ON A DISCONTINUOUS NONLOCAL CONSERVATION LAW MODELING MATERIAL FLOW ON CONVEYOR BELTS

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ABSTRACT. We prove existence of solutions to a conservation law with nonlocal discontinuous flux modeling material flow on a conveyor belt. The discontinuity is with respect to the unknown function and arises in a dynamic velocity field which is active only at high densities and takes into account the effect of colliding parts though the nonlocal operator. The strategy of the proof is based on the vanishing viscosity method. We smooth the discontinuity and add a second-order regularizing term. The key tools used to establish the convergence of the sequence of solutions of the approximating problems are a BV estimate "away from the discontinuity", a suitable application of Murat's compact embedding, and a diagonal argument.

1. INTRODUCTION

1.1. **A non-local model for material flow on conveyor belts.** We consider the problem of modeling material flow on conveyor belts by nonlocal conservation laws with a discontinuity in the flux function, as suggested in [35, 24]. In one spatial dimension, the model leads to the following conservation law:

$$\begin{cases} \partial_t\rho + \partial_x(\rho v(x) - \alpha\rho H(\rho-\rho_c)f(K' * \rho)) = 0, & t>0,\ x\in\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where we assume

$$v\in C^2(\mathbb{R})\cap W^{2,\infty}(\mathbb{R}),\quad K\in C^3(\mathbb{R})\cap W^{3,\infty}(\mathbb{R}),\quad \rho_c,\ \alpha>0, \tag{1.2}$$

$$\rho_0\in L^1(\mathbb{R})\cap BV(\mathbb{R}),\quad \rho_0\geq 0, \tag{1.3}$$

H denotes the *Heaviside function*, i.e.

$$H(\xi) = \begin{cases} 1 & \text{if } \xi>0, \\ 0 & \text{if } \xi\leq 0, \end{cases} \tag{1.4}$$

and

$$f(\xi) = \frac{\xi}{\sqrt{1+\xi^2}}, \qquad \xi\in\mathbb{R}. \tag{1.5}$$

The function $\rho:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is the density of the transported parts, v is the velocity field induced by the conveyor belt (which is constant in time); the term

$$-\alpha H(\rho-\rho_c)f(K'*\rho)$$

2010 *Mathematics Subject Classification.* 35L65.
Key words and phrases. Conservation laws, discontinuous flux, nonlocal flux, material flow, vanishing viscosity.

represents a dynamic velocity field which is active only at high densities and accounts the effect of colliding parts though the nonlocal operator

$$I[\rho] := f(K' * \rho) = \alpha \frac{K' * \rho}{\sqrt{1 + |K' * \rho|^2}}.$$

pushing the mass towards lower density regions. This collision operator was introduced in [17], where the authors studied the well-posedness of the model assuming $H \equiv 1$.

More recently, in [35], the authors smoothed the discontinuity and were able to apply classical methods to obtain existence and uniqueness of solutions. In particular, they proved the convergence of a finite volume approximation scheme to the entropy solution and L^1-Lipschitz continuous dependence on the initial data, thus ensuring the uniqueness. Although a uniform L^∞ bound on the solution holds with regard to this smoothing, the question whether one can define and obtain solutions for the discontinuous problem itself has not yet been answered. This is what we address in the present contribution: we extend the results in [35], by proving the existence of (a suitable notion of) solutions to the discontinuous problem (1.1).

1.2. **Conservation laws with nonlocal discontinuous flux in the literature.** Nonlocal conservation laws have been intensively studied because of their applicability in various fields, e.g. pedestrian and traffic flow modeling [17, 6, 30], sedimentation [5], slow erosion of granular matter [2, 16], and opinion formation [34, 29].

Existence and uniqueness of solutions for rather general classes of nonlocal conservation laws has been proved in several papers. In the L^p setting, classical results rely on numerical methods (e.g. Lax-Friedrichs', see [6, 5]) or on the vanishing viscosity technique (see [16, 12]) and on prescribing an entropy condition to ensure uniqueness; more recently, it was possible to establish existence and uniqueness of weak solutions (see [28, 29, 30]) via fixed-point methods and without needing an entropy condition. For certain types of smooth kernels, there are well-posedness results even in the measure-valued setting (see [18]).

However, so far nonlocal conservation laws with discontinuities have only been considered in the recent paper [12], where the authors study the well-posedness of solutions for a (single) discontinuity in the spatial variable, and in [3], which deals with another class of nonlocal conservation laws with flux being discontinuous in the unknown arising from a gradient-constraint problem.

Local conservation laws with discontinuities, on the other hand, have been subject to more studies over the last decades. The literature on the case of a flux being discontinuous with respect to the unknown is still rather limited though. In the series of papers [21, 20, 19], the authors considered a particular discontinuous flux function that can be associated to the limit case of a phase transition model. They proved the existence of weak solutions (in a suitable sense) by using the vanishing viscosity method. They also provided a definition of entropy weak solution and proved an estimate based on Kružkov's techniques, but ultimately gave a counterexample to uniqueness. They also computed the solution to the associated Riemann problem by a Lax-Friedrichs approximation and provided some applications to a class of discontinuous p-systems.

In [11], Carrillo studied the initial boundary value problem for a first order quasilinear equation $\partial_t u + \operatorname{div} \phi(u) \ni f$ in some bounded domain $\Omega \subset \mathbb{R}^N$, with zero boundary data. The flux vector $\phi(u)$ is supposed to have finite number of jump discontinuities. The author introduced the notion of entropy solution and, using the nonlinear semigroups theory, established existence and uniqueness results. In [31], the same problem was also investigated numerically.

In [9, 23, 8, 7], the authors carried out a systematic study of conservation laws with discontinuities in the unknown. To handle the discontinuities, they work in the framework of re-parametrization of the flux and the source functions – which is somewhat inspired by the key observation for the proofs in [11].

On the other hand, conservation laws with discontinuities with respect to the space or time variable have received more attention – mostly motivated by spatially heterogeneous physical models: for example, traffic flow model with a discontinuously varying road surface (see [32, 27, 1]). See also [10] and references therein for a more detailed review of the literature on this topic.

1.3. **Outline of the paper.** The paper is organized as follows. In Section 2, we introduce notions of weak and entropy solutions to problem (1.1) which take into account the discontinuity of the flux (inspired by the contributions about local conservation laws contained in [21, 11]).

Our main result is that such solutions exist; however, we are not able to prove a uniqueness theorem. The existence proof relies on the vanishing viscosity technique: we smooth the Heaviside function and add a second-order regularizing term; we then prove the convergence of the solutions of the approximating problems to an entropy solution of problem (1.1). In Section 3, we start by obtaining several useful a priori estimates on the sequence of approximate solutions: namely, non-negativity, (uniform) L^1 and L^∞ bounds, and an improved L^2 estimate involving also the H^1 norm of the regularized solutions. With these estimates, we obtain a bound on the total variation "away from the discontinuity" and, using Murat's compact embedding, we deduce the L^p-compactness of the approximating sequence.

In Section 4, we complete the proof of the main theorem by showing the convergence (up to sub-sequences) of the approximate solutions to an entropy solution of problem (1.1) by a diagonal argument.

2. Notions of solutions for a conservation law with discontinuous flux and main results

We consider the following definition of weak solution for problem (1.1).

Definition 2.1 (Weak solutions)**.** *A function $\rho : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, with $\rho \in L^\infty((0,T)\times\mathbb{R})$ for every $T > 0$, is a* weak solution *of* (1.1) *if there exist a function $h_\rho \in L^\infty((0,\infty)\times\mathbb{R})$ such that*

$$h_\rho(t,x) \begin{cases} = 1 & \text{if } \rho(t,x) > \rho_c, \\ \in [0,1] & \text{if } \rho(t,x) = \rho_c, \\ = 0 & \text{if } \rho(t,x) < \rho_c, \end{cases} \tag{2.1}$$

and, for every test function $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \left(\rho \partial_t \varphi + (\rho v(x) - \alpha \rho h_\rho(t,x) f(K' * \rho))\partial_x \varphi\right) \mathrm{d}t\, \mathrm{d}x \\ + \int_{\mathbb{R}} \rho_0(x)\varphi(0,x)\,\mathrm{d}x = 0.\end{aligned} \tag{2.2}$$

As noted in [21, 20, 19] for the local case, this kind of definition of weak solutions does not seem to be enough to guarantee a uniqueness result. For this reason, inspired by [11], we propose the following definition of entropy solution. It still does not seem to provide uniqueness for problem (1.1), although is it possible to use it to obtain the well-posedness of a different class of nonlocal discontinuous conservation laws (see [3]).

Definition 2.2 (Entropy solutions)**.** *A function* $\rho : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, *with* $\rho \in L^\infty((0,T) \times \mathbb{R})$ *for every* $T > 0$, *is an* entropy solution *of* (1.1) *if, for every* $c \in \mathbb{R}$, *there exist a function* $h_\rho : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *satisfying* (2.1) *and, for every non-negative test function* $\varphi \in C_c^\infty(\mathbb{R}^2)$, *the following inequality holds:*

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}} (|\rho - c|\partial_t \varphi + |\rho - c| v(x) \partial_x \varphi)\,\mathrm{d}t\,\mathrm{d}x \\ &- \int_0^\infty \int_{\mathbb{R}} \operatorname{sign}(\rho - c) c v'(x) \varphi\,\mathrm{d}t\,\mathrm{d}x \\ &- \alpha \int_0^\infty \int_{\mathbb{R}} \operatorname{sign}(\rho - c)(\rho h_\rho(t,x) - cH(c - \rho_c)) f(K' * \rho)\partial_x\varphi\,\mathrm{d}t\,\mathrm{d}x \\ &+ \alpha \int_0^\infty \int_{\mathbb{R}} \operatorname{sign}(\rho - c) c H(c - \rho_c) f'(K' * \rho)(K'' * \rho)\varphi\,\mathrm{d}t\,\mathrm{d}x \\ &+ \int_0^\infty \int_{\mathbb{R}} |\rho_0(x) - c|\varphi(0,x)\,\mathrm{d}x \geq 0.\end{aligned} \tag{2.3}$$

Remark 2.1 (Formal motivation of the entropy condition)**.** *To formally deduce* (2.3), *we fix* $c \in \mathbb{R}$ *and multiply* (1.1) *by* $\operatorname{sign}(\rho - c)$ *as in the classical Kružkov's entropy argument:*

$$\begin{aligned}\partial_t |\rho - c| + \underbrace{\operatorname{sign}(\rho - c)\partial_x(\rho v(x))}_{E_1} \\ \underbrace{- \alpha \operatorname{sign}(\rho - c)\partial_x(\rho H(\rho - \rho_c) f(K' * \rho))}_{E_2} = 0.\end{aligned} \tag{2.4}$$

Then we observe that

$$\begin{aligned}E_1 =& \partial_x(|\rho - c| v(x)) + (\operatorname{sign}(\rho - c)\rho - |\rho - c|)v'(x), \\ E_2 =& -\alpha \partial_x(\operatorname{sign}(\rho - c)(\rho H(\rho - \rho_c) - cH(c - \rho_c)) f(K' * \rho)) \\ & - \alpha \operatorname{sign}(\rho - c) c H(c - \rho_c) f'(K' * \rho)(K'' * \rho) \\ & - \underbrace{\alpha \delta_{\{\rho = c\}} \left(\rho H(\rho - \rho_c) f(K' * \rho) - cH(c - \rho_c) f(K' * \rho)\right)}_{=0}.\end{aligned}$$

The main theorem of this paper is the following existence result.

Theorem 2.1. *Let us assume that* (1.2) *and* (1.3) *hold. Then the Cauchy problem* (1.1) *admits an entropy solution* $\rho : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *in the sense of Definition* 2.2.

Our argument in based on passing to the limit in the following second order approximation of (1.1):

$$\begin{cases}\partial_t\rho_\varepsilon + \partial_x(\rho_\varepsilon v(x) - \alpha\rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f(K' * \rho_\varepsilon)) = \varepsilon\partial_{xx}^2\rho_\varepsilon, & t > 0,\ x \in \mathbb{R},\\ \rho_\varepsilon(0, x) = \rho_{0,\varepsilon}(x), & x \in \mathbb{R},\end{cases} \tag{2.5}$$

where we assume

$$\{\rho_{0,\varepsilon}\}_{\varepsilon>0} \subset C^\infty(\mathbb{R}), \qquad \{H_\varepsilon\}_{\varepsilon>0} \subset C^\infty(\mathbb{R}), \tag{2.6}$$

$$\rho_{0,\varepsilon} \to \rho_0 \quad \text{in } L^p(\mathbb{R}),\ 1 \le p < \infty, \text{ and a.e. in } \mathbb{R} \text{ as } \varepsilon \to 0, \tag{2.7}$$

$$H_\varepsilon(\xi) = \begin{cases} 1 & \text{if } \xi \ge \varepsilon,\\ 0 & \text{if } \xi \le 0,\end{cases} \qquad \xi \in \mathbb{R}, \tag{2.8}$$

$$\|\rho_{0,\varepsilon}\|_{L^1(\mathbb{R})} \le \|\rho_0\|_{L^1(\mathbb{R})}, \quad \|\rho_{0,\varepsilon}\|_{L^2(\mathbb{R})} \le \|\rho_0\|_{L^2(\mathbb{R})}, \tag{2.9}$$

$$0 \le \rho_{0,\varepsilon} \le \|\rho_0\|_{L^\infty(\mathbb{R})}, \quad 0 \le H_\varepsilon \le 1, \quad 0 \le H'_\varepsilon \le \frac{2}{\varepsilon}. \tag{2.10}$$

We remark that this kind of results could be extended to more general nonlocal terms, as long as f is non-decreasing and satisfies the estimates in Lemma 3.3. To be consistent with the existent literature, we focus only on the function f defined in (1.5).

3. A priori estimates and compactness results

In this section, we prove several a priori estimates on the solution ρ_ε of (2.5). In what follows, we denote with $c \ge 0$ the constants independent from $\varepsilon > 0$ and $t \ge 0$.

Lemma 3.1 (Non-negativity and smoothness of ρ_ε)**.** *For every $\varepsilon > 0$, there exists a unique non-negative smooth solution $\rho_\varepsilon \in C^\infty([0,\infty) \times \mathbb{R}) \cap W^2((0,\infty) \times \mathbb{R})$ of the Cauchy problem* (2.5)*,*

$$\rho_\varepsilon \ge 0, \qquad t \ge 0,\ x \in \mathbb{R}. \tag{3.1}$$

Proof. The existence and uniqueness of smooth solutions of (2.5) can be proved following the same strategy as [13, 14, 15]. We focus on showing the non-negativity of solutions. To this end, we consider the function

$$\eta(\xi) = -\xi\chi_{(-\infty,0]}(\xi), \qquad \xi \in \mathbb{R},$$

and observe that

$$\eta'(\xi) = -\chi_{(-\infty,0]}(\xi), \qquad \eta''(\xi) = \delta_{\{\xi=0\}} \ge 0. \tag{3.2}$$

Multiplying (2.5) by $\eta'(\rho_\varepsilon)$, integrating over $\mathbb{R}$, and using [4, Lemma 2] yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \eta(\rho_\varepsilon)\,\mathrm{d}x &= \int_{\mathbb{R}} \partial_t\rho_\varepsilon\eta'(\rho_\varepsilon)\,\mathrm{d}x\\ &= \varepsilon\int_{\mathbb{R}} \partial_{xx}^2\rho_\varepsilon\eta'(\rho_\varepsilon)\,\mathrm{d}x\\ &\quad - \int_{\mathbb{R}} \partial_x(\rho_\varepsilon v(x) - \alpha\rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c)f(K' * \rho_\varepsilon))\eta'(\rho_\varepsilon)\,\mathrm{d}x\\ &= -\varepsilon\int_{\mathbb{R}} \underbrace{(\partial_x\rho_\varepsilon)^2\eta''(\rho_\varepsilon)}_{\ge 0}\,\mathrm{d}x\end{aligned}$$

$$+\int_{\mathbb{R}}(v(x)-\alpha H_\varepsilon(\rho_\varepsilon-\rho_c)f(K'*\rho_\varepsilon)\partial_x\rho_\varepsilon\underbrace{\rho_\varepsilon\eta''(\rho_\varepsilon)}_{=0}\,\mathrm{d}x\le 0.$$

Integrating over $(0,t)$ and using (2.10) and (3.2), we have

$$0\le\int_{\mathbb{R}}\eta(\rho_\varepsilon(t,x))\,\mathrm{d}x\le\int_{\mathbb{R}}\eta(\rho_{0,\varepsilon}(x))\,\mathrm{d}x=0.$$

Therefore,

$$\eta(\rho_\varepsilon)\equiv 0,$$

which proves (3.1). □

In the following lemma, we establish uniform L^1 bounds on the solution.

Lemma 3.2 (Uniform L^1 estimate on ρ_ε)**.** *If ρ_ε is the solution of (2.5), then the following estimate holds:*

$$\|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})}\le\|\rho_0\|_{L^1(\mathbb{R})},\qquad t\ge 0. \tag{3.3}$$

Proof. Differentiating the L^1 norm of the ρ_ε (and using 3.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|\rho_\varepsilon(t,x)|\,\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\rho_\varepsilon(t,x)\,\mathrm{d}x=\int_{\mathbb{R}}\partial_t\rho_\varepsilon(t,x)\,\mathrm{d}x=0.$$

As there is this no change of the L^1 norm over time, we obtain

$$\|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})}=\|\rho_{0,\varepsilon}\|_{L^1(\mathbb{R})},\qquad t\ge 0,$$

and (2.9) gives the claim. □

Next, we deduce uniform bounds on the nonlocal term in terms of $\|\rho_0\|_{L^1(\mathbb{R})}$.

Lemma 3.3 (L^∞ estimates on the nonlocal term)**.** *The nonlocal term $K'*\rho_\varepsilon$, where ρ_ε solves (2.5), admits the following bounds:*

$$\begin{aligned}
\|K'*\rho_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}&\le\|K'\|_{L^\infty(\mathbb{R})}\|\rho_0\|_{L^1(\mathbb{R})},\\
\|K''*\rho_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}&\le\|K''\|_{L^\infty(\mathbb{R})}\|\rho_0\|_{L^1(\mathbb{R})},\\
\|K'''*\rho_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}&\le\|K'''\|_{L^\infty(\mathbb{R})}\|\rho_0\|_{L^1(\mathbb{R})},\\
\|f(K'*\rho_\varepsilon)\|_{L^\infty((0,T)\times\mathbb{R})}&\le\|K'\|_{L^\infty(\mathbb{R})}\|\rho_0\|_{L^1(\mathbb{R})},\\
\|f'\|_{L^\infty(\mathbb{R})}&\le 1,\\
\|f''(K'*\rho_\varepsilon)\|_{L^\infty((0,T)\times\mathbb{R})}&\le 3\|K'\|_{L^\infty(\mathbb{R})}\|\rho_0\|_{L^1(\mathbb{R})}.
\end{aligned} \tag{3.4}$$

Proof. The proof is a direct consequence of (1.2), (1.5), and Lemma 3.2. □

We use Lemma 3.4 to establish a uniform bound of the L^∞ norm of ρ_ε.

Lemma 3.4 (L^∞ estimate on ρ_ε)**.** *For every $\varepsilon>0$, the solution ρ_ε of the Cauchy problem (2.5) satisfies the following estimate:*

$$\|\rho_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\le\|\rho_0\|_{L^\infty(\mathbb{R})}e^{\kappa_0 t},\qquad t\ge 0, \tag{3.5}$$

with

$$\kappa_0\coloneqq\|v'\|_{L^\infty(\mathbb{R})}+\alpha\|K''\|_{L^\infty(\mathbb{R})}\|\rho_0\|_{L^1(\mathbb{R})}. \tag{3.6}$$

Proof. We know that ρ_ε is a solution of the Cauchy problem

$$\begin{cases} \partial_t r + \partial_x r\big(v(x) - \alpha(H_\varepsilon(r-\rho_c) + rH'_\varepsilon(r-\rho_c))f(K' * r)\big) \\ \quad + r\big(v'(x) - \alpha H_\varepsilon(r-\rho_c)f'(K' * r)(K'' * r)\big) = \varepsilon \partial^2_{xx} r, & t>0,\, x\in\mathbb{R}, \\ r(0,x) = \rho_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases}$$

We claim that $w(t,x) = \|\rho_0\|_{L^\infty(\mathbb{R})} e^{\kappa_0 t}$, for $t \geq 0$ and $x \in \mathbb{R}$, is a super-solution of the same problem. Indeed

$$\begin{aligned} &\bigg[\partial_t r + \partial_x r\big(v(x) - \alpha(H_\varepsilon(r-\rho_c) + rH'_\varepsilon(r-\rho_c))f(K' * r)\big) \\ &\qquad + r\big(v'(x) - \alpha H_\varepsilon(r-\rho_c)f'(K' * r)(K'' * r)\big) - \varepsilon \partial^2_{xx} r\bigg]_{r=w} \\ &\quad = w\Big(\kappa_0 + v' - \alpha H_\varepsilon(\rho_\varepsilon - \rho_c) f'(K' * \rho_\varepsilon)(K'' * \rho_\varepsilon)\Big) \geq 0, \\ &w(0,x) = \|\rho_0\|_{L^\infty(\mathbb{R})} \geq \rho_{0,\varepsilon}(x) \end{aligned}$$

Therefore, (3.5) follows from (3.1) and the comparison principle for parabolic equations (see [22]). Alternatively, we can show that the claimed L^∞ estimate holds by arguing as in the proof of Lemma 3.1. □

Although the previous two lemmas already provide us also with a uniform L^2 bound of the solution, we state a stronger uniform L^2 estimate involving also the H^1 norm of the smoothed solution.

Lemma 3.5 (Improved L^2 estimate on ρ_ε)**.** *The solution ρ_ε of the Cauchy problem* (2.5) *satisfies*

$$\|\rho_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^{\kappa_0 t}\int_0^t e^{-\kappa_0 s}\,\|\partial_x\rho_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \leq \|\rho_0\|^2_{L^2(\mathbb{R})} e^{\kappa_0 t}, \tag{3.7}$$

for $t \geq 0$, $\varepsilon > 0$, and κ_0 chosen as in (3.6)*.*

Proof. Consider the function

$$G_\varepsilon(\xi) = \int_0^\xi \zeta H_\varepsilon(\zeta - \rho_c)\,\mathrm{d}\zeta, \qquad \xi \in \mathbb{R},$$

and observe that

$$|G_\varepsilon(\xi)| \leq \frac{\xi^2}{2}, \qquad \xi \in \mathbb{R}.$$

Since, due to (3.4),

$$|v'(x)| + \alpha|f'(K' * \rho_\varepsilon(t,x))(K'' * \rho_\varepsilon(t,x))| \leq \kappa_0, \qquad t \geq 0,\ x \in \mathbb{R},$$

we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{\rho_\varepsilon^2}{2}\,\mathrm{d}x &= \int_{\mathbb{R}} \rho_\varepsilon \partial_t \rho_\varepsilon\,\mathrm{d}x \\ &= \varepsilon\int_{\mathbb{R}} \rho_\varepsilon \partial^2_{xx}\rho_\varepsilon\,\mathrm{d}x - \int_{\mathbb{R}} \rho_\varepsilon \partial_x\big(\rho_\varepsilon v(x) - \alpha\rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f(K' * \rho_\varepsilon)\big)\,\mathrm{d}x \\ &= -\varepsilon\int_{\mathbb{R}} (\partial_x\rho_\varepsilon)^2\,\mathrm{d}x + \int_{\mathbb{R}} \partial_x\rho_\varepsilon\big(\rho_\varepsilon v(x) - \alpha\rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f(K' * \rho_\varepsilon)\big)\,\mathrm{d}x \\ &= -\varepsilon\int_{\mathbb{R}} (\partial_x\rho_\varepsilon)^2\,\mathrm{d}x + \int_{\mathbb{R}} \partial_x\left(\frac{\rho_\varepsilon^2}{2}\right) v(x)\,\mathrm{d}x - \alpha\int_{\mathbb{R}} \partial_x G_\varepsilon(\rho_\varepsilon) f(K' * \rho_\varepsilon)\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= -\varepsilon \int_{\mathbb{R}} (\partial_x \rho_\varepsilon)^2 \,\mathrm{d}x - \int_{\mathbb{R}} \frac{\rho_\varepsilon^2}{2} v'(x) \,\mathrm{d}x + \alpha \int_{\mathbb{R}} G_\varepsilon(\rho_\varepsilon) f'(K' * \rho_\varepsilon)(K'' * \rho_\varepsilon) \,\mathrm{d}x \\
&\leq -\varepsilon \int_{\mathbb{R}} (\partial_x \rho_\varepsilon)^2 \,\mathrm{d}x + \int_{\mathbb{R}} \frac{\rho_\varepsilon^2}{2} \left(|v'(x)| + \alpha |f'(K' * \rho_\varepsilon)(K'' * \rho_\varepsilon)| \right) \mathrm{d}x \\
&\leq -\varepsilon \int_{\mathbb{R}} (\partial_x \rho_\varepsilon)^2 \,\mathrm{d}x + \kappa_0 \int_{\mathbb{R}} \frac{\rho_\varepsilon^2}{2} \,\mathrm{d}x.
\end{aligned}
$$

Therefore, using Gronwall's inequality we get the claim. □

The core of our compactness argument is inspired by the one of [12]. We consider the sequence

$$\{\rho_\varepsilon \chi_\sigma(\rho_\varepsilon)\}_{\varepsilon,\sigma>0},$$

where χ_σ is a cut-off function (see [26, Section 2]) such that

$$0 \leq \chi_\sigma \leq 1, \quad \chi_\sigma \in C^2(\mathbb{R}), \quad \chi_\sigma(\xi) = \begin{cases} 1 & \text{if } |\xi - \rho_c| \geq \sigma, \\ 0 & \text{if } |\xi - \rho_c| \leq \sigma/2, \end{cases} \tag{3.8}$$

and

$$\chi_\sigma{}'(\xi) \leq c_\sigma \chi_\sigma(\xi). \tag{3.9}$$

for some $c_\sigma > 0$. Moreover, since we first send $\varepsilon \to 0$ and then $\sigma \to 0$, we can always assume that $\varepsilon < \sigma/2$ and

$$|\chi_\sigma(\xi) \xi H_\varepsilon'(\xi - \rho_c)| = 0 \tag{3.10}$$

We start by establishing the following BV estimate.

Lemma 3.6. *There exists a constant $c_\sigma > 0$ independent on ε and t such that*

$$\|\partial_x(\rho_\varepsilon \chi_\sigma(\rho_\varepsilon))(t,\cdot)\|_{L^1(\mathbb{R})} \leq c_\sigma e^{c_\sigma t} \qquad t \geq 0, \tag{3.11}$$

where ρ_ε is the solution of (2.5).

Proof. We begin by observing that, from (3.9), it follows that there exists $c_\sigma > 0$ such that

$$|\partial_x(\rho_\varepsilon \chi_\sigma(\rho_\varepsilon))| \leq |\partial_x \rho_\varepsilon| \chi_\sigma(\rho_\varepsilon) + |\partial_x \rho_\varepsilon| \rho_\varepsilon |\chi_\sigma'(\rho_\varepsilon)| \leq c_\sigma |\partial_x \rho_\varepsilon| \chi_\sigma(\rho_\varepsilon), \tag{3.12}$$

for $t > 0$, $x \in \mathbb{R}$. We compute

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \chi_\sigma(\rho_\varepsilon) \,\mathrm{d}x &= \int_{\mathbb{R}} \left(|\partial_x \rho_\varepsilon| \chi_\sigma'(\rho_\varepsilon) \partial_t \rho_\varepsilon + \operatorname{sign}(\partial_x \rho_\varepsilon) \chi_\sigma(\rho_\varepsilon) \partial_{tx}^2 \rho_\varepsilon \right) \mathrm{d}x \\
&= I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &\coloneqq \varepsilon \int_{\mathbb{R}} \left(|\partial_x \rho_\varepsilon| \chi_\sigma'(\rho_\varepsilon) \partial_{xx}^2 \rho_\varepsilon + \operatorname{sign}(\partial_x \rho_\varepsilon) \chi_\sigma(\rho_\varepsilon) \partial_{xxx}^3 \rho_\varepsilon \right) \mathrm{d}x, \\
I_2 &\coloneqq - \int_{\mathbb{R}} \left(|\partial_x \rho_\varepsilon| \chi_\sigma'(\rho_\varepsilon) \partial_x(\rho_\varepsilon v(x)) + \operatorname{sign}(\partial_x \rho_\varepsilon) \chi_\sigma(\rho_\varepsilon) \partial_{xx}^2(\rho_\varepsilon v(x)) \right) \mathrm{d}x, \\
I_3 &\coloneqq \alpha \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \chi_\sigma'(\rho_\varepsilon) \partial_x(\rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f(K' * \rho_\varepsilon)) \,\mathrm{d}x \\
&\quad + \alpha \int_{\mathbb{R}} \operatorname{sign}(\partial_x \rho_\varepsilon) \chi_\sigma(\rho_\varepsilon) \partial_{xx}^2(\rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f(K' * \rho_\varepsilon)) \,\mathrm{d}x.
\end{aligned}
$$

Using [4, Lemma 2], we obtain

$$I_1 = \underbrace{\varepsilon \int_{\mathbb{R}} \Big(|\partial_x \rho_\varepsilon| \chi'_\sigma(\rho_\varepsilon) \partial^2_{xx}\rho_\varepsilon - \operatorname{sign}(\partial_x \rho_\varepsilon)\, \chi'_\sigma(\rho_\varepsilon)\partial_x\rho_\varepsilon \partial^2_{xx}\rho_\varepsilon \Big)\,\mathrm{d}x}_{=0} \underbrace{- \varepsilon \int_{\mathbb{R}} \delta_{\{\partial_x\rho_\varepsilon = 0\}} \chi_\sigma(\rho_\varepsilon)(\partial^2_{xx}\rho_\varepsilon)^2 \,\mathrm{d}x}_{\leq 0},$$

that is

$$I_1 \leq 0.$$

Using again [4, Lemma 2], we obtain

$$\begin{aligned}
I_2 = &- \int_{\mathbb{R}} \Big(|\partial_x \rho_\varepsilon| \chi'_\sigma(\rho_\varepsilon)\rho_\varepsilon v'(x) + |\partial_x \rho_\varepsilon| \chi'_\sigma(\rho_\varepsilon)\partial_x\rho_\varepsilon v(x) \Big)\,\mathrm{d}x \\
&- \int_{\mathbb{R}} \operatorname{sign}(\partial_x\rho_\varepsilon)\, \chi_\sigma(\rho_\varepsilon)\partial_x(\partial_x\rho_\varepsilon v(x) + \rho_\varepsilon v'(x)) \Big)\,\mathrm{d}x \\
= &- \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \chi'_\sigma(\rho_\varepsilon)\rho_\varepsilon v'(x)\,\mathrm{d}x \\
&- \int_{\mathbb{R}} \Big(\underbrace{|\partial_x \rho_\varepsilon| \chi'_\sigma(\rho_\varepsilon)\partial_x\rho_\varepsilon v(x) - \operatorname{sign}(\partial_x\rho_\varepsilon)\, \chi'_\sigma(\rho_\varepsilon)(\partial_x\rho_\varepsilon)^2 v(x)}_{=0} \Big)\,\mathrm{d}x \\
&- \int_{\mathbb{R}} \underbrace{\delta_{\{\partial_x\rho_\varepsilon=0\}} \chi_\sigma(\rho_\varepsilon)\partial^2_{xx}\rho_\varepsilon \partial_x\rho_\varepsilon v(x)}_{=0}\,\mathrm{d}x \\
&+ \int_{\mathbb{R}} \Big(|\partial_x \rho_\varepsilon| \chi_\sigma(\rho_\varepsilon) v'(x) - \operatorname{sign}(\partial_x\rho_\varepsilon)\, \chi_\sigma(\rho_\varepsilon)\rho_\varepsilon v''(x)) \Big)\,\mathrm{d}x \\
\leq &\; c_\sigma \int_{\mathbb{R}} |\partial_x\rho_\varepsilon| \chi_\sigma(\rho_\varepsilon)\,\mathrm{d}x + \|v''\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \rho_\varepsilon\,\mathrm{d}x.
\end{aligned}$$

Due to (3.3) and (3.12),

$$I_2 \leq c_\sigma \int_{\mathbb{R}} |\partial_x\rho_\varepsilon| \chi_\sigma(\rho_\varepsilon)\,\mathrm{d}x + \|v''\|_{L^\infty(\mathbb{R})} \|\rho_0\|_{L^1(\mathbb{R})}.$$

We split I_3 as

$$I_3 = I_{3,1} + I_{3,2} + I_{3,3},$$

where

$$\begin{aligned}
I_{3,1} \coloneqq &\; \alpha \int_{\mathbb{R}} |\partial_x\rho_\varepsilon| \chi'_\sigma(\rho_\varepsilon)\partial_x\rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f(K' * \rho_\varepsilon)\,\mathrm{d}x \\
&+ \alpha \int_{\mathbb{R}} \operatorname{sign}(\partial_x\rho_\varepsilon)\, \chi_\sigma(\rho_\varepsilon)\partial_x(\partial_x\rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f(K' * \rho_\varepsilon))\,\mathrm{d}x, \\
I_{3,2} \coloneqq &\; \alpha \int_{\mathbb{R}} |\partial_x\rho_\varepsilon| \chi'_\sigma(\rho_\varepsilon)\rho_\varepsilon H'_\varepsilon(\rho_\varepsilon - \rho_c)\partial_x\rho_\varepsilon f(K' * \rho_\varepsilon)\,\mathrm{d}x \\
&+ \alpha \int_{\mathbb{R}} \operatorname{sign}(\partial_x\rho_\varepsilon)\, \chi_\sigma(\rho_\varepsilon)\partial_x(\rho_\varepsilon H'_\varepsilon(\rho_\varepsilon - \rho_c)\partial_x\rho_\varepsilon f(K' * \rho_\varepsilon))\,\mathrm{d}x, \\
I_{3,3} = &\; \alpha \int_{\mathbb{R}} |\partial_x\rho_\varepsilon| \chi'_\sigma(\rho_\varepsilon)\rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f'(K' * \rho_\varepsilon)(K'' * \rho_\varepsilon)\,\mathrm{d}x \\
&+ \alpha \int_{\mathbb{R}} \operatorname{sign}(\partial_x\rho_\varepsilon)\, \chi_\sigma(\rho_\varepsilon)\partial_x(\rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f'(K' * \rho_\varepsilon)(K'' * \rho_\varepsilon))\,\mathrm{d}x.
\end{aligned}$$

We have that

$$I_{3,1} = I_{3,2} = 0.$$

Indeed,

$$\begin{aligned} I_{3,1} =& \alpha \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \chi'_\sigma(\rho_\varepsilon) \partial_x \rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f(K' * \rho_\varepsilon)\,\mathrm{d}x \\ &- \alpha \int_{\mathbb{R}} \operatorname{sign}(\partial_x \rho_\varepsilon)\, \chi'_\sigma(\rho_\varepsilon)(\partial_x \rho_\varepsilon)^2 H_\varepsilon(\rho_\varepsilon - \rho_c) f(K' * \rho_\varepsilon)\,\mathrm{d}x \\ &- \alpha \int_{\mathbb{R}} \delta_{\{\partial_x \rho_\varepsilon = 0\}} \partial^2_{xx} \rho_\varepsilon \chi_\sigma(\rho_\varepsilon) \partial_x \rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f(K' * \rho_\varepsilon)\,\mathrm{d}x, \\ I_{3,2} =& \alpha \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \chi'_\sigma(\rho_\varepsilon) \rho_\varepsilon H'_\varepsilon(\rho_\varepsilon - \rho_c) \partial_x \rho_\varepsilon f(K' * \rho_\varepsilon)\,\mathrm{d}x \\ &- \alpha \int_{\mathbb{R}} \operatorname{sign}(\partial_x \rho_\varepsilon)\, \chi'_\sigma(\rho_\varepsilon) \rho_\varepsilon H'_\varepsilon(\rho_\varepsilon - \rho_c)(\partial_x \rho_\varepsilon)^2 f(K' * \rho_\varepsilon)\,\mathrm{d}x \\ &- \alpha \underbrace{\int_{\mathbb{R}} \delta_{\{\partial_x \rho_\varepsilon = 0\}} \partial^2_{xx} \rho_\varepsilon \chi_\sigma(\rho_\varepsilon) \rho_\varepsilon H'_\varepsilon(\rho_\varepsilon - \rho_c) \partial_x \rho_\varepsilon f(K' * \rho_\varepsilon)\,\mathrm{d}x}_{=0}. \end{aligned}$$

Then, we estimate $I_{3,3}$ as follows (thanks to (3.3), (3.4), (3.5), and (3.10)):

$$\begin{aligned} I_{3,3} =& \alpha \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \underbrace{\chi'_\sigma(\rho_\varepsilon) \rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f'(K' * \rho_\varepsilon)(K'' * \rho_\varepsilon)}_{\le c_\sigma \chi_\sigma(\rho_\varepsilon)}\,\mathrm{d}x \\ &+ \alpha \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \chi_\sigma(\rho_\varepsilon) \underbrace{H_\varepsilon(\rho_\varepsilon - \rho_c) f'(K' * \rho_\varepsilon)(K'' * \rho_\varepsilon)}_{\le c}\,\mathrm{d}x \\ &+ \alpha \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \underbrace{\chi_\sigma(\rho_\varepsilon) \rho_\varepsilon H'_\varepsilon(\rho_\varepsilon - \rho_c) f'(K' * \rho_\varepsilon)(K'' * \rho_\varepsilon)}_{=0}\,\mathrm{d}x \\ &+ \alpha \int_{\mathbb{R}} \underbrace{\operatorname{sign}(\partial_x \rho_\varepsilon)\, \chi_\sigma(\rho_\varepsilon) \rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f''(K' * \rho_\varepsilon)(K'' * \rho_\varepsilon)^2}_{\le c\rho_\varepsilon}\,\mathrm{d}x \\ &+ \alpha \int_{\mathbb{R}} \underbrace{\operatorname{sign}(\partial_x \rho_\varepsilon)\, \chi_\sigma(\rho_\varepsilon) \rho_\varepsilon H_\varepsilon(\rho_\varepsilon - \rho_c) f'(K' * \rho_\varepsilon)(K''' * \rho_\varepsilon)}_{\le c\rho_\varepsilon}\,\mathrm{d}x \\ \le& c_\sigma \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \chi_\sigma(\rho_\varepsilon)\,\mathrm{d}x + c \int_{\mathbb{R}} \rho_\varepsilon\,\mathrm{d}x \\ \le& c_\sigma \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \chi_\sigma(\rho_\varepsilon)\,\mathrm{d}x + c. \end{aligned}$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \chi_\sigma(\rho_\varepsilon)\,\mathrm{d}x \le c_\sigma \int_{\mathbb{R}} |\partial_x \rho_\varepsilon| \chi_\sigma(\rho_\varepsilon)\,\mathrm{d}x + c$$

and Gronwall's lemma concludes the proof. □

With this preparation, we are ready to prove the compactness of the family $\{\rho_\varepsilon \chi_\sigma(\rho_\varepsilon)\}_{\varepsilon>0}$.

Lemma 3.7. *For every $\sigma > 0$, the family $\{\rho_\varepsilon \chi_\sigma(\rho_\varepsilon)\}_{\varepsilon>0}$, with χ_σ as in (3.8) and ρ_ε solution of (2.5), is compact in $L^p_{loc}((0,\infty) \times \mathbb{R})$, $1 \le p < \infty$.*

The proof of this result is based on Murat's compact embedding (see [33]).

Theorem 3.1 (Murat's compact embedding)**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. Suppose the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

Proof of Lemma 3.7. Consider the functions

$$\eta_\sigma(\xi) = \xi\chi_\sigma(\xi), \qquad q_{\varepsilon,\sigma}(\xi) = \int_0^\xi \eta'_\sigma(\zeta)(H_\varepsilon(\zeta-\rho_c) + \zeta H'_\varepsilon(\zeta-\rho_c))\,\mathrm{d}\zeta.$$

Multiplying (2.5) by $\eta'_\sigma(\rho_\varepsilon)$, we obtain

$$\begin{aligned}\partial_t\eta_\sigma(\rho_\varepsilon) =& \varepsilon\partial^2_{xx}\eta_\sigma(\rho_\varepsilon) - \varepsilon\eta''_\sigma(\rho_\varepsilon)(\partial_x\rho_\varepsilon)^2 - \partial_x(\eta_\sigma(\rho_\varepsilon)v(x))\\ &- (\eta'_\sigma(\rho_\varepsilon)\rho_\varepsilon - \eta_\sigma(\rho_\varepsilon))v'(x) + \alpha\partial_x(q_{\varepsilon,\sigma}(\rho_\varepsilon)f(K'*\rho_\varepsilon))\\ &+ \alpha(\eta'_\sigma(\rho_\varepsilon)\rho_\varepsilon H_\varepsilon(\rho_\varepsilon-\rho_c) - q_{\varepsilon,\sigma}(\rho_\varepsilon))f'(K'*\rho_\varepsilon)(K''*\rho_\varepsilon).\end{aligned} \tag{3.13}$$

Using Lemmas 3.2, 3.4, 3.4, 3.5, 3.6 we have that

$$\{\partial_t\eta_\sigma(\rho_\varepsilon)\}_{\varepsilon>0}$$

is bounded in $W^{-1,\infty}_{loc}((0,\infty)\times\mathbb{R})$,

$$\{\varepsilon\partial^2_{xx}\eta_\sigma(\rho_\varepsilon)\}_{\varepsilon>0}$$

is compact in $H^{-1}_{loc}((0,\infty)\times\mathbb{R})$, and

$$\begin{aligned}&\{\partial_x(\eta_\sigma(\rho_\varepsilon)v)\}_{\varepsilon>0}, \qquad \{(\eta'_\sigma(\rho_\varepsilon)\rho_\varepsilon - \eta_\sigma(\rho_\varepsilon))v'\}_{\varepsilon>0},\\ &\{\alpha\partial_x(q_{\varepsilon,\sigma}(\rho_\varepsilon)f(K'*\rho_\varepsilon))\}_{\varepsilon>0},\\ &\{\alpha(\eta'_\sigma(\rho_\varepsilon)H_\varepsilon(\rho_\varepsilon-\rho_c) - q_{\varepsilon,\sigma}(\rho_\varepsilon))f'(K'*\rho_\varepsilon)(K''*\rho_\varepsilon)\}_{\varepsilon>0}\end{aligned}$$

are bounded in $L^1_{loc}((0,\infty)\times\mathbb{R})$. Indeed, for $T>0$,

$$\begin{aligned}&\|\eta_\sigma(\rho_\varepsilon)\|_{L^\infty((0,T)\times\mathbb{R})} \leq \|\rho_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leq \|\rho_0\|_{L^\infty(\mathbb{R})} e^{\kappa_0 T},\\ &\varepsilon\partial^2_{xx}\eta_\sigma(\rho_\varepsilon) = \partial_x\Big(\varepsilon\partial_x\eta_\sigma(\rho_\varepsilon)\Big),\\ &\|\varepsilon\partial_x\eta_\sigma(\rho_\varepsilon)\|^2_{L^2((0,T)\times\mathbb{R})} \leq \varepsilon^2\int_0^T \|\partial_x\eta_\sigma(\rho_\varepsilon)(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\\ &\qquad \leq \varepsilon^2\|\eta'_\sigma\|^2_{L^\infty(\mathbb{R})}\int_0^T \|\partial_x\rho_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \leq \varepsilon c \to 0,\\ &\|\partial_x(\eta_\sigma(\rho_\varepsilon)v)\|_{L^1((0,T)\times\mathbb{R})}\\ &\qquad \leq c\Big(\|\rho_\varepsilon\|_{L^1((0,T)\times\mathbb{R})} + \|\partial_x\rho_\varepsilon\chi_\sigma(\rho_\varepsilon)\|_{L^1((0,T)\times\mathbb{R})}\Big) \leq c_{T,\sigma},\\ &\|(\eta'_\sigma(\rho_\varepsilon)\rho_\varepsilon - \eta_\sigma(\rho_\varepsilon))v'\|_{L^1((0,T)\times\mathbb{R})} \leq c\|\rho_\varepsilon\|_{L^1((0,T)\times\mathbb{R})} \leq c_{T,\sigma},\\ &\|\alpha\partial_x(q_{\varepsilon,\sigma}(\rho_\varepsilon)f(K'*\rho_\varepsilon))\|_{L^1((0,T)\times\mathbb{R})}\\ &\qquad \leq c\Big(\|\rho_\varepsilon\|_{L^1((0,T)\times\mathbb{R})} + \|\partial_x\rho_\varepsilon\chi_\sigma(\rho_\varepsilon)\|_{L^1((0,T)\times\mathbb{R})}\Big) \leq c_{T,\sigma},\\ &\|\alpha(\eta'_\sigma(\rho_\varepsilon)H_\varepsilon(\rho_\varepsilon-\rho_c) - q_{\varepsilon,\sigma}(\rho_\varepsilon))f'(K'*\rho_\varepsilon)(K''*\rho_\varepsilon)\|_{L^1((0,T)\times\mathbb{R})}\\ &\qquad \leq c\|\rho_\varepsilon\|_{L^1((0,T)\times\mathbb{R})} \leq c_{T,\sigma}.\end{aligned}$$

Theorem 3.1 guarantees that the family $\{\partial_t \eta_\sigma(\rho_\varepsilon)\}_{\varepsilon>0}$ is compact in $H^{-1}_{loc}((0,\infty)\times\mathbb{R})$, namely $\{\eta_\sigma(\rho_\varepsilon)\}_{\varepsilon>0}$ is compact in $L^2_{loc}((0,\infty)\times\mathbb{R})$. Thanks to the L^∞ estimate (3.5), we conclude the proof. □

4. Proof of the main theorem

Having established the compactness results of the previous section, we have all the tools needed to prove our main theorem.

Proof of Theorem 2.1. We claim that there exists a function $\rho \in L^\infty((0,T)\times\mathbb{R})$, $T>0$, and a subsequence $\{\varepsilon_n\}_n$, $\varepsilon_n \to 0$, such that

$$\rho_{\varepsilon_n} \to \rho, \qquad \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty, \text{ and a.e. in } (0,\infty)\times\mathbb{R}). \tag{4.1}$$

Consider the sequence $\{\rho_\varepsilon\chi_\sigma(\rho_\varepsilon)\}_{\varepsilon>0}$. Thanks to Lemma 3.7, for every $\sigma>0$, we can find a sequence $\{\varepsilon_n^\sigma\}_n$, $\varepsilon_n^\sigma\to 0$, and a function $\rho_\sigma \in L^\infty((0,T)\times\mathbb{R})$, $T>0$, such that

$$\rho_{\varepsilon_n^\sigma} \to \rho_\sigma, \quad \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty, \text{ and a.e. in } (0,\infty)\times\mathbb{R}.$$

Fix $\sigma'<\sigma$ and consider $\{\rho_{\varepsilon_n^\sigma}\chi_{\sigma'}(\rho_{\varepsilon_n^\sigma})\}_{\varepsilon>0}$. Using again Lemma 3.7, we can find a subsequence $\{\varepsilon_n^{\sigma'}\}_n$ of $\{\varepsilon_n^\sigma\}_n$ and a function $\rho_{\sigma'}\in L^\infty((0,T)\times\mathbb{R})$, $T>0$, such that

$$\rho_{\varepsilon_n^{\sigma'}} \to \rho_{\sigma'}, \quad \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty, \text{ and a.e. in } (0,\infty)\times\mathbb{R}.$$

Since

$$\rho_{\varepsilon_n^\sigma} = \rho_{\varepsilon_n^{\sigma'}} \quad \text{in } \{|\rho_{\varepsilon_n^\sigma}-\rho_c|>\sigma\},$$

we have that

$$\rho_\sigma = \rho_{\sigma'} \quad \text{in } \{|\rho_\sigma-\rho_c|>\sigma\}.$$

Considering smaller and smaller values of σ, we can define a function $\rho\in L^\infty((0,T)\times\mathbb{R})$, $T>0$, such that

$$\rho=\rho_\sigma \quad \text{in } \{|\rho_\sigma-\rho_c|>\sigma\} \text{ for every } \sigma>0.$$

In other words, using a diagonal argument, we find a subsequence $\{\varepsilon_n\}_n$, $\varepsilon_n\to 0$, such that (4.1) holds.

Lebesgue's dominated convergence theorem guarantees that

$$\begin{aligned} f(K'*\rho_{\varepsilon_n}) \to f(K'*\rho), \qquad & \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty,\\ & \text{and a.e. in } (0,\infty)\times\mathbb{R}. \end{aligned} \tag{4.2}$$

Finally, there exists a function $h_\rho\in L^\infty((0,\infty)\times\mathbb{R})$ such that

$$H_{\varepsilon_n}(\rho_{\varepsilon_n}-\rho_c) \rightharpoonup h_\rho \qquad \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty. \tag{4.3}$$

We have to verify that h_ρ satisfies (2.1).

Since

$$0\le H_{\varepsilon_n}(\rho_{\varepsilon_n}-\rho_c)\le 1,$$

we have

$$0\le h_\rho\le 1.$$

Given $\sigma>0$, since

$$H_{\varepsilon_n}(\rho_{\varepsilon_n}-\rho_c)\chi_{\sigma+}(\rho_{\varepsilon_n}) \rightharpoonup h_\rho\chi_\sigma(\rho) \qquad \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty,$$

and

$$H_{\varepsilon_n}(\rho_{\varepsilon_n}-\rho_c)\chi_{\sigma+}(\rho_{\varepsilon_n})\equiv 1,$$

where $\chi_{\sigma+}$ is a cut-off function such that

$$0 \leq \chi_{\sigma+} \leq 1, \quad \chi_{\sigma+} \in C^\infty(\mathbb{R}), \quad \chi_{\sigma+}(\xi) = \begin{cases} 1 & \text{if } \xi - \rho_c \geq \sigma, \\ 0 & \text{if } \xi - \rho_c \leq \sigma/2, \end{cases}$$

we have

$$h\chi_{\sigma+}(\rho) \equiv 1.$$

Sending $\sigma \to 0$ we obtain that $h = 1$ on $\{\rho > \rho_c\}$. In the same way we can prove that $h = 0$ on $\{\rho < \rho_c\}$.

This implies that ρ is a weak solution of (1.1) in the sense of Definition 2.1. We have to verify that ρ satisfies (2.3). Let $c \in \mathbb{R}$ be given. For every $\nu > 0$ we consider the functions

$$\operatorname{sign}_\nu(\xi) = \begin{cases} 1 & \text{if } \xi > \nu, \\ \xi/\nu & \text{if } |\xi| < \nu, \\ -1 & \text{if } \xi < -\nu, \end{cases} \qquad |\xi|_\nu = \int_0^\xi \operatorname{sign}_\nu(\zeta)\,\mathrm{d}\zeta, \qquad \xi \in \mathbb{R}.$$

Multiplying (2.5) by $\operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)$, we get

$$\begin{aligned} &\partial_t|\rho_{\varepsilon_n} - c|_\nu + \underbrace{\operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)\partial_x(\rho_{\varepsilon_n} v(x))}_{E_1} \\ &\qquad \underbrace{-\alpha \operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)\partial_x(\rho_{\varepsilon_n} H_{\varepsilon_n}(\rho_{\varepsilon_n} - \rho_c) f(K' * \rho_{\varepsilon_n}))}_{E_2} \\ &= \underbrace{\varepsilon_n \operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)\partial^2_{xx}\rho_{\varepsilon_n}}_{E_3}. \end{aligned} \tag{4.4}$$

Since

$$\begin{aligned} E_1 =& \partial_x(|\rho_{\varepsilon_n} - c|_\nu v(x)) + (\operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)\rho_{\varepsilon_n} - |\rho_{\varepsilon_n} - c|_\nu)v'(x), \\ E_2 =& -\alpha\partial_x(\operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)(\rho_{\varepsilon_n} H_{\varepsilon_n}(\rho_{\varepsilon_n} - \rho_c) - cH_{\varepsilon_n}(c - \rho_c))f(K' * \rho_{\varepsilon_n})) \\ &- \alpha \operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)cH_{\varepsilon_n}(c - \rho_c)f'(K' * \rho_{\varepsilon_n})(K'' * \rho_{\varepsilon_n}) \\ &- \alpha\partial_x(r_{\nu,\varepsilon_n}(\rho_\varepsilon)f(K' * \rho_{\varepsilon_n})) + \alpha r_{\nu,\varepsilon_n}(\rho_\varepsilon)f'(K' * \rho_{\varepsilon_n})(K'' * \rho_{\varepsilon_n}), \\ E_3 =& \varepsilon_n\partial^2_{xx}|\rho_{\varepsilon_n} - c|_\nu \underbrace{-\varepsilon_n(\partial_x\rho_{\varepsilon_n})^2 \operatorname{sign}'_\nu(\rho_{\varepsilon_n} - c)}_{\leq 0} \leq \varepsilon_n\partial^2_{xx}|\rho_{\varepsilon_n} - c|_\nu, \end{aligned}$$

where

$$r_{\nu,\varepsilon_n}(\xi) = -\int_c^\xi \operatorname{sign}'_\nu(\zeta - c)(\zeta H_{\varepsilon_n}(\zeta - \rho_c) - cH_{\varepsilon_n}(c - \rho_c))\,\mathrm{d}\zeta,$$

(4.4) gives

$$\begin{aligned} &\partial_t|\rho_{\varepsilon_n} - c|_\nu + \partial_x(|\rho_{\varepsilon_n} - c|_\nu v(x)) \\ &\quad + (\operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)\rho_{\varepsilon_n} - |\rho_{\varepsilon_n} - c|_\nu)v'(x) \\ &\quad - \alpha\partial_x(\operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)(\rho_{\varepsilon_n} H_{\varepsilon_n}(\rho_{\varepsilon_n} - \rho_c) - cH_{\varepsilon_n}(c - \rho_c))f(K' * \rho_{\varepsilon_n})) \\ &\quad - \alpha \operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)cH_{\varepsilon_n}(c - \rho_c)f'(K' * \rho_{\varepsilon_n})(K'' * \rho_{\varepsilon_n}) \\ &\quad - \alpha\partial_x(r_{\nu,\varepsilon_n}(\rho_\varepsilon)f(K' * \rho_{\varepsilon_n})) + \alpha r_{\nu,\varepsilon_n}(\rho_\varepsilon)f'(K' * \rho_{\varepsilon_n})(K'' * \rho_{\varepsilon_n}) \\ &\leq \varepsilon_n\partial^2_{xx}|\rho_{\varepsilon_n} - c|_\nu. \end{aligned} \tag{4.5}$$

Let $\varphi \in C_c^2(\mathbb{R}^2)$ be a non-negative test function. Testing (4.5) against φ yields

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} (|\rho_{\varepsilon_n} - c|_\nu \partial_t \varphi + |\rho_{\varepsilon_n} - c|_\nu v(x) \partial_x \varphi) \,\mathrm{d}t \,\mathrm{d}x \\
&- \int_0^\infty \int_{\mathbb{R}} (\operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)\rho_{\varepsilon_n} - |\rho_{\varepsilon_n} - c|_\nu) v'(x) \varphi \,\mathrm{d}t \,\mathrm{d}x \\
&- \alpha \int_0^\infty \int_{\mathbb{R}} \operatorname{sign}_\nu(\rho_{\varepsilon_n} - c)(\rho_{\varepsilon_n} H_{\varepsilon_n}(\rho_{\varepsilon_n} - \rho_c) - c H_{\varepsilon_n}(c - \rho_c)) f(K' * \rho_{\varepsilon_n}) \partial_x \varphi \,\mathrm{d}t \,\mathrm{d}x \\
&+ \alpha \int_0^\infty \int_{\mathbb{R}} \operatorname{sign}_\nu(\rho_{\varepsilon_n} - c) c H_{\varepsilon_n}(c - \rho_c) f'(K' * \rho_{\varepsilon_n})(K'' * \rho_{\varepsilon_n}) \varphi \,\mathrm{d}t \,\mathrm{d}x \\
&- \alpha \int_0^\infty \int_{\mathbb{R}} (r_{\nu,\varepsilon_n}(\rho_\varepsilon) f(K' * \rho_{\varepsilon_n}) \partial_x \varphi + \alpha r_{\nu,\varepsilon_n}(\rho_\varepsilon) f'(K' * \rho_{\varepsilon_n})(K'' * \rho_{\varepsilon_n}) \varphi) \,\mathrm{d}t \,\mathrm{d}x \\
&+ \int_0^\infty \int_{\mathbb{R}} |\rho_{0,\varepsilon_n}(x) - c|_\nu \varphi(0,x) \,\mathrm{d}x \geq -\varepsilon_n |\rho_{\varepsilon_n} - c|_\nu \partial_{xx}^2 \varphi \,\mathrm{d}t \,\mathrm{d}x.
\end{aligned}$$

As $n \to \infty$, $H_{\varepsilon_n}(c - \rho_c) \to H(c - \rho_c)$. Therefore, thanks to (4.1), (4.2), and (4.3), we deduce

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} (|\rho - c|_\nu \partial_t \varphi + |\rho - c|_\nu v(x) \partial_x \varphi) \,\mathrm{d}t \,\mathrm{d}x \\
&- \int_0^\infty \int_{\mathbb{R}} (\operatorname{sign}_\nu(\rho - c)\rho - |\rho - c|_\nu) v'(x) \varphi \,\mathrm{d}t \,\mathrm{d}x \\
&- \alpha \int_0^\infty \int_{\mathbb{R}} \operatorname{sign}_\nu(\rho - c)(\rho h_\rho(t,x) - c H(c - \rho_c)) f(K' * \rho) \partial_x \varphi \,\mathrm{d}t \,\mathrm{d}x \\
&+ \alpha \int_0^\infty \int_{\mathbb{R}} \operatorname{sign}_\nu(\rho - c) c H(c - \rho_c) f'(K' * \rho)(K'' * \rho) \varphi \,\mathrm{d}t \,\mathrm{d}x \\
&- \alpha \int_0^\infty \int_{\mathbb{R}} (r_\nu(\rho) f(K' * \rho) \partial_x \varphi + \alpha r_\nu(\rho) f'(K' * \rho)(K'' * \rho) \varphi) \,\mathrm{d}t \,\mathrm{d}x \\
&+ \int_0^\infty \int_{\mathbb{R}} |\rho_0(x) - c|_\nu \varphi(0,x) \,\mathrm{d}x \geq 0.
\end{aligned}$$

Sending $\nu \to 0$, we prove (2.3). □

Remark 4.1 (Uniqueness of entropy solutions)**.** *The entropy condition used in this work does not seem to be enough to guarantee uniqueness. The key difficulty in applying the classical doubling of variables argument – which is pivotal in Kružkov's well-posedness theory for (local) scalar conservation laws (see* [25, Proposition 2.10]*) – is illustrated below. Indeed, condition* (2.3) *is equivalent to*

$$\begin{aligned}
&\partial_t |\rho - c| + \partial_x \Big(|\rho - c| v(x) - \alpha \operatorname{sign}(\rho - c)\,(\rho h_\rho - c H(c - \rho_c)) f(K' * \rho) \Big) \\
&\qquad \leq - \operatorname{sign}(\rho - c)\, c v'(x) + \alpha \operatorname{sign}(\rho - c)\, c H(c - \rho_c) \partial_x f(K' * \rho)
\end{aligned}$$

and, if ρ and r are two entropy solutions, using the doubling of variables technique, we obtain

$$\begin{aligned}
&2\partial_t |\rho - r| + 2\partial_x \Big(|\rho - r| v(x) \Big) \\
&\quad - \alpha \partial_x \Big(\operatorname{sign}(\rho - r)\,(\rho h_\rho - r H(r - \rho_c)) f(K' * \rho) \Big) \\
&\quad + \alpha \partial_x \Big(\operatorname{sign}(\rho - r)\,(r h_r - \rho H(\rho - \rho_c)) f(K' * r) \Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq - |\rho - r| v'(x) - \alpha \operatorname{sign}(\rho - c) \Big(\rho H(\rho - \rho_c) - r H(r - \rho_c)\Big) \partial_x f(K' * \rho) \\
&\quad + \alpha \operatorname{sign}(\rho - c)\, r H(r - \rho_c) \partial_x \Big(f(K' * \rho) - f(K' * r)\Big) \\
&= - |\rho - r| v'(x) - \alpha \Big|\rho H(\rho - \rho_c) - r H(r - \rho_c)\Big| f'(K' * \rho)(K'' * \rho) \\
&\quad + \alpha \operatorname{sign}(\rho - c)\, r H(r - \rho_c) \Big(f'(K' * \rho) - f'(K' * r)\Big)(K'' * \rho) \\
&\quad + \alpha \operatorname{sign}(\rho - c)\, r H(r - \rho_c) f'(K' * r)(K'' * (\rho - r)) \\
&\leq C|\rho - r| - \underbrace{\alpha \Big|\rho H(\rho - \rho_c) - r H(r - \rho_c)\Big| f'(K' * \rho)(K'' * \rho)}_{I_1} \\
&\quad + C r \left\|\rho(t, \cdot) - r(t, \cdot)\right\|_{L^1(\mathbb{R})}.
\end{aligned}
$$

We would need I_1 to be controlled in terms of $\|\rho(t,\cdot) - r(t,\cdot)\|_{L^1(\mathbb{R})}$ or to be non-negative in order to deduce a uniqueness result. However, supposing $K'' \leq 0$ is against the assumptions (1.2) *and supposing $K' = 0$ or $f' = 0$ would reduce the problem to a "local" conservation law (treated in* [11]*). On the other hand, we remark that this kind of doubling of variable argument does work in* [3] *for a different class of nonlocal discontinuous conservation laws.*

5. Conclusion

In this contribution, we introduced a notion of weak solution for a class of nonlocal conservation laws with discontinuous flux (with respect to the unknown) that describes material flow on a conveyor belt and we proved an existence result by relying on the vanishing viscosity method. Furthermore, we showed that an entropy-type inequality holds for the solutions obtained by this limit process. It is an open problem to prove or disprove that such entropy condition guarantees uniqueness.

Acknowledgments

N. De Nitti and A. Keimer acknowledge the kind hospitality of the Centro De Ciencias De Benasque Pedro Pascual, where part of this work was discussed during the workshop "VIII Partial differential equations, optimal design, and numerics".

G. M. Coclite is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). N. De Nitti is partially supported by the Alexander von Humboldt Fundation.

We thank Paulo Amorim for helpful discussions on the topic of this work.

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