Maximal L^p regularity for elliptic equations with unbounded coefficients [1]

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Abstract: We consider an elliptic equation on R^n of the form $\lambda\varphi - \frac{1}{2}\,\Delta\varphi + \langle DU, D\varphi\rangle = f$ with the potential U regular but unbounded. We prove a maximal regularity result in the space $L^p(R^n, \nu)$ where ν is an invariant measure.

Key words: elliptic equations, unbounded coefficients, maximal regularity, invariant measures.

1 Introduction and setting of the problem

We are concerned with the following equation in $\mathbb{R}^n$:

$$\lambda\varphi - N_0\varphi = f, \quad \lambda > 0, \tag{1.1}$$

where N_0 is the elliptic operator defined by

$$N_0\varphi = \frac{1}{2}\,\Delta\varphi - \langle DU, D\varphi\rangle,\ \varphi \in C_0^\infty(\mathbb{R}^n),$$

and $U : \mathbb{R}^n \to \mathbb{R}$ is a function satisfying the following assumption.

Hypothesis 1.1 *(i) U is of class C^3.*

(ii) $\displaystyle\int_{\mathbb{R}^n} e^{-2U(y)}dy < +\infty.$

It is important to notice that N_0 is a symmetric operator in the space $L^2(\mathbb{R}^n, \nu)$, where ν is the Borel probability measure on $\mathbb{R}^n$ defined as

$$\nu(dx) = \frac{e^{-2U(x)}dx}{\int_{\mathbb{R}^n} e^{-2U(y)}dy}.$$

Indeed the following identity can be easily checked:

$$\int_{\mathbb{R}^n} N_0\varphi\psi d\nu = -\frac{1}{2}\int_{\mathbb{R}^n}\langle D\varphi, D\psi\rangle d\nu, \tag{1.2}$$

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for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$. See Lemma 1.2 below.

The operator N_0 naturally arises in studying a dynamical system described by a differential equation of potential type perturbed by a white noise,

$$dX = -DU(X)dt + dW(t),\ X(0) = x,$$

where $W(t)$ is a standard Brownian motion on $\mathbb{R}^n$.

In fact, let us consider the transition semigroup defined by

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))],\ \varphi \in C_b(\mathbb{R}^n),$$

where $C_b(\mathbb{R}^n)$ is the space of all unformly continuous and bounded functions on $\mathbb{R}^n$, $X(t,x)$ is the solution of the stochastic equation above and $\mathbb{E}$ denotes the expectation. Then, in a sense to be precised, N_0 is the infinitesimal generator of P_t, and ν is an invariant measure for P_t :

$$\int_{\mathbb{R}^n} P_t\varphi d\nu = \int_{\mathbb{R}^n} \varphi d\nu,\ \varphi \in C_b(\mathbb{R}^n).$$

If DU is continuous and bounded then wellposedness for the elliptic equation (1.1), or for the corresponding parabolic equation, is well known, see [17]. However the boundedness assumption for DU is too restrictive for the applications. A general existence result for equation (1.1) with DU was proved by S. Ito, see [15]; however uniqueness of the solutions to (1.1) is not trivial, see e.g. [16].

Some cases of wellposedness (for an equation similar to (1.1) but including an additional term $V\varphi$) in suitable weighted spaces, are presented in [4], [6], [7].

In the past few years many papers have been devoted to study equation (1.1), even for non regular potentials U and also in infinite dimensional spaces H. We recall the approach through the Dirichlet forms theory, see [1], [2], [19] and the references therein, and through the parabolic criterion of Berezanskii, [3].

Another approach was introduced in [11], where essential self–adjointness of N_0 was proved by showing the density of the range of $\lambda I - N_0$ for any $\lambda > 0$.

From Hypothesis 1.1 it follows that N_0 is essentially self–adjoint on $L^2(\mathbb{R}^n, \nu)$, see e.g. [12]. We shall denote by N_2 its closure on $L^2(\mathbb{R}^n, \nu)$. Moreover from (1.2) it follows that the domain of the square root of $-N_2$, is $W^{1,2}(\mathbb{R}^n, \nu)$.

A more difficult task is to characterize the domain of N_2. At our knowledge such a characterization is known in two cases: when DU is linear (see [18],[9]) and when U is convex (see [10], [12]). In both cases the domain of N_2 is given by

$$D(N_2) = W^{2,2}(\mathbb{R}^n, \nu).$$

Let now $p > 1$. Then N_0 is dissipative in $L^p(\mathbb{R}^n, \nu)$, see Lemma 1.2 below, and so it is closable. Let us denote by N_p its closure. The main aim of this paper is to prove that N_p is m–dissipative and that its domain is given by

$$D(N_p) = W^{2,p}(\mathbb{R}^n, \nu). \tag{1.3}$$

This result seems to be unknown, unless U is a quadratic form; see the paper [8], where (1.3) was proved (even in an infinite dimensional Hilbert space H) by using a generalization of the Littlewood–Paley–Stein Theory.

We are able to prove (1.3) under quite strong assumptions on U. Rougly speaking we can cover the case when U and its derivatives up to the second order have polynomial growth.

Let us explain our method. Consider equation (1.1) where $\lambda > 0$, $f \in L^p(\mathbb{R}^n, \nu)$ and $p > 1$. Setting

$$v(x) = e^{-\frac{2}{p}U(x)}\varphi(x),\ g(x) = e^{-\frac{2}{p}U(x)}f(x),\ \ x \in \mathbb{R}^n,$$

equation (1.1) reduces to

$$\lambda v - \frac{1}{2}\,\Delta v + \frac{p-2}{p}\,\langle Dv, DU\rangle + G_p v = g, \tag{1.4}$$

where

$$G_p = \frac{2(p-1)}{p^2}\,|DU|^2 - \frac{1}{p}\,\Delta U.$$

We shall prove, under suitable assumptions, existence and uniqueness of a solution of equation (1.4) together with maximal regularity, using a result due to P. Cannarsa and V. Vespri, see [6]. Then, going back to equation (1.1), we are able to show that N_p is m–dissipative and that (1.3) holds.

We end this section by proving the above mentionned lemma and recalling the result of [6].

Lemma 1.2 *Assume that Hypothesis 1.1 holds. Then for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ the following identity holds.*

$$\int_{\mathbb{R}^n}(N_0\varphi)\psi d\nu = -\frac{1}{2}\int_{\mathbb{R}^n}\langle D\varphi, D\psi\rangle d\nu. \tag{1.5}$$

Moreover, for $p > 2$, and for any $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}(N_0\varphi)|\varphi|^{p-2}\varphi d\nu = -\frac{1}{2}\int_{\mathbb{R}^n}|D\varphi|^2|\varphi|^{p-2}d\nu. \tag{1.6}$$

Proof—Set

$$\rho(x) = \frac{e^{-2U(x)}}{\int_{\mathbb{R}^n} e^{-2U(y)}dy},\ x \in H.$$

Then we have

$$\int_{\mathbb{R}^n} (N_0\varphi)\psi d\nu = \frac{1}{2}\int_{\mathbb{R}^n} \Delta\varphi(\psi\rho)dx - \int_{\mathbb{R}^n} \langle DU, D\varphi\rangle \psi d\nu$$

$$= -\frac{1}{2}\int_{\mathbb{R}^n} \langle D\varphi, D\psi\rangle d\nu - \frac{1}{2}\int_{\mathbb{R}^n} \langle D\varphi, D\log\rho\rangle \psi d\nu - \int_{\mathbb{R}^n} \langle DU, D\varphi\rangle \psi d\nu$$

$$= -\frac{1}{2}\int_{\mathbb{R}^n} \langle D\varphi, D\psi\rangle d\nu.$$

Thus (1.5) is proved. Now (1.6) follows setting $\psi = |\varphi|^{p-2}\varphi$ in (1.5) . ∎

Remark 1.3 From (1.5) it follows immediately that N_0 is symmetric on $L^2(\mathbb{R}^n, \nu)$. Moreover, using the duality mapping on $L^p(\mathbb{R}^n, \nu)$ and formula (1.6), we can show easily that N_p is dissipative on $L^p(\mathbb{R}^n, \nu)$ for $p \geq 2$. Moreover, by an argument due to A. Eberle, see [14], we see that N_p is also dissipative on $L^p(\mathbb{R}^n, \nu)$ for $p \in [1, 2]$.

To state the result of [6] we need the following assumption.

Hypothesis 1.4

(i) There exists $k > 0$ such that $G_p(x) + k \geq 1,\ x \in \mathbb{R}^n$, and

$$|DG_p(x)| = o(|G_p(x) + k|^{3/2}),\ x \in \mathbb{R}^n,$$

that is for any $\varepsilon > 0$ there exists $k_\varepsilon > 0$ such that

$$|DG_p(x)| \leq \varepsilon(|G_p(x) + k|^{3/2}) + k_\varepsilon,\ x \in \mathbb{R}^n.$$

(ii) There exists $b \in [0, 2)$ such that

$$|DU(x)|^2 \leq b^2(G_p(x) + k),\ x \in \mathbb{R}^n.$$

The following result was proved in [6].

Theorem 1.5 *Assume that DU is continuous and that Hypothesis 1.4 holds. Then for all $\lambda > k$ there exists a unique solution $v \in W^{2,2}(\mathbb{R}^n, dx)$ of (1.3). Moreover*

$$D_i v\sqrt{G_p(x) + k},\ \text{and}\ D_iD_j(G_p(x) + k) \in L^p(\mathbb{R}^n, dx), \tag{1.7}$$

for all $i, j = 1, ..., n$.

2 Results

We shall assume, besides Hypotheses 1.1 and 1.4, that

Hypothesis 2.1 *There exists $k_1 > 0$ such that*

$$|D_i D_j U(x)| \leq k_1(G_p(x) + k), \quad x \in H, \ i,j = 1,...,n.$$

Roughly speaking Hypothesis 2.1 means that $U, D_i U, D_{i,j}U$, $i,j = 1,...,n$, behave like a polynomial, see Examples 2.4 and 2.8 below.

We can now prove the main result of the paper.

Theorem 2.2 *Assume that Hypotheses 1.1, 1.4 and 2.1 hold. Then N_p is m–dissipative in $L^p(\mathbb{R}^n, \nu)$ and we have $D(N_p) = W^{2,p}(\mathbb{R}^n, \nu)$.*

Proof—We are clearly in position to apply Theorem 1.5. Thus if $\lambda > k$, there is a unique solution $v \in W^{2,2}(\mathbb{R}^n, dx)$ of (1.3), and (1.7) holds. Moreover, using Hypothesis 2.1 and the obvious identities

$$D_i \varphi = \left[D_i v + \frac{2}{p}\, (D_i U) v \right] e^{\frac{2}{p}U}, \ i = 1,...,n,$$

and

$$D_i\varphi D_j \varphi = \left[D_i D_j v + \frac{2}{p}\, (D_i U D_j v + D_j U D_i v) + \frac{4}{p^2}\, D_i U D_j U v \right] e^{\frac{2}{p}U},$$

we see that $\varphi(x) = e^{\frac{2}{p}U(x)} v(x)$ is a solution of (1.1) and belongs to $W^{2,p}(\mathbb{R}^n, \nu)$.

Moreover, φ belongs to the domain of the closure N_p of N_0 on $L^p(\mathbb{R}^n, \nu)$. In fact from the proof of [6] it follows that $C_0^\infty(\mathbb{R}^n)$ is a core for the linear operator

$$Lv = \frac{1}{2}\, \Delta v + \frac{p-2}{p}\, \langle Dv, DU \rangle + G_p v$$

on $L^p(\mathbb{R}^n, dx)$. Since the transformation

$$u(x) = e^{\frac{2}{p}U(x)} \varphi(x), \ x \in \mathbb{R}^n,$$

maps $C_0^\infty(\mathbb{R}^n)$ into itself, it follows that $C_0^\infty(\mathbb{R}^n)$ is also a core for N_p.

To end the proof it remains to remove the condition $\lambda > k$. This can be easily done because N_0 is dissipative on $L^p(\mathbb{R}^n, \nu)$ by Lemma 1.2. The proof is complete. ■

Remark 2.3 Assume that $p = 2$, and that Hypotheses 1.1, 1.4–(i)–(ii) and 2.1 hold. Then Hypothesis 1.4–(iii) also holds. Under these assumptions U is not necessarily convex. Thus the conclusion of the theorem seems to be new.

Let us give now an example.

Example 2.4 Let $n = 1$ and

$$U(x) = \frac{1}{4}\, x^4, \;\; x \in H.$$

Then equations (1.1) and (1.4) read as follows

$$\lambda\varphi - \frac{1}{2}\,\varphi_{xx} + \frac{1}{2}\, x^3 \varphi_x = f,$$

and

$$\lambda v - \frac{1}{2}\, v_{xx} + \frac{p-2}{p}\, x^3 v_x + \frac{1}{p^2}\, \Big(2(p-1)x^6 - 3px^2\Big) = g,$$

respectively. Thus it is easy to check that Hypotheses 1.1, 1.4–(i)–(ii), and 2.1 are fulfilled. As for as Hypothesis 1.4–(iii) is concerned, it holds provided

$$4 - \sqrt{8} < p < 4 + \sqrt{8}.$$

This example can be obviously generalized taking for U any polynomial of even degree having positive leading coefficient.

As shown by the example, the assumption 1.4–(iii) is quite restrictive. As we shall see, it can be removed provided an estimate of the $L^p(\mathbb{R}^n, \nu)$ norm of $G_p v$ be available.

From now on we shall assume that $p \geq 2$ for simplicity. Let us start with a lemma.

Lemma 2.5 *Let v be a solution of (1.4). Then the following identity holds:*

$$\begin{aligned}
&(\lambda - k)\int_{\mathbb{R}^n} (G_p + k)^{p-1} |v|^p dx + \frac{p-1}{2} \int_{\mathbb{R}^n} |Dv|^2 |v|^{p-2} (G_p + k)^{p-1} dx \\
&+ \int_{\mathbb{R}^n} \Bigg[(G_p + k)^p - \frac{p-1}{2p}\, \mathrm{div}\, \left((G_p + k)^{p-2} D(G_p + k)\right) \\
&- \frac{p-2}{p^2}\, \mathrm{div}\, \left((G_p + k)^{p-1} DU\right) \Bigg] |v|^p dx = \int_{\mathbb{R}^n} g\, (G_p + k)^{p-1} |v|^{p-2} v dx.
\end{aligned} \tag{2.1}$$

Proof—We will assume for simplicity that $k = 0$. Multiplying both sides of (1.4) by $G_p^{p-1}|v|^{p-2}v$, we obtain

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^n} G_p^{p-1} |v|^p dx - \frac{1}{2} \int_{\mathbb{R}^n} \Delta v |v|^{p-2} v G_p^{p-1} dx \\
&+ \frac{p-2}{p} \int_{\mathbb{R}^n} \langle DU, DG_p \rangle G_p^{p-1} |v|^{p-2} v) dx + \int_{\mathbb{R}^n} G_p^p |v|^p dx \\
&= \int_{\mathbb{R}^n} g\, G_p^{p-1} |v|^{p-2} v dx.
\end{aligned}$$

An integration by parts in the second integral of the left hand side gives

$$
\begin{aligned}
&\lambda \int_{\mathbb{R}^n} G_p^{p-1}|v|^p dx + \frac{p-1}{2} \int_{\mathbb{R}^n} |Dv|^2 |v|^{p-2} G_p^{p-1} dx \\
&+\frac{p-2}{p} \int_{\mathbb{R}^n} \langle DU, DG_p\rangle G_p^{p-1}|v|^{p-2}v) dx + \int_{\mathbb{R}^n} G_p^p |v|^p dx \qquad (2.2)\\
&= \int_{\mathbb{R}^n} g G_p^{p-1}|v|^{p-2} v dx.
\end{aligned}
$$

Now two other integrations by parts give

$$
\frac{p-2}{p} \int_{\mathbb{R}^n} \langle DU, DG_p\rangle G_p^{p-1}|v|^{p-2} v dx = -\frac{p-1}{2p} \int_{\mathbb{R}^n} \text{div}\,(G_p^{p-2} DG_p)\, |v|^p dx, \qquad (2.3)
$$

$$
\frac{p-2}{p} \int_{\mathbb{R}^n} \langle DU, DG_p\rangle G_p^{p-1}|v|^{p-2} v dx = -\frac{p-2}{p^2} \int_{\mathbb{R}^n} \text{div}\,(G_p^{p-1} DU)\, |v|^p dx. \qquad (2.4)
$$

Substituting (2.3) and (2.4) in (2.2) gives the conclusion. ∎

From Lemma 2.5 it follows that we can estimate the $L^p(\mathbb{R}^n, dx)$ norm of $G_p v$ provided the following assumption holds

Hypothesis 2.6 *There exists $\alpha \in [0,1)$ such that*

$$
\begin{aligned}
&(G_p+k)^p - \frac{p-1}{2p} \text{div}\,((G_p+k)^{p-2} DG_p) \\
&-\frac{p-2}{p^2} \text{div}\,((G_p+k)^{p-1} DU) \leq \alpha (G_p+k)^p.
\end{aligned} \qquad (2.5)
$$

Notice that this assumption is fulfilled in the Example 2.8.

We can prove finally the result

Theorem 2.7 *Assume that Hypotheses 1.1, 1.4–(i)–(iii), 2.1 and 2.6 holds and that $p \geq 2$. Then N_p is m–dissipative in $L^p(\mathbb{R}^n, \nu)$ and we have $D(N_p) = W^{2,p}(\mathbb{R}^n, \nu)$.*

Proof—Let v be a variational solution of equation 1.4. By Lemma 2.5 we get that

$$
(1-\alpha) \int_{\mathbb{R}^n} (G_p+k)^p |v|^p dx \leq \int_{\mathbb{R}^n} |g|\, (G_p+k)^{p-1} |v|^{p-1} dx.
$$

Consequently, by the Hölder inequality,

$$
\int_{\mathbb{R}^n} (G_p+k)^p |v|^p dx \leq \left(\int_{\mathbb{R}^n} |g|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} (G_p+k)^p |v|^p dx\right)^{\frac{p-1}{p}},
$$

and so

$$\int_{\mathbb{R}^n} |g|^p dx \le (1-\alpha)^{-p} \int_{\mathbb{R}^n} |g|^p dx.$$

Therefore there exists $c > 0$ such that if $\lambda > k$ we have

$$\int_{\mathbb{R}^n} (G_p + k)^p |v|^p dx \le c.$$

Then for any $M > 0$ we have obviously

$$(\lambda - k)v - \frac{1}{2}\,\Delta v + \frac{p-2}{p}\,\langle Dv, DU\rangle + M(G_p + k)v = g + (M-1)G_p v. \quad (2.6)$$

In view of Lemma 2.5 we have $g + (M-1)G_p v \in L^p(\mathbb{R}^n, dx)$. Moreover, given $b \in [0,2)$, by an appropriate choice of M, we get

$$|DU(x)|^2 \le b^2 M(G_p(x) + k),\ x \in H.$$

Now we can apply Theorem 1.5 and conclude that for any $\lambda > k$ equation (1.4) has a unique solution $v \in W^{2,p}(\mathbb{R}^n, dx)$, such that (1.7) holds. Consequently equation (1.1) has a unique solution in $W^{2,p}(H,\mu)$. Finally the restriction $\lambda > k$ can be removed as before since N_p is dissipative in $L^p(\mathbb{R}^n, \nu)$. ■

Example 2.8 Let $H = \mathbb{R}^n$, $n, m \in \mathbb{N}$, and let

$$U(x) = \frac{1}{4}\,|x|^{2m}.$$

Then it is easy to see that all hypotheses of Theorem 2.7 are fulfilled.

Remark 2.9 Assume that the assumptions of Theorem 2.2 or 2.7 hold. Then $P_t = e^{tN_2}$ is a Markov semigroup on $L^2(\mathbb{R}^n, \nu)$. By [13, Theorem 1.4.2] it follows that e^{tN_p} is an analytic semigroup in $L^p(\mathbb{R}^n, \nu)$ for any $p > 1$.

Remark 2.10 Let us consider equation (1.1) where N_0 is the elliptic operator

$$N_0 \varphi = \frac{1}{2}\,\text{div}\,[AD\varphi] - \langle A^{-1}DU, D\varphi\rangle\ \varphi \in C_0^\infty(\mathbb{R}^n),$$

where $U : \mathbb{R}^n \to \mathbb{R}$ is a function fulfilling Hypothesis 1.1 and $A \in C(\mathbb{R}^n; L(\mathbb{R}^n))$ is such that

$$\langle A\xi, \xi\rangle \ge \alpha|\xi|^2,\ \ \xi \in \mathbb{R}^n,$$

for some $\alpha > 0$. Then it is easy to see that N_0 is again a symmetric operator in the space $L^2(\mathbb{R}^n, \nu)$, where in this case ν is the Borel probability measure on H defined as

$$\nu(dx) = \frac{e^{-2U(x)}dx}{\int_{\mathbb{R}^n} e^{-2U(y)}dy},$$

and that all previous considerations can be adapted to cover this more general situation.

Remark 2.11 Let us consider equation (1.1), where N_0 is the elliptic operator

$$N_0\varphi = \frac{1}{2}\,\mathrm{div}\,[AD\varphi] - \langle B(x), D\varphi\rangle,\ \ \varphi \in C_0^\infty(\mathbb{R}^n),$$

B is a regular vector field, and A is as in the previous remark. Then the existence of an invariant measure ν, absolutely continuous with respect to the Lebesgue measure and such that

$$\int_H N_0\varphi\varphi d\nu = -\frac{1}{2}\int_H \langle AD\varphi, D\varphi\rangle d\nu, \tag{2.7}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, is well known, see e.g. [12]. However in this case we do not know explicity the density $\rho = d\nu/dx$. Moreover only local regularity results for ρ are available, see [5]. So it seems difficult to generalize the previous results to cover this case.

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