EXISTENCE OF MINIMIZERS FOR THE SDRI MODEL

SHOKHRUKH YU. KHOLMATOV AND PAOLO PIOVANO

Abstract. The SDRI model introduced in [39] in the framework of the theory on Stress-Driven Rearrangement Instabilities for the morphology of crystalline material under stress [2, 37] is considered. The existence of solutions is established in dimension two in the absence of graph-like assumptions and of the restriction to a finite number m of connected components for the free boundary of the region occupied by the crystalline material, thus extending previous available results for epitaxially-strained thin films and material cavities [5, 29, 30, 39]. Due to the lack of compactness and lower semicontinuity even for sequences of m-minimizers, i.e., energy minimizers among configurations with a fixed number m of connected boundary-components, the minimizing candidate of the SDRI model is directly constructed. By means of uniform density estimates for the local decay of the energy at the m-minimizers' boundaries, such candidate is then shown to be a minimizer also in view of the convergence of the energy at m-minimizers to the energy infimum as $m \to \infty$. Finally, regularity properties for the morphology of any minimizer are deduced.

1. Introduction

In this paper we establish existence and regularity properties for the solutions of the model for Stress-Driven Rearrangement Instabilities (SDRI) [2, 21, 37] that was introduced in [39]. Under the name of Stress-Driven Rearrangement Instabilities are included all those material morphologies, such as boundary irregularities, cracks, filaments, and surface patterns, which a crystalline material may exhibit in the presence of external forces, such as in particular the chemical bonding forces with adjacent materials. In order to release the induced stresses, atoms rearrange from the material optimal crystalline order and instabilities may develop.

The main leap forward of the results contained in this manuscript with respect to [39] is the absence of the unphysical restriction on the number of connected components for the boundary of the region occupied by the crystalline material, by both avoiding graph-like assumptions for such boundaries assumed for the specific settings of epitaxially-strained thin films in [5, 14, 29] and material voids in [30], and the extension to unphysical bulk displacements, namely displacements allowed to attain infinite value on sets of positive measure (and technically assigning a zero cost to the elastic-energy contribution related to those sets), considered in [14].

The SDRI model of [39] is a variational model introduced in the framework of the SDRI theory initiated in the seminal papers of [2] and [37], and on the basis of the subsequent analytical descriptions provided in context of epitaxially-strained thin films [5, 22, 23, 29], crystal cavities [7, 30], capillarity droplets [24, 25], fractures [6, 10, 15, 17, 31], and boundary debonding and delamination [3, 41]. All such settings are included and can be

Date: August 24, 2020.

2010 *Mathematics Subject Classification.* 49J45,35R35,74G65.

Key words and phrases. Minimal configurations, elastic energy, surface energy, existence, regularity, density estimates, SDRI, interface instabilities, thin films, crystal cavities, fractures.

treated simultaneously in the SDRI model [39]. In agreement with [2, 37] since SDRI morphologies relates to the boundary of crystalline materials and depend on the bulk rearrangements, the energy $\mathcal{F}$ characterizing the SDRI model displays both an *elastic bulk energy* and a *surface energy* denoted by $\mathcal{W}$ and $\mathcal{S}$, respectively. More precisely, the energy $\mathcal{F}$ is defined as

$$\mathcal{F}(A,u) := \mathcal{S}(A,u) + \mathcal{W}(A,u) \tag{1.1}$$

for any admissible *configurational pair* (A,u) consisting of a set A that represents the region occupied by the crystalline material in a fixed *container* $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, i.e., $A \subset \Omega$, such that the topological boundary ∂A is $\mathcal{H}^{d-1}$-rectifiable with finite area, and of a displacement function of the bulk materials (with respect to the optimal crystal arrangement)

$$u \in GSBD^2(\mathrm{Int}(A \cup S \cup \Sigma); \mathbb{R}^d) \cap H^1_{\mathrm{loc}}(\mathrm{Int}(A) \cup S; \mathbb{R}^d),$$

where $S \subset \mathbb{R}^d \setminus \Omega$ represents the region occupied by possibly a different material and it is denoted as *substrate* in analogy with the setting of thin film, while

$$\Sigma := \partial S \cap \partial \Omega$$

is the *contact surface* between the container Ω and the substrate S. In the following we refer to $\mathcal{C}$ as the *configurational space* and to any configuration $(A,u) \in \mathcal{C}$ as a *free crystal* with A and u as the *free-crystal region* and the *free-crystal displacement*, respectively (see Figure 1).

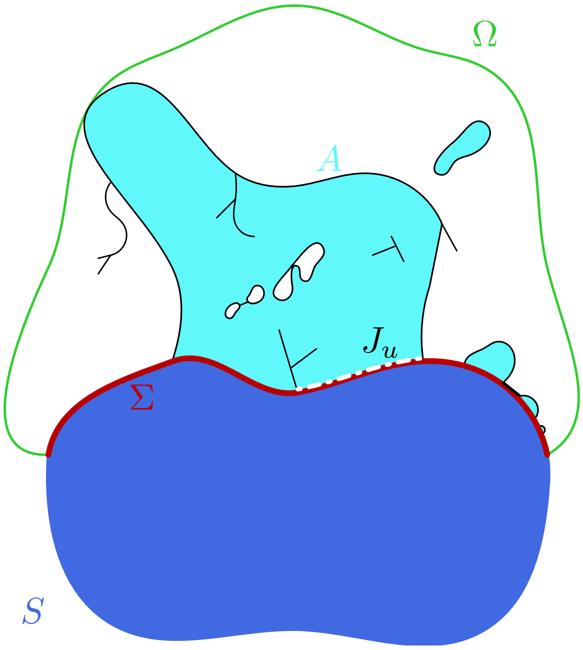

FIGURE 1. An admissible free-crystal region A is displayed in light blue in the container Ω, while the substrate S is represented in dark blue. The boundary of A (with the cracks) is depicted in black, the container boundary in green, the contact surface Σ in red (thicker line) while the free-crystal delamination region J_u with a white dashed line.

We consider the case $d = 2$ as in [39], with the fixed sets Ω and S bounded Lipschitz open, connected, and such that Σ is a Lipschitz 1-manifold. For $d \geq 3$ there are presently no available existence results for the SDRI model. They are available for the isotropic Griffith model with L^p-fidelity term (of the type (2.11)) in [10] and with Dirichlet conditions for the displacements at the boundary in [11]. Moreover, the SDRI energy introduced in [39] was subsequently found for the wetting regime (i.e., the case where free crystals are assumed to always cover the substrate) in [14] as a relaxation formula for both thin films and material voids (without external filaments). The existence results in [14] are achieved by working with displacements in the functional space $GSBD^p_\infty$, $p > 1$, that includes displacements attaining the infinite value in a set of finite perimeter (on which their strain $e(u)$ is defined to be zero, see [14, Page 1055]), which appear to go beyond the classical framework of small displacements of linearized elasticity, and for the thin-film setting (where indeed minimizing displacements are proven to be in $GSBD^p$) under a bounded-graph assumption on film profiles and without considering delamination.

The bulk elastic energy $\mathcal{W}$ in (1.1) is defined by

$$\mathcal{W}(A,u) = \int_{A\cup S} W(z, e(u) - M_0)\, dz,$$

where the *elastic density* W is given by

$$W(z,M) := \mathbb{C}(z)M : M \tag{1.2}$$

for any $z \in \Omega \cup S$ and any 2×2-symmetric matrix $M \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$, and for a positive-definite elasticity tensor $\mathbb{C}$, and attains its minimum value zero for any z at a fixed strain $M_0 \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ in the following referred to as *mismatch strain*. The inclusion in (1.2) of M_0 that is defined by

$$M_0 := \begin{cases} e(u_0) & \text{in } \Omega, \\ 0 & \text{in } S, \end{cases}$$

for a fixed $u_0 \in H^1(\Omega;\mathbb{R}^2)$, together with the fact that both M_0 and $\mathbb{C}$ are let free of jumping across Σ, allows to model the presence of two different materials in the substrate and in the free crystals, and in particular to take into account the *lattice mismatch* that can be present between their optimal crystalline lattices that is crucial, e.g., in the setting of heteroepitaxy [22, 23].

The surface energy $\mathcal{S}$ in (1.1) is defined as

$$\mathcal{S}(A,u) = \int_{\partial A} \psi(z,u,\nu)\, d\mathcal{H}^{d-1},$$

where the *surface tension* ψ is given by

$$\psi(z,u,\nu) := \begin{cases} \varphi(z,\nu_A(z)) & z \in \Omega \cap \partial^* A, \\ 2\varphi(z,\nu_A(z)) & z \in \Omega \cap (A^{(1)} \cup A^{(0)}) \cap \partial A, \\ \varphi(z,\nu_S(z)) + \beta(z) & \Sigma \cap A^{(0)} \cap \partial A, \\ \beta(z) & z \in \Sigma \cap \partial^* A \setminus J_u, \\ \varphi(z,\nu_S(z)) & J_u, \end{cases} \tag{1.3}$$

with $\varphi \in C(\overline{\Omega} \times \mathbb{R}^d; [0,+\infty))$ being a Finsler norm such that $c_1|\xi| \le \varphi(x,\xi) \le c_2|\xi|$ for some $c_1, c_2 > 0$ and representing the *anisotropy* of the free-crystal material, β denoting the *relative adhesion coefficient* on Σ such that, as for capillarity problems [24, 25],

$$|\beta(z)| \le \varphi(z,\nu_S(z))$$

for $z \in \Sigma$, ν coinciding with the exterior normal on the reduced boundary $\partial^* A$, and $A^{(\delta)}$ denoting the set of points of A with density $\delta \in [0,1]$.

The anisotropic form of ψ in (1.3) distinguishes the various portions of the free-crystal topological boundary ∂A, that are the *free boundary* $\partial^* A \cap \Omega$, the set of *internal cracks* $A^{(1)} \cap \Omega \cap \partial A$, the set of *external filaments* $A^{(0)} \cap \Omega \cap \partial A$, the *delaminated region* J_u that coincides with the debonding area, i.e., the portion on the contact surface Σ where there is no bonding between the free crystal and the substrate (even if they are adjacent), the *adhesion area* where the free-crystal displacement is continuous through Σ, i.e., $\Sigma \cap \partial^* A \setminus J_u$, and the *wetting layer* represented by the filaments on Σ, i.e., $\Sigma \cap A^{(0)}$. In particular, ψ weights the different portions of ∂A in relation to the active chemical bondings present at each portion: φ when there is no extra chemical bonding, such as at the free profile and at the delaminated region, and β at the adhesion contact area with the substrate, while both the cracks and at external filaments are counted 2φ and the wetting layer sees the contribution of both ψ and β.

As shown in [39, Theorem 2.8] such specific weights are crucial to obtain the lower semicontinuity of the energy $\mathcal{F}$ in the subfamily $\mathcal{C}_m$ of configurations with free-crystal regions presenting a fixed number $m \in \mathbb{N}$ of boundary connected components, namely

$$\mathcal{C}_m := \Big\{(A,u) \in \mathcal{C} : \ \partial A \text{ has at most } m \text{ connected components}\Big\},$$

with respect to a properly selected topology $\tau_{\mathcal{C}}$, i.e.,

$$\liminf_{k\to\infty} \mathcal{F}(A_k,u_k) \geq \mathcal{F}(A,u)$$

for every $\{(A_k,u_k)\} \subset \mathcal{C}_m$ and $(A,u) \in \mathcal{C}_m$ such that $\mathcal{H}^1(\partial A_k)$ are equibounded, $\mathrm{sdist}(\cdot,\partial A_k) \to \mathrm{sdist}(\cdot,\partial A)$ locally uniformly in $\mathbb{R}^2$ with *sdist* representing the *signed distance* function (recall definition at (2.1)), and $u_n \to u$ a.e. in $\mathrm{Int}(A)\cup S$. In particular, the restriction to the subfamily $\mathcal{C}_m$, which represented already an extension of the more restrictive graph condition assumed in [29] for the particular setting of epitaxially-strained thin films and the starshapedness condition in [30] for material cavities, was needed in [39] not only to prove lower semicontinuity, but also to prove compactness with respect to $\tau_{\mathcal{C}}$, which fails in $\mathcal{C}$ (see Remark 2.3). This enabled in [39, Theorem 2.6] to prove the existence of minimizers $(A_m,u_m) \in \mathcal{C}_m$ of $\mathcal{F}$ among all configurations in $\mathcal{C}_m$, in the following referred to as *m-minimizer* of $\mathcal{F}$, by means of the *direct method* of the calculus of variations.

The aim of the investigation in this paper is to recover the full generality avoiding any extra hypothesis on the admissible free-crystal regions apart from having an $\mathcal{H}^1$-rectifiable and finite-length topological boundary (or displacements u allowed to be infinite on sets with positive measure as in [14]). This is achieved by retrieving compactness with respect to the free-crystal regions at least for any sequence of m-minimizers $(A_m,u_m) \in \mathcal{C}_m$, and by combining the strategies of [20] and [39]. More precisely, the use in [39] of the Golab-type Theorem [34] is avoided for the compactness of the free-crystal regions by adapting to our setting classical *density-estimate* arguments first introduced for surface energies and the Mumford-Shah functional (see, e.g., [1, 43]), and then extended to the Griffith functional [11, 17]. We notice that even though our approach stems from the strategy employed in [20], in our setting there is the further difficulty that compactness and lower semicontinuity along sequences of m-minimizers (with respect to the topology used to find such m-minimizers through the direct method) are missing. Therefore, we directly construct a minimizing candidate, prove that it belongs to $\mathcal{C}$, and finally show a "lower-semicontinuity inequality" (see (1.6) below) with respect to the m-minimizers (see Subsection 1.1 for more details).

The result of this paper are twofold: An existence result in Theorem 2.5, and regularity results in Theorem 2.6. More precisely, in Theorem 2.5 we prove the existence of a minimum configuration of $\mathcal{F}$ among all configurations in $\mathcal{C}$ with free-crystal region satisfying a volume constraint, i.e., we solve the minimum problem

$$\inf_{(A,u)\in\mathcal{C},\ |A|=\mathtt{v}} \mathcal{F}(A,u) \tag{1.4}$$

for a fixed volume parameter $\mathtt{v} \in (0,|\Omega|)$, and we show that the unconstrained problem consisting in minimizing in $\mathcal{C}$ the *volume-penalized functional*

$$\mathcal{F}^{\lambda}(A,u) := \mathcal{F}(A,u) + \lambda\big||A| - \mathtt{v}\big| \tag{1.5}$$

for a *penalization constant* $\lambda > 0$, is equivalent to the minimization problem (1.4) provided that $\lambda \geq \lambda_0$ for some $\lambda_0 > 0$ (independent of (A, u)).

In Theorem 2.6 some regularity properties shared by minimizers of (1.4) are found. Notice that we cannot directly apply the arguments of [29, 30] based on the *external sphere condition* introduced in [13] because of the absence of graph and star-shapedness assumptions on admissible free-crystal regions. As a byproduct of Theorem 2.5 we prove that minimizing free-crystal regions are open sets with cracks coinciding in Ω with the jump set of the corresponding minimizing free-crystal displacements, and that their boundary satisfies uniform upper and lower density estimates. Furthermore, we also observe that, given a minimizer (A, u) any connected component E of A that do not intersect $\Sigma \setminus J_u$ (up to $\mathcal{H}^1$-negligible sets), must have a sufficiently large area, i.e.,

$$|E| \geq (c_1\sqrt{4\pi}/\lambda_0)^2,$$

and must satisfy $u = u_0$ in E.

1.1. **Paper organization and detail of the proofs.** The paper is organized in 5 sections. In Section 2 we introduce the mathematical setting, recall the SDRI model from [39], and carefully state the main results of the paper.

In Section 3 we prove the upper and lower density estimates for the local decay of the energy $\mathcal{F}$ on any sequence of m-minimizers $(A_m, u_m) \in \mathcal{C}_m$ (see Theorem 3.1) by considering a local version of $\mathcal{F}^\lambda$ (see (2.4)) and adapting arguments of [1, 11, 17] to our setting with not only displacements, but displacements paired with free-crystal regions, and paying extra care to the fact that $\mathbb{C}$ is possibly not constant (but in $L^\infty(\Omega \cup S) \cap C^0(\Omega)$).

In Section 4 we begin by establishing in Proposition 4.1 the compactness (up to extracting a subsequence) of the free-crystal regions A_m of the m-minimizers (A_m, u_m) with limit a set of finite perimeter $A \subset \Omega$ by means of both the Blaschke-type selection principle [39, Proposition 3.1] and the density estimates established in Section 3. Then, in Proposition 4.3, we further extend the (already generalized) Golab-type Theorem [34, Theorem 4.2] to a priori not connected $\mathcal{H}^1$-measurable sets satisfying uniform density estimates (see [20] for the isotropic case). Finally, we prove Theorem 2.5. This is achieved by first considering the free-crystal configuration $(\widetilde{A}, \widetilde{u})$ with free-crystal region $\widetilde{A}$ defined as

$$\widetilde{A} := \operatorname{Int}(A \cup \overline{\{x \in A^{(1)} \cap \partial A : \text{ one-sided traces of } \widetilde{u} \text{ at } x \text{ exist and are equal}\}}),$$

where the displacement $\widetilde{u}$ is defined as the weak H^1_{loc}-limit of the displacements u_m in the regular substrate S and in those components of A where such limit exists in H^1_{loc} (up to extracting subsequences and adding infinitesimal rigid displacements), and coincides with the function u_0 on all other components E of A, where we observe that

$$\liminf_{m\to\infty} |u_m(x)| = +\infty$$

for a.e. x in E. The configuration $(\widetilde{A}, \widetilde{u}) \in \mathcal{C}$ (not necessarily to any $\mathcal{C}_m$) and the property

$$\liminf_{h\to\infty} \mathcal{F}(A_{m_h}, u_{m_h}) \geq \mathcal{F}(\widetilde{A}, \widetilde{u}). \tag{1.6}$$

follows from the blow-up method differently performed for each portion of the $\partial\widetilde{A}$. In particular extra care is needed for $\widetilde{A}$-cracks (where by definition of $\widetilde{A}$ there is no bound on the number of connected components) where we need to extend some ideas from [39, Proposition 4.1]. This yields the assertion in view of property

$$\inf_{(A,u)\in\mathcal{C},\ |A|=\mathtt{v}} \mathcal{F}(A, u) = \inf_{(A,u)\in\mathcal{C}} \mathcal{F}^\lambda(A, u) = \lim_{m\to\infty} \inf_{(A,u)\in\mathcal{C}_m,\ |A|=\mathtt{v}} \mathcal{F}(A, u).$$

following from [39, Theorem 2.6].

In Section 5 we prove Theorem 2.6, whose regularity properties are direct consequence of the density estimates of Section 3, comparison arguments, the isoperimetric inequality in $\mathbb{R}^2$, and the equivalence of the constrained minimum problem (1.4) and the unconstrained penalized minimum problem related to (1.5).

2. Mathematical setting

In this section we recall the SDRI model from [39], collect all the definitions and hypothesis and state the main results of the paper. Since our model is two-dimensional, unless otherwise stated, all sets we consider are subsets of $\mathbb{R}^2$. We choose the standard basis $\{\mathbf{e_1} = (1,0), \mathbf{e_2} = (0,1)\}$ in $\mathbb{R}^2$ and denote the coordinates of $x \in \mathbb{R}^2$ with respect to this basis by (x_1, x_2). We denote by $\mathrm{Int}(A)$ the interior of $A \subset \mathbb{R}^2$. Given a Lebesgue measurable set E, we denote by χ_E its characteristic function and by $|E|$ its Lebesgue measure. The set

$$E^{(\alpha)} := \Big\{x \in \mathbb{R}^2 : \lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \alpha\Big\}, \qquad \alpha \in [0,1],$$

where $B_r(x)$ denotes the ball in $\mathbb{R}^2$ centered at x of radius $r > 0$,is called the set of points of density α of E. Clearly, $E^{(\alpha)} \subset \partial E$ for any $\alpha \in (0,1)$, where

$$\partial E := \{x \in \mathbb{R}^2 : \ B_r(x) \cap E \neq \emptyset \text{ and } B_r(x) \setminus E \neq \emptyset \text{ for any } r > 0\}$$

is the topological boundary. The set $E^{(1)}$ is the *Lebesgue set* of E and $|E^{(1)} \Delta E| = 0$. We denote by $\partial^* E$ the *reduced* boundary of a finite perimeter set E [1, 35], i.e.,

$$\partial^* E := \Big\{x \in \mathbb{R}^2 : \ \exists \nu_E(x) := -\lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}, \quad |\nu_E(x)| = 1\Big\}.$$

The vector $\nu_E(x)$ is called the *measure-theoretic* normal to ∂E.

The symbol $\mathcal{H}^s$, $s \geq 0$, stands for the s-dimensional Hausdorff measure. An $\mathcal{H}^1$-measurable set K with $0 < \mathcal{H}^1(K) < \infty$ is called $\mathcal{H}^1$-*rectifiable* if $\theta^*(K,x) = \theta_*(K,x) = 1$ for $\mathcal{H}^1$-a.e. $x \in K$, where

$$\theta^*(K,x) := \limsup_{r\to 0^+} \frac{\mathcal{H}^1(B_r(x) \cap K)}{2r}, \qquad \theta_*(K,x) := \liminf_{r\to 0^+} \frac{\mathcal{H}^1(B_r(x) \cap K)}{2r}.$$

By [28, Theorem 2.3] any $\mathcal{H}^1$-measurable set K with $0 < \mathcal{H}^1(K) < \infty$ satisfies $\theta^*(K,x) = 1$ for $\mathcal{H}^1$-a.e. $x \in K$.

Remark 2.1 ([35])**.** If E is a finite perimeter set, then

(a) $\overline{\partial^* E} = \partial E^{(1)}$;
(b) $\partial^* E \subseteq E^{(1/2)}$ and $\mathcal{H}^1(E^{(1/2)} \setminus \partial^* E) = 0$;
(c) $P(E,B) = \mathcal{H}^1(B \cap \partial^* E) = \mathcal{H}^1(B \cap E^{(1/2)})$ for any Borel set E.

The notation $\mathrm{dist}(\cdot, E)$ stands for the distance function from the set $E \subset \mathbb{R}^2$ with the convention that $\mathrm{dist}(\cdot, \emptyset) \equiv +\infty$. Given a set $A \subset \mathbb{R}^2$, we consider also signed distance function from ∂A, negative inside, defined as

$$\mathrm{sdist}(x, \partial A) := \begin{cases} \mathrm{dist}(x, A) & \text{if } x \in \mathbb{R}^2 \setminus A, \\ -\mathrm{dist}(x, \mathbb{R}^2 \setminus A) & \text{if } x \in A. \end{cases} \tag{2.1}$$

Remark 2.2. The following assertions are equivalent:

(a) $\mathrm{sdist}(x, \partial E_k) \to \mathrm{sdist}(x, \partial E)$ locally uniformly in $\mathbb{R}^2$;
(b) $E_k \overset{\mathcal{K}}{\to} \overline{E}$ and $\mathbb{R}^2 \setminus E_k \overset{\mathcal{K}}{\to} \mathbb{R}^2 \setminus \mathrm{Int}(E)$, where $\mathcal{K}$–Kuratowski convergence of sets [18].

Moreover, either assumption implies $\partial E_k \overset{\mathcal{K}}{\to} \partial E$.

Given $r > 0$, $\nu \in \mathbb{S}^1$ and $x \in \mathbb{R}^2$ we denote by $U_{r,\nu}(x)$ the square of sidelength $2r$ centered at x whose sides are either parallel or perpendicular to ν. When $\nu = \mathbf{e_2}$ or $\nu = \mathbf{e_1}$, drop the dependence on ν and write $U_r(x)$ if in addition $x = 0$, we write just U_r. Given $x \in \mathbb{R}^2$ and $r > 0$, the blow-up map $\sigma_{x,r}$ is defined as

$$\sigma_{x,r}(y) = \frac{y - x}{r}. \tag{2.2}$$

The blow-up $K \subset \mathbb{R}^2$ is defined as $\sigma_{x,r}(K)$.

2.1. **The SDRI model.** Given two nonempty open sets $\Omega \subset \mathbb{R}^2$ and $S \subset \mathbb{R}^2 \setminus \Omega$, we define the family of admissible regions for the *free crystal* and the space of *admissible configurations* by

$$\mathcal{A} := \{A \subset \overline{\Omega} : \ \partial A \text{ is } \mathcal{H}^1\text{-rectifiable and } \mathcal{H}^1(\partial A) < \infty\}$$

and

$$\mathcal{C} := \big\{(A,u) : A \in \mathcal{A}, \\ u \in GSBD^2(\mathrm{Int}(A \cup S \cup \Sigma); \mathbb{R}^2) \cap H^1_{\mathrm{loc}}(\mathrm{Int}(A) \cup S; \mathbb{R}^2)\big\},$$

respectively, where $\Sigma := \partial S \cap \partial\Omega$ and $GSBD^2(E, \mathbb{R}^2)$ is the collection of all generalized special functions of bounded deformation [12, 19]. Given a displacement field $u \in GSBD^2(\mathrm{Int}(A \cup S \cup \Sigma); \mathbb{R}^2) \cap H^1_{\mathrm{loc}}(\mathrm{Int}(A) \cup S; \mathbb{R}^2)$ we denote by $e(u(\cdot))$ the density of $\mathbf{e}(u) = (Du + (Du)^T)/2$ with respect to Lebesgue measure $\mathcal{L}^2$ and by J_u the jump set of u. Recall that $e(u) \in L^2(A \cup S)$ and J_u is $\mathcal{H}^1$-rectifiable. Notice also that assumption $u \in H^1_{\mathrm{loc}}(\mathrm{Int}(A) \cup S; \mathbb{R}^2)$ implies $J_u \subset \Sigma \cap \overline{\partial^* A}$.

Unless otherwise stated, in what follows Ω is a bounded Lipchitz open set, $\Sigma \subsetneq \partial\Omega$ is a Lipschitz 1-manifold and $S \subset \mathbb{R}^2 \setminus \Omega$ is a nonempty connected bounded Lipschitz open set.

The *energy* of admissible configurations is given by $\mathcal{F} : \mathcal{C} \to [-\infty, +\infty]$,

$$\mathcal{F} := \mathcal{S} + \mathcal{W},$$

where $\mathcal{S}$ and $\mathcal{W}$ are the surface and elastic energies of the configuration, respectively. The surface energy of $(A,u) \in \mathcal{C}$ is defined as

$$\begin{aligned} \mathcal{S}(A,u) := &\int_{\Omega \cap \partial^* A} \varphi(x, \nu_A(x)) d\mathcal{H}^1(x) \\ &+ \int_{\Omega \cap (A^{(1)} \cup A^{(0)}) \cap \partial A} \big(\varphi(x, \nu_A(x)) + \varphi(x, -\nu_A(x))\big) d\mathcal{H}^1(x) \\ &+ \int_{\Sigma \cap A^{(0)} \cap \partial A} \big(\varphi(x, \nu_\Sigma(x)) + \beta(x)\big) d\mathcal{H}^1(x) \\ &+ \int_{\Sigma \cap \partial^* A \setminus J_u} \beta(x) d\mathcal{H}^1(x) + \int_{J_u} \varphi(x, -\nu_\Sigma(x))\, d\mathcal{H}^1(x), \end{aligned} \tag{2.3}$$

where $\varphi : \overline{\Omega} \times \mathbb{S}^1 \to [0, +\infty)$ and $\beta : \Sigma \to \mathbb{R}$ are Borel functions denoting the *anisotropy* of crystal and the *relative adhesion* coefficient of the substrate, respectively, and $\nu_\Sigma := \nu_S$. In the following we refer to the first term in (2.3) as the *free-boundary energy*, to the second as the *energy of internal cracks and external filaments*, to the third as the *wetting-layer energy*, to the fourth as the *contact energy*, and to the last as the *delamination energy*. In applications instead of $\varphi(x, \cdot)$ it is more convenient to use its positively one-homogeneous extension $|\xi|\varphi(x, \xi/|\xi|)$. With a slight abuse of notation we denote this extension also by φ.

The elastic energy of $(A,u) \in \mathcal{C}$ is defined as

$$\mathcal{W}(A,u) := \int_{A \cup S} W(x, e(u(x)) - M_0(x)) dx,$$

where the elastic density W is determined as the quadratic form

$$W(x,M) := \mathbb{C}(x)M : M,$$

by the so-called *stress-tensor*, a measurable function $x \in \Omega \cup S \to \mathbb{C}(x)$, where $\mathbb{C}(x)$ is a nonnegative fourth-order tensor in the Hilbert space $\mathbb{M}^{2\times 2}_{\rm sym}$ of all 2×2-symmetric matrices with the natural inner product

$$M : N = \sum_{i,j=1}^{2} M_{ij}N_{ij}$$

for $M = (M_{ij})_{1\le i,j\le 2}$, $N = (N_{ij})_{1\le i,j\le 2} \in \mathbb{M}^{2\times 2}_{\rm sym}$.

The *mismatch strain* $x \in \Omega \cup S \mapsto M_0(x) \in \mathbb{M}^{2\times 2}_{\rm sym}$ is given by

$$M_0 := \begin{cases} e(u_0) & \text{in } \Omega,\\ 0 & \text{in } S, \end{cases}$$

for a fixed $u_0 \in H^1(\Omega;\mathbb{R}^2)$.

Given $m \ge 1$, let $\mathcal{A}_m$ be a collection of all subsets A of $\overline{\Omega}$ such that ∂A has at most m connected components and let

$$\mathcal{C}_m := \Big\{(A,u) \in \mathcal{C} : \ A \in \mathcal{A}_m\Big\}$$

to be the set of constrained admissible configurations.

Remark 2.3. The reason to introduce $\mathcal{C}_m$ is that $\mathcal{C}_m$ is both closed under $\tau_{\mathcal{C}}$-convergence (see [39, Definition 2.5]) and $\mathcal{F}$ is lower semicontinuous with respect to $\tau_{\mathcal{C}}$ in $\mathcal{C}_m$ (see [39, Theorems 2.7 and 2.8]). Such two properties do not apply instead to $\mathcal{C}$ as the following examples show.

We begin by recalling that a sequence $\{(A_k,u_k)\} \subset \mathcal{C}$ is said to $\tau_{\mathcal{C}}$-converge to $(A,u) \subset \mathcal{C}$ and we denote by $(A_k,u_k) \xrightarrow{\tau_{\mathcal{C}}} (A,u)$, if

- $\sup\limits_{k\ge 1} \mathcal{H}^1(\partial A_k) < \infty$,
- $\mathrm{sdist}(\cdot,\partial A_k) \to \mathrm{sdist}(\cdot,\partial A)$ locally uniformly in $\mathbb{R}^2$ as $k\to\infty$,
- $u_k \to u$ a.e. in $\mathrm{Int}(A)\cup S$.

Let $X := \{x_n\}$ be a countable dense set in Ω and $A \in \mathcal{A}$ such that $|A| = \mathtt{v} \in (0,|\Omega|]$. Then the sets $A_k := A \setminus \{x_1,\dots,x_k\} \in \mathcal{A}$, $k\in\mathbb{N}$, are such that $|A_k| = \mathtt{v} \in (0,|\Omega|)$, $\mathcal{H}^1(\partial A_k) = \mathcal{H}^1(\partial A)$, and $(A_k,0) \xrightarrow{\tau_{\mathcal{C}}} (A\setminus X,0)$ as $k\to\infty$, but $A\setminus X \notin \mathcal{A}$ since $\partial(A\setminus X) = \overline{A}$. Therefore, compactness with respect to $\tau_{\mathcal{C}}$ fails in $\mathcal{C}$.

Furthermore, let $\Gamma \subset A$ be a segment such that $\mathcal{H}^1(\Gamma) > 0$, $B := A\setminus\Gamma$, $B_k := A\setminus(\Gamma\cap\{x_1,\dots,x_k\})$ for every $k\in\mathbb{N}$, and assume that X is dense in Γ. We notice that $\{(B_k,0)\}\subset\mathcal{C}$, $(B,0)\in\mathcal{C}$, $|B_k| = |B| = |A|$, $(B_k,0) \xrightarrow{\tau_{\mathcal{C}}} (B,0)$ as $k\to\infty$. However,

$$\mathcal{F}(B_k,0) = \mathcal{F}(A,0) < \mathcal{F}(A\setminus\Gamma,0) = \mathcal{F}(B,0).$$

Therefore, lower semicontinuity of $\mathcal{F}$ with respect to $\tau_{\mathcal{C}}$ fails in $\mathcal{C}$.

2.2. **Localized energy.** In this subsection we introduce the notion of quasi minimizers of $\mathcal{F}$ in Ω and the localized version $\mathcal{F}(\cdot;O) : \mathcal{C}_m \to \mathbb{R}$ of $\mathcal{F}$ for open sets $O\subset\Omega$ and for $m\in\mathbb{N}\cup\{\infty\}$ with the convention $\mathcal{C}_\infty := \mathcal{C}$. We define

$$\mathcal{F}(A,u;O) := \mathcal{S}(A;O) + \mathcal{W}(A,u;O), \tag{2.4}$$

where

$$\mathcal{S}(A;O) := \int_{O\cap\partial^*A} \varphi(y,\nu_A)\,d\mathcal{H}^1 + 2\int_{O\cap(A^{(1)}\cup A^{(0)})\cap\partial A} \varphi(y,\nu_A)\,d\mathcal{H}^1$$

and

$$\mathcal{W}(A,u;O) = \int_{O\cap A} \mathbb{C}(y)e(u-u_0) : e(u-u_0)dy$$

are the localized versions of the surface and the elastic energy, respectively. Note that since $\mathcal{F}(\cdot;O)$ does not "see" the substrate S, $\mathcal{S}(\cdot;O)$ does not depend on u.

Definition 2.4. Given $\Lambda \geq 0$ and $m \in \mathbb{N}\cup\{\infty\}$, the configuration $(A,u) \in \mathcal{C}_m$ is a local (Λ,m)-*minimizer* of $\mathcal{F} : \mathcal{C}_m \to \mathbb{R}$ in O if

$$\mathcal{F}(A,u;O) \leq \mathcal{F}(B,v;O) + \theta|A\Delta B|$$

whenever $(B,v) \in \mathcal{C}_m$ with $A\Delta B \subset\subset O$ and $\operatorname{supp}(u-v) \subset\subset O$. Furthermore, we define

$$\Phi(A,u;O) := \inf\Big\{\mathcal{F}(B,v;O) : (B,v) \in \mathcal{C}_m,\quad B\Delta A \subset\subset O,\ \operatorname{supp}(u-v) \subset\subset O\Big\} \tag{2.5}$$

and

$$\Psi(A,u;O) := \mathcal{F}(A,u;O) - \Phi(A,u;O) \tag{2.6}$$

for every $(A,u) \in \mathcal{C}_m$ and every open set $O \subset\subset \Omega$.

2.3. **Main results.** We begin by stating the hypotheses which will be assumed throughout the paper:

(H1) $\varphi \in C(\overline{\Omega}\times\mathbb{R}^2)$ and is a Finsler norm, i.e., there exist $c_2 \geq c_1 > 0$ such that for every $x \in \overline{\Omega}$, $\varphi(x,\cdot)$ is a norm in $\mathbb{R}^2$ satisfying

$$c_1|\xi| \leq \varphi(x,\xi) \leq c_2|\xi| \tag{2.7}$$

for any $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^2$;

(H2) $\beta \in L^\infty(\Sigma)$ and satisfies

$$-\varphi(x,\nu_\Sigma(x)) \leq \beta(x) \leq \varphi(x,\nu_\Sigma(x)) \tag{2.8}$$

for $\mathcal{H}^1$-a.e. $x \in \Sigma$;

(H3) $\mathbb{C} \in L^\infty(\Omega\cup S)\cap C^0(\Omega)$ and there exists $c_4 \geq c_3 > 0$ such that

$$2c_3\, M : M \leq \mathbb{C}(x)M : M \leq 2c_4\, M : M \tag{2.9}$$

for any $x \in \Omega\cup S$ and $M \in \mathbb{M}^{2\times2}_{\rm sym}$.

The first result of the paper is the *global existence* in $\mathcal{C}$.

Theorem 2.5 (**Existence**). *Assume* (H1)-(H3). *Let either* $\mathtt{v} \in (0,|\Omega|)$ *or* $S = \emptyset$. *Then the minimum problem*

$$\inf_{(A,u)\in\mathcal{C},\ |A|=\mathtt{v}} \mathcal{F}(A,u) \tag{2.10}$$

admits a solution. Moreover, there exists $\lambda_0 > 0$ *such that* $(A,u) \in \mathcal{C}$ *is a solution to* (2.10) *if and only if it solves*

$$\inf_{(A,u)\in\mathcal{C}} \mathcal{F}^\lambda(A,u)$$

for any $\lambda \geq \lambda_0$, *where*

$$\mathcal{F}^\lambda(A,u) := \mathcal{F}(A,u) + \lambda\big||A| - \mathtt{v}\big|;$$

For simplicity we call the solutions of (2.10) the *global minimizers*.

The second result of the paper is a *regularity result.*

Theorem 2.6 (**Properties of global minimizers**). *Assume* (H1)-(H3). *Let either* $\mathtt{v} \in (0,|\Omega|)$ *or* $S = \emptyset$, *and let* $(A,u) \in \mathcal{C}$ *be a solution of* (2.10). *Then:*

(A) there exists an open set $G \subset A$ such that $\mathcal{H}^1(\partial A \Delta \partial G) = 0$ and $\mathcal{F}(G,u) = \mathcal{F}(A,u)$, hence A can be supposed to be open;

(B) the closure of the set of all $x \in A^{(1)} \cap \partial A$ at which one-sided traces $u^+(x)$ and $u^-(x)$ exist and are equal, is $\mathcal{H}^1$-negligible, hence cracks are jump set for the deformation u;

(C) for any $x \in \Omega$ and $r \in (0, \min\{1, \mathrm{dist}(x, \partial\Omega)\})$,

$$\frac{\mathcal{H}^1(U_r(x) \cap \partial A)}{r} \le \frac{16c_2 + 4\lambda_0}{c_1},$$

where c_1, c_2 are given by (2.7);

(D) there exist $\varsigma_0 = \varsigma_0(c_3, c_4) \in (0,1)$ and $R_0 = R_0(c_1, c_2, c_3, c_4) > 0$, where c_3, c_4 are given by (2.9), with the following property: if $x \in \Omega \cap \partial A$ is such that $\theta_*(\partial A, x) > 0$, then

$$\frac{\mathcal{H}^1(U_r(x) \cap \partial A)}{r} \ge \varsigma_0$$

for any ball $U_r(x) \subset\subset \Omega$ with $r \in (0, R_0)$;

(E) if $E \subset A$ is any connected component of A with $\mathcal{H}^1(\partial E \cap \Sigma \setminus J_u) = 0$, then $|E| \ge (c_1\sqrt{4\pi}/\lambda_0)^2$ and $u = u_0$ in E, where $\lambda_0 > 0$ is given in Theorem 2.5.

Notice that properties (A)-(D) of Theorem 2.6 hold also for every (Λ, ∞)-minimizers of $\mathcal{F}$ (see Definition 2.4). In what follows we refer to the estimates in (C) and (D) as the (uniform) *upper and lower density estimate*, respectively.

2.4. **Applications.** We recall that the SDRI model introduced in [39] includes the settings of various free boundary problems, some of which are outlined below.

– *Epitaxially-strained thin films, e.g.,* [5, 7, 22, 23, 29, 33]: $\Omega := (a,b) \times (0, +\infty)$, $S := (a,b) \times (-\infty, 0)$ for some $a < b$, free crystals in the subfamily

$$\mathcal{A}_{\mathrm{subgraph}} := \{A \subset \Omega :\ \exists h \in BV(\Sigma; [0,\infty)) \text{ and l.s.c. such that } A = A_h\} \subset \mathcal{A}_1,$$

where $A_h := \{(x^1, x^2)\ :\ 0 < x^2 < h(x^1)\}$, and admissible configurations in the subspace

$$\mathcal{C}_{\mathrm{subgraph}} := \{(A,u) :\ A \in \mathcal{A}_{\mathrm{subgraph}},\ u \in H^1_{\mathrm{loc}}(\mathrm{Int}(A \cup S \cup \Sigma); \mathbb{R}^2)\} \subset \mathcal{C}_1$$

(see also [4, 36]).

– *Crystal cavities, e.g.,* [30, 32, 45, 46]: $\Omega \subset \mathbb{R}^2$ smooth set containing the origin, $S := \mathbb{R}^2 \setminus \Omega$, free crystals in the subfamily

$$\mathcal{A}_{\mathrm{starshaped}} := \{A \subset \Omega :\ \text{open, starshaped w.r.t. } (0,0), \text{ and } \partial\Omega \subset \partial A\} \subset \mathcal{A}_1,$$

and the space of admissible configurations

$$\mathcal{C}_{\mathrm{starshaped}} := \{(A,u) :\ \ A \in \mathcal{A}_{\mathrm{starshaped}},\ u \in H^1_{\mathrm{loc}}(\mathrm{Int}(A \cup S \cup \Sigma); \mathbb{R}^2)\} \subset \mathcal{C}_1.$$

– *Capillarity droplets, e.g.,* [8, 24, 25]: $\Omega \subset \mathbb{R}^2$ is a bounded open set (or a cylinder), $\mathbb{C} = 0$, $S = \emptyset$, and admissible configurations in the collection

$$\mathcal{C}_{\mathrm{capillary}} := \{(A, 0) :\ \ A \in \mathcal{A}\} \subset \mathcal{C}.$$

– *Griffith fracture model, e.g.,* [6, 9, 10, 15, 17, 31]: $S = \Sigma = \emptyset$, $E_0 \equiv 0$, and the space of configurations

$$\mathcal{C}_{\mathrm{Griffith}} := \{(\Omega \setminus K, u) :\ K \text{ closed, } \mathcal{H}^1\text{-rectifiable, } u \in H^1_{\mathrm{loc}}(\Omega \setminus K; \mathbb{R}^2)\} \subset \mathcal{C}.$$

We notice that the same arguments employed to prove Theorem 2.5 seems to be adaptable to the case where a *fidelity term*, namely

$$\kappa \int_{\Omega \setminus \partial A} |u - g|^p dx \tag{2.11}$$

for $p \in (1,\infty)$, $\kappa > 0$, and $g \in L^\infty(\Omega;\mathbb{R}^2)$ if $p \in (1,2]$, and $g \in W^{1,p}(\Omega;\mathbb{R}^2)$ if $p \in (2,\infty)$, is added to the energy $\mathcal{F}$, providing, in particular, a different proof (and an extension to the anisotropic case) of [17, Theorem 1.2].

– *Mumford-Shah model, e.g.,* [1, 20, 42]: $S = \Sigma = \emptyset$, $E_0 = 0$, $\mathbb{C}$ is such that the elastic energy $\mathcal{W}$ reduces to the Dirichlet energy, and the space of configurations

$$\mathcal{C}_{\mathrm{Mumfard-Shah}} := \{(\Omega \setminus K, u) \in \mathcal{C}_{\mathrm{Griffith}} :\ u = (u_1, 0)\} \subset \mathcal{C}.$$

– *Boundary delaminations, e.g.,* [3, 26, 38, 40, 41, 47]: the SDRI model includes also the setting of debonding and edge delamination in composites [47]. The focus is here on the 2-dimensional film and substrate vertical section, while in [3, 40, 41] a reduced model for the horizontal interface between the film and the substrate is derived.

3. Decay estimates

In this section we always assume $m \in \mathbb{N}$ apart from Lemma 3.2, and we consider either $\mathrm{v} \in (0,|\Omega|)$ or $S = \emptyset$. We recall that by [39, Theorem 2.6] under the hypotheses (H1)-(H3) both the volume-contrained minimum problem

$$\inf_{(A,u)\in\mathcal{C}_m,\ |A|=\mathrm{v}} \mathcal{F}(A,u),$$

and the unconstrained minimum problem

$$\inf_{(A,u)\in\mathcal{C}_m} \mathcal{F}^\lambda(A,u)$$

admit a solution. Moreover, by [39, Theorem 2.6] there exists $\lambda_0 > 0$ such that

$$\inf_{(A,u)\in\mathcal{C},\ |A|=\mathrm{v}} \mathcal{F}(A,u) = \inf_{(A,u)\in\mathcal{C}} \mathcal{F}^\lambda(A,u) = \lim_{m\to\infty} \inf_{(A,u)\in\mathcal{C}_m,\ |A|=\mathrm{v}} \mathcal{F}(A,u) \tag{3.1}$$

for every $\lambda \geq \lambda_0$.

The main results of this section are the following density estimates for the minimizers of $\mathcal{F}$ in $\mathcal{C}_m$.

Theorem 3.1 (**Density estimates**). *There exist $\varsigma_* = \varsigma_*(c_3,c_4) \in (0,1)$ and $R_* = R_*(c_1,c_2,c_3,c_4) > 0$, where c_i are given by (2.9), such that if $(A,u) \in \mathcal{C}_m$ is a (Λ,m)-minimizer of $\mathcal{F}$ in $\mathcal{C}_m$, then for any $x \in \Omega$ and $r \in (0,\min\{1,\mathrm{dist}(x,\partial\Omega)\})$,*

$$\frac{\mathcal{H}^1(U_r(x)\cap\partial A)}{r} \leq \frac{16c_2 + 4\Lambda}{c_1}. \tag{3.2}$$

Moreover, if $x \in \Omega \cap \partial A$ is such that $\theta_*(\partial A, x) > 0$, then

$$\frac{\mathcal{H}^1(U_r(x)\cap\partial A)}{r} \geq \varsigma_* \tag{3.3}$$

for any ball $U_r(x) \subset\subset \Omega$ with $r \in (0,R_*)$.

Note that by (3.1) the minimizers of $\mathcal{F}$ in $\mathcal{C}_m$ are also (λ_0,m)-minimizers of $\mathcal{F}$. Indeed, since (A,u) is a minimizer of $\mathcal{F}^{\lambda_0}$ in $\mathcal{C}_m$, for any open set $O \subset \Omega$ and $(B,v) \in \mathcal{C}_m$ with $A\Delta B \subset\subset O$ and $\mathrm{supp}\,(u-v) \subset\subset O$ we have

$$\mathcal{F}(A,u;O) \leq \mathcal{F}(B,v;O) + \lambda_0\big||A| - |B|\big| \leq \mathcal{F}(B,v;O) + \lambda_0\big|A\Delta B\big|.$$

Hence, (A,u) is (λ_0,m)-minimizer of $\mathcal{F}(\cdot;\Omega)$ in Ω.

To prove Theorem 3.1 we start with the following adaptation of [10, Theorem 3] to our setting (of set-function pairs).

Lemma 3.2. *Let* $\eta \in (0, 1/32)$ *and* $c_0 > 1$ *be given by* [10, Theorem 3], *and* $m \in \mathbb{N} \cup \{\infty\}$. *Then for any admissible* $(A, u) \in \mathcal{C}_m$ *and a square* $U_R(x_0) \subset \Omega$ *of sidelength* $2R > 0$ *with*

$$\delta := R^{-1/2}\mathcal{H}^1(U_R(x_0) \cap \partial A)^{1/2} < \eta$$

there exist $v \in GSBD^2(\mathrm{Int}(\Omega \cup S \cup \Sigma); \mathbb{R}^2)$, $B \in \mathcal{A}$ *with* $(B, v\big|_B) \in \mathcal{C}_m$ *and a Lebesgue measurable set* $\omega \subset\subset U_R(x_0)$ *such that*

(1) $v \in C^\infty(U_{R(1-\sqrt{\delta})})$, $A\Delta B \subset\subset U_R \setminus U_{R(1-\sqrt{\delta})}(x_0)$ *and* $\mathrm{supp}\,(\widetilde{u} - v) \subset\subset U_R(x_0)$, *where*

$$\widetilde{u} := u\chi_{U_R(x_0)\cap A} + \xi\chi_{U_R(x_0)\setminus A},$$

where $\xi \in U_R$ *is chosen such that* $U_R \cap \partial^* A \subset J_{\widetilde{u}}$;

(2) $\mathcal{H}^1(\partial B \setminus \partial A) \le c_0 \sqrt{\delta}\, \mathcal{H}^1([U_R(x_0) \setminus U_{R(1-\sqrt{\delta})}(x_0)] \cap \partial A)$;

(3) $|\omega| \le c_0\delta\, \mathcal{H}^1(U_R(x_0) \cap \partial A)$ *and*

$$\int_{U_R(x_0)\setminus\omega} |v - \widetilde{u}|^2 dx \le c_0\delta^2 R^2 \int_{U_R(x_0)} |e(\widetilde{u})|^2 dx;$$

(4) *for any* $\psi \in \mathrm{Lip}(U_R; [0,1])$ *and elasticity tensor* $\mathbb{C} \in L^\infty(U_R)$ *with*

$$d_1 M : M \le \mathbb{C}(x)M : M \le d_2 M : M, \qquad (x, M) \in U_R \times \mathbb{M}^{2\times 2}_{\mathrm{sym}}, \tag{3.4}$$

there exist $d_3 := d_3(c_0, d_1, d_2) > 0$ *and* $s := s(c_0, d_1, d_2) \in (0, 1/2)$ *such that*

$$\int_{U_R} \psi\mathbb{C}(x)e(v) : e(v)dx \le \int_{U_R\cap A} \psi\mathbb{C}(x)e(u) : e(u)dx + d_3\,\delta^s\,(1 + R\,\mathrm{Lip}(\psi)) \int_{U_R\cap A} |e(u)|^2 dx.$$

Proof. Without loss of generality we assume that $x_0 = 0$. Possibly perturbing $\widetilde{u}$ by a $W^{1,\infty}$-function with small norm supported near $U_R \cap \partial A$ we suppose that $J_{\widetilde{u}} = U_R \cap \partial A$ up to a $\mathcal{H}^1$-negligible set. We rescale everything by factor R, i.e., let

$$A_R := \frac{A}{R}, \qquad \widetilde{u}_R(x) := \widetilde{u}(Rx), \qquad u_R(x) := u(Rx);$$

then

$$\delta := \Big(\frac{\mathcal{H}^1(U_R \cap \partial A)}{R}\Big)^{1/2} = \mathcal{H}^1(U_1 \cap \partial A_R)^{1/2} = \mathcal{H}^1(U_1 \cap J_{\widetilde{u}_R})^{1/2}.$$

We variate the arguments of [10, Theorem 3]. Set $N := [1/\delta]$ so that $(-N\delta, N\delta)^2 \subset U_1$. For $i := 0, 1, \ldots, N-1$ let $U^i := (-(N-i)\delta, (N-i)\delta)^2$ and $C^i := U^i \setminus U^{i+1}$ (assuming $C^{N-1} := U^{N-1}$). Up to a slight translation of U^i we assume that $\mathcal{H}^1(\partial A_R \cap \partial U^i) = 0$ for all i. Let $i_0 \ge 1$ be such that

$$\begin{cases} \int_{C^{i_0}\cup C^{i_0+1}} |e(\widetilde{u}_R)|^2 dx \le 8\sqrt{\delta} \int_{U_1\setminus U_{1-\sqrt{\delta}}} |e(\widetilde{u}_R)|^2 dx, \\ \mathcal{H}^1(\partial A_R \cap (C^{i_0} \cup C^{i_0+1})) \le 8\sqrt{\delta}\, \mathcal{H}^1(\partial A_R \cap (U_1 \setminus U_{1-\delta})) \end{cases}$$

(the existence of such i^0 follows from [10, Lemma 3.3]). We partition U^{i_0+1} into pairwise disjoint squares with sidelength δ and divide the slice C^{i_0} into dyadic slices

$$G_j := (-(N - i_0 - 2^{-j})\delta, (N - i_0 - 2^{-j})\delta)^2 \setminus (-(N - i_0 - 2^{-j+1})\delta, (N - i_0 - 2^{-j+1})\delta)^2,$$

then we partition each slice G_j into pairwise disjoint squares $U_{j,l}$ of sidelength $2^{-j}\delta$ whose sides parallel to the coordinate axis. Let $\mathcal{V}_0$ be the collection of all squares of sidelength δ which cover the central square U^{i_0+1} and let $\mathcal{V}$ be the union of $\mathcal{V}_0$ and the collection of all $U_{j,l}$. Let us call a square $U \in \mathcal{V}$ "good" if

$$\mathcal{H}^1(U''' \cap \partial A_R) \le \eta\delta_U, \tag{3.5}$$

where U''' is the square with the same center as U and dilated by 7/6, and $\delta_U := \delta$ if $U \in \mathcal{W}_0$ and $\delta_U := 2^{-j}\delta$ if $U \subset S_j$. We say U "bad" if it does not satisfy (3.5). Since $\delta^2 = \mathcal{H}^1(U_1 \cap \partial A_R) < \eta\delta$, by definition, all squares in $\mathcal{V}_0$ are good and by [10, Eq. 3.10] sum of perimeters of all bad cubes satisfies

$$\sum_{U\in\mathcal{W}, U \text{ bad}} P(U) \le c_0\sqrt{\delta}\,\mathcal{H}^1((U_1\setminus U_{1-\sqrt{\delta}})\cap\partial A_R) \tag{3.6}$$

for some $c_0 > 0$. Since $\delta < \eta$, by [10, Theorem 3] there exist $\widetilde{v}_R \in GSBD^2(U_1;\mathbb{R}^2)$, $r \in (1-\sqrt{\delta}, 1)$ and a Lebesgue measurable set $\widetilde{\omega}_R \subset\subset U_r$ such that

(a1) $\widetilde{v}_R \in C^\infty(U_{1-\sqrt{\delta}})$, $\widetilde{u}_R = \widetilde{v}_R$ in $U_1\setminus U_r$ and $\mathcal{H}^1(J_{\widetilde{u}_R}\cap\partial U_r) = \mathcal{H}^1(J_{\widetilde{v}_R}\cap\partial U_r) = 0$;
(a2) $\mathcal{H}^1(J_{\widetilde{v}_R}\setminus J_{\widetilde{u}_R}) \le c_0\sqrt{\delta}\,\mathcal{H}^1((U_1\setminus U_{1-\sqrt{\delta}})\cap J_{\widetilde{u}_R})$;
(a3) $|\widetilde{\omega}_R| \le c_0\delta\mathcal{H}^1(U_r\cap\partial A_R)$ and $\int_{U_1\setminus\widetilde{\omega}_R}|\widetilde{v}_R-\widetilde{u}_R|^2dx \le c_0\delta^2\int_{U_1}|e(\widetilde{u}_R)|^2dx$;
(a4) for any $\psi \in \mathrm{Lip}(U_1;[0,1])$ and elasticity tensor $\mathbb{C}\in L^\infty(U_1)$ satisfying (3.4) there exists $d_3 := d_3(c_0,d_1,d_2) > 0$ such that

$$\int_{U_1}\psi\mathbb{C}(x)e(\widetilde{v}_R):e(\widetilde{v}_R)dx \le \int_{U_1}\psi\mathbb{C}(x)e(\widetilde{u}_R):e(\widetilde{u}_R)dx + d_3\,(1+\mathrm{Lip}(\psi))\int_{U_1}|e(\widetilde{u}_R)|^2dx;$$

(a4) $J_{\widetilde{v}_R}\subset\partial^*D\cup(J_{\widetilde{u}_R}\setminus U^{i_0+1})$ and $J_{\widetilde{u}_R}\setminus J_{\widetilde{v}_R}\subset\partial^*D$, where D is the union of all bad squares.

Let

$$B_R := (A_R\setminus\overline{D})\cup\partial D,\qquad v_R := \widetilde{v}_R\chi_{U_1} + \widetilde{u}_R\chi_{(\Omega\cup S)_R\setminus U_1},$$

where $(\Omega\cup S)_R := \frac{\Omega\cup S}{R}$. We claim that

$$B := R\,B_R,\qquad v(x) := v_R(x/R),\qquad \omega := R\widetilde{\omega}_R$$

satisfy all assertions of the lemma.

Indeed, from (a3) applied with $\psi\equiv 1$ and $\mathbb{C} = I$ it follows that $v_R \in GSBD^2(\mathrm{Int}(B_R);\mathbb{R}^2)$. Moreover, by (a4) $v_R \in H^1_{\mathrm{loc}}(\mathrm{Int}(B_R);\mathbb{R}^2)$, thus, $(B,v)\in\mathcal{C}$. Note that by definition, any bad square U crosses ∂A_R, therefore, either ∂A crosses ∂D or some connected components of ∂A is compactly contained in some bad U. Since we do not remove ∂A from good squares (as a difference from [10, Theorem 3]), by the definition of B_R, in both cases we do not increase the number of connected components ∂A_R. Hence, if $(A,u)\in\mathcal{C}_m$ for some $m\in\mathbb{N}$, then $(B,v\big|_B)\in\mathcal{C}_m$.

Note that since $v(x) = \widetilde{v}_R(x/R)$ in U_R, by (a1) it follows that $v\in C^\infty(U_{R(1-\sqrt{\delta})})$. Moreover, by the definition of B_R, $A_R\Delta B_R\subset\subset U_1\setminus U_{1-\sqrt{\delta}}$. Thus, $A\Delta B\subset\subset U_R\setminus U_{R(1-\sqrt{\delta})}$. Also, by (a1) supp$\,(\widetilde{u}_R-\widetilde{v}_R)\subset\subset U_1$ so that supp$\,(\widetilde{u}-v)\subset\subset U_R$, and (1) follows.

Moreover, since $\partial A_R\Delta\partial B_R\subset\partial D$, by (3.6)

$$\mathcal{H}^1(\partial B_R\setminus\partial A_R)\le P(D)\le\sum_{U\in\mathcal{W},U\text{ bad}}P(U)\le c_0\sqrt{\delta}\mathcal{H}^1((U_1\setminus U_{1-\sqrt{\delta}})\cap\partial A_R),$$

hence, $\mathcal{H}^1(\partial B\setminus\partial A)\le c_0\sqrt{\delta}\mathcal{H}^1((U_R\setminus U_{R(1-\sqrt{\delta})})\cap\partial A)$, and (2) follows.

Next, by (a3) $|\omega|\le c_0\delta\mathcal{H}^1(U_R\cap\partial A)$, and also using the change of variables $x = Ry$

$$\begin{aligned}\int_{U_R\setminus\omega}|v(x)-\widetilde{u}(x)|^2dx =& R^2\int_{U_1\setminus\widetilde{\omega}_R}|\widetilde{v}_R(y)-\widetilde{u}_R(y)|^2dy \le c_0\delta^2R^2\int_{U_1}|e(\widetilde{u}_R)|^2(y)dy\\ =& c_0\delta^2R^4\int_{U_1}|e(\widetilde{u})|^2(Ry)dy = c_0\delta^2R^2\int_{U_1}|e(\widetilde{u})|^2(x)dx.\end{aligned}$$

Finally, by (a4) and the definition of B_R, for any $\psi \in \mathrm{Lip}(U_R)$ and $\mathbb{C} \in L^\infty(U_R)$ satisfying (3.4) we have

$$\begin{aligned}\int_{U_R} \psi(x)\mathbb{C}(x)e(v):e(v)dx &\le \int_{U_1} \psi(Rx)\mathbb{C}(Rx)e(\widetilde{v}_R):e(\widetilde{v}_R)dx\\ &\le \int_{U_1} \psi(Rx)\mathbb{C}(Rx)e(\widetilde{u}_R):e(\widetilde{u}_R)dx + d_3\,(1+R\,\mathrm{Lip}(\psi))\int_{U_1}|e(\widetilde{u}_R)|^2dx\\ &= \int_{U_R\cap A} \psi(x)\mathbb{C}(x)e(u):e(u)dx + d_3\,(1+R\,\mathrm{Lip}(\psi))\int_{U_R\cap A}|e(u)|^2dx,\end{aligned}$$

since u_R is constant in $U_1 \setminus A_R$. Hence, (a4) follows. □

The following proposition is a generalization to our setting of [10, Theorem 4] established for the Griffith model. Note that since $\mathcal{F}(\cdot,\cdot;O)$ for every $O \subset \Omega$ does not "see" the substrate, we can assume without loss of generality in the remaining part of the section that $u_0 = 0$.

Proposition 3.3. *Let $U_R(x_0) \subset \Omega$ be a square of side length $2R > 0$. Consider sequences of integers $\{m_h\} \subset \mathbb{N}$, Finsler norms $\{\varphi_h\}$ and ellipticity tensors $\{\mathbb{C}_h\}$ such that $\{\mathbb{C}_h\}$ is equicontinuous in $\overline{U_R(x_0)}$ and there exist $d_3, d_4, d_5 > 0$ with*

$$d_3M:M \le \mathbb{C}_h(x)M:M \le d_4M:M \quad \text{for all } (x,M) \in \overline{U_R(x_0)} \times \mathbb{M}^{2\times2}_{\mathrm{sym}}, \tag{3.7}$$

and

$$d_5 \sup_{(x,\nu)\in U_1\times\mathbb{S}^1} \phi_h(x,\nu) \le \inf_{(x,\nu)\in U_1\times\mathbb{S}^1} \phi_h(x,\nu), \tag{3.8}$$

and define $\mathcal{F}_h$ and Ψ_h in $\mathcal{C}_{m_h}$ as in (2.4) and (2.6), respectively, with φ_h, $\mathbb{C}_h$ and m_h in places of φ, $\mathbb{C}$ and m. Let $\{(A_h,u_h)\} \subset \mathcal{C}_{m_h}$ be such that

$$\lim_{h\to\infty} \Psi_h(A_h,u_h;U_R(x_0)) = 0, \tag{3.9}$$

$$\lim_{h\to\infty} \mathcal{H}^1(U_R(x_0)\cap\partial A_h) = 0, \tag{3.10}$$

$$\sup_{h\ge1} \mathcal{F}_h(A_h,u_h;U_R(x_0)) =: M < \infty. \tag{3.11}$$

Then there exist $u \in H^1(U_R(x_0))$, an elasticity tensor $\mathbb{C} \in C^{0,1}(\overline{U_R(x_0)};\mathbb{M}^{2\times2}_{\mathrm{sym}})$, sequences $\{\xi_j\} \subset (0,1)^2$ of vectors and $\{a_j\}$ of rigid displacements and subsequences $\{(A_{h_j},u_{h_j})\}$, $\{\varphi_{h_j}\}$ and $\{\mathbb{C}_{h_j}\}$ such that

(a) $\mathbb{C}_{h_j} \to \mathbb{C}$ uniformly in $\overline{U_R(x_0)}$ and $w_j := u_{h_j}\chi_{U_R(x_0)\cap A_{h_j}} + \xi_j\chi_{U_R(x_0)\setminus A_{h_j}} - a_j \to u$ pointwise a.e. in $U_R(x_0)$, and $e(w_j) \rightharpoonup e(u)$ in $L^2(U_R(x_0))$ as $j\to\infty$;

(b) for all $v \in u + H^1_0(U_R(x_0))$

$$\int_{U_R(x_0)} \mathbb{C}(y)e(u):e(u)\,dy \le \int_{U_R(x_0)} \mathbb{C}(y)e(v):e(v)\,dy; \tag{3.12}$$

(c) for any $r \in (0,R]$

$$\lim_{j\to\infty} \mathcal{F}_h(A_{h_j},u_{h_j};U_r(x_0)) = \int_{U_r(x_0)} \mathbb{C}(x)e(u):e(u)\,dx. \tag{3.13}$$

Proof. Without loss of generality, we suppose $R = 1$ and $x_0 = 0$. Let

$$c_{1,h} := \inf_{(x,\nu)\in U_1\times\mathbb{S}^1} \phi_h(x,\nu), \qquad c_{2,h} := \sup_{(x,\nu)\in U_1\times\mathbb{S}^1} \phi_h(x,\nu); \tag{3.14}$$

by (3.8) we have $d_5c_{2,h} \le c_{1,h}$. Since $\sup_h \mathcal{H}^1(U_1\cap\partial A_h) < \infty$, for every $h \ge 1$ there exists $\xi_h \in (0,1)^2$ such that

$$\mathcal{H}^1(\{y \in U_1\cap\partial A_h:\ \text{trace of } u_h \text{ exists and equals to } \xi_h \text{ at } y\}) = 0$$

(see e.g. [43, Proposition 2.16]). Therefore

$$\widetilde{u}_h := \begin{cases} u_h & \text{in } U_1 \cap A_h, \\ \xi_h & \text{in } U_1 \setminus A_h \end{cases} \tag{3.15}$$

belongs to $GSBD^2(U_1;\mathbb{R}^2)$ with $J_{\widetilde{u}_h} \subset U_1 \cap \partial A_h$ and

$$\lim_{h\to\infty} \mathcal{H}^1(J_{\widetilde{u}_h}) = 0 \tag{3.16}$$

in view of (3.10). Further we suppose $\mathcal{H}^1(J_{\widetilde{u}_h}) < 1/4$ for any $h \ge 1$.

By [9, Proposition 2] and (3.7), there exist a constant c (depending only on d_3) and sequences $\{\widetilde{\omega}_h\}$ of a Lebesgue measurable subsets U_1 with $|\widetilde{\omega}_h| \le c\mathcal{H}^1(U_1 \cap \partial A_h)$ and $\{a_h\}$ of rigid motions such that

$$\int_{U_1\setminus\widetilde{\omega}_h} |\widetilde{u}_h - a_h|^2 dx \le c \int_{U_1} \mathbb{C}_h(x) e(\widetilde{u}_h) : e(\widetilde{u}_h) dx. \tag{3.17}$$

By (3.7) and (3.11), there exists $u \in L^2(U_1)$ such that up to a subsequence $(\widetilde{u}_h - a_h)\chi_{U_1\setminus\omega_h} \rightharpoonup u$ weakly in $L^2(U_1)$. Furthermore from (3.7) and (3.11) we obtain

$$\sup_{h\ge1} \int_{U_1} |e(\widetilde{u}_h - a_h)|^2\, dx + \mathcal{H}^1(J_{\widetilde{u}_h}) < \infty,$$

and hence, by [12, Theorem 1.1] there exist a subsequence still denoted by $\{\widetilde{u}_h - a_h\}$ for which the set

$$E := \{y \in U_1 :\ \lim_{h\to\infty} |\widetilde{u}_h(y) - a_h(y)| \to \infty\}$$

has finite perimeter and $\widetilde{u} \in GSBD^2(U_1 \setminus E;\mathbb{R}^2)$ with $\widetilde{u} = 0$ in E such that

$$\begin{aligned} \widetilde{u}_h - a_h \to \widetilde{u} \qquad & \text{a.e. in } U_1 \setminus E \\ e(\widetilde{u}_h - a_h) \rightharpoonup e(\widetilde{u}) \qquad & \text{in } L^2(U_1 \setminus E; \mathbb{M}^{2\times2}_{\rm sym}), \\ \mathcal{H}^1((U_1 \setminus E) \cap J_{\widetilde{u}}) + \mathcal{H}^1(U_1 \cap \partial^* E) \le \liminf_{h\to+\infty} & \mathcal{H}^1(J_{\widetilde{u}_h}) = 0. \end{aligned} \tag{3.18}$$

By the definition of E, (3.10), the uniform $L^2(U_1)$-boundedness of $\{(\widetilde{u}_h - a_h)\chi_{U_1\setminus\widetilde{\omega}_h}\}$ which is a consequence of (3.17) and (3.11), and Fatou's Lemma it follows that $|E| = 0$. Hence, all relations in (3.18) hold in U_1 and $\widetilde{u} = u$ a.e. in U_1. In particular, since $\mathcal{H}^1(J_u) = 0$, by the Poincaré-Korn inequality $u \in H^1(U_1;\mathbb{R}^2)$. In view of the fact that our elastic energy is invariant under rigid deformations, we suppose $a_h = 0$ for any $h \ge 1$.

Next we prove (3.12). Let $v \in H^1(U_1;\mathbb{R}^2)$ be such that $\operatorname{supp}(u - v) \subset\subset U_r$ for some $r \in (0, 1)$. Let $\psi \in C^1_c(U_r; [0, 1])$ be a cut-off function with $\{0 < \psi < 1\} \subset \{u = v\} \cap U_{r'}$ and $\operatorname{supp}(u - v) \subseteq \{\psi \equiv 1\} \subseteq U_{r''}$ for some $r'' < r' < r$. By (3.16) and Lemma 3.2 applied with (A_h, u_h) and U_r there exist $\widetilde{v}_h \in GSBD^2(\mathrm{Int}(\Omega \cup S \cup \Sigma);\mathbb{R}^2)$, $B_h \in \mathcal{A}_{m_h}$ with $(B_h, \widetilde{v}_h\big|_{B_h}) \in \mathcal{C}_{m_h}$ and a Lebesgue measurable set $\omega_h \subset\subset U_r$ such that

- (a1) $\widetilde{v}_h \in C^\infty(U_{r(1-\sqrt{\delta_h})})$, $A_h \Delta B_h \subset\subset U_r \setminus U_{r(1-\sqrt{\delta_h})}$ and $\operatorname{supp}(\widetilde{u}_h - \widetilde{v}_h) \subset\subset U_r$;
- (a2) $\mathcal{H}^1(\partial B_h \setminus \partial A_h) \le c_0 \sqrt{\delta_h}\, \mathcal{H}^1([U_r \setminus U_{r(1-\sqrt{\delta_h})}] \cap \partial A_h)$;
- (a3) $|\omega_h| \le c_0\delta_h\, \mathcal{H}^1(U_r \cap \partial A_h)$ and
$$\int_{U_r\setminus\omega_h} |\widetilde{v}_h - \widetilde{u}_h|^2 dx \le c_0\delta_h^2 r^2 \int_{U_r\cap A_h} |e(u)|^2 dx;$$
- (a4) for any $\eta \in \mathrm{Lip}(U_r; [0, 1])$
$$\begin{aligned} \int_{U_r} \eta\mathbb{C}e(\widetilde{v}_h) : e(\widetilde{v}_h) dx \le & \int_{U_r\cap A_h} \eta\mathbb{C}e(u_h) : e(u_h) dx \\ & + d_3\, \delta_h^s\, (1 + r\,\mathrm{Lip}(\eta)) \int_{U_r\cap A_h} |e(u_h)|^2 dx, \end{aligned} \tag{3.19}$$

where $\delta_h := r^{-1/2}\mathcal{H}^1(U_r\cap\partial A_h)^{1/2}\to 0$, and d_3 and s are constants. Set

$$v_h := (1-\psi)\widetilde{v}_h + \psi v. \tag{3.20}$$

By (a1) and the definition of v_h $\mathrm{supp}\,(u_h - v_h) \subset\subset U_r$, and hence (B_h, v_h) is an admissible configuration in (2.5). Therefore from (3.9) and the definition of deviation it follows that

$$\mathcal{F}_h(A_h, u_h; U_1) \le \mathcal{F}_h(B_h, v_h; U_1) + o(1), \tag{3.21}$$

where $o(1)\to 0$ as $h\to\infty$. By (a1), (a2), (3.14), (3.8) , the definition of $\mathcal{S}(A_h;U_1)$ and (3.11)

$$\begin{aligned}\mathcal{S}_h(B_h;U_1) - \mathcal{S}_h(A_h;U_1) &\le \mathcal{S}_h(B_h;U_r\setminus \overline{U_{r(1-\sqrt{\delta_h})}}) \le 2c_{2,h}\mathcal{H}^1(\partial B_h\setminus\partial A_h)\\ &\le 2c_0c_{2,h}\sqrt{\delta_h}\mathcal{H}^1([U_r\setminus U_{r(1-\sqrt{\delta_h})}]\cap\partial A_h) \le \frac{2c_0\sqrt{\delta_h}}{d_5}\mathcal{S}_h(A_h;U_1) = o(1)\end{aligned}$$

as $h\to+\infty$. Thus, (3.21) is rewritten as

$$\mathcal{W}_h(A_h, u_h; U_1) \le \mathcal{W}_h(B_h, v_h; U_1) + o(1). \tag{3.22}$$

Note that by (a1), (a3), (3.16), (3.18) and Fatou's Lemma $\widetilde{v}_h\to u$ a.e. in U_1. We claim that $\widetilde{v}_h\to u$ strongly in $L^2_{\rm loc}(U_r)$. Indeed, from (3.7), (3.11) and (3.19) as well as the Poincaré-Korn inequality we deduce that for any $\rho\in(0,r)$ there exists $h_\rho>1$ such that

$$\sup_{h>h_\rho}\int_{U_\rho}|\nabla\widetilde{v}_h|^2dx<\infty.$$

Since $\widetilde{v}_h\in H^1(U_\rho;\mathbb{R}^2)$, by the Poincaré inequality and Rellich-Kondrachov Theorem, there exist $z\in H^1(U_\rho;\mathbb{R}^2)$ and not relabelled subsequence such that $\widetilde{v}_h - C_h\to z$ in $L^2(U_\rho;\mathbb{R}^2)$, where

$$C_h = |U_\rho|^{-1}\int_{U_\rho}\widetilde{v}_h dx.$$

Since $\widetilde{v}_h\to u$ a.e. in U_ρ and $u\in H^1(U_1;\mathbb{R}^2)$, one has $C_h\to C$ some constant $C\in\mathbb{R}$. Therefore,

$$\limsup_{h\to\infty}\|\widetilde{v}_h - u\|_{L^2(U_\rho)} \le \limsup_{h\to\infty}\|\widetilde{v}_h - C_h - u + C\|_{L^2(U_\rho)} + \limsup_{h\to\infty}\|C_h - C\|_{L^2(U_\rho)} = 0,$$

and the claim follows.

Since $u=v$ out of $\{\psi=1\}$, the claim implies $\widetilde{v}_h\to v$ strongly in $L^2(\{0<\psi<1\})$, and hence,

$$\lim_{h\to\infty}\int_{U_r}|\nabla\psi\odot(v-\widetilde{v}_h)\big|_{A_h}|^2 \le \liminf_{h\to\infty}\int_{\{0<\psi<1\}}|\nabla\psi\odot(v-\widetilde{v}_h)|^2 = 0, \tag{3.23}$$

where $X\odot Y = (X\otimes Y + Y\otimes X)/2$, Thus, by the definition of v_h and the equality

$$e(v_h)\big|_{B_h} = (1-\psi)e(\widetilde{v}_h)\big|_{B_h} + \psi e(v)\big|_{B_h} + \nabla\psi\odot(v-\widetilde{v}_h)\big|_{B_h},$$

we estimate

$$\begin{aligned}&\int_{U_r}\mathbb{C}_h e(v_h):e(v_h)dx\\ =&\int_{U_r}(1-\psi)^2\mathbb{C}_h e(\widetilde{v}_h):e(\widetilde{v}_h)dx + \int_{U_r}\psi^2\mathbb{C}_h e(v):e(v)dx\\ &+\int_{U_r}\mathbb{C}_h(\nabla\psi\odot(v-\widetilde{v}_h)):(\nabla\psi\odot(v-\widetilde{v}_h))dx\\ &+\int_{U_r}(1-\psi)\mathbb{C}_h e(\widetilde{v}_h):(\nabla\psi\odot(v-\widetilde{v}_h))dx\\ &+\int_{U_r}\psi\mathbb{C}_h e(v):(\nabla\psi\odot(v-\widetilde{v}_h))dx\end{aligned}$$

$$= \int_{U_r} (1-\psi)^2 \mathbb{C}_h e(\widetilde{v}_h) : e(\widetilde{v}_h) dx + \int_{U_r} \psi^2 \mathbb{C}_h e(v) : e(v) dx + o(1)$$
$$\leq \int_{U_r \cap A_h} (1-\psi)^2 \mathbb{C}_h e(u_h) : e(u_h) dx + \int_{U_r} \psi^2 \mathbb{C}_h e(v) : e(v) dx + o(1), \tag{3.24}$$

where in the second equality we use (3.11), (3.19) with $\eta \equiv 1$, (3.23), (3.7) and the Hölder inequality, while in the last inequality we use (3.19) with $\eta = (1-\psi)^2$ and (3.15). Now (3.22), (3.24) and (3.15) implies

$$\int_{U_r} (2\psi - \psi^2) \mathbb{C}_h e(\widetilde{u}_h) : e(\widetilde{u}_h) dx \leq \int_{U_r} \psi^2 \mathbb{C}_h e(v) : e(v) dx + o(1). \tag{3.25}$$

Since $\{\mathbb{C}_h\}$ is equibounded (see (3.7)) and equicontinuous, by the Arzela-Ascoli Theorem, there exist a (not relabelled) subsequence and an elasticity tensor $\mathbb{C} \in C^0(U_1; \mathbb{M}^{2\times 2}_{\rm sym})$ such that $\mathbb{C}_h \to \mathbb{C}$ uniformly in U_1. Hence, letting $h \to \infty$ in (3.25) and using the convexity of the elastic energy and (3.18), we obtain

$$\int_{U_r} (2\psi - \psi^2) \mathbb{C}(y) e(u) : e(u) dy \leq \int_{U_r} \psi^2 \mathbb{C}(y) e(v) : e(v)\, dy. \tag{3.26}$$

By the choice of ψ, (3.26) implies

$$\int_{U_{r''}} \mathbb{C}(y) e(u) : e(u) dy \leq \int_{U_r} \mathbb{C}(y) e(v) : e(v)\, dy. \tag{3.27}$$

Since r'' is arbitrary, letting $r'' \nearrow r$ we deduce that (3.27) holds also with $r'' = r$. Since $\operatorname{supp}(u-v) \subset\subset U_r$, this implies (3.12).

It remains to prove (3.13). If we take $v = u$ in (3.25) and use $0 \leq \psi \leq 1$ and $\psi = 1$ in $U_{r''}$ we get

$$\int_{U_{r''}} \mathbb{C} e(u) : e(u) dx \leq \liminf_{h\to\infty} \int_{U_{r''}} \mathbb{C}_h e(\widetilde{u}_h) : e(\widetilde{u}_h) dx$$
$$\leq \limsup_{h\to\infty} \int_{U_{r''}} \mathbb{C}_h e(\widetilde{u}_h) : e(\widetilde{u}_h) dx \leq \int_{U_r} \mathbb{C} e(u) : e(u) dx.$$

Since r'' is arbitrary, letting $r'' \nearrow r$ we deduce

$$\lim_{h\to\infty} \int_{U_r} \mathbb{C}_h e(\widetilde{u}_h) : e(\widetilde{u}_h) dx = \int_{U_r} \mathbb{C} e(u) : e(u) dx. \tag{3.28}$$

Now we prove that

$$\lim_{h\to\infty} \mathcal{S}_h(A_h; U_r) = 0 \tag{3.29}$$

for any $r \in (0,1)$. By (3.10), we can find $h_r > 0$ such that $\mathcal{H}^1(U_1 \cap \partial A_h) < (1-r)/5$ for any $h > h_r$, and hence there is no connected component of ∂A_h intersecting both ∂U_r and ∂U_1. Also by the relative isoperimetric inequality, passing to further subsequence we suppose that either

$$\lim_{h\to\infty} |U_1 \cap A_h| = 0 \tag{3.30}$$

or

$$\lim_{h\to\infty} |U_1 \setminus A_h| = 0. \tag{3.31}$$

First assume that (3.30) holds. Let $E_h \subset A_h$ be the set consisting of all connected components of $\overline{A_h}$ not intersecting ∂U_1. Then, $(A_h \setminus E_h, u_h|_{A_h\setminus E_h})$ is an admissible configuration in (2.5), thus,

$$\mathcal{F}_h(A_h, u_h; U_1) \leq \Phi_h(A_h, u_h; U_1) + o(1) \leq \mathcal{F}_h(A_h \setminus E_h, u_h; U_1) + o(1), \tag{3.32}$$

where in the first inequality we use (3.9) and in the second we use the definition of Φ_h. From the definition of E_h and (3.32) it follows that

$$\mathcal{S}(A_h;U_r) \le \mathcal{S}_h(A_h;U_1) - \mathcal{S}_h(A_h\setminus E_h;U_1) \le o(1)$$

and hence, (3.29) follows.

Now assume that (3.31) holds. Let $\delta_h := \sqrt{\mathcal{H}^1(J_{\widetilde u_h})}$ and ψ, and $r''<r'<r$ and v_h be as in (3.20) with $v=u$. Fix any $\rho\in(0,r)$. By (3.10), we can can find $h_{r,\rho}>0$ such that $\delta_h<\min\{1-r,r-\rho\}/5$ for any $h>h_{r,\rho}$. Since $\partial A_h\in\mathcal{A}_{m_h}$, no connected component of ∂A_h intersect both ∂U_r and ∂U_ρ. Let $E_h\subset U_1\setminus A_h$ be the union of all connected components F of $\overline{U_1\setminus A_h}$ lying strictly inside U_1 (so E_h is a union of "holes" and $\partial F\subset\partial A_h$). Let $\psi\in C_c^1(U_r;[0,1])$ be a cut-off function with $\{0<\psi<1\}\subset\underline{\{u=v\}}\cap U_{r'}$ and $\operatorname{supp}(u-v)\subseteq\{\psi\equiv1\}\subseteq U_{r''}$ for some $r''<r'<r$. Set $A_h':=A_h\cup\overline{E_h}$. Applying Lemma 3.2 with $(A_h',\widetilde u_h\big|_{A_h'})$, U_r and $m=m_h$ we find $\widetilde v_h'\in GSBD^2(\mathrm{Int}(\Omega\cup S\cup\Sigma);\mathbb{R}^2)$, $B_h'\in\mathcal{A}_{m_h}$ with $(B_h',\widetilde v_h'\big|_{B_h})\in\mathcal{C}_{m_h}$ and a Lebesgue measurable set $\omega_h'\subset\subset U_r$ such that

(b1) $\widetilde v_h'\in C^\infty(U_{r(1-\sqrt{\delta_h})})$, $A_h'\Delta B_h'\subset\subset U_r\setminus U_{r(1-\sqrt{\delta_h})}$ and $\operatorname{supp}(\widetilde u_h-\widetilde v_h')\subset\subset U_r$;

(b2) $\mathcal{H}^1(\partial B_h'\setminus\partial A_h')\le c_0\sqrt{\delta_h}\,\mathcal{H}^1([U_r\setminus U_{r(1-\sqrt{\delta_h})}]\cap\partial A_h')$;

(b3) $|\omega_h'|\le c_0\delta_h\,\mathcal{H}^1(U_r\cap\partial A_h')$ and

$$\int_{U_r\setminus\omega_h'}|\widetilde v_h'-\widetilde u_h|^2dx\le c_0\delta_h^2r^2\int_{U_r\cap A_h'}|e(u_h)|^2dx;$$

(b4) for any $\eta\in\mathrm{Lip}(U_r;[0,1])$

$$\int_{U_r}\eta\mathbb{C}e(\widetilde v_h'):e(\widetilde v_h')dx\le\int_{U_r\cap A_h}\eta\mathbb{C}e(u_h):e(u_h)dx + d_3\,\delta_h^s\,(1+r\mathrm{Lip}(\eta))\int_{U_r\cap A_h}|e(u_h)|^2dx,$$

where $\delta_h:=r^{-1/2}\mathcal{H}^1(U_r\cap\partial A_h)^{1/2}\to0$, and d_3 and s are constants. Set

$$v_h':=(1-\psi)\widetilde v_h'+\psi u.$$

By the definition of A_h' and (b1) $(B_h',v_h'\big|_{B_h'})$ is an admissible configuration for $\Phi_h(A_h,u_h;U_1)$ in (2.5). Thus from (3.9) and (3.31)

$$\mathcal{F}_h(A_h,u_h;U_1)\le\mathcal{F}_h(B_h',v_h'\big|_{B_h'};U_1)+o(1). \tag{3.33}$$

Now as in the proof of (3.25)

$$\begin{aligned}&\mathcal{W}_h(B_h',v_h'\big|_{B_h'};U_1)-\mathcal{W}_h(A_h,u_h;U_1)\\ &\le\int_{U_r}\psi^2\mathbb{C}_he(u):e(u)dx-\int_{U_r}(2\psi-\psi^2)\mathbb{C}_he(\widetilde u_h):e(\widetilde u_h)dx+o(1)\\ &\le\int_{U_r}\mathbb{C}_he(u):e(u)dx-\int_{U_{r''}}\mathbb{C}_he(\widetilde u_h):e(\widetilde u_h)dx+o(1).\end{aligned} \tag{3.34}$$

Moreover, by the definition of A_h', (b1), (b2), the inclusion $\partial A_h'\subset\partial A_h$, , (3.14), (3.8) , the definition of $\mathcal{S}(A_h;U_1)$ and (3.11)

$$\begin{aligned}&\mathcal{S}_h(B_h';U_1)-\mathcal{S}_h(A_h;U_1)=\mathcal{S}_h(B_h';U_1)-\mathcal{S}_h(A_h';U_1)+\mathcal{S}_h(A_h';U_1)-\mathcal{S}_h(A_h;U_1)\\ \le&\mathcal{S}_h(B_h';U_r\setminus U_{r(1-\sqrt{\delta_h})})-\mathcal{S}_h(A_h;U_\rho)\le2c_{2,h}\mathcal{H}^1(\partial B_h'\setminus\partial A_h')-\mathcal{S}_h(A_h;U_\rho)\\ \le&2c_0c_{2,h}\sqrt{\delta_h}\mathcal{H}^1([U_r\setminus U_{r(1-\sqrt{\delta_h})}]\cap\partial A_h)-\mathcal{S}_h(A_h;U_\rho)\\ \le&\frac{2c_0\sqrt{\delta_h}}{d_5}\mathcal{S}_h(A_h;U_1)-\mathcal{S}_h(A_h;U_\rho)=o(1)-\mathcal{S}_h(A_h;U_\rho),\end{aligned} \tag{3.35}$$

where we used $\mathcal{S}_h(A_h;U_1) \ge \mathcal{S}_h(A_h';U_1) + \mathcal{S}_h(A_h;U_\rho)$. Hence, (3.33), (3.34) and (3.35) imply

$$\mathcal{S}_h(A_h;U_\rho) + \int_{U_{r''}} \mathbb{C}_h e(\widetilde{u}_h) : e(\widetilde{u}_h)dx \le \int_{U_r} \mathbb{C}_h e(u) : e(u)dx + o(1).$$

Thus, letting $h \to \infty$ and using (3.28) we get

$$\limsup_{h\to\infty} \mathcal{S}_h(A_h;U_\rho) + \int_{U_{r''}} \mathbb{C}e(u) : e(u)dx \le \int_{U_r} \mathbb{C}e(u) : e(u)dx. \tag{3.36}$$

Now letting $r'' \to r$ we get

$$\limsup_{h\to\infty} \mathcal{S}_h(A_h;U_\rho) = 0. \tag{3.37}$$

Since the function $B \mapsto \mathcal{S}_h(A_h;B)$ defined for Borel sets $B \subset U_1$ extends to a bounded nonnegative Radon measure μ_h in U_1. Since (3.37) holds for any $\rho \in (0,r)$, μ_h converges to 0 in the weak* sense, and thus (3.29) follows. □

Recall that by [16, Proposition 3.4] if the elasticity tensor $\mathbb{C}$ is constant, then for any $\gamma \in (0,2)$ there exists $c_\gamma := c_\gamma(c_3,c_4) > 0$ such that for every local minimizer $(\Omega,u) \in \mathcal{C}$ of $\mathcal{F}(\cdot;O)$, u is analytic in O and for any square $U_R(x) \subset\subset O$ and $r \in (0,R)$,

$$\int_{U_r(x)} \mathbb{C}e(u) : e(u)\,dx \le c_\gamma \left(\frac{r}{R}\right)^{2-\gamma} \int_{U_R(x)} \mathbb{C}e(u) : e(u)\,dx. \tag{3.38}$$

Given $\gamma \in (0,1)$ let

$$\tau_0 = \tau_0(\gamma,c_3,c_4) := \min\{1, \tfrac{1}{2}c_\gamma^{-\frac{1}{4-2\gamma}}\},$$

where c_γ is the constant appearing in (3.38). Using Proposition 3.3 and repeating similar arguments of [11, 17] we get the following decay property of the functional $\mathcal{F}$.

Proposition 3.4. *For any $\tau \in (0,\tau_0)$ there exist $\varsigma = \varsigma(\tau) \in (0,1)$, $\vartheta := \vartheta(\tau) > 0$ and $R := R(\tau) > 0$ such that if $(A,u) \in \mathcal{C}_m$ satisfies*

$$\mathcal{H}^1(U_\rho(x) \cap \partial A) < 2\varsigma\rho \quad \textit{and} \quad \mathcal{F}(A,u;U_\rho(x)) \le (1+\vartheta)\Phi(A,u;U_\rho(x))$$

for some $m \ge 1$ and $U_\rho(x) \subset\subset \Omega$ with $0 < \rho < R$, then

$$\mathcal{F}(A,u;U_{\tau\rho}(x)) \le \tau^{2-\gamma}\mathcal{F}(A,u;U_\rho(x)).$$

Proof. Assume by contradiction that there exist $\tau \in (0,\tau_0)$, positive real numbers $\varsigma_h, \vartheta_h, \rho_h \to 0$, natural numbers $m_h \in \mathbb{N}$, squares $U_{\rho_h}(x_h) \subset\subset \Omega$, and admissible configurations $(A_h,u_h) \in \mathcal{C}_{m_h}$ such that

$$\mathcal{H}^1(U_{\rho_h}(x_h) \cap \partial A_h) \le 2\varsigma_h\rho_h \tag{3.39}$$

$$\mathcal{F}(A_h,u_h;U_{\rho_h}(x_h)) \le (1+\vartheta_h)\Phi(A_h,u_h;U_{\rho_h}(x_h)), \tag{3.40}$$

but

$$\mathcal{F}(A_h,u_h;U_{\tau\rho_h}(x_h)) > \tau^{2-\gamma}\mathcal{F}(A_h,u_h;U_{\rho_h}(x_h)) \tag{3.41}$$

for any h. Note that $\mathcal{F}(A_h,u_h;U_{\rho_h}(x_h)) > 0$. Let us define the rescaled energy $\mathcal{F}_h(\cdot;U_1) : \mathcal{C}_{m_h} \to \mathbb{R}$ as in (2.4) with

$$\varphi_h(y,\nu) := \frac{\rho_h\varphi(x_h+\rho_h y,\nu)}{\mathcal{F}(A_h,u_h;U_{\rho_h}(x_h))}$$

in place of $\varphi(y,\nu)$ and

$$\mathbb{C}_h(y) := \mathbb{C}(x_h+\rho_h y)$$

in place of $\mathbb{C}(y)$, for $y \in U_1$. We notice that

$$\mathcal{F}_h(E_h,v_h;U_1) = 1 \tag{3.42}$$

for

$$E_h := \sigma_{x_h,\rho_h}(A_h)$$

(see definition of blow-up map $\sigma_{x,r}$ at (2.2)) and

$$v_h(y) := \frac{\rho_h u_h(x_h + \rho_h y)}{\sqrt{\mathcal{F}(A_h, u_h; B_{\rho_h}(x_h))}}.$$

By (3.39) we obtain

$$\mathcal{H}^1(U_1 \cap \partial E_h) < 2\varsigma_h$$

while (3.40) and (3.42) entails

$$\Psi_h(E_h, v_h; U_1) \le \vartheta_h \Phi_h(E_h, v_h; U_1) \le \vartheta_h \mathcal{F}_h(E_h, v_h; U_1) = \vartheta_h,$$

where Φ_h and Ψ_h are defined as in (2.5) and (2.6) (again with φ_h and $\mathbb{C}_h$ in places of φ and $\mathbb{C}$, respectively). By (2.9) $\{\mathbb{C}_h\}$ is equibounded. Since Ω is bounded, there exists $x_0 \in \overline{\Omega}$ such that, up to extracting a subsequence, $x_h \to x_0$ as $h \to +\infty$. As $\rho_h \to 0$, one has $x_h + \rho_h y \to x_0$ for every $y \in \overline{U_1}$. Thus $\{\mathbb{C}_h\}$ is also equicontinuous and $\mathbb{C}_h \to \mathbb{C}_0 := \mathbb{C}(x_0)$ uniformly in $\overline{U_1}$. In view of (3.39), (3.40) and (3.42), we can apply Proposition 3.3 to find $u \in H^1(U_1; \mathbb{R}^2)$, vectors $\xi_h \in (0,1)^2$, and infinitesimal rigid displacements a_h such that, up to a subsequence,

$$w_h := v_h \chi_{U_1 \cap E_h} + \xi_h \chi_{U_1 \setminus E_h} - a_h \to v$$

pointwise a.e. in U_1, $e(w_h) \rightharpoonup e(v)$ in $L^2(U_1)$ as $h \to +\infty$, and

$$\lim_{h\to+\infty} \mathcal{F}_h(E_h, v_h; U_r) = \lim_{h\to+\infty} \mathcal{F}_h(E_h, w_h; U_r) = \int_{U_r} \mathbb{C}_0(x) e(v) : e(v) dx \tag{3.43}$$

for any $r \in (0,1]$. In particular, from (3.43) and (3.41) it follows that

$$\begin{aligned}\int_{U_\tau} \mathbb{C}_0(x) e(v) : e(v) dx &= \lim_{h\to+\infty} \mathcal{F}(E_h, v_h; U_\tau) \\ &\ge \lim_{h\to+\infty} \tau^{2-\gamma} \mathcal{F}(E_h, v_h; U_1) = \tau^{2-\gamma} \int_{U_1} \mathbb{C}_0(x) e(v) : e(v) dx.\end{aligned}$$

Since $\mathbb{C}_0$ is constant, applying (3.38) with $r := \tau$ and $R := 1$ we get

$$\begin{aligned}c_\gamma \tau^{2-\gamma} \int_{U_1} \mathbb{C}_0(x) e(v) : e(v) dx &\ge \int_{U_\tau} \mathbb{C}_0(x) e(v) : e(v) dx \\ &\ge \tau^{\gamma-2} \int_{U_1} \mathbb{C}_0(x) e(v) : e(v) dx.\end{aligned}$$

Now recalling that $\mathcal{F}_h(E_h, v_h; U_1) = 1$, by (3.43) we get $\int_{U_1} \mathbb{C}_0(x) e(v) : e(v) dx = 1$, thus, $\tau^{2-\gamma} \ge c_\gamma^{-1/2} > \tau_0^{2-\gamma}$, which a contradiction. □

By employing the arguments of [44, Section 4.3] and using Proposition 3.4 we establish the following lower bound for $\mathcal{F}$.

Proposition 3.5. *Given $\tau \in (0, \tau_0)$, let $\varsigma := \varsigma(\tau) \in (0,1)$ and $R := R(\tau) > 0$ be as in Proposition 3.4. Then if $(A, u) \in \mathcal{C}_m$ is a (Λ, m)-minimizer of $\mathcal{F}$ in $U_{r_0}(x_0)$ for some $r_0 > 0$, then for any $x \in U_{r_0}(x_0) \cap \partial A$ with $\theta_*(\partial A, x) > 0$ one has*

$$\mathcal{F}(A, u; U_\rho(x)) \ge 2c_1 \varsigma \rho \tag{3.44}$$

for any ball $U_\rho(x) \subset U_{r_0}(x_0)$ with $\rho \in (0, R_0)$, where

$$R_0 := R_0(\Lambda, \tau, c_1) := \min\left\{ R(\tau), \frac{\sqrt{\pi}\, c_1 \vartheta}{\Lambda(2 + \vartheta)} \right\}.$$

Proof. Note that for any $(C,w),(D,v)\in\mathcal{C}_m$ and $O\subset\Omega$ with $C\Delta D\subset\subset O$

$$\begin{aligned}\sqrt{4\pi}\,|C\Delta D|^{1/2} \le& \mathcal{H}^1(\partial^*(C\Delta D))\le \mathcal{H}^1(O\cap\partial^* C)+\mathcal{H}^1(O\cap\partial^* D)\\ \le& \frac{\mathcal{S}(C,O)+\mathcal{S}(D,O)}{c_1}\le \frac{\mathcal{F}(C,w;O)+\mathcal{F}(D,v;O)}{c_1},\end{aligned}\tag{3.45}$$

where in the first inequality we used the isoperimetric inequality, in the second $\partial^*(C\Delta D)\subset O\cap(\partial^*C\cup\partial^*D)$, in the third (2.7) and the definition of $\mathcal{S}(\cdot;O)$ and in the last the nonnegativitiy of $\mathcal{W}(\cdot;O)$. Thus, from the (Λ,m)-minimality of (A,u) in $U_{r_0}(x_0)$ we deduce that

$$\begin{aligned}\mathcal{F}(A,u;U_r(x)) \le& \mathcal{F}(B,v;U_r(x))+\Lambda|A\Delta B|^{\frac12}|A\Delta B|^{\frac12}\\ \le& \mathcal{F}(B,v;U_r(x))+\frac{\Lambda r}{\sqrt{\pi}\,c_1}\Big(\mathcal{F}(A,u;U_r(x))+\mathcal{F}(B,v;U_r(x))\Big)\end{aligned}\tag{3.46}$$

for any $U_\rho(x)\subset U_{r_0}(x_0)$ and $(B,v)\in\mathcal{C}_m$ with $A\Delta B\subset\subset U_r(x)$ and $\operatorname{supp}(u-v)\subset\subset U_r(x)$, where in the last inequality we used (3.45) and the inequality $|A\Delta B|\le|U_r|=4r^2$. Let $r>0$ be small enough so that $\frac{\Lambda r}{\sqrt{\pi}\,c_1}\le\frac{\vartheta}{2+\vartheta}$, where $\vartheta:=\vartheta(\tau)\in(0,1)$ is given by Proposition 3.4. From (3.46) we obtain

$$\mathcal{F}(A,u;U_r(x))\le(1+\vartheta)\mathcal{F}(B,v;U_r(x)),$$

which by the arbitrariness of (B,v) is equivalent to

$$\mathcal{F}(A,u;U_r(x))\le(1+\vartheta)\Phi(A,u;U_r(x)).\tag{3.47}$$

We are ready to prove (3.44). Let

$$J_A^*:=\{y\in U_{r_0}(x_0)\cap\partial A:\ \theta^*(\partial A,y)=\theta_*(\partial A,y)=1\}.$$

and $x\in J_A^*$. For simplicity we suppose that $x=0$. Assume by contradiction that

$$\mathcal{F}(A,u;U_\rho)<2c_1\varsigma\rho$$

for some $U_\rho\subset\subset U_{r_0}(x_0)$ with $\rho\in(0,R_0)$. Then by the nonnegativity of the elastic energy and (2.7),

$$2c_1\varsigma\rho>\mathcal{F}(A,u;U_\rho)\ge\int_{U_\rho\cap\partial A}\varphi(x,\nu_A)d\mathcal{H}^1\ge c_1\mathcal{H}^1(U_\rho\cap\partial A)$$

so that

$$\mathcal{H}^1(U_\rho\cap\partial A)<2\varsigma\rho.\tag{3.48}$$

By (3.48) and (3.47) we can apply Proposition 3.4 and obtain that

$$\mathcal{F}(A,u;U_{\tau\rho})\le\tau^{2-\gamma}\mathcal{F}(A,u;U_\rho)\le 2c_1\varsigma\tau^{2-\gamma}\rho<2c_1\tau\varsigma\rho$$

since $\gamma,\tau\in(0,1)$, so that

$$\mathcal{H}^1(U_{\tau\rho}\cap\partial A)<2\varsigma\tau\rho.$$

Thus, by induction,

$$\mathcal{H}^1(U_{\tau^n\rho}\cap\partial A)<2\varsigma\tau^n\rho.$$

However, by the choice of x

$$1=\lim_{n\to+\infty}\frac{\mathcal{H}^1(U_{\tau^n\rho}\cap\partial A)}{2\tau^n\rho}\le\frac{2c_1\varsigma}{2c_1}=\varsigma<1,$$

a contradiction. This contradiction implies (3.44) for $x\in J_A^*$. Note that the map $\mathcal{F}(A,u;\cdot)$, defined for open sets $O\subset\subset U_{r_0}(x_0)$ extends to a positive Borel measure in $U_{r_0}(x_0)$, hence, (3.44) is valid also for all $x\in U_{r_0}(x_0)\cap\overline{J_A^*}$. Since ∂A has at most m connected components, $\theta_*(\partial A,x)=0$ if and only if $x\in\partial A$ is isolated point. Since (A,u) is a minimizer, points x with $\theta_*(\partial A,x)=0$ are removable singularities for u so that u is a H^1-function in a small neighborhood of x, and hence,

$$U_{r_0}\cap\overline{J_A^*}=\{x\in U_{r_0}(x_0)\cap\partial A:\ \theta_*(\partial A,x)>0\}.$$

□

Now we are ready to prove (3.2) and (3.3).

Proof of Theorem 3.1. We begin by establishing (3.2). Let $x \in \Omega$, $r \in (0, \min\{1, \mathrm{dist}(x, \partial\Omega)\})$, and $U_r := U_r(x)$. Since $E := (A \setminus \overline{U_r}) \cup \partial U_r \in \mathcal{A}_m$, by the (Λ, m)-minimality of (A, u) we obtain

$$\mathcal{F}(A, u; U_r) \le \mathcal{F}(E, u; U_r) + \Lambda|U_r|,$$

so that

$$\int_{U_r\cap\partial A} \varphi(x, \nu_A) d\mathcal{H}^1 \le 2 \int_{\partial U_r} \varphi(x, \nu_{U_r}) d\mathcal{H}^1 + 4\Lambda r^2,$$

in view of the nonnegativity of $\mathcal{W}(A, u; U_r)$, which by (2.7) entails (3.2). In particular, since $E\Delta A \subset\subset U_\rho$ for every $\rho \in (r, \mathrm{dist}(x, \partial\Omega))$, we also have

$$\begin{aligned}\mathcal{F}(A, u; U_\rho) \le& \mathcal{F}(E, u; U_\rho) + \Lambda|U_r| = \mathcal{F}(E, u; U_\rho \setminus \overline{U_r}) + \mathcal{S}(E, u; \overline{U_r}) + 4\Lambda r^2\\ \le& \mathcal{F}(E, u; U_\rho \setminus \overline{U_r}) + 2\int_{\partial U_r} \varphi(x, \nu_{U_r}) d\mathcal{H}^1 + 4\Lambda r^2\\ \le& \mathcal{F}(E, u; U_\rho \setminus \overline{U_r}) + 16c_2 r + 4\Lambda r^2\end{aligned}$$

and hence, letting $\rho \searrow r$ and using $r \le 1$ we get

$$\mathcal{F}(A, u; \overline{U_r}) \le (16c_2 + 4\Lambda)r. \tag{3.49}$$

Now we prove (3.3). Let $\tau_o := \frac{\tau_0}{2}$, let $\varsigma_o = \varsigma(\tau_o) \in (0, 1)$ and $R_o = R_0(\tau_o, \phi, \Lambda) > 0$ be as in Proposition 3.5. Then by (3.44),

$$\mathcal{F}(A, u; U_{\tau r}) \ge 2c_1\varsigma_o\tau r \tag{3.50}$$

for $\tau \in (0, 1]$ and for any square $U_r \subset \Omega$ with $r \in (0, R_o)$. We consider $\varsigma_* := \varsigma(\tau_*)$, $\vartheta_* := \vartheta(\tau^*)$, and $R_* := \min\{R(\tau_*), R_o\}$ as given by Proposition 3.4 for $\tau_* := \min\{\frac{\tau_0}{2}, \big(\frac{c_1\varsigma_o}{16c_2+4\Lambda}\big)^{\frac{1}{1-\gamma}}\}$. By contradiction, if $\mathcal{H}^1(U_r \cap \partial A) < \varsigma_* r$, then by applying (3.47) with $\tau = \tau_*$ we obtain

$$\mathcal{F}(A, u; U_r) \le (1 + \vartheta_*)\Phi(A, u; U_r).$$

Hence, by Proposition 3.4,

$$\mathcal{F}(A, u; U_{\tau_* r}) \le \tau_*^{2-\gamma}\mathcal{F}(A, u; U_r)$$

so that by (3.50) and (3.49)

$$\tau_*^{1-\gamma} \ge \frac{2c_1\varsigma_o}{16c_2 + 4\Lambda},$$

which is a contradiction. □

4. Proof of Theorem 2.5

We postpone the proof after several propositions.

Proposition 4.1 (Compactness of free crystals). *For any $m \in \mathbb{N}$ let $(A_m, u_m) \in \mathcal{C}_m$ be a minimizer of $\mathcal{F}$ in $\mathcal{C}_m$ such that ∂A_m does not contain isolated points. Then there exist $A \in \mathcal{A}$ and a sequence $\{A_{m_h}\}$ such that $\mathrm{sdist}(\cdot, \partial A_{m_h}) \to \mathrm{sdist}(\cdot, \partial A)$ locally uniformly in $\mathbb{R}^2$ as $h \to \infty$. Moreover, for any $x \in \Omega \cap \partial A$ and $r \in (0, \min\{R_*, \mathrm{dist}(x, \partial\Omega)\})$*

$$\varsigma_* \le \frac{\mathcal{H}^1(U_r(x)) \cap \partial A}{r} \le 2\pi c_2,$$

where $\varsigma_* := \varsigma_*(c_3, c_4, \mathrm{Lip}(\mathbb{C})) \in (0, 1)$ and $R_* := R_*(c_1, c_2, c_3, c_4, \mathrm{Lip}(\mathbb{C})) > 0$ are given in Theorem 3.1.

Proof. By [39, Proposition 3.1], there exist $A \subset \Omega$ and a subsequence $\{(A_{m_h}, u_{m_h})\}$ such that $\mathrm{sdist}(\cdot, \partial A_{m_h}) \to \mathrm{sdist}(\cdot, \partial A)$ as $h \to \infty$. Consider the sequence $\mu_h := \mathcal{H}^1 \llcorner \partial A_{m_h}$ of positive Radon measures. By Theorem 3.1

$$\frac{\varsigma_*}{2} \le \frac{\mu_h(U_r(x))}{2r} \le \frac{2\pi c_2}{c_1} \tag{4.1}$$

for any $x \in \Omega \cap \partial A_{m_h}$ and $U_r(x) \subset\subset \Omega$ with $r \in (0, R_*)$, where c_1, c_2 are given by (2.7) and c_3, c_4 is given by (2.9). By (2.7), (2.8) and (3.1),

$$\begin{aligned} \mu_h(\mathbb{R}^2) = \mathcal{H}^1(\partial A_{m_h}) \le & \mathcal{H}^1(\partial\Omega) + \frac{\mathcal{F}(A_{m_h}, u_{m_h}) + 2c_2\mathcal{H}^1(\Sigma)}{c_1} \\ \le & \mathcal{H}^1(\partial\Omega) + \frac{\mathcal{F}(A_1, u_1) + 2c_2\mathcal{H}^1(\Sigma)}{c_1}, \end{aligned}$$

hence, by compactness, there exist a not relabelled subsequence and a positive Radon measure μ in $\mathbb{R}^2$ such that $\mu_h \rightharpoonup^* \mu$ as $h \to \infty$. We claim that

$$\overline{\Omega \cap \partial A} \subseteq \mathrm{supp}\, \mu \subseteq \partial A. \tag{4.2}$$

Indeed, let $x \in \Omega \cap \partial A$ and $r \in (0, \min\{\mathrm{dist}(x, \partial\Omega), R_*\})$. By the sdist-convergence, there exists $x_h \in U_r(x) \cap \partial A_{m_h}$ with $x_h \to x$ such that $U_{r/2}(x_h) \subset B_r(x)$ and hence, by the weak* convergence and (4.1),

$$\mu(\overline{U_r(x)}) \ge \limsup_{h\to\infty} \mu_h(\overline{U_r(x)}) \ge \limsup_{h\to\infty} \mu_h(U_{r/2}(x_h)) \ge \varsigma_* r.$$

This implies $x \in \mathrm{supp}\, \mu$. Conversely, if, by contradiction, there exists $x \in \mathrm{supp}\, \mu \setminus \partial A$, then we can find $r > 0$ such that $U_r(x) \cap \partial A = \emptyset$. From the sdist-convergence it follows that $U_{r/2}(x) \cap \partial A_{m_h} = \emptyset$ for h large enough, and hence,

$$0 < \mu(U_{r/2}(x)) \le \liminf_{h\to\infty} \mu_h(U_{r/2}(x)) = 0,$$

which is a contradiction.

From (4.1) it follows that

$$\frac{\varsigma_*}{2} \le \frac{\mu(U_r(x))}{2r} \le \frac{2\pi c_2}{c_1} \tag{4.3}$$

for any $x \in \Omega \cap \mathrm{supp}\, \mu$ any $r \in (0, R_*)$ with $U_r(x) \subset\subset \Omega$. Indeed, let $x \in \Omega \cap \mathrm{supp}\, \mu$ and $r \in (0, \min\{R_*, \mathrm{dist}(x, \partial\Omega)\})$. Then by the sdist-convergence for any $\delta \in (0, \frac{R_* - r}{4})$ there exists $x_h \in U_r(x) \cap \partial A_{m_h}$ with $x_h \to x$ such that $U_r(x) \subset U_{r+\delta}(x_h)$ so that by weak* convergence and (4.1),

$$\mu(U_r(x)) \le \liminf_{h\to\infty} \mu_h(U_r(x)) \le \limsup_{h\to\infty} \mu_h(U_{r+\delta}(x_h)) \le \frac{4\pi c_2}{c_1}(r + \delta).$$

Now letting $\delta \to 0$ implies the upper density estimate in (4.3). On the other hand, for any $\delta \in (0, \frac{r}{4})$ there exists $x_h \in U_r(x) \cap \partial A_{m_h}$ with $x_h \to x$ such that $U_{r-\delta}(x_h) \subset\subset U_r(x)$ so that by weak* convergence and (4.1),

$$\mu(\overline{U_r(x)}) \ge \limsup_{h\to\infty} \mu_h(\overline{U_r(x)}) \ge \limsup_{h\to\infty} \mu_h(U_{r-\delta}(x_h)) \ge \varsigma_* (r - \delta).$$

Hence, letting $\delta \to 0$ yields $\mu(\overline{U_r(x)}) \ge \varsigma_* r$. Since μ is a finite Radon measure, this implies the lower density estimate in (4.3).

From (4.3) and [1, Theorem 2.56] it follows that

$$\varsigma_* \mathcal{H}^1 \llcorner (\Omega \cap \mathrm{supp}\, \mu) \le \mu \llcorner \Omega \le \frac{4\pi c_2}{c_1} \mathcal{H}^1 \llcorner (\Omega \cap \mathrm{supp}\, \mu). \tag{4.4}$$

Thus, μ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner (\Omega \cap \operatorname{supp} \mu)$, $\operatorname{supp} \mu$ is $\mathcal{H}^1$-rectifiable, and $\mathcal{H}^1(\operatorname{supp} \mu) < \infty$. By (4.4),

$$\mathcal{H}^1(\partial A) \le \mathcal{H}^1(\Omega \cap \partial A) + \mathcal{H}^1(\partial\Omega \cap \partial A) \le \frac{1}{\varsigma_*}\,\mu(\Omega) + \mathcal{H}^1(\partial\Omega) < \infty.$$

Since $\partial\Omega$ is Lipschitz, (4.2) implies that ∂A is $\mathcal{H}^1$-rectifiable. Therefore, $A \in \mathcal{A}$. □

Lemma 4.2. *Let $\{A_{m_h}\}$ and A be as in Proposition 4.1. Then $A_{m_h} \to A$ in $L^1(\mathbb{R}^2)$ as $h \to \infty$.*

Proof. Since $\mathcal{H}^1(\partial A) < \infty$ and $A_{m_h} \overset{K}{\to} A$ as $h \to \infty$, one has $\chi_{A_{m_h}}(x) \to \chi_A(x)$ as $h \to \infty$ for a.e. $x \in \mathbb{R}^2$. Now Lemma 4.2 follows from the Dominated Convergence Theorem. □

The following result generalizes [34, Theorem 4.2] since it applies to set Γ a priori not connected but satisfying uniform density estimates.

Proposition 4.3. *Let $\Gamma \subset \mathbb{R}^2$ be a $\mathcal{H}^1$-rectifiable closed set such that for some $r_0, c, C > 0$ and for all $x \in \Gamma$*

$$c \le \frac{\mathcal{H}^1(U_r(x))}{2r} \le C, \qquad r \in (0, r_0). \tag{4.5}$$

Then for a.e. $x \in \Gamma$,

$$\overline{U_{1,\nu_\Gamma(x)}(x)} \cap \sigma_{x,\rho}(\Gamma) \overset{K}{\to} \overline{U_{1,\nu_\Gamma(x)}(x)} \cap T_x \tag{4.6}$$

and

$$\mathcal{H}^1 \llcorner (\sigma_{x,\rho}(\Gamma)) \overset{*}{\rightharpoonup} \mathcal{H}^1 \llcorner T_x \tag{4.7}$$

as $\rho \to 0$, where $\sigma_{x,r}$ denotes the blow-up map defined in (2.2) and T_x is the generalized tangent line to Γ at x. Moreover, for any $\mathcal{H}^1$-measurable $\Gamma' \subset \Gamma$ and $\mathcal{H}^1$-a.e. $x \in \Gamma'$ the relations (4.6) and (4.7) hold with Γ' in place of Γ.

Proof. By [28, Theorem 3.3], Γ (and hence Γ') has a approximate tangent line at $\mathcal{H}^1$-a.e. x, therefore, (4.7) follows from [1, Remark 2.80]. To prove (4.6) with Γ choose $x \in \Gamma$ such that $\theta_*(\Gamma, x) = \theta^*(\Gamma, x) = 1$ and T_x exists. Without loss of generality we assume that $x = 0$, $\mathcal{H}^1(\Gamma) < \infty$ and $\nu_\Gamma(x) = \mathbf{e_2}$ is the unit normal to T_x. First we prove

$$\sigma_{0,r}(\Gamma) \overset{K}{\to} T_0 \tag{4.8}$$

as $r \searrow 0$. Indeed, let $\mu_r := \mathcal{H}^1 \llcorner (\sigma_{0,r}(\Gamma))$ and $\mu_0 := \mathcal{H}^1 \llcorner T_0$. Given $r > 0$, since $\mu_r(U_\rho(x)) = \frac{\mathcal{H}^1(U_{\rho r}(rx))}{r}$, by (4.5) for all $x \in \sigma_{0,r}(\Gamma)$ and $\rho \in (0, r_0/r)$ one has

$$c \le \frac{\mu_r(U_\rho(x))}{2\rho} \le C. \tag{4.9}$$

Let $r_k \searrow 0$ be any sequence. By compactness of sets in the Kuratowski convergence, passing to a further not relabelled subsequence if necessary we suppose that

$$\sigma_{0,r_k}(\Gamma) \overset{K}{\to} L \tag{4.10}$$

for some closed set $L \subset \mathbb{R}^2$ as $k \to \infty$. We claim that $L = T_0$. If there exists $x \in T_0 \setminus L$, then for some $\rho > 0$, $U_\rho(x) \cap L = \emptyset$. By (4.10), $U_{\rho/2}(x) \cap \sigma_{0,r_k}(\Gamma) = \emptyset$ for all large k so that $\mu_{r_k}(U_{\rho/2}(x)) = 0$. Then by (4.7)

$$0 = \lim_{k\to\infty} \mu_{r_k}(U_{\rho/2}(x)) \ge \mu_0(U_{\rho/2}(x)) > \rho,$$

a contradiction. If there exists $x \in L \setminus T_0$, then for some $U_\rho(x) \cap T_0 = \emptyset$ for some $\rho > 0$ and there exists a sequence $x_k \in \sigma_{0,r_k}(\Gamma)$ such that $x_k \to x$. Then $U_{\rho/2}(x_k) \subset U_\rho(x)$ for all large k so that by (4.7) and (4.9),

$$0 = \mu_0(\overline{U_\rho(x)}) \ge \limsup_{k\to\infty} \mu_{r_k}(\overline{U_\rho(x)}) \ge \limsup_{k\to\infty} \mu_{r_k}(U_{\rho/2}(x_k)) \ge c\rho,$$

a contradiction. Thus, $L = T_0$. Since the sequence $r_k \searrow 0$ is arbitrary, (4.8) follows. Now (4.6) is obvious. $\square$

Now we are ready to prove the existence of global minimizers of $\mathcal{F}$.

Proof of Theorem 2.5. The proof of the second assertion can be done using the first one and also following the arguments of [27, Theorem 1.1] and [39, Proposition A.6]. Hence, we prove only the first assertion. Let $(A_{m_h}, u_{m_h}) \in \mathcal{C}_{m_h}$ and $A \in \mathcal{C}$ be as in Proposition 4.1. By Lemma 4.2, $|A| = \mathrm{v}$. Since S is connected and Lipschitz, by the Korn-Poincaré inequality and the Rellich-Kondrachov Theorem there exists a further not relabelled subsequence $\{u_h\}$, a sequence $\{a_h\}$ of infinitesimal rigid displacements and $v_0 \in H^1(S;\mathbb{R}^2)$ such that $u_{m_h} + a_h \to v_0$ weakly in $H^1_{\rm loc}(S;\mathbb{R}^2)$ and a.e. in S. Let $\{E_i\}_{i\in\Lambda}$ be all connected components of $\mathrm{Int}(A)$. By a diagonal argument we choose a further not relabelled subsequence $\{u_{m_h}\}$ and the subset Λ' of indices $i \in \Lambda$ such that there exists a ball $B_i \subset\subset E_i$ such that the limit of $u_{m_h}(x) + a_h(x)$ is finite for a.e. $x \in B_i$. Then by [39, Proposition 3.7] for all $i \in \Lambda'$ there exists $v_i \in H^1_{\rm loc}(E_i;\mathbb{R}^2)$ such that $u_{m_h} + a_h \to v_i$ weakly in $H^1_{\rm loc}(E_i;\mathbb{R}^2)$ and a.e. in E_i as $h \to \infty$. By the choice of Λ', for any $j \in \Lambda \setminus \Lambda'$ one has $|u_{m_h} + a_h| \to +\infty$ a.e. in E_j as $h \to +\infty$. Define

$$u := v_0\chi_S + \sum_{i\in\Lambda'} v_i\chi_{E_i} + \sum_{j\in\Lambda\setminus\Lambda'} u_0\chi_{E_j}. \tag{4.11}$$

By construction, $u_{m_h} + a_h \to u$ weakly in $H^1_{\rm loc}(\cup_{i\in\Lambda'} E_i;\mathbb{R}^2)$ and a.e. in $\cup_{i\in\Lambda'} E_i$, and $|u_{m_h} + a_h| \to +\infty$ a.e. in $\cup_{j\in\Lambda\setminus\Lambda'} E_j$. In particular, $e(u_{m_h}) = e(u_{m_h} + a_h) \rightharpoonup e(u)$ in $L^2_{\rm loc}(\cup_{i\in\Lambda'} E_i;\mathbb{R}^2)$ Therefore, by the convexity of the elastic enery density $W(x, M)$ in M, for any $D \subset\subset \mathrm{Int}(A) \cup S$ we have

$$\begin{aligned}
&\liminf_{h\to\infty} \mathcal{W}(A_{m_h}, u_{m_h})\\
&\ge \liminf_{h\to\infty} \Big(\int_{D\cap S} W(x, e(u_{m_h}) - M_0)dx + \sum_{j\in\Lambda'} \int_{D\cap E_i} W(x, e(u_{m_h}) - M_0)dx \Big)\\
&\ge \int_{D\cap S} W(x, e(u) - M_0)dx + \sum_{j\in\Lambda'} \int_{D\cap E_i} W(x, e(u) - M_0)dx.
\end{aligned}$$

Taking into account $e(u) - E_0 = 0$ a.e. in $\cup_{j\in\Lambda\setminus\Lambda'} E_j$, this inequality can also be rewritten as

$$\liminf_{h\to\infty} \mathcal{W}(A_{m_h}, u_{m_h}) \ge \int_{D\cap(A\cup S)} W(x, e(u) - M_0)dx.$$

In particular, letting $D \searrow \mathrm{Int}(A) \cup S$ and using $|A \setminus \mathrm{Int}(A)| = 0$ we get

$$\liminf_{h\to\infty} \mathcal{W}(A_{m_h}, u_{m_h}) \ge \mathcal{W}(A, u). \tag{4.12}$$

We now introduce the open subset G of $A^{(1)}$, which we refer as *reduced set* G constructed from A. Let

$$\Gamma := \{x \in A^{(1)} \cap \partial A : \text{ one-sided traces of } u \text{ at } x \text{ exist and are equal}\}.$$

By Proposition 4.1 all points of Γ satisfies the uniform density estimates, and therefore, $\theta^*(\Gamma, x) = \theta_*(\Gamma, x) = 1$ at $\mathcal{H}^1$-a.e. $x \in \Gamma$. In particular, $\mathcal{H}^1(\overline{\Gamma} \setminus \Gamma) = 0$. Let

$$G := \mathrm{Int}(A \cup \overline{\Gamma}).$$

We claim that (G, u) is a minimizer of $\mathcal{F}$. Indeed, note that $|G| = |A|$ and $(G, u) \in \mathcal{C}$. In view of (3.1), it suffices to prove that

$$\liminf_{h\to\infty} \mathcal{F}(A_{m_h}, u_{m_h}) \ge \mathcal{F}(G, u). \tag{4.13}$$

Since $|G\Delta A| = 0$, by (4.12) we have

$$\liminf_{h\to\infty} \mathcal{W}(A_{m_h}, u_{m_h}) \ge \mathcal{W}(G,u).$$

thus to prove (4.13) it suffices to establish

$$\liminf_{h\to\infty} \mathcal{S}(A_{m_h}, u_{m_h} + a_h) \ge \mathcal{S}(G,u). \tag{4.14}$$

To prove (4.14) we follow the arguments of [39, Proposition 4.1]. Let $g \in L^\infty(\Sigma\times\{0,1\})$ be such that

$$g(x,s) := \varphi(x, \nu_\Sigma(x)) + s\beta(x)$$

and define

$$J_{A_{m_h}} := J_{u_{m_h}} = J_{u_{m_h}+a_h} \qquad \text{and} \qquad J_G := J_u \cup \bigcup_{j\in\Lambda\setminus\Lambda'} (\Sigma\cap\partial E_j).$$

Then by (2.8), $g \ge 0$ and

$$|g(x,1) - g(x,0)| \le \varphi(x,\nu_\Sigma(x)) \qquad \text{for a.e. } x\in\Sigma. \tag{4.15}$$

Let μ_h be the sequence of positive Radon measures defined at Borel sets $B\subset\mathbb{R}^2$ as

$$\begin{aligned} \mu_h(B) := & \int_{B\cap\Omega\cap\partial^* A_{m_h}} \varphi(x,\nu_{A_{m_h}})d\mathcal{H}^1 + 2\int_{B\cap\Omega\cap(A^{(1)}_{m_h}\cup A^{(0)}_{m_h})\cap\partial A_{m_h}} \varphi(x,\nu_{A_{m_h}})d\mathcal{H}^1 \\ & + \int_{B\cap\Sigma\cap A^{(0)}_{m_h}\cap\partial A_{m_h}} \big[\varphi(x,\nu_\Sigma) + g(x,1)\big] d\mathcal{H}^1(x) + \int_{B\cap\Sigma\setminus\partial A_{m_h}} g(x,0)d\mathcal{H}^1 \\ & + \int_{B\cap\Sigma\cap\partial^* A_{m_h}\setminus J_{A_{m_h}}} g(x,1)d\mathcal{H}^1 + \int_{B\cap J_{A_{m_h}}} \big[g(x,0)+\varphi(x,\nu_\Sigma)\big]\, d\mathcal{H}^1 \end{aligned}$$

and let μ be the positive measure defined at Borel sets $B\subset\mathbb{R}^2$ as

$$\begin{aligned} \mu(B) := & \int_{B\cap\Omega\cap\partial^* G} \varphi(x,\nu_G)d\mathcal{H}^1 + 2\int_{B\cap\Omega\cap G^{(1)}\cap\partial G} \varphi(x,\nu_G)d\mathcal{H}^1 \\ + & \int_{B\cap\Sigma\setminus\partial G} g(x,0)d\mathcal{H}^1 + \int_{B\cap\Sigma\cap\partial^* G\setminus J_G} g(x,1)d\mathcal{H}^1 + \int_{B\cap J_G} \big[g(x,0)+\varphi(x,\nu_\Sigma)\big]\, d\mathcal{H}^1. \end{aligned}$$

Recalling $\mathcal{H}^1(G^{(0)}\cap\partial G) = 0$ we note that μ is defined as μ_h with A_{m_h} replaced by E. Note that

$$\mu_h(\mathbb{R}^2) = \mathcal{S}(A_{m_h}, u_{m_h}) + \int_\Sigma \varphi(x,\nu_\Sigma)d\mathcal{H}^1$$

and

$$\mu(\mathbb{R}^2) = \mathcal{S}(G,u) + \int_\Sigma \varphi(x,\nu_\Sigma)d\mathcal{H}^1.$$

Hence, it suffices to prove

$$\liminf_{h\to\infty} \mu_h(\mathbb{R}^2) \ge \mu(\mathbb{R}^2). \tag{4.16}$$

Note that since $\sup_h \mu_h(\mathbb{R}^2) < +\infty$, by compactness, there exists a positive Radon measure μ_0 in $\mathbb{R}^2$ such that (up to a subsequence) $\mu_h \rightharpoonup^* \mu_0$ as $h\to\infty$. We prove

$$\mu_0 \ge \mu \tag{4.17}$$

and we observe that (4.16) immediately follows from (4.17). To establish (4.17) it suffices to prove

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\Omega\cap\partial^* G)}(x) \ge \varphi(x,\nu_G(x)) \qquad \text{for a.e. } x\in\Omega\cap\partial^* G, \tag{4.18a}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\Omega\cap G^{(1)}\cap\partial G)}(x) \ge 2\varphi(x,\nu_G(x)) \qquad \text{for a.e. } x\in\Omega\cap G^{(1)}\cap\partial G, \tag{4.18b}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\Sigma \cap \partial^* G)}(x) \geq g(x,1) \qquad \text{for a.e. } x \in \Sigma \cap \partial^* G, \tag{4.18c}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\Sigma \cap J_G)}(x) \geq \varphi(x, \nu_\Sigma(x)) + g(x,0) \qquad \text{for a.e. } x \in \Sigma \cap J_G, \tag{4.18d}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\Sigma \setminus \partial G)}(x) \geq g(x,0) \qquad \text{for a.e. } x \in \Sigma \setminus \partial G. \tag{4.18e}$$

Since $\partial^* A = \partial^* G$ (up to a $\mathcal{H}^1$-negligible set) and $A_k \to G$ in $L^1(\mathbb{R}^2)$, the proofs of (4.18a)and (4.18e) can be done following the arguments of [39, Eq. 4.40a] and [39, Eq. 4.40d], respectively. Moreover, for any $x_0 \in \Sigma \setminus \partial G$ and a square $U_r := U_{r,\nu_\Sigma(x_0)}(x_0) \subset\subset \mathbb{R}^2 \setminus \overline{G}$ one has $U_r \cap \overline{\operatorname{Int}(A_{m_h})} = \emptyset$ for all large h, and thus, by the nonnegativeity of g and (4.15),

$$\mu_h(U_r) \geq \int_{U_r} g(x,0) d\mathcal{H}^1,$$

and hence (4.18c) follows. Furthermore, the proof of (4.18d) for $x \in J_G \cap J_u$ can be done following [39, Eq. 4.40g] and for $x \in J_G \setminus J_u$ can be done adapting the arguments of [39, Proposition 3.9].

It remains to show (4.18b). We notice that the argument of [39, Eq. 4.40c] cannot be used in this setting, as it hinges on the uniform bound on the number of connected components which we do not have here. We instead adapt the arguments employed in [39, Eq. 4.40g]. We prove (4.18b) for all $x \in G^{(1)} \cap \partial G$ satisfying:

(b1) $\theta_*(\partial G, x) = \theta^*(\partial G, x) = \theta_*(G^{(1)} \cap \partial G, x) = \theta^*(G^{(1)} \cap \partial G, x) = 1$ and $\nu_{\partial G}(x)$ exists;
(b2) $\theta_*(\partial A, x) = \theta^*(\partial A, x) = \theta_*(A^{(1)} \cap \partial A, x) = \theta^*(A^{(1)} \cap \partial A, x) = 1$ and $\nu_{\partial A}(x)$ exists;
(b3) $\overline{U_{1,\nu_{\partial G}}(x)} \cap \sigma_{\rho,x}(\partial G) \xrightarrow{K} U_{1,\nu_{\partial G}}(x) \cap T_x$ as $\rho \to 0$, where T_x is the generalized tangent line to ∂G at x;
(b4) one-sided traces $u^+(x)$ and $u^-(x)$ of u w.r.t. $G^{(1)} \cap \partial G$ exist and are not equal;
(b5) $\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (G^{(1)} \cap \partial G)}(x)$ exists and is finite.

We notice that the set of points of $G^{(1)} \cap \partial G$ not satisfying (b1)-(b5) is $\mathcal{H}^1$-negligible, since ∂G and ∂A are $\mathcal{H}^1$-rectifiable, the definition and $\mathcal{H}^1$-rectifiability of J_u [19, Definition 2.4 and Theorem 6.2], both the statements $G^{(1)} \cap \partial G = J_u$ and $G^{(1)} \cap \partial G \subseteq A^{(1)} \cap \partial A$ hold up to a $\mathcal{H}^1$-negligible set, and the fact that we can apply Proposition 4.3 with $\Gamma = \partial A$ and $\Gamma' = \partial G$, and the Besicovitch Differentiation Theorem. Notice that $\nu_{\partial G}(x) = \nu_{\partial A}(x)$ for every $x \in \partial G$, and assume, without loss of generality, that $x = 0$ and $\nu_{\partial G}(x) = \mathbf{e_2}$. By the construction of G, the definition of $G^{(1)} \cap \partial G \ni 0$, and the definition (4.11) of u we have three cases:

(a) there exists $i_0 \in \Lambda'$ such that

$$\theta^*(E_{i_0}^{(1)} \cap \partial E_{i_0}, 0) = \theta_*(E_{i_0}^{(1)} \cap \partial E_{i_0}, 0) = 1$$

and $u_{m_h} + a_{m_h} \to u$ a.e. in E;

(b) there exist $i_0 \in \Lambda'$ and $j_0 \in \Lambda \setminus \Lambda'$ such that $0 \in \partial^* E_{i_0} \cap \partial E_{j_0}^*$ and $u_{m_h} + a_{m_h} \to u$ a.e. in E_{i_0} and $|u_{m_h} + a_{m_h}| \to \infty$ a.e. in E_{j_0}. Without loss of generality we assume that $\mathbf{e_2}$ is the inner normal of E_{i_0};

(c) there exist $i_1, i_2 \in \Lambda'$ such that $0 \in \partial^* E_{i_1} \cap \partial E_{i_2}^*$ and $u_{m_h} + a_{m_h} \to u$ a.e. in $E_{i_1} \cup E_{i_2}$.

First we observe the following. Let $4r_0 := \operatorname{dist}(0, \partial\Omega)$. By weak convergence,

$$\lim_{h \to \infty} \mu_h(\overline{U_r}) = \mu_0(U_r) \tag{4.19}$$

for a.e. $r \in (0,r_0)$. Let $\xi \in (0,1)^2$ be such that the set of all $x \in \partial A_{m_h}$ (resp. $x \in \partial G$) with one-sided traces of $u_{m_h} + a_h$ (resp. of u) existing and equal to ξ is $\mathcal{H}^1$-negligible. Define

$$w_h := (u_{m_h} + a_h)\chi_{A_{m_h}} + \xi\chi_{\Omega\setminus A_{m_h}}$$

and

$$w := u\chi_G + \xi\chi_{\Omega\setminus G}.$$

We recall that by the definition (4.11) of u the set of $x \in \Omega$ such that $w_h(x)$ does not converge to $w(x)$ could have positive measure.

By the construction of w_h, $J_{w_h} \subset \partial A_{m_h}$ and

$$\sup_{h\ge1}\ \mathcal{H}^1(J_{w_h}) + \int_\Omega |e(w_h)|^2 dx < \infty.$$

Moreover, by (b4), [19, Definition 2.4] and [19, Remark 2.2] separately applied to U_1^+ and $U_1 \setminus U_1^+$ we have

$$\lim_{r\to0}\int_{U_1} |\tau(w(rx)) - \tau(w_0(x))|dx = 0, \tag{4.20}$$

where $\tau(x_1,x_2) = (\arctan x_1, \arctan x_2)$,

$$w_0 := u^+(0)\chi_{U_1^+} + u^-(0)\chi_{U_1\setminus U_1^+},$$

and $u^+(0)$ and $u^-(0)$ are given by (b4). For every $r \in (0,r_0)$ let

$$U_r^\infty := \{x \in U_1 :\ \liminf_{h\to\infty} |w_h(rx)| = +\infty\}.$$

Unlike the proof of [39, Eq. 4.40g], U_r^∞ can be non-empty; more precisely, $rU_r^\infty \subset \cup_{j\in\Lambda\setminus\Lambda'} E_j$ (see the definition (4.11) of u). Since $w_h(rx) \to w(rx)$ a.e. in $U_1 \setminus U_r^\infty$, one has

$$\lim_{h\to\infty}\int_{U_1\setminus U_r^\infty} |\tau(w_h(rx)) - \tau(w(rx))|dx = 0. \tag{4.21}$$

By Proposition 4.3

$$U_4 \cap \sigma_r(\partial A) \overset{K}{\to} I_4 \quad \text{and} \quad \mathcal{H}^1 \llcorner (U_4 \cap \sigma_r(\partial A)) \overset{*}{\rightharpoonup} \mathcal{H}^1 \llcorner I_4, \tag{4.22a}$$

$$U_4 \cap \sigma_r(\partial E_{i_0}) \overset{K}{\to} I_4 \quad \text{and} \quad \mathcal{H}^1 \llcorner (U_4 \cap \sigma_r(\partial E_{i_0})) \overset{*}{\rightharpoonup} \mathcal{H}^1 \llcorner I_4, \tag{4.22b}$$

$$U_4 \cap \sigma_r(\partial[E_{i_0} \cup E_{j_0}]) \overset{K}{\to} I_4 \quad \text{and} \quad \mathcal{H}^1 \llcorner (U_4 \cap \sigma_r(\partial[E_{i_0} \cup E_{j_0}])) \overset{*}{\rightharpoonup} \mathcal{H}^1 \llcorner I_4 \tag{4.22c}$$

$$U_4 \cap \sigma_r(\partial[E_{i_1} \cup E_{i_2}]) \overset{K}{\to} I_4 \quad \text{and} \quad \mathcal{H}^1 \llcorner (U_4 \cap \sigma_r(\partial[E_{i_1} \cup E_{i_2}])) \overset{*}{\rightharpoonup} \mathcal{H}^1 \llcorner I_4 \tag{4.22d}$$

as $r \to 0$, where we apply Proposition 4.3 with $\Gamma := \partial A$ and with $\Gamma' := \partial E_{i_0}$ in (4.22b), with[1] $\Gamma' := \partial[E_{i_0} \cup E_{j_0}] = \partial E_{i_0} \cup \partial E_{j_0}$ in (4.22c) and with $\Gamma' := \partial[E_{i_1} \cup E_{i_2}] = \partial E_{i_1} \cup \partial E_{i_2}$ in (4.22d). Furthermore, by (4.22a)-(4.22d) and [39, Proposition A.5]

$$\mathrm{sdist}(\cdot, \sigma_r(\partial A)) \to -\mathrm{dist}(\cdot, T_0), \tag{4.23a}$$

$$\mathrm{sdist}(\cdot, \sigma_r(\partial E_{i_0})) \to -\mathrm{dist}(\cdot, T_0), \tag{4.23b}$$

$$\mathrm{sdist}(\cdot, \sigma_r(\partial[E_{i_0} \cup E_{j_0}])) \to -\mathrm{dist}(\cdot, T_0), \tag{4.23c}$$

$$\mathrm{sdist}(\cdot, \sigma_r(\partial[E_{i_1} \cup E_{i_2}])) \to -\mathrm{dist}(\cdot, T_0) \tag{4.23d}$$

locally uniformly in $\overline{U_4}$ as $r \to 0$. Hence,

$$\lim_{r\to0} |U_r^\infty \Delta U_0^\infty| = 0, \tag{4.24}$$

[1]Note that if if $P, Q \subset \mathbb{R}^n$ are open and disjoint, then $\partial(P \cup Q) = \partial P \cup \partial Q$.

where by (4.23b)-(4.23d)

$$U_0^\infty = \begin{cases} \emptyset & \text{in cases (a) and (c),} \\ U_1 \setminus U_1^+ & \text{in case (b).} \end{cases} \tag{4.25}$$

Now we choose sequences $h_k \to \infty$ and $r_k \searrow 0$ as follows. By (4.19), (4.23a), (4.20) and (4.24) for any $k \in \mathbb{N}$ there exists $r_k \in (0, \frac{1}{k})$ such that (4.19) holds with $r = r_k$ and

$$\|\mathrm{sdist}(\cdot, \sigma_{r_k}(U_{4r_k} \cap \partial A)) + \mathrm{dist}(\cdot, U_4 \cap T_x)\|_{L^\infty(U_{3/2})} < \frac{1}{k^2}, \tag{4.26a}$$

$$\int_{U_1} |\tau(w(r_k x)) - \tau(w_0(x))| dx < \frac{1}{k^2}, \tag{4.26b}$$

$$|U_{r_k}^\infty \Delta U_0^\infty| < \frac{1}{k^2}. \tag{4.26c}$$

Given $k \geq 1$ and r_k, since A_{m_h} $\tau_{\mathcal{A}}$-converges to A and τ is bounded, by (4.21), (4.26c) and (4.19) we can choose h_k such that

$$\frac{1}{h_k r_k} < \frac{1}{k}, \tag{4.27a}$$

$$\|\mathrm{sdist}(\cdot, U_4 \cap \sigma_{r_k}(\partial A_{m_{h_k}})) - \mathrm{sdist}(\cdot, U_4 \cap \sigma_{r_k}(\partial A))\|_{L^\infty(U_{3/2})} < \frac{1}{k}, \tag{4.27b}$$

$$\int_{U_1 \setminus U_0^\infty} |\tau(w_{h_k}(r_k x)) - \tau(w(r_k x))| dx < \frac{1}{k}, \tag{4.27c}$$

$$\mu_{h_k}(\overline{U_{r_k}}) \leq \mu_0(U_{r_k}) + r_k^2. \tag{4.27d}$$

Notice that by (4.27a), $h_k \to \infty$ as $k \to \infty$.

Let

$$D_k := \sigma_{r_k}(U_{4r_k} \cap A_{m_{h_k}}) = U_4 \cap \sigma_{r_k}(A_{m_{h_k}}).$$

By (4.27b) and (4.26a),

$$U_{3/2} \cap \partial D_k \overset{K}{\to} I_{3/2} = \overline{U_{3/2}} \cap T_0 \qquad \text{as } k \to \infty.$$

In particular, for any $\delta \in (0,1)$ there exists $k_\delta > 1$ such that $\overline{U_1 \cap \partial D_k} \subset [-1,1] \times [-\delta, \delta]$. Furthermore, by (4.27d) and finiteness of

$$\frac{d\mu_0}{\mathcal{H}^1 \llcorner (G^{(1)} \cap \partial G)}(0) = \lim_{k \to \infty} \frac{\mu_0(U_{r_k})}{2r_k}$$

we may suppose for $C := \frac{d\mu_0}{\mathcal{H}^1 \llcorner (G^{(1)} \cap \partial G)}(0) + 1$,

$$\mathcal{H}^1(U_1 \cap \partial D_k) = \frac{\mathcal{H}^1(U_{r_k} \cap \partial A_{m_{h_k}})}{r_k} \leq \frac{\mu_{h_k}(\overline{U_{r_k}})}{r_k} \leq C \tag{4.28}$$

for all $k \geq 1$. Furthermore, since φ is uniformly continuous, we suppose also that

$$|\varphi(x, \nu) - \varphi(0, \nu)| < \delta, \qquad x \in \overline{U_{r_k}}, \ \nu \in \mathbb{S}^1,$$

thus, by the definition of μ_{h_k} and D_k and (4.28),

$$\frac{\mu_{h_k}(\overline{U_{r_k}})}{r_k} \geq \int_{U_1 \cap \partial^* D_k} \phi(\nu_{D_k}) d\mathcal{H}^1 + 2 \int_{U_1 \cap D_k^{(1)} \cap \partial D_k} \phi(\nu_{D_k}) d\mathcal{H}^1 - 2C\delta, \tag{4.29}$$

where $\phi(\nu) = \varphi(0, \nu)$. Also, since the number of connected components of $\partial A_{m_{h_k}}$ is at most m_{h_k}, the number of connected components of $[U_1 \cap \partial D_k] \cup (\{\pm 1\} \times [-\delta, \delta])$ does not exceed $m_{h_k} + 2$. Let us index them with the index set Λ'' and denote their closures by $\{L_i^k\}_{i \in \Lambda''}$. As in the proof of [39, Lemma 4.7] we will replace L_i^k with a pairwise disjoint finite family $\{V_j^k\}_j$ of closed connected sets containing them.

First we partition Λ'' into finitely many pairwise disjoint non-empty subsets $\{\Lambda_j^k\}_{j=1}^{N_k}$ by induction as follows. In the following we denote by $co\,(S)$ the closed convex hull of a set S.

We begin by defining Λ_1^k as the set of indices $\{i_1,\dots,i_{n_1^k}\}$ in Λ'' for some $n_1^k\ge 1$ such that:

(c1) $i_1:=1$;
(c2) i_l for any $1<l\le n_1^k$ is the smallest index $q\in\Lambda''\setminus\{i_1,\dots,i_{l-1}\}$ such that $co\left(L_q^k\right)$ intersects $co\left(\cup_{j=1}^{l-1}L_{i_j}\right)$,
(c3) n_1^k is maximal, i.e., given $q\in\Lambda''\setminus\Lambda_1^k$ such that $co\left(L_q^k\right)$ does not intersects $co\left(\cup_{j=1}^{n_1^k}L_{i_j}\right)$.

Suppose that for $j\ge 2$ the non-empty sets $\Lambda_1^k,\dots,\Lambda_{j-1}^k$ are defined. If $\Lambda_j:=\Lambda''\setminus\cup_{i=1}^{j-1}\Lambda_i^k\neq\emptyset$, then we define $\Lambda_j^k:=\{i_1^j,\dots,i_{n_j^k}^j\}$ as the set with i_1^j the smallest element of Λ_j and satisfying (c2) and (c3) with i_l, n_1^k and $\Lambda''\setminus\Lambda_1^k$ replaced by i_l^j, n_j^k and Λ_j, respectively. Since $\Lambda''\subset\{1,\dots,m_{h_k}+2\}$, the cardinality of $\{\Lambda_j^k\}_j$, which we denote by N_k, satisfies $N_k\le m_{h_k}+2$.

Now define

$$V_j^k:=\ co\left(\bigcup_{i\in\Lambda_j^k}L_i^k\right),\qquad j=1,\dots,N_k.$$

Notice that if C_i^k denotes the convex hull of L_i^k, then V_j^k is also the convex hull of $\bigcup\limits_{i\in\Lambda_j^k}C_i^k$.

By applying (recursively if needed) the anisotropic minimality of segments,

$$\int_{L_i^k\cap\partial^*D_k}\phi(\nu_{L_i^k})d\mathcal{H}^1+2\int_{L_i^k\cap D_k^{(1)}\cap\partial D_k}\phi(\nu_{L_i^k})d\mathcal{H}^1 + \int_{L_i^k\cap(\{\pm1\}\times[-\delta,\delta])}\phi(\mathbf{e_1})d\mathcal{H}^1\ge\int_{\partial C_i^k}\phi(\nu_{C_i^k})d\mathcal{H}^1$$

for any $i\ge 1$ and

$$\sum_{i\in\Lambda_j^k}\int_{\partial C_i^k}\phi(\nu_{C_i^k})d\mathcal{H}^1\ge\int_{\partial V_j^k}\phi(\nu_{V_j^k})d\mathcal{H}^1$$

for any $j\in\{1,\dots,N_k\}$. Note that if V_j^k is a segment, then $\partial V_j^k=V_j^k$, and in this case we replace V_j^k with a very thin rectangle $\hat V_j^k$ such that $V_j^k\subset\hat V_j^k$, $\mathrm{dist}(\hat V_j^k,V_i^k)>0$ for $i\neq j$ and

$$2\int_{V_j^k}\phi(\nu_{V_i^k})d\mathcal{H}^1\ge\int_{\partial\hat V_j^k}\phi(\nu_{\hat V_j^k})d\mathcal{H}^1-\frac{c_2\delta\rho_k}{2^j}\tag{4.30}$$

for some sequence $N_k\rho_k\searrow 0$. Such $\hat V_j^h$ always exists since $N_k<\infty$. For shortness we abuse the notation V_j^h also for $\hat V_j^h$. Then by (4.29),

$$\begin{aligned}\frac{\mu_h(\overline{U_{r_k}})}{r_k}&\ge\sum_{i\in\Lambda''}\Big(\int_{L_i^k\cap\partial^*D_k}\phi(\nu_{D_k})d\mathcal{H}^1+2\int_{D_k^{(1)}\cap\partial D_k}\phi(\nu_{D_k})d\mathcal{H}^1\Big)-2C\delta\\&\ge\sum_{j=1}^{N_k}\int_{\partial V_j^k}\phi(\nu_{V_j^k})d\mathcal{H}^1-(5c_2+2C)\delta,\end{aligned}\tag{4.31}$$

where $5c_2\delta$ is related to the surface energy of vertical segments $\{\pm 1\} \times [-\delta, \delta]$ and the error in (4.30).

Notice that $x \in U_1 \mapsto w_{h_k}(r_k x)$ belongs to $H^1(U_1 \setminus \bigcup_{j=1}^{N_k} V_j^k; \mathbb{R}^2)$. Hence, for any fixed $\xi_1 \in (0,1)^2$ if we define

$$v_k(x) := w_{h_k}(r_k x)\chi_{U_1 \setminus \bigcup_{j=1}^{N_k} V_j^k} + \xi_1 \chi_{\bigcup_{j=1}^{N_k} V_j^k}, \qquad x \in U_1,$$

then $v_k \in GSBD^2(U_1; \mathbb{R}^2)$. Further we choose $\xi_1 \in (0,1)^2$ such that $J_{v_k} = \bigcup_i \partial V_i^k$ (up to a $\mathcal{H}^1$-negligible set). Since

$$\int_{U_1} |e(v_k)|^2 dx \le \int_{U_{r_k} \cap A_{m_{h_k}}} |e(u_{m_{h_k}})|^2 dx$$

and $\{(A_{m_h}, u_{m_h})\}$ is a minimizing sequence, it follows that

$$\sup_{k \ge 1} \int_{U_1} |e(v_k)|^2 dx + \mathcal{H}^1(J_{v_k}) < \infty.$$

By (4.27c) and (4.26b) and from the estimate $|\tau(y)| \le 2\pi$ it follows that

$$\int_{U_1 \setminus U_0^\infty} |\tau(v_k(x)) - \tau(w_0(x))| dx \le \frac{2}{k} + 2\pi \sum_{j \ge 1}^{N_k} |V_j^k|. \tag{4.32}$$

As $\overline{U_1} \cap \partial D_k \xrightarrow{K} I_1$,

$$\lim_{k \to \infty} \sum_{j \ge 1}^{N_k} |V_j^k| = 0.$$

Thus, from (4.32) passing to further not relabelled subsequence, we get $v_k \to w_0$ a.e. in $U_1 \setminus U_0^\infty$ and $|v_k| \to +\infty$ a.e. in U_0^∞. By (4.25), U_0^∞ is either empty or the lower part of the unit square. Thus repeating the same arguments of the proof of [39, Eq. 4.35] we get

$$\begin{aligned}\liminf_{k \to \infty} \int_{J_{v_k}} \phi(\nu_{J_{v_k}}) d\mathcal{H}^1 &= \liminf_{k \to \infty} \sum_{i \ge 1} \int_{\partial V_i^k} \phi(\nu_{V_i^k}) d\mathcal{H}^1 \\ &\ge 2 \int_{U_1 \cap (J_{w_0} \cup \partial^* U_0^\infty)} \phi(\mathbf{e_2}) d\mathcal{H}^1 = 2 \int_{I_1} \phi(\mathbf{e_2}) d\mathcal{H}^1.\end{aligned}$$

In particular, there exists $k_\delta > 0$ such that

$$\sum_{i \ge 1} \int_{\partial V_i^k} \phi(\nu_{V_i^k}) d\mathcal{H}^1 > 2 \int_{I_1} \phi(\mathbf{e_2}) d\mathcal{H}^1 - \delta$$

for all $k > k_\delta$. Furthermore, from (4.31) and (4.27d) it follows that

$$\frac{\mu_0(U_{r_k})}{2r_k} + \frac{r_k}{2} \ge \int_{I_1} \phi(\mathbf{e_2}) d\mathcal{H}^1 - \frac{5c_2 + 2C + 1}{2}\delta.$$

Now first letting $k \to \infty$ and then $\delta \to 0$ we get (4.18b). □

5. Proof of Theorem 2.6

In this section we study the properties of the minimizers of $\mathcal{F}$ in $\mathcal{C}$ provided by Theorem 2.5.

Proof of Theorem 2.6. Properties (C) and (D) follow from the same arguments used to prove (3.2) and (3.3) of Theorem 3.1, respectively, with the only difference of applying the assertion of Lemma 3.2 for $\mathcal{C}$ instead of $\mathcal{C}_m$. Since ∂A satisfies the uniform density estimates, the reduced set G constructed in the proof of Theorem 2.5 satisfies (A) and (B). Hence, we can suppose that $A = G$.

Notice that if $E \subset A$ is a connected component of A with $\mathcal{H}^1(\partial E \cap \Sigma \setminus J_u) = 0$, then for (A, u') with $u' = u\chi_{(A\cup S)\setminus E} + u_0\chi_E$ we have

$$\mathcal{S}(A,u) \geq \mathcal{S}(A,u')$$

and

$$\mathcal{W}(A,u) \geq \mathcal{W}(A,u'), \tag{5.1}$$

where in (5.1) equality holds if and only of $u = u_0$ in E. Therefore, by the minimality of (A, u) it follows that $u = u_0$ in E. It remains to prove

$$|E| \geq 4\pi \left(\frac{c_1}{\lambda_0}\right)^2. \tag{5.2}$$

Consider the competitor $(A \setminus E, u) \in \mathcal{C}$. By minimality and Theorem 2.5, $\mathcal{F}^\lambda(A, u) \leq \mathcal{F}^\lambda(A \setminus E, u)$, so that by (5.1) and the additivity of the surface energy, $\mathcal{S}(E, u) \leq \lambda|E|$. Then by (2.7) and the isoperimetric inequality in $\mathbb{R}^2$

$$\lambda|E| \geq c_1\mathcal{H}^1(\partial E) \geq c_1\sqrt{4\pi}|E|^{1/2}.$$

Hence, $|E| \geq (c_1\sqrt{4\pi}/\lambda)^2$. Since $\lambda > \lambda_0$ is arbitrary, (5.2) follows. □

Acknowledgments

Sh. Kholmatov acknowledges support from the Austrian Science Fund (FWF) project M 2571-N32. P. Piovano acknowledges support from the Austrian Science Fund (FWF) project P 29681, from the Vienna Science and Technology Fund (WWTF) together with the City of Vienna and Berndorf Privatstiftung through Project MA16-005, and from BMBWF through the OeAD-WTZ project HR 08/2020.

References

[1] L. Ambrosio, N. Fusco, D. Pallara: Functions of Bounded Variation and Free Discontinuity problems. Oxford University Press, New York 2000.
[2] R. Asaro, W. Tiller: Interface morphology development during stress corrosion cracking: Part I. Via surface diffusion. Metall. Trans. **3** (1972), 1789–1796.
[3] J.-F. Babadjian, D. Henao: Reduced models for linearly elastic thin films allowing for fracture, debonding or delamination. Interface Free Bound. **18** (2016), 545–578.
[4] P. Bella, M. Goldman, B. Zwicknagl: Study of island formation in epitaxially strained films on unbounded domains. Arch. Rational Mech. Anal. **218** (2015), 163–217.
[5] E. Bonnetier, A. Chambolle: Computing the equilibrium configuration of epitaxially strained crystalline films. SIAM J. Appl. Math. **62** (2002), 1093–1121.
[6] B. Bourdin, G. Francfort, J.-J. Marigo: The variational approach to fracture. Springer, Amsterdam, 2008.
[7] A. Braides, A. Chambolle, M. Solci: A relaxation result for energies defined on pairs set-function and applications. ESAIM: Control Optim. Calc. Var. **13** (2007), 717–734.
[8] L. Caffarelli, A. Mellet: Capillary drops: contact angle hysteresis and sticking drops. Calc. Var. Partial Differential Equations **29** (2007), 141–160.
[9] A. Chambolle, S. Conti, G. Francfort: Korn-Poincaré inequalities for functions with a small jump set. Indiana Univ. Math. J. **65** (2016), 1373–1399.

[10] A. Chambolle, S. Conti, F. Iurlano: Approximation of functions with small jump sets and existence of strong minimizers of Griffith's energy. J. Math. Pures Appl. **128** (2019), 119–139.

[11] A. Chambolle, V. Crismale: Existence of strong solutions to the Dirichlet problem for the Griffith energy. Calc. Var. Partial Differential Equations **58** (2019), 136.

[12] A. Chambolle, V. Crismale: Compactness and lower-semicontinuity in $GSBD$. To appear in J. Eur. Math. Soc. (JEMS).

[13] A. Chambolle, Ch.J. Larsen: C^∞-regularity of the free boundary for a two-dimensional optimal compliance problem. Calc. Var. Partial Differential Equations **18** (2003), 77–94.

[14] V. Crismale, M. Friedrich: Equilibrium configurations for epitaxially strained films and material voids in three-dimensional linear elasticity. To appear in Arch. Rational Mech. Anal.

[15] S. Conti, M. Focardi, F. Iurlano: Existence of minimizers for the 2d stationary Griffith fracture model. C.R. Math. **354** (2016), 1055–1059.

[16] S. Conti, M. Focardi, F. Iurlano: A note on the Hausdorff dimension of the singular set of solutions to elasticity type systems. Commun. Contemp. Math. **21** (2019), 1950026.

[17] S. Conti, M. Focardi, F. Iurlano: Existence of strong minimizers for the Griffith static fracture model in dimension two. Ann. Inst. H. Poincaré Anal. Non Linéare **36** (2019), 455–474.

[18] G. Dal Maso: An Introduction to Γ-Convergence. Birkhäuser, Boston, 1993.

[19] G. Dal Maso: Generalised functions of bounded deformation. J. Eur. Math. Soc. **15** (2013), 1943–1997.

[20] G. Dal Maso, J. Morel, S. Solimini: A variational method in image segmentation: Existence and approximation results. Acta Math. **168** (1992), 89–151.

[21] A. Danescu: The Asaro-Tiller-Grinfeld instability revisited. Int. J. Solids Struct. **38** (2001), 4671–4684.

[22] E. Davoli, P. Piovano: Analytical validation of the Young-Dupré law for epitaxially-strained thin films. Math. Models Methods Appl. Sci. **29** (2019), 2183–2223.

[23] E. Davoli, P. Piovano: Derivation of a heteroepitaxial thin-film model. Interface Free Bound. **22** (2020), 1–26.

[24] P.-G. De Gennes, F. Brochard-Wyart, D. Quéré: Capillarity and Wetting Phenomena: Drops, Bubbles, Pearls, Waves. Springer, New York, 2004.

[25] G. De Philippis, F. Maggi: Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. Arch. Rational Mech. Anal. **216** (2015), 473–568.

[26] X. Deng: Mechanics of debonding and delamination in composites: Asymptotic studies. Compos. Eng. **5** (1995), 1299–1315.

[27] L. Esposito, N. Fusco: A remark on a free interface problem with volume constraint. J. Convex Anal. **18** (2011), 417–426.

[28] K. Falconer: The Geometry of Fractal Sets. Cambridge University Press, Cambridge, 1985.

[29] I. Fonseca, N. Fusco, G. Leoni, M. Morini: Equilibrium configurations of epitaxially strained crystalline films: existence and regularity results. Arch. Rational Mech. Anal. **186** (2007), 477–537.

[30] I. Fonseca, N. Fusco, G.Leoni, V. Millot: Material voids in elastic solids with anisotropic surface energies. J. Math. Pures Appl. **96** (2011), 591–639.

[31] G. Francfort, A. Giacomini, O. Lopez-Pamies: Fracture with healing: a first step towards a new view of cavitation. Anal. PDE **12** (2019), 417–447.

[32] N. Fusco, V. Julin, M. Morini: The surface diffusion flow with elasticity in the plane. Commun. Math. Phys. **362** (2018), 571–607.

[33] N. Fusco, M. Morini: Equilibrium configurations of epitaxially strained elastic films: second order minimality conditions and qualitative properties of solutions. Arch. Rational Mech. Anal. **203** (2012), 247–327.

[34] A. Giacomini: A generalization of Golab's theorem and applications to fracture mechanics. Math. Models Methods Appl. Sci. **12** (2002), 1245–1267.

[35] E. Giusti: Minimal Surfaces and Functions of Bounded Variation. Birkhäuser, Boston, 1984.

[36] M. Goldman, B. Zwicknagl: Scaling law and reduced models for epitaxially strained crystalline films. SIAM J. Math. Anal. **46** (2014), 1–24.

[37] M.A. Grinfeld: The stress driven instabilities in crystals: mathematical models and physical manifestations. J. Nonlinear Sci. **3** (1993), 35–83.

[38] J. Hutchinson, Z. Suo: Mixed mode cracking in layered materials. Adv. Appl. Mech. **29** (1991), 63–191.

[39] Sh. Kholmatov, P. Piovano: A unified model for stress-driven rearrangement instabilities. Arch. Rational Mech. Anal., **238** (2020), 415–488.

[40] A.A. León Baldelli, B. Bourdin, J.-J. Marigo, C. Maurini: Fracture and debonding of a thin film on a stiff substrate: analytical and numerical solutions of a one-dimensional variational model. Cont. Mech. Thermodyn. **25** (2013), 243–268.

[41] A.A. León Baldelli, J.-F. Babadjian, B. Bourdin, D. Henao, C. Maurini: A variational model for fracture and debonding of thin films under in-plane loadings. J. Mech. Phys. Solids **70** (2014), 320–348.
[42] F. Maddalena, S. Solimini: Lower smeicontinuity properties of functionals with free discontinnuities. Arch. Rational Mech. Anal. **159** (2001), 273–294.
[43] F. Maggi: Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory. Cambridge University Press, Cambridge, 2012.
[44] P. Piovano: Evolution and Regularity Results for Epitaxially Strained Thin Films and Material Voids. Ph.D. Thesis - Carnegie Mellon University, 2012.
[45] M. Siegel, M. Miksis, P. Voorhees: Evolution of material voids for highly anisotropic surface energy. J. Mech. Phys. Solids **52** (2004), 1319–1353.
[46] H. Wang, Z. Li: The instability of the diffusion-controlled grain-boundary void in stressed solid. Acta Mech. Sinica **19** (2003), 330–339.
[47] Z.C. Xia, J.W. Hutchinson: Crack patterns in thin films. J. Mech. Phys. Solids **48** (2000), 1107–1131.

(Shokhrukh Yu. Kholmatov) Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern Platz 1, 1090 Wien (Austria)

Email address, Sh. Kholmatov: shokhrukh.kholmatov@univie.ac.at

(Paolo Piovano) Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern Platz 1, 1090 Wien (Austria)

Email address, P. Piovano: paolo.piovano@univie.ac.at