

**TRACES AND EXTENSIONS OF CERTAIN WEIGHTED
SOBOLEV SPACES ON \mathbb{R}^n AND BESOV FUNCTIONS ON
AHLFORS REGULAR COMPACT SUBSETS OF \mathbb{R}^n**

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Dedicated with gratitude to Professor Pekka Koskela on his 59th birthday

ABSTRACT. The focus of this paper is on Ahlfors Q -regular compact sets $E \subset \mathbb{R}^n$ such that, for each $Q - 2 < \alpha \leq 0$, the weighted measure μ_α given by integrating the density $\omega(x) = \text{dist}(x, E)^\alpha$ yields a Muckenhoupt \mathcal{A}_p -weight in a ball B containing E . For such sets E we show the existence of a bounded linear trace operator acting from $W^{1,p}(B, \mu_\alpha)$ to $B_{p,p}^\theta(E, \mathcal{H}^Q|_E)$ when $0 < \theta < 1 - \frac{\alpha+n-Q}{p}$, and the existence of a bounded linear extension operator from $B_{p,p}^\theta(E, \mathcal{H}^Q|_E)$ to $W^{1,p}(B, \mu_\alpha)$ when $1 - \frac{\alpha+n-Q}{p} \leq \theta < 1$. We illustrate these results with E as the Sierpiński carpet, the Sierpiński gasket, and the von Koch snowflake.

Key words and phrases: Besov space, weighted Sobolev space, Ahlfors regular sets, Sierpiński carpet, gasket, von Koch snowflake, trace, extension.

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1. INTRODUCTION

The Sobolev classes $W^{1,p}(\Omega)$, with Ω a Euclidean domain, are associated with (potentially degenerate) elliptic partial differential equations related to a strongly local Dirichlet form. In considering Dirichlet boundary value problems on a Euclidean domain in \mathbb{R}^n related to elliptic differential operators, solutions are known to exist if the prescribed boundary datum, namely a function that is given on the boundary of the domain, arises as the trace of a Sobolev function that is defined on the domain. The works of Jonsson and Wallin [JW, JW2] identify certain Besov spaces of functions on a compact d -set as traces on that set of Sobolev functions on \mathbb{R}^n . Here, a set is a d -set if it is Ahlfors d -regular, namely, $\mathcal{H}^d(B(x, r)) \simeq r^d$ whenever x is a point in that set and $r > 0$ is no larger than the diameter of that set. If Ω is a bounded Lipschitz domain in \mathbb{R}^n , then its boundary is an $(n - 1)$ -set. It

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was shown in [JW, Page 228] that the trace of $W^{1,p}(\mathbb{R}^n)$ in a d -set $E \subset \mathbb{R}^n$ is the Besov space $B_{p,p}^{1-(n-d)/p}(E)$. Thus for each $p > 1$ a specific value of $\theta = 1 - \frac{n-d}{p}$ was chosen amongst all possible values of $0 < \theta < 1$ for which the Besov space $B_{p,p}^\theta(E)$ is identified with the Sobolev space $W^{1,p}(\mathbb{R}^n)$.

On the other hand, [B2S, Theorem 1.1] tells us that each compact doubling metric measure space Z is the boundary of a uniform space (a locally compact non-complete metric space that is a uniform domain in its completion) X_ε and that for each choice of $p > 1$ and $0 < \theta < 1$ there is a choice of a doubling measure μ_β , $\beta = \beta(\theta)$, on the uniform space such that $B_{p,p}^\theta(Z)$ is the trace space of the Newton-Sobolev class $N^{1,p}(X_\varepsilon, \mu_\beta)$. The measures μ_β are obtained as weighted measures, that is, $d\mu_\beta(x) = \omega_\beta(x) d\mu(x)$ for some underlying measure μ on X_ε . With this perspective in mind, it is then natural to ask whether, given $p > 1$ and $0 < \theta < 1$, there is a weighted measure μ_α on \mathbb{R}^n , with α depending perhaps on θ and p , such that $B_{p,p}^\theta(E, \nu)$ is the trace space of the weighted Sobolev class $W^{1,p}(\mathbb{R}^n, \mu_\alpha)$. Here ν is the Hausdorff measure on E of dimension $\dim_H(E)$. There is evidence for this to be plausible, see for example [Ma] where E was considered to be the boundary of a uniform domain. The paper [Ba, Theorem 1.5] identified each $B_{p,p}^\theta(\partial\Omega)$, $1 < p < \infty$, $0 < \theta < 1$, with certain weighted Sobolev classes of functions on the Lipschitz domain $\Omega \subset \mathbb{R}^n$, while the papers [DFM1, DFM2] consider the case of $p = 2$ with $\mathbb{R}^n \setminus E$ a domain satisfying a one-sided NTA domain condition. In this note we do not assume that $\mathbb{R}^n \setminus E$ is a domain (the example of E being a Sierpiński carpet does not have its complement in \mathbb{R}^2 be a domain) nor do we restrict ourselves to the case of $p = 2$. Indeed, when $n \geq 3$, in considering \mathbb{R}^n , the choice of $p = n$ is related to the quasi-conformal geometry of $\mathbb{R}^n \setminus E$. In many cases, including the three examples considered here as well as the setting considered in [Ma, DFM1, DFM2], the weight $\omega_\beta(x) \simeq \text{dist}(x, E)^\beta$.

In this paper we expand on this question and study the following setting. With $0 < Q < n$, let $E \subset \mathbb{R}^n$ be an Ahlfors Q -regular compact set with $\text{diam}(E) \leq 1$, and B be a ball in \mathbb{R}^n such that $E \subset \frac{1}{2}B$. We also assume that for each $\alpha \leq 0$ there is a measure μ_α on B such that whenever $\alpha + n - Q > 0$ and $x \in E$, and $0 < r < 2$ such that $r > \text{dist}(x, E)/9$, the comparison $\mu_\alpha(B(x, r)) \simeq r^{n+\alpha}$ holds. We also assume that the ball B , equipped with the Euclidean metric d and the measure μ_α , is doubling. Furthermore, we assume that for each $p > 1$ there exists such α so that μ_α supports a p -Poincaré inequality. We prove the following two theorems in this paper. In what follows, $\nu = \mathcal{H}^Q|_E$ and for each real number α there is a Borel regular measure μ_α on \mathbb{R}^n that is absolutely continuous with respect to the Lebesgue measure m and satisfying $\mu_\alpha(B(x, r)) \simeq r^{n+\alpha}$ when $x \in E$ and $0 < r \leq 10$.

Theorem 1.1. *Let $p > 1$ and $0 < \theta < 1$ such that $p\theta < 1$. Let $\alpha \leq 0$ such that $\alpha + n - Q > 0$ and $\theta < 1 - \frac{\alpha + n - Q}{p}$ such that μ_α is doubling and supports*

a p -Poincaré inequality. Then there is a bounded linear trace operator

$$T : N^{1,p}(B, \mu_\alpha) \rightarrow B_{p,p}^\theta(E, \nu)$$

such that $Tu = u|_E$ when u is a Lipschitz function on B .

Theorem 1.2. *Let $p > 1$ and $0 < \theta < 1$, and let $\alpha \leq 0$ such that $\alpha + n - Q > 0$ and $\theta \geq 1 - \frac{\alpha + n - Q}{p}$. Then there is a bounded linear extension operator*

$$S : B_{p,p}^\theta(E, \nu) \rightarrow N^{1,p}(B, \mu_\alpha)$$

such that if u is a Lipschitz function on E , then Su is a Lipschitz function on B with $u = Su|_E$.

As one might note above, there is a lack of sharpness in the range of allowable θ in Theorem 1.1, which then prevents us from identifying $B_{p,p}^\theta(E, \nu)$ as the trace space of $N^{1,p}(B, \mu_\alpha)$ when $\theta = 1 - \frac{\alpha + n - Q}{p}$. This hurdle is overcome if E is the boundary of a uniform domain, as shown by Malý in [Ma], see also [DFM1]. We do not know whether we can take $\theta = 1 - \frac{\alpha + n - Q}{p}$ in Theorem 1.1.

In this note we also show that the fractals such as the standard Sierpiński carpet, the standard Sierpiński gasket, and the von Koch snowflake curve in \mathbb{R}^2 satisfy the conditions imposed on the set E above, with $n = 2$. For such sets, the measure μ_α is a weighted Lebesgue measure given in (1) below. The trace and extension theorems for the example of the von Koch snowflake curve follow from the work of [Ma] once a family of measures μ_α , $0 \geq \alpha > -\beta_0$ for a suitable β_0 are constructed and shown to satisfy the hypotheses of [Ma, Theorem 1.1]. We do this construction in Section 4. We show also that for the Sierpiński carpet and the Sierpiński gasket, this weight $\omega(x) = \text{dist}(x, E)^\alpha$ is a Muckenhoupt \mathcal{A}_q -weight for each $q > 1$ when $\alpha + 2 > Q$, where Q is the Hausdorff dimension of the fractal set. Observe that if a weight is an \mathcal{A}_q -weight, then the associated weighted measure is doubling and supports a q -Poincaré inequality. However, not all weights that give a doubling measure supporting a q -Poincaré inequality are \mathcal{A}_q -weights, see the discussion in [HKM].

Another interesting nonlocal space is the so-called Hajlasz space, see for example [HM, H-]. In [HM, Theorem 9] it is shown that if $\Omega \subset \mathbb{R}^n$ satisfies an $A(c)$ -condition (a porosity condition at the boundary), then the trace of Sobolev spaces $W^{1,p}(\Omega)$ are Hajlasz spaces of functions on E when E is equipped with an appropriately snow-flaked metric and a doubling Borel measure. We refer the interested reader to [JW, JW2, Bes, Maz, KSW, Ba, HM] for more on Sobolev spaces and Besov spaces of functions on Euclidean domains and sets, to [GKS, GKZ, Ma] for more on Besov spaces of functions on subsets of certain metric measure spaces, but this is not an exhaustive list of papers on these topics in the current literature.

2. BACKGROUND

In this section we describe the notions used throughout this note. With $0 < Q < n$, let $E \subset \mathbb{R}^n$ be an Ahlfors Q -regular compact set with $\text{diam}(E) \leq 1$ and let B be a fixed ball in \mathbb{R}^n such that $E \subset \frac{1}{2}B$. We set $\nu = \mathcal{H}^Q|_E$. We also consider the measure μ_α , obtained as a weighted measure with respect to the Lebesgue measure on \mathbb{R}^n as in [HKM], namely $d\mu_\alpha(x) = \omega(x) dm(x)$ with m the canonical Lebesgue measure on \mathbb{R}^n .

There are two basic function spaces under consideration here: the weighted Sobolev space $W^{1,p}(B, \mu_\alpha)$ and the Besov space $B_{p,p}^\theta(E, \nu)$ with $1 < p < \infty$. Recall from [HKM] that a function $f \in L^p(B, \mu_\alpha)$ is in the weighted Sobolev space $W^{1,p}(B, \mu_\alpha)$ if f has a weak derivative $\nabla f : B \rightarrow \mathbb{R}^n$ such that $|\nabla f| \in L^p(B, \mu_\alpha)$. It was shown in [HKM] that when μ_α is doubling and satisfies a p -Poincaré inequality near on B , then $W^{1,p}(B, \mu_\alpha)$ is a Banach space. Here, $W^{1,p}(B, \mu_\alpha)$ is equipped with the norm

$$\|f\|_{N^{1,p}(B, \mu_\alpha)} := \|f\|_{L^p(B, \mu_\alpha)} + \|\nabla f\|_{L^p(B, \mu_\alpha)}.$$

Recall that μ_α is doubling on B if there is a constant $C \geq 1$ such that whenever $x \in B$ and $0 < r \leq 10$,

$$\mu_\alpha(B(x, 2r)) \leq C \mu_\alpha(B(x, r)),$$

and we say that μ_α supports a p -Poincaré inequality on B if there is a constant $C > 1$ such that whenever f is in $W^{1,p}(B, \mu_\alpha)$ and B_0 is a ball with $B_0 \cap B \neq \emptyset$ and $\text{rad}(B_0) \leq 10$, then

$$\int_{B_0} |f - f_{B_0}| d\mu_\alpha \leq C \text{rad}(B_0) \left(\int_{B_0} |\nabla f|^p d\mu_\alpha \right)^{1/p}.$$

Here

$$f_{B_0} := \int_{B_0} f d\mu_\alpha = \frac{1}{\mu_\alpha(B_0)} \int_{B_0} f d\mu_\alpha.$$

A function $u \in L^p(E, \nu)$ is in the Besov space $B_{p,p}^\theta(E, \nu)$ for a fixed $0 < \theta < 1$ if

$$\|u\|_{B_{p,p}^\theta(E, \nu)}^p := \int_E \int_E \frac{|u(y) - u(x)|^p}{d(x, y)^{\theta p} \nu(B(x, d(x, y)))} d\nu(y) d\nu(x)$$

is finite.

The next two notions are related to the specific examples considered in this paper. Information about Muckenhoupt weights can be found for example in [MW1, MW2, HKM], while information about uniform domains can be found for example in [MS, HrK, BS]; these example references barely scratch the surface of the current literature on these topics and therefore should not be considered to be even an almost exhaustive list.

Definition 2.1. For $p > 1$, a weight $\omega : \mathbb{R}^n \rightarrow [0, \infty)$ is said to be a *Muckenhoupt \mathcal{A}_p -weight near B* if ω is a locally integrable function such that

$$\sup_{B_0} \left(\frac{1}{m(B_0)} \int_{B_0} \omega dm \right) \left(\frac{1}{m(B_0)} \int_{B_0} \omega^{-\frac{q}{p}} dm \right)^{\frac{p}{q}} < \infty$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and B_0 ranges over balls in \mathbb{R}^n intersecting B with $\text{rad}(B_0) \leq 10$. In this case we write $\omega \in \mathcal{A}_p$.

Definition 2.2. A domain $\Omega \subset \mathbb{R}^n$ is said to be a uniform domain if $\Omega \neq \mathbb{R}^n$ and there is a constant $A \geq 1$ such that for each distinct pair of points $x, y \in \Omega$ there is a curve γ in Ω with end points x, y with length $\ell(\gamma) \leq Ad(x, y)$ and $\min\{\ell(\gamma_{z,x}), \ell(\gamma_{z,y})\} \leq A \text{dist}(z, \partial\Omega)$ whenever z is a point on γ . Here $\gamma_{z,y}$ denotes each subcurve of γ with end points z, y .

3. THE CARPET AND GASKET EXAMPLES

In this section we first consider the weighted measure corresponding to the weight $\omega_\alpha(x) = \text{dist}(x, \mathbb{S})^\alpha$, with \mathbb{S} the standard Sierpiński carpet with outer perimeter $\partial[0, 1]^2$. We will show that when $\alpha > Q - 2 = \frac{\log(8)}{\log(3)} - 2$, the weighted measure is doubling. We also show that if in addition $p > \frac{\alpha+2-Q}{2-Q}$, then the weight is a Muckenhoupt \mathcal{A}_p -weight.

For $\alpha \in \mathbb{R}$, we define a Borel measure μ_α with density $\omega_\alpha(x) = \text{dist}(x, \mathbb{S})^\alpha$ outside of \mathbb{S} . That is, for Borel sets $A \subset \mathbb{R}^2$ we have

$$\mu_\alpha(A) = \int_A \text{dist}(x, \mathbb{S})^\alpha dx. \quad (1)$$

Note that \mathbb{S} has Lebesgue measure zero. We now investigate for which values of α the measure μ_α is doubling.

Notation. The carpet \mathbb{S} can be written as $[0, 1]^2 \setminus \bigcup_i H_i$ where the collection $\bigcup_i H_i$ consists of pairwise disjoint open squares $H_i \subset [0, 1]^2$. We call the open squares H_i holes. Each hole has sidelength 3^{-k} for some $k \in \mathbb{N}$. If the sidelength of a hole H is 3^{-k} , then we say H belongs to generation k . For each $k \in \mathbb{N}$ there are 8^{k-1} holes in generation k .

Lemma 3.1. *Let H be a hole in generation k . If $\alpha > -1$, then $\mu_\alpha(H) = c_\alpha 3^{-k(\alpha+2)}$ where $c_\alpha = \frac{8}{(\alpha+1)(\alpha+2)} 2^{-(\alpha+2)}$. Otherwise, $\mu_\alpha(H) = \infty$.*

Proof. By symmetry, we have

$$\mu_\alpha(H) = 8 \int_0^{2^{-1}3^{-k}} \int_0^x y^\alpha dy dx.$$

If $\alpha \leq -1$, then $\mu_\alpha(H)$ is infinite. Otherwise, we have $\alpha + 1 > 0$ and so

$$\mu_\alpha(H) = \frac{8}{\alpha + 1} \int_0^{2^{-1}3^{-k}} x^{\alpha+1} dx = \frac{8 \cdot 2^{-(\alpha+2)}}{(\alpha + 1)(\alpha + 2)} (3^{-k})^{\alpha+2} = c_\alpha 3^{-k(\alpha+2)}.$$

□

For our analysis we use squares instead of Euclidean balls. For $x \in \mathbb{R}^2$ and $s > 0$, we set

$$S(x, s) = \{y \in \mathbb{R}^2 : \|x - y\|_\infty < \frac{s}{2}\},$$

the open square in \mathbb{R}^2 centered at x with sidelength s . For $\tau > 0$, we set $\tau S(x, s) = S(x, \tau s)$ and estimate $\mu_\alpha(S(x, s))$. To do this, we introduce families of squares that have easy to compute μ_α mass. For $k \in \mathbb{N}$, let \mathcal{S}^k be the set of (open) squares of the form $S(x, 3^{-k})$ with x of the form $((m + \frac{1}{2})3^{-k}, (n + \frac{1}{2})3^{-k})$ with $m, n \in \mathbb{Z}$.

Lemma 3.2. *Let $S = S(x, s) \in \mathcal{S}^k$ and $\alpha > \frac{\log(8)}{\log(3)} - 2$. If $\overline{9S} \cap \mathbb{S} \neq \emptyset$, then $\mu_\alpha(S) \simeq s^{\alpha+2}$. Otherwise, $\mu_\alpha(S) \simeq s^2 d(x, \mathbb{S})^\alpha$. In particular, if $c > 9$ is the smallest integer such that $c\overline{S} \cap \mathbb{S} \neq \emptyset$, then $\mu_\alpha(S) \simeq c^\alpha s^{\alpha+2}$*

Observe that when $\alpha > \frac{\log(8)}{\log(3)} - 2$, we automatically have $\alpha > -1$.

Proof. For $\alpha = 0$ the claim is clear, so we assume that $\alpha \neq 0$ for the remainder of the proof. Note that $s = 3^{-k}$ as $S \in \mathcal{S}^k$. First suppose that $\overline{9S} \cap \mathbb{S} \neq \emptyset$. We examine three cases: (i) $3^k(\overline{S} \cap \mathbb{S})$ is isometric to the carpet, (ii) S is a hole as above, or (iii) neither of these cases.

Case (i): Using Lemma 3.1, we compute $\mu_\alpha(S)$ exactly. For each $j \in \mathbb{N}_0$, we see that S contains 8^j holes of generation $k + j + 1$. By assumption, $3^{\alpha+2} > 8$. As S is a scaled copy of the carpet, it follows from Lemma 3.1 that

$$\begin{aligned} \mu_\alpha(S) &= \sum_{j=0}^{\infty} 8^j c_\alpha (3^{\alpha+2})^{-k-j-1} = c_\alpha (3^{\alpha+2})^{-k-1} \sum_{j=0}^{\infty} 8^j 3^{-j(\alpha+2)} \\ &= c_\alpha 3^{-k(\alpha+2)} \frac{1}{3^{\alpha+2} - 8} = \left(\frac{c_\alpha}{3^{\alpha+2} - 8} \right) s^{\alpha+2}. \end{aligned}$$

Case (ii): In this case S is a hole in generation k , so by Lemma 3.1 we have that $\mu_\alpha(S) = c_\alpha s^{\alpha+2}$.

Case (iii): First assume that $\alpha < 0$. From our choice of \mathcal{S}^k , in this case we must have that $S \cap \mathbb{S} = \emptyset$. It is clear that if H is a hole of generation k and $\iota: S \rightarrow H$ is an isometry given by translation, then for all $y \in S$ we have $d(y, \mathbb{S}) \geq d(\iota(y), \mathbb{S})$. As $\alpha < 0$, it follows that $d(y, \mathbb{S})^\alpha \leq d(\iota(y), \mathbb{S})^\alpha$ and so $\mu_\alpha(S) \leq \mu_\alpha(H) \simeq s^{\alpha+2}$. On the other hand, $\overline{9S} \cap \mathbb{S} \neq \emptyset$, so $d(y, \mathbb{S}) \leq 11 \cdot 3^{-k} = 11s$ for all $y \in S$. Hence, as $\alpha < 0$ we have $d(y, \mathbb{S})^\alpha \geq (11s)^\alpha$. It follows that

$$\mu_\alpha(S) \geq \int_S (11s)^\alpha = 11^\alpha s^{\alpha+2}.$$

If $\alpha > 0$ instead, then the lower and upper bounds above are reversed but the conclusion is the same.

Now suppose that $\overline{9S} \cap \mathbb{S} = \emptyset$. It follows that $d(x, \mathbb{S}) \geq 3s$. Then, if $y \in S$, we have

$$\frac{1}{3}d(x, \mathbb{S}) \leq d(x, \mathbb{S}) - s \leq d(y, \mathbb{S}) \leq d(x, \mathbb{S}) + s \leq 2d(x, \mathbb{S}).$$

The result follows immediately.

For the last part of the lemma, if $c > 9$ is the smallest integer such that $c\overline{S} \cap \mathbb{S} \neq \emptyset$, then $d(x, \mathbb{S}) \simeq cs$ and so $\mu_\alpha(S) \simeq c^\alpha s^{\alpha+2}$. \square

We now use Lemma 3.2 to prove the same result for general squares.

Lemma 3.3. *Let $S = S(x, s)$ with $s \leq 9$. Let $\alpha > \frac{\log(8)}{\log(3)} - 2$. If $\overline{9S} \cap \mathbb{S} \neq \emptyset$, then $\mu_\alpha(S) \simeq s^{\alpha+2}$. Otherwise, $\mu_\alpha(S) \simeq s^2 d(x, \mathbb{S})^\alpha$. In particular, if $c > 9$ is the smallest integer such that $\overline{cS} \cap \mathbb{S} \neq \emptyset$, then $\mu_\alpha(S) \simeq c^\alpha s^{\alpha+2}$*

Proof. The proof that if $\overline{9S} \cap \mathbb{S} = \emptyset$, then $\mu_\alpha(S) \simeq s^2 d(x, \mathbb{S})^\alpha$ is the same as in Lemma 3.2. For the first part of the statement of the lemma, suppose that $\overline{9S} \cap \mathbb{S} \neq \emptyset$. Let $k \in \mathbb{N}$ be the smallest integer with $3^{-k} < s$. As $s \leq 9$ we have $\frac{s}{3^{-k}} \leq 3$. It follows that there is a subset $\{S_i\}_{i \in I} \subseteq S_k$ with $S \subseteq \cup_{i \in I} S_i$ and $|I| \leq 25$. Write $S_i = S(x_i, s_k)$ with $s_k = 3^{-k}$. For each S_i we have

$$d(x_i, \mathbb{S}) \leq s_k + s + d(x, \mathbb{S})$$

and, as $s_k \simeq s$ and $d(x, \mathbb{S}) \simeq s$, there is a constant $a > 0$ independent of i and S such that $\overline{aS_i} \cap \mathbb{S} \neq \emptyset$ for each $i \in I$. Hence, we may apply Lemma 3.2 to each S_i (with different squares S_i potentially falling into different cases of Lemma 3.2) and conclude that $\mu_\alpha(S_i) \simeq s^{\alpha+2}$. Hence, as $|I| \leq 25$ we have $\mu_\alpha(S) \lesssim s^{\alpha+2}$.

For the lower bound, we again choose the smallest integer k with $3^{-k} < s$ and then note that there is a square $S' \in \mathcal{S}^{k+2}$ with $S' \subseteq S$. An argument similar to the above shows us that $\mu_\alpha(S') \simeq s^{\alpha+2}$, and so $\mu_\alpha(S) \gtrsim s^{\alpha+2}$. \square

We now show that μ_α is doubling for small squares.

Lemma 3.4 (Doubling). *Let $S = S(x, s)$ be a square such that $s \leq 9$. Then, $\mu_\alpha(3S) \lesssim \mu_\alpha(S)$.*

Proof. First suppose that $\overline{27S} \cap \mathbb{S} = \emptyset$. Then, by Lemma 3.3 we know that $\mu_\alpha(3S) \simeq (3s)^2 d(x, \mathbb{S})^\alpha$ and $\mu_\alpha(S) \simeq s^2 d(x, \mathbb{S})^\alpha$.

Now, suppose that $\overline{27S} \cap \mathbb{S} \neq \emptyset$. Then, by Lemma 3.3, we know that $\mu_\alpha(3S) \simeq (3s)^{\alpha+2}$. We also see from Lemma 3.3 that $\mu_\alpha(S) \simeq s^{\alpha+2}$ (either $\overline{3S} \cap \mathbb{S} \neq \emptyset$, or we have $c \leq 27$ in the last part of the statement of Lemma 3.3). \square

Note that $q = \frac{p}{p-1}$. We investigate conditions that guarantee that the weight ω_α given by $\omega_\alpha(x) = \text{dist}(x, \mathbb{S})^\alpha$ is in the Muckenhoupt class \mathcal{A}_p . It is clear that in the definition of \mathcal{A}_p we may replace the use of balls B with that of squares S .

Lemma 3.5 (\mathcal{A}_p weights). *The function $\omega_\alpha(x) = \text{dist}(x, \mathbb{S})^\alpha$ is a Muckenhoupt \mathcal{A}_p -weight near B when $\frac{\log(8)}{\log(3)} - 2 < \alpha < (p-1)(2 - \frac{\log(8)}{\log(3)})$.*

Proof. Let $\frac{\log(8)}{\log(3)} - 2 < \alpha < (p-1)(2 - \frac{\log(8)}{\log(3)})$. Let $S = S(x, s)$ be a square with $s < 9$. First, assume that $\overline{9S} \cap \mathbb{S} \neq \emptyset$. Then, by Lemma 3.3,

$$\frac{1}{m(B)} \int_B \omega_\alpha dm \lesssim \frac{1}{s^2} s^{\alpha+2} = s^\alpha.$$

We see $\frac{q}{p} = \frac{1}{p-1}$. As $-\frac{\alpha}{p-1} > \frac{\log(8)}{\log(3)} - 2$, by Lemma 3.3 we have

$$\begin{aligned} \left(\frac{1}{m(B)} \int_B \omega_\alpha^{-\frac{q}{p}} dm \right)^{\frac{p}{q}} &= \left(\frac{1}{m(B)} \int_B \text{dist}(x, \mathbb{S})^{-\frac{\alpha}{p-1}} dx \right)^{p-1} \\ &\lesssim \left(\frac{1}{s^2} s^{-\frac{\alpha}{p-1}+2} \right)^{p-1} = s^{-\alpha}. \end{aligned}$$

It follows that the \mathcal{A}_p bound holds for these squares for α in the above range.

If, instead, we have $\overline{9S} \cap \mathbb{S} = \emptyset$, then with α in the same range we have

$$\frac{1}{m(B)} \int_B \omega_\alpha dm \lesssim \frac{1}{s^2} s^2 \text{dist}(x, \mathbb{S})^\alpha = \text{dist}(x, \mathbb{S})^\alpha$$

and

$$\begin{aligned} \left(\frac{1}{m(B)} \int_B \omega_\alpha^{-\frac{q}{p}} dm \right)^{\frac{p}{q}} &= \left(\frac{1}{m(B)} \int_B \text{dist}(x, \mathbb{S})^{-\frac{\alpha}{p-1}} dx \right)^{p-1} \\ &\lesssim \left(\frac{1}{s^2} s^2 \text{dist}(x, \mathbb{S})^{-\frac{\alpha}{p-1}} \right)^{p-1} = \text{dist}(x, \mathbb{S})^{-\alpha}. \end{aligned}$$

It again follows that for $\frac{\log(8)}{\log(3)} - 2 < \alpha < (p-1)(2 - \frac{\log(8)}{\log(3)})$ the \mathcal{A}_p bound holds. \square

Now, let \mathbb{G} denote the Sierpiński gasket for which the points $(0, 0)$, $(1, 0)$, and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ are the vertices of its boundary triangle. Consider the measures

$$\mu'_\alpha(A) = \int_A \text{dist}(x, \mathbb{G})^\alpha dx.$$

As for the carpet, for the correct values of α the measures μ'_α are doubling and the functions $\text{dist}(x, \mathbb{G})^\alpha$ are \mathcal{A}_p weights. The argument for this is similar, and we summarize the differences below.

First, squares are no longer the natural objects for integration. Instead, it is easier to work with equilateral triangles. The grids \mathcal{S}^k are replaced by grids of equilateral triangles with side lengths 2^{-k} . If $T = T(x, s)$ is such a triangle (centered at x with side length s), then for $\alpha > \frac{\log(3)}{\log(2)} - 2$ we can estimate $\mu'_\alpha(T)$ as in Lemma 3.2. For grid triangles far from the gasket relative to their side lengths, the estimate $\mu'_\alpha(T) \simeq s^2 \text{dist}(x, \mathbb{G})^\alpha$ still holds. For grid triangles near the gasket relative to their side length but which are neither holes nor scaled versions of the gasket, the estimate $\mu'_\alpha(T) \simeq s^{2+\alpha}$ still holds by comparison with $\mu'_\alpha(H)$ for hole-triangles H . For a single hole triangle T with side length $s = 2^{-k}$, we have

$$\mu'_\alpha(T) = 6 \int_0^{s/2} \int_0^{x/\sqrt{3}} y^\alpha dy dx \simeq s^{\alpha+2}.$$

For triangles $T = T(x, s)$ where $T \cap \mathbb{G}$ is a scaled copy of the gasket, we see that if $s = 2^{-k}$ then for each $j \in \mathbb{N}_0$, the triangle T contains 3^j hole

triangles of side length 2^{-k-j-1} . Hence, in this case

$$\mu'_\alpha(T) \simeq \sum_{j=0}^{\infty} 3^j 2^{(-k-j-1)(\alpha+2)} \simeq s^{\alpha+2}$$

where the series is finite as $2^{\alpha+2} > 3$.

As in Lemma 3.3, one again estimates the μ'_α measure of arbitrary triangles from the grid triangles. Once this is done, the doubling property and \mathcal{A}_p weight condition are as before. For the function $\text{dist}(x, \mathbb{G})^\alpha$ to be an \mathcal{A}_p weight, we require $\frac{\log(3)}{\log(2)} - 2 < \alpha < (p-1)(2 - \frac{\log(3)}{\log(2)})$.

4. THE VON KOCH SNOWFLAKE CURVE

In this section we consider the von Koch snowflake curve K (with an equilateral triangle with side-lengths 1 as a “zeroth” iteration) that is the boundary of the von Koch snowflake domain. We will show that Ω will satisfy the hypotheses of our paper with K playing the role of E and the ball B replaced by Ω . However, in this case we obtain a sharper result by combining the results of [BS] with [Ma, Theorem 1.1] to obtain the full range $0 < \theta \leq 1 - \frac{\alpha+n-Q}{p}$ in Theorem 1.1.

From [Ah, Theorem 1] we know that the snowflake curve is a quasicircle. Therefore, from [MS, Theorem 2.15] or [Hr, Theorem 1.2] it follows that the von Koch snowflake domain is a uniform domain; this sets us up to use the results of [BS].

Definition 4.1 (Definition 2.6 in [BS]). Let $\Omega \subset \mathbb{R}^n$ be a domain and $\beta > 0$. We say that Ω satisfies a local β -shell condition if there is a constant $C > 0$ such that for all $x \in \overline{\Omega}$ and $0 < \rho \leq r \leq \text{diam}(\Omega)$ we have

$$m(\{y \in B(x, r) \cap \Omega : \delta_\Omega(y) \leq \rho\}) \leq C \left(\frac{\rho}{r}\right)^\beta m(B(x, r) \cap \Omega) \quad (2)$$

where $\delta_\Omega(y) = \text{dist}(y, \partial\Omega)$. Recall here that m is the n -dimensional Lebesgue measure on \mathbb{R}^n .

Definition 4.2. We say that Ω satisfies a strong local β -shell condition if it satisfies a local β -shell condition and in addition,

$$m(\{y \in B(x, r) \cap \Omega : \delta_\Omega(y) \leq \rho\}) \simeq \left(\frac{\rho}{r}\right)^\beta m(B(x, r) \cap \Omega) \quad (3)$$

whenever $x \in \partial\Omega$ and $0 < \rho \leq r \leq \text{diam}(\Omega)$.

By [BS, Lemma 2.7], if $\Omega \subset \mathbb{R}^n$ is a bounded domain that satisfies the above local β -shell condition for some $\beta > 0$, then for all $\alpha > -\beta$ the measure $d\mu_\alpha(y) = \delta_\Omega(y)^\alpha dm(y)$ is doubling on Ω . Combining this with [BS, Theorem 4.4] and noting that the Newton-Sobolev space discussed there is the same as the standard Sobolev space $W^{1,p}(\mathbb{R}^2)$ in our setting (see for example [H-]) tells us that when Ω is the von Koch snowflake domain,

the metric measure space (Ω, d, μ_α) is doubling and supports a 1-Poincaré inequality. Here d is the Euclidean metric. Moreover, with $\nu = \mathcal{H}^Q|_K$ where Q is the Hausdorff dimension of K , we would also have that $\nu(B(x, r)) \simeq r^Q$ whenever $x \in K$ and $0 < r \leq 10$.

Lemma 4.3. *Suppose that $\Omega \subset \mathbb{R}^n$ a bounded domain that satisfies a strong local β -shell condition for some $\beta > 0$ and that $K = \partial\Omega$ is Ahlfors Q -regular for some $n > Q > 0$. For $-\beta < \alpha \leq 0$ we set μ_α to be the measure on Ω given by $d\mu_\alpha(y) = \delta_\Omega(y)^\alpha dm(y)$. Moreover, assume that for each $x \in K$ and $0 < r \leq 10$ we have $m(B(x, r) \cap \Omega) \simeq r^n$. Then whenever $0 < r \leq 10$ and $x \in K$, we have*

$$\mu_\alpha(B(x, r) \cap \Omega) \simeq r^{n+\alpha-Q} \mathcal{H}^Q(B(x, r) \cap K).$$

Proof. Fix $x \in K$ and $0 < r \leq 10$, and without loss of generality assume that $\alpha < 0$. Then by the Cavalieri principle,

$$\begin{aligned} \mu_\alpha(B(x, r) \cap \Omega) &= \int_{B(x, r) \cap \Omega} \frac{1}{\delta_\Omega(y)^{|\alpha|}} dm(y) \\ &= \int_0^\infty m(\{y \in B(x, r) \cap \Omega : \delta_\Omega(y)^{-|\alpha|} > t\}) dt \\ &\simeq \int_0^M Cr^n dt + \int_M^\infty m(\{y \in B(x, r) \cap \Omega : \delta_\Omega(y) < t^{-1/|\alpha|}\}) dt \end{aligned}$$

where $M = r^{-|\alpha|} > 0$. Note that as $\beta < \alpha < 0$, we have $\beta/|\alpha| > 1$. Using $M = r^{-|\alpha|}$ and the local shell property, we obtain

$$\begin{aligned} \mu_\alpha(B(x, r) \cap \Omega) &\simeq r^{n-|\alpha|} + \int_M^\infty \left(\frac{1}{t^{1/|\alpha|r}} \right)^\beta r^n dt \\ &\simeq r^{n+\alpha} + r^{n-\beta} M^{1-\beta/|\alpha|} \simeq r^{n+\alpha}. \end{aligned}$$

Since $\mathcal{H}^Q(B(x, r) \cap K) \simeq r^Q$, the conclusion follows. \square

Hence if the von Koch snowflake domain satisfies a strong local β -shell condition, then (Ω, μ_α) is doubling and supports a 1-Poincaré inequality when $\alpha > -\beta$, and in addition from Lemma 4.3 we know that $\nu = \mathcal{H}^Q|_K$ is $2 + \alpha - Q$ -codimension regular with respect to μ_α , and so by [Ma, Theorem 1.1] the conclusions of Theorem 1.1 and Theorem 1.2 hold for the von Koch domain and its boundary.

In light of the above discussion, it only remains to verify the strong local β -shell condition for Ω for the choice of $0 < \beta = \beta_0 := 2 - \frac{\log(4)}{\log(3)} = 2 - Q$.

For each non-negative integer n , let K_n denote the n -th iteration of the von Koch snowflake, so K_n consists of $3 \cdot 4^n$ line segments of length 3^{-n} . Let $x \in \bar{\Omega}$, $0 < r < \frac{1}{2}$, and choose a non-negative integer k such that $3^{-k-1} \leq 2r < 3^{-k}$. For $0 < \rho < r$ choose a non-negative integer j such that $3^{-k-j-1} \leq \rho < 3^{-k-j}$, and so $\rho \simeq 3^{-j}r$.

There is a constant M , independent of x and r , such that the number of the line segments in K_{k+1} intersecting $B(x, 2r)$ is at most M , for the set of endpoints of the segments of K_{k+1} are 3^{-k-1} -separated and $r \simeq 3^{-k-1}$.

For $m \in \mathbb{N}$, let A_m be the bounded component of $\mathbb{R}^2 \setminus K_m$. Clearly $A_m \subset A_{m+1}$. Moreover,

$$m(\{y \in \Omega : \delta_\Omega(y) \leq \rho\} \cap B(x, r)) \simeq m(\{y \in A_{k+j+1} : \delta_\Omega(y) \leq \rho\} \cap B(x, r)).$$

Also, each line segment L of length 3^{-k-1} that makes up the construction of K_{k+1} is modified j times to obtain the set K_{k+j+1} by replacing L with 4^j number of line segments, each of length 3^{-j-k-1} . If ℓ is one of these line segments, then

$$m(\{y \in A_{k+j+1} : \text{dist}(y, \ell) \leq \delta_\Omega(y) \leq \rho\}) \simeq \rho^2,$$

and therefore

$$m(\{y \in A_{k+j+1} : \delta_\Omega(y) \leq \rho\} \cap B(x, r)) \lesssim M \times 4^j \times \rho^2,$$

with \lesssim actually being \simeq if $x \in K$. From the fact that $\rho \simeq 3^{-j}r$, it follows that

$$m(\{y \in \Omega : \delta_\Omega(y) \leq \rho\} \cap B(x, r)) \lesssim \left(\frac{4}{9}\right)^j r^2 \simeq \left(\frac{4}{9}\right)^j m(B(x, r) \cap \Omega),$$

again with \lesssim actually being \simeq if $x \in K$. Set $\beta_0 = 2 - \frac{\log(4)}{\log(3)} = 2 - Q$ and observe that $\rho/r \simeq 3^{-j}$. Then we have that

$$\left(\frac{4}{9}\right)^j = (3^{-j})^{\beta_0} \simeq \left(\frac{\rho}{r}\right)^{\beta_0}$$

as desired, proving that the snowflake domain satisfies the strong β_0 -shell condition.

5. TRACE OF WEIGHTED SOBOLEV FUNCTIONS ARE IN BESOV SPACES, OR HOW THE SURROUNDING LEAVES A MARK ON SUBSETS

The goal of this section is to study trace of $N^{1,p}(B, \mu_\alpha)$ on the set E and relate it to the Besov classes $B_{p,p}^\theta(E, \nu)$ and prove Theorem 1.1. We recall the setting considered here (see the introduction for more on this).

With $0 < Q < n$, let $E \subset \mathbb{R}^n$ be an Ahlfors Q -regular compact set with $\text{diam}(E) \leq 1$. Let B be a ball in \mathbb{R}^n such that $E \subset \frac{1}{2}B$. We also assume that for each $\alpha \leq 0$ there is a measure μ_α on B such that whenever $\alpha + n - Q > 0$ and $x \in B$, and $0 < r < 2$ such that $r > \text{dist}(x, E)/9$, the comparison $\mu_\alpha(B(x, r)) \simeq r^{n+\alpha}$ holds. We also assume that the ball B , equipped with the Euclidean metric d and the measure μ_α , is doubling and supports a q -Poincaré inequality for each $q > 1$.

Lemma 5.1. *Suppose that B is a ball in \mathbb{R}^n such that $E \subset \frac{1}{2}B$, where E is compact and is Ahlfors Q -regular. Let μ_α be as in (1) such that $\nu = \mathcal{H}^Q|_E$ is $\alpha + n - Q$ -codimensional with respect to μ_α . Suppose that $\gamma > 0$ such*

that $\gamma \geq \alpha + n - Q$. Then there is a constant $C > 0$ such that whenever $h \in L^1_{loc}(\mathbb{R}^n, \mu_\alpha)$, we have for all $t > 0$,

$$\nu(\{M_\gamma h > t\}) \leq \frac{C}{t} \int_B h d\mu_\alpha,$$

where M_γ is the fractional Hardy-Littlewood maximal function operator given by

$$M_\gamma h(x) = \sup_{\text{rad}(B) \leq 1, x \in B} \text{rad}(B)^\gamma \int_B h d\mu_\alpha.$$

Proof. This is a variant of the standard proof, the variant being that the measure with respect to which the maximal operator functions, μ_α , is not the same as the measure ν with respect to which the superlevel sets are measured. For this reason we give the complete proof here.

Let $t > 0$ and set $E_t := \{x \in E : M_\gamma h(x) > t\}$. Then for each $x \in E_t$ there is a ball B_x of radius $r_x > 0$ such that $x \in B_x$ and

$$r_x^\gamma \int_{B_x} h d\mu_\alpha > t.$$

It follows that

$$\nu(E \cap B_x) \simeq r_x^Q < \frac{r_x^{\gamma+Q-(\alpha+n)}}{t} \int_{B_x} h d\mu_\alpha.$$

Recalling that $\alpha + n > Q$, we set $\eta = \gamma - (\alpha + n - Q)$. Then $\eta \geq 0$. The balls B_x , $x \in E_t$, cover E_t . Therefore, by the 5-covering theorem (which is applicable here to E as ν is doubling on E), we obtain a pairwise disjoint countable subfamily of balls B_i with radii r_i such that $E_t \subset \bigcup_i 5B_i$. Then

$$\begin{aligned} \nu(E_t) &\leq \sum_i \nu(5B_i) \leq C \sum_i r_i^Q \leq \frac{C}{t} \sum_i r_i^\eta \int_{B_i} h d\mu_\alpha \leq \frac{C}{t} \sum_i \int_{B_i} h d\mu_\alpha \\ &\leq \frac{C}{t} \int_B h d\mu_\alpha, \end{aligned}$$

where we have used the facts that $r_i \leq 1$, $\eta \geq 0$, and that the balls B_i are pairwise disjoint. \square

Lemma 5.2. *Suppose that $0 \leq g \in L^p(B, \mu_\alpha)$ where B is the ball as in Lemma 5.1 and $1 < p < \infty$. Fix $1 \leq q < p$. Then*

$$\int_E (M_\gamma(g^q))^{p/q} d\nu \leq C \int_B g^p d\mu_\alpha.$$

Proof. Recall from the Cavalieri principle that

$$\int_E (M_\gamma(g^q))^{p/q} d\nu = \frac{p}{q} \int_0^\infty t^{\frac{p}{q}-1} \nu(\{M_\gamma g^q > t\}) dt.$$

For $t > 0$, we can write $g^q = G_1 + G_2$, where

$$G_1 = g^q \chi_{\{g^q \leq t/2\}}, \quad G_2 = g^q \chi_{\{g^q > t/2\}}.$$

Then

$$M_\gamma g^q \leq M_\gamma G_1 + M_\gamma G_2 \leq \frac{t}{2} + M_\gamma G_2.$$

As $M_\gamma g^q(z) > t$ when $z \in E_t$, it follows that $\{M_\gamma g^q > t\} \subset \{M_\gamma G_2 > t/2\}$. Hence by Lemma 5.1,

$$\begin{aligned} \int_E (M_\gamma g^q)^{p/q} d\nu &\leq \frac{p}{q} \int_0^\infty t^{\frac{p}{q}-1} \nu(\{M_\gamma G_2 > t/2\}) dt \\ &\leq C \int_0^\infty t^{\frac{p}{q}-2} \int_B G_2 d\mu_\alpha dt \\ &= C \int_0^\infty t^{\frac{p}{q}-2} \int_{B \cap \{g^q > t/2\}} g^q d\mu_\alpha dt \\ &= \int_0^\infty t^{\frac{p}{q}-2} \left[\frac{t}{2} \mu_\alpha(B \cap \{g^q > t/2\}) + \int_{t/2}^\infty \mu_\alpha(B \cap \{g^q > s\}) ds \right] dt \\ &= C_1 \int_0^\infty (t/2)^{\frac{p}{q}-1} \mu_\alpha(B \cap \{g^q > t/2\}) dt \\ &\quad + C \int_0^\infty \int_0^\infty t^{\frac{p}{q}-2} \chi_{(t/2, \infty)}(s) \mu_\alpha(B \cap \{g^q > s\}) ds dt \\ &= C_2 \int_B g^p d\mu_\alpha \\ &\quad + C \int_0^\infty \left(\int_0^\infty t^{\frac{p}{q}-2} \chi_{(0, 2s)}(t) dt \right) \mu_\alpha(B \cap \{g^q > s\}) ds \\ &= C_2 \int_B g^p d\mu_\alpha + C_3 \int_0^\infty s^{\frac{p}{q}-1} \mu_\alpha(B \cap \{g^q > s\}) ds, \end{aligned}$$

where we also used the Cavalieri principle and Tonelli's theorem in obtaining the last few lines above. By the Cavalieri principle again, we obtain the desired result. \square

Now we are ready to prove Theorem 1.1. For the convenience of the reader, we state an expanded version of this theorem now.

Theorem 5.3. *Let E be an Ahlfors Q -regular compact subset of $\frac{1}{2}B$ where B is a ball in \mathbb{R}^2 . Let $p > 1$ and $0 < \theta < 1$ be such that $p\theta < 1$. Let $\alpha \leq 0$ be such that $\alpha + n - Q > 0$ and $\theta < 1 - \frac{\alpha + n - Q}{p}$. Then there exists $C \geq 1$ and a linear trace operator*

$$T : N^{1,p}(B, \mu_\alpha) \rightarrow B_{p,p}^\theta(E, \nu)$$

with

$$\|Tu\|_{B_{p,p}^\theta(E, \nu)} \leq C \|\nabla u\|_{L^p(B, \mu_\alpha)}$$

and

$$\|Tu\|_{L^p(E, \nu)}^p \leq C \|u\|_{N^{1,p}(B, \mu_\alpha)}^p.$$

Moreover, if $u \in N^{1,p}(B, \mu_\alpha)$ is Lipschitz continuous in a neighborhood of E , then $Tu = u|_E$.

Note that if $p\theta < 1$, then we can always choose $\alpha \leq 0$ satisfying the hypotheses of the above theorem. Moreover, if we only know that there is a fixed $\alpha > Q - n$ such that μ_α is doubling and supports a p -Poincaré inequality for some $p > 1$, then the conclusion of the above theorem holds true as long as there exists $1 \leq q < p$ such that μ_α supports a q -Poincaré inequality and $\alpha + n - Q < q(1 - \theta)$. The support of a q -Poincaré inequality for some $1 \leq q < p$ is guaranteed by the self-improvement property of Poincaré inequality, see [KZ, H-].

Proof. We first prove the above claim for Lipschitz functions in $N^{1,p}(B, \mu_\alpha)$. As μ_α is doubling and supports a p -Poincaré inequality, we know that Lipschitz functions are dense in $N^{1,p}(B, \mu_\alpha)$, and hence we get the estimates for all functions in $N^{1,p}(B, \mu_\alpha)$. So, in the following we will assume that u is Lipschitz continuous. It follows that every point in B is a μ_α -Lebesgue point of u . We denote $g_u := |\nabla u|$.

Let $x, y \in E$ and set $B_0 = B(x, 2d(x, y))$, and for positive integers k we set $B_k = B(x, 2^{1-k}d(x, y))$ and $B_{-k} = B(y, 2^{1-k}d(x, y))$. We also set $r_k = \text{rad}(B_k)$ for $k \in \mathbb{Z}$. Then by the following standard telescoping argument and by the q -Poincaré inequality, we obtain

$$\begin{aligned} |u(y) - u(x)| &\leq \sum_{k \in \mathbb{Z}} |u_{B_k} - u_{B_{k+1}}| \\ &\leq C \sum_{k \in \mathbb{Z}} \int_{2B_k} |u - u_{2B_k}| d\mu_\alpha \\ &\leq C \sum_{k \in \mathbb{Z}} r_k \left(\int_{2B_k} g_u^q d\mu_\alpha \right)^{1/q} \\ &= C \sum_{k \in \mathbb{Z}} r_k^{1-\gamma/q} \left(r_k^\gamma \int_{2B_k} g_u^q d\mu_\alpha \right)^{1/q} \\ &= C d(x, y)^{1-\gamma/q} \sum_{k \in \mathbb{Z}} 2^{-|k|(1-\gamma/q)} \left(r_k^\gamma \int_{2B_k} g_u^q d\mu_\alpha \right)^{1/q}. \end{aligned}$$

By assumption, we have $p\theta < 1$. Hence we can choose α with $Q - 2 < \alpha \leq 0$ such that $p\theta < p - (\alpha + n - Q)$. Since $\alpha + n - Q > 0$, we can choose $\gamma = \alpha + n - Q$ in the above. Then the condition on α as described above reads as $p\theta < p - \gamma$, and so $0 < \theta < 1 - \gamma/p$. Hence we can choose $1 < q < p$ such that $\theta < 1 - \gamma/q$, whence by this choice of q we have that $1 - \gamma/q > 0$. It follows that $\sum_{k \in \mathbb{Z}} 2^{-|k|(1-\gamma/q)} < \infty$, and so

$$|u(y) - u(x)| \leq C d(x, y)^{1-\gamma/q} \left[M_\gamma g_u^q(x)^{1/q} + M_\gamma g_u^q(y)^{1/q} \right].$$

Hence, for $0 < \theta < 1$, we obtain

$$\begin{aligned} \frac{|u(y) - u(x)|^p}{d(x, y)^{\theta p \nu(B(x, d(x, y)))}} &\simeq \frac{|u(y) - u(x)|^p}{d(x, y)^{Q + \theta p}} \\ &\leq C d(x, y)^{p - \frac{\gamma}{q} p - \theta p - Q} \left[M_\gamma g_u^q(x)^{p/q} + M_\gamma g_u^q(y)^{p/q} \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u\|_{B_{p,p}^\theta(E,\nu)}^p &\leq C \int_E \int_E d(x, y)^{p - \frac{\gamma}{q} p - \theta p - Q} \left[M_\gamma g_u^q(x)^{p/q} + M_\gamma g_u^q(y)^{p/q} \right] d\nu(x) d\nu(y) \\ &= C [I_1 + I_2], \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_E \int_E d(x, y)^{p - \frac{\gamma}{q} p - \theta p - Q} M_\gamma g_u^q(x)^{p/q} d\nu(x) d\nu(y) \\ I_2 &= \int_E \int_E d(x, y)^{p - \frac{\gamma}{q} p - \theta p - Q} M_\gamma g_u^q(y)^{p/q} d\nu(x) d\nu(y). \end{aligned}$$

Thanks to Tonelli's theorem, any estimate we obtain for I_1 is valid also for I_2 , so we consider I_1 only next. Note that for $x \in E$,

$$\begin{aligned} \int_E d(x, y)^{p - \frac{\gamma}{q} p - \theta p - Q} d\nu(y) &= \sum_{j=0}^{\infty} \int_{B(x, 2^{-j}) \setminus B(x, 2^{-j-1})} d(x, y)^{p - \frac{\gamma}{q} p - \theta p - Q} d\nu(y) \\ &\leq C \sum_{j=0}^{\infty} 2^{-j[p - \frac{\gamma}{q} p - \theta p - Q]} 2^{-jQ} = C \sum_{j=0}^{\infty} 2^{-jp[1 - \frac{\gamma}{q} - \theta]}. \end{aligned}$$

As we had chosen $1 < q < p$ such that $\theta < 1 - \gamma/q$, it follows that the above series is finite. Then we get from Lemma 5.2 that

$$I_1 \leq C \int_E M_\gamma g_u^q(x)^{p/q} d\nu(x) \leq C \int_B g_u^p d\mu_\alpha.$$

It then follows that

$$\|u\|_{B_{p,p}^\theta(E,\nu)} \leq C \|g_u\|_{L^p(B,\mu_\alpha)} = C \|\nabla u\|_{L^p(B,\mu_\alpha)}.$$

In the above computation, if we fix $x \in E$ to be a μ_α -Lebesgue point of u and consider only the balls $B_k = B(x, 2^{-k})$ for $k \geq 0$, then we obtain

$$\begin{aligned} |u(x)| &\leq |u(x) - u_{B_0}| + |u_{B_0}| \leq \sum_{k=0}^{\infty} |u_{B_k} - u_{B_{k+1}}| + C \int_B |u| d\mu_\alpha \\ &\leq \sum_{k=0}^{\infty} r_k \left(\int_{2B_k} g_u^q d\mu_\alpha \right)^{1/q} + C \left(\int_B |u|^p d\mu_\alpha \right)^{1/p} \\ &\leq C \sum_{k=0}^{\infty} r_k^{1 - \gamma/q} M_\gamma g_u^q(x)^{1/q} + C \left(\int_B |u|^p d\mu_\alpha \right)^{1/p} \\ &= C M_\gamma g_u^q(x)^{1/q} + C \left(\int_B |u|^p d\mu_\alpha \right)^{1/p}. \end{aligned}$$

Therefore

$$|u(x)|^p \leq CM_\gamma g_u^q(x)^{p/q} + C \int_B |u|^p d\mu_\alpha.$$

Integrating the above over E with respect to the measure ν and applying Lemma 5.2 we obtain

$$\|u\|_{L^p(E,\nu)}^p \leq C \int_B g_u^p d\mu_\alpha + C_B \int_B |u|^p d\mu_\alpha,$$

from which the claim now follows. \square

6. EXTENSION OF BESOV FUNCTIONS ARE IN SOBOLEV SPACES, OR HOW SUBSETS INFLUENCE THEIR SURROUNDINGS

In this section we show that we can extend functions from the Besov class for E to Newtonian class for $(\mathbb{R}^2, \mu_\alpha)$ by proving Theorem 1.2.

Since E is compact and $E \subset \frac{1}{2}B$, we can construct a Whitney cover $B_{i,j}$, $i \in \mathbb{N}$ and $j = 1, \dots, M_i$, of $B \setminus E$. Such a cover is described in [HM, Section 2] and in [H-, Proposition 4.1.15], where the construction did not need the open set Ω to be connected, and so their construction is available in our setting as well. We can ensure that with $B_{i,j} = B(x_{i,j}, r_{i,j})$ we have $r_{i,j} = \text{dist}(x_{i,j}, E) = 2^{-i}$ and that for each $T \geq 1$ there exists $N_T \in \mathbb{N}$ such that for each $i \in \mathbb{N}$,

$$\sum_{j=1}^{M_i} \chi_{TB_{i,j}} \leq N_T. \quad (4)$$

Let $\varphi_{i,j}$ be a Lipschitz partition of unity subordinate to the cover $B_{i,j}$, that is, each $\varphi_{i,j}$ is $2^i C$ -Lipschitz continuous, $0 \leq \varphi_{i,j} \leq 1$, $\text{supp}(\varphi_{i,j}) \subset 2B_{i,j}$ and

$$\sum_{i,j} \varphi_{i,j} = \chi_{B \setminus E}.$$

Moreover, there exist $N_1, N_2 \in \mathbb{N}$ such that if $2B_{i,j}$ and $2B_{m,n}$ intersect, then $|i - m| < N_1$ and there are at most N_2 balls $B_{m,n}$ satisfying the above when i, j are fixed. For the convenience of the reader, we state an expanded version of Theorem 1.2 below.

Theorem 6.1. *With $E, B, \nu = \mathcal{H}^Q|_E$ and $0 < Q < 2$ as above, let $p > 1$ and $0 < \theta < 1$. We fix $\alpha \leq 0$ such that $\alpha + n - Q > 0$ and $\theta \geq 1 - \frac{\alpha + n - Q}{p}$. Then there is a constant $C \geq 1$ and a linear extension operator*

$$S : B_{p,p}^\theta(E, \nu) \rightarrow N^{1,p}(B, \mu_\alpha)$$

such that

$$\int_B |\nabla Su|^p d\mu_\alpha \leq C \|u\|_{B_{p,p}^\theta(E,\nu)}^p, \quad \int_B |Su|^p d\mu_\alpha \leq C \int_E |u|^p d\nu.$$

Moreover, if u is L -Lipschitz on E , then Su is CL -Lipschitz on B .

Note that in the trace theorem, Theorem 5.3, we were not able to gain control of $\int_E |Tu|^p d\nu$ solely in terms of $\int_B |u|^p d\mu_\alpha$. The above extension theorem however does allow us these separate controls.

Proof. From [B2S, Proposition 13.4] we know that Lipschitz functions are dense in $B_{p,p}^\theta(E, \nu)$ for each $0 < \theta < 1$ and $p \geq 1$. We fix our attention on $p > 1$ and $0 < \theta < 1$. We will first extend Lipschitz functions in $B_{p,p}^\theta(E, \nu)$ to $N^{1,p}(B, \mu_\alpha)$ and use this to conclude that every function in $B_{p,p}^\theta(E, \nu)$ has an extension lying in $N^{1,p}(B, \mu_\alpha)$. To this end, let $u \in B_{p,p}^\theta(E, \nu)$ be Lipschitz continuous, and for $x \in B \setminus E$ we set

$$Su(x) = \sum_{i,j} u_{2B_{i,j}} \varphi_{i,j}(x),$$

where $u_{2B_{i,j}} := \int_{2B_{i,j}} u d\nu$. We extend Su to E by setting $Su(x) = u(x)$ when $x \in E$. If $x \in E$ and $y \in B_{i_0, j_0}$, then

$$\begin{aligned} |Su(y) - u(x)| &= \left| \sum_{i,j} [u_{2B_{i,j}} - u(x)] \varphi_{i,j}(y) \right| \\ &\leq \sum_{i,j} \varphi_{i,j}(y) \int_{2B_{i,j}} |u(w) - u(x)| d\nu(w) \\ &\leq \sum_{i,j} \varphi_{i,j}(y) L 2^{1-i} \leq CL d(y, x). \end{aligned}$$

It follows that for each $x \in E$, we have

$$\limsup_{r \rightarrow 0^+} \int_{B(x,r) \setminus E} |Su(y) - u(x)|^q d\mu_\alpha(y) = 0$$

for all $1 \leq q < \infty$.

If $x, y \in B_{i_0, j_0}$, then by the properties of the Whitney cover listed above,

$$\begin{aligned} |Su(y) - Su(x)| &= \left| \sum_{i,j} [u_{2B_{i,j}} - u_{2B_{i_0, j_0}}] [\varphi_{i,j}(y) - \varphi_{i,j}(x)] \right| \\ &\leq \frac{C d(y, x)}{r_{i_0}} \sum_{i,j; 2B_{i,j} \cap B_{i_0, j_0} \neq \emptyset} |u_{2B_{i,j}} - u_{2B_{i_0, j_0}}| \\ &\leq \frac{C d(y, x)}{r_{i_0}} \sum_{i,j; 2B_{i,j} \cap B_{i_0, j_0} \neq \emptyset} \int_{2B_{i,j}} \int_{2B_{i_0, j_0}} |u(w) - u(v)| d\nu(v) d\nu(w) \\ &\leq \frac{C d(y, x)}{r_{i_0}} \int_{CB_{i_0, j_0}} \int_{CB_{i_0, j_0}} |u(w) - u(v)| d\nu(v) d\nu(w) \quad (5) \\ &\leq \frac{C d(y, x)}{r_{i_0}} 2CL r_{i_0} \leq CL d(x, y). \quad (6) \end{aligned}$$

It follows that Su is CL -Lipschitz on B and hence is in $N^{1,p}(B, \mu_\alpha)$. It now only remains to obtain norm bounds.

Recall from [GKS, Theorem 5.2 and equation (5.1)] that

$$\|u\|_{B_{p,p}^\theta(E,\nu)}^p \simeq \sum_{n=0}^{\infty} \int_E \int_{B(x,2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} d\nu(y) d\nu(x). \quad (7)$$

As E is Ahlfors Q regular for some $Q < 2$, it follows that $\mathcal{H}^2(E) = 0$, and hence $\mu_\alpha(E) = 0$. Let $z \in B_{i_0,j_0}$. Setting $x = z$ and letting $y \rightarrow z$, by applying the Hölder inequality to (5) we have that

$$\begin{aligned} \text{Lip } Su(z)^p &= \left(\limsup_{y \rightarrow z} \frac{|Su(y) - Su(z)|}{d(y,z)} \right)^p \\ &\leq \frac{C}{r_{i_0}^p} \int_{CB_{i_0,j_0}} \int_{CB_{i_0,j_0}} |u(w) - u(v)|^p d\nu(v) d\nu(w) \\ &= \frac{C}{2^{-i_0(1-\theta)p}} \int_{CB_{i_0,j_0}} \int_{CB_{i_0,j_0}} \frac{|u(w) - u(v)|^p}{2^{-i_0\theta p}} d\nu(w) d\nu(v) \\ &\leq \frac{C}{2^{-i_0(Q+(1-\theta)p)}} \int_{CB_{i_0,j_0}} \int_{B(v,2^{k_0-i_0})} \frac{|u(w) - u(v)|^p}{2^{-i_0\theta p}} d\nu(w) d\nu(v), \end{aligned}$$

where k_0 is the smallest positive integer such that $2^{k_0} \geq 2C$; note that k_0 is independent of i_0, j_0, v . Here we also used the fact that $r_{i_0} \simeq \text{dist}(z, E) \simeq 2^{-i_0}$. Integrating the above over B_{i_0,j_0} , we obtain

$$\begin{aligned} \int_{B_{i_0,j_0}} \text{Lip } Su(z)^p d\mu_\alpha(z) &= \int_{B_{i_0,j_0}} |\nabla Su(z)|^p d\mu_\alpha(z) \\ &\leq C 2^{-i_0(\alpha+2-Q-(1-\theta)p)} \int_{CB_{i_0,j_0}} \int_{B(v,2^{k_0-i_0})} \frac{|u(w) - u(v)|^p}{2^{-i_0\theta p}} d\nu(w) d\nu(v). \end{aligned}$$

Summing the above over $j_0 = 1, \dots, M_{i_0}$ and noting by (4) that $\sum_{j=1}^{M_{j_0}} \chi_{CB_{i_0,j}} \leq N_C$ with $E \subset \bigcup_{j=1}^{M_{i_0}} CB_{i_0,j}$, and then summing over i_0 , we obtain

$$\begin{aligned} &\int_B \text{Lip } Su(z)^p d\mu_\alpha(z) \\ &\leq C \sum_{i_0=0}^{\infty} 2^{-i_0(\alpha+2-Q-(1-\theta)p)} \int_E \int_{B(v,2^{k_0-i_0})} \frac{|u(w) - u(v)|^p}{2^{-i_0\theta p}} d\nu(w) d\nu(v) \\ &\leq \sum_{i_0=0}^{\infty} \int_E \int_{B(v,2^{k_0-i_0})} \frac{|u(w) - u(v)|^p}{2^{-i_0\theta p}} d\nu(w) d\nu(v) \end{aligned}$$

provided that $\alpha + n - Q - (1 - \theta)p \geq 0$. So if $\alpha \leq 0$ is chosen such that $\alpha + n - Q > 0$ and

$$\theta \geq 1 - \frac{\alpha + n - Q}{p},$$

then by (7) we have that

$$\int_B |\nabla Su|^p d\mu_\alpha = \int_B \text{Lip } Su^p d\mu_\alpha \leq C \|u\|_{B_{p,p}^\theta(E,\nu)}^p.$$

To complete the argument, we next obtain control of $\|Su\|_{L^p(B, \mu_\alpha)}$. For $x \in B_{i_0, j_0}$,

$$\begin{aligned} |Su(x)| &= \left| \sum_{i,j: 2B_{i,j} \cap B_{i_0, j_0} \neq \emptyset} \varphi_{i,j}(x) \int_{2B_{i,j}} u(y) d\nu(y) \right| \\ &\leq C \int_{C_0 B_{i_0, j_0}} |u(y)| d\nu(y) \\ &\leq C \left(\int_{C_0 B_{i_0, j_0}} |u(y)|^p d\nu(y) \right)^{1/p}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{B_{i_0, j_0}} |Su(x)|^p d\mu_\alpha(x) &\leq C \mu_\alpha(B_{i_0, j_0}) \int_{C_0 B_{i_0, j_0}} |u(y)|^p d\nu(y) \\ &\leq C 2^{-i_0(\alpha+n-Q)} \int_{C_0 B_{i_0, j_0}} |u(y)|^p d\nu(y). \end{aligned}$$

As before, summing over $j_0 = 1, \dots, M_{i_0}$ and then over i_0 gives

$$\int_B |Su(x)|^p d\mu_\alpha(x) \leq C \sum_{i_0=0}^{\infty} 2^{-i_0(\alpha+n-Q)} \int_E |u(y)|^p d\nu(y).$$

As $\alpha + n - Q > 0$, it follows that

$$\int_B |Su(x)|^p d\mu_\alpha(x) \leq C \int_E |u(y)|^p d\nu(y)$$

as desired. □

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