ASYMPTOTIC ANALYSIS OF A SECOND-ORDER SINGULAR PERTURBATION MODEL FOR PHASE TRANSITIONS

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Abstract. We study the asymptotic behavior, as ε tends to zero, of the functionals F_ε^k introduced by Coleman and Mizel in the theory of nonlinear second-order materials; *i.e.*,

$$F_\varepsilon^k(u) := \int_I \left(\frac{W(u)}{\varepsilon} - k\,\varepsilon\,(u')^2 + \varepsilon^3 (u'')^2 \right) dx, \quad u \in W^{2,2}(I),$$

where $k > 0$ and $W : \mathbb{R} \to [0, +\infty)$ is a double-well potential with two potential wells of level zero at $a, b \in \mathbb{R}$. By proving a new nonlinear interpolation inequality, we show that there exists a positive constant k_0 such that, for $k < k_0$, and for a class of potentials W, (F_ε^k) $\Gamma(L^1)$-converges to

$$F^k(u) := \mathbf{m}_k \,\#(S(u)), \quad u \in BV(I; \{a, b\}),$$

where $\mathbf{m}_k$ is a constant depending on W and k. Moreover, in the special case of the classical potential $W(s) = \frac{(s^2-1)^2}{2}$, we provide an upper bound on the values of k such that the minimizers of F_ε^k cannot develop oscillations on some fine scale and a lower bound on the values for which oscillations occur, the latter improving a previous estimate by Mizel, Peletier and Troy.

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1. Introduction

In this note we address some features of the limiting behavior of the minimizers of a class of second-order singular perturbation energies. The model we analyze was introduced in 1984 by Coleman and Mizel in the context of the theory of second-order materials and was then studied in [2] in collaboration with Marcus.

Coleman and Mizel proposed a model for nonlinear materials in which the free energy depends on both first and second order spatial derivatives of the mass density. In this way they expected to prove the occurrence of layered structures of the ground states (as observed in concentrated soap solutions and metallic alloys) without appealing to non-local energies (such as, for example, the Otha-Kawasaki functional [15]). Specifically, they introduced the free-energy functional $F_\varepsilon^k : L^1(I) \longrightarrow (-\infty, +\infty]$ given by

$$F_\varepsilon^k(u, I) = \begin{cases} \displaystyle\int_I \left(\frac{W(u)}{\varepsilon} - k\,\varepsilon\,(u')^2 + \varepsilon^3 (u'')^2 \right) dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{if } u \in L^1(I) \setminus W^{2,2}(I), \end{cases} \tag{1.1}$$

where $I \subset \mathbb{R}$ is a bounded open interval, u (the mass density) is the order parameter of the system, $\varepsilon, k > 0$, and $W\colon \mathbb{R} \to [0,+\infty)$ is a double-well potential with two potential wells of level zero at $a, b \in \mathbb{R}$.

As ε goes to zero, the functional (1.1) accounts for the energy stored by a one-dimensional physical system occupying the interval I. This model can be viewed as a scaled second-order Landau expansion of the classical Cahn-Hillard model for sharp phase transitions; *i.e.,*

$$\min\left\{\int_I W(u)\,dx:\; u\in L^1(I),\; \fint_I u\,dx = \lambda\, a + (1-\lambda)\, b\right\},\qquad 0<\lambda<1.$$

For the Cahn-Hillard model the lack of uniqueness is usually solved in the context of first-order gradient theory of phase transitions considering the simplest diffuse phase-transitions model; *i.e.,* the Van der Waals model. The latter is obtained by adding a singular gradient perturbation to the previous functional. After scaling, the new functional $\mathcal{F}_\varepsilon\colon L^1(I) \longrightarrow [0,+\infty]$ reads as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_I \left(\frac{W(u)}{\varepsilon} + \varepsilon(u')^2\right) dx & \text{if } u\in W^{1,2}(I),\\ +\infty & \text{if } u \in L^1(I)\setminus W^{1,2}(I).\end{cases}$$

If W grows at least linearly at infinity, Modica and Mortola [11, 12] proved that sequences (u_ε) with equi-bounded energy (*i.e.* such that $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$) cannot oscillate as, up to subsequences, they converge in $L^1(I)$ to a function $u \in BV(I;\{a,b\})$. Moreover, the $\Gamma(L^1)$-limit of $\mathcal{F}_\varepsilon$ is given by

$$\mathcal{F}(u) = \begin{cases} \mathbf{m}\#(S(u)) & \text{if } u \in BV(I;\{a,b\}),\\ +\infty & \text{otherwise in } L^1(I),\end{cases} \tag{1.2}$$

for a suitable constant $\mathbf{m}$ depending on the double-well potential W.

The above phenomenon characterizes first order phase transitions of every material having positive surface energy.

On the other hand, in nature there are materials that relieve energy whenever the measure of their surface is increased. These materials have a so-called negative surface energy. To give a mathematical description of this kind of materials within the framework of the gradient theory of phase transitions, Coleman and Mizel introduced the energy F^k_ε.

The requirement for an energy of the form of F^k_ε to be bounded from below forces the coefficient in front of the highest gradient squared to be nonnegative. On the other hand different phenomena can occur depending on the coefficient k in front of $\varepsilon\,(u')^2$. Specifically, for negative constants k, different authors showed that this model leads to the same asymptotic behavior of the first order perturbation, avoiding oscillations and converging to a sharp interface functional. The case $k<0$ was settled by Hilhorst, Peletier and Schätzle in [6], where the authors proved that the functionals F^k_ε $\Gamma(L^1)$-converge to a limit functional of type (1.2). The case $k=0$ was instead considered by Fonseca and Mantegazza. In [3] the authors established the same limit behavior thanks to a compactness result for sequences with equi-bounded energy obtained exploiting some *a priori* bounds given by the growth assumption on the double-well potential W and by a Gagliardo-Nirenberg interpolation inequality.

In this paper we investigate the case $k > 0$.

The presence of a negative contribution due to the term involving the first order derivative makes the problem quite unusual in the context of higher-order models of phase transitions.

In particular, since the three different terms in the energy are of the same order, their competition's outcomes strongly depend on the value of k.

Heuristically, large values of k make the phases highly unstable favoring oscillations between them and correspond to negative surface tensions.

This was rigorously proved by Mizel, Peletier and Troy in [10]. The authors considered the classical potential $W(s) = \frac{1}{2}(s^2-1)^2$ and showed that, for $k > 0.9481$, $\lim_{\varepsilon\to 0} \min F_\varepsilon^k = -\infty$ and that there exists a class of minimizers of F_ε^k which are non-constant periodic functions oscillating between the two potential wells. Finer properties of these minimizers have been studied also in [9].

It is worth mentioning here that analogous results have been obtained for the non-local perturbations of the Van der Waals model in one-dimensional space, as the already mentioned Otha–Kawasaki model (see, for example, the forerunner study of Müller [13] in the context of coherent solid-solid phase transitions). These energies, when viewed as functionals of a suitable primitive of the order parameter of the system, become second-order functionals with a potential constraint on the first derivative, and lead to similar results.

What was left open by the analysis carried out in [10] is the case of "small", positive constants k. We prove here that, under the assumptions that the potential $W(s)$ is quadratic in a neighborhood of the wells (to fix the ideas, suppose from now on $a = -1, b = 1$) and grows at least as s^2 at infinity (both hypothesis being necessary as discussed in Section 3), small values of k make the phases stable and correspond to positive surface tensions; *i.e.,* the asymptotic behavior of F_ε^k is again described by a sharp interface limit as in (1.2).

The main difficulty in the achievement of the above result lies in the proof of a compactness theorem analog to the one obtained in the case of the Modica–Mortola functional. Indeed, the negative term in the energy F_ε^k when $k > 0$ gives no *a priori* bounds on minimizing sequences. Here we solve this problem showing the existence of constants $k_0, \varepsilon_0 > 0$ such that a new nonlinear interpolation inequality holds (see Lemma 3.1 and Proposition 3.3):

$$k \int_I \varepsilon (u')^2 \, dx \leq \int_I \left(\frac{W(u)}{\varepsilon} + \varepsilon^3 (u'')^2 \right) dx, \tag{1.3}$$

for every $k < k_0$, $u \in W^{2,2}(I)$ and $\varepsilon \leq \varepsilon_0$. This inequality enables us to estimate from below our functionals with F_ε^0 (the one corresponding to $k = 0$) for which a compactness result has been proved in [3]. Therefore, for $k < k_0$ every sequence of functions with equi-bounded energy F_ε^k converges in $L^1(I)$ (up to subsequences) to a function $u \in BV(I; \{\pm 1\})$ (see Proposition 3.4).

On account of this result, we then complete the Γ-convergence analysis of the family of functionals F_ε^k by proving in Theorem 4.1 that, for every $k < k_0$, the functionals F_ε^k $\Gamma(L^1)$-converge to

$$F^k(u) := \begin{cases} \mathbf{m}_k \, \#(S(u)) & \text{if } u \in BV(I; \{\pm 1\}), \\ +\infty & \text{otherwise in } L^1(I), \end{cases} \tag{1.4}$$

where $\mathbf{m}_k > 0$ is given by the following optimal profile problem (whose solution's existence is part of the result),

$$\mathbf{m}_k := \min\left\{\int_{\mathbb{R}} (W(u) - k(u')^2 + (u'')^2)\,dx\colon u \in W^{2,2}_{\rm loc}(\mathbb{R}),\right.$$
$$\left.\lim_{x\to-\infty} u(x) = -1,\ \lim_{x\to+\infty} u(x) = 1\right\}.$$

In the last part of the paper we address the problem of estimating the constants k for which there are no oscillations in the asymptotic minimizers. In order to compare our results with those in [10], we focus here on the explicit potential $W(s) = \frac{(s^2-1)^2}{2}$ they considered. Since the estimate we could derive for k_0 are very rough, in Section 5 we investigate the following different problem,

$$k_1 := \inf_{L>0} \inf\left\{R^L_{-L}(u), u \in W^{2,2}(-L, L)\colon u'(\pm L) = 0, u' \neq 0\right\}, \tag{1.5}$$

where for every interval (α, β) and every $u \in W^{2,2}(\alpha, \beta)$, $R^\beta_\alpha(u)$ is the Rayleigh quotient defined as

$$R^\beta_\alpha(u) := \begin{cases} \dfrac{\displaystyle\int_\alpha^\beta (W(u) + (u'')^2)\,dx}{\displaystyle\int_\alpha^\beta (u')^2\,dx} & \text{if } \displaystyle\int_\alpha^\beta (u')^2\,dx > 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.6}$$

Problem (1.5) is clearly related to the computation of the optimal constant in the nonlinear interpolation inequality (1.3) and seems to be a challenging open problem.

Clearly, for $k \geq k_1$, minimizers of F^k_ε exhibit an oscillating behavior. On the other hand, we are able to show that for $k < \min\{k_1, \sqrt{2}/2\}$ there are no oscillations. Indeed, for these values of k, we prove a L^1 compactness result in $BV(I; \{\pm 1\})$ for sequences of functions equi-bounded in energy and having at least one zero of the first derivative (see Proposition 5.1 and notice that this condition is fulfilled by any sequence of functions which is supposed to oscillate). This compactness result, although analogous to the one provided for $k < k_0$, is actually more difficult, because in this last case the energy is not everywhere positive, but can be in principle negative near the boundary (see Lemma 5.4).

The benefit of this refined compactness result is that we can provide an upper bound and a lower bound on k_1, having the same order of magnitude. The lower bound we obtain for k_1 follows by carefully tracing the constants in the linear interpolation inequality and amounts to 1/8. The upper bound $k_1 < 0,9385$, instead, is an improvement of the estimate given in [10] ($k_1 \leq 0,9481$) and is obtained by testing profiles made by suitably combined linear and quadratic parts.

What is not yet comprised in the present analysis is a better understanding of the interpolation constants k_0 and k_1. We think, indeed, that for every $k < k_1$ the functionals F^k_ε do not develop microstructures and Γ-converge to a sharp interface functional of type (1.2) up to an additive constant depending on the presence of possible boundary layers' energies.

Finally, a comment on the n-dimensional case is in order. In Remark 3.5, we briefly discuss the main idea of the interpolation inequality for smooth domains in $\mathbb{R}^n$, which gives the compactness in any space dimension. After the submission of our paper, a preprint [1] dealing with the n-dimensional version of F_ε^k appeared. In [1] the authors generalize Theorem 4.1 to any space dimension by means of blow-up arguments, while the investigation on the bounds of the interpolation constant is not addressed. Furthermore, we point out that a satisfactory description of the dependence of the limit interface energy density $\mathbf{m}_n$ (using the notation of [1]) on the space dimension n is still missing. Specifically, it would be worth investigating if $\mathbf{m}_n = \mathbf{m}_1$, as it happens, for instance, for the classical Modica–Mortola functional [11, 12] and for F_ε^k when $k = 0$ as shown in [3].

2. Notation and preliminaries

In this section we set our notation and we recall some preliminary results we employ in the sequel.

With $I \subset \mathbb{R}$ we always denote an open bounded interval and with ε, k two positive constants. Moreover, we fix a class of double-well potentials with the following properties: $W : \mathbb{R} \to [0, +\infty)$ is continuous, $W^{-1}(\{0\}) = \{\pm 1\}$ (the location of the wells clearly can be fixed arbitrarily), and satisfies

(w) there exists $c_W > 0$ such that $W(s) \geq c_W\,(s \mp 1)^2$ for $\pm s \geq 0$.

Note that in particular the standard double-well potential $W(s) = \frac{(s^2-1)^2}{2}$ belongs to this class.

We consider the functionals F_ε^k defined in (1.1) and, whenever the domain of integration is clear from the context, we simply write $F_\varepsilon^k(u)$ in place of $F_\varepsilon^k(u, I)$. We denote by $E_\varepsilon = F_\varepsilon^0$ the functional introduced in [3]; that is

$$E_\varepsilon(u, I) := \begin{cases} \displaystyle\int_I \left(\frac{W(u)}{\varepsilon} + \varepsilon^3 (u'')^2 \right) dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{if } u \in L^1(I) \setminus W^{2,2}(I). \end{cases}$$

As we heavily use it in the sequel, we recall here the statement of one of the main results of [3] (see [3, Proposition 2.7]).

Proposition 2.1. *Let $(u_\varepsilon) \subset W^{2,2}(I)$ satisfy $\sup_\varepsilon E_\varepsilon(u_\varepsilon, I) < +\infty$. Then, there exist a subsequence (not relabeled) and a function $u \in BV(I, \{\pm 1\})$ such that $u_\varepsilon \to u$ in $L^1(I)$.*

We also recall two classical interpolation inequalities (see [7, Theorem 1.2 and (1.22) pag. 10] and [4, 14]).

Proposition 2.2. *For every $a, b \in \mathbb{R}$, $a < b$, and every function $u \in W^{2,2}(a, b)$, the following inequalities hold:*

(i) (optimal constant)

$$\|u'\|_{L^2(a,b)} \leq c\,\|u''\|_{L^2(a,b)} + k(c)\,\|u\|_{L^2(a,b)}, \tag{2.1}$$

for every $c > 0$, with $k(c) = \frac{1}{c} + \frac{12}{(b-a)^2}$;

(ii) there exists a constant $c > 0$ such that

$$\|u'\|_{L^{\frac{4}{3}}(a,b)} \leq c \left(\|u\|_{L^1(a,b)}^{\frac{1}{2}} \|u''\|_{L^2(a,b)}^{\frac{1}{2}} + \|u\|_{L^1(a,b)} \right). \tag{2.2}$$

Finally, we prove the following interpolation inequality with boundary terms.

Proposition 2.3 (Interpolation with boundary terms)**.** *For every $a,b\in\mathbb{R}$ with $a<b$, $u\in W^{2,2}(a,b)$ and $c>0$, it holds*

$$c\int_a^b (u')^2 \le c^3\int_a^b (u'')^2 + \int_a^b \frac{(u\pm 1)^2}{c} + (c\,u'(b)+u(b)\pm 1)^2 - (c\,u'(a)+u(a)\pm 1)^2. \tag{2.3}$$

Proof. We have the following identity

$$c^2(u')^2 + (c^2u'' + cu' + u\pm 1)^2 \\ = c^4(u'')^2 + (u\pm 1)^2 + 2c(cu'+u\pm 1)(cu''+u'). \tag{2.4}$$

Then, integrating both sides of (2.4) we find

$$\int_a^b c^2(u')^2\,dx + \int_a^b (c^2u''+cu'+u\pm 1)^2\,dx \\ = \int_a^b (c^4(u'')^2 + (u\pm 1)^2)\,dx + c\big((cu'(b)+u(b)\pm 1)^2 - (cu'(a)+u(a)\pm 1)^2\big).$$

Hence, dividing by $c>0$ we get the thesis. □

3. Compactness

In this section we prove one of the main result of this paper, namely the existence of a constant $k_0>0$ such that, for every $k<k_0$, the functional F_ε^k satisfy the same compactness property of Proposition 2.1. As an easy consequence, we then obtain the existence of the solution to the optimal profile problem for F_ε^k.

3.1. **Nonlinear interpolation and compactness.**

In this subsection we prove a nonlinear version of the standard L^2-interpolation inequality of type (i) Proposition 2.2.

Lemma 3.1 (Nonlinear interpolation)**.** *There exists a constant $k_0>0$ such that*

$$k_0\int_a^b (u')^2\,dx \le \frac{1}{(b-a)^2}\int_a^b W(u)\,dx + (b-a)^2\int_a^b (u'')^2\,dx, \tag{3.1}$$

for every $u\in W^{2,2}(a,b)$ and for every $a,b\in\mathbb{R}$ with $a<b$.

Proof. Up to translations and rescalings, it is enough to prove (3.1) for $(a,b)=(0,1)$. To this end, we set

$$m := \int_0^1 u'\,dx.$$

From the fundamental theorem of calculus, it follows that,

$$|u'-m| \le \int_0^1 |u''|\,dx, \tag{3.2}$$

and hence

$$\int_0^1 (u')^2\,dx \le 2\int_0^1 (u'')^2\,dx + 2\,m^2.$$

Therefore, to prove (3.1) it is enough to show the existence of a constant $c>0$ such that

$$m^2 \le c\int_0^1 \Big(W(u) + (u'')^2\Big)\,dx. \tag{3.3}$$

If $m^2 \leq 4\int_0^1 (u'')^2\,dx$, then (3.3) clearly follows. If this is not the case, applying Jensen's inequality, we have

$$|m| \geq 2\int_0^1 |u''|\,dx.$$

Then, from (3.2), we get

$$\frac{|m|}{2} \leq |u'|. \tag{3.4}$$

This implies that u is strictly monotone in $(0,1)$ and, therefore, u does not vanish in at least one of the two intervals $(0,1/2)$ and $(1/2,1)$. To fix the ideas, we assume that $u>0$ in $(0,1/2)$ (the case $u<0$ is analogous). Hence, by (2.1) applied for instance with $c=1$ (we set below $C := 2\cdot 7^4$) and hypothesis (w), we have

$$\int_0^{\frac{1}{2}} (u')^2\,dx \leq \int_0^{\frac{1}{2}} \left(C\,(u-1)^2 + 2\,(u'')^2\right)\,dx \leq \int_0^1 \left(\frac{C}{c_W}W(u) + 2\,(u'')^2\right)\,dx. \tag{3.5}$$

Since (3.4) implies $m^2 \leq 8\int_0^{\frac{1}{2}}(u')^2\,dx$, from (3.5) we get (3.3) and thus the thesis. $\square$

Remark 3.2. Dividing $\mathbb{R}$ into disjoint intervals of length 1 and applying (3.1) we may deduce

$$k_0\int_{\mathbb{R}} (u')^2\,dx \leq \int_{\mathbb{R}} (W(u) + (u'')^2)\,dx, \tag{3.6}$$

for every $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R})$ with $k_0>0$ as in Lemma 3.1.

Now we prove that Lemma 3.1 together with a simple decomposition argument yield a lower bound for F^k_ε in terms of the functional E_ε.

Proposition 3.3. *For every interval I and $\delta>0$, there exists $\varepsilon_0>0$ such that, for every $k>0$, $0<\varepsilon\leq\varepsilon_0$, and $u\in L^1(I)$,*

$$\left(1-\frac{k}{k_0}-\delta\right)E_\varepsilon(u,I) \leq F^k_\varepsilon(u,I). \tag{3.7}$$

Proof. A change of variable gives

$$F^k_\varepsilon(u,I) = \int_{I/\varepsilon} (W(v) - k(v')^2 + (v'')^2)\,dx,$$

where $I/\varepsilon = \{x\in\mathbb{R} : \varepsilon\, x \in I\}$ and $v(x) := u(\varepsilon x)$. Set $n_\varepsilon := \left[\frac{|I|}{\varepsilon}\right]$; we divide the interval I/ε into n_ε pairwise disjoint open intervals I^i_ε, $i=1,\dots,n_\varepsilon$, of length $\frac{|I|}{\varepsilon\, n_\varepsilon}$. Then, by applying (3.1) on each interval I^i_ε we get

$$\begin{aligned}
F^k_\varepsilon(u,I) &= \sum_{i=1}^{n_\varepsilon}\int_{I^i_\varepsilon} (W(v) - k(v')^2 + (v'')^2)\,dx \\
&\geq \left(1-\frac{k}{k_0}\frac{\varepsilon^2 n_\varepsilon^2}{|I|^2}\right)\int_{I/\varepsilon} W(v)\,dx + \left(1-\frac{k}{k_0}\frac{|I|^2}{\varepsilon^2 n_\varepsilon^2}\right)\int_{I/\varepsilon} (v'')^2\,dx.
\end{aligned}$$

Since

$$\lim_{\varepsilon\to 0}\frac{\varepsilon n_\varepsilon}{|I|} = 1,$$

we get the thesis just by unscaling. $\square$

The following compactness result is now an immediate consequence of Proposition 3.3 and Proposition 2.1.

Proposition 3.4 (Compactness)**.** *Let $k < k_0$ and let $(u_\varepsilon) \subset W^{2,2}(I)$ be a sequence satisfying $\sup_\varepsilon F_\varepsilon^k(u_\varepsilon) < +\infty$. Then there exist a subsequence (not relabeled) and a function $u \in BV(I; \{\pm 1\})$ such that $u_\varepsilon \to u$ in $L^1(I)$.*

Remark 3.5. Proposition 3.3 can be easily generalized to any space dimension n. Namely, an immediate consequence of it is the existence of a constant $k_n \geq k_0/n > 0$ such that, for every smooth bounded domain $\Omega \subset \mathbb{R}^n$, $u \in W^{2,2}(\Omega)$, $k < k_n$, and ε small

$$k \int_\Omega \varepsilon\, |\nabla u|^2 \, dx \leq \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon^3 \, |\nabla^2 u|^2 \right) dx. \tag{3.8}$$

Indeed, by a standard covering argument it is enough to discuss the case of a rectangle $\Omega = I_1 \times \cdots \times I_n$ and then the conclusion follows by an easy application of Fubini's Theorem. Let $\hat{I}_i = I_1 \times \cdots \times I_{i-1} \times I_{i+1} \times \cdots \times I_n$ and $k = \frac{k_0 - \delta}{n}$, we have

$$\begin{aligned}
\int_\Omega \frac{k_0 - \delta}{n} \, \varepsilon \, |\nabla u|^2 &= \sum_{i=1}^n \int_{\hat{I}_i} \int_{I_i} \frac{k_0 - \delta}{n} \, \varepsilon \, |\partial_i u|^2 \, dx_i d\hat{x}_i \\
&\overset{(3.7)}{\leq} \sum_{i=1}^n \int_{\hat{I}_i} \int_{I_i} \left(\frac{W(u)}{n\, \varepsilon} + \frac{\varepsilon^3}{n} \, |\partial_{ii} u|^2 \right) dx_i d\hat{x}_i \\
&\leq \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon^3 \, |\nabla^2 u|^2 \right) dx.
\end{aligned}$$

Here we briefly comment on the assumption (w) on the double-well potential W. We show with two explicit examples that the two following conditions

(i) $\liminf_{|s| \to +\infty} \frac{W(s)}{s^2} > 0$,
(ii) $\liminf_{s \to 0} \frac{W(\pm 1 + s)}{s^2} > 0$,

(which together are equivalent to (w)) are necessary to establish (3.7).

Indeed, let $l, \alpha > 0$ be two parameters to be fixed later and such that $(6\, l\, \varepsilon)^{-1} \in \mathbb{N}$. Consider the two families of periodic functions, of period $(6\, l\, \varepsilon)^{-1}$, (u_ε) and (v_ε) defined in $(0, 1)$ in the following way. On a half period, both u_ε and v_ε are defined by a line of slope α/ε, an arc of parabola, and another line of slope $-\alpha/\varepsilon$, as in the Figure 1; moreover, $u_\varepsilon(0) = 0$ and $v_\varepsilon(0) = 1$.

For the sake of simplicity, to shorten the present computation, assume that W is monotone on the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, +\infty)$ (note that this hypothesis is not necessary).

It is readily verified that:

(a) $|u''_\varepsilon| = |v''_\varepsilon| \leq \frac{2\,\alpha}{l\, \varepsilon^2}$ always and $|u'_\varepsilon| = |v'_\varepsilon| = \frac{\alpha}{\varepsilon}$ on a set of measure $\frac{2}{3}$,

(b) $|u_\varepsilon| \leq 2\, l\, \alpha$, so that, for $l\, \alpha$ large enough, we have $W(u_\varepsilon) \leq W(2\, l\, \alpha)$,

(c) and $|v_\varepsilon - 1| \leq 2\, l\, \alpha$, so that $W(v_\varepsilon) \leq W(1 + 2\, l\, \alpha)$.

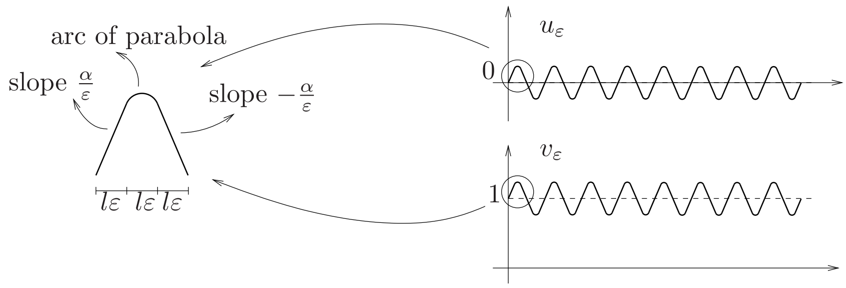

FIGURE 1. The functions u_ε and v_ε.

Now, if (3.7) holds true, from estimates (a), (b), (c) it follows that

$$k_0 \le \frac{\displaystyle\int_0^1 \Big(\varepsilon^3\,(u_\varepsilon'')^2 + \frac{W(u_\varepsilon)}{\varepsilon}\Big)\,dx}{\displaystyle\int_0^1 \varepsilon\,(u_\varepsilon')^2\,dx} \le \frac{\varepsilon^3\left(\dfrac{2\,\alpha}{l\,\varepsilon^2}\right)^2 + \dfrac{W(2\,l\,\alpha)}{\varepsilon}}{\varepsilon\left(\dfrac{\alpha}{\varepsilon}\right)^2\dfrac{2}{3}} = \frac{6}{l^2} + \frac{3}{2}\,\frac{W(2\,l\,\alpha)}{\alpha^2},$$

$$k_0 \le \frac{\displaystyle\int_0^1 \Big(\varepsilon^3\,(v_\varepsilon'')^2 + \frac{W(v_\varepsilon)}{\varepsilon}\Big)\,dx}{\displaystyle\int_0^1 \varepsilon\,(v_\varepsilon')^2\,dx} \le \frac{\varepsilon^3\left(\dfrac{2\,\alpha}{l\,\varepsilon^2}\right)^2 + \dfrac{W(1+2\,l\,\alpha)}{\varepsilon}}{\varepsilon\left(\dfrac{\alpha}{\varepsilon}\right)^2\dfrac{2}{3}}$$
$$= \frac{6}{l^2} + \frac{3}{2}\,\frac{W(1+2\,l\,\alpha)}{\alpha^2}.$$

Then, if (i) does not hold true, taking the limit as α goes to $+\infty$ and then as l goes to $+\infty$ gives contradiction. Similarly, if (ii) is not satisfied, taking the limit as α goes to 0 and then l tends to $+\infty$ yields a contradiction as well.

3.2. **Optimal profile problem.** As a consequence of Lemma 3.1, we prove here the existence of a solution to the optimal profile problem for F_ε^k, with $k < k_0$. Specifically, we consider the following set of functions

$$\mathcal{A} := \big\{f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R})\colon f(x) = 1 \text{ if } x > T,\ f(x) = -1 \text{ if } x < -T,\ \text{for some } T > 0\big\}$$

and we define

$$\mathbf{m}_k := \inf\left\{\int_{\mathbb{R}} (W(f) - k(f')^2 + (f'')^2)\,dx\colon f \in \mathcal{A}\right\}. \tag{3.9}$$

We have the following result.

Proposition 3.6 (Existence of an optimal profile)**.** *Let k_0 be as in Lemma 3.1. For every $k < k_0$ the constant $\mathbf{m}_k$ is positive and*

$$\mathbf{m}_k = \min\left\{\int_{\mathbb{R}} (W(f) - k(f')^2 + (f'')^2)\,dx\colon f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}),\right.$$
$$\left.\lim_{x\to-\infty} f(x) = -1,\ \lim_{x\to+\infty} f(x) = 1\right\}.$$

Before proving Proposition 3.6, we introduce the functions $G^k,\ H^k\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$G^k(w,z) := \inf\Big\{\int_0^1 \Big(W(g) - k(g')^2 + (g'')^2\Big)\,dx\colon g \in C^2([0,1]),$$
$$g(0) = w,\ g(1) = 1,\ g'(0) = z,\ g'(1) = 0\Big\}$$

and

$$H^k(w,z) := \inf\Big\{\int_0^1 \Big(W(h) - k(h')^2 + (h'')^2\Big)\,dx\colon h \in C^2([0,1]),$$
$$h(0) = -1,\ h(1) = w,\ h'(0) = 0,\ h'(1) = z\Big\}.$$

If $G := G^0$ and $H := H^0$ are the corresponding functions for $k = 0$ it is easy to check (see also [3, Section 2]) that

$$\lim_{(w,z)\to(1,0)} G(w,z) = \lim_{(w,z)\to(-1,0)} H(w,z) = 0.$$

Then, by the positivity of k and by virtue of (3.1) we have

$$\Big(1 - \frac{k}{k_0}\Big) G \le G^k \le G \quad\text{and}\quad \Big(1 - \frac{k}{k_0}\Big) H \le H^k \le H,$$

which lead immediately to

$$\lim_{(w,z)\to(1,0)} G^k(w,z) = \lim_{(w,z)\to(-1,0)} H^k(w,z) = 0 \quad \forall\, k < k_0. \tag{3.10}$$

Proof of Proposition 3.6. By virtue of the nonlinear interpolation inequality of Lemma 3.1, the proof of this proposition is an easy modification of that of [3, Lemma 2.5].

The positivity of $\mathbf{m}_k$ follows from Remark 3.2 and [3, Lemma 2.5], since

$$\mathbf{m}_k = \inf_{f\in\mathcal{A}} \int_{\mathbb{R}} (W(f) - k(f')^2 + (f'')^2)\,dx \ge \Big(1 - \frac{k}{k_0}\Big) \inf_{f\in\mathcal{A}} \int_{\mathbb{R}} (W(f) + (f'')^2)\,dx > 0.$$

Now we prove that $\mathbf{m}_k = \tilde{\mathbf{m}}_k$, where

$$\tilde{\mathbf{m}}_k := \inf\Big\{\int_{\mathbb{R}} (W(f) - k(f')^2 + (f'')^2)\,dx\colon f \in W^{2,2}_{\rm loc}(\mathbb{R}),$$
$$\lim_{x\to-\infty} f(x) = -1,\ \lim_{x\to+\infty} f(x) = 1\Big\}.$$

Clearly, $\mathbf{m}_k \ge \tilde{\mathbf{m}}_k$. For the converse inequality, fix $\sigma > 0$ and let f be an admissible function for $\tilde{\mathbf{m}}_k$ such that

$$\int_{\mathbb{R}} (W(f) - k(f')^2 + (f'')^2)\,dx \le \tilde{\mathbf{m}}_k + \sigma.$$

We show that it is possible to find two sequences (x_j) and (y_j) converging to $+\infty$ and $-\infty$ respectively, and such that

$$|f'(x_j)| + |f'(y_j)| + |f(x_j) - 1| + |f(y_j) + 1| \to 0,$$

as $j \to +\infty$. Indeed, in view of Remark 3.2 we have

$$(k_0 - k) \int_{\mathbb{R}} (f')^2\,dx \le \int_{\mathbb{R}} (W(f) - k(f')^2 + (f'')^2)\,dx \le \tilde{\mathbf{m}}_k + \sigma.$$

Thus, since $k < k_0$ we deduce that $f' \in L^2(\mathbb{R})$ and so there exist two sequences of points $x_j \to +\infty$ and $y_j \to -\infty$ such that

$$\lim_{j\to+\infty} f'(x_j) = \lim_{j\to-\infty} f'(y_j) = 0.$$

Let g and h be two admissible functions for $G^k(f(x_j), f'(x_j))$ and $H^k(f(y_i), f'(y_i))$, respectively, such that

$$\int_0^1 (W(g) - k(g')^2 + (g'')^2)\, dx \le G^k(f(x_j), f'(x_j)) + \sigma,$$

$$\int_0^1 (W(h) - k(h')^2 + (h'')^2)\, dx \le H^k(f(y_j), f'(y_j)) + \sigma,$$

and set

$$g_j(x) := g(x - x_j), \quad h_j(x) := h(x - y_j + 1).$$

We define

$$f_j(x) := \begin{cases} 1 & \text{if} \quad x \ge x_j + 1, \\ g_j(x) & \text{if} \quad x_j \le x \le x_j + 1, \\ f(x) & \text{if} \quad y_j \le x \le x_j, \\ h_j(x) & \text{if} \quad y_j - 1 \le x \le y_j, \\ -1 & \text{if} \quad x \le y_j - 1. \end{cases}$$

Clearly, f_j is a test function for $\mathbf{m}_k$ and for $k < k_0$ we have

$$\begin{aligned} \tilde{\mathbf{m}}_k + \sigma \quad &\ge \quad \int_{\mathbb{R}} (W(f) - k(f')^2 + (f'')^2)\, dx \ge \int_{y_j}^{x_j} (W(f) - k(f')^2 + (f'')^2)\, dx \\ &= \quad \int_{\mathbb{R}} (W(f_j) - k(f_j')^2 + (f_j'')^2)\, dx - \int_{x_j}^{x_j+1} (W(g_j) - k(g_j')^2 + (g_j'')^2)\, dx \\ &\quad\quad - \int_{y_j-1}^{y_j} (W(h_j) - k(h_j')^2 + (h_j'')^2)\, dx \\ &\ge \quad \mathbf{m}_k - G^k(f(x_j), f'(x_j)) - H^k(f(y_j), f'(y_j)) - 2\sigma. \end{aligned}$$

Hence we conclude letting $j \to +\infty$ and appealing to (3.10).

Finally, it remains to prove that $\tilde{\mathbf{m}}_k$ admits a minimizer. To this end, let $(f_n) \subset W^{2,2}_{\mathrm{loc}}(\mathbb{R})$ be a sequence which realizes $\tilde{\mathbf{m}}_k$. Then, by Remark 3.2 we have

$$\lim_{n\to+\infty} \Big(1 - \frac{k}{k_0}\Big) \int_{\mathbb{R}} (W(f_n) + (f_n'')^2)\, dx \le \lim_{n\to+\infty} \int_{\mathbb{R}} (W(f_n) - k(f_n')^2 + (f_n'')^2)\, dx = \tilde{\mathbf{m}}_k.$$

Hence, again by interpolation and appealing to the Sobolev embedding theorem, we deduce that (up to subsequence) the sequence of C^1 functions (f_n) converges in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ to a C^1 function f with

$$\int_{\mathbb{R}} (W(f) + (f'')^2)\, dx < +\infty.$$

By (3.6), it follows that

$$0 \le \int_{\mathbb{R}} (W(f) - k(f')^2 + (f'')^2)\, dx < +\infty. \tag{3.11}$$

For every $T>0$, by the $W^{1,\infty}_{\rm loc}(\mathbb{R})$-convergence of (f_n), Fatou Lemma and the lower semicontinuity of the L^2-norm of the second derivative, we have

$$\begin{aligned}\int_{-T}^{T}(W(f)-k(f')^2+(f'')^2)\,dx &\le \liminf_{n\to+\infty}\int_{-T}^{T}(W(f_n)-k(f_n')^2+(f_n'')^2)\,dx\\ &\le \liminf_{n\to+\infty}\int_{\mathbb{R}}(W(f_n)-k(f_n')^2+(f_n'')^2)\,dx, \quad (3.12)\end{aligned}$$

where the last inequality in (3.12) is a consequence of (3.6) written for the two half lines $(-\infty,T)$ and $(T,+\infty)$. Then, taking into account (3.11) and passing to the sup on $T>0$ we get

$$\int_{\mathbb{R}}(W(f)-k(f')^2+(f'')^2)\,dx \le \liminf_{n\to+\infty}\int_{\mathbb{R}}(W(f_n)-k(f_n')^2+(f_n'')^2)\,dx=\tilde{\mathbf{m}}_k. \quad (3.13)$$

Thus, it remains only to show that the limit function f is admissible. Since this is a direct consequence of the third step of the proof of [3, Lemma 2.5], we leave some minor details to the reader and conclude the proof. □

4. Γ-convergence

On account of the compactness result Proposition 3.4, in this section we compute the Γ-limit of the functionals F^k_ε when $k<k_0$.

Theorem 4.1. *For every $k<k_0$, the sequence (F^k_ε) $\Gamma(L^1)$-converges to the functional $F^k\colon L^1(I)\longrightarrow[0,+\infty]$ given by*

$$F^k(u):=\begin{cases}\mathbf{m}_k\,\#(S(u)) & \text{if } u\in BV(I;\{\pm1\}),\\ +\infty & \text{if } u\in L^1(I)\setminus BV(I;\{\pm1\}),\end{cases} \quad (4.1)$$

where $\#(S(u))$ is the number of jumps of u in I and $\mathbf{m}_k$ is as in (3.9).

Proof. We divide the proof into two parts, proving the Γ-liminf and the Γ-limsup inequality, respectively.

Part I: Γ-liminf. Let $u\in L^1(I)$ and $(u_\varepsilon)\subset L^1$ such that $u_\varepsilon\to u$. We want to show that

$$\liminf_{\varepsilon\to0}F^k_\varepsilon(u_\varepsilon)\ge\mathbf{m}_k\#(S(u)). \quad (4.2)$$

Clearly, it is enough to consider the case $\lim_{\varepsilon\to0}F^k_\varepsilon(u_\varepsilon)=\liminf_{\varepsilon\to0}F^k_\varepsilon(u_\varepsilon)<+\infty$. Therefore, by virtue of Proposition 3.4, $u\in BV(I;\{\pm1\})$. Moreover, from (3.7) we immediately deduce $\|u_\varepsilon''\|_{L^2(I)}\le c\varepsilon^{-\frac{3}{2}}$, so that (2.2) gives

$$\varepsilon\,u_\varepsilon'\to0 \quad\text{in}\quad L^1(I). \quad (4.3)$$

Let $\#(S(u)):=N$, $S(u):=\{s_1,\dots,s_N\}$ with $s_1<s_2<\dots<s_N$, and set $\delta_0:=\min\{s_{i+1}-s_i\colon i=1,\dots N-1\}$. Fix $0<\delta<\delta_0/2$. Then (up to subsequences)

$$u_\varepsilon\to u,\quad \varepsilon u_\varepsilon'\to0\quad\text{a.e. in}\quad B(s_i,\delta),$$

for every $i=1,\dots,N$. Hence if we let $\sigma>0$, for every $i=1,\dots,N$ we may find two points $x^+_{\varepsilon,i},x^-_{\varepsilon,i}\in B(s_i,\delta)$ such that, for sufficiently small $\varepsilon>0$,

$$|u_\varepsilon(x^+_{\varepsilon,i})-1|<\sigma,\ |u_\varepsilon(x^-_{\varepsilon,i})+1|<\sigma,\ |\varepsilon u_\varepsilon'(x^+_{\varepsilon,i})|<\sigma,\ |\varepsilon u_\varepsilon'(x^+_{\varepsilon,i})|<\sigma. \quad (4.4)$$

To fix the ideas, without loss of generality, suppose $x^-_{\varepsilon,i}<x^+_{\varepsilon,i}$ and set

$$\hat g_{\varepsilon,i}(x):=g_{\varepsilon,i}\Big(x-\frac{x^+_{\varepsilon,i}}{\varepsilon}\Big)\quad\text{and}\quad \hat h_{\varepsilon,i}(x):=h_{\varepsilon,i}\Big(x-\frac{x^-_{\varepsilon,i}+1}{\varepsilon}\Big),$$

with $g_{\varepsilon,i}$ and $h_{\varepsilon,i}$ admissible for $G^k(u_\varepsilon(x^+_{\varepsilon,i}),\varepsilon u'_\varepsilon(x^+_{\varepsilon,i}))$ and $H^k(u_\varepsilon(x^-_{\varepsilon,i}),\varepsilon u'_\varepsilon(x^-_{\varepsilon,i}))$, respectively, and satisfying

$$\int_0^1 \Big(W(g_{\varepsilon,i}) - k(g'_{\varepsilon,i})^2 + (g''_{\varepsilon,i})^2\Big)\,dx \le G^k(u_\varepsilon(x^+_{\varepsilon,i}),\varepsilon u'_\varepsilon(x^+_{\varepsilon,i})) + \frac{\varepsilon}{2}$$

and

$$\int_0^1 \Big(W(h_{\varepsilon,i}) - k(h'_{\varepsilon,i})^2 + (h''_{\varepsilon,i})^2\Big)\,dx \le H^k(u_\varepsilon(x^-_{\varepsilon,i}),\varepsilon u'_\varepsilon(x^-_{\varepsilon,i})) + \frac{\varepsilon}{2}.$$

Now we suitably modify the sequence (u_ε) "far" from each jump point s_i. To this end, for every $i = 1,\dots,N$ we define on $\mathbb{R}$ the functions $v_{\varepsilon,i}$ as

$$v_{\varepsilon,i}(x) := \begin{cases} 1 & \text{if } x \ge \frac{x^+_{\varepsilon,i}}{\varepsilon} + 1 \\ \hat{g}_{\varepsilon,i}(x) & \text{if } \frac{x^+_{\varepsilon,i}}{\varepsilon} \le x \le \frac{x^+_{\varepsilon,i}}{\varepsilon} + 1 \\ u_\varepsilon(\varepsilon x) & \text{if } \frac{x^-_{\varepsilon,i}}{\varepsilon} \le x \le \frac{x^+_{\varepsilon,i}}{\varepsilon} \\ \hat{h}_{\varepsilon,i}(x) & \text{if } \frac{x^-_{\varepsilon,i}}{\varepsilon} - 1 \le x \le \frac{x^-_{\varepsilon,i}}{\varepsilon} \\ -1 & \text{if } x \le \frac{x^+_{\varepsilon,i}}{\varepsilon} - 1. \end{cases}$$

Since each $v_{\varepsilon,i}$ is a test function for $\mathbf{m}_k$, we have

$$\begin{aligned}\mathbf{m}_k &\le \int_{\mathbb{R}} (W(v_{\varepsilon,i}) - k(v'_{\varepsilon,i})^2 + (v''_{\varepsilon,i})^2)\,dx = \int_{\frac{x^-_{\varepsilon,i}}{\varepsilon}-1}^{\frac{x^+_{\varepsilon,i}}{\varepsilon}+1} (W(v_{\varepsilon,i}) - k(v'_{\varepsilon,i})^2 + (v''_{\varepsilon,i})^2)\,dx \\ &\le \int_{x^-_{\varepsilon,i}}^{x^+_{\varepsilon,i}} \Big(\frac{W(u_\varepsilon)}{\varepsilon} - k\varepsilon(u'_\varepsilon)^2 + \varepsilon^3(u''_\varepsilon)^2\Big)\,dx + \\ &\quad + G^k(u_\varepsilon(x^+_{\varepsilon,i}),\varepsilon u'_\varepsilon(x^+_{\varepsilon,i})) + H^k(u_\varepsilon(x^-_{\varepsilon,i}),\varepsilon u'_\varepsilon(x^-_{\varepsilon,i})) + \varepsilon.\end{aligned}$$

Then, as the intervals $(x^-_{\varepsilon,i}, x^+_{\varepsilon,i})$ are pairwise disjoint for $i = 1,\dots,N$, in view of the non-negative character of F^k_ε for $k < k_0$, we get

$$\begin{aligned}\lim_{\varepsilon\to0} F^k_\varepsilon(u_\varepsilon) &\ge \liminf_{\varepsilon\to0} \sum_{i=1}^N \int_{x^-_{\varepsilon,i}}^{x^+_{\varepsilon,i}} \Big(\frac{W(u_\varepsilon)}{\varepsilon} - k\varepsilon(u'_\varepsilon)^2 + \varepsilon^3(u''_\varepsilon)^2\Big)\,dx \\ &\ge N\,\mathbf{m}_k - \limsup_{\varepsilon\to0} \sum_{i=1}^N \Big(G^k(u_\varepsilon(x^+_{\varepsilon,i}),\varepsilon u'_\varepsilon(x^+_{\varepsilon,i})) + H^k(u_\varepsilon(x^-_{\varepsilon,i}),\varepsilon u'_\varepsilon(x^-_{\varepsilon,i}))\Big).\end{aligned}$$

Finally, letting $\sigma \to 0^+$, we conclude by (3.10).

Part II: Γ-limsup. Let $u \in BV(I;\{\pm1\})$ with $S(u)$ as in Part I, and set $s_0 := \alpha$, $s_{N+1} := \beta$. For $i = 1,\dots,N$ define $I_i := [\frac{s_{i-1}+s_i}{2}, \frac{s_i+s_{i+1}}{2}]$ and $\delta_0 := \min_i\{s_{i+1}-s_i\}$.

Fix $0 < \delta < \delta_0$ and $f \in \mathcal{A}$ such that $f(x) = 1$ if $x > T$, $f(x) = -1$ if $x < -T$, for some $T > 0$, and

$$\int_{\mathbb{R}} (W(f) - k(f')^2 + (f'')^2)\,dx \le \mathbf{m}_k + \frac{\delta}{N}.$$

Starting from this f we construct a recovery sequence (u_ε) for our Γ-limit.

There exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we have $\frac{\delta}{2\varepsilon} > T$. For $\varepsilon < \varepsilon_0$, we define

$$u_\varepsilon := \begin{cases} f\Big(\dfrac{x-s_i}{\varepsilon}\Big) & \text{if } x \in I_i \text{ and } [u](s_i) > 0, \\ f\Big(-\dfrac{x-s_i}{\varepsilon}\Big) & \text{if } x \in I_i \text{ and } [u](s_i) < 0, \\ u(x) & \text{otherwise,} \end{cases}$$

where $[u](s_i) := u(s_i) - u(s_{i-1})$, for $i = 2, \dots, N$.

It can be easily shown that $(u_\varepsilon) \subset W^{2,2}(I)$ and $u_\varepsilon \to u$ in $L^1(I)$. Moreover,

$$\begin{aligned} \lim_{\varepsilon\to 0} F^k_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon\to 0} \sum_{i=1}^N \int_{I_i} \Big(\frac{W(u_\varepsilon)}{\varepsilon} - k\varepsilon(u'_\varepsilon)^2 + \varepsilon^3(u''_\varepsilon)^2\Big)\,dx \\ &= \lim_{\varepsilon\to 0} \Big\{ \sum_{i=1,\dots,N\colon [u](s_i)>0} \int_{I_i/\varepsilon} (W(f(x)) - k(f'(x))^2 + (f''(x))^2)\,dx \\ &\quad + \sum_{i=1,\dots,N\colon [u](s_i)<0} \int_{I_i/\varepsilon} (W(f(-x)) - k(f'(-x))^2 + (f''(-x))^2)\,dx \Big\} \\ &\le \mathbf{m}_k N + \delta = \mathbf{m}_k \# S(u) + \delta, \end{aligned}$$

hence we conclude by the arbitrariness of $\delta > 0$. □

5. Phase transitions vs. oscillations

Throughout the last two sections we fix $W(s) = \frac{(s^2-1)^2}{2}$.

In the spirit of Mizel, Peletier and Troy [10], in this section we provide a compactness result, alternative to that of Proposition 3.4, which asserts the existence of a range of values of k such that sequences with equi-bounded energy F^k_ε, whose derivative vanishes at least in one point of I, do not develop oscillations. The reason for this new compactness result, as explained in the Introduction, is to give reasonable bounds on these values of k.

The key parameter for our analysis is the following

$$k_1 := \inf_{L>0} \inf_{u\in\mathcal{X}^L_0} R^L_0(u), \tag{5.1}$$

where

$$\mathcal{X}^L_0 := \Big\{u \in W^{2,2}(0,L)\colon \lim_{x\to 0} u'(x) = \lim_{x\to L} u'(x) = 0, u' > 0\Big\},$$

and for every interval (a,b) and every $u \in W^{2,2}(a,b)$, $R^b_a(u)$ is the Rayleigh quotient defined as

$$R^b_a(u) := \begin{cases} \dfrac{\displaystyle\int_a^b (W(u) + (u'')^2)\,dx}{\displaystyle\int_a^b (u')^2\,dx} & \text{if } \displaystyle\int_a^b (u')^2\,dx > 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2}$$

We note that for $k > k_1$ there are functions for which the functionals F^k_ε are non-positive. Indeed, given $k > k_1$, there exist $L > 0$ and $u \in \mathcal{X}^L_0$ such that

$$R^L_0(u) < k. \tag{5.3}$$

Let

$$v(x) := \begin{cases} u(x) & \text{in } (0, L), \\ u(2\,L - x) & \text{in } (L, 2L). \end{cases}$$

We denote by w the $(2L)$-periodic extension of v to $\mathbb{R}$. Note that by (5.3), for every $m \in \mathbb{Z}$,

$$F_\varepsilon^k\Big(w\Big(\frac{x}{\varepsilon}\Big), (m\,\varepsilon\, L, (m+1)\,\varepsilon\, L)\Big) < 0.$$

Then, it follows that for any interval (a, b)

$$\lim_{\varepsilon\to 0} F_\varepsilon^k\Big(w\Big(\frac{x}{\varepsilon}\Big), (a, b)\Big) = \lim_{\varepsilon\to 0} \frac{b-a}{\varepsilon\, L} F_\varepsilon^k\left(w\left(\frac{x}{\varepsilon}\right), (0, \varepsilon\, L)\right) = -\infty.$$

That is, there are minimizers of F_ε^k developing an oscillating structure, finer and finer as ε approaches 0. A thorough study of oscillating minimizers has been carried out in [10].

As a consequence, we have that $k_1 \geq k_0 > 0$, where k_0 is as in Lemma 3.1. Indeed, if this is not the case, we can take k with $k_1 < k < k_0$. Then, reasoning as above, since $k > k_1$, there exists $w \in W^{2,2}_{\text{loc}}(\mathbb{R})$ satisfying

$$\lim_{\varepsilon\to 0} F_\varepsilon^k\Big(w\Big(\frac{x}{\varepsilon}\Big), I\Big) = -\infty, \qquad \text{for every interval } I;$$

while, by Proposition 3.3, since $k < k_0$, we also have

$$\liminf_{\varepsilon\to 0} F_\varepsilon^k\Big(w\Big(\frac{x}{\varepsilon}\Big), I\Big) \geq 0,$$

and thus a contradiction.

On the other hand, if we prescribe suitable boundary conditions (as, for instance, periodic or homogeneous Neumann boundary conditions), in view of the analysis performed in the previous sections, we can derive analogous Γ-convergence results also for $k < k_1$. To see this, consider a bounded interval I and a function $u \in W^{2,2}(I)$. As $W^{2,2}(I) \subset C^{1,\frac{1}{2}}(I)$, we can write

$$I = \{u' = 0\} \cup \bigcup_{i\in\mathbb{N}} (a_i, b_i),$$

where (a_i, b_i) are the connected components of $\{u' > 0\} \cup \{u' < 0\}$. Consider any of these components (a_i, b_i): thanks to the boundary conditions, we have

$$u'(a_i) = u'(b_i) = 0.$$

Assume without loss of generality that $u' > 0$ in (a_i, b_i) (the case $u' < 0$ is analogous). Since $k < k_1$, it follows that

$$k\,\varepsilon \int_{a_i}^{b_i} (u')^2\, dx \leq \int_{a_i}^{b_i} \Big(\frac{W(u)}{\varepsilon} + \varepsilon^3 (u'')^2\Big)\, dx.$$

Therefore, summing over i, we have

$$\begin{aligned} k\,\varepsilon \int_I (u')^2\, dx &= \sum_{i\in\mathbb{N}} k\,\varepsilon \int_{a_i}^{b_i} (u')^2\, dx \\ &\leq \sum_{i\in\mathbb{N}} \int_{a_i}^{b_i} \Big(\frac{W(u)}{\varepsilon} + \varepsilon^3 (u'')^2\Big)\, dx \\ &\leq \int_I \Big(\frac{W(u)}{\varepsilon} + \varepsilon^3 (u'')^2\Big)\, dx. \end{aligned}$$

This implies that

$$F^k_\varepsilon(u,I) \geq \Big(1 - \frac{k}{k_1}\Big) E_\varepsilon(u,I). \tag{5.4}$$

Hence, reasoning as in Theorem 4.1, we can rule out the development of oscillations and prove an analogous Γ-convergence result.

On the contrary, if we do not impose any boundary conditions on u, the estimate of the energy of u in a neighborhood of the extrema of the interval I requires a further investigation.

Such investigation is the main issue of this section. The principal result asserts that for $k < \min\{k_1, \sqrt{2}/2\}$, even without prescribing boundary conditions, minimizers of F^k_ε cannot develop an oscillatory structure.

Proposition 5.1. *Let $k < \min\{k_1, \frac{\sqrt{2}}{2}\}$; let $(u_\varepsilon) \subset W^{2,2}(I)$ be a sequence such that $u'_\varepsilon = 0$ at least in one point of I and satisfying $\liminf_{\varepsilon\to 0} F^k_\varepsilon(u_\varepsilon) < +\infty$, then there exist a subsequence (not relabeled) and a function $u \in BV(I;\{\pm 1\})$ such that $u_\varepsilon \to u$ in $L^1(0,1)$.*

The proof of Proposition 5.1 is a straightforward consequence of Proposition 5.3 below and of Proposition 2.1.

Remark 5.2. Unfortunately, at this stage it is still unclear if in Proposition 5.1 taking the minimum between k_1 and $\sqrt{2}/2$ is really necessary or it is a technical hypothesis.

Proposition 5.3. *For every $k < \min\{k_1, \frac{\sqrt{2}}{2}\}$ there exist two constants $C_k, C > 0$ such that*

$$F^k_\varepsilon(u) \geq C_k E_\varepsilon(u) - C, \tag{5.5}$$

for every $\varepsilon > 0$ and for every $u \in W^{2,2}(I)$ such that u' vanishes at least in one point of I.

The following lemma is the main ingredient in the proof of Proposition 5.3.

Lemma 5.4. *Let $u \in W^{2,2}(I)$ and suppose that u' vanishes at least in one point of $I := (\alpha,\beta)$. Let $b \in I$ be the smallest point such that $u'(b) = 0$. Then, there exists $s > 0$ such that, for every $\eta \in (0,1)$ we have*

$$F^k_\varepsilon(u,(\alpha,b)) \geq \Big(1 - \frac{2}{\sqrt{2}}(1+\eta)k\Big) E_\varepsilon(u,(\alpha,b)) - \frac{1}{\eta^s}. \tag{5.6}$$

Proof. By symmetry, we assume without loss of generality, that $u' > 0$ in (α,b).

We start proving a preliminary inequality. We distinguish two cases:

(1) $u > 0$;
(2) there exists $a \in (\alpha,b)$ such that $u(a) = 0$.

In case (1), we employ the interpolation inequality (2.3) on the interval (α,b), with $c^2 = \sqrt{2}\varepsilon^2$, obtaining

$$\frac{\varepsilon}{\sqrt{2}}\int_\alpha^b (u')^2\,dx \leq \varepsilon^3\int_\alpha^b (u'')^2 + \int_\alpha^b \frac{(u-1)^2}{2\varepsilon} + \frac{\sqrt[4]{2}}{2}(u(b)-1)^2. \tag{5.7}$$

In case (2), let $\alpha < a < b < \beta$ and

$$u > 0 \ \text{ in } \ (a,b), \quad u' > 0 \ \text{ in } \ (\alpha,b), \quad u(a) = u'(b) = 0.$$

(see also Figure 2). We prove that an analogous of (5.7) holds true.

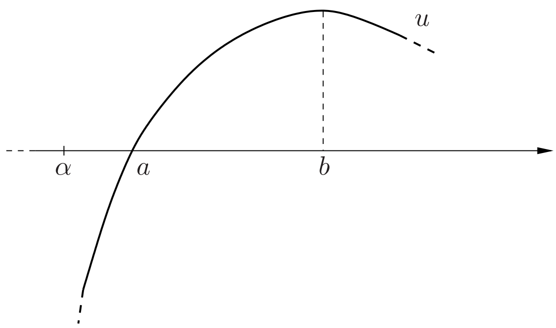

FIGURE 2. The function u in a neighborhood of α.

Application of the interpolation inequality (2.3) on the interval (α, a), with $c^2 = \sqrt{2}\varepsilon^2$, gives

$$\frac{\varepsilon}{\sqrt{2}} \int_\alpha^a (u')^2 \, dx \leq \varepsilon^3 \int_\alpha^a (u'')^2 + \int_\alpha^a \frac{(u+1)^2}{2\varepsilon} + \frac{\sqrt[4]{2}}{2} (\sqrt[4]{2}\varepsilon u'(a) + 1)^2, \tag{5.8}$$

while the same computation on (a, b) yields

$$\frac{\varepsilon}{\sqrt{2}} \int_a^b (u')^2 \, dx \leq \varepsilon^3 \int_a^b (u'')^2 + \int_a^b \frac{(u-1)^2}{2\varepsilon} + \frac{\sqrt[4]{2}}{2} (u(b)-1)^2 - \frac{\sqrt[4]{2}}{2} (\sqrt[4]{2}\varepsilon u'(a) - 1)^2. \tag{5.9}$$

From (5.9) we deduce

$$\frac{\sqrt[4]{2}}{2} (\sqrt[4]{2}\varepsilon u'(a) - 1)^2 \leq \varepsilon^3 \int_a^b (u'')^2 - \frac{\varepsilon}{\sqrt{2}} \int_a^b (u')^2 \, dx + \int_a^b \frac{(u-1)^2}{2\varepsilon} + \frac{\sqrt[4]{2}}{2} (u(b)-1)^2. \tag{5.10}$$

Since for every $\delta \in (0, 1)$ we have $(A+B)^2 \leq (1+\delta)A^2 + (1+\frac{1}{\delta})B^2$, we may write

$$(\sqrt[4]{2}\varepsilon u'(a) + 1)^2 \leq (1+\delta)(\sqrt[4]{2}\varepsilon u'(a) - 1)^2 + 4\Big(1 + \frac{1}{\delta}\Big). \tag{5.11}$$

Then, gathering (5.8) and (5.11), we find

$$\begin{aligned} \frac{\varepsilon}{\sqrt{2}} \int_\alpha^a (u')^2 \, dx \leq \varepsilon^3 \int_\alpha^a (u'')^2 + \int_\alpha^a \frac{(u+1)^2}{2\varepsilon} + \\ + \frac{\sqrt[4]{2}}{2} (1+\delta)(\sqrt{2}\varepsilon u'(a) - 1)^2 + 2\sqrt[4]{2}\Big(1 + \frac{1}{\delta}\Big). \end{aligned} \tag{5.12}$$

By estimating in (5.12) the quantity $(\sqrt[4]{2}\varepsilon u'(a) - 1)^2$ with (5.10), we get

$$\begin{aligned} \frac{\varepsilon}{\sqrt{2}} \int_\alpha^a (u')^2 \, dx \leq (1+\delta) \int_\alpha^b \Big(\frac{(u-1)^2}{2\varepsilon} + \varepsilon^3 (u'')^2\Big) \, dx + \\ - \frac{\varepsilon}{2} \int_a^b (u')^2 \, dx + C\,(u(b)-1)^2 + \frac{C}{\delta}. \end{aligned}$$

Thus finally

$$\frac{\varepsilon}{\sqrt{2}} \int_\alpha^b (u')^2 \, dx \leq (1+\delta) \int_\alpha^b \Big(\frac{(u-1)^2}{2\varepsilon} + \varepsilon^3 (u'')^2\Big) \, dx + C\,u(b)^2 + \frac{C}{\delta}. \tag{5.13}$$

Then, in view of (5.7) and (5.13), to get the thesis for $\eta = c\delta$ (for a suitable constant $c > 0$) it suffices to show that

$$u^2(b) \le \delta \int_\alpha^b \Big(\frac{W(u)}{\varepsilon} + \varepsilon^3 (u'')^2\Big)\,dx + \frac{1}{\delta} \tag{5.14}$$

for every $\delta \in (0,1)$.

We prove (5.14). Consider $\nu \in (0,1)$ to be fixed later. If $b-a \le \nu\varepsilon$, by exploiting $u'(b) = 0$ and the fundamental theorem of calculus, we have

$$\begin{aligned} u^2(b) = (u(b) - u(a))^2 &\le (b-a)\int_a^b (u')^2\,dx \le (b-a)^3 \int_\alpha^b (u'')^2\,dx \\ &\le \nu^3\varepsilon^3 \int_\alpha^b (u'')^2\,dx, \end{aligned}$$

from which (5.14) follows with $\delta = \nu^3$.

So now suppose $b - a > \nu\varepsilon$. Again we distinguish two cases. If

$$(u(b) - u(b-\nu\varepsilon))^2 > \frac{u^2(b)}{4}, \tag{5.15}$$

then, arguing as above we find

$$(u(b) - u(b-\nu\varepsilon))^2 \le \nu^3\varepsilon^3 \int_\alpha^b (u'')^2\,dx,$$

thus (5.15) directly yields (5.14) again with $\delta = \nu^3$.

If (5.15) does not hold true, then, from $u^2(b)/2 \le (u(b)-u(b-\nu\varepsilon))^2 + u^2(b-\nu\varepsilon)$, to get (5.14) it is enough to estimate $u^2(b-\nu\varepsilon)$.

By the Young Inequality $\nu^2 A^2 + \frac{B^2}{\nu^2} \ge 2AB$ we have

$$\nu^2 \int_\alpha^b \frac{W(u)}{\varepsilon}\,dx + \frac{1}{\nu^2} \ge 2\Big(\int_\alpha^b \frac{W(u)}{\varepsilon}\,dx\Big)^{1/2} \ge 2\Big(\int_{b-\nu\varepsilon}^b \frac{(u^2-1)^2}{2\varepsilon}\,dx\Big)^{1/2}.$$

On the other hand, using the Jensen Inequality we find

$$\int_{b-\nu\varepsilon}^b (u^2-1)^2\,dx \ge \frac{1}{\nu\varepsilon}\Big(\int_{b-\nu\varepsilon}^b (u^2-1)\,dx\Big)^2.$$

Therefore

$$\nu^2 \int_\alpha^b \frac{W(u)}{\varepsilon}\,dx + \frac{1}{\nu^2} \ge \frac{\sqrt{2}}{\nu^{1/2}} \int_{b-\nu\varepsilon}^b \frac{u^2-1}{\varepsilon}\,dx \ge \sqrt{2}\nu^{1/2}(u^2(b-\nu\varepsilon) - 1),$$

where in the last inequality we also used the fact that $b - \nu\varepsilon > a$, and $u, u' > 0$ in (a,b). Eventually we have

$$u^2(b-\nu\varepsilon) \le \frac{\nu^{3/2}}{\sqrt{2}} \int_\alpha^b \frac{W(u)}{\varepsilon}\,dx + \frac{1}{\nu^{5/2}} + 1.$$

Taking $\delta = c\,\nu^{3/2}$ for a suitable constant $c > 0$, (5.14) follows and thus the thesis. □

Proof of Proposition 5.3. The proof is straightforward combining Lemma 5.4 and (5.4). Indeed, let $I = (\alpha, \beta)$. Since $W^{2,2}(I) \subset C^{1,\frac{1}{2}}(I)$, we write

$$I = \{u' = 0\} \cup \bigcup_{i \in \mathbb{N}} (a_i, b_i),$$

where (a_i, b_i) are the connected components of $\{u' > 0\} \cup \{u' < 0\}$. Note that by hypothesis $\{u' = 0\} \neq \emptyset$. If we consider an interval (a_i, b_i) three cases can occur:

(1) $a_i = \alpha$;
(2) $b_i = \beta$;
(3) $\alpha < a_i < b_i < \beta$, and hence $u'(a_i) = u'(b_i) = 0$.

Notice that both cases (1) and (2) can occur at most for one value of i.

In cases (1) and (2), we use (5.6) (and its analogous involving a neighborhood of β) and the hypothesis $k < \frac{\sqrt{2}}{2}$ to deduce that there exists a positive constant $C_k > 0$ such that

$$F_\varepsilon^k(u, (a_i, b_i)) \geq \Big(1 - \frac{2}{\sqrt{2}}(1+\eta)k\Big) E_\varepsilon(u, (a_i, b_i)) - \frac{1}{\eta^s} \geq C_k\, E_\varepsilon(u, (a_i, b_i)) - C.$$

In case (3), by definition of k_1, since $u'(a_i) = u'(b_i) = 0$, we immediately infer

$$F_\varepsilon^k(u, (a_i, b_i)) \geq \Big(1 - \frac{k}{k_1}\Big)\, E_\varepsilon(u, (a_i, b_i)).$$

Summing over i and combining the two previous estimates, we deduce (5.3). □

6. Estimates on the interpolation constant k_1

In order to compare our results with those by Mizel, Peletier and Troy [10], in this section we provide two estimates, one from below and one from above, on the interpolation constant k_1, the one from above improving the bound in [10].

To establish an estimate from below on k_1, the idea is to use the interpolation inequality (i) of Proposition 2.2, which gives a good bound on k_1 on "large" intervals, and to combine it with an inequality of Jensen type which is good on "small" intervals.

Let $u \in \mathcal{X}_0^L$; since $u'(L) = 0$ we get

$$\int_0^L (u')^2\, dx \leq \frac{L^2}{2} \int_0^L (u'')^2\, dx. \tag{6.1}$$

Indeed, for every $x \in (0, L)$ we have

$$|u'(x)|^2 \leq \Big(\int_x^L |u''(t)|\, dt\Big)^2 \leq (L - x) \int_0^L |u''(t)|^2\, dt,$$

thus integrating on $(0, L)$ gives (6.1).

Then, recalling the definition of $R_0^L(u)$, by (6.1) we deduce the first bound

$$\inf_u R_0^L(u) \geq \frac{2}{L^2}, \quad \text{for every } L > 0. \tag{6.2}$$

Now let $u \in \mathcal{X}_0^L$ and assume moreover that $u > 0$ in $(0, L)$ (the case $u < 0$ being analogous). Then proposition 2.2 (i) gives

$$\int_0^L (u')^2\, dx \leq 4\Big(\frac{1}{c} + \frac{12}{L^2}\Big)^2 \int_0^L \frac{(u-1)^2}{2}\, dx + 2\, c^2 \int_0^L (u'')^2\, dx. \tag{6.3}$$

Hence, (6.3) yields

$$R_0^L(u) \geq \left(\max\left\{2\, c^2, 4\Big(\frac{1}{c} + \frac{12}{L^2}\Big)^2\right\}\right)^{-1}, \quad \text{for every } c, L > 0. \tag{6.4}$$

Now we prove (6.4) for a generic $u \in \mathcal{X}_0^L$ without any extra assumption on its sign. Assume that there exits $a \in (0, L)$ such that $u(a) = 0$. Then, we claim that

$$R_0^L(u) \geq \min\left\{R_0^a(u), R_a^L(u)\right\}. \tag{6.5}$$

Indeed, set

$$I_1 := \int_0^a \left(W(u) + (u'')^2\right) dx, \quad I_2 := \int_a^L \left(W(u) + (u'')^2\right) dx,$$
$$J_1 := \int_0^a (u')^2\, dx, \quad J_2 := \int_a^L (u')^2\, dx,$$

and note that $J_1,\, J_2 \neq 0$. A straightforward algebraic computation gives

$$R_0^L(u) = \frac{I_1 + I_2}{J_1 + J_2} \geq \min\left\{\frac{I_1}{J_1}, \frac{I_2}{J_2}\right\},$$

from which (6.5) follows. Since $u|_{(0,a)}$ and $u|_{(a,L)}$ are now functions with constant sign, we infer (6.4) for every $u \in \mathcal{X}_0^L$.

Then, gathering (6.2) and (6.4), we conclude that

$$\inf_{u \in \mathcal{X}_0^L} R_0^L(u) \geq \max\left\{\frac{2}{L^2}, \left(\max\left\{2c^2, 4\Big(\frac{1}{c} + \frac{12}{L^2}\Big)^2\right\}\right)^{-1}\right\}, \quad \forall\, c > 0. \tag{6.6}$$

Optimizing on $c, L > 0$, and an explicit calculation yields

$$k_1 \geq \inf_{c,L>0} \max\left\{\frac{2}{L^2}, \left(\max\left\{2\,c^2, 4\Big(\frac{1}{c} + \frac{12}{L^2}\Big)^2\right\}\right)^{-1}\right\} = 0,141467.$$

Concerning the estimate from above, we test the value of the Rayleigh quotient R_0^L on functions u which satisfy

$$u'(0) = u(L) = 0 \quad \text{and} \quad u' < 0 \quad \text{in } (0, L).$$

Indeed, if we set

$$v(x) := \begin{cases} -u(x) & \text{in } (0, L), \\ u(2L - x) & \text{in } (0, L), \end{cases}$$

then, $v \in \mathcal{X}_0^{2L}$ and $R_0^{2L}(v) = R_0^L(u)$. To make the formula easier, we consider a piecewise defined function consisting of an arc of parabola and a line:

$$u(x) = \begin{cases} -\frac{a}{2}\, x^2 + h & \text{if } 0 \leq x \leq \delta \\ -a\,\delta\, x + h + \frac{a\,\delta^2}{2} & \text{if } \delta \leq x \leq \frac{h}{a\,\delta} + \frac{\delta}{2}. \end{cases}$$

It is easy to verify that u is a continuous differentiable function in $(0, L)$, with $L = \frac{h}{a\,\delta} + \frac{\delta}{2}$, and $u'(0) = 0 = u(L)$.

Straightforward (but long) computations of the different terms in the energy lead to the following result:

$$
\begin{aligned}
I_1 &:= \int_0^L W(u) = \frac{h}{2\,a\,\delta} + \frac{\delta}{4} - \frac{\delta\,h^2}{2} + \frac{a\,\delta^3\,h}{12} + \frac{\delta\,h^4}{4} - \frac{a\,\delta^3\,h^3}{12} + \frac{a^2\,\delta^5\,h^2}{40} + \\
&\quad - \frac{a^3\,\delta^7\,h}{224} + \frac{a^4\,\delta^9}{2880} - \frac{a^2\,\delta^5}{120} - \frac{h^3}{3\,a\,\delta} + \frac{h^5}{10\,\alpha\,\delta};\\
I_2 &:= \int_0^L (u'')^2 = a^2\,\delta;\\
I_3 &:= \int_0^L (u')^2 = \frac{a\,\delta}{6}\,(6\,h - a\,\delta^2).
\end{aligned}
$$

Now, minimizing the ratio $\frac{I_1+I_2}{I_3}$ in $a, \delta, h > 0$, the definition of k_1 and an explicit computation yield

$$k_1 = \inf_{a,L,h>0} \frac{I_1 + I_2}{I_3} \leq 0.9385,$$

which improves the estimate contained in [10] (0.9481) of the lower bound for which oscillations occur.

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References

[1] M. Chermisi, G. Dal Maso, I. Fonseca, G. Leoni: Singular perturbation models in phase transitions for second order materials, *preprint* (2010).

[2] B.D. Coleman, M. Marcus, V.J. Mizel: On the thermodynamics of Periodic Phases, *Arch. Rational Mech. Anal.* **117** (1992), 321-347.

[3] I. Fonseca, C. Mantegazza: Second order singular perturbation models for phase transitions, *SIAM J. Math. Anal.* **31** (2000) no. 5, 1121-1143.

[4] E. Gagliardo: Proprietà di alcune classi di funzioni in più variabili, *Ricerche Mat.* **7** (1958), 102-137.

[5] M.E. Gurtin: Some results and conjectures in the gradient theory of phase transitions, in "Metastability and incompletely posed problems, Proc. Workshop, Minneapolis/Minn. 1984/85", *IMA Vol. Math. Appl.* **3** (1987), 135-146.

[6] D. Hilhorst, L.A. Peletier, R. Schätzle: Γ-limit for the extended Fischer-Kolmogorov equation, *Proc. Roy. Soc. Edinburgh Sect. A* **132** (2002), no. 1, 141-162.

[7] M.K. Kwong, A. Zettl, *Norm Inequalities for Derivatives and Differences*, Lecture Notes in Mathematics 1536, Springer-Verlag, Berlin, 1992.

[8] A. Leizarowitz, V. J. Mizel: One-dimensional infinite-horizon variational problems arising in continuum mechanics. *Arch. Rational Mech. Anal.* **106** (1989), no. 2, 161-194.

[9] M. Marcus: Universal properties of stable states of a free energy model with small parameter, *Calc. Var. Partial Differential Equations* **6** (1998), 123-142.

[10] V. J. Mizel, L. A. Peletier, W. C. Troy: Periodic phases in second-order materials. *Arch. Ration. Mech. Anal.* **145** (1998), no. 4, 343-382.

[11] L. Modica: The gradient theory of phase transitions and the minimal interface criterion, *Arch. Rational Mech. Anal.* **98** (1987), 123-142.

[12] L. Modica, S. Mortola: Un esempio di Γ-convergenza, *Boll. Un. Mat. Ital.* **14-B** (1977), 285-299.

[13] S. Müller: Singular perturbations as a selection criterion for periodic minimizing sequences, *Calc. Var. Partial Differential Equations* **1** (1993), 169-204.

[14] L. Nirenberg: On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa* (3) **13** (1959), 115-162.

[15] T. Otha, K. Kawasaki: Equilibrium morphology of diblock copolymer melts, *Macromolecules* **19** (1986), 1621-2632.

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