omega| + \|\mathcal{A}_n^{-1}(f)\|_V^p\right]^{\frac{p-2}{p}p'} \|\mathcal{A}_n^{-1}(f)\|_V^{p'},
\end{aligned}$$

i.e.,

$$\|A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f))\|_{L^{p'}(\Omega, H\mathbb{G})} \leq \beta \left[|\Omega| + \|\mathcal{A}_n^{-1}(f)\|_V^p\right]^{\frac{p-2}{p}} \|\mathcal{A}_n^{-1}(f)\|_V .$$

Therefore, by Proposition 3.2.7 (b), we get

$$\|A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f))\|_{L^{p'}(\Omega, H\mathbb{G})} \leq \frac{\beta}{\alpha^{\frac{1}{p-1}}} \left[|\Omega| + \left(\frac{1}{\alpha}\right)^{p'} \|f\|_{V^*}^{p'}\right]^{\frac{p-2}{p}} \|f\|_{V^*}^{\frac{1}{p-1}},$$

that is, the sequence $\{A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f))\}_n$ is bounded in $L^{p'}(\Omega, H\mathbb{G})$. Then, there exists a diagonal subsequence of $\{A^n(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_n^{-1}(f))\}_n$ weakly convergent in $L^{p'}(\Omega, H\mathbb{G})$ to

$$M(f) := \lim_{m\to\infty} A^m(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_m^{-1}(f)) \quad \text{for any } f \in X.$$

We conclude by extending M to the whole space V^*. Let $f, g \in X$. By Proposition 3.2.7, we have

$$\begin{aligned}\|A^m(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_m^{-1}(f)) - A^m(\cdot, \nabla_{\mathbb{G}}\mathcal{A}_m^{-1}(g))\|_{L^{p'}(\Omega,H\mathbb{G})}\\ \leq \frac{\beta}{\alpha^{\frac{1}{p-1}}}\left[|\Omega| + \left(\frac{1}{\alpha}\right)^{p'}\|f\|_{V^*}^{p'} + \left(\frac{1}{\alpha}\right)^{p'}\|g\|_{V^*}^{p'}\right]^{\frac{p-2}{p}}\|f-g\|_{V^*}^{\frac{1}{p-1}}.\end{aligned}$$

Then, by the lower semicontinuity of the norm $\|\cdot\|_{L^{p'}(\Omega,H\mathbb{G})}$, we finally have

$$\|M(f) - M(g)\|_{L^{p'}(\Omega,H\mathbb{G})} \leq \frac{\beta}{\alpha^{\frac{1}{p-1}}}\left[|\Omega| + \left(\frac{1}{\alpha}\right)^{p'}\|f\|_{V^*}^{p'} + \left(\frac{1}{\alpha}\right)^{p'}\|g\|_{V^*}^{p'}\right]^{\frac{p-2}{p}}\|f-g\|_{V^*}^{\frac{1}{p-1}}.$$

□

In order to complete the proof of Theorem 3.2.9, we need a last step. The following result was given by Baldi, Franchi, Tchou and Tesi in [13, Theorem 5.1]. The definitions of weights and $\text{curl}_{\mathbb{G}}$, omitted here, can be found in [13, Section 5].

Theorem 3.2.12 (Div-curl lemma)**.** *Let $\Omega \subset \mathbb{G}$ be an open set and let $p, q > 1$ be a Hölder conjugate pair. Moreover, following the notations of [13], if $\sigma \in \mathcal{I}_0^2$ (where $\mathcal{I}_0^2$ is defined in [13, (20)]), let $a(\sigma) > 1$ and $b > 1$ be such that*

$$a(\sigma) > \frac{Qp}{Q + (\sigma - 1)p} \quad \textit{and} \quad b > \frac{Qq}{Q+q}.$$

Taking into account [13, Definition 2.3], let us now consider the sequences of horizontal vector fields $\{E^n\}_n \subset L^p_{loc}(\Omega, H\mathbb{G})$ and $\{D^n\}_n \subset L^q_{loc}(\Omega, H\mathbb{G})$, weakly convergent, respectively, to $E \in L^p_{loc}(\Omega, H\mathbb{G})$ and $D \in L^q_{loc}(\Omega, H\mathbb{G})$ and such that:

(i) *the components of $\{\text{curl}_{\mathbb{G}} E^n\}_n$ of weight σ are bounded in $L^{a(\sigma)}_{loc}(\Omega, H\mathbb{G})$;*

(ii) $\{\mathrm{div}_{\mathbb{G}} D^n\}_n$ *is bounded in* $L^b_{loc}(\Omega, H\mathbb{G})$.

Then

$$\langle D^n, E^n\rangle_{\mathbb{R}^m} \to \langle D, E\rangle_{\mathbb{R}^m} \ \textit{in } \mathcal{D}'(\Omega)\,,$$

that is,

$$\int_\Omega \langle D^n(x), E^n(x)\rangle_{\mathbb{R}^m}\varphi(x)\,dx \to \int_\Omega \langle D(x), E(x)\rangle_{\mathbb{R}^m}\varphi(x)\,dx$$

for any $\varphi \in \mathcal{D}(\Omega)$.

Proof of Theorem 3.2.9. Given the operators $\mathcal{A}_\infty$ and M, defined in the proofs of Lemma 3.2.10 and Lemma 3.2.11, let us consider the following composition

$$C := M \circ \mathcal{A}_\infty : W^{1,p}_{\mathbb{G},0}(\Omega) \to L^{p'}(\Omega, H\mathbb{G})$$

and let us show the existence of $A^{\mathrm{eff}} \in \mathcal{M}(\alpha, \beta; \Omega)$, such that

$$C(u) = A^{\mathrm{eff}}(x, \nabla_{\mathbb{G}}\mathcal{A}^{-1}_\infty(f))$$

for every $f \in W^{-1,p'}_{\mathbb{G}}(\Omega)$ and for any $u \in W^{1,p}_{\mathbb{G},0}(\Omega)$ such that $\mathcal{A}_\infty(u) = f$ a.e. $x \in \Omega$.

Let $f \in W^{-1,p'}_{\mathbb{G}}(\Omega)$, let ω be an open set such that $\overline{\omega} \subset \Omega$ and let $v \in W^{1,p}_{\mathbb{G},0}(\Omega)$ be the solution of

$$-\mathrm{div}_{\mathbb{G}}(A(x, \nabla_{\mathbb{G}} v)) = f \quad \text{in } \Omega.$$

We define $A^{\mathrm{eff}} : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ as

$$A^{\mathrm{eff}}(x, \xi) := C(v) \quad \text{if } \nabla_{\mathbb{G}} v(x) = \xi \text{ a.e. } x \in \omega\,.$$

The previous definition makes sense if

$$A^{\mathrm{eff}}(x, \xi_1) = A^{\mathrm{eff}}(x, \xi_2) \quad \text{for any } \xi_1 = \xi_2 \in \mathbb{R}^m \text{ a.e. } x \in \omega_1 \cap \omega_2\,,$$

where ω_1, ω_2 are open sets such that $\overline{\omega_1}, \overline{\omega_2} \subset \Omega$.

For $i = 1, 2$, let us consider the following spaces, functions and sequences:

(a) ω_i open sets such that $\overline{\omega_i} \subset \Omega$;

(b) $\varphi_i \in \mathbf{C}_c^1(\Omega)$ such that $\varphi_i|_{\omega_i} = 1$;

(c) $\xi_i \in \mathbb{R}^m$;

(d) $\{v_{i,n}\}_n \subset W^{1,p}_{\mathbb{G},0}(\Omega)$ weakly convergent, up to subsequences, in $W^{1,p}_{\mathbb{G},0}(\Omega)$ to

$$v_{i,\infty}(x) = \varphi_i(x)\langle \xi_i, \pi(x)\rangle_{\mathbb{R}^m}$$

where $\pi(x) = (x_1, .., x_m)$ for every $x = (x_1, .., x_n) \in \Omega$;

(e) $f_i \in W^{-1,p'}_{\mathbb{G}}(\Omega)$ such that $f_i = -\mathrm{div}_{\mathbb{G}} C(v_{i,\infty})$;

(f) $\{D_i^n\}_n := \{A^n(\cdot, \nabla_{\mathbb{G}} v_{i,n})\}_n \subset L^{p'}(\Omega, H\mathbb{G})$;

(g) $\{E_i^n\}_n := \{\nabla_{\mathbb{G}} v_{i,n}\}_n \subset L^p(\Omega, H\mathbb{G})$.

Notice that, by definition

$$\nabla_{\mathbb{G}} v_{i,\infty} = \xi_i \quad \text{in } \omega_i, i = 1, 2.$$

Moreover, there exist $\{D_i^m\}_m$ and $\{E_i^m\}_m$ diagonal subsequences of $\{D_i^n\}_n$ and $\{E_i^n\}_n$ and there exist $D_i \in L^{p'}(\Omega, H\mathbb{G})$ and $E_i \in L^p(\Omega, H\mathbb{G})$ such that

$$D_i^m \rightharpoonup D_i \quad \text{weakly in } L^{p'}(\Omega, H\mathbb{G}) ,$$
$$E_i^m \rightharpoonup E_i \quad \text{weakly in } L^p(\Omega, H\mathbb{G}) .$$

Since, for any $m \in \mathbb{N}$ and $i = 1, 2$

$$\mathrm{curl}_{\mathbb{G}}(E_i^m) = 0,\ D_i(\cdot) = A^{\mathrm{eff}}(\cdot, \xi_i) \text{ and } E_i = \xi_i \text{ in } \omega_i,$$

then, by Theorem 3.2.12

$$\int_\Omega \langle D_2^m - D_1^m, E_2^m - E_1^m\rangle_{\mathbb{R}^m} \varphi \, dx \to \int_\Omega \langle D_2 - D_1, E_2 - E_1\rangle_{\mathbb{R}^m} \varphi \, dx$$

for any $\varphi \in \mathcal{D}(\omega_1 \cap \omega_2)$, i.e.,

$$\int_\Omega \langle A^m(x, \nabla_{\mathbb{G}} v_{2,m}) - A^m(x, \nabla_{\mathbb{G}} v_{1,m}), \nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m}\rangle_{\mathbb{R}^m} \varphi(x)\, dx \\ \to \int_\Omega \langle A^{\text{eff}}(x, \xi_2) - A^{\text{eff}}(x, \xi_1), \xi_2 - \xi_1\rangle_{\mathbb{R}^m} \varphi(x)\, dx$$

for any $\varphi \in \mathcal{D}(\omega_1 \cap \omega_2)$.

Let $\varphi \in \mathcal{D}(\omega_1 \cap \omega_2)$ be such that $\varphi \geq 0$. Since $\{A^m\}_m \subset \mathcal{M}(\alpha, \beta; \Omega)$, then

$$\begin{aligned} &\int_\Omega \langle A^{\text{eff}}(x, \xi_2) - A^{\text{eff}}(x, \xi_1), \xi_2 - \xi_1\rangle_{\mathbb{R}^m} \varphi(x)\, dx \\ &\qquad \geq \liminf_{m\to\infty} \int_\Omega \langle A^m(x, \nabla_{\mathbb{G}} v_{2,m}) - A^m(x, \nabla_{\mathbb{G}} v_{1,m}), \nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m}\rangle_{\mathbb{R}^m} \varphi(x)\, dx \\ &\qquad \geq \liminf_{m\to\infty} \alpha \int_\Omega |\nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m}|^p \varphi(x)\, dx \\ &\qquad \geq \alpha \int_\Omega |\nabla_{\mathbb{G}} v_{2,\infty} - \nabla_{\mathbb{G}} v_{1,\infty}|^p \varphi(x)\, dx = \alpha \int_\Omega |\xi_2 - \xi_1|^p \varphi(x)\, dx \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega \langle A^{\text{eff}}(x, \xi_2) - A^{\text{eff}}(x, \xi_1), \xi_2 - \xi_1\rangle_{\mathbb{R}^m} \varphi(x)\, dx \geq \liminf_{m\to\infty} \alpha \int_\Omega |\nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m}|^p \varphi(x)\, dx \\ &\geq \liminf_{m\to\infty} \frac{\alpha}{\beta^p} \int_\Omega [1 + |\nabla_{\mathbb{G}} v_{2,m}|^p + |\nabla_{\mathbb{G}} v_{1,m}|^p]^{2-p} |A^m(x, \nabla_{\mathbb{G}} v_{2,m}) - A^m(x, \nabla_{\mathbb{G}} v_{1,m})|^p \varphi(x)\, dx \\ &\geq \frac{\alpha}{\beta^p} \int_\Omega [1 + |\nabla_{\mathbb{G}} v_{2,\infty}|^p + |\nabla_{\mathbb{G}} v_{1,\infty}|^p]^{2-p} |A^{\text{eff}}(x, \nabla_{\mathbb{G}} v_{2,\infty}) - A^{\text{eff}}(x, \nabla_{\mathbb{G}} v_{1,\infty})|^p \varphi(x)\, dx \\ &= \frac{\alpha}{\beta^p} \int_\Omega [1 + |\xi_2|^p + |\xi_1|^p]^{2-p} |A^{\text{eff}}(x, \xi_2) - A^{\text{eff}}(x, \xi_1)|^p \varphi(x)\, dx\,. \end{aligned}$$

Restricting the integrals to $\omega_1 \cap \omega_2$ and varying φ, we get

(a) $\langle A^{\text{eff}}(x, \xi_2) - A^{\text{eff}}(x, \xi_1), \xi_2 - \xi_1\rangle_{\mathbb{R}^m} \geq \alpha |\xi_2 - \xi_1|^p$;

(b) $\langle A^{\text{eff}}(x, \xi_2) - A^{\text{eff}}(x, \xi_1), \xi_2 - \xi_1\rangle_{\mathbb{R}^m} \geq \dfrac{\alpha}{\beta^p} [1 + |\xi_2|^p + |\xi_1|^p]^{2-p} |A^{\text{eff}}(x, \xi_2) - A^{\text{eff}}(x, \xi_1)|^p$

a.e. $x \in \omega_1 \cap \omega_2$.

For $\xi_1 = \xi_2$, we obtain $A^{\text{eff}}(x, \xi_1) = A^{\text{eff}}(x, \xi_2)$ a.e. $x \in \omega_1 \cap \omega_2$, while, taking $\xi_1 \neq \xi_2$, and recalling that $A^{\text{eff}}(\cdot, 0) = 0$ by definition, we deduce that

$$A^{\text{eff}} \in \mathcal{M}(\alpha, \beta; \Omega).$$

Remark 3.2.13. For the sake of completeness, we show below the validity of condition (*iii*) of Definition 3.2.1. Observe that

$$\begin{aligned}
&\int_\Omega |\xi_2 - \xi_1|^p \varphi(x)\,dx \geq \liminf_{m\to\infty} \int_\Omega |\nabla_{\mathbb{G}} v_{2,m} - \nabla_{\mathbb{G}} v_{1,m}|^p \varphi(x)\,dx \\
&\geq \liminf_{m\to\infty} \frac{1}{\beta^p} \int_\Omega [1 + |\nabla_{\mathbb{G}} v_{2,m}|^p + |\nabla_{\mathbb{G}} v_{1,m}|^p]^{2-p} \, |A^m(x, \nabla_{\mathbb{G}} v_{2,m}) - A^m(x, \nabla_{\mathbb{G}} v_{1,m})|^p \varphi(x)\,dx \\
&\geq \frac{1}{\beta^p} \int_\Omega [1 + |\nabla_{\mathbb{G}} v_{2,\infty}|^p + |\nabla_{\mathbb{G}} v_{1,\infty}|^p]^{2-p} \, |A^{\mathrm{eff}}(x, \nabla_{\mathbb{G}} v_{2,\infty}) - A^{\mathrm{eff}}(x, \nabla_{\mathbb{G}} v_{1,\infty})|^p \varphi(x)\,dx \\
&= \frac{1}{\beta^p} \int_\Omega [1 + |\xi_2|^p + |\xi_1|^p]^{2-p} \, |A^{\mathrm{eff}}(x, \xi_2) - A^{\mathrm{eff}}(x, \xi_1)|^p \varphi(x)\,dx \,.
\end{aligned}$$

Let now $u_\infty \in W^{1,p}_{\mathbb{G},0}(\Omega)$ be the solution of

$$\mathcal{A}_\infty(u) = f \quad \text{in } \Omega$$

and let $\{u_m\}_m \subset W^{1,p}_{\mathbb{G},0}(\Omega)$ be weakly convergent to u_∞ in $W^{1,p}_{\mathbb{G},0}(\Omega)$. We conclude the proof of the theorem by showing that

$$C(u_\infty) = A^{\mathrm{eff}}(x, \nabla_{\mathbb{G}} u_\infty) \quad \text{a.e. } x \in \Omega.$$

For $D_2^m = A^m(x, \nabla_{\mathbb{G}} u_m)$, $E_2^m = \nabla_{\mathbb{G}} u_m$ and for every $\varphi \in \mathcal{D}(\omega_1)$ such that $\varphi \geq 0$, we have, by Theorem 3.2.12

$$\begin{aligned}
&\int_\Omega \langle A^m(x, \nabla_{\mathbb{G}} u_m) - A^m(x, \nabla_{\mathbb{G}} v_{1,m}), \nabla_{\mathbb{G}} u_m - \nabla_{\mathbb{G}} v_{1,m} \rangle_{\mathbb{R}^m} \varphi(x)\,dx \\
&\qquad \to \int_\Omega \langle C(u_\infty) - A^{\mathrm{eff}}(x, \xi_1), \nabla_{\mathbb{G}} u_\infty - \xi_1 \rangle_{\mathbb{R}^m} \varphi(x)\,dx \,.
\end{aligned}$$

Then, following the techniques of the first part of the proof, we get

(*c*) $\langle C(u_\infty) - A^{\mathrm{eff}}(x, \xi_1), \nabla_{\mathbb{G}} u_\infty - \xi_1 \rangle_{\mathbb{R}^m} \geq \alpha |\nabla_{\mathbb{G}} u_\infty - \xi_1|^p$;

(*d*) $\langle C(u_\infty) - A^{\mathrm{eff}}(x, \xi_1), \nabla_{\mathbb{G}} u_\infty - \xi_1 \rangle_{\mathbb{R}^m} \geq \dfrac{\alpha}{\beta^p} [1 + |\nabla_{\mathbb{G}} u_\infty|^p + |\xi_1|^p]^{2-p} \, |C(u_\infty) - A^{\mathrm{eff}}(x, \xi_1)|^p$.

It follows that

$$\left| C(u_\infty) - A^{\mathrm{eff}}(x, \xi_1) \right| \leq \beta \, [1 + |\nabla_{\mathbb{G}} u_\infty|^p + |\xi_1|^p]^{\frac{p-2}{p}} \, |\nabla_{\mathbb{G}} u_\infty - \xi_1| \quad \text{a.e. } x \in \omega_1.$$

Therefore, varying ω_1 and ξ_1, we get the thesis. □

Chapter Four

Asymptotic behaviours in fractional Orlicz-Sobolev spaces depending on vector fields

The last chapter of the thesis is devoted to generalizations of Bourgain-Brezis-Mironescu and Maz'ya-Shaposhnikova formulas in the setting of Orlicz-Sobolev spaces. In the first part of the chapter, we prove a Bourgain-Brezis-Mironescu formula in the framework of Carnot groups (see Section 1.4.1 for details), while a Maz'ya-Shaposhnikova formula in the magnetic setting is provided in the second part (see Section 1.4.2). This last result follows from Theorem 4.2.1, which is a generalization of the Hardy-type inequality, given by Maz'ya and Shaposhnikova in [93, Theorem 2].

4.1 Bourgain-Brezis-Mironescu formula for fractional Orlicz-Sobolev spaces on Carnot groups

The main result of this section is the following generalization of the classic Bourgain-Brezis-Mironescu formula [28, Theorem 2] to the framework of Orlicz-Sobolev spaces on Carnot groups, introduced in Section 1.4.1.

Theorem 4.1.1. *Let φ be an Orlicz function satisfying condition* (L) *and let $\tilde{\varphi}$ be as in* (1.23)*. Then, for any $u \in L^{\varphi}(\mathbb{R}^n)$, it holds that*

$$\lim_{s\uparrow 1}(1-s)\iint_{\mathbb{G}\times\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^{s}}\right)\frac{dx\,dy}{|y^{-1}\cdot x|_{\mathbb{G}}^{Q}} = \int_{\mathbb{G}} \tilde{\varphi}(|\nabla_{\mathbb{G}} u|_{\mathbb{R}^m})\,dx\,,$$

with the convention that $\Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}} u|_{\mathbb{R}^m}) = \infty$ if $u \notin W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$.

The proof of Theorem 4.1.1 relies on the application of the following two lemmas.

Lemma 4.1.2. *Let $u \in W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$. Then, for any $0 < s < 1$, it holds that*

$$\Phi_{s,\varphi}(u) \leq \frac{QC_b}{p^-}\left(\frac{1}{1-s}\Phi_{\varphi}(|\nabla_{\mathbb{G}} u|_{\mathbb{R}^m}) + \frac{\mathbf{C}}{s}\Phi_{\varphi}(u)\right),$$

where $\mathbf{C}$ is the Δ_2-constant given in (1.22)*, p^- is given in* (L) *and C_b denotes the Lebesgue measure of the unit ball $B(0,1)$.*

Proof. Let $u \in \mathbf{C}^2_c(\mathbb{R}^n)$. By definition, we can split $\Phi_{s,\varphi}$ as

$$\Phi_{s,\varphi}(u) = I_1 + I_2\,,$$

where, for $h := y^{-1}\cdot x$,

$$I_1 := \int_{\mathbb{G}}\left(\int_{\{|h|_{\mathbb{G}}<1\}} \varphi\left(\frac{|u(x\cdot h)-u(x)|}{|h|_{\mathbb{G}}^{s}}\right)\frac{dh}{|h|_{\mathbb{G}}^{Q}}\right)dx$$
$$I_2 := \int_{\mathbb{G}}\left(\int_{\{|h|_{\mathbb{G}}\geq 1\}} \varphi\left(\frac{|u(x\cdot h)-u(x)|}{|h|_{\mathbb{G}}^{s}}\right)\frac{dh}{|h|_{\mathbb{G}}^{Q}}\right)dx.$$

Since, in virtue of Theorem 1.3.8, $u \in \mathbf{C}^2_c(\mathbb{R}^n)$ implies that u is Pansu differentiable, then, defining the auxiliary function $\xi(t) := u(x\cdot\delta_t h)$, and noticing that

$$u(x\cdot h)-u(x) = \xi(1)-\xi(0) = \int_0^1 \frac{d}{dt}\xi(t)\,dt = \int_0^1 \langle \nabla_{\mathbb{G}} u(x\cdot\delta_t h), h'\rangle_{\mathbb{R}^m}\,dt,$$

by the monotonicity and convexity of φ, we get

$$\begin{aligned}
\varphi\left(\frac{|u(x\cdot h)-u(x)|}{|h|_{\mathbb{G}}^{s}}\right) &\leq \varphi\left(\int_0^1 \frac{|\langle \nabla_{\mathbb{G}} u(x\cdot\delta_t h), h'\rangle_{\mathbb{R}^m}|}{|h|_{\mathbb{G}}^{s}}\,dt\right)\\
&\leq \int_0^1 \varphi\left(\frac{|\langle \nabla_{\mathbb{G}} u(x\cdot\delta_t h), h'\rangle_{\mathbb{R}^m}|}{|h|_{\mathbb{G}}^{s}}\right)dt \qquad (4.1)\\
&\leq \int_0^1 \varphi(|\nabla_{\mathbb{G}} u(x\cdot\delta_t h)|_{\mathbb{R}^m}|h|_{\mathbb{G}}^{1-s})\,dt.
\end{aligned}$$

Thus, by (φ_1) and Proposition 1.3.14

$$\begin{aligned}
I_1 &\le \int_{\mathbb{G}} \left(\int_{\{|h|_{\mathbb{G}}<1\}} \left(\int_0^1 \varphi(|\nabla_{\mathbb{G}} u(x\cdot\delta_t h)|_{\mathbb{R}^m} |h|_{\mathbb{G}}^{1-s})\, dt \right) \frac{dh}{|h|_{\mathbb{G}}^Q} \right) dx \\
&\le \int_{\mathbb{G}} \left(\int_{\{|h|_{\mathbb{G}}<1\}} \left(\int_0^1 \varphi(|\nabla_{\mathbb{G}} u(x\cdot\delta_t h)|_{\mathbb{R}^m})\, dt \right) \frac{|h|_{\mathbb{G}}^{(1-s)p^-}}{|h|_{\mathbb{G}}^Q}\, dh \right) dx \\
&= \int_{\{|h|_{\mathbb{G}}<1\}} |h|_{\mathbb{G}}^{(1-s)p^- - Q}\, dh \int_{\mathbb{G}} \varphi(|\nabla_{\mathbb{G}} u(x)|_{\mathbb{R}^m})\, dx \\
&= QC_b \int_0^1 r^{(1-s)p^- -1}\, dr\, \Phi_\varphi(|\nabla u|_{\mathbb{R}^m}) = \frac{QC_b}{(1-s)p^-} \Phi_\varphi(|\nabla u|_{\mathbb{R}^m}).
\end{aligned}$$

Moreover, by (φ_1), (φ_2), Proposition 1.3.14, the monotonicity of φ and by a change of variables, we have

$$\begin{aligned}
I_2 &\le \int_{\mathbb{G}} \left(\int_{\{|h|_{\mathbb{G}}\ge 1\}} \varphi\left(|u(x\cdot h)| + |u(x)|\right) \frac{dh}{|h|_{\mathbb{G}}^{sp^-+Q}} \right) dx \\
&\le \frac{\mathbf{C}}{2} \int_{\mathbb{G}} \left(\int_{\{|h|_{\mathbb{G}}\ge 1\}} \varphi\left(|u(x\cdot h)|\right) \frac{dh}{|h|_{\mathbb{G}}^{sp^-+Q}} \right) dx + \frac{\mathbf{C}}{2} \int_{\mathbb{G}} \left(\int_{\{|h|_{\mathbb{G}}\ge 1\}} \varphi\left(|u(x)|\right) \frac{dh}{|h|_{\mathbb{G}}^{sp^-+Q}} \right) dx \\
&= \mathbf{C} \int_{\mathbb{G}} \left(\int_{\{|h|_{\mathbb{G}}\ge 1\}} \varphi\left(|u(x)|\right) \frac{dh}{|h|_{\mathbb{G}}^{sp^-+Q}} \right) dx = \mathbf{C} \int_{\{|h|_{\mathbb{G}}\ge 1\}} \frac{dh}{|h|_{\mathbb{G}}^{sp^-+Q}} \int_{\mathbb{G}} \varphi\left(|u(x)|\right) dx \\
&= \mathbf{C} QC_b \int_1^{+\infty} r^{-sp^- -1}\, dr\, \Phi_\varphi(u) = \mathbf{C} \frac{QC_b}{sp^-} \Phi_\varphi(u).
\end{aligned}$$

Let us now fix $u \in W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$ and let $\{u_k\}_k \subset \mathbf{C}^2_c(\mathbb{R}^n)$ be convergent to u in $W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$. Then, by Fatou's Lemma and the continuity of φ, we finally have

$$\begin{aligned}
\Phi_{s,\varphi}(u) &\le \liminf_{k\to\infty} \Phi_{s,\varphi}(u_k) \le \lim_{k\to\infty} \left[\frac{QC_b}{p^-} \left(\frac{1}{1-s} \Phi_\varphi(|\nabla_{\mathbb{G}} u_k|_{\mathbb{R}^m}) + \frac{\mathbf{C}}{s} \Phi_\varphi(u_k) \right) \right] \\
&= \frac{QC_b}{p^-} \left(\frac{1}{1-s} \Phi_\varphi(|\nabla_{\mathbb{G}} u|_{\mathbb{R}^m}) + \frac{\mathbf{C}}{s} \Phi_\varphi(u) \right).
\end{aligned}$$

□

Lemma 4.1.3. *Let φ be an Orlicz function such that $\tilde{\varphi}$ exists and let $u \in \mathbf{C}^2_c(\mathbb{R}^n)$. Then, for every fixed $x \in \mathbb{G}$, we have that*

$$\lim_{s\uparrow 1} (1-s) \int_{\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s} \right) \frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^Q} = \tilde{\varphi}(|\nabla_{\mathbb{G}} u|_{\mathbb{R}^m}).$$

Proof. As before, for any $x \in \mathbb{G}$, let us split

$$\int_{\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^{s}}\right)\frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^{Q}} = I_1 + I_2\,,$$

where

$$I_1 := \int_{\{|y^{-1}\cdot x|_{\mathbb{G}}<1\}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^{s}}\right)\frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^{Q}}$$
$$I_2 := \int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\geq 1\}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^{s}}\right)\frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^{Q}}.$$

Let us first notice that

$$\lim_{s\uparrow 1}(1-s)I_2 = 0.$$

In fact, by (φ_1) and Proposition 1.3.14, we have

$$\begin{aligned}\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\geq 1\}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^{s}}\right)\frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^{Q}} &\leq \varphi(2\|u\|_\infty)\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\geq 1\}}\frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^{sp^-+Q}}\\ &= \varphi(2\|u\|_\infty)QC_b\int_1^{+\infty} r^{-sp^- -1}\,dr\\ &= \frac{QC_b}{sp^-}\varphi(2\|u\|_\infty).\end{aligned}$$

Moreover, by the local Lipschitzianity of φ, for any $x, y \in \mathbb{G}$ such that $x \neq y$, we have

$$\left|\varphi\left(\frac{|u(x)-u(y)|}{|h|_{\mathbb{G}}^{s}}\right) - \varphi\left(\frac{|\langle\nabla_{\mathbb{G}}u(x), h'\rangle_{\mathbb{R}^m}|}{|h|_{\mathbb{G}}^{s}}\right)\right| \leq L\frac{|u(x)-u(y)-\langle\nabla_{\mathbb{G}}u(x), h'\rangle_{\mathbb{R}^m}|}{|h|_{\mathbb{G}}^{s}} \leq C|h|_{\mathbb{G}}^{2-s}$$

where L is the Lipschitz constant of φ in the interval $[0, \|\nabla_{\mathbb{G}}u\|_\infty]$, C is a constant depending on the $\mathbf{C}^2$-norm of u and $h := y^{-1}\cdot x$. The last inequality follows from standard results about the Taylor polynomial that can be found, for instance, in [27, Chapter 20].

Since, by Proposition 1.3.14

$$\int_{\{|h|_{\mathbb{G}}<1\}} |h|_{\mathbb{G}}^{2-s}\frac{dy}{|h|_{\mathbb{G}}^{Q}} = QC_b\int_0^1 r^{1-s}\,dr = \frac{QC_b}{2-s}\,,$$

then

$$\begin{aligned}&\lim_{s\uparrow 1}(1-s)\int_{\{|y^{-1}\cdot x|_{\mathbb{G}}<1\}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^{s}}\right)\frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^{Q}}\\ &= \lim_{s\uparrow 1}(1-s)\int_{\{|h|_{\mathbb{G}}<1\}} \varphi\left(\frac{|\langle\nabla_{\mathbb{G}}u(x), h'\rangle_{\mathbb{R}^m}|}{|h|_{\mathbb{G}}^{s}}\right)\frac{dy}{|h|_{\mathbb{G}}^{Q}}\,.\end{aligned}$$

Finally, by Proposition 1.3.15 and the invariance of $|\cdot|_{\mathbb{G}}$ under horizontal rotations, we have

$$\begin{aligned}\int_{\{|h|_{\mathbb{G}}<1\}} \varphi\left(\frac{|\langle \nabla_{\mathbb{G}} u(x), h'\rangle_{\mathbb{R}^m}|}{|h|^s_{\mathbb{G}}}\right)\frac{dy}{|h|^Q_{\mathbb{G}}} &= \int_0^1 \left(\int_S \varphi\left(\frac{|\langle \nabla_{\mathbb{G}} u(x), \delta_r z'\rangle_{\mathbb{R}^m}|}{|\delta_r z|^s_{\mathbb{G}}}\right)\frac{d\sigma(z)}{|\delta_r z|^Q_{\mathbb{G}}}\right) r^{Q-1}\,dr \\ &= \int_0^1 \left(\int_S \varphi\left(\frac{|\langle \nabla_{\mathbb{G}} u(x), z'\rangle_{\mathbb{R}^m}|}{r^s|z|^s_{\mathbb{G}}} r\right)\frac{d\sigma(z)}{|z|^Q_{\mathbb{G}}}\right) \frac{r^{Q-1}}{r^Q}\,dr \\ &= \int_0^1 \left(\int_S \varphi(|\nabla_{\mathbb{G}} u(x)|_{\mathbb{R}^m}|z'|_{\mathbb{R}^m} r^{1-s}) d\sigma(z)\right)\frac{dr}{r}\,,\end{aligned}$$

i.e.,

$$\lim_{s\uparrow 1}(1-s)I_1 = \tilde{\varphi}(|\nabla_{\mathbb{G}} u(x)|_{\mathbb{R}^m}).$$

□

We are now in the position to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. We divide the proof of the theorem in three steps.

1st step. First, let us prove the result for any $u \in \mathbf{C}^2_c(\mathbb{R}^n)$.

Let $R > 1$ and let $u \in \mathbf{C}^2_c(\mathbb{R}^n)$ be such that $\mathrm{supp}(u) \subset B(0, R)$. For any $0 < s < 1$ and $x \in \mathbb{G}$, let us define

$$F_s(x) := \int_{\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|^s_{\mathbb{G}}}\right)\frac{dy}{|y^{-1}\cdot x|^Q_{\mathbb{G}}}\,.$$

If $|x|_{\mathbb{G}} < 2R$, we can split F_s as

$$F_s(x) = I_1 + I_2\,,$$

where

$$\begin{aligned} I_1 &:= \int_{\{|y^{-1}\cdot x|_{\mathbb{G}}<1\}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|^s_{\mathbb{G}}}\right)\frac{dy}{|y^{-1}\cdot x|^Q_{\mathbb{G}}} \\ I_2 &:= \int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\ge 1\}} \varphi\left(\frac{|u(x)-u(y)|}{|y^{-1}\cdot x|^s_{\mathbb{G}}}\right)\frac{dy}{|y^{-1}\cdot x|^Q_{\mathbb{G}}}\,. \end{aligned}$$

By (4.1), (φ_1), Proposition 1.3.14 and the monotonicity of φ, named $h := y^{-1}\cdot x$, we have

$$\begin{aligned} I_1 &\leq \int_{\{|h|_{\mathbb{G}}<1\}} \left(\int_0^1 \varphi(|\nabla_{\mathbb{G}} u(x\cdot \delta_t h)|_{\mathbb{R}^m} |h|_{\mathbb{G}}^{1-s})\, dt\right) \frac{dh}{|h|_{\mathbb{G}}^Q} \\ &\leq \int_{\{|h|_{\mathbb{G}}<1\}} \left(\int_0^1 \varphi(|\nabla_{\mathbb{G}} u(x\cdot \delta_t h)|_{\mathbb{R}^m})\, dt\right) \frac{|h|_{\mathbb{G}}^{(1-s)p^-}}{|h|_{\mathbb{G}}^Q}\, dh \\ &\leq \varphi(||\nabla_{\mathbb{G}} u||_\infty) Q C_b \int_0^1 r^{(1-s)p^- -1}\, dr = \frac{QC_b}{(1-s)p^-}\varphi(||\nabla_{\mathbb{G}} u||_\infty) \end{aligned}$$

and

$$\begin{aligned} I_2 &\leq \int_{\{|y^{-1}\cdot x|_{\mathbb{G}}\geq 1\}} \varphi\left(|u(x)|+|u(y)|\right) \frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^{sp^-+Q}} \\ &\leq \varphi(2||u||_\infty) Q C_b \int_1^\infty r^{-sp^- -1}\, dr = \frac{QC_b}{sp^-}\varphi(2||u||_\infty)\,, \end{aligned}$$

that is,

$$F_s(x) \leq \frac{QC_b}{(1-s)p^-}\varphi(||\nabla_{\mathbb{G}} u||_\infty) + \frac{QC_b}{sp^-}\varphi(2||u||_\infty)\,. \tag{4.2}$$

If $|x|_{\mathbb{G}} \geq 2R$, since $\mathrm{supp}(u) \subset B(0,R)$, then

$$F_s(x) = \int_{\{|y|_{\mathbb{G}}\leq R\}} \varphi\left(\frac{|u(y)|}{|y^{-1}\cdot x|_{\mathbb{G}}^s}\right) \frac{dy}{|y^{-1}\cdot x|_{\mathbb{G}}^Q}$$

and, by the triangular inequality, it holds that

$$|y^{-1}\cdot x|_{\mathbb{G}} \geq |x|_{\mathbb{G}} - |y|_{\mathbb{G}} \geq |x|_{\mathbb{G}} - R \geq \frac{1}{2}|x|_{\mathbb{G}} \quad \text{for any } |y|_{\mathbb{G}} \leq R\,.$$

Therefore, for any $s \geq \frac{1}{2}$, by the monotonicity of φ, the Δ_2-condition and (φ_1), we get

$$\begin{aligned} F_s(x) &\leq \int_{\{|y|_{\mathbb{G}}\leq R\}} \varphi\left(\frac{|u(y)|}{(\frac{1}{2}|x|_{\mathbb{G}})^s}\right) \frac{dy}{(\frac{1}{2}|x|_{\mathbb{G}})^Q} \leq 2^Q \int_{\{|y|_{\mathbb{G}}\leq R\}} \varphi\left(2^s|u(y)|\right) \frac{dy}{|x|_{\mathbb{G}}^{sp^-+Q}} \\ &\leq \mathbf{C}\frac{2^Q}{|x|_{\mathbb{G}}^{sp^-+Q}} \int_{\{|y|_{\mathbb{G}}\leq R\}} \varphi\left(|u(y)|\right)\, dy \leq \mathbf{C}\frac{2^Q}{|x|_{\mathbb{G}}^{\frac{1}{2}p^-+Q}} \int_{\{|y|_{\mathbb{G}}\leq R\}} \varphi\left(|u(y)|\right)\, dy\,, \end{aligned}$$

that is,

$$F_s(x) \leq \frac{K}{|x|_{\mathbb{G}}^{\frac{1}{2}p^-+Q}}\,, \tag{4.3}$$

where K is a constant independent of s.

Combining (4.2) and (4.3), we finally have

$$F_s(x) \leq \left(\frac{QC_b}{(1-s)p^-}\varphi(||\nabla_{\mathbb{G}}u||_\infty) + \frac{QC_b}{sp^-}\varphi(2||u||_\infty) \right) \chi_{B(0,2R)}(x) + \frac{K}{|x|_{\mathbb{G}}^{\frac{1}{2}p^- + Q}} \chi_{\mathbb{G}\setminus B(0,2R)}(x) =: H(x)\,,$$

that is,

$$(1-s)F_s(x) \leq (1-s)H(x) \in L^1(\mathbb{R}^n).$$

The thesis follows by the dominated convergence theorem and Lemma 4.1.3.

2nd step. We now extend the result to any $u \in W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$.

Let $u \in W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$, $\varepsilon > 0$ and let $\{u_k\}_k \subset \mathbf{C}^2_c(\mathbb{G})$ be convergent to u in $W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$. Since

$$\begin{aligned}|(1-s)\Phi_{s,\varphi}(u) - \Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}}u|_{\mathbb{R}^m})| \leq & |(1-s)\Phi_{s,\varphi}(u) - (1-s)\Phi_{s,\varphi}(u_k)| \\ & + |(1-s)\Phi_{s,\varphi}(u_k) - \Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}}u_k|_{\mathbb{R}^m})| \\ & + |\Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}}u_k|_{\mathbb{R}^m}) - \Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}}u|_{\mathbb{R}^m})|\,,\end{aligned}$$

then, in virtue of the first step of the proof, we show the existence of $\overline{k} \in \mathbb{N}$ such that

$$(1-s)|\Phi_{s,\varphi}(u) - \Phi_{s,\varphi}(u_k)| + |\Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}}u_k|_{\mathbb{R}^m}) - \Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}}u|_{\mathbb{R}^m})| < \varepsilon \quad \text{for any } k \geq \overline{k}\,. \tag{4.4}$$

By Theorem 1.4.8, there exists $k_0 \in \mathbb{N}$ such that

$$|\Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}}u_k|_{\mathbb{R}^m}) - \Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}}u|_{\mathbb{R}^m})| < \frac{\varepsilon}{2} \quad \text{for any } k \geq k_0$$

and, by Lemma 1.4.3, for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\varphi(s+t) \leq C_\delta \varphi(s) + (1+\delta)^{p^+}\varphi(t) \text{ for any } s, t \geq 0.$$

Moreover, there exists $\overline{\delta} > 0$ such that $(1+\delta)^{p^+} \leq 1 + \overline{\delta}$. It follows that

$$\begin{aligned}|\Phi_{s,\varphi}(u) - \Phi_{s,\varphi}(u_k)| \leq & \iint_{\mathbb{G}\times\mathbb{G}} \left| \varphi\left(\frac{|(u-u_k)(x) - (u-u_k)(y)|}{|y^{-1}\cdot x|^s_{\mathbb{G}}} + \frac{|u_k(x) - u_k(y)|}{|y^{-1}\cdot x|^s_{\mathbb{G}}} \right) \right. \\ & \left. - \varphi\left(\frac{|u_k(x) - u_k(y)|}{|y^{-1}\cdot x|^s_{\mathbb{G}}} \right) \right| \frac{dx\,dy}{|y^{-1}\cdot x|^Q_{\mathbb{G}}} \leq C_\delta \Phi_{s,\varphi}(u-u_k) + \overline{\delta}\Phi_{s,\varphi}(u_k).\end{aligned}$$

By Lemma 4.1.2, there exist $k_1 \in \mathbb{N}$ and a positive constant M such that

$$(1-s)\Phi_{s,\varphi}(u-u_k) \leq \frac{\varepsilon}{4C_\delta}$$

and

$$\Phi_{s,\varphi}(u_k) \leq M \quad \text{for any } k \geq k_1 \, .$$

Therefore, taking $\overline{\delta} \leq \dfrac{\varepsilon}{4M(1-s)}$, we get (4.4) assuming $\overline{k} := \max\{k_0, k_1\}$.

3rd step. Let $u \in L^\varphi(\mathbb{R}^n)$. If $\Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}} u|_{\mathbb{R}^m}) = \infty$, then the thesis trivially follows. To conclude the proof of the theorem, we show that if

$$\liminf_{s\uparrow 1}(1-s)\Phi_{s,\varphi}(u) < \infty \, , \tag{4.5}$$

then $u \in W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$.

Let the approximating family $\{u_{k,\varepsilon}\}_{k,\varepsilon} \subset \mathbf{C}^\infty_c(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $\varepsilon > 0$, be defined as

$$u_{k,\varepsilon} := \rho_\epsilon * (\eta_k u) \, ,$$

where $\{\rho_\varepsilon\}_\varepsilon$ is a family of mollifiers and $\{\eta_k\}_k$ is a family of cut-off functions. By (4.5), Lemma 1.4.12 and Lemma 1.4.14, there exists $N > 0$, independent of k and ε, such that

$$\liminf_{s\uparrow 1}(1-s)\Phi_{s,\varphi}(u_{k,\varepsilon}) < N. \tag{4.6}$$

Then, by (4.6) and the first step of the proof,

$$\Phi_{\tilde{\varphi}}(|\nabla_{\mathbb{G}} u_{k,\varepsilon}|_{\mathbb{R}^m})| = \liminf_{s\uparrow 1}(1-s)\Phi_{s,\varphi}(u_{k,\varepsilon}) < \infty \, ,$$

that is, the sequence $\{u_{k,\varepsilon}\}_{k,\varepsilon}$ is bounded in $W^{1,\tilde{\varphi}}_{\mathbb{G}}(\mathbb{R}^n)$ and then, in virtue of Proposition 1.4.5, $\{u_{k,\varepsilon}\}_{k,\varepsilon}$ is bounded in $W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$.

Therefore, by the reflexivity of the space $W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$, there exists $\tilde{u} \in W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$ such that, up to subsequences,

$$u_{k,\varepsilon} \rightharpoonup \tilde{u} \text{ weakly in } W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$$

as $k \uparrow \infty$ and $\varepsilon \downarrow 0$ and, since $u_{k,\varepsilon} \to u$ in $L^\varphi(\mathbb{R}^n)$, then $\tilde{u} = u$ in $W^{1,\varphi}_{\mathbb{G}}(\mathbb{R}^n)$.

The thesis follows by the second step of the proof. □

4.2 Magnetic Hardy-type inequality

In this section we prove the following Hardy-type inequality in the magnetic setting

Theorem 4.2.1. *Given a Orlicz function φ satisfying* (L) *and $s \in (0,1)$ such that $s < \frac{n}{p^+}$, then there exists a constant $C = C(n, s, p^{\pm})$ such that*

$$\int_{\mathbb{R}^n} \varphi\left(\frac{|u(x)|}{|x|^s}\right) dx \leq C \iint_{\mathbb{R}^{2n}} \varphi\left(\left|\frac{u(x) - e^{\mathrm{i}(x-y)A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^s}\right|\right) \frac{dx\, dy}{|x-y|^n} \tag{4.7}$$

for any $u \in W^{s,\varphi}_{A,0}(\mathbb{R}^n; \mathbb{C})$.

The proof of Theorem 4.2.1 comes out as a combination of the Hardy-type inequality proved in [3] with the fractional diamagnetic inequality, which in turn heavily relies upon the so-called *diamagnetic inequality.* The latter is well-known in the classical setting, see e.g. [83, Theorem 7.21], and it reads as follows:

Proposition 4.2.2. *Let $A\colon \Omega \to \mathbb{R}^n$ be a measurable magnetic potential such that $|A| < \infty$ a.e. in Ω and let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C})$. Then*

$$|\nabla |u|(x)| \leq |\nabla u(x) - \mathrm{i}A(x)u(x)|, \tag{4.8}$$

for a.e. $x \in \Omega$.

The fractional analogue of (4.8) was provided in [48, Lemma 3.1 and Remark 3.2]:

Proposition 4.2.3. *Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a measurable magnetic potential such that $|A| < \infty$ a.e. in $\mathbb{R}^n$ and let $u\colon \mathbb{R}^n \to \mathbb{C}$ be a measurable function such that $|u| < \infty$ a.e. in $\mathbb{R}^n$. Then*

$$\big||u(x)| - |u(y)|\big| \leq \left|u(x) - e^{\mathrm{i}(x-y)A\left(\frac{x+y}{2}\right)}u(y)\right|, \tag{4.9}$$

for a.e. $x, y \in \mathbb{R}^n$.

Remark 4.2.4. Observe that, using the compact notation introduced in Section 1.4.2, the fractional diamagnetic inequality (4.9) can be re-stated as

$$\big|D_s|u|(x,y)\big| \leq \big|D_s^A u(x,y)\big| \quad \text{for a.e. } x, y \in \mathbb{R}^n, \tag{4.10}$$

where $D_s v(x,y) = \frac{v(x)-v(y)}{|x-y|^s}$.

With this at hand, we are now ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. Given an Orlicz function φ, by [3, Theorem 5.1], there exist an Orlicz function $\hat{\varphi}$ and a positive constant $C = C(n,s) > 0$ such that

$$\int_{\mathbb{R}^n} \hat{\varphi}\left(\frac{|u(x)|}{|x|^s}\right) dx \leq (1-s) \iint_{\mathbb{R}^{2n}} \varphi(C|D_s u(x,y)|)\, d\mu. \tag{4.11}$$

Moreover, in light of [42, Propositions 5.1 and 5.2], since $s < \frac{n}{p^+}$, then $\hat{\varphi}$ and φ are equivalent Orlicz functions, i.e., there exist two positive constants $c_1 < c_2$ such that

$$\varphi(c_1 t) \leq \hat{\varphi}(t) \leq \varphi(c_2 t) \quad \text{for all } t \in \mathbb{R}_0^+.$$

Therefore, by using inequality (4.11) on $|u|$ and the Diamagnetic inequality (4.9), it holds that

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\left(c_1 \frac{|u(x)|}{|x|^s}\right) dx &\leq (1-s) \iint_{\mathbb{R}^{2n}} \varphi\left(C|D_s|u|(x,y)|\right) d\mu \\ &\leq (1-s) \iint_{\mathbb{R}^{2n}} \varphi\left(C|D_s^A u(x,y)|\right) d\mu.\end{aligned}$$

Thus, the result follows by (φ_1). □

Remark 4.2.5. Let us notice that, accordingly to the Hardy-type inequality proved by Maz'ya-Shaposhnikova, see [93, Theorem 2], and to the analogous result holding in the case of Orlicz spaces, see [3], inequality (4.7) holds just for small values of the fractional parameter s, meaning that $s < \frac{n}{p^+}$. Here, the term p^+, which is defined in (L), is bigger or equal to the upper Matuszewska-Orlicz index $I(\varphi)$, taken into account in [4]. We stress that, if one is interested in proving a Hardy-type inequality for *all* $s \in (0,1)$, one has to reinforce the assumptions on the Orlicz function, see e.g. [3, Section 5]. On the other hand, as explained in [4], the validity of condition (L) ensures that for s small enough such assumptions are satisfied by all Orlicz functions. Finally, we notice that, as a consequence of Theorem 4.2.1 and the equivalence of φ and $\overline{\varphi}$ (see (1.25) for details), it holds that

$$\int_{\mathbb{R}^n} \overline{\varphi}\left(\alpha \frac{|u(x)|}{|x|^s}\right) dx \leq \int_{\mathbb{R}^n} \varphi\left(\alpha \frac{|u(x)|}{|x|^s}\right) dx < \infty \tag{4.12}$$

for any $u \in W^{s,\varphi}_{A,0}(\mathbb{R}^n;\mathbb{C})$ and for any $\alpha > 0$.

4.3 Maz'ya-Shaposhnikova formula for magnetic fractional Orlicz-Sobolev spaces

In this final section of the thesis we are going to prove to following Maz'ya-Shaposhnikova-type formula for magnetic fractional Orlicz-Sobolev spaces.

Theorem 4.3.1. *Let φ be an Orlicz function satisfying* (L) *and let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a magnetic field. If $u \in \bigcup_{s\in(0,1)} W^{s,\varphi}_{A,0}(\mathbb{R}^n;\mathbb{C})$, then*

$$\lim_{s\downarrow 0} s \iint_{\mathbb{R}^n\times\mathbb{R}^n} \varphi\left(\left|\frac{u(x)-e^{\mathrm{i}(x-y)A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^s}\right|\right)\frac{dx\,dy}{|x-y|^n} = \frac{2\,\omega_n}{n}\int_{\mathbb{R}^n}\overline{\varphi}(|u(x)|)\,dx\,. \tag{4.13}$$

The proof of Theorem 4.3.1 follows from the combination of two estimates, the first one for the liminf and the other one for the limsup.

Lemma 4.3.2 (Liminf estimate)**.** *Let $u \in \bigcup_{s\in(0,1)} W^{s,\varphi}_{A,0}(\mathbb{R}^n;\mathbb{C})$. Then*

$$\liminf_{s\downarrow 0} s\,\Phi^A_{s,\varphi}(u) \geq \frac{2\omega_n}{n}\,\Phi_{\overline{\varphi}}(|u|).$$

Proof. If $\liminf_{s\downarrow 0} s\,\Phi^A_{s,\varphi}(u) = \infty$, the result follows. Otherwise, there exists a sequence $\{s_k\}_k \subset (0,1)$, $k\in\mathbb{N}$, such that $s_k \downarrow 0$ and

$$\liminf_{s\downarrow 0} s\,\Phi^A_{s,\varphi}(u) = \lim_{k\to\infty} s_k\,\Phi^A_{s_k,\varphi}(u).$$

By (4.10) and the monotonicity of φ, we get

$$s_k \iint_{\mathbb{R}^{2n}} \varphi(|D_{s_k}|u||)\,d\mu \leq s_k \iint_{\mathbb{R}^{2n}} \varphi(|D^A_{s_k}u|)\,d\mu\,,$$

that is, for any $k\in\mathbb{N}$,

$$s_k\,\Phi_{s_k,\varphi}(|u|) \leq s_k\,\Phi^A_{s_k,\varphi}(u).$$

Thus, taking the limit as $k\to\infty$ in the last inequality and applying [4, Theorem 1.1] to $|u|$, we get

$$\frac{2\omega_n}{n}\,\Phi_{\overline{\varphi}}(|u|) \leq \lim_{s_k\to\infty} s_k\,\Phi^A_{s_k,\varphi}(u),$$

which concludes the proof. $\square$

We can now move to the upper estimate for the limsup.

Lemma 4.3.3 (Limsup estimate)**.** *Let* $u \in \bigcup_{s\in(0,1)} W^{s,\varphi}_{A,0}(\mathbb{R}^n;\mathbb{C})$*. Then*

$$\limsup_{s\downarrow 0} s\,\Phi^A_{s,\varphi}(u) \le \frac{2\omega_n}{n}\,\Phi_{\overline{\varphi}}(|u|).$$

Proof. Let us first observe that, by Fubini's Theorem and a change of variables

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\int_{\{|y|<|x|\}}\varphi\left(\left|\frac{u(x)-e^{i(x-y)A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^s}\right|\right)\frac{dy}{|x-y|^n}\right)dx\\
&=\int_{\mathbb{R}^n}\left(\int_{\{|x|>|y|\}}\varphi\left(\left|\frac{u(x)-e^{i(x-y)A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^s}\right|\right)\frac{dx}{|x-y|^n}\right)dy\\
&=\int_{\mathbb{R}^n}\left(\int_{\{|y|>|x|\}}\varphi\left(\left|\frac{u(y)-e^{-i(x-y)A\left(\frac{x+y}{2}\right)}u(x)}{|x-y|^s}\right|\right)\frac{dy}{|x-y|^n}\right)dx.
\end{aligned}$$

Since

$$\left|\left(\frac{u(y)-e^{-i(x-y)A\left(\frac{x+y}{2}\right)}u(x)}{|x-y|^s}\right)\frac{e^{i(x-y)A\left(\frac{x+y}{2}\right)}}{e^{i(x-y)A\left(\frac{x+y}{2}\right)}}\right| = \left|\frac{u(x)-e^{i(x-y)A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^s}\right|,$$

then

$$\begin{aligned}
s\,\Phi^A_{s,\varphi}(u) &= s\int_{\mathbb{R}^n}\left(\int_{\{|y|\ge|x|\}}\varphi(|D^A_s u(x,y)|)\frac{dy}{|x-y|^n}\right)dx\\
&\quad + s\int_{\mathbb{R}^n}\left(\int_{\{|y|<|x|\}}\varphi(|D^A_s u(x,y)|)\frac{dy}{|x-y|^n}\right)dx\\
&= 2s\int_{\mathbb{R}^n}\left(\int_{\{|y|\ge|x|\}}\varphi(|D^A_s u(x,y)|)\frac{dy}{|x-y|^n}\right)dx.
\end{aligned}$$

Let us now fix $\varepsilon>0$. By the monotonicity and convexity of φ, we can split the previous integral as

$$\begin{aligned}
s\,\Phi^A_{s,\varphi}(u) &= 2s\int_{\mathbb{R}^n}\left(\int_{\{|y|\ge 2|x|\}}\varphi(|D^A_s u(x,y)|)\frac{dy}{|x-y|^n}\right)dx\\
&\quad + 2s\int_{\mathbb{R}^n}\left(\int_{\{|x|\le|y|<2|x|\}}\varphi(|D^A_s u(x,y)|)\frac{dy}{|x-y|^n}\right)dx \qquad (4.14)\\
&\le \frac{2s}{1+\varepsilon}J_1+\frac{2s\varepsilon}{1+\varepsilon}J_2+2sJ_3\,,
\end{aligned}$$

where

$$J_1 := \int_{\mathbb{R}^n} \left(\int_{\{|y|\geq 2|x|\}} \varphi\left((1+\varepsilon) \frac{|u(x)|}{|x-y|^s} \right) \frac{dy}{|x-y|^n} \right) dx$$
$$J_2 := \int_{\mathbb{R}^n} \left(\int_{\{|y|\geq 2|x|\}} \varphi\left(\frac{1+\varepsilon}{\varepsilon} \frac{|u(y)|}{|x-y|^s} \right) \frac{dy}{|x-y|^n} \right) dx$$
$$J_3 := \int_{\mathbb{R}^n} \left(\int_{\{|x|\leq |y| < 2|x|\}} \varphi(|D_s^A u(x,y)|) \frac{dy}{|x-y|^n} \right) dx\,.$$

Taking into account [4, (2.22) and (2.23)], it holds that

$$J_1 \leq \frac{w_n}{ns} \int_{\mathbb{R}^n} \overline{\varphi}\left((1+\varepsilon)\frac{|u(x)|}{|x|^s} \right) dx \tag{4.15}$$

and

$$J_2 \leq w_n \int_{\mathbb{R}^n} \varphi\left(\frac{1+\varepsilon}{\varepsilon} 2^s \frac{|u(y)|}{|y|^s} \right) dy\,. \tag{4.16}$$

To conclude the proof, we only need to provide a suitable upper estimate from above for J_3. We claim that, fixed $N > 3$, there exists $\overline{s} \in (0,1)$ such that

$$J_3 \leq \iint_{\mathbb{R}^n \times E} \varphi\left(N^{\overline{s}-s} |D_{\overline{s}}^A u(x,y)| \right) d\mu + \frac{\varepsilon}{s} \tag{4.17}$$

for any $s \in (0, \overline{s})$, where $E := \{|x| \leq |y| < 2|x|, |x-y| \leq N\}$.

In order to prove (4.17), we first notice that J_3 can be written as

$$J_3 = (i) + (ii),$$

where, denoting by $F := \{|x| \leq |y| < 2|x|,\ |x-y| > N\}$, we write

$$(i) := \iint_{\mathbb{R}^n \times E} \varphi(|D_s^A u(x,y)|)\, d\mu\,, \quad (ii) := \iint_{\mathbb{R}^n \times F} \varphi(|D_s^A u(x,y)|)\, d\mu\,.$$

Since $u \in \bigcup_{s\in(0,1)} W^{s,\varphi}_{A,0}(\mathbb{R}^n;\mathbb{C})$, then there exists $\overline{s} \in (0,1)$ such that $u \in W^{\overline{s},\varphi}_{A,0}(\mathbb{R}^n;\mathbb{C})$. Let now $s < \overline{s}$. Then

$$(i) = \iint_{\mathbb{R}^n \times E} \varphi\left(|x-y|^{\overline{s}-s} |D_{\overline{s}}^A u(x,y)| \right) d\mu \leq \iint_{\mathbb{R}^n \times E} \varphi\left(N^{\overline{s}-s} |D_{\overline{s}}^A u(x,y)| \right) d\mu\,. \tag{4.18}$$

Moreover, since $|x| \leq |y| < 2|x|$ and $|x-y| > N$ imply that

$$|x| > \frac{N}{3} \text{ and } |y| > \frac{N}{3}, \tag{4.19}$$

then, by (4.19), the monotonicity and convexity of φ, by a change of variable and taking into account that $\left|e^{i(x-y)A\left(\frac{x+y}{2}\right)}\right| = 1$, we get

$$\begin{aligned}
(ii) &\leq \frac{1}{2}\iint_{\mathbb{R}^n\times F} \varphi\left(\frac{2|u(x)|}{|x-y|^s}\right)\, d\mu + \frac{1}{2}\iint_{\mathbb{R}^n\times F} \varphi\left(\frac{2|u(y)|}{|x-y|^s}\right)\, d\mu \\
&\leq \frac{1}{2}\int_{\{|x|>\frac{N}{3}\}} \left(\int_{\{|x-y|>N\}} \varphi\left(\frac{2|u(x)|}{|x-y|^s}\right)\, \frac{dy}{|x-y|^n}\right)\, dx \\
&\quad + \frac{1}{2}\int_{\{|y|>\frac{N}{3}\}} \left(\int_{\{|x-y|>N\}} \varphi\left(\frac{2|u(y)|}{|x-y|^s}\right)\, \frac{dx}{|x-y|^n}\right)\, dy \\
&= \int_{\{|x|>\frac{N}{3}\}} \left(\int_{\{|x-y|>N\}} \varphi\left(\frac{2|u(x)|}{|x-y|^s}\right)\, \frac{dy}{|x-y|^n}\right)\, dx\,.
\end{aligned}$$

Finally, the last inequality and a change of variables gives that

$$\begin{aligned}
(ii) &\leq \frac{w_n}{n}\int_{\{|x|>\frac{N}{3}\}} \left(\int_N^{+\infty} \varphi\left(\frac{2|u(x)|}{r^s}\right)\, \frac{dr}{r}\right)\, dx = \frac{w_n}{ns}\int_{\{|x|>\frac{N}{3}\}} \left(\int_0^{\frac{2|u(x)|}{N^s}} \varphi(\tau)\frac{d\tau}{\tau}\right)\, dx \\
&= \frac{w_n}{ns}\int_{\{|x|>\frac{N}{3}\}} \overline{\varphi}\left(\frac{2|u(x)|}{N^s}\right)\, dx < \frac{w_n}{ns}\int_{\{|x|>\frac{N}{3}\}} \overline{\varphi}(2|u(x)|)\, dx < \frac{\varepsilon}{s}
\end{aligned}$$

for N sufficiently large. From this, (4.17) easily follows.

In order to justify the passage to the limsup, we can argue as in [4] once again. In this way, by the arbitrariness of ε, by gathering (4.14), (4.15), (4.16) and (4.17), by (4.12) and by using Fatou's Lemma, we close the proof. □

Proof of Theorem 4.3.1. Combining the lower bounds obtained in Lemma 4.3.2 with the upper bounds provided by Lemma 4.3.3, we finally get (4.13). □

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