HOMOGENIZATION LIMIT AND ASYMPTOTIC DECAY FOR ELECTRICAL CONDUCTION IN BIOLOGICAL TISSUES IN THE HIGH RADIOFREQUENCY RANGE

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Abstract. We derive a macroscopic model of electrical conduction in biological tissues in the high radio-frequency range, which is relevant in applications like electric impedance tomography. This model is derived via a homogenization limit by a microscopic formulation, based on Maxwell's equations, taking into account the periodic geometry of the microstructure. We also study the asymptotic behavior of the solution for large times. Our results imply that periodic boundary data lead to an asymptotically periodic solution.

1. Introduction

Recent developments in diagnostic techniques are drawing attention to the problem of modeling the response of biological tissues to the injection of electrical current [9]. For example, diagnosis of pulmonary emboli, monitoring of heart function and blood flow, and breast cancer detection can benefit from a measurement of the dielectric properties of the living tissue. Indeed, Electric Impedance Tomography (EIT) is the inverse problem of determining the impedance in the interior of a body, given simultaneous measurements of direct or alternating electric currents and voltages at the boundary [11]. Clearly, an effective numerical reconstruction must be based on a reliable mathematical model of electric conduction.

In practice quite different frequency ranges of alternating currents are employed, calling for different modelling set-ups. Most of the models available in the literature rely on a quasi-static assumption, implying that the variation in time of the magnetic field may be neglected [12], so that the electric field is given by the gradient of an electric potential. Even under this general assumption, different equations for the potential are derived in different frequency ranges: for frequencies up to 1 MHz the behavior of the intra and extra cellular phases is of Ohmic type, i.e., the current density is proportional to the gradient of the electric potential. For higher frequencies, also the electric displacement current, which is proportional to the time derivative of the gradient, must be taken into account. This is the case we deal with here; namely we consider the equation for the electric potential given by (2-2). A peculiar feature of biological tissues is that the intra and extra-cellular phases are separated by an interface, that is the cell membrane, displaying a capacitive behavior. This leads to a dynamical jump condition for the electric potential across the interface [2, 4] (see equations (2-3)–(2-4)).

The geometrical and functional complexity of the problem at the microscopic cellular scale, as opposed to the macroscopical scale of clinical measurements, suggests to perform a homogenization limit, by letting the characteristic cellular length ε go to zero.

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The homogenization of the Maxwell equations has been treated extensively in [24], where however no interfaces, and therefore no jumps, are allowed, and only homogeneous initial data are considered. This motivates a rigorous mathematical investigation of this problem in the framework of the homogenization with active interfaces ([21, 22, 2, 4, 5, 13]).

This study is based on the method due to Tartar [27] of the "oscillating test functions", which in this case must be determined in a peculiar way, due to the presence of both the time derivative of the gradient in (2-2) and the dynamical interface condition (2-4).

Of course, equation (2-2) must be complemented with an initial datum (see (2-6)), which, according to the energy inequality (3-11), is naturally chosen as the initial value of the electric field (i.e. the gradient of the solution). It turns out that the initial datum cannot be arbitrarily assigned, but a compatibility condition as assumption (H1) of Section 2 must be imposed, as pointed out in Remark 4.7.

We prove that the homogenization of (2-2)-(2-7) leads to an equation with memory, similar to the one derived in [4], with the relevant difference that now the time derivative of the gradient (see (4-47)) appears under the divergence operator.

In view of the applications, it is also of interest to study the evolution in time of the homogenized potential. From a mathematical point of view, the asymptotic behavior of evolutive equations with memory is a classical problem [15, 26, 14, 20], currently drawing much interest in the literature, e.g. [16, 19, 17, 23, 8].

In [7] the exponential decay of the homogenized potential with homogeneus Dirichlet boundary data is proved, however the most interesting case in applications involves periodic boundary data. Indeed, experimental measurements are currently performed by assigning time-harmonic boundary data and assuming that the resulting electric potential is time-harmonic, too. This assumption, which is often referred to as the limiting amplitude principle, leads to the commonly accepted mathematical model based on the complex elliptic problem (5-22)–(5-23) for the electric potential.

In this paper we prove that this assumption is essentially correct, since time-periodic, not necessarily time-harmonic, boundary data elicit a time-periodic solution for large times, also in the high radio-frequency range. The time derivative of the gradient appearing in the present homogenized equation requires new estimates in order to extend to the present case the argument in [6], where the low radio-frequency range was investigated.

The paper is organized as follows: in Section 2 we present the problem and state our main results (Theorems 2.1 and 2.3); in Section 3 we prove some preliminary results of existence and compactness; in Section 4 we prove Theorem 2.1, i.e. the homogenization result, and finally in Section 5 we establish Theorem 2.3, i.e. the asymptotic behaviour of the solution.

2. Position of the problem and main results

Let Ω be an open bounded subset of $\boldsymbol{R}^N$. Following [4], [6], [7], we introduce a periodic open subset E of $\boldsymbol{R}^N$, so that $E + z = E$ for all $z \in \boldsymbol{Z}^N$. We assume that Ω, E have regular boundary, say of class C^∞ for the sake of simplicity. We also employ the notation $Y = (0,1)^N$, and $E_1 = E \cap Y$, $E_2 = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$. We stipulate that E_1 is a connected smooth subset of Y such that $\operatorname{dist}(\overline{E_1}, \partial Y) > 0$. Some generalizations may be possible, but we do not dwell on this point here. We introduce the set:

$$\boldsymbol{Z}^N_\varepsilon := \{ z \in \boldsymbol{Z}^N : \varepsilon(Y + z) \subseteq \Omega \} \,. \tag{2-1}$$

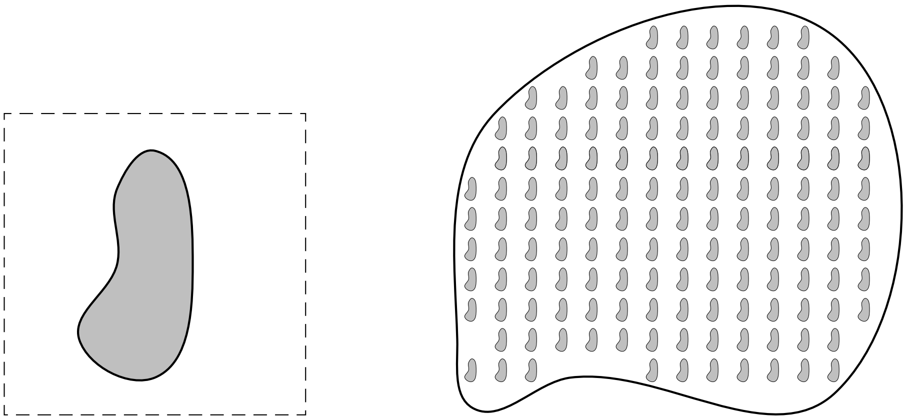

FIGURE 1. *Left*: an example of admissible periodic unit cell $Y = E_1 \cup E_2 \cup \Gamma$ in $\boldsymbol{R}^2$. Here E_1 is the light gray region and Γ is its boundary. The remaining part of Y (the white region) is E_2. *Right*: the corresponding domain $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$. Here Ω_1^ε is the light gray region and Γ^ε is its boundary. The remaining part of Ω (the white region) is Ω_2^ε.

For all $\varepsilon > 0$ we define

$$\Omega_1^\varepsilon = \bigcup_{z \in \boldsymbol{Z}_\varepsilon^N} \varepsilon(E_1 + z) , \qquad \Omega_2^\varepsilon = \Omega \setminus \overline{\Omega_1^\varepsilon} , \qquad \Gamma^\varepsilon = \partial \Omega_1^\varepsilon .$$

Clearly, $\operatorname{dist}(\Gamma^\varepsilon, \partial\Omega) > \gamma\varepsilon$ for some constant $\gamma > 0$ independent of ε since, by the choice of $\boldsymbol{Z}_\varepsilon^N$, we dropped all the inclusions contained in the cells $\varepsilon(Y + z)$, $z \in \boldsymbol{Z}^N$ which intersect $\partial\Omega$. The typical geometry we have in mind is depicted in Figure 1.

We look at the homogenization limit ($\varepsilon \searrow 0$) of the following problem for $u_\varepsilon(x,t)$:

$$-\operatorname{div}(\kappa \nabla u_{\varepsilon t} + \sigma \nabla u_\varepsilon) = 0 , \qquad \text{in } (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times (0, +\infty); \tag{2-2}$$

$$[\![(\kappa \nabla u_{\varepsilon t} + \sigma \nabla u_\varepsilon) \cdot \boldsymbol{\nu}]\!] = 0 , \qquad \text{on } \Gamma^\varepsilon \times (0, +\infty); \tag{2-3}$$

$$\frac{\alpha}{\varepsilon} \frac{\partial}{\partial t} [\![u_\varepsilon]\!] + \frac{\beta}{\varepsilon} [\![u_\varepsilon]\!] = ((\kappa \nabla u_{\varepsilon t} + \sigma \nabla u_\varepsilon) \cdot \boldsymbol{\nu})^{(2)} , \qquad \text{on } \Gamma^\varepsilon \times (0, +\infty); \tag{2-4}$$

$$u_\varepsilon(x,t) = 0 , \qquad \text{on } \partial\Omega \times (0, +\infty); \tag{2-5}$$

$$\nabla u_\varepsilon(x,0) = \boldsymbol{g}_\varepsilon(x) , \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{2-6}$$

$$[\![u_\varepsilon(x,0)]\!] = s_\varepsilon(x) , \qquad \text{on } \Gamma^\varepsilon. \tag{2-7}$$

The operators div and ∇ act with respect to the space variable x. Moreover, we assume that:

$$\begin{aligned} \alpha > 0 ; \qquad \beta \geq 0 ; \qquad & \kappa = \kappa_1 > 0 , \ \sigma = \sigma_1 > 0 \quad \text{in } \Omega_1^\varepsilon; \\ & \kappa = \kappa_2 > 0 , \ \sigma = \sigma_2 > 0 \quad \text{in } \Omega_2^\varepsilon; \end{aligned} \tag{2-8}$$

where κ_1, κ_2, σ_1, σ_2, α and β are constants. From a physical point of view, Γ^ε represents the cell membranes, having capacitance α/ε and conductance β/ε per unit area [5], whereas Ω_1^ε (resp., Ω_2^ε) is the intracellular (resp., extracellular) space, having permittivity κ_1 (resp., κ_2) and conductivity σ_1 (resp., σ_2).

Since u_ε is not in general continuous across Γ^ε we have set

$$u_\varepsilon^{(2)} := \text{ trace of } u_{\varepsilon|\Omega_2^\varepsilon} \text{ on } \Gamma^\varepsilon\,, \quad u_\varepsilon^{(1)} := \text{ trace of } u_{\varepsilon|\Omega_1^\varepsilon} \text{ on } \Gamma^\varepsilon\,, \quad \text{and} \quad [\![u_\varepsilon]\!] := u_\varepsilon^{(2)} - u_\varepsilon^{(1)}\,.$$

A similar convention is employed for the current flux density across the membrane $(\kappa\nabla u_{\varepsilon t} + \sigma\nabla u_\varepsilon)\cdot\boldsymbol{\nu}$.

We assume that the initial data s_ε and $\boldsymbol{g}_\varepsilon$ satisfy:

(H1) $s_\varepsilon \in H^{1/2}(\Gamma^\varepsilon)$ and $\boldsymbol{g}_\varepsilon \in L^2(\Omega;\boldsymbol{R}^N)$ such that $s_\varepsilon = [\![z_\varepsilon]\!]$ and, for $i=1,2$, $\boldsymbol{g}_\varepsilon|_{\Omega_i^\varepsilon} = \nabla z_\varepsilon|_{\Omega_i^\varepsilon}$, for some scalar function $z_\varepsilon \in H^1(\Omega_i^\varepsilon)$ with null trace on $\partial\Omega$;

(H2) there exists a constant γ independent of ε such that

$$\int_\Omega |\boldsymbol{g}_\varepsilon(x)|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\Gamma^\varepsilon} s_\varepsilon^2(x)\,\mathrm{d}\sigma \le \gamma\,;$$

(H3) $\|\boldsymbol{g}_0\|_{L^\infty(\Omega\times Y)} < \infty$, $\boldsymbol{g}_0(x,y)$ is continuous in x, uniformly over $y\in Y$, and periodic in y, for each $x\in\Omega$;

(H4) $\|s_1\|_{L^\infty(\Omega\times\Gamma)} < \infty$, $s_1(x,y)$ is continuous in x, uniformly over $y\in\Gamma$, and periodic in y, for each $x\in\Omega$;

where $\boldsymbol{g}_0 : \Omega\times E_i \to \boldsymbol{R}$, $i=1,2$, and $s_1 : \Omega\times\Gamma\to\boldsymbol{R}$ are the leading order terms in the two scale expansion of $\boldsymbol{g}_\varepsilon$ and s_ε (see (4-4) and (4-5)).

Our main result concerning the homogenization of problem (2-2)–(2-7) is stated in the following theorem.

Theorem 2.1. *Let $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon$ be as before. Assume that hypothesis (H1)–(H4) are satisfied and that* (2-8) *holds. Let u_ε be the solution of* (2-2)*–*(2-7)*. Then $u_\varepsilon \to u_0$ strongly in $L^2(\Omega\times(0,T))$, for any $T>0$, where u_0 is the solution of*

$$\begin{aligned} -\operatorname{div}\left(\boldsymbol{K}\nabla_x u_{0t} + \boldsymbol{A}\nabla_x u_0 + \int_0^t \boldsymbol{B}(t-\tau)\nabla_x u_0(\cdot,\tau)\,\mathrm{d}\tau - \mathcal{F}\right) &= 0\,, \\ \nabla_x u_{0|t=0} &= \overline{\boldsymbol{g}}_0(x) + \int_\Gamma s_1(x,y)\boldsymbol{\nu}\,\mathrm{d}\sigma\,, \\ u_{0|\partial\Omega}(x,t) &= 0\,, \end{aligned} \tag{2-9}$$

where we set $\overline{\boldsymbol{g}}_0(x) = \int_Y \boldsymbol{g}_0(x,y)\,\mathrm{d}y$ and the matrices $\boldsymbol{K}$, $\boldsymbol{A}$, $\boldsymbol{B}(t)$ and the vector $\mathcal{F}(x,t)$ are defined in (4-48)-(4-51).

The limit function u_0 introduced above satisfies the following exponential time-decay [7].

Theorem 2.2. *Let $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon$ be as before. Assume that hypothesis (H1)–(H2) are satisfied and that* (2-8) *holds. Then*

$$\|u_0(\cdot,t)\|_{L^2(\Omega)} \le \gamma\,\mathrm{e}^{-\lambda t} \quad a.e.\ in\ (0,+\infty). \tag{2-10}$$

We note that, up to take into account some additional sources in (2-2)–(2-4), we may permit non-homogeneous boundary data in (2-5). Then we look at the asymptotic behaviour of the solution of the following problem

$$-\operatorname{div}(\kappa\nabla u_{\varepsilon t}+\sigma\nabla u_\varepsilon)=0\,,\qquad \text{in } (\Omega_1^\varepsilon\cup\Omega_2^\varepsilon)\times(0,+\infty); \tag{2-11}$$

$$[\![(\kappa\nabla u_{\varepsilon t}+\sigma\nabla u_\varepsilon)\cdot\boldsymbol{\nu}]\!]=0\,,\qquad \text{on } \Gamma^\varepsilon\times(0,+\infty); \tag{2-12}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[\![u_\varepsilon]\!]+\frac{\beta}{\varepsilon}[\![u_\varepsilon]\!]=((\kappa\nabla u_{\varepsilon t}+\sigma\nabla u_\varepsilon)\cdot\boldsymbol{\nu})^{(2)}\,,\qquad \text{on } \Gamma^\varepsilon\times(0,+\infty); \tag{2-13}$$

$$u_\varepsilon(x,t)=\Psi(x)\Phi(t)\,,\qquad \text{on } \partial\Omega\times(0,+\infty); \tag{2-14}$$

$$\nabla u_\varepsilon(x,0)=\boldsymbol{g}_\varepsilon(x)\,,\qquad \text{in } \Omega_1^\varepsilon\cup\Omega_2^\varepsilon; \tag{2-15}$$

$$[\![u_\varepsilon]\!](x,0)=s_\varepsilon(x)\,,\qquad \text{on } \Gamma^\varepsilon; \tag{2-16}$$

which corresponds to (2-2)–(2-7), where (2-5) has been replaced by (2-14). We assume that

$$\Phi(t)\in H^1_\#(\boldsymbol{R})\,. \tag{2-17}$$

Here and in the following a subscript # denotes a space of $T^\#$-periodic functions, for some fixed $T^\#>0$. In addition, we assume that Ψ is the trace on $\partial\Omega$ of a function, still denoted by Ψ, such that

$$\Psi(x)\in H^1(\boldsymbol{R}^N)\,,\qquad \Delta\Psi=0 \text{ in } \Omega. \tag{2-18}$$

The homogenization limit of problem (2-11)–(2-16), still denoted by u_0 with a slight abuse of notation, exhibits the following asymptotic behavior.

Theorem 2.3. *Let $\Omega_1^\varepsilon,\Omega_2^\varepsilon,\Gamma^\varepsilon$ be as before. Assume that hypothesis (H1)–(H2) is satisfied and that* (2-8), (2-17) *and* (2-18) *hold. Then the following estimate holds:*

$$\|u_0(\cdot,t)-u_0^\#(\cdot,t)\|_{L^2(\Omega)}\le\gamma\,\mathrm{e}^{-\lambda t}\quad a.e.\ in\ (1,+\infty), \tag{2-19}$$

where γ and λ are positive constants and $u_0^\#\in H^1_\#(\boldsymbol{R};H^1(\Omega))$ solves

$$-\operatorname{div}\Big(\boldsymbol{K}\nabla u_{0t}^\#+\boldsymbol{A}\nabla u_0^\#+\int_0^{+\infty}\boldsymbol{B}(\tau)\nabla u_0^\#(x,t-\tau)\,\mathrm{d}\tau\Big)=0\,,\qquad in\ \Omega\times\boldsymbol{R}; \tag{2-20}$$

$$u_0^\#=\Psi(x)\Phi(t)\,,\qquad on\ \partial\Omega\times\boldsymbol{R}. \tag{2-21}$$

Here and in the following, we denote by γ a generic positive constant (independent of ε), taking in principle different values in different occurrences.

3. Preliminary results

We introduce the space

$$H^1_\varepsilon(\Omega):=\{v\in L^2(\Omega)\ :\ v_{|\Omega_i^\varepsilon}\in H^1(\Omega_i^\varepsilon),\ i=1,2;\ v=0 \text{ on } \partial\Omega\}\,. \tag{3-1}$$

Lemma 3.1 (Poincaré's inequality, see [18, 4]). *Let v belong to the space $H^1_\varepsilon(\Omega)$ introduced in* (3-1). *Then,*

$$\int_\Omega v^2\,\mathrm{d}x\le\gamma\Big\{\int_\Omega|\nabla v|^2\,\mathrm{d}x+\varepsilon^{-1}\int_{\Gamma^\varepsilon}[\![v]\!]^2\,\mathrm{d}\sigma\Big\}\,. \tag{3-2}$$

Here γ depends only on Ω and E.

3A. **Existence.** The weak formulation of problem (2-2)–(2-7) is: find a function u_ε such that for all T positive

$$u_{\varepsilon|\Omega_i^\varepsilon} \in H^1(0,T;H^1_\varepsilon(\Omega)), \tag{3-3}$$

satisfying (2-6)–(2-7) in the sense of trace such that

$$\begin{aligned}\int_0^t\int_\Omega \kappa\nabla u_{\varepsilon t}(x,\tau)\nabla\phi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_\Omega \sigma\nabla u_\varepsilon(x,\tau)\nabla\phi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ + \frac{\alpha}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon} [\![u_{\varepsilon\tau}(x,\tau)]\!][\![\phi(x,\tau)]\!]\,\mathrm{d}\sigma\,\mathrm{d}\tau + \frac{\beta}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon} [\![u_\varepsilon(x,\tau)]\!][\![\phi(x,\tau)]\!]\,\mathrm{d}\sigma\,\mathrm{d}\tau = 0\,,\end{aligned} \tag{3-4}$$

for every $t \in (0,T)$ and for each $\phi \in L^2(0,T;H^1_\varepsilon(\Omega))$, such that $[\![\phi]\!] \in H^1(0,T;H^{1/2}(\Gamma^\varepsilon))$.

Theorem 3.2. *Let Ω, Ω_1^ε, Ω_2^ε, Γ^ε be as in Section 2 and assume that* (2-8) *and assumption (H1) hold. Then for every $\varepsilon > 0$ problem* (2-2)–(2-7) *admits a unique solution in the sense of* (3-3)–(3-4)*.*

Proof. The proof of the existence is only sketched here, since it is quite standard.
Choosing ϕ independent of t in (3-4), we obtain

$$\begin{aligned}\int_\Omega \kappa\nabla u_\varepsilon(x,t)\nabla\phi(x)\,\mathrm{d}x + \int_\Omega \sigma\nabla\left(\int_0^t u_\varepsilon(x,\tau)\,\mathrm{d}\tau\right)\nabla\phi(x)\,\mathrm{d}x \\ + \frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon} [\![u_\varepsilon(x,t)]\!][\![\phi(x)]\!]\,\mathrm{d}\sigma + \frac{\beta}{\varepsilon}\int_{\Gamma^\varepsilon} [\![\int_0^t u_\varepsilon(x,\tau)\,\mathrm{d}\tau]\!][\![\phi(x)]\!]\,\mathrm{d}\sigma \\ = \int_\Omega \kappa\boldsymbol{g}_\varepsilon(x)\nabla\phi(x)\,\mathrm{d}x + \frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon} [\![s_\varepsilon(x)]\!][\![\phi(x)]\!]\,\mathrm{d}\sigma\,,\end{aligned} \tag{3-5}$$

which, taking t as a parameter, can be regarded as an elliptic problem. To study its solvability, for every $\varepsilon > 0$ and for a.e. $t \in (0,T)$, we firstly consider the problem

$$\begin{aligned}\int_\Omega \kappa\nabla u_\varepsilon(x,t)\nabla\phi(x)\,\mathrm{d}x + \frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon} [\![u_\varepsilon(x,t)]\!][\![\phi(x)]\!]\,\mathrm{d}\sigma \\ = -\int_\Omega \sigma\nabla\left(\int_0^t f(x,\tau)\,\mathrm{d}\tau\right)\nabla\phi(x)\,\mathrm{d}x - \frac{\beta}{\varepsilon}\int_{\Gamma^\varepsilon} [\![\int_0^t f(x,\tau)\,\mathrm{d}\tau]\!][\![\phi(x)]\!]\,\mathrm{d}\sigma \\ + \int_\Omega \kappa\boldsymbol{g}_\varepsilon(x)\nabla\phi(x)\,\mathrm{d}x + \frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon} [\![s_\varepsilon(x)]\!][\![\phi(x)]\!]\,\mathrm{d}\sigma\,,\end{aligned} \tag{3-6}$$

where $f \in L^2(0,T;H^1_\varepsilon(\Omega))$. Due to inequality (3-2), for a.e. $t \in (0,T)$, existence and uniqueness in $H^1_\varepsilon(\Omega)$ for problem (3-6) is a consequence of a standard application of Lax-Milgram Lemma. Moreover, having in mind that $f \in L^2(0,T;H^1_\varepsilon(\Omega))$, it easily follows that the solution belongs to $L^2(0,T;H^1_\varepsilon(\Omega))$. Then, for $\boldsymbol{g}_\varepsilon$ and s_ε fixed, it is enough to apply a contraction principle to the operator $\mathcal{L} : L^2(0,T;H^1_\varepsilon(\Omega)) \to L^2(0,T;H^1_\varepsilon(\Omega))$ defined by $\mathcal{L}(f) = u$, where $u \in L^2(0,T;H^1_\varepsilon(\Omega))$ is the unique solution of (3-6). Indeed, $\mathcal{L}$ turns out to be a contraction map in $L^2(0,\overline{T};H^1_\varepsilon(\Omega))$, for $\overline{T}$ sufficiently small. Such a number $\overline{T}$ does not depend on the initial data, so that we can cover the whole interval $(0,T)$, by repeating the previous procedure a finite number of times. The weak differentiability of the solution with respect to t (for positive t) is standard, so that equation (3-4) follows by differentiating (3-6) with respect to time (where $f = u_\varepsilon$, of course), then by replacing $\phi(x)$ with $\phi(x,t)$, such that $\phi \in L^2(0,T;H^1_\varepsilon(\Omega))$, and finally integrating in time. In order to have (2-6) and (2-7) satisfied in the sense of trace, we let $t \to 0$ in (3-6), obtaining the weak formulation of the following elliptic problem

$$-\operatorname{div}(\kappa \nabla u_\varepsilon(0)) = -\operatorname{div}(\kappa \boldsymbol{g}_\varepsilon)\,, \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{3-7}$$

$$[\![\kappa \nabla u_\varepsilon(0) \cdot \boldsymbol{\nu}]\!] = [\![\kappa \boldsymbol{g}_\varepsilon \cdot \boldsymbol{\nu}]\!]\,, \qquad \text{on } \Gamma^\varepsilon; \tag{3-8}$$

$$[\![u_\varepsilon(0)]\!] = [\![s_\varepsilon]\!]\,, \qquad \text{on } \Gamma^\varepsilon; \tag{3-9}$$

$$u_\varepsilon(0) = 0\,, \qquad \text{on } \partial\Omega. \tag{3-10}$$

Problem (3-7)–(3-10) uniquely determines $u_\varepsilon(0)$. This solution needs not satisfy (2-6), since problem (3-7)–(3-10) does not require equality of gradients; indeed $u_\varepsilon(0)$ does not vary if we add to $\kappa \boldsymbol{g}_\varepsilon$ a solenoidal vector field. However, if assumption (H1) is fulfilled, then uniqueness of problem (3-7)–(3-10) implies that $u_\varepsilon(0) = z_\varepsilon$ and so (2-6) is satisfied.
Finally, the uniqueness of (2-2)–(2-7) follows from the energy estimate below. □

3B. **Energy estimate.** Taking $\phi = u_\varepsilon$ in (3-4) and using (2-6)–(2-7), we arrive, for a.e. $t > 0$, to the energy estimate

$$\begin{aligned}\int_\Omega \frac{\kappa}{2} |\nabla u_\varepsilon(x,t)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega \sigma |\nabla u_\varepsilon(x,\tau)|^2 \,\mathrm{d}x \,\mathrm{d}\tau + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [\![u_\varepsilon(x,t)]\!]^2 \,\mathrm{d}\sigma \\ + \frac{\beta}{\varepsilon} \int_0^t \int_{\Gamma^\varepsilon} [\![u_\varepsilon(x,\tau)]\!]^2 \,\mathrm{d}\sigma \,\mathrm{d}\tau = \int_\Omega \frac{\kappa}{2} |\boldsymbol{g}_\varepsilon(x)|^2 \,\mathrm{d}x + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} s_\varepsilon^2(x) \,\mathrm{d}\sigma\,. \end{aligned} \tag{3-11}$$

Moreover, using $u_{\varepsilon t}$ as a testing function in (3-4) and using again (2-6)–(2-7), it follows

$$\begin{aligned}\int_0^t \int_\Omega \kappa |\nabla u_{\varepsilon t}(x,\tau)|^2 \,\mathrm{d}x \,\mathrm{d}\tau + \int_\Omega \frac{\sigma}{2} |\nabla u_\varepsilon(x,t)|^2 \,\mathrm{d}x + \frac{\alpha}{\varepsilon} \int_0^t \int_{\Gamma^\varepsilon} [\![u_{\varepsilon t}(x,\tau)]\!]^2 \,\mathrm{d}\sigma \,\mathrm{d}\tau \\ + \frac{\beta}{2\varepsilon} \int_{\Gamma^\varepsilon} [\![u_\varepsilon(x,t)]\!]^2 \,\mathrm{d}\sigma = \int_\Omega \frac{\sigma}{2} |\boldsymbol{g}_\varepsilon(x)|^2 \,\mathrm{d}x + \frac{\beta}{2\varepsilon} \int_{\Gamma^\varepsilon} s_\varepsilon^2(x) \,\mathrm{d}\sigma\,, \end{aligned} \tag{3-12}$$

for a.e. $t > 0$.

In fact (3-11) and (3-12), coupled with Poincaré's inequality (Lemma 3.1 and Remark 3.3), are the main tools in the rigorous proof of convergence of u_ε to its limit. In particular, up to a subsequence, u_ε and $u_{\varepsilon t}$ weakly converge in $L^2(\Omega \times (0,T))$ as $\varepsilon \to 0$ to u_0 and u_{0t}, respectively, for every $T > 0$. The equation satisfied by u_0 will be derived via a homogenization procedure in Sections 4A and 4C.

In order to prove that actually $u_\varepsilon \to u_0$ strongly in $L^2(\Omega \times (0,T))$ as $\varepsilon \to 0$, we need the following compactness results.

Remark 3.3. *Applying inequality* (3-2) *to $v = u_\varepsilon$ and $v = u_{\varepsilon t}$, respectively, integrating in time in $(0,T)$ and recalling* (3-11), (3-12) *and (H2), we obtain*

$$\int_0^T \int_\Omega u_\varepsilon^2 \,\mathrm{d}x \,\mathrm{d}t \le \gamma \Big\{ \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}t + \varepsilon^{-1} \int_0^T \int_{\Gamma^\varepsilon} [\![u_\varepsilon]\!]^2 \,\mathrm{d}\sigma \,\mathrm{d}t \Big\} \le \gamma \tag{3-13}$$

$$\int_0^T \int_\Omega u_{\varepsilon t}^2 \,\mathrm{d}x \,\mathrm{d}t \le \gamma \Big\{ \int_0^T \int_\Omega |\nabla u_{\varepsilon t}|^2 \,\mathrm{d}x \,\mathrm{d}t + \varepsilon^{-1} \int_0^T \int_{\Gamma^\varepsilon} [\![u_{\varepsilon t}]\!]^2 \,\mathrm{d}\sigma \,\mathrm{d}t \Big\} \le \gamma\,. \tag{3-14}$$

Proposition 3.4 (see [18]). *Let $0 \le s < 1/2$. Then there exists a constant γ, depending on s but independent of ε, such that for every $\varepsilon > 0$ and all $v \in H^1_\varepsilon(\Omega)$ we have*

$$\|v\|^2_{H^s(\Omega)} \le \gamma \left(\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \frac{1}{\varepsilon} \int_{\Gamma^\varepsilon} [\![v]\!]^2 \,\mathrm{d}\sigma \right)\,. \tag{3-15}$$

Lemma 3.5 (see [25])**.** *Let X, H, Y be Banach spaces such that $X \subseteq H$ with compact embedding and $H \subseteq Y$ with continuous embedding. Assume that $\{u_h\}$ is an equibounded sequence in $L^p(0,T;X)$, $1 < p < +\infty$, such that $\{u'_h\}$ is equibounded in $L^1(0,T;Y)$. Then there exists $u \in L^p(0,T;X)$ such that, up to a subsequence, $u_h \to u$ strongly in $L^p(0,T;H)$.*

Lemma 3.6. *For $\varepsilon > 0$, let $u_\varepsilon \in L^2(0,T;H^1(\Omega_i^\varepsilon))$, $i = 1,2$ be the solution of* (2-2)–(2-7)*. Then, extracting a subsequence if needed, u_ε strongly converges in $L^2(\Omega \times (0,T))$.*

Proof. Let us choose $p = 2$, $X = H^s(\Omega)$, with $0 \le s < 1/2$, and $H = Y = L^2(\Omega)$ in Lemma 3.5. Writing (3-15) for $v = u_\varepsilon(\cdot,t)$, for a.e. $t \in (0,T)$, integrating in $(0,T)$ and using (3-11) and (H2), we obtain that $\|u_\varepsilon\|^2_{L^2(0,T;H^s(\Omega))} \le \gamma$. Moreover, by (3-14), the sequence $\{u_{\varepsilon t}\}$ is equibounded in $L^2(\Omega \times (0,T))$ (and hence in $L^1(0,T;L^2(\Omega))$), so that by Lemma 3.5 the thesis follows. □

4. Homogenization

4A. **Formal homogenization.** We summarize here, to establish the notation, some well known asymptotic expansions needed in the two-scale method (see, e.g., [10], [24]). Introduce the microscopic variables $y \in Y$, $y = x/\varepsilon$, assuming

$$u_\varepsilon = u_\varepsilon(x,y,t) = u_0(x,y,t) + \varepsilon u_1(x,y,t) + \varepsilon^2 u_2(x,y,t) + \dots \,. \tag{4-1}$$

Note that u_0, u_1, u_2 are periodic in y, and u_1, u_2 are assumed to have zero integral average over Y. Recalling that

$$\operatorname{div} = \frac{1}{\varepsilon}\operatorname{div}_y + \operatorname{div}_x \,, \qquad \nabla = \frac{1}{\varepsilon}\nabla_y + \nabla_x \,, \tag{4-2}$$

we compute, e.g.,

$$\nabla u_\varepsilon = \frac{1}{\varepsilon}\nabla_y u_0 + \big(\nabla_x u_0 + \nabla_y u_1\big) + \varepsilon\big(\nabla_y u_2 + \nabla_x u_1\big) + \dots \,. \tag{4-3}$$

We also stipulate

$$\boldsymbol{g}_\varepsilon = \boldsymbol{g}_\varepsilon(x,y) = \boldsymbol{g}_0(x,y) + \varepsilon \boldsymbol{g}_1(x,y) + \varepsilon^2 \boldsymbol{g}_2(x,y) + \dots \,; \tag{4-4}$$

$$s_\varepsilon = s_\varepsilon(x,y) = \varepsilon s_1(x,y) + \varepsilon^2 s_2(x,y) + \dots \,, \tag{4-5}$$

where the restrictions of $\boldsymbol{g}_0(x,\cdot)$, $\boldsymbol{g}_1(x,\cdot)$, $\boldsymbol{g}_2(x,\cdot)$, … to E_1 and E_2 are the gradient of scalar fields. The terms $\varepsilon^{-1}\boldsymbol{g}_{-1}(x,y)$, $s_0(x,y)$ respectively expected in the previous expansions are ruled out by assumption (H2), recalling that $|\Gamma^\varepsilon|_{N-1} \sim 1/\varepsilon$.

According to assumption (H1) in Section 2, we may consider also the expansion of the function z_ε which is given by

$$z_\varepsilon = z_\varepsilon(x,y) = z_0(x,y) + \varepsilon z_1(x,y) + \varepsilon^2 z_2(x,y) + \dots \,. \tag{4-6}$$

Identifying the terms in (4-4) and (4-5) with the expansion (4-6), we obtain that $\boldsymbol{g}_{-1} \equiv 0$ and $s_0 \equiv 0$ imply that $z_0(x,y) = z_0(x)$ with null jump and that

$$\boldsymbol{g}_0(x,y) = \nabla_x z_0(x) + \nabla_y z_1(x,y) \,; \qquad s_1(x,y) = [\![z_1(x,y)]\!] \,. \tag{4-7}$$

For the sake of brevity, we introduce the operator:

$$\mathcal{D} := \kappa \frac{\partial}{\partial t} + \sigma \,. \tag{4-8}$$

Applying (4-2)–(4-3) to Problem (2-2)–(2-7), one readily obtains by matching corresponding powers of ε, that u_0 solves,

$$-\mathcal{D}\,\Delta_y u_0 = 0\,, \qquad \text{in } (E_1 \cup E_2) \times (0,+\infty); \tag{4-9}$$

$$[\![\mathcal{D}\nabla_y u_0 \cdot \boldsymbol{\nu}]\!] = 0\,, \qquad \text{on } \Gamma \times (0,+\infty); \tag{4-10}$$

$$\alpha \frac{\partial [\![u_0]\!]}{\partial t} + \beta [\![u_0]\!] = (\mathcal{D}\nabla_y u_0 \cdot \boldsymbol{\nu})^{(2)}\,, \qquad \text{on } \Gamma \times (0,+\infty). \tag{4-11}$$

$$\nabla_y u_{0|t=0} = 0\,, \qquad \text{on } E_1 \cup E_2; \tag{4-12}$$

$$[\![u_0]\!]_{|t=0} = 0\,, \qquad \text{on } \Gamma. \tag{4-13}$$

Reasoning as in Section 3B we obtain an energy estimate for (4-9)–(4-13), which implies that $[\![u_0]\!] = 0$ for all times, and

$$u_0 = u_0(x,t)\,.$$

Next we find for u_1:

$$-\mathcal{D}\,\Delta_y u_1 = 0\,, \qquad \text{in } (E_1 \cup E_2) \times (0,+\infty); \tag{4-14}$$

$$[\![\mathcal{D}(\nabla_y u_1 + \nabla_x u_0) \cdot \boldsymbol{\nu}]\!] = 0\,, \qquad \text{on } \Gamma \times (0,+\infty); \tag{4-15}$$

$$\alpha \frac{\partial [\![u_1]\!]}{\partial t} + \beta [\![u_1]\!] = (\mathcal{D}(\nabla_y u_1 + \nabla_x u_0) \cdot \boldsymbol{\nu})^{(2)}\,, \qquad \text{on } \Gamma \times (0,+\infty); \tag{4-16}$$

$$\nabla_y u_{1|t=0} + \nabla_x u_{0|t=0} = \boldsymbol{g}_0\,, \qquad \text{on } E_1 \cup E_2; \tag{4-17}$$

$$[\![u_1]\!]_{|t=0} = s_1\,, \qquad \text{on } \Gamma. \tag{4-18}$$

Taking into account (4-18) and integrating over Y the function $\nabla u_{1|t=0}$, we obtain

$$\begin{aligned}\int_Y \nabla u_1(x,y;0)\,\mathrm{d}y &= \int_{E_1} \nabla u_1(x,y;0)\,\mathrm{d}y + \int_{E_2} \nabla u_1(x,y;0)\,\mathrm{d}y \\ &= -\int_\Gamma [\![u_1(x,y;0)]\!]\boldsymbol{\nu}\,\mathrm{d}\sigma = -\int_\Gamma s_1(x,y)\boldsymbol{\nu}\,\mathrm{d}\sigma\,.\end{aligned} \tag{4-19}$$

Hence, integrating equation (4-17) over Y and recalling that u_0 does not depend on y, equation (4-19) implies

$$\nabla_x u_{0|t=0} = \overline{\boldsymbol{g}}_0(x) + \int_\Gamma s_1(x,y)\boldsymbol{\nu}\,\mathrm{d}\sigma\,, \tag{4-20}$$

where we set $\overline{\boldsymbol{g}}_0(x) = \int_Y \boldsymbol{g}_0(x,y)\,\mathrm{d}y$. This implies that (4-17) can be replaced by

$$\nabla_y u_{1|t=0} = \boldsymbol{g}_0 - \overline{\boldsymbol{g}}_0(x) - \int_\Gamma s_1(x,y)\boldsymbol{\nu}\,\mathrm{d}\sigma\,. \tag{4-21}$$

Note that by (4-18), (4-21) and (4-7), we obtain that $[\![u_1]\!]_{|t=0} = [\![z_1]\!]$ and $\nabla_y u_{1|t=0} = \nabla_y z_1$, indeed

$$\begin{aligned}&\nabla_y u_{1|t=0} = \boldsymbol{g}_0 - \overline{\boldsymbol{g}}_0(x) - \int_\Gamma s_1(x,y)\boldsymbol{\nu}\,\mathrm{d}\sigma = \\ &\quad \nabla_x z_0(x) + \nabla_y z_1(x,y) - \int_Y [\nabla_x z_0(x) + \nabla_y z_1(x,y)]\,\mathrm{d}y - \int_\Gamma s_1(x,y)\boldsymbol{\nu}\,\mathrm{d}\sigma = \\ &\qquad \nabla_y z_1(x,y) + \int_\Gamma s_1(x,y)\boldsymbol{\nu}\,\mathrm{d}\sigma - \int_\Gamma s_1(x,y)\boldsymbol{\nu}\,\mathrm{d}\sigma = \nabla_y z_1(x,y)\end{aligned} \tag{4-22}$$

where, in the third equality we use the Gauss Lemma. This, according to Theorem 3.2, guarantees the well-posedness of the problem (4-14)–(4-18).

Remark 4.1. *Note that, if $\boldsymbol{g}_0$ and s_1 do not depend on y (i.e., $s_\varepsilon/\varepsilon$ and $\boldsymbol{g}_\varepsilon$ strongly converge in L^2), then $\nabla_y u_{1|t=0} = 0$ and $\nabla_x u_{0|t=0} = \boldsymbol{g}_0(x)$, see* [7].

In order to represent u_1 in a suitable way, let $\boldsymbol{g} \in L^2(Y)$ and $s \in H^{1/2}(\Gamma)$ be assigned, such that $s = [\![z]\!]$ and, for $i = 1, 2$, $\boldsymbol{g}|_{E_i} = \nabla z|_{E_i}$, for some scalar periodic function $z \in H^1(E_i)$, and consider the problem

$$
\begin{aligned}
-\mathcal{D}\,\Delta_y v &= 0\,, && \text{in } (E_1 \cup E_2) \times (0, +\infty); && \text{(4-23)}\\
[\![\mathcal{D}\nabla_y v \cdot \boldsymbol{\nu}]\!] &= 0\,, && \text{on } \Gamma \times (0, +\infty); && \text{(4-24)}\\
\alpha \frac{\partial [\![v]\!]}{\partial t} + \beta [\![v]\!] &= (\mathcal{D}\nabla_y v \cdot \boldsymbol{\nu})^{(2)}\,, && \text{on } \Gamma \times (0, +\infty); && \text{(4-25)}\\
\nabla_y v_{|t=0} &= \boldsymbol{g}\,, && \text{on } E_1 \cup E_2; && \text{(4-26)}\\
[\![v]\!]_{|t=0} &= s\,, && \text{on } \Gamma. && \text{(4-27)}
\end{aligned}
$$

where v is a periodic function in Y, such that $\int_Y v(y,t)\,\mathrm{d}y = 0$. Define the transform $\mathcal{T}$ by

$$
\mathcal{T}(\boldsymbol{g}, s)(y,t) = v(y,t)\,, \qquad y \in Y\,, t > 0\,. \tag{4-28}
$$

Then, introduce the cell functions $\boldsymbol{\chi}^0 : Y \to \boldsymbol{R}^N$ and $\boldsymbol{\chi}^1 : Y \times (0, +\infty) \to \boldsymbol{R}^N$, whose components χ^0_h and $\chi^1_h(\cdot, t)$, $h = 1, \dots, N$, are required to be periodic functions with vanishing integral average over Y for $t \geq 0$. The components χ^0_h of the function $\boldsymbol{\chi}^0$ satisfy

$$
\begin{aligned}
-\kappa\,\Delta_y(\chi^0_h - y_h) &= 0\,, && \text{in } E_1 \cup E_2; && \text{(4-29)}\\
[\![\kappa(\nabla_y \chi^0_h - \boldsymbol{e}_h) \cdot \boldsymbol{\nu}]\!] &= 0\,, && \text{on } \Gamma; && \text{(4-30)}\\
\alpha [\![\chi^0_h]\!] &= (\kappa(\nabla_y \chi^0_h - \boldsymbol{e}_h) \cdot \boldsymbol{\nu})^{(2)}\,, && \text{on } \Gamma. && \text{(4-31)}
\end{aligned}
$$

The initial value $\chi^1_h(\cdot, 0)$ of the components of $\boldsymbol{\chi}^1$ satisfies

$$
\begin{aligned}
-\kappa\,\Delta_y\, \chi^1_h(\cdot,0) - \sigma\,\Delta_y(\chi^0_h - y_h) &= 0\,, && \text{in } E_1 \cup E_2; && \text{(4-32)}\\
[\![(\kappa \nabla_y \chi^1_h(\cdot,0) + \sigma(\nabla_y \chi^0_h - \boldsymbol{e}_h)) \cdot \boldsymbol{\nu}]\!] &= 0\,, && \text{on } \Gamma; && \text{(4-33)}\\
((\kappa \nabla_y \chi^1_h(\cdot,0) + \sigma(\nabla_y \chi^0_h - \boldsymbol{e}_h)) \cdot \boldsymbol{\nu})^{(2)} &= \alpha [\![\chi^1_h(\cdot,0)]\!] + \beta [\![\chi^0_h]\!]\,, && \text{on } \Gamma. && \text{(4-34)}
\end{aligned}
$$

Finally, χ^1_h is defined for $t > 0$ by

$$
\chi^1_h = \mathcal{T}(\nabla_y \chi^1_h(\cdot,0), [\![\chi^1_h(\cdot,0)]\!])\,. \tag{4-35}
$$

Straightforward calculations show that u_1 may be written in the form

$$
u_1(x,y,t) = -\boldsymbol{\chi}^0(y) \cdot \nabla_x u_0(x,t) - \int_0^t \boldsymbol{\chi}^1(y, t-\tau) \cdot \nabla_x u_0(x,\tau)\,\mathrm{d}\tau + \mathcal{T}(\tilde{\boldsymbol{g}}_0, \tilde{s}_1)(x,y,t)\,, \tag{4-36}
$$

where we define

$$
\tilde{\boldsymbol{g}}_0(x,y) = \boldsymbol{g}_0(x,y) - (\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0(y))^t \Big[\overline{\boldsymbol{g}}_0(x) + \int_\Gamma s_1(x,\tilde{y}) \boldsymbol{\nu}\,\mathrm{d}\sigma\Big]\,, \tag{4-37}
$$

$$
\tilde{s}_1(x,y) = s_1(x,y) + [\![\boldsymbol{\chi}^0(y)]\!] \cdot \Big[\overline{\boldsymbol{g}}_0(x) + \int_\Gamma s_1(x,\tilde{y}) \boldsymbol{\nu}\,\mathrm{d}\sigma\Big]\,. \tag{4-38}
$$

Next we find for u_2:

$$-\mathcal{D}(\Delta_y u_2 + 2\frac{\partial^2 u_1}{\partial x_j \partial y_j} + \Delta_x u_0) = 0\,, \qquad \text{in } (E_1 \cup E_2) \times (0, +\infty); \tag{4-39}$$

$$[\![\mathcal{D}(\nabla_y u_2 + \nabla_x u_1) \cdot \boldsymbol{\nu}]\!] = 0\,, \qquad \text{on } \mathit{\Gamma} \times (0, +\infty); \tag{4-40}$$

$$(\mathcal{D}(\nabla_y u_2 + \nabla_x u_1) \cdot \boldsymbol{\nu})^{(2)} = \alpha \frac{\partial [\![u_2]\!]}{\partial t} + \beta [\![u_2]\!]\,, \quad \text{on } \mathit{\Gamma} \times (0, +\infty); \tag{4-41}$$

$$\nabla_y u_{2|t=0} + \nabla_x u_{1|t=0} = \boldsymbol{g}_1\,, \qquad \text{on } E_1 \cup E_2; \tag{4-42}$$

$$[\![u_2]\!]_{|t=0} = s_2\,, \qquad \text{on } \mathit{\Gamma}. \tag{4-43}$$

Let us find the solvability conditions for this problem. Integrating by parts the partial differential equations (4-39) solved by u_2, both in E_1 and in E_2, adding the two contributions, and using (4-40), we get

$$\left[\int_{E_1} + \int_{E_2} \right] \mathcal{D} \left\{ \Delta_x u_0(x,t) + 2\frac{\partial^2 u_1}{\partial x_j \partial y_j} \right\} \mathrm{d}y = - \int_{\mathit{\Gamma}} [\![\mathcal{D} \nabla_x u_1 \cdot \boldsymbol{\nu}]\!] \, \mathrm{d}\sigma\,. \tag{4-44}$$

Thus we obtain

$$\left(\overline{\kappa} \frac{\partial}{\partial t} + \overline{\sigma} \right) \Delta_x u_0 = 2 \int_{\mathit{\Gamma}} [\![\mathcal{D} \nabla_x u_1 \cdot \boldsymbol{\nu}]\!] \, \mathrm{d}\sigma - \int_{\mathit{\Gamma}} [\![\mathcal{D} \nabla_x u_1 \cdot \boldsymbol{\nu}]\!] \, \mathrm{d}\sigma = \int_{\mathit{\Gamma}} [\![\mathcal{D} \nabla_x u_1 \cdot \boldsymbol{\nu}]\!] \, \mathrm{d}\sigma\,, \tag{4-45}$$

where

$$\overline{\kappa} = \kappa_1 |E_1| + \kappa_2 |E_2|\,; \quad \overline{\sigma} = \sigma_1 |E_1| + \sigma_2 |E_2|\,. \tag{4-46}$$

Then we substitute the representation (4-36) into equation (4-45) and, after simple algebra, obtain the homogenized equation for u_0 in $\mathit{\Omega} \times (0, +\infty)$ as

$$- \operatorname{div} \left(\boldsymbol{K} \nabla_x u_{0t} + \boldsymbol{A} \nabla_x u_0 + \int_0^t \boldsymbol{B}(t-\tau) \nabla_x u_0(\cdot, \tau) \, \mathrm{d}\tau - \mathcal{F} \right) = 0\,, \tag{4-47}$$

where the matrices $\boldsymbol{K}$, $\boldsymbol{A}$, $\boldsymbol{B}(t)$ and the vector $\mathcal{F}(x,t)$ are defined as follows:

$$\boldsymbol{K} = \overline{\kappa} \boldsymbol{I} + \int_{\mathit{\Gamma}} [\![\kappa \boldsymbol{\chi}^0(y)]\!] \otimes \boldsymbol{\nu} \, \mathrm{d}\sigma\,, \tag{4-48}$$

$$\boldsymbol{A} = \overline{\sigma} \boldsymbol{I} + \int_{\mathit{\Gamma}} [\![\kappa \boldsymbol{\chi}^1(y,0) + \sigma \boldsymbol{\chi}^0(y)]\!] \otimes \boldsymbol{\nu} \, \mathrm{d}\sigma\,, \tag{4-49}$$

$$\boldsymbol{B}(t) = \int_{\mathit{\Gamma}} [\![(\mathcal{D} \boldsymbol{\chi}^1)(y,t)]\!] \otimes \boldsymbol{\nu} \, \mathrm{d}\sigma\,, \tag{4-50}$$

$$\mathcal{F}(x,t) = \int_{\mathit{\Gamma}} [\![\mathcal{D} \mathcal{T}(\tilde{\boldsymbol{g}}_0, \tilde{s}_1)(x,y,t)]\!] \boldsymbol{\nu} \, \mathrm{d}\sigma\,. \tag{4-51}$$

Equation (4-47) is complemented with the initial condition (4-20).

Finally, integrating in time equation (4-47), changing the order in the double integral thus appearing and using (4-20), we obtain also the following formulation

$$- \operatorname{div} \Bigg(\boldsymbol{K} \nabla_x u_0 + \int_0^t \Big(\boldsymbol{A} + \int_0^{t-s} \boldsymbol{B}(\tau) \, \mathrm{d}\tau \Big) \nabla_x u_0(\cdot, s) \, \mathrm{d}s - \boldsymbol{K} \Big(\overline{\boldsymbol{g}}_0 + \int_{\mathit{\Gamma}} s_1(\cdot, y) \otimes \boldsymbol{\nu} \, \mathrm{d}\sigma(y) \Big) - \int_0^t \mathcal{F}(\cdot, \tau) \, \mathrm{d}\tau \Bigg) = 0\,, \tag{4-52}$$

which shows that the homogenized equation has exactly the form of an equation with memory of the type derived in [2, 4] and studied in [3].

Remark 4.2. *We note that in the definition of the function u_1 (see* (4-36)*), the cell function $\chi^0 : Y \to \boldsymbol{R}^N$ is standard. In addition to this function, a new cell function $\chi^1 : Y \to \boldsymbol{R}^N$ is required, owing to the dynamical terms in equations* (4-14)*–*(4-16)*. The definition of such a function involves a transform $\mathcal{T}$, which plays an essential role. From the point of view of physics, the transform $\mathcal{T}$ associates to the initial data the evolution of the potential itself, in the process determining the discharge of the membrane in the unit cell Y under periodic boundary conditions. Memory effects appear in the homogenized equation (see* (4-47) *and* (4-52)*) just as a consequence of the transform $\mathcal{T}$.*

4B. **The structure of the limit equation.**

Lemma 4.3. *Problem* (4-29)*–*(4-31)*, problem* (4-32)*–*(4-34) *and problem* (4-23)*–*(4-27) *have respectively Y-periodic solutions with vanishing integral average over Y*

$$\chi^0|_{E_i},\ \chi^1(\cdot,0)\,|_{E_i}\in H^1(E_i) \qquad \textit{and} \qquad v\,|_{E_i}\in H^1(0,T;H^1(E_i))$$

for $i = 1, 2$.

Proof. Existence and regularity of χ^0, $\chi^1(\cdot,0)$ follow by standard application of Lax-Milgram Lemma, whereas existence and regularity of v can be obtained reasoning as in the proof of Theorem 3.2. □

Owing to previous lemma, the matrices $\boldsymbol{K}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are well defined and $\boldsymbol{B} \in H^1(0,T)$. Moreover, due to (4-56), (H3) and (H4), $\mathcal{F} \in H^1(0,T;\mathcal{C}^0(\Omega))$.

Proposition 4.4. *The matrices $\boldsymbol{K}$, $\boldsymbol{A}$, $\boldsymbol{B}(t)$ and the vector $\mathcal{F}(x,t)$, defined in* (4-48)*–*(4-51)*, can be alternatively expressed as follows. Of course we assume here that $\boldsymbol{g}_0(x,\cdot)$ and $s_1(x,\cdot)$ have the regularity mentioned in Lemma 4.6 below, so that $\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)$ is well defined.*

$$\boldsymbol{K} = \int_Y \kappa(\boldsymbol{I}-\nabla_y\boldsymbol{\chi}^0(y))(\boldsymbol{I}-\nabla_y\boldsymbol{\chi}^0(y))^t\,\mathrm{d}y + \int_\Gamma \alpha[\![\boldsymbol{\chi}^0(y)]\!]\otimes[\![\boldsymbol{\chi}^0(y)]\!]\,\mathrm{d}\sigma\,, \tag{4-53}$$

$$\boldsymbol{A} = \int_Y \sigma(\boldsymbol{I}-\nabla_y\boldsymbol{\chi}^0(y))(\boldsymbol{I}-\nabla_y\boldsymbol{\chi}^0(y))^t\,\mathrm{d}y + \int_\Gamma \beta[\![\boldsymbol{\chi}^0(y)]\!]\otimes[\![\boldsymbol{\chi}^0(y)]\!]\,\mathrm{d}\sigma\,, \tag{4-54}$$

$$\boldsymbol{B}(t) = -\int_Y \kappa\nabla_y\boldsymbol{\chi}^1(y,t)(\nabla_y\boldsymbol{\chi}^1(y,0))^t\,\mathrm{d}y - \int_\Gamma \alpha[\![\boldsymbol{\chi}^1(y,t)]\!]\otimes[\![\boldsymbol{\chi}^1(y,0)]\!]\,\mathrm{d}\sigma\,, \tag{4-55}$$

$$\mathcal{F}(x,t) = -\int_Y \kappa\nabla_y\boldsymbol{\chi}^1(y,t)\boldsymbol{g}_0(x,y)\,\mathrm{d}y - \int_\Gamma \alpha[\![\boldsymbol{\chi}^1(y,t)]\!]\,s_1(x,y)\,\mathrm{d}\sigma\,. \tag{4-56}$$

Proof. Equation (4-53) is obtained from (4-29)–(4-31) and the Gauss Lemma, as follows:

$$\begin{aligned}\boldsymbol{K} = \overline{\kappa}\boldsymbol{I} + \int_\Gamma [\![\kappa\boldsymbol{\chi}^0(y)]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma &= \int_Y \kappa(\boldsymbol{I}-\nabla_y\boldsymbol{\chi}^0(y))\,\mathrm{d}y \\ &= \int_Y \kappa(\boldsymbol{I}-\nabla_y\boldsymbol{\chi}^0(y))(\boldsymbol{I}-\nabla_y\boldsymbol{\chi}^0(y))^t\,\mathrm{d}y + \int_Y \kappa(\boldsymbol{I}-\nabla_y\boldsymbol{\chi}^0(y))(\nabla_y\boldsymbol{\chi}^0(y))^t\,\mathrm{d}y \\ &= \int_Y \kappa(\boldsymbol{I}-\nabla_y\boldsymbol{\chi}^0(y))(\boldsymbol{I}-\nabla_y\boldsymbol{\chi}^0(y))^t\,\mathrm{d}y + \int_\Gamma \alpha[\![\boldsymbol{\chi}^0(y)]\!]\otimes[\![\boldsymbol{\chi}^0(y)]\!]\,\mathrm{d}\sigma\,.\end{aligned} \tag{4-57}$$

Analogously, (4-54) is obtained from (4-29)–(4-31), (4-32)–(4-34) and the Gauss Lemma, as follows:

$$\begin{aligned}\boldsymbol{A} &= \overline{\sigma}\boldsymbol{I} + \int_\Gamma [\![\kappa\boldsymbol{\chi}^1(y,0) + \sigma\boldsymbol{\chi}^0(y)]\!] \otimes \boldsymbol{\nu}\,\mathrm{d}\sigma = \int_Y [\sigma(\boldsymbol{I} - \nabla_y\boldsymbol{\chi}^0(y)) - \kappa\nabla_y\boldsymbol{\chi}^1(y,0)]\,\mathrm{d}y \\ &= \int_Y \sigma(\boldsymbol{I} - \nabla_y\boldsymbol{\chi}^0(y))(\boldsymbol{I} - \nabla_y\boldsymbol{\chi}^0(y))^t\,\mathrm{d}y + \int_Y [\sigma(\boldsymbol{I} - \nabla_y\boldsymbol{\chi}^0(y)) - \kappa\nabla_y\boldsymbol{\chi}^1(y,0)](\nabla_y\boldsymbol{\chi}^0(y))^t\,\mathrm{d}y \\ &\quad - \int_Y \kappa\nabla_y\boldsymbol{\chi}^1(y,0)(\boldsymbol{I} - \nabla_y\boldsymbol{\chi}^0(y))^t\,\mathrm{d}y = \int_Y \sigma(\boldsymbol{I} - \nabla_y\boldsymbol{\chi}^0(y))(\boldsymbol{I} - \nabla_y\boldsymbol{\chi}^0(y))^t\,\mathrm{d}y \\ &\quad + \int_\Gamma (\alpha[\![\boldsymbol{\chi}^1(y,0)]\!] + \beta[\![\boldsymbol{\chi}^0(y)]\!]) \otimes [\![\boldsymbol{\chi}^0(y)]\!]\,\mathrm{d}\sigma - \int_\Gamma \alpha[\![\boldsymbol{\chi}^1(y,0)]\!] \otimes [\![\boldsymbol{\chi}^0(y)]\!]\,\mathrm{d}\sigma\,. \end{aligned} \tag{4-58}$$

The derivation of (4-56) is a little bit lengthier. From (4-51), using the Gauss Lemma, it turns out that:

$$\begin{aligned}\mathcal{F}(x,t) &= -\int_Y \mathcal{D}\nabla_y\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)\,\mathrm{d}y = -\int_Y \sigma\nabla_y\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)\,\mathrm{d}y \\ &\quad - \int_Y \kappa\nabla_y\boldsymbol{\chi}^0(y)\frac{\partial}{\partial t}\nabla_y\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)\,\mathrm{d}y - \int_Y \kappa(\boldsymbol{I} - \nabla_y\boldsymbol{\chi}^0(y))\frac{\partial}{\partial t}\nabla_y\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)\,\mathrm{d}y\,.\end{aligned} \tag{4-59}$$

Using again the Gauss Lemma and (4-31), it follows that:

$$\begin{aligned}\mathcal{F}(x,t) &= -\int_Y \sigma\nabla_y\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)\,\mathrm{d}y - \int_Y \kappa\nabla_y\boldsymbol{\chi}^0(y)\frac{\partial}{\partial t}\nabla_y\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)\,\mathrm{d}y \\ &\quad - \int_\Gamma \alpha[\![\boldsymbol{\chi}^0(y)]\!]\frac{\partial}{\partial t}[\![\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)]\!]\,\mathrm{d}\sigma\,.\end{aligned} \tag{4-60}$$

On the other hand, using $\chi^0_h(y)$ as test function in (4-23)–(4-25), it follows that:

$$\int_Y \nabla_y\boldsymbol{\chi}^0(y)\mathcal{D}\nabla_y\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)\,\mathrm{d}y + \int_\Gamma [\![\boldsymbol{\chi}^0(y)]\!]\Big\{\alpha\frac{\partial}{\partial t} + \beta\Big\}[\![\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)]\!]\,\mathrm{d}\sigma = 0\,. \tag{4-61}$$

Hence, adding (4-60) and (4-61) yields:

$$\mathcal{F}(x,t) = -\int_Y \sigma(\boldsymbol{I} - \nabla_y\boldsymbol{\chi}^0(y))\nabla_y\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)\,\mathrm{d}y + \int_\Gamma \beta[\![\boldsymbol{\chi}^0(y)]\!][\![\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)]\!]\,\mathrm{d}\sigma\,. \tag{4-62}$$

Adding and subtracting $\int_Y \kappa\nabla_y\boldsymbol{\chi}^1(y,0)\nabla_y\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)\,\mathrm{d}y$ at the right-hand side of the previous equation, using the Gauss Lemma, and recalling (4-32)–(4-34), it turns out that:

$$\mathcal{F}(x,t) = -\int_Y \kappa\nabla_y\boldsymbol{\chi}^1(y,0)\nabla_y\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)\,\mathrm{d}y - \int_\Gamma \alpha[\![\boldsymbol{\chi}^1(y,0)]\!][\![\mathcal{T}(\tilde{\boldsymbol{g}}_0,\tilde{s}_1)(x,y,t)]\!]\,\mathrm{d}\sigma\,. \tag{4-63}$$

Then, recalling (4-35) and using Lemma 4.6 below, equation (4-56) is obtained:

$$\begin{aligned}\mathcal{F}(x,t) &= -\int_Y \kappa\nabla_y\boldsymbol{\chi}^1(y,t)\tilde{\boldsymbol{g}}_0(x,y)\,\mathrm{d}y - \int_\Gamma \alpha[\![\boldsymbol{\chi}^1(y,t)]\!]\,\tilde{s}_1(x,y)\,\mathrm{d}\sigma \\ &= -\int_Y \kappa\nabla_y\boldsymbol{\chi}^1(y,t)\boldsymbol{g}_0(x,y)\,\mathrm{d}y - \int_\Gamma \alpha[\![\boldsymbol{\chi}^1(y,t)]\!]\,s_1(x,y)\,\mathrm{d}\sigma\,.\end{aligned} \tag{4-64}$$

The last equality follows by noting that, by (4-37), $\tilde{\boldsymbol{g}}_0(x,y) - \boldsymbol{g}_0(x,y) = -(\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0(y))^t \boldsymbol{v}(x)$, and $\tilde{s}_1(x,y) - s_1(x,y) = [\![\boldsymbol{\chi}^0(y)]\!] \cdot \boldsymbol{v}(x)$, with $\boldsymbol{v}(x) = \overline{\boldsymbol{g}}_0(x) + \int_\Gamma s_1(x,y)\boldsymbol{\nu}\,\mathrm{d}\sigma$, and that:

$$\int_Y \kappa \nabla_y \boldsymbol{\chi}^1(y,t)(\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0(y))^t \,\mathrm{d}y - \int_\Gamma \alpha [\![\boldsymbol{\chi}^1(y,t)]\!] \otimes [\![\boldsymbol{\chi}^0(y)]\!] \,\mathrm{d}\sigma = 0\,, \tag{4-65}$$

by (4-29)–(4-31), using the Gauss Lemma.

A similar, and indeed simpler, argument leads to (4-55), since $\boldsymbol{\chi}^1$ is a $\mathcal{T}$-transform, too, by (4-35).

□

Proposition 4.5. *$\boldsymbol{K}$, $\boldsymbol{A}$, $\boldsymbol{B}$ are symmetric matrices. $\boldsymbol{K}$ and $\boldsymbol{A}$ are positive definite.*

Proof. The symmetry of $\boldsymbol{K}$ and $\boldsymbol{A}$ follows from (4-53) and (4-54), respectively. The symmetry of $\boldsymbol{B}$ follows from (4-55), using Lemma 4.6 below. The positive definiteness of $\boldsymbol{K}$ is proved as follows. For $\boldsymbol{\zeta} \in \boldsymbol{R}^N$, from (4-53),

$$\boldsymbol{K}\boldsymbol{\zeta} \cdot \boldsymbol{\zeta} = \int_Y \kappa |\nabla_y \psi(y)|^2 \,\mathrm{d}y + \int_\Gamma \alpha [\![\psi(y)]\!]^2 \,\mathrm{d}\sigma\,, \tag{4-66}$$

where $\psi = (\boldsymbol{y} - \boldsymbol{\chi}^0) \cdot \boldsymbol{\zeta}$. If $\boldsymbol{K}\boldsymbol{\zeta} \cdot \boldsymbol{\zeta} = 0$, then ψ is a constant ψ_0 by (4-66) and, recalling that $\boldsymbol{\chi}^0$ is Y-periodic, also the function $\boldsymbol{y} \mapsto \boldsymbol{y} \cdot \boldsymbol{\zeta} = \boldsymbol{\chi}^0 \cdot \boldsymbol{\zeta} + \psi_0$ is Y-periodic. This implies that $\boldsymbol{\zeta} = 0$, and the positive-definiteness of $\boldsymbol{K}$ follows. Analogously, the positive definiteness of $\boldsymbol{A}$ is obtained from (4-54). □

Lemma 4.6. *Let $\boldsymbol{\gamma}_1,\ \boldsymbol{\gamma}_2 \in L^2(Y)$ and $\zeta_1,\ \zeta_2 \in H^{1/2}(\Gamma)$ be assigned such that $\zeta_{1,2} = [\![z_{1,2}]\!]$ and, for $i = 1, 2$, $\boldsymbol{\gamma}_{1,2}|_{E_i} = \nabla z_{1,2}|_{E_i}$, for some scalar periodic functions $z_{1,2} \in H^1(E_i)$. Then, for all $t > 0$:*

$$\begin{aligned}\int_Y \kappa \nabla_y \mathcal{T}(\boldsymbol{\gamma}_1,\zeta_1)(t)\, \nabla_y \mathcal{T}(\boldsymbol{\gamma}_2,\zeta_2)(0) \,\mathrm{d}y + \int_\Gamma \alpha [\![\mathcal{T}(\boldsymbol{\gamma}_1,\zeta_1)(t)]\!]\, [\![\mathcal{T}(\boldsymbol{\gamma}_2,\zeta_2)(0)]\!] \,\mathrm{d}\sigma = \\ \int_Y \kappa \nabla_y \mathcal{T}(\boldsymbol{\gamma}_1,\zeta_1)(0)\, \nabla_y \mathcal{T}(\boldsymbol{\gamma}_2,\zeta_2)(t) \,\mathrm{d}y + \int_\Gamma \alpha [\![\mathcal{T}(\boldsymbol{\gamma}_1,\zeta_1)(0)]\!]\, [\![\mathcal{T}(\boldsymbol{\gamma}_2,\zeta_2)(t)]\!] \,\mathrm{d}\sigma\,.\end{aligned} \tag{4-67}$$

Proof. Let us define $v_h = \mathcal{T}(\boldsymbol{\gamma}_h,\zeta_h)$, and, for a fixed $T > 0$,

$$\hat{v}_h(y,t) = v_h(y, T-t)\,, \qquad 0 < t < T\,. \tag{4-68}$$

Using $\hat{v}_2(y,t)$ as a test function in (4-23)–(4-25) written for $v_1(y,t)$, and integrating in time over $(0,T)$, we obtain:

$$\int_0^T \int_Y \mathcal{D} \nabla v_1 \nabla \hat{v}_2 \,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Gamma \Big(\alpha [\![v_1]\!]_t + \beta [\![v_1]\!] \Big) [\![\hat{v}_2]\!] \,\mathrm{d}\sigma\,\mathrm{d}t = 0\,. \tag{4-69}$$

Exchanging the role of v_1 and v_2 and subtracting the resulting equation from the previous one we obtain:

$$\int_0^T \int_Y (\kappa \nabla v_{1t} \nabla \hat{v}_2 + \kappa \nabla v_1 \nabla \hat{v}_{2t}) \,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Gamma (\alpha [\![v_1]\!]_t [\![\hat{v}_2]\!] + \alpha [\![v_1]\!] [\![\hat{v}_2]\!]_t) \,\mathrm{d}\sigma\,\mathrm{d}t = 0\,. \tag{4-70}$$

Whence the assert follows, explicitly evaluating the time integrals. □

4C. **The homogenization limit.** Introduce for $i = 1, \dots, N$, and any $T > 0$ arbitrarily fixed the functions

$$w_i^\varepsilon(x,t) = x_i - \varepsilon\chi_i^0\big(\frac{x}{\varepsilon}\big) - \varepsilon\int_t^T \chi_i^1\big(\frac{x}{\varepsilon}, \tau - t\big)\,\mathrm{d}\tau\,, \tag{4-71}$$

so that explicit calculations reveal

$$\begin{aligned}
-\operatorname{div}\mathcal{D}^*\nabla w_i^\varepsilon &= 0\,, && \text{in } (\Omega_1^\varepsilon\cup\Omega_2^\varepsilon)\times(0,T); && \text{(4-72)}\\
[\![\mathcal{D}^*\nabla w_i^\varepsilon\cdot\boldsymbol{\nu}]\!] &= 0\,, && \text{on } \Gamma^\varepsilon\times(0,T); && \text{(4-73)}\\
-\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[\![w_i^\varepsilon]\!] + \frac{\beta}{\varepsilon}[\![w_i^\varepsilon]\!] &= \mathcal{D}^*\nabla w_i^\varepsilon\cdot\boldsymbol{\nu}\,, && \text{on } \Gamma^\varepsilon\times(0,T); && \text{(4-74)}\\
\nabla w_i^\varepsilon(x,T) &= \boldsymbol{e}_i - \nabla_y\chi_i^0\big(\frac{x}{\varepsilon}\big)\,, && \text{on } \Gamma^\varepsilon\times(0,T); && \text{(4-75)}\\
[\![w_i^\varepsilon(x,T)]\!] &= -\varepsilon[\![\chi_i^0\big(\frac{x}{\varepsilon}\big)]\!]\,, && \text{on } \Gamma^\varepsilon\times(0,T); && \text{(4-76)}
\end{aligned}$$

where

$$\mathcal{D}^* := -\kappa\frac{\partial}{\partial t} + \sigma\,. \tag{4-77}$$

Let $\varphi\in C_o^\infty(\Omega)$, select $w_i^\varepsilon\varphi$ as a testing function in the weak formulation (3-4) written for $t = T$ and integrate by parts in time. Then, select $u_\varepsilon\varphi$ as a testing function in the weak formulation of (4-72)–(4-76); in this second step, no integration by parts in t is needed. Finally, subtract the latter equation thus obtained from the former one, and find,

$$\begin{aligned}
-\int_0^T\int_\Omega \kappa\nabla u_\varepsilon\cdot\nabla\varphi\, w_{i\,t}^\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \sigma\nabla u_\varepsilon\cdot\nabla\varphi\, w_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}t&\\
+\int_0^T\int_\Omega \kappa\nabla w_{i\,t}^\varepsilon\cdot\nabla\varphi\, u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \sigma\nabla w_i^\varepsilon\cdot\nabla\varphi\, u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t&\\
+\int_\Omega \kappa\nabla u_\varepsilon(T)\cdot\big[\boldsymbol{e}_i - \nabla_y\chi_i^0\big(\frac{x}{\varepsilon}\big)\big]\varphi\,\mathrm{d}x + \int_\Omega \kappa\nabla u_\varepsilon(T)\cdot\nabla\varphi\big[x_i - \varepsilon\chi_i^0\big(\frac{x}{\varepsilon}\big)\big]\,\mathrm{d}x&\\
+\int_{\Gamma^\varepsilon}\frac{\alpha}{\varepsilon}[\![u_\varepsilon(T)]\!]\big(-\varepsilon[\![\chi_i^0\big(\frac{x}{\varepsilon}\big)]\!]\big)\varphi\,\mathrm{d}\sigma = \int_\Omega \kappa\boldsymbol{g}_\varepsilon\cdot\nabla w_i^\varepsilon(0)\varphi\,\mathrm{d}x&\\
+\int_\Omega \kappa\boldsymbol{g}_\varepsilon\cdot\nabla\varphi\, w_i^\varepsilon(0)\,\mathrm{d}x + \int_{\Gamma^\varepsilon}\frac{\alpha}{\varepsilon}s_\varepsilon[\![w_i^\varepsilon(0)]\!]\varphi\,\mathrm{d}\sigma\,.&
\end{aligned}\tag{4-78}$$

Integrating by parts in time the first term on the left-hand side, the above equation is transformed into:

$$\begin{aligned}
\int_0^T\int_\Omega (\mathcal{D}\,\nabla u_\varepsilon)\cdot\nabla\varphi\, w_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega (\mathcal{D}^*\,\nabla w_i^\varepsilon)\cdot\nabla\varphi\, u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t&\\
+\int_\Omega \kappa\nabla u_\varepsilon(T)\cdot\big[\boldsymbol{e}_i - \nabla_y\chi_i^0\big(\frac{x}{\varepsilon}\big)\big]\varphi\,\mathrm{d}x - \int_{\Gamma^\varepsilon}\alpha[\![u_\varepsilon(T)]\!][\![\chi_i^0\big(\frac{x}{\varepsilon}\big)]\!]\varphi\,\mathrm{d}\sigma&\\
= \int_\Omega \kappa\boldsymbol{g}_\varepsilon\cdot\nabla w_i^\varepsilon(0)\varphi\,\mathrm{d}x + \int_{\Gamma^\varepsilon}\frac{\alpha}{\varepsilon}s_\varepsilon[\![w_i^\varepsilon(0)]\!]\varphi\,\mathrm{d}\sigma\,,&
\end{aligned}\tag{4-79}$$

which, by using the Gauss Lemma on the third term on the left-hand side and (4-29)–(4-31), further simplifies into:

$$\int_0^T \int_\Omega (\mathcal{D}\,\nabla u_\varepsilon) \cdot \nabla\varphi\, w_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega (\mathcal{D}^*\,\nabla w_i^\varepsilon) \cdot \nabla\varphi\, u_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ - \int_\Omega \kappa u_\varepsilon(T)\big[\boldsymbol{e}_i - \nabla_y \chi_i^0(\frac{x}{\varepsilon})\big] \cdot \nabla\varphi \,\mathrm{d}x \\ = \int_\Omega \kappa \boldsymbol{g}_\varepsilon \cdot \nabla w_i^\varepsilon(0)\varphi \,\mathrm{d}x + \int_{\Gamma^\varepsilon} \frac{\alpha}{\varepsilon} s_\varepsilon [\![w_i^\varepsilon(0)]\!] \varphi \,\mathrm{d}\sigma\,. \quad (4\text{-}80)$$

We rely next on the energy inequalitis (3-11) and (3-12) which, together with Lemma 3.6, imply that, extracting subsequences if needed, we may assume

$$-\mathcal{D}\,\nabla u_\varepsilon \rightharpoonup \boldsymbol{\xi}_0\,, \qquad \text{weakly in } L^2(\Omega\times(0,T)), \quad (4\text{-}81)$$

$$u_\varepsilon \to u_0\,, \qquad \text{strongly in } L^2(\Omega\times(0,T)), \quad (4\text{-}82)$$

for some $\boldsymbol{\xi}_0 \in L^2(\Omega\times(0,T))^N$, $u_0 \in L^2(\Omega\times(0,T))$. On the other hand, $w_i^\varepsilon \to x_i$ strongly in $L^2(\Omega\times(0,T))$, as $\varepsilon\to 0$. Moreover, due to the periodicity of the functions χ^0, χ^1, one gets

$$\sigma\nabla w_i^\varepsilon \rightharpoonup \overline{\sigma}\boldsymbol{e}_i - \int_Y \sigma\nabla_y\chi_i^0(y)\,\mathrm{d}y - \int_t^T\int_Y \sigma\nabla_y\chi_i^1(y,\tau-t)\,\mathrm{d}y\,\mathrm{d}\tau\,, \quad (4\text{-}83)$$

weakly in $L^2(\Omega\times(0,T))$, and, in the same weak sense,

$$\kappa\nabla w_{i\,t}^\varepsilon \rightharpoonup \int_Y \kappa\nabla_y\chi_i^1(y,0)\,\mathrm{d}y + \int_t^T\int_Y \kappa\nabla_y\chi_{i\tau}^1(y,\tau-t)\,\mathrm{d}y\,\mathrm{d}\tau\,. \quad (4\text{-}84)$$

Thus, taking the limit $\varepsilon\to 0$ in (4-80) and recalling (4-49), (4-50), we obtain

$$-\int_0^T\int_\Omega \boldsymbol{\xi}_0\cdot\nabla\varphi\, x_i\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \Big[\boldsymbol{A} + \int_t^T \boldsymbol{B}(\tau-t)\,\mathrm{d}\tau\Big]^t \boldsymbol{e}_i\cdot\nabla\varphi\, u_0\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega u_0(T)\boldsymbol{K}^t\boldsymbol{e}_i\cdot\nabla\varphi\,\mathrm{d}x \\ = \int_\Omega \varphi\int_Y \kappa\boldsymbol{g}_0\cdot\Big[\boldsymbol{e}_i - \nabla_y\chi_i^0 - \int_0^T \nabla_y\chi_i^1(\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}y\,\mathrm{d}x \\ - \int_\Omega\varphi\int_\Gamma \alpha s_1(x,y)\Big[[\![\chi_i^0]\!] + \int_0^T [\![\chi_i^1(\tau)]\!]\,\mathrm{d}\tau\Big]\,\mathrm{d}\sigma\,\mathrm{d}x\,. \quad (4\text{-}85)$$

As usual, next we take φx_i as a testing function in (3-4). This test essentially does not detect the boundary Γ^ε, due to (2-3); on letting $\varepsilon\to 0$

$$\int_0^T\int_\Omega \boldsymbol{\xi}_0\cdot\nabla\varphi\, x_i\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \boldsymbol{\xi}_0\cdot\boldsymbol{e}_i\,\varphi\,\mathrm{d}x\,\mathrm{d}t = 0\,. \quad (4\text{-}86)$$

We substitute (4-86) in (4-85) and get

$$\int_0^T\int_\Omega \boldsymbol{\xi}_0\cdot\boldsymbol{e}_i\,\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \Big[\boldsymbol{A} + \int_t^T \boldsymbol{B}(\tau-t)\,\mathrm{d}\tau\Big]^t \boldsymbol{e}_i\cdot\nabla\varphi\, u_0\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0(T)\boldsymbol{K}^t\boldsymbol{e}_i\cdot\nabla\varphi\,\mathrm{d}x \\ + \int_\Omega \varphi\int_Y \kappa\boldsymbol{g}_0\cdot\Big[\boldsymbol{e}_i - \nabla_y\chi_i^0 - \int_0^T \nabla_y\chi_i^1(\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}y\,\mathrm{d}x \\ - \int_\Omega\varphi\int_\Gamma \alpha s_1(x,y)\Big[[\![\chi_i^0]\!] + \int_0^T [\![\chi_i^1(\tau)]\!]\,\mathrm{d}\tau\Big]\,\mathrm{d}\sigma\,\mathrm{d}x\,. \quad (4\text{-}87)$$

Then we change the integration order with respect to τ and t in the first term at the right-hand side, and differentiate in T the resulting equality; in fact the choice of T is essentially arbitrary in this setting. We obtain, reverting to t as the time variable, for a.e. t,

$$\int_{\Omega} \boldsymbol{\xi}_0 \, \varphi \, \mathrm{d}x = \int_{\Omega} \Big[\boldsymbol{A} \, u_0 + \int_0^t \boldsymbol{B}(t-\tau) \, u_0(\cdot,\tau) \, \mathrm{d}\tau + \boldsymbol{K} u_{0t}(t) \Big] \nabla\varphi \, \mathrm{d}x + \int_{\Omega} \varphi \mathcal{F} \, \mathrm{d}x \,, \tag{4-88}$$

where we have used (4-56). From (4-88) and Proposition 4.5, it follows that $u_0 \in H^1(0,T;H^1(\Omega))$. Indeed the Gronwall argument of Lemma 7.2 of [4] carries over to the present case, which however deals with a second order equation, and hence needs the L^2-estimate on $\nabla u_0(\cdot,0)$ implied by (4-92) and (4-93). Note that (4-92) and (4-93) are independent of the sought after regularity for t positive. Thus,

$$\boldsymbol{\xi}_0(x,t) = -\boldsymbol{K} \nabla u_{0t}(x,t) - \boldsymbol{A} \nabla u_0(x,t) - \int_0^t \boldsymbol{B}(t-\tau) \nabla u_0(x,\tau) \, \mathrm{d}\tau + \mathcal{F} \,, \tag{4-89}$$

a.e. (x,t). Clearly $\operatorname{div} \boldsymbol{\xi}_0 = 0$ in the sense of distributions (see e.g., (4-86) above). This shows that (4-47) is in force.

In order to obtain an initial condition for (4-47), we consider again (4-87) and let $T \to 0$ there. We get:

$$0 = \int_{\Omega} u_0(0) \boldsymbol{K} \nabla\varphi \, \mathrm{d}x + \int_{\Omega} \varphi \Big\{ \int_Y \kappa \Big[\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0 \Big] \boldsymbol{g}_0(x,y) \, \mathrm{d}y - \int_{\Gamma} \alpha s_1(x,y) [\![\boldsymbol{\chi}^0]\!] \, \mathrm{d}\sigma \Big\} \, \mathrm{d}x \,. \tag{4-90}$$

Then, using the Gauss Lemma, the positive definiteness of $\boldsymbol{K}$ and (4-31), we get:

$$\nabla u_0(x,0) = \boldsymbol{K}^{-1} \Big\{ \int_Y \kappa \Big[\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0 \Big] \boldsymbol{g}_0(x,y) \, \mathrm{d}y - \int_{\Gamma} \kappa \Big[\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0 \Big] \boldsymbol{\nu} s_1(x,y) \, \mathrm{d}\sigma \Big\} \,. \tag{4-91}$$

Now, using again the Gauss Lemma and taking into account equations (4-29)–(4-30), and (4-7), (4-48) (or (4-57)), we may rewrite equality (4-91) in the form

$$\begin{aligned} \nabla u_0(x,0) &= \boldsymbol{K}^{-1} \Big\{ \int_Y \kappa \Big[\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0 \Big] [\nabla_x z_0(x) + \nabla_y z_1(x,y)] \, \mathrm{d}y - \int_{\Gamma} \kappa \Big[\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0 \Big] \boldsymbol{\nu} s_1(x,y) \, \mathrm{d}\sigma \Big\} \\ &= \nabla_x z_0(x) \boldsymbol{K}^{-1} \int_Y \kappa \Big[\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0 \Big] \, \mathrm{d}y - \boldsymbol{K}^{-1} \Big\{ \int_Y \operatorname{div}_y \Big(\kappa \Big[\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0 \Big] \Big) z_1(x,y) \, \mathrm{d}y \\ &\quad + \int_{\Gamma} \kappa \Big[\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0 \Big] \boldsymbol{\nu} [\![z_1(x,y)]\!] \, \mathrm{d}\sigma - \int_{\Gamma} \kappa \Big[\boldsymbol{I} - \nabla_y \boldsymbol{\chi}^0 \Big] \boldsymbol{\nu} s_1(x,y) \, \mathrm{d}\sigma \Big\} = \nabla_x z_0(x) \,. \end{aligned} \tag{4-92}$$

Finally, we note that (4-7) yields

$$\overline{\boldsymbol{g}}_0(x) = \int_Y \boldsymbol{g}_0(x,y) \, \mathrm{d}y = \int_Y [\nabla_x z_0(x) + \nabla_y z_1(x,y)] \, \mathrm{d}y = \nabla_x z_0(x) - \int_{\Gamma} [\![z_1(x,y)]\!] \nu \, \mathrm{d}\sigma \,;$$

i.e.,

$$\nabla_x z_0(x) = \overline{\boldsymbol{g}}_0(x) + \int_{\Gamma} [\![z_1(x,y)]\!] \nu \, \mathrm{d}\sigma = \overline{\boldsymbol{g}}_0(x) + \int_{\Gamma} s_1(x,y) \nu \, \mathrm{d}\sigma \,. \tag{4-93}$$

Hence, the initial condition (4-91) reduces to (4-20), which was formally obtained by an integration over Y of (4-17).

Finally, the boundary data prescribed in (2-9) can be proven to be attained following the approach of Subsection 5.1 of [4].

Remark 4.7. *As pointed out in the Introduction, the choice* (2-6) *of the initial data, amounting to assigning the initial value of the electric field, is the most natural from the physical point of view (see* [24]*) and moreover it is suggested by the energy estimate* (3-11)*. Let us show how in this connection assumption (H1) of Section 2 is fundamental. First, note that in the proof of Theorem 3.2 only the divergence of $\kappa\boldsymbol{g}_\varepsilon$ plays a role. Namely, if we only assume $\boldsymbol{g}_\varepsilon$ to be any gradient field, we still arrive at problem* (3-7)*–*(3-10)*, but only assumption (H1) guarantees that condition* (2-6) *holds, as shown in Subsection 3A. Therefore in practice this amounts to assuming as initial data the function z_ε itself. In the same spirit we remark that only assumption (H1) allows us to transform condition* (4-91) *into* (4-20)*, thereby reconciling the formal and rigorous approaches to homogenization.*

Finally, from an inspection of the differential equation in (2-2)*, one might infer that from the mathematical point of view the most natural initial data to be assigned is* $\operatorname{div}(\kappa\nabla u_\varepsilon)$ *in the sense of distributions. However, by the remarks above, this amounts again to prescribing z_ε.*

5. Asymptotic decay

We recall that in [7] the following theorem has been proven, concerning the time-asymptotic decay of the solution u_ε of problem (2-2)–(2-7).

Theorem 5.1. *Under the assumptions of Theorem 2.2, we have*

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le \gamma(\varepsilon + \mathrm{e}^{-\lambda t}) \quad \text{a.e. in } (0,+\infty), \tag{5-1}$$

where γ and λ are positive constants independent of ε. Moreover, if $\beta > 0$, or else if s_ε has null mean average over each connected component of Γ^ε, it follows that

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le \gamma\,\mathrm{e}^{-\lambda t} \quad \text{a.e. in } (0,+\infty). \tag{5-2}$$

The main idea in order to prove Theorem 2.3 is to apply the previous result to the function

$$w_\varepsilon = u_\varepsilon - u_\varepsilon^{\#}\,,$$

which satisfies a homogeneous Dirichlet boundary condition on $\partial\Omega \times (0,+\infty)$, where $u_\varepsilon^{\#}(x,t)$ solves the following time-periodic version of the microscopic differential scheme introduced above:

$$-\operatorname{div}(\kappa\nabla u_{\varepsilon t}^{\#} + \sigma\nabla u_\varepsilon^{\#}) = 0\,, \qquad \text{in } (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times \boldsymbol{R}; \tag{5-3}$$

$$[\![\kappa\nabla u_{\varepsilon t}^{\#} + \sigma\nabla u_\varepsilon^{\#} \cdot \boldsymbol{\nu}]\!] = 0\,, \qquad \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; \tag{5-4}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[\![u_\varepsilon^{\#}]\!] + \frac{\beta}{\varepsilon}[\![u_\varepsilon^{\#}]\!] = (\kappa\nabla u_{\varepsilon t}^{\#} + \sigma\nabla u_\varepsilon^{\#} \cdot \boldsymbol{\nu})^{(2)}\,, \quad \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; \tag{5-5}$$

$$u_\varepsilon^{\#}(x,t) = \Psi(x)\Phi(t)\,, \qquad \text{on } \partial\Omega \times \boldsymbol{R}; \tag{5-6}$$

$$u_\varepsilon^{\#}(x,\cdot) \text{ is } T^{\#}\text{-periodic}, \qquad \forall x \in \Omega\,. \tag{5-7}$$

Indeed, this problem is derived from (2-11)–(2-16), replacing equation (2-15)-(2-16) with (5-7). We emphasize that, when $\beta > 0$, system (5-3)–(5-7) uniquely determines the periodic solution $u_\varepsilon^{\#}$. Moreover, by (5-3)–(5-5), this solution must satisfy

$$\int_{\Gamma_i^\varepsilon} [\![u_\varepsilon^{\#}(x,t)]\!]\,\mathrm{d}\sigma = \mathrm{e}^{-(\beta/\alpha)t}\int_{\Gamma_i^\varepsilon} [\![u_\varepsilon^{\#}(x,0)]\!]\,\mathrm{d}\sigma\,, \quad \forall i = 1,\dots,d\,, \tag{5-8}$$

where Γ_i^ε, $i = 1,\dots,d$, are the connected component of Γ^ε. Condition (5-8), jointly with (5-7), implies that $[\![u_\varepsilon^{\#}(x,t)]\!]$ has null average at every time t over each connected component of Γ^ε. On the contrary, when $\beta = 0$, system (5-3)–(5-7) determines the periodic solution $u_\varepsilon^{\#}$, up to an additive constant, which can be arbitrarily chosen in each connected component of Ω_1^ε; hence, it must be complemented with another condition which guarantees the uniqueness. It seems natural to impose the following condition:

$$[\![u_\varepsilon^\#(\cdot,t)]\!] - s_\varepsilon(\cdot) \quad \text{has null average over each connected component of } \Gamma^\varepsilon . \tag{5-9}$$

Equation (5-9) is suggested by the observation that $[\![u_\varepsilon(\cdot,t)]\!] - s_\varepsilon(\cdot)$ has null average over each connected component of Γ^ε, as a consequence of (2-11)–(2-13), (2-16).

We can prove the following result, which easily implies Theorem 2.3.

Theorem 5.2. *Let $\{u_\varepsilon\}$ and $\{u_\varepsilon^\#\}$ be, respectively, the sequences of the solutions of* (2-11)–(2-16) *and* (5-3)–(5-7)*, complemented with* (5-9) *for $\beta = 0$. Under the assumptions of Theorem 2.3 we have*

$$\|u_\varepsilon(\cdot,t) - u_\varepsilon^\#(\cdot,t)\|_{L^2(\Omega)} \le \gamma\, \mathrm{e}^{-\lambda t} \quad \textit{a.e. in } (1,+\infty), \tag{5-10}$$

where γ and λ are positive constants, independent of ε.

To solve Problem (5-3)–(5-7), we express the function Φ by means of its Fourier series, i.e.,

$$\Phi(t) = \sum_{k=-\infty}^{+\infty} c_k\, \mathrm{e}^{i\omega_k t} \tag{5-11}$$

where $\omega_k = 2k\pi/T^\#$ is the k-th circular frequency, and we represent the solution $u_\varepsilon^\#(x,t)$ as follows:

$$u_\varepsilon^\#(x,t) = \sum_{k=-\infty}^{+\infty} v_{\varepsilon k}(x)\, \mathrm{e}^{i\omega_k t} , \tag{5-12}$$

where the complex-valued functions $v_{\varepsilon k}(x) \in L^2(\Omega)$ are such that $v_{\varepsilon k}|_{\Omega_i^\varepsilon} \in H^1(\Omega_i^\varepsilon)$, $i = 1,2$ and for $k \neq 0$ satisfy the problem

$$-\operatorname{div}\big((i\omega_k\kappa + \sigma)\nabla v_{\varepsilon k}\big) = 0 , \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{5-13}$$

$$[\![(i\omega_k\kappa + \sigma)\nabla v_{\varepsilon k} \cdot \boldsymbol{\nu}]\!] = 0 , \qquad \text{on } \Gamma^\varepsilon; \tag{5-14}$$

$$\frac{i\omega_k\alpha + \beta}{\varepsilon}[\![v_{\varepsilon k}]\!] = \big((i\omega_k\kappa + \sigma)\nabla v_{\varepsilon k} \cdot \boldsymbol{\nu}\big)^{(2)} , \qquad \text{on } \Gamma^\varepsilon; \tag{5-15}$$

$$v_{\varepsilon k} = c_k\Psi , \qquad \text{on } \partial\Omega; \tag{5-16}$$

whereas for $k = 0$ $v_{\varepsilon 0}$ satisfies

$$-\operatorname{div}(\sigma\nabla v_{\varepsilon 0}) = 0 , \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{5-17}$$

$$[\![\sigma\nabla v_{\varepsilon 0} \cdot \boldsymbol{\nu}]\!] = 0 , \qquad \text{on } \Gamma^\varepsilon; \tag{5-18}$$

$$(\sigma\nabla v_{\varepsilon 0} \cdot \boldsymbol{\nu})^{(2)} = \frac{\beta}{\varepsilon}[\![v_{\varepsilon 0}]\!] , \text{ on } \Gamma^\varepsilon; \tag{5-19}$$

$$v_{\varepsilon 0} = c_0\Psi , \qquad \text{on } \partial\Omega. \tag{5-20}$$

Note that any solution of Problem (5-13)–(5-16) and, in the case $\beta > 0$, of Problem (5-17)–(5-20) has jump $[\![v_{\varepsilon k}]\!]$, $k \in \boldsymbol{Z}$, with null mean average over each connected component of Γ^ε. On the contrary, in order to assure well-posedness, Problem (5-17)–(5-20), when $\beta = 0$, must be complemented with the additional condition

$$[\![v_{\varepsilon 0}]\!] - s_\varepsilon(\cdot) \quad \text{has null average over each connected component of } \Gamma^\varepsilon , \tag{5-21}$$

according to (5-9).

We prove the following homogenization result:

Theorem 5.3. *Let* $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon$ *be as before and assume that* (2-8) *and* (2-18) *hold. Then, for* $k \in \boldsymbol{Z} \setminus \{0\}$ *[respectively,* $k = 0$*, under the further assumption (H2), if* $\beta = 0$*], the solution* $v_{\varepsilon k}$ *of Problem* (5-13)–(5-16) *[respectively, Problem* (5-17)–(5-21)*] strongly converges in* $L^2(\Omega)$ *to a function* $v_{0k} \in H^1(\Omega)$ *which is the unique solution of the problem*

$$-\operatorname{div}(A^{\omega_k} \nabla v_{0k}) = 0\,, \qquad \text{in } \Omega; \tag{5-22}$$

$$v_{0k} = c_k \Psi\,, \qquad \text{on } \partial\Omega; \tag{5-23}$$

where

$$A^{\omega_k} = i\omega_k \boldsymbol{K} + \boldsymbol{A} + \int_0^{+\infty} \boldsymbol{B}(t)\, \mathrm{e}^{-i\omega_k t}\, \mathrm{d}t\,, \tag{5-24}$$

with $\boldsymbol{K}$*,* $\boldsymbol{A}$ *and* $\boldsymbol{B}(t)$ *defined in* (4-48), (4-49) *and* (4-50).

The case $k \neq 0$ is dealt with in §5A, where the subscript k is dropped throughout for the sake of simplicity, and an alternative expression for A^{ω_k} is given (equation (5-41)). The case $k = 0$ is dealt with in §5B.

In §5C we study Problem (5-3)–(5-9), and establish:

Theorem 5.4. *Let* $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon$ *be as before and assume that* (2-8), (2-17), (2-18) *hold, complemented with (H2) when* $\beta = 0$*. Then,*

i) *the series at the right-hand side of equation* (5-12) *strongly converges, uniformly with respect to* ε*, in* $H^1_\#(\boldsymbol{R}; L^2(\Omega))$ *and in* $H^1_\#(\boldsymbol{R}; H^1(\Omega_i^\varepsilon))$*,* $i = 1, 2$*, to the unique solution* $u_\varepsilon^\#(x,t)$ *of Problem* (5-3)–(5-9)*;*

ii) *the sequence* $\{u_\varepsilon^\#(x,t)\}$ *strongly converges in* $H^1_\#(\boldsymbol{R}; L^2(\Omega))$ *as* $\varepsilon \to 0$ *to a function* $u_0^\#(x,t)$*,* $T^\#$*-periodic in time, which can be represented by means of the following Fourier series:*

$$u_0^\#(x,t) = \sum_{k=-\infty}^{+\infty} v_{0k}(x)\, \mathrm{e}^{i\omega_k t}\,, \tag{5-25}$$

in turn strongly converging in $H^1_\#(\boldsymbol{R}; H^1(\Omega))$*;*

iii) *the function* $u_0^\#(x,t)$ *is the unique solution* $T^\#$*-periodic in time of the problem* (2-20), (2-21).

Remark 5.5. *We note that, with a change of variables, equation* (2-20) *can be recast as follows:*

$$-\operatorname{div}\Big(\boldsymbol{K}\nabla u_{0t}^\# + \boldsymbol{A}\nabla u_0^\# + \int_{-\infty}^t \boldsymbol{B}(t-\tau)\nabla u_0^\#(x,\tau)\,\mathrm{d}\tau\Big) = 0\,, \qquad \text{in } \Omega \times \boldsymbol{R}, \tag{5-26}$$

which closely resembles the first equation in (2-9)*. In fact, equation* (2-9) *involves a time integration over* $(0,t)$ *and contains a source* $\mathcal{F}$ *accounting for the initial data of the original Problem* (2-11)–(2-16)*, whereas equation* (5-26) *involves a time integration over* $(-\infty, t)$ *and is relevant to periodic functions, i.e., to situations where any transient phenomenon has elapsed.*

5A. **Homogenization limit of time-harmonic solutions: case** $k \neq 0$**.** In this Section we prove Theorem 5.3 in the case $k \neq 0$. For the sake of simplicity, we omit here the subscript k and set

$$\psi(x) := c_k \Psi(x)\,. \tag{5-27}$$

Firstly, we establish the following energy estimate:

$$\int_\Omega (|\omega|\kappa + \sigma)|\nabla v_\varepsilon|^2\,\mathrm{d}x + \frac{|\omega|\alpha + \beta}{\varepsilon}\int_{\Gamma^\varepsilon} |[\![v_\varepsilon]\!]|^2\,\mathrm{d}\sigma \le \gamma \int_\Omega (|\omega|\kappa + \sigma)|\nabla\psi|^2\,\mathrm{d}x\,, \tag{5-28}$$

where γ is independent of ε and ω. This estimate, together with Poincaré's inequality in [18, 4] imply the following L^2 estimate:

$$\int_\Omega v_\varepsilon^2 \,\mathrm{d}x \le \gamma \int_\Omega |\nabla\psi|^2 \,\mathrm{d}x \,. \tag{5-29}$$

In order to carry out the proof, which is quite standard, we set

$$z_\varepsilon = v_\varepsilon - \psi \,. \tag{5-30}$$

The complex-valued function $z_\varepsilon(x,t)$ satisfies the equations

$$-\operatorname{div}((i\omega\kappa+\sigma)\nabla z_\varepsilon) = 0 \,, \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{5-31}$$

$$[\![(i\omega\kappa+\sigma)\nabla z_\varepsilon \cdot \boldsymbol{\nu}]\!] = -[\![(i\omega\kappa+\sigma)]\!] \nabla\psi \cdot \boldsymbol{\nu} \,, \qquad \text{on } \Gamma^\varepsilon; \tag{5-32}$$

$$\frac{i\omega\alpha+\beta}{\varepsilon} [\![z_\varepsilon]\!] = ((i\omega\kappa+\sigma)\nabla z_\varepsilon \cdot \boldsymbol{\nu})^{(2)} + (i\omega\kappa_2+\sigma_2)\nabla\psi \cdot \boldsymbol{\nu} \,, \qquad \text{on } \Gamma^\varepsilon; \tag{5-33}$$

$$z_\varepsilon = 0 \,, \qquad \text{on } \partial\Omega. \tag{5-34}$$

We multiply (5-31) by $\overline{z}_\varepsilon$, integrate over $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$, use the Gauss-Green identity and equation (5-34), and arrive at

$$\int_\Omega (i\omega\kappa+\sigma)|\nabla z_\varepsilon|^2 \,\mathrm{d}x + \int_{\Gamma^\varepsilon} [\![\overline{z}_\varepsilon (i\omega\kappa+\sigma)\nabla z_\varepsilon \cdot \boldsymbol{\nu}]\!] \,\mathrm{d}\sigma = 0 \,. \tag{5-35}$$

Using equations (5-32)–(5-33), and then the Gauss-Green identity and equations (2-18) and (5-34), we obtain

$$\begin{aligned} \int_\Omega (i\omega\kappa+\sigma)|\nabla z_\varepsilon|^2 \,\mathrm{d}x + \frac{i\omega\alpha+\beta}{\varepsilon} \int_{\Gamma^\varepsilon} |[\![z_\varepsilon]\!]|^2 \,\mathrm{d}\sigma &= \int_{\Gamma^\varepsilon} [\![\overline{z}_\varepsilon (i\omega\kappa+\sigma)\nabla\psi \cdot \boldsymbol{\nu}]\!] \,\mathrm{d}\sigma \\ &= -\int_\Omega (i\omega\kappa+\sigma)\nabla\overline{z}_\varepsilon \cdot \nabla\psi \;\mathrm{d}x \,. \end{aligned} \tag{5-36}$$

Taking the real and imaginary parts of equation (5-36), adding them and using Young's inequality, we get

$$\int_\Omega (|\omega|\kappa+\sigma)|\nabla z_\varepsilon|^2 \,\mathrm{d}x + \frac{|\omega|\alpha+\beta}{\varepsilon} \int_{\Gamma^\varepsilon} |[\![z_\varepsilon]\!]|^2 \,\mathrm{d}\sigma \le \frac{1}{2} \int_\Omega (|\omega|\kappa+\sigma)|\nabla z_\varepsilon|^2 \;\mathrm{d}x + 2 \int_\Omega (|\omega|\kappa+\sigma)|\nabla\psi|^2 \;\mathrm{d}x \,, \tag{5-37}$$

whence equation (5-28) follows.

The proof of the existence of solution to Problem (5-31)–(5-34), for the unknown z_ε defined in equation (5-30), in the class

$$\mathcal{H} = \{\varphi \in L^2(\Omega) \,; \quad \varphi_{|\Omega_i^\varepsilon} \in H^1(\Omega_i^\varepsilon) \,, \ i = 1\,,2 \,; \quad \varphi_{|\partial\Omega} = 0\} \,, \tag{5-38}$$

can be obtained in a standard way, using the Lax-Milgram Theorem [28, Ch. 6, Th. 1.4], applied to the bilinear form

$$a(\varphi,\phi) := \int_\Omega (i\omega\kappa+\sigma)\nabla\varphi \cdot \nabla\overline{\phi} \,\mathrm{d}x + \frac{i\alpha\omega+\beta}{\varepsilon} \int_{\Gamma^\varepsilon} [\![\varphi]\!] [\![\overline{\phi}]\!] \,\mathrm{d}\sigma \,, \quad \forall \varphi,\phi \in \mathcal{H} \,, \tag{5-39}$$

taking into account that the weak formulation of problem (5-31)–(5-34) is given by

$$a(z_\varepsilon,\phi) = \int_{\Gamma^\varepsilon} [\![\overline{\phi} \, (i\omega\kappa+\sigma)\nabla\psi \cdot \boldsymbol{\nu}]\!] \,\mathrm{d}\sigma, \quad \forall \phi \in \mathcal{H}. \tag{5-40}$$

Moreover, by means of standard homogenization techniques, it follows that $v_\varepsilon \to v_0$ strongly in $L^2(\Omega)$, where v_0 satisfies (5-22)–(5-23), the matrix A^ω is given by (here the superscript t denotes transposition)

$$A^\omega = (i\omega\overline{\kappa} + \overline{\sigma})I + \int_\Gamma [\![(i\omega\kappa + \sigma)\chi^\omega]\!] \otimes \boldsymbol{\nu} \, \mathrm{d}\sigma = (i\omega\overline{\kappa} + \overline{\sigma})I - \int_Y (i\omega\kappa + \sigma)\nabla^t \chi^\omega \, \mathrm{d}y \,, \tag{5-41}$$

and the cell function $\chi^\omega : Y \to \boldsymbol{C}^N$, is such that its components χ^ω_h, $h = 1, \dots, N$, satisfy

$$-(i\omega\kappa + \sigma)\, \Delta_y \chi^\omega_h = 0\,, \qquad \text{in } E_1,\, E_2; \tag{5-42}$$

$$[\![(i\omega\kappa + \sigma)(\nabla_y \chi^\omega_h - \boldsymbol{e}_h) \cdot \boldsymbol{\nu}]\!] = 0\,, \qquad \text{on } \Gamma; \tag{5-43}$$

$$(i\omega\alpha + \beta)[\![\chi^\omega_h]\!] = ((i\omega\kappa + \sigma)(\nabla_y \chi^\omega_h - \boldsymbol{e}_h) \cdot \boldsymbol{\nu})^{(2)}\,, \qquad \text{on } \Gamma; \tag{5-44}$$

and are periodic functions with vanishing integral average over Y. For more details see [6].

Finally, we show that equations (5-24) and (5-41) yield the same matrix A^ω. To this end, we set

$$\theta^\omega = \chi^0 + \int_0^{+\infty} \chi^1(\cdot, t)\, \mathrm{e}^{-i\omega t} \, \mathrm{d}t\,. \tag{5-45}$$

It follows that θ^ω satisfies equations (5-42)–(5-44). Indeed, recalling (4-23), (4-29), (4-32) and (4-35), it follows

$$\begin{aligned}
-\operatorname{div}\left((i\omega\kappa + \sigma)\nabla\theta^\omega_h\right) &= -\operatorname{div}\left((i\omega\kappa + \sigma)\nabla\big(\chi^0_h + \int_0^{+\infty} \chi^1_h(\cdot, t)\, \mathrm{e}^{-i\omega t} \, \mathrm{d}t\big)\right) \\
&= -i\omega \operatorname{div}(\kappa\nabla\chi^0_h) - \operatorname{div}(\sigma\nabla\chi^0_h) - \int_0^{+\infty} \operatorname{div}\left(\mathcal{D}\big(\nabla\chi^1_h(\cdot, t)\big)\, \mathrm{e}^{-i\omega t}\right) \mathrm{d}t \\
&\qquad + \operatorname{div}\left(\kappa\nabla\Big(\int_0^{+\infty} \frac{\partial(\chi^1_h(\cdot, t)\, \mathrm{e}^{-i\omega t})}{\partial t} \, \mathrm{d}t\Big)\right) \\
&= -i\omega\kappa\Delta_y\chi^0_h - \kappa\Delta_y\chi^1_h(\cdot, 0) - \sigma\Delta_y\chi^0_h - \int_0^{+\infty} \mathcal{D}\big(\Delta_y\chi^1_h(\cdot, t)\big)\, \mathrm{e}^{-i\omega t} \, \mathrm{d}t = 0\,,
\end{aligned}$$

where we used (5-75). Analogously,

$$\begin{aligned}
[\![(i\omega\kappa+\sigma)(\nabla_y\theta^\omega_h - \boldsymbol{e}_h)\cdot\boldsymbol{\nu}]\!] &= [\![(i\omega\kappa+\sigma)(\nabla_y\chi^0_h - \boldsymbol{e}_h)\cdot\boldsymbol{\nu}]\!] + [\![(i\omega\kappa+\sigma)\nabla_y \int_0^{+\infty} \chi^1_h(\cdot, t)\, \mathrm{e}^{-i\omega t} \, \mathrm{d}t\cdot\boldsymbol{\nu}]\!] \\
&= i\omega[\![\kappa(\nabla_y\chi^0_h - \boldsymbol{e}_h)\cdot\boldsymbol{\nu}]\!] + [\![\sigma(\nabla_y\chi^0_h - \boldsymbol{e}_h)\cdot\boldsymbol{\nu}]\!] + [\![\sigma\nabla_y \int_0^{+\infty} \chi^1_h(\cdot, t)\, \mathrm{e}^{-i\omega t} \, \mathrm{d}t\cdot\boldsymbol{\nu}]\!] \\
&\quad + [\![\kappa\nabla_y \int_0^{+\infty} \frac{\partial\chi^1_h(\cdot, t)}{\partial t}\, \mathrm{e}^{-i\omega t} \, \mathrm{d}t\cdot\boldsymbol{\nu}]\!] - [\![\kappa\nabla_y \int_0^{+\infty} \frac{\partial\left(\chi^1_h(\cdot, t)\, \mathrm{e}^{-i\omega t}\right)}{\partial t} \, \mathrm{d}t\cdot\boldsymbol{\nu}]\!] = \\
i\omega[\![\kappa(\nabla_y\chi^0_h - \boldsymbol{e}_h)\cdot\boldsymbol{\nu}]\!] &+ [\![\big(\kappa\nabla_y\chi^1_h(\cdot, 0) + \sigma(\nabla_y\chi^0_h - \boldsymbol{e}_h)\big)\cdot\boldsymbol{\nu}]\!] + \int_0^{+\infty} [\![\mathcal{D}\nabla_y\chi^1_h(\cdot, t)\cdot\boldsymbol{\nu}]\!]\, \mathrm{e}^{-i\omega t} \, \mathrm{d}t = 0\,,
\end{aligned}$$

where we used (4-30), (4-33), (4-24) and (5-74). Finally,

$$
\begin{aligned}
&((i\omega\kappa+\sigma)(\nabla_y\theta_h^\omega-\boldsymbol{e}_h)\cdot\boldsymbol{\nu})^{(2)}= \\
&\quad\big((i\omega\kappa+\sigma)(\nabla_y\chi_h^0-\boldsymbol{e}_h)\cdot\boldsymbol{\nu}\big)^{(2)}+\int_0^{+\infty}\big((i\omega\kappa+\sigma)(\nabla_y\chi^1(\cdot,t))\cdot\boldsymbol{\nu}\big)^{(2)}\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t= \\
&i\omega\big(\kappa(\nabla_y\chi_h^0-\boldsymbol{e}_h)\cdot\boldsymbol{\nu}\big)^{(2)}+\big(\sigma(\nabla_y\chi_h^0-\boldsymbol{e}_h)\cdot\boldsymbol{\nu}\big)^{(2)}+\int_0^{+\infty}\big(\sigma\nabla_y\chi^1(\cdot,t)\cdot\boldsymbol{\nu}\big)^{(2)}\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t \\
&\quad+\int_0^{+\infty}\big(\kappa\frac{\partial\nabla_y\chi^1(\cdot,t)}{\partial t}\cdot\boldsymbol{\nu}\big)^{(2)}\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t-\int_0^{+\infty}\big(\kappa\frac{\partial(\nabla_y\chi^1(\cdot,t)\,\mathrm{e}^{-i\omega t})}{\partial t}\cdot\boldsymbol{\nu}\big)^{(2)}\,\mathrm{d}t= \\
&i\omega\big(\kappa(\nabla_y\chi_h^0-\boldsymbol{e}_h)\cdot\boldsymbol{\nu}\big)^{(2)}+\big((\kappa\nabla_y\chi_h^1(\cdot,0)+\sigma(\nabla_y\chi_h^0-\boldsymbol{e}_h))\cdot\boldsymbol{\nu}\big)^{(2)}+\int_0^{+\infty}\big(\mathcal{D}\nabla_y\chi^1(\cdot,t)\cdot\boldsymbol{\nu}\big)^{(2)}\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t \\
&\quad=i\omega\alpha[\![\chi_h^0]\!]+\alpha[\![\chi_h^1(\cdot,0)]\!]+\beta[\![\chi_h^0]\!]+\int_0^{+\infty}\Big(\alpha\frac{\partial}{\partial t}[\![\chi_h^1(\cdot,t)]\!]+\beta[\![\chi_h^1(\cdot,t)]\!]\Big)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t \\
&\quad=(i\omega\alpha+\beta)[\![\chi_h^0]\!]+\int_0^{+\infty}i\omega\alpha[\![\chi_h^1(\cdot,t)\,\mathrm{e}^{-i\omega t}]\!]\,\mathrm{d}t+\int_0^{+\infty}\beta[\![\chi_h^1(\cdot,t)]\!]\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t \\
&\qquad+\alpha[\![\chi_h^1(\cdot,0)]\!]+\int_0^{+\infty}\alpha\frac{\partial}{\partial t}[\![\chi_h^1(\cdot,t)\,\mathrm{e}^{-i\omega t}]\!]\,\mathrm{d}t=(i\omega\alpha+\beta)[\![\theta_h^\omega]\!]
\end{aligned}
$$

where we used (4-31), (4-34), (4-25), (5-70) and (5-73). Thus $\theta_h^\omega=\chi_h^\omega$, since both of them satisfy Problem (5-42)–(5-44), which admits a unique solution in the class $\widehat{H}^1(Y)$ defined by

$$
\begin{aligned}
\widehat{H}^1(Y):=\{f\in L^2(\boldsymbol{R}^N)\colon f|_{E_i}\in H^1(E_i),\ i=1,2,\ f\text{ is }Y\text{-periodic} \\
\text{with vanishing integral average over }Y\}\,. \qquad (5\text{-}46)
\end{aligned}
$$

This implies the equivalence between equations (5-24) and (5-41). Indeed, replacing the right-hand side of (5-45) in (5-41), we obtain, in particular, that

$$
\begin{aligned}
&\int_\Gamma[\![(i\omega\kappa+\sigma)\chi^\omega]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma=\int_\Gamma[\![(i\omega\kappa+\sigma)\Big(\chi^0+\int_0^{+\infty}\chi^1(y,t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t\Big)]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma \\
&=i\omega\int_\Gamma[\![\kappa\chi^0]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma+\int_\Gamma[\![\sigma\chi^0]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma \\
&\quad+\int_\Gamma[\![\int_0^{+\infty}\sigma\chi^1(y,t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma-\int_\Gamma[\![\int_0^{+\infty}\kappa\chi^1(y,t)\,\frac{d\,\mathrm{e}^{-i\omega t}}{dt}\,\mathrm{d}t]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma \\
&=i\omega\int_\Gamma[\![\kappa\chi^0]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma+\int_\Gamma[\![\sigma\chi^0]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma+\int_\Gamma[\![\int_0^{+\infty}\sigma\chi^1(y,t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma \\
&\quad-\int_\Gamma[\![\int_0^{+\infty}\kappa\frac{\partial(\chi^1(y,t)\,\mathrm{e}^{-i\omega t})}{\partial t}\,\mathrm{d}t]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma+\int_\Gamma[\![\int_0^{+\infty}\kappa\chi_t^1(y,t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma \\
&=i\omega\int_\Gamma[\![\kappa\chi^0]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma+\int_\Gamma[\![\kappa\chi^1(y,0)+\sigma\chi^0]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma+\int_0^{+\infty}\Big(\int_\Gamma[\![\mathcal{D}\chi^1(y,t)]\!]\otimes\boldsymbol{\nu}\,\mathrm{d}\sigma\Big)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t\,,
\end{aligned}
$$

where we used (5-72). Hence, recalling (4-48)–(4-50), the assertion follows.

Moreover, following the proof of Proposition 15 in [6] and taking into account (5-24) and Proposition 5.8, we can state the following result.

Proposition 5.6. *A^ω is symmetric; its real part and its imaginary part are positive definite; $|A^\omega_{hj}|$, $h,j=1,\dots,N$, is uniformly bounded with respect to ω. Moreover, $\Re(A^\omega\zeta,\zeta)\geq\gamma|\zeta|^2$, for* *all $\zeta \in \boldsymbol{C}^N$, where $(\cdot,\cdot)$ is the scalar product in $\boldsymbol{C}^N$ and γ is a positive constant, independent of ω.*

In particular, as a consequence, it follows that the problem

$$-\operatorname{div}(A^\omega \nabla v) = 0\,, \qquad \text{in } \Omega; \tag{5-47}$$

$$v = \psi\,, \qquad \text{on } \partial\Omega, \tag{5-48}$$

is uniformly elliptic with respect to k and admits a unique solution $v \in H^1(\Omega)$. Moreover, the function $v_0 = \lim_{\varepsilon\to 0} v_\varepsilon$, which was proved to satisfy the problem above, coincides with v. Hence, $v_0 \in H^1(\Omega)$ and the following estimate holds:

$$\int_\Omega (|v_0|^2 + |\nabla v_0|^2)\,\mathrm{d}x \le \gamma \int_\Omega |\nabla\psi|^2\,\mathrm{d}x\,, \tag{5-49}$$

for a constant γ independent of k. We note that the uniqueness of v_0 also implies that actually the whole sequence $\{v_\varepsilon\}$ converges to v_0.

5B. **Homogenization limit of time-harmonic solutions: the case $k = 0$.** Here we prove Theorem 5.3 in the case $k = 0$. Let us distinguish the cases: $\beta = 0$ and $\beta \neq 0$.
In the first case, we have to study problem (5-17)–(5-21), where the third equation corresponds now to a homogeneous conditions. This problem is exactly the same treated in [6], where Theorem 5.3 was already proved.
On the contrary, in the case $\beta \neq 0$, we have to study problem (5-17)–(5-20). This problem is the scalar version of the one considered in [21]. There, the authors proved that $v_{\varepsilon 0} \to v_{00}$ strongly in $L^2(\Omega)$, where v_{00} satisfies (5-22)–(5-23), with

$$A^\omega = \overline{\sigma} I + \int_\Gamma [\![\sigma\chi^\omega]\!] \otimes \boldsymbol{\nu}\,\mathrm{d}\sigma \tag{5-50}$$

and χ^ω satisfies the cell problem

$$-\sigma\,\Delta_y\,\chi^\omega_h = 0\,, \qquad \text{in } E_1 \cup E_2; \tag{5-51}$$

$$[\![\sigma(\nabla_y\chi^\omega_h - \boldsymbol{e}_h)\cdot\boldsymbol{\nu}]\!] = 0\,, \qquad \text{on } \Gamma; \tag{5-52}$$

$$\beta[\![\chi^\omega_h]\!] = (\sigma(\nabla_y\chi^\omega_h - \boldsymbol{e}_h)\cdot\boldsymbol{\nu})^{(2)}\,, \qquad \text{on } \Gamma. \tag{5-53}$$

In order to prove (5-24), it is enough to set

$$\theta^\omega = \chi^0 + \int_0^{+\infty} \chi^1(\cdot,t)\,\mathrm{d}t\,,$$

and show that $\theta^\omega = \chi^\omega$. This can be done taking into account (4-23)–(4-50), Remark 5.9 and reasoning as in Section 5A. Hence, Theorem 5.3 is achieved for any choice of $\beta \ge 0$. Moreover, reasoning as in Section 5A and recalling the lower semicontinuity of the norm, we obtain

$$\begin{aligned} \|v_{\varepsilon 0}\|^2_{L^2(\Omega)} &\le \gamma \int_\Omega |\nabla\psi|^2\,\mathrm{d}x \qquad \text{and} \qquad \|\nabla v_{\varepsilon 0}\|^2_{L^2(\Omega)} \le \gamma \int_\Omega |\nabla\psi|^2\,\mathrm{d}x\,, \\ \|v_{00}\|^2_{L^2(\Omega)} &\le \gamma \int_\Omega |\nabla\psi|^2\,\mathrm{d}x \qquad \text{and} \qquad \|\nabla v_{00}\|^2_{L^2(\Omega)} \le \gamma \int_\Omega |\nabla\psi|^2\,\mathrm{d}x\,, \end{aligned} \tag{5-54}$$

where the constant γ does not depend on ε (see also [6] and [21]).

5C. **Time-periodic solutions.** In this Section we prove Theorem 5.4.

Firstly, we prove Theorem 5.4, Part i). In order to show the convergence in $H^1_\#(\boldsymbol{R}; L^2(\Omega))$ of the series on the right-hand side of equation (5-12), we use the Parseval identity and equations (2-17), (2-18), (5-11), (5-27), (5-29), (5-54), and we get

$$\|u_\varepsilon^\#\|_{H^1_\#(\boldsymbol{R};L^2(\Omega))} = \int_0^{T^\#} \int_\Omega \left[\left| \sum_{k=-\infty}^{+\infty} v_{\varepsilon k}(x)\, \mathrm{e}^{i\omega_k t} \right|^2 + \left| \sum_{k=-\infty}^{+\infty} i\omega_k v_{\varepsilon k}(x)\, \mathrm{e}^{i\omega_k t} \right|^2 \right] \mathrm{d}x\, \mathrm{d}t$$

$$= T^\# \int_\Omega \sum_{k=-\infty}^{+\infty} (1+\omega_k^2)|v_{\varepsilon k}(x)|^2\, \mathrm{d}x \le \gamma \sum_{k=-\infty}^{+\infty} (1+\omega_k^2)|c_k|^2 < +\infty\,.$$

The convergence in $H^1_\#(\boldsymbol{R}; H^1(\Omega_i^\varepsilon))$, $i = 1, 2$ can be shown analogously, using (5-28) instead of (5-29). Now, exactly as in [6], it is easy to prove that the function $u_\varepsilon^\#(x,t)$ defined in (5-12) solves Problem (5-3)–(5-9), using the weak formulation of this problem, and the linearity and continuity of the trace operator in the space $H^1_\#(\boldsymbol{R}; H^1(\Omega_2^\varepsilon))$.

Concerning the proof of Theorem 5.4, Part ii), note that the strong convergence in $H^1_\#(\boldsymbol{R}; H^1(\Omega))$ of the series on the right-hand side of (5-25) can be obtained exactly as for $u_\varepsilon^\#$, using Parseval identity, equations (5-27), (5-49), (5-54) and assumptions (2-17), (2-18) and (5-11).

In order to show that $\{u_\varepsilon^\#\}$ strongly converges in $H^1_\#(\boldsymbol{R}; L^2(\Omega))$ as $\varepsilon \to 0$ to $u_0^\#$, we compute, for $k_0 \in \boldsymbol{N}$ fixed,

$$\int_0^{T^\#} \int_\Omega \left[|u_\varepsilon^\#(x,t) - u_0^\#(x,t)|^2 + |u_{\varepsilon t}^\#(x,t) - u_{0t}^\#(x,t)|^2 \right] \mathrm{d}x\, \mathrm{d}t$$

$$= T^\# \int_\Omega \sum_{k=-\infty}^{+\infty} (1+\omega_k^2)|v_{\varepsilon k}(x) - v_{0k}(x)|^2\, \mathrm{d}x$$

$$= T^\# \sum_{|k|\le k_0} (1+\omega_k^2)\|v_{\varepsilon k} - v_{0k}\|^2_{L^2(\Omega)} + T^\# \sum_{|k|>k_0} (1+\omega_k^2)\|v_{\varepsilon k} - v_{0k}\|^2_{L^2(\Omega)} =: I_1 + I_2\,,$$

where we used the monotone convergence theorem. Using equations (5-29), (5-49), (5-54) and (5-27) we compute

$$|I_2| \le \gamma \sum_{|k|>k_0} (1+\omega_k^2)(\|v_{\varepsilon k}\|^2_{L^2(\Omega)} + \|v_{0k}\|^2_{L^2(\Omega)}) \le \gamma \sum_{|k|>k_0} (1+\omega_k^2)|c_k|^2\,.$$

By hypothesis (2-17), the right-hand term of the above inequality can be made arbitrarily small by choosing k_0 sufficiently large. For such fixed k_0, I_1 can be made arbitrarily small letting $\varepsilon \to 0$, by virtue of the strong L^2 convergence of $v_{\varepsilon k}$ to v_{0k} as $\varepsilon \to 0$, and the assertion follows.

Finally, Theorem 5.4, Part iii) easily follows from equations (5-22), (5-23), (5-11) and the $H^1_\#(\boldsymbol{R}; H^1(\Omega))$-convergence of the series (5-25), having in mind the weak formulation of equation (2-20) and taking into account (5-49), (5-54) and Proposition 5.8.

5D. **Stability result.** In this Section we prove Theorems 5.2 and 2.3. Let u_ε and $u_\varepsilon^\#$ be the solutions of Problem (2-11)–(2-16) and Problem (5-3)–(5-9), respectively. We set

$$w_\varepsilon = u_\varepsilon - u_\varepsilon^\#\,. \tag{5-55}$$

Since w_ε satisfies Problem (2-11)–(2-16) with homogeneous Dirichlet boundary data on $\partial\Omega \times (0,+\infty)$, i.e. $\Psi \equiv 0$, and with $\boldsymbol{g}_\varepsilon(x)$ replaced by $\boldsymbol{g}_\varepsilon(x) - \nabla u^{\#}_\varepsilon(x,0)$ and s_ε replaced by $s_\varepsilon - [\![u^{\#}_\varepsilon(\cdot,0)]\!]$ (which, for $\beta = 0$, has null mean average over each connected component of Γ^ε), the assertions of Theorems 5.2 and 2.3 respectively follow from Theorem 5.1 and Theorem 2.2, after proving that (H2) with $\boldsymbol{g}_\varepsilon$ and s_ε replaced by $\nabla u^{\#}_\varepsilon(\cdot,0)$ and $[\![u^{\#}_\varepsilon(\cdot,0)]\!]$, respectively.

To this end, we firstly observe that the classical trace inequality implies that

$$\int_\Omega |\nabla u^{\#}_\varepsilon(x,0)|^2 \,\mathrm{d}x \le \gamma \int_0^{T^\#} \int_\Omega \left(|\nabla u^{\#}_\varepsilon|^2 + |\nabla u^{\#}_{\varepsilon t}|^2 \right) \mathrm{d}x\,\mathrm{d}t\,, \tag{5-56}$$

$$\frac{1}{\varepsilon}\int_{\Gamma^\varepsilon} [\![u^{\#}_\varepsilon(x,0)]\!]^2 \,\mathrm{d}\sigma \le \frac{\gamma}{\varepsilon} \int_0^{T^\#} \int_{\Gamma^\varepsilon} \left(|[\![u^{\#}_\varepsilon]\!]|^2 + |[\![u^{\#}_{\varepsilon t}]\!]|^2 \right) \mathrm{d}\sigma\,\mathrm{d}t\,. \tag{5-57}$$

Then we estimate

$$\int_0^{T^\#} \int_\Omega \left(|\nabla u^{\#}_\varepsilon|^2 + |\nabla u^{\#}_{\varepsilon t}|^2 \right) \mathrm{d}x\,\mathrm{d}t = \sum_{k=-\infty}^{+\infty} \int_\Omega |\nabla v_{\varepsilon k}|^2 (1+\omega_k^2)\,\mathrm{d}x \le \gamma \sum_{k=-\infty}^{+\infty} |c_k|^2 (1+\omega_k^2)\,, \tag{5-58}$$

and

$$\frac{\gamma}{\varepsilon}\int_0^{T^\#} \int_{\Gamma^\varepsilon} \left(|[\![u^{\#}_\varepsilon]\!]|^2 + |[\![u^{\#}_{\varepsilon t}]\!]|^2 \right) \mathrm{d}\sigma\,\mathrm{d}t = \frac{\gamma}{\varepsilon} \sum_{k=-\infty}^{+\infty} \int_{\Gamma^\varepsilon} |[\![v_{\varepsilon k}]\!]|^2 (1+\omega_k^2)\,\mathrm{d}\sigma \le \gamma \sum_{k=-\infty}^{+\infty} |c_k|^2 (1+\omega_k^2)\,, \tag{5-59}$$

using equation (5-12), the Parseval identity, (5-27), (5-28), (5-54) and (5-13)–(5-20) when $\beta \neq 0$ (respectively, [6, inequality (5.10)] for $\beta = 0$).

The assertion follows since the right-hand terms of (5-58) and (5-59) are estimated by a constant independent of ε, by (5-11) and (2-17).

5E. **Decay estimates.** This section is devoted to prove the decay estimates satisfied by the cell functions χ^1_h, used above.

Lemma 5.7. *Assume that* (2-8) *holds. Let* $\boldsymbol{g} \in L^2(Y)$ *and* $s \in H^{1/2}(\Gamma)$ *be assigned, such that* $s = [\![z]\!]$ *and, for* $i = 1, 2$, $\boldsymbol{g}|_{E_i} = \nabla z|_{E_i}$, *for some scalar periodic function* $z \in H^1(E_i)$. *Moreover, if* $\beta = 0$, *assume also that* $\int_\Gamma s\,\mathrm{d}\sigma = 0$. *Then the function* $\mathcal{T}(s)(y,t)$ *defined in equation* (4-28) *satisfies the following estimates, for some constants* $\gamma, \lambda > 0$*:*

$$\|[\![\mathcal{T}(s)(\cdot,t)]\!]\|_{L^2(\Gamma)} \le \gamma\,\mathrm{e}^{-\lambda t}\,; \tag{5-60}$$

$$\|\nabla\mathcal{T}(s)(\cdot,t)\|_{L^2(Y)} \le \gamma\,\mathrm{e}^{-\lambda t}\,. \tag{5-61}$$

Proof. By (4-23)–(4-27), we have

$$\int_Y \frac{\kappa}{2} |\nabla\mathcal{T}(s)(y,t)|^2 \,\mathrm{d}y + \int_0^t\int_\Omega \sigma |\nabla\mathcal{T}(s)(y,\tau)|^2 \,\mathrm{d}y\,\mathrm{d}\tau + \frac{\alpha}{2}\int_\Gamma [\![\mathcal{T}(s)(y,t)]\!]^2 \,\mathrm{d}\sigma$$
$$+ \beta \int_0^t\int_\Gamma [\![\mathcal{T}(s)(y,\tau)]\!]^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau = \int_Y \frac{\kappa}{2} |\boldsymbol{g}(y)|^2 \,\mathrm{d}y + \frac{\alpha}{2}\int_\Gamma [\![s(y)]\!]^2 \,\mathrm{d}\sigma \le \gamma\,. \tag{5-62}$$

In [7] Section 4, it was proved that this energy estimate, jointly with assumption (2-8) and the differential version of Gronwall's Lemma, implies (5-60) and (5-61). □

Proposition 5.8. *The function* χ^1 *satisfies the estimates*

$$\|[\![\chi^1_h(\cdot,t)]\!]\|_{L^2(\Gamma)} \le \gamma\,\mathrm{e}^{-\lambda t}\,, \qquad h,j=1,\dots,N\,; \tag{5-63}$$

$$\|\nabla\chi^1_h(\cdot,t)\|_{L^2(Y)} \le \gamma\,\mathrm{e}^{-\lambda t}\,, \qquad h,j=1,\dots,N\,; \tag{5-64}$$

$$\left|(\nabla\chi^1_h(\cdot,t)\cdot\boldsymbol{\nu})^{(2)}\right| \le \gamma\,\mathrm{e}^{-\lambda t}\,, \quad h,j=1,\dots,N\,, \quad \text{on } \Gamma, \text{ in the sense of distributions;} \tag{5-65}$$

$$\left|[\![\kappa\nabla\chi^1_h(\cdot,t)\cdot\boldsymbol{\nu}]\!]\right| \le \gamma\,\mathrm{e}^{-\lambda t}\,, \quad h,j=1,\dots,N\,, \quad \text{on } \Gamma, \text{ in the sense of distributions.} \tag{5-66}$$

The matrix $\boldsymbol{B}(t)$ *satisfies the estimate*

$$|\boldsymbol{B}_{hj}(t)| \le \gamma\,\mathrm{e}^{-\lambda t}\,, \quad h,j=1,\dots,N\,, \tag{5-67}$$

where γ *and* λ *are positive constants.*

Proof. Estimates (5-63), (5-64) on χ^1 follow from (4-35) and Lemma 5.7 applied to $\boldsymbol{g}=\nabla\chi^1_h(\cdot,0)$ and $s=[\![\chi^1_h(\cdot,0)]\!]$. In order to prove (5-65) and (5-66), we remark that equation (4-23) implies $\Delta_y\chi^1_h(\cdot,t)^{(2)}=\Delta_y\chi^1_h(\cdot,0)^{(2)}\,\mathrm{e}^{-\sigma_2 t/\kappa_2}$, in the sense of distribution in Y; hence, it follows

$$\begin{aligned}
&\left|\int_\Gamma \big(\kappa\nabla_y\chi^1_h(y,t)\cdot\boldsymbol{\nu}\big)^{(2)}\phi(y)\,\mathrm{d}\sigma\right| \\
&\quad\le \left|\int_{E_2}\big(\kappa\Delta_y\chi^1_h(y,t)\big)^{(2)}\phi(y)\,\mathrm{d}y\right| + \left|\int_{E_2}\big(\kappa\nabla_y\chi^1_h(y,t)\big)^{(2)}\nabla\phi(y)\,\mathrm{d}y\right| \\
&\quad\le \gamma\left(\left|\int_{E_2}\big(\Delta_y\chi^1_h(y,0)\big)^{(2)}\phi(y)\,\mathrm{d}y\right|\mathrm{e}^{-\sigma_2 t/\kappa_2}\right. \\
&\qquad\left. + \left(\int_{E_2}|\nabla_y\chi^1_h(y,t)|^2\,\mathrm{d}y\right)^{1/2}\left(\int_{E_2}|\nabla\phi(y)|^2\,\mathrm{d}y\right)^{1/2}\right)
\end{aligned} \tag{5-68}$$

for every $\phi\in\boldsymbol{C}^\infty_\#(Y)$. An analogous estimate holds also for $\int_\Gamma\big(\kappa\nabla_y\chi^1_h(y,t)\cdot\boldsymbol{\nu}\big)^{(1)}\phi(y)\,\mathrm{d}\sigma$. Now, recalling that, by [1] Section 10, $\chi^0_h\in\boldsymbol{C}^\infty_\#(\boldsymbol{R}^N)$ so that also $\chi^1_h(\cdot,0)\in\boldsymbol{C}^\infty_\#(\boldsymbol{R}^N)$, in view of (5-64), estimates (5-65) and (5-66) follow.

It remains to prove equation (5-67). To this purpose, we note that multiplying (4-23), written for $\chi^1_h(\cdot,0)$, by $\frac{\partial\chi^1_h}{\partial t}(\cdot,0)=\chi^1_{ht}(\cdot,0)$, integrating by parts, using (4-24), (4-25) and finally applying Young's inequality, we obtain

$$\int_Y|\nabla_y\chi^1_{ht}(y,0)|^2\,\mathrm{d}y+\int_\Gamma|[\![\chi^1_{ht}(y,0)]\!]|^2\,\mathrm{d}\sigma\le\gamma\left(\int_Y|\nabla_y\chi^1_h(y,0)|^2\,\mathrm{d}y+\int_\Gamma|[\![\chi^1_h(y,0)]\!]|^2\,\mathrm{d}\sigma\right)\le\gamma\,, \tag{5-69}$$

for $h=1,\dots,N$, where the last inequality follows from (4-29)–(4-31) and (4-32)–(4-34). Hence, by Lemma 5.7 it follows that (5-63) and (5-64) hold also with χ^1_h replaced by χ^1_{ht}. Now, reasoning as in Remark 5.9 below, we obtain

$$\|[\![\sigma\chi^1_h(\cdot,t)]\!]\|_{L^2(\Gamma)}\le\gamma\,\mathrm{e}^{-\lambda t}\,,\qquad \|[\![\kappa\chi^1_{ht}(\cdot,t)]\!]\|_{L^2(\Gamma)}\le\gamma\,\mathrm{e}^{-\lambda t}\,,\quad h,j=1,\dots,N\,,$$

so that, recalling (4-50) the assertion follows. □

Remark 5.9. *Easy calculations show that* (5-63) *implies*

$$\int_0^{+\infty}\frac{\partial}{\partial t}[\![\chi^1_h(\cdot,t)\,\mathrm{e}^{i\omega t}]\!]\,\mathrm{d}t=-[\![\chi^1(\cdot,0)]\!] \qquad \text{on } \Gamma, \text{ in the sense of distributions.} \tag{5-70}$$

As a consequence of the standard trace inequality, jointly with the Poincarè's inequality ([4], [18]) and estimates (5-63)–(5-64), we obtain

$$\|\big(\chi_h^1(\cdot,t)\big)^{(2)}\|_{L^2(\Gamma)} \le \gamma\,\mathrm{e}^{-\lambda t}\,, \quad h,j=1,\dots,N\,, \tag{5-71}$$

so that

$$\|[\![\kappa\chi_h^1(\cdot,t)]\!]\|_{L^2(\Gamma)} \le \gamma\,\mathrm{e}^{-\lambda t}\,, \quad h,j=1,\dots,N\,. \tag{5-72}$$

By (5-65)*, it follows*

$$\int_0^{+\infty} \Big(\kappa\frac{\partial\left(\nabla_y\chi^1(\cdot,t)\,\mathrm{e}^{-i\omega t}\right)}{\partial t}\cdot\boldsymbol{\nu}\Big)^{(2)}\,\mathrm{d}t = -\big(\kappa\nabla\chi^1(\cdot,0)\cdot\boldsymbol{\nu}\big)^{(2)} \quad \text{on } \Gamma\text{, in the sense of distributions.} \tag{5-73}$$

By (5-66)*, it follows*

$$[\![\kappa\nabla_y\int_0^{+\infty}\frac{\partial\left(\chi^1(\cdot,t)\,\mathrm{e}^{-i\omega t}\right)}{\partial t}\,\mathrm{d}t\cdot\boldsymbol{\nu}]\!] = -[\![\kappa\nabla_y\chi^1(\cdot,0)\cdot\boldsymbol{\nu}]\!] \qquad \text{on } \Gamma\text{, in the sense of distributions.} \tag{5-74}$$

Finally, easy calculations show that (5-65) *and* (5-66) *imply*

$$\kappa\Delta_y\left(\int_0^{+\infty}\frac{\partial\left(\chi^1(\cdot,t)\,\mathrm{e}^{-i\omega t}\right)}{\partial t}\,\mathrm{d}t\right) = -\kappa\Delta_y\chi^1(\cdot,0) \qquad \text{in } E_1\cup E_2\text{, in the sense of distributions.} \tag{5-75}$$

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