Singular limits with vanishing viscosity for nonlocal conservation laws

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Abstract

We consider a class of nonlocal conservation laws with a second-order viscous regularization term which finds an application in modelling macroscopic traffic flow. The velocity function depends on a weighted average of the density ahead, where the averaging kernel is of exponential type. We show that, as the nonlocal reach and the viscosity parameter simultaneously tend to zero (under a suitable balance condition), the solution of the nonlocal problem converges to the entropy solution of the corresponding local conservation law. The key idea of our proof is to observe that the nonlocal term satisfies a third-order equation with diffusive and dispersive effects and to deduce a suitable energy estimate on the nonlocal term. The convergence result is then based on the compensated compactness theory.

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1. Introduction

1.1. A class of nonlocal conservation laws modeling traffic flow

Nonlocal balance and conservation laws have been studied in the last decade quite intensively as they model many phenomena in applications more realistically than corresponding local equations. For instance, in macroscopic traffic flow modelling, the typical "local" first-order dynamics is given by

$$\begin{cases} \partial_t \rho + \partial_x \big(V(\rho)\rho\big) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

However, at a given space-time point, such a model does not change its velocity based on the traffic ahead (nonlocal), but only based on the density at the given space-time point. This is one reason for introducing a nonlocal variant of this model, which can be written as

$$\begin{cases} \partial_t \rho + \partial_x \big(V(W[\rho])\rho\big) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

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with

$$W[\rho](t,x) := (\gamma * \rho)(t,x) := \tfrac{1}{\varepsilon}\int_x^\infty \gamma\big(\tfrac{s-x}{\varepsilon}\big)\rho(t,s)\,\mathrm{d}s, \tag{1.3}$$

for a nonlocal impact $\varepsilon > 0$ (see [23, 21, 1, 4]). The kernel γ then influences how the nonlocal term integrates the current density nonlocally. For traffic flow modelling it is indeed reasonable to assume that γ is monotonically decreasing, meaning that one takes traffic density further ahead not as much into account than traffic close to the considered position. Under suitable assumptions, this model class exhibits unique weak solutions (no entropy condition required) and also a maximum principle, as established in [23].

The natural question of whether one can recover the solution of the local equation when the nonlocal impact ε approaches zero – i.e. as the corresponding weight γ approaches a Dirac distribution – has been discussed for special cases via various approaches (see [5, 24, 16, 18]). Although a lot of numerical evidence therein suggests this to be correct in more generality, a theory of proving this convergence does not exist.

In this contribution, we approach this problem for a specific nonlocal kernel γ of exponential type and add an artificial viscosity. We show that when the nonlocal term together with the viscosity approaches zero, the sequence of solutions converges to the local entropy solution. Although this is not a result for the original proposed problem of nonlocal to local limit but for an approximated version, it might help to better understand what we can and cannot expect in the limit (without viscosity) to be true.

We assume throughout the paper the following natural conditions to be satisfied:

$$\rho_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad 0 \le \rho_0 \le 1; \tag{1.4}$$

$$V \in W^{1,\infty}(\mathbb{R}) \cap C^2(\mathbb{R}), \quad V \ge 0, \quad V' \le 0, \tag{1.5}$$

$$f : \rho \mapsto \rho V(\rho) \text{ is genuinely nonlinear, i.e. } |\{(\rho V(\rho))'' = 0\}| = 0. \tag{1.6}$$

Additionally, we choose as nonlocal kernel – as required in Eq. (1.3) – one of exponential type:

$$\gamma_\varepsilon(x) = \tfrac{1}{\varepsilon}\exp(-\tfrac{x}{\varepsilon})\chi_{[0,\infty)}(x), \;\; x \in \mathbb{R},$$

for $\varepsilon > 0$. Thanks to this structure, the nonlocal term $\gamma_\varepsilon * \rho$ satisfies the identity

$$\partial_x\big(\gamma_\varepsilon * \rho\big) \equiv \tfrac{1}{\varepsilon}\big(\gamma_\varepsilon * \rho - \rho\big),$$

which can be used to derive some reformulations as presented in Remark 1.1. As mentioned above, we do not consider the nonlocal conservation law (1.2) but instead its viscosity approximation which reads for $\nu \in \mathbb{R}_{>0}$

$$\begin{cases} \partial_t \rho_{\varepsilon,\nu} + \partial_x\big(V(W_\varepsilon[\rho_{\varepsilon,\nu}])\rho_{\varepsilon,\nu}\big) = \nu\partial_{xx}^2\rho_{\varepsilon,\nu}, & (t,x) \in (0,\infty)\times\mathbb{R},\\ \rho_{\varepsilon,\nu}(0,x) = \rho_{0,\nu}(x), & x \in \mathbb{R},\\ W_\varepsilon[\rho_{\varepsilon,\nu}](t,x) := \frac{1}{\varepsilon}\int_x^\infty e^{-(s-x)/\varepsilon}\rho_{\varepsilon,\nu}(t,s)\,\mathrm{d}s, & (t,x) \in (0,\infty)\times\mathbb{R}. \end{cases} \tag{1.7}$$

We smooth initial datum in the following way:

$$\{\rho_{0,\nu}\}_{\nu>0} \subset C_{\mathrm{c}}^\infty(\mathbb{R}), \tag{1.8}$$

$$\rho_{0,\nu} \overset{\nu\to 0}{\longrightarrow} \rho_0 \text{ a.e. and in } L^p_{\mathrm{loc}}(\mathbb{R}), \;\; p \in [1,\infty), \tag{1.9}$$

$$0 \le \rho_{0,\nu} \le 1, \quad \nu > 0. \tag{1.10}$$

Remark 1.1 (Reformulations of (1.7))**.** *As pointed out before, assuming sufficient regularity the nonlocal conservation law with viscosity term (1.7) can also be reformulated into*

$$\begin{cases} \partial_t \rho_{\varepsilon,\nu} + \partial_x\big(V(W_{\varepsilon,\nu})\rho_{\varepsilon,\nu}\big) = \nu\partial_{xx}^2\rho_{\varepsilon,\nu}, & (t,x) \in (0,\infty)\times\mathbb{R},\\ -\varepsilon\partial_x W_{\varepsilon,\nu} + W_{\varepsilon,\nu} = \rho_{\varepsilon,\nu}, & (t,x) \in (0,\infty)\times\mathbb{R},\\ \rho_{\varepsilon,\nu}(0,x) = \rho_{0,\nu}(x), & x \in \mathbb{R},\\ W_{\varepsilon,\nu}(0,x) = \frac{1}{\varepsilon}\int_x^\infty e^{-(s-x)/\varepsilon}\rho_{0,\nu}(s)\,\mathrm{d}s, & x \in \mathbb{R}, \end{cases} \tag{1.11}$$

or as a higher-order equation with competing diffusion and dispersion effects,

$$\begin{cases} \partial_t W_{\varepsilon,\nu} + \partial_x (V(W_{\varepsilon,\nu}) W_{\varepsilon,\nu}) = \varepsilon \partial^2_{tx} W_{\varepsilon,\nu} \\ +\nu \partial^2_{xx} W_{\varepsilon,\nu} + \varepsilon \partial_x (V(W_{\varepsilon,\nu}) \partial_x W_{\varepsilon,\nu}) - \varepsilon\nu \partial^3_{xxx} W_{\varepsilon,\nu}, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ W(0,x) = \frac{1}{\varepsilon} \int_x^\infty e^{-(s-x)/\varepsilon} \rho_{0,\nu}(s) \, \mathrm{d}s, & x \in \mathbb{R}. \end{cases} \tag{1.12}$$

1.2. Singular limits in the literature

A "nonlocal-to-local convergence" result can be seen as "closing the gap" between local and nonlocal modelling with conservation laws and unifying both theories. It would also provide another way for defining the proper (entropy) solutions for local conservation laws as limits of weak solutions to nonlocal conservation laws, which usually do not require an entropy condition for uniqueness (see [19, 23, 25, 26]). Eventually, such a convergence result would also give additional insights into questions related to control theory (see [3]).

Positive result on the nonlocal-to-local convergence were obtained in [31], provided that the limit entropy solution is smooth and the convolution kernel is even; and in [24] for a large class of nonlocal conservation laws under the assumption of having monotone initial data. In [5], Bressan and Shen considered the inviscid case of the equation with exponential kernel considered in the present paper, i.e. (1.2). Provided that the initial datum is bounded away from zero and has bounded total variation (but without monotonicity assumptions), they proved that, as $\varepsilon \to 0^+$, the family $\{\rho_\varepsilon\}_{\varepsilon>0}$ converges (up to subsequences) to a weak solution of the corresponding local conservation law; they also show that the limit is the unique entropy solution under the additional assumption that V is an affine function.

The positive effect of viscosity in the nonlocal-to-local approximation process was previously studied in [18, 16] for more general compactly supported kernels (see also [6] in the case of more regular initial data and linear velocity). On the other hand, for the inviscid case, in [18], the authors also exhibit counterexamples showing that, under rather general assumptions, the convergence of the solutions does not hold. Under more restrictive assumptions that the initial datum has bounded total variation, is bounded away from zero and satisfies a one-sided Lipschitz condition, a positive result was obtained in [17]. In the same paper, the authors also showed that, if the initial data is not bounded away from zero, a total variation blow-up may occur, which is a key difficulty to prove a convergence result.

Considering viscous perturbations is relevant because some of the numerical tests showing convergence are performed by using a Lax-Friedrichs type scheme involving some kind of numerical viscosity (see [15] for an analysis of the effect of numerical viscosity in the study of the nonlocal-to-local limit). However, we also remark that the numerical simulations performed in [24, Section 7], which are based on the method of characteristics and do not introduce artificial viscosity, show that we should expect convergence of the nonlocal solution to the local solution.

Our approach differs from the previous contributions mentioned above. Indeed, the proof in [18] is based on a priori estimates obtained by extensively using energy estimates for the heat kernel and the Duhamel representation formula. On the other hand, we establish an energy estimate on the nonlocal term $W_{\varepsilon,\nu}(t,\cdot)$ by relying just on the structure of (1.12); we then apply Tatar's compensated compactness technique to deduce the L^p compactness of the family $\{\rho_{\varepsilon,\nu}\}_{\varepsilon,\nu>0}$. In our more particular setting (in which a maximum principle holds), we obtain convergence under much milder assumptions on the ratio ε/ν (i.e. we assume $\varepsilon/\nu \to 0$ instead of $\varepsilon \le e^{-C\nu^{-\beta}}$, with C, $\beta > 0$).

Our approach is somewhat inspired by the strategy used to study the singular limit problem for the Camassa-Holm equation (see [12, 9]), which contains nonlinear dispersive effects as well as fourth order dissipative effects, or of the Ostrovski-Hunter equation (see [10, 8]). In both cases, the approximating equations are higher-order ones which can be equivalently rewritten as parabolic-elliptic systems or (using the Helmholtz kernel) as a conservation law with a nonlocal perturbation, which is useful in establishing compactness estimates.

These papers on singular limits are, in turn, inspired by the pioneering results by Schonbek on the zero diffusion-dispersion limit for the Korteweg-de Vries-Burgers and Benjamin-Bona-Mahony equation (see [29]). Equation (1.12) differs substantially from the equations appearing in these papers for several reasons: a mixed derivative $\partial_{tx}^2 W_{\varepsilon,\nu}$ appears; the system in (1.2) is a PDE-ODE coupling instead of a parabolic-elliptic PDE system; and we do not need to rely on the L^p compensated compactness developed by Schonbek because, from the formulation (1.7) of our problem, we are able to deduce a L^∞ estimate, which allows us to apply the standard L^∞ compensated compactness theorem by Tartar (see [27, 30]).

1.3. Outline of the paper

The paper is organized as follows. In Section 2, we recall the notions of solutions for conservation laws and state our main convergence result.

In Section 3, we prove the well-posedness of (1.7) – which, in turn, implies the rigorous equivalence between (1.7) and (1.2) – and the a priori estimates required for the study of the singular limit. More specifically, we establish L^∞ bounds on $\rho_{\varepsilon,\nu}$ and $W_{\varepsilon,\nu}$ and an L^2 estimate on $W_{\varepsilon,\nu}(t,\cdot)$ which also involves the H^2 norm of $W_{\varepsilon,\nu}(t,\cdot)$.

In Section 4, we use the previous estimates to prove that the family $\{\rho_{\varepsilon,\nu}\}_{\varepsilon,\nu>0}$ is compact in L^p. To this end, we rely on Tartar's compensated compactness technique and show that the family $\{\partial_t \eta(\rho_{\varepsilon,\nu}) + \partial_x q(\rho_{\varepsilon,\nu})\}_{\varepsilon,\nu>0}$, for every convex entropy-entropy flux pair, is compact in $H^{-1}_{\text{loc}}((0,\infty)\times\mathbb{R})$. Finally, we check that the limit function ρ is an entropy solution of (1.1). We remark that in the compactness estimates and in the verification of the entropy condition, the assumption $\varepsilon = o(\nu)$ is crucial.

2. Main results

Let us recall the notion of an entropy solutions for the Cauchy problem (1.1).

Definition 2.1 (Entropy solutions). *A function $\rho : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ is an* entropy solution *of the Cauchy problem (1.1) if $\rho \in L^\infty((0,T)\times\mathbb{R})$ for every $T>0$ and, for every non-negative test function $\varphi \in C_c^\infty(\mathbb{R}^2)$, we have*

$$\int_0^\infty \int_{\mathbb{R}} \Big(\eta(\rho)\partial_t\varphi + q(V(\rho)\rho)\partial_x\varphi\Big)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \eta(u_0(x))\varphi(0,x)\,\mathrm{d}x \geq 0 \tag{2.1}$$

for every entropy η with entropy flux q, i.e. $\eta, q \in C^2(\mathbb{R})$, $\eta'' \geq 0$, $\eta'(V(\rho)\rho)' = q'(\rho)$ for all $\rho\in\mathbb{R}$.

For a modern presentation of the proof of the well-posedness of entropy solutions for scalar conservation laws, we refer the reader to the monographs [22, 14].

Our main theorem is the following convergence result.

Theorem 2.1 (Nonlocal-to-local limit). *Let $\{\rho_{\varepsilon,\nu}\}_{\varepsilon,\nu}$ be a family of classical solutions of the Cauchy problem (1.7). Then, for all*

$$(\varepsilon,\nu) \subset \mathbb{R}^2_{>0} \quad \text{such that} \quad (\varepsilon,\nu)\to(0,0) \ \wedge\ \tfrac{\varepsilon}{\nu}\to 0, \tag{2.2}$$

there exists $\rho \in L^\infty((0,\infty)\times\mathbb{R})$ such that

$$\rho_{\varepsilon,\nu} \to \rho \quad a.e.\ and\ in\ L^p_{loc}((0,\infty)\times\mathbb{R}),\ with\ p\in[1,\infty),$$

and ρ is the entropy solution of the Cauchy problem (1.1).

Remark 2.1 (Long-time behavior of periodic solutions). *We note that the singular limit problem considered in this paper is strictly related to the analysis of the long-time behavior of* periodic *solutions to the problem*

$$\begin{cases} \partial_t\rho + \partial_x(\rho V(W)) = 0, & (t,x)\in(0,\infty)\times\mathbb{R},\\ -\alpha\partial_x W + W = \rho, & (t,x)\in(0,\infty)\times\mathbb{R},\\ \rho(0,x) = \rho_0(x), & x\in\mathbb{R}, \end{cases}$$

where $\alpha>0$ is a fixed parameter and ρ_0 is 1-periodic. Indeed, following [7, 13], we introduce the functions

$$\rho_T = \rho(Tt, Tx), \quad W_T(t,x) = W(Tt,Tx), \quad T,t\geq 0,\ x\in\mathbb{R}.$$

The functions ρ_T and P_T are T-periodic and solve the rescaled system

$$\begin{cases} \partial_t \rho_T + \partial_x(\rho_T V(W_T)) = 0, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ -\alpha \dfrac{1}{T}\partial_x W_T + W_T = \rho_T, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ \rho_T(0,x) = \rho_0(Tx), & x \in \mathbb{R}. \end{cases}$$

Due to the periodicity assumption, the study of the asymptotic behavior of ρ is equivalent to the study of the limit of the family $\{\rho_T\}_{T>0}$ (see [7, Theorem 3.2]). Setting $\varepsilon = \alpha/T$, this reduces to the nonlocal-to-local singular limit problem studied in [5]. In particular, under the assumptions of [5, Theorem 3], we can show that there exists a subsequence $\{\rho_{T_k}\}_{k>0}$, $T_k \to \infty$ and a limit function $\bar\rho \in L^\infty((0,\infty)\times\mathbb{R})$ such that, as $k \to \infty$,

$$\rho_{T_k} \to \bar\rho \text{ a.e. and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty$$

and $\bar\rho$ is a weak (actually, classical) solution of

$$\begin{cases} \partial_t \bar\rho + \partial_x(V(\bar\rho)\bar\rho) = 0, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ \bar\rho(0,x) = \int_0^1 \rho_0(x)\,\mathrm{d}x, & x \in \mathbb{R}, \end{cases}$$

namely $\bar\rho(t,x) = \int_0^1 \rho_0(x)\,\mathrm{d}x$.

In other words, we have observed that the nonlocal conservation law and the corresponding local one (whose decay properties were established in [7]) have the same long-time behavior.

3. A priori estimates

We start by proving the well-posedness of classical solutions of (1.7), their non-negativity, and an upper bound in terms of the L^∞ norm of the initial data. This, in turn, implies an L^∞ estimate on $W_{\varepsilon,\nu}$.

Lemma 3.1 (Well-posedness and L^∞-estimate)**.** *For every $\varepsilon, \nu > 0$, there exists a unique non-negative smooth solution $\rho_{\varepsilon,\nu} \in C^\infty([0,\infty)\times\mathbb{R}) \cap W^2((0,\infty)\times\mathbb{R})$ of the Cauchy problem (1.7) such that*

$$0 \le \rho_{\varepsilon,\nu},\, W_{\varepsilon,\nu} \le 1.$$

Proof. Since $\rho_{0,\nu} \in W^2(\mathbb{R})$, the existence and uniqueness of smooth solutions of (1.7) can be proved arguing similarly to [18, Theorem 2.1] or [11]. We focus on showing the L^∞ bound on the solutions. To prove $\rho_{\varepsilon,\nu} \ge 0$, we consider the function

$$\eta(\xi) = -\xi\chi_{(-\infty,0]}(\xi), \qquad \xi \in \mathbb{R},$$

which satisfies

$$\eta'(\xi) = -\chi_{(-\infty,0]}(\xi), \qquad \eta''(\xi) = \delta_{\{\xi=0\}} \ge 0. \tag{3.1}$$

Multiplying Eq. (1.7) by $\eta'(\rho_{\varepsilon,\nu})$, integrating over $\mathbb{R}$, and using [2, Lemma 2] yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \eta(\rho_{\varepsilon,\nu})\,\mathrm{d}x &= \int_{\mathbb{R}} \partial_t\rho_{\varepsilon,\nu}\eta'(\rho_{\varepsilon,\nu})\,\mathrm{d}x \\ &= \nu\int_{\mathbb{R}} \partial^2_{xx}\rho_{\varepsilon,\nu}\eta'(\rho_{\varepsilon,\nu})\,\mathrm{d}x - \int_{\mathbb{R}} \partial_x(V(W_{\varepsilon,\nu})\rho_{\varepsilon,\nu}))\eta'(\rho_{\varepsilon,\nu})\,\mathrm{d}x \\ &= -\nu\int_{\mathbb{R}} \underbrace{(\partial_x\rho_{\varepsilon,\nu})^2\eta''(\rho_{\varepsilon,\nu})}_{\ge 0}\,\mathrm{d}x + \int_{\mathbb{R}} V(W_{\varepsilon,\nu})\partial_x\rho_{\varepsilon,\nu}\ \underbrace{\rho_{\varepsilon,\nu}\eta''(\rho_{\varepsilon,\nu})}_{=0\text{ (see Eq. (3.1))}}\,\mathrm{d}x \\ &\le 0. \end{aligned}$$

Integrating over $(0,t)$ and using Eq. (1.10) and Eq. (3.1), we have

$$0 \leq \int_{\mathbb{R}} \eta(\rho_{\varepsilon,\nu}(t,x))\,\mathrm{d}x \leq \int_{\mathbb{R}} \eta(\rho_{0,\nu}(x))\,\mathrm{d}x = 0.$$

Therefore, $\eta(\rho_{\varepsilon,\nu}) \equiv 0$ and, that is

$$\rho_{\varepsilon,\nu}(t,x) \geq 0. \tag{3.2}$$

To prove $\rho_{\varepsilon,\nu} \leq 1$, we follow the argument in [26, Corollary 5.9]. For $t \geq 0$, let

$$X_{max}(t) := \left\{ x \in \mathbb{R} : \rho_{\varepsilon,\nu}(t,x) = \|\rho_{\varepsilon,\nu}(t,\cdot)\|_{L^\infty(\mathbb{R})} \right\}.$$

For $x \in X_{max}(t)$ and a.e. $t > 0$, we have

$$\begin{aligned}
\partial_t \rho_{\varepsilon,\nu}(t,x) = &-\underbrace{\partial_x \rho_{\varepsilon,\nu}(t,x) V(W_{\varepsilon,\nu}(t,x))}_{=0} - \tfrac{\rho_{\varepsilon,\nu}(t,x)}{\varepsilon} V'(W_{\varepsilon,\nu}(t,x)) W_{\varepsilon,\nu}(t,x) \\
&+ \tfrac{\rho^2_{\varepsilon,\nu}(t,x)}{\varepsilon} V'(W_{\varepsilon,\nu}(t,x)) + \underbrace{\nu \partial^2_{xx} \rho_{\varepsilon,\nu}(t,x)}_{\leq 0} \\
\leq & \underbrace{\tfrac{\rho_{\varepsilon,\nu}(t,x)}{\varepsilon} V'(W_{\varepsilon,\nu}(t,x))(\rho_{\varepsilon,\nu}(t,x) - W_{\varepsilon,\nu}(t,x))}_{\leq 0},
\end{aligned}$$

where we have used Eq. (3.2), Eq. (1.5) and the fact that, for $x \in X_{max}(t)$,

$$\begin{aligned}
&\partial_x \rho_{\varepsilon,\nu}(t,x) = 0, \\
W_{\varepsilon,\nu}(t,x) - \rho_{\varepsilon,\nu}(t,x) &= \tfrac{1}{\varepsilon} \int_x^\infty e^{-(s-x)/\varepsilon} \big(\rho_{\varepsilon,\nu}(t,s) - \rho_{\varepsilon,\nu}(t,x)\big)\,\mathrm{d}s \leq 0.
\end{aligned}$$

We have thus shown that, for all maximal points $x \in X_{max}(t)$,

$$\partial_t \rho_{\varepsilon,\nu}(t,x) \leq 0,$$

which implies

$$\|\rho_{\varepsilon,\nu}(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|\rho_{\varepsilon,\nu}(0,\cdot)\|_{L^\infty(\mathbb{R})} \leq 1.$$

The L^∞-estimate for $W_{\varepsilon,\nu}$ then follows from the one for $\rho_{\varepsilon,\nu}$ thanks to Eq. (1.3). □

From the regularity of $\rho_{\varepsilon,\nu}$, we deduce that problems (1.7), (1.2), and (1.12) are indeed equivalent and $W_{\varepsilon,\nu}$ is also smooth.

Relying on (1.12), we obtain an energy estimate for $W_{\varepsilon,\nu}(t,\cdot)$. In the proof, a key role is played by the assumption (2.2) on the ratio ε/ν.

Lemma 3.2 (Energy estimate)**.** *If $W_{\varepsilon,\nu}$ is the solution of (1.12), then the following estimate holds:*

$$\begin{aligned}
&\|W_{\varepsilon,\nu}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon^2 \|\partial_x W_{\varepsilon,\nu}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad + \nu \int_0^t \|\partial_x W_{\varepsilon,\nu}(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + \varepsilon^2 \nu \int_0^t \left\|\partial^2_{xx} W_{\varepsilon,\nu}\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \leq C,
\end{aligned}$$

for some constant $C > 0$ independent from ε and ν and for every $t \geq 0$. In particular,

$$\begin{aligned}
&\{W_{\varepsilon,\nu}\}_{\varepsilon,\nu>0},\ \{\varepsilon \partial_x W_{\varepsilon,\nu}\}_{\varepsilon,\nu>0} \text{ are bounded in } L^\infty(0,\infty; L^2(\mathbb{R})), \\
&\{\sqrt{\nu} \partial_x W_{\varepsilon,\nu}\}_{\varepsilon,\nu>0},\ \{\varepsilon \sqrt{\nu} \partial^2_{xx} W_{\varepsilon,\nu}\}_{\varepsilon,\nu>0} \text{ are bounded in } L^2((0,\infty) \times \mathbb{R}).
\end{aligned}$$

Proof. We differentiate the L^2-norm $\frac{1}{2}\|W_{\varepsilon,\nu}(t,\cdot)\|_{L^2(\mathbb{R})}$ with respect to time and, using Eq. (1.12), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{W_{\varepsilon,\nu}^2}{2}\,\mathrm{d}x &= \int_{\mathbb{R}} W_{\varepsilon,\nu}\partial_t W_{\varepsilon,\nu}\,\mathrm{d}x \\
&= -\int_{\mathbb{R}} \partial_x(V(W_{\varepsilon,\nu})W_{\varepsilon,\nu})W_{\varepsilon,\nu}\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}} W_{\varepsilon,\nu}\partial^2_{tx}W_{\varepsilon,\nu}\,\mathrm{d}x \\
&\quad + \nu\int_{\mathbb{R}} W_{\varepsilon,\nu}\partial^2_{xx}W_{\varepsilon,\nu}\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}} W_{\varepsilon,\nu}\partial_x(V(W_{\varepsilon,\nu})\partial_x W_{\varepsilon,\nu})\,\mathrm{d}x - \varepsilon\nu\int_{\mathbb{R}} W_{\varepsilon,\nu}\partial^3_{xxx}W_{\varepsilon,\nu}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} V(W_{\varepsilon,\nu})W_{\varepsilon,\nu}\partial_x W_{\varepsilon,\nu}\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}} W_{\varepsilon,\nu}\partial^2_{tx}W_{\varepsilon,\nu}\,\mathrm{d}x \\
&\quad + \nu\int_{\mathbb{R}} W_{\varepsilon,\nu}\partial^2_{xx}W_{\varepsilon,\nu}\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}} W_{\varepsilon,\nu}\partial_x(V(W_{\varepsilon,\nu})\partial_x W_{\varepsilon,\nu})\,\mathrm{d}x + \varepsilon\nu\int_{\mathbb{R}} \partial_x W_{\varepsilon,\nu}\partial^2_{xx}W_{\varepsilon,\nu}\,\mathrm{d}x \\
&= \underbrace{\int_{\mathbb{R}} \partial_x\left(\int_0^{W_{\varepsilon,\nu}} V(\xi)\xi\,\mathrm{d}\xi\right)\mathrm{d}x}_{=0} - \varepsilon\int_{\mathbb{R}} \partial_t W_{\varepsilon,\nu}\partial_x W_{\varepsilon,\nu}\,\mathrm{d}x \\
&\quad - \nu\int_{\mathbb{R}} (\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x - \varepsilon\int_{\mathbb{R}} V(W_{\varepsilon,\nu})(\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x + \varepsilon\nu\underbrace{\int_{\mathbb{R}} \partial_x\left(\tfrac{(\partial_x W_{\varepsilon,\nu})^2}{2}\right)\mathrm{d}x}_{=0} \\
&= -\varepsilon\int_{\mathbb{R}} \partial_t W_{\varepsilon,\nu}\partial_x W_{\varepsilon,\nu}\,\mathrm{d}x - \nu\int_{\mathbb{R}} (\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x - \varepsilon\int_{\mathbb{R}} V(W_{\varepsilon,\nu})(\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x.
\end{aligned}$$

Using again Eq. (1.12),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{W_{\varepsilon,\nu}^2}{2}\,\mathrm{d}x + \nu\int_{\mathbb{R}} (\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x =&\varepsilon\int_{\mathbb{R}} \partial_x(V(W_{\varepsilon,\nu})W_{\varepsilon,\nu})\partial_x W_{\varepsilon,\nu}\,\mathrm{d}x - \varepsilon^2\int_{\mathbb{R}} \partial_x W_{\varepsilon,\nu}\partial^2_{tx}W_{\varepsilon,\nu}\,\mathrm{d}x \\
&- \varepsilon\nu\underbrace{\int_{\mathbb{R}} \partial_x W_{\varepsilon,\nu}\partial^2_{xx}W_{\varepsilon,\nu}\,\mathrm{d}x}_{=0} - \varepsilon^2\int_{\mathbb{R}} \partial_x W_{\varepsilon,\nu}\partial_x(V(W_{\varepsilon,\nu})\partial_x W_{\varepsilon,\nu})\,\mathrm{d}x \\
&+ \varepsilon^2\nu\int_{\mathbb{R}} \partial_x W_{\varepsilon,\nu}\partial^3_{xxx}W_{\varepsilon,\nu}\,\mathrm{d}x - \varepsilon\int_{\mathbb{R}} V(W_{\varepsilon,\nu})(\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x \\
=&\varepsilon\int_{\mathbb{R}} V(W_{\varepsilon,\nu})(\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}} V'(W_{\varepsilon,\nu})W_{\varepsilon,\nu}(\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x \\
&- \varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{(\partial_x W_{\varepsilon,\nu})^2}{2}\,\mathrm{d}x + \varepsilon^2\int_{\mathbb{R}} V(W_{\varepsilon,\nu})\partial_x W_{\varepsilon,\nu}\partial^2_{xx}W_{\varepsilon,\nu}\,\mathrm{d}x \\
&- \varepsilon^2\nu\int_{\mathbb{R}} (\partial^2_{xx}W_{\varepsilon,\nu})^2\,\mathrm{d}x - \varepsilon\int_{\mathbb{R}} V(W_{\varepsilon,\nu})(\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x.
\end{aligned}$$

Using the L^∞-bound established in Lemma 3.1 and Young's inequality (see [20, Appendix B]), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{W_{\varepsilon,\nu}^2 + \varepsilon^2(\partial_x W_{\varepsilon,\nu})^2}{2}\,\mathrm{d}x + \nu\int_{\mathbb{R}} (\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x + \varepsilon^2\nu\int_{\mathbb{R}} (\partial^2_{xx}W_{\varepsilon,\nu})^2\,\mathrm{d}x \\
=&\varepsilon\int_{\mathbb{R}} V'(W_{\varepsilon,\nu})W_{\varepsilon,\nu}(\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x + \varepsilon^2\int_{\mathbb{R}} V(W_{\varepsilon,\nu})\partial_x W_{\varepsilon,\nu}\partial^2_{xx}W_{\varepsilon,\nu}\,\mathrm{d}x \\
\leq&\varepsilon\int_{\mathbb{R}} V'(W_{\varepsilon,\nu})W_{\varepsilon,\nu}(\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x \\
&+ \frac{\varepsilon^2}{2\nu}\int_{\mathbb{R}} (V(W_{\varepsilon,\nu})\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x + \frac{\varepsilon^2\nu}{2}\int_{\mathbb{R}} (\partial^2_{xx}W_{\varepsilon,\nu})^2\,\mathrm{d}x \\
\leq&\varepsilon\left(\|V'\|_{L^\infty(0,1)} + \frac{\varepsilon}{\nu}\|V\|^2_{L^\infty(0,1)}\right)\int_{\mathbb{R}} (\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x + \frac{\varepsilon^2\nu}{2}\int_{\mathbb{R}} (\partial^2_{xx}W_{\varepsilon,\nu})^2\,\mathrm{d}x.
\end{aligned}$$

Thanks to Eq. (2.2), when ε and ν are small, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}}\frac{W_{\varepsilon,\nu}^2+\varepsilon^2(\partial_x W_{\varepsilon,\nu})^2}{2}\,\mathrm{d}x+\frac{\nu}{2}\int_{\mathbb{R}}(\partial_x W_{\varepsilon,\nu})^2\,\mathrm{d}x+\frac{\varepsilon^2\nu}{2}\int_{\mathbb{R}}(\partial_{xx}^2 W_{\varepsilon,\nu})^2\,\mathrm{d}x\leq 0.$$

Finally, due to Eq. (1.8), Eq. (1.9), and Eq. (1.10), we conclude the proof by integrating over $(0,t)$. □

4. Compensated compactness framework and proof of the convergence result

In this section, we use Tartar's compensated compactness method (see [27, 30]) to obtain strong convergence of a subsequence of solutions of (1.7) to the unique entropy solution of (1.1).

Lemma 4.1 (Tartar's compensated compactness)**.** *Let $f\in C^2(\mathbb{R})$ be a genuinely nonlinear function, i.e. $|\{f''=0\}|=0$, and $\{\rho_\delta\}_{\delta>0}$ be a measurable family of functions defined on $\mathbb{R}_+\times\mathbb{R}$ such that*

$$\|\rho_\delta\|_{L^\infty((0,T)\times\mathbb{R})}\leq M_T,\qquad T,\,\delta>0,$$

and the family

$$\{\partial_t\eta(\rho_\delta)+\partial_x q(\rho_\delta)\}_{\delta>0}$$

is compact in $H^{-1}_{loc}(\mathbb{R}_+\times\mathbb{R})$, for every convex $\eta\in C^2(\mathbb{R})$, where $q'=f'\eta'$. Then there exist a sequence $\{\delta_n\}_{n\in\mathbb{N}}\subset(0,\infty)$, $\delta_n\to 0$, and a map $\rho\in L^\infty((0,T)\times\mathbb{R})$, $T>0$, such that

$$\rho_{\delta_n}\longrightarrow\rho\quad \textit{a.e. and in } L^p_{loc}(\mathbb{R}_+\times\mathbb{R}),\ 1\leq p<\infty.$$

To check that the family $\{\rho_{\varepsilon,\nu}\}_{\varepsilon,\nu>0}$ satisfies the assumptions of Lemma 4.1, we rely on Murat's compact embedding (see [28]).

Lemma 4.2 (Murat's compact embedding)**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N\geq 2$. Suppose the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,p}(\Omega)$ for some $2<p\leq\infty$. Suppose also that*

$$\mathcal{L}_n=\mathcal{L}_{1,n}+\mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $L^1_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

Proof of Theorem 2.1. First, we observe the following:

$$\begin{aligned}\partial_t\rho_{\varepsilon,\nu}+\partial_x(V(\rho_{\varepsilon,\nu})\rho_{\varepsilon,\nu})&=\nu\partial_{xx}^2\rho_{\varepsilon,\nu}+\partial_x\big((V(\rho_{\varepsilon,\nu})-V(W_{\varepsilon,\nu}))\rho_{\varepsilon,\nu}\big)\\&=\nu\partial_{xx}^2\rho_{\varepsilon,\nu}+\partial_x\big(b(t,x)(\rho_{\varepsilon,\nu}-W_{\varepsilon,\nu})\rho_{\varepsilon,\nu}\big)\\&=\nu\partial_{xx}^2\rho_{\varepsilon,\nu}-\varepsilon\partial_x\big(b(t,x)\partial_x W_{\varepsilon,\nu}\rho_{\varepsilon,\nu}\big),\end{aligned}\tag{4.1}$$

where

$$b(t,x):=\int_0^1 V'(\theta\rho_{\varepsilon,\nu}(t,x)+(1-\theta)W_{\varepsilon,\nu}(t,x))\,\mathrm{d}\theta,\qquad (t,x)\in\mathbb{R}_{>0}\times\mathbb{R}.\tag{4.2}$$

Let $\eta,q:\mathbb{R}\to\mathbb{R}$ be a C^2 convex entropy-entropy flux pair for the conservation law (1.1), i.e. $\eta,q\in C^2(\mathbb{R})$, $\eta''\geq 0$, $\eta'f'=q'$, where $f:\rho\mapsto V(\rho)\rho$. Multiplying Eq. (4.1) by $\eta'(\rho)$ yields

$$\begin{aligned}\partial_t\eta(\rho_{\varepsilon,\nu})+\partial_x q(\rho_{\varepsilon,\nu})&=\nu\eta'(\rho_{\varepsilon,\nu})\partial_{xx}^2\rho_{\varepsilon,\nu}-\varepsilon\eta'(\rho_{\varepsilon,\nu})\partial_x(b\,\partial_x W_{\varepsilon,\nu}\rho_{\varepsilon,\nu})\\&=\nu\partial_{xx}^2\eta(\rho_{\varepsilon,\nu})-\nu\eta''(\rho_{\varepsilon,\nu})(\partial_x\rho_{\varepsilon,\nu})^2-\varepsilon\partial_x(\eta'(\rho_{\varepsilon,\nu})b\,\partial_x W_{\varepsilon,\nu}\rho_{\varepsilon,\nu})\\&\quad+\varepsilon\eta''(\rho_{\varepsilon,\nu})b\,\partial_x W_{\varepsilon,\nu}\rho_{\varepsilon,\nu}\partial_x\rho_{\varepsilon,\nu}.\end{aligned}$$

To apply Tartar compensated compactness, we show that the right-hand side is compact in $H^{-1}_{\text{loc}}((0,\infty)\times\mathbb{R})$. By Lemma 3.2 and Lemma 3.1, we have for $T>0$:

$$\begin{aligned}\|\nu\eta'(\rho_{\varepsilon,\nu})\partial_x\rho_{\varepsilon,\nu}\|_{L^2((0,T)\times\mathbb{R})} &= \|\nu\eta'(\rho_{\varepsilon,\nu})\partial_x W_{\varepsilon,\nu} - \varepsilon\nu\eta'(\rho_{\varepsilon,\nu})\partial^2_{xx}W_{\varepsilon,\nu}\|_{L^2((0,T)\times\mathbb{R})}\\ &\le \sqrt{\nu}\,\|\eta'(\rho_{\varepsilon,\nu})\|_{L^\infty((0,T)\times\mathbb{R})}\,\|\sqrt{\nu}\partial_x W_{\varepsilon,\nu} - \varepsilon\sqrt{\nu}\partial_x W_{\varepsilon,\nu}\|_{L^2((0,T)\times\mathbb{R})}\\ &\le \sqrt{\nu}\,c_T \to 0;\end{aligned}\tag{4.3}$$

Additionally, we obtain

$$\begin{aligned}\|\nu\eta''(\rho_{\varepsilon,\nu})(\partial_x\rho_{\varepsilon,\nu})^2\|_{L^1((0,T)\times\mathbb{R})} &= \|\nu\eta''(\rho_{\varepsilon,\nu})\Big(\partial_x W_{\varepsilon,\nu} - \varepsilon\partial^2_{xx}W_{\varepsilon,\nu}\Big)^2\|_{L^1((0,T)\times\mathbb{R})}\\ &= \|\nu\eta''(\rho_{\varepsilon,\nu})\Big(\nu(\partial_x W_{\varepsilon,\nu})^2 - 2\varepsilon\nu\partial_x W_{\varepsilon,\nu}\partial^2_{xx}W_{\varepsilon,\nu}\\ &\qquad + \nu\varepsilon^2(\partial^2_{xx}W_{\varepsilon,\nu})^2\Big)\|_{L^1((0,T)\times\mathbb{R})}\\ &\le c_T\end{aligned}\tag{4.4}$$

as well as

$$\begin{aligned}\|\varepsilon\eta'(\rho_{\varepsilon,\nu})b(t,x)\partial_x W_{\varepsilon,\nu}\rho_{\varepsilon,\nu}\|_{L^2((0,T)\times\mathbb{R})} &= \tfrac{\varepsilon}{\sqrt{\nu}}\|\eta'(\rho_{\varepsilon,\nu})b(t,x)\rho_{\varepsilon,\nu}\|_{L^\infty((0,T)\times\mathbb{R})}\int_0^T\sqrt{\nu}\|\partial_x W_{\varepsilon,\nu}\|_{L^2(\mathbb{R})}\,\mathrm{d}t\\ &\le \tfrac{\varepsilon}{\sqrt{\nu}}c_T \to 0\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}&\|\varepsilon\eta''(\rho_{\varepsilon,\nu})b\,\partial_x W_{\varepsilon,\nu}\rho_{\varepsilon,\nu}\partial_x\rho_{\varepsilon,\nu}\|_{L^1((0,T)\times\mathbb{R})}\\ &\quad= \|\varepsilon\eta''(\rho_{\varepsilon,\nu})b\,\rho_{\varepsilon,\nu}\partial_x W_{\varepsilon,\nu}\Big(\partial_x W_{\varepsilon,\nu} - \varepsilon\partial^2_{xx}W_{\varepsilon,\nu}\Big)\|_{L^1((0,T)\times\mathbb{R})}\\ &\quad= \|\eta''(\rho_{\varepsilon,\nu})b\,\rho_{\varepsilon,\nu}\Big(\varepsilon(\partial_x W_{\varepsilon,\nu})^2 - \varepsilon^2\partial_x W_{\varepsilon,\nu}\partial^2_{xx}W_{\varepsilon,\nu}\Big)\|_{L^1((0,T)\times\mathbb{R})}\\ &\quad\le \|\eta''(\rho_{\varepsilon,\nu})b\,\rho_{\varepsilon,\nu}\Big(\varepsilon(\partial_x W_{\varepsilon,\nu})^2 + \varepsilon^2|\partial_x W_{\varepsilon,\nu}\partial^2_{xx}W_{\varepsilon,\nu}|\Big)\|_{L^1((0,T)\times\mathbb{R})}\\ &\quad\le \|\eta''(\rho_{\varepsilon,\nu})b\,\rho_{\varepsilon,\nu}\|_{L^\infty((0,T)\times\mathbb{R})}\times\\ &\qquad\qquad\left\|\frac{\varepsilon}{\nu}\nu(\partial_x W_{\varepsilon,\nu})^2 + \frac{\varepsilon^2\nu}{2}(\partial^2_{xx}W_{\varepsilon,\nu})^2 + \frac{\varepsilon^2}{2\nu^2}\nu(\partial_x W_{\varepsilon,\nu})^2\right\|_{L^1((0,T)\times\mathbb{R})}\\ &\quad\le c_T.\end{aligned}\tag{4.6}$$

Then, by Lemma 4.2, we deduce that $\{\partial_t\eta(\rho_{\varepsilon,\nu}) + \partial_x q(\rho_{\varepsilon,\nu})\}_{\varepsilon,\nu>0}$ is compact in $H^{-1}_{\text{loc}}((0,\infty)\times\mathbb{R})$. Therefore, by Lemma 4.1, we conclude that given (2.2) there exists a function $\rho\in L^\infty((0,T)\times\mathbb{R})$, $T>0$, such that

$$\rho_{\varepsilon_n,\nu_n} \longrightarrow \rho \quad \text{in } L^p_{\text{loc}}((0,\infty)\times\mathbb{R}),\ p\in[1,\infty), \text{ and a.e. in } (0,\infty)\times\mathbb{R}.$$

By the Lebesgue dominated convergence theorem, we have that ρ is a weak solution of (1.1). It remains to show that ρ is an entropy solution. We start by observing that

$$\begin{aligned}\partial_t\eta(\rho_{\varepsilon_n,\nu_n}) + \partial_x q(\rho_{\varepsilon_n,\nu_n}) &= \nu_n\partial^2_{xx}\eta(\rho_{\varepsilon_n,\nu_n}) - \underbrace{\nu_n\eta''(\rho_{\varepsilon_n,\nu_n})(\partial_x\rho_{\varepsilon_n,\nu_n})^2}_{\ge 0}\\ &\quad - \varepsilon_n\partial_x(\eta'(\rho_{\varepsilon_n,\nu_n})b(t,x)\partial_x W_{\varepsilon_n,\nu_n}\rho_{\varepsilon_n,\nu_n})\\ &\quad + \varepsilon_n\eta''(\rho_{\varepsilon_n,\nu_n})b(t,x)\partial_x W_{\varepsilon_n,\nu_n}\rho_{\varepsilon_n,\nu_n}\partial_x\rho_{\varepsilon_n,\nu_n}\\ &\le \nu_n\partial^2_{xx}\eta(\rho_{\varepsilon_n,\nu_n}) - \varepsilon_n\partial_x(\eta'(\rho_{\varepsilon_n,\nu_n})b(t,x)\partial_x W_{\varepsilon_n,\nu_n}\rho_{\varepsilon_n,\nu_n})\end{aligned}$$

$$+\,\varepsilon_n\eta''(\rho_{\varepsilon_n,\nu_n})b(t,x)\partial_x W_{\varepsilon_n,\nu_n}\rho_{\varepsilon_n,\nu_n}\partial_x\rho_{\varepsilon_n,\nu_n}$$

Let us consider a non-negative test function $\varphi \in C_c^\infty(\mathbb{R}^2)$. Then,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \Big(\eta(\rho_{\varepsilon_n,\nu_n})\partial_t\varphi + q(\rho_{\varepsilon_n,\nu_n})\partial_x\varphi\Big)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \eta(\rho_{0,\nu_n}(x))\varphi(0,x)\,\mathrm{d}x\\
&\geq \nu_n \int_0^\infty \int_{\mathbb{R}} \eta(\rho_{\varepsilon_n,\nu_n})\partial_{xx}^2\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \varepsilon_n \int_0^\infty \int_{\mathbb{R}} \Big(\eta'(\rho_{\varepsilon_n,\nu_n})b(t,x)\partial_x W_{\varepsilon_n,\nu_n}\rho_{\varepsilon_n,\nu_n}\Big)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \varepsilon_n \int_0^\infty \int_{\mathbb{R}} \eta''(\rho_{\varepsilon_n,\nu_n})b(t,x)\rho_{\varepsilon_n,\nu_n}\partial_x W_{\varepsilon_n,\nu_n}\Big(\partial_x W_{\varepsilon_n,\nu_n} - \varepsilon_n\partial_{xx}^2 W_{\varepsilon_n,\nu_n}\Big)\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&= \underbrace{\nu_n \int_0^\infty \int_{\mathbb{R}} \eta(\rho_{\varepsilon_n,\nu_n})\partial_{xx}^2\varphi\,\mathrm{d}x\,\mathrm{d}t}_{I_1}\\
&\quad \underbrace{+\,\varepsilon_n \int_0^\infty \int_{\mathbb{R}} \Big(\eta'(\rho_{\varepsilon_n,\nu_n})b(t,x)\partial_x W_{\varepsilon_n,\nu_n}\rho_{\varepsilon_n,\nu_n}\Big)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t}_{I_2}\\
&\quad \underbrace{+\,\varepsilon_n \int_0^\infty \int_{\mathbb{R}} \eta''(\rho_{\varepsilon_n,\nu_n})b(t,x)\rho_{\varepsilon_n,\nu_n}(\partial_x W_{\varepsilon_n,\nu_n})^2\,\mathrm{d}x}_{I_3}\\
&\quad \underbrace{-\,\tfrac{\varepsilon_n}{\nu_n} \int_0^\infty \int_{\mathbb{R}} \eta''(\rho_{\varepsilon_n,\nu_n})b(t,x)\rho_{\varepsilon_n,\nu_n}\sqrt{\nu_n}\partial_x W_{\varepsilon_n,\nu_n}\varepsilon_n\sqrt{\nu_n}\partial_{xx}^2 W_{\varepsilon_n,\nu_n}\varphi\,\mathrm{d}x\,\mathrm{d}t}_{I_4}.
\end{aligned}$$

Due to the estimates done in the first part of the proof, we have

$$\begin{aligned}
I_1 &\leq \nu_n\|\eta\|_{L^\infty(\mathbb{R})}\|\partial_{xx}^2\varphi\|_{L^1(\mathbb{R}^2)};\\
I_2 &\leq \tfrac{\varepsilon_n}{\sqrt{\nu_n}}\|\eta'(\rho_{\varepsilon_n,\nu_n})b\,\rho_{\varepsilon_n,\nu_n}\|_{L^\infty((0,T)\times\mathbb{R})}\int_0^T \sqrt{\nu_n}\|\partial_x W_{\varepsilon_n,\nu_n}\|_{L^2(\mathbb{R})}\,\mathrm{d}t\\
&\leq \tfrac{\varepsilon_n}{\sqrt{\nu_n}}c_T;\\
I_3 &\leq \tfrac{\varepsilon_n}{\nu_n}\|\eta''(\rho_{\varepsilon_n,\nu_n})b\,\rho_{\varepsilon_n,\nu_n}\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\nu_n(\partial_x W_{\varepsilon_n,\nu_n})^2\right\|_{L^1((0,T)\times\mathbb{R})} \leq \tfrac{\varepsilon_n}{\nu_n}c_T;\\
I_4 &\leq \tfrac{\varepsilon_n}{\nu_n}\|\eta''(\rho_{\varepsilon_n,\nu_n})b\,\rho_{\varepsilon_n,\nu_n}\|_{L^\infty((0,T)\times\mathbb{R})}\|\sqrt{\nu_n}\partial_x W_{\varepsilon_n,\nu_n}\|_{L^2((0,T)\times\mathbb{R})}\times\\
&\qquad \|\varepsilon_n\sqrt{\nu_n}\partial_{xx}^2 W_{\varepsilon_n,\nu_n}\|_{L^2((0,T)\times\mathbb{R})}\|\varphi\|_{L^\infty(\mathbb{R}^2)} \leq \tfrac{\varepsilon_n}{\nu_n}c_T.
\end{aligned}$$

Passing to the limit, thanks to assumption Eq. (2.2) and to the Lebesgue dominated convergence theorem, we obtain Eq. (2.1).

The fact that, in the statement of the theorem, the family $\{\rho_{\varepsilon,\nu}\}_{\varepsilon,\nu>0}$ converges to ρ and not just up to subsequences follows from the uniqueness of entropy solutions of (1.1) and from the Urysohn property, i.e. $\rho_{\varepsilon,\nu} \to \rho$ if and only if for all subsequences $\{\rho_{\varepsilon_n,\nu_n}\}_{n\in\mathbb{N}}$, there exists an extract $\{\rho_{\varepsilon_{n_k},\nu_{n_k}}\}_{k\in\mathbb{N}}$ such that $\rho_{\varepsilon_{n_k},\nu_{n_k}} \to \rho$. □

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