AN EXTENSION RESULT FOR GENERALISED SPECIAL FUNCTIONS OF BOUNDED DEFORMATION

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Abstract. We show an extension result for functions in $GSBD^p$, for every $p > 1$ and any dimension $n \geq 2$. The proof is based on a recent result in [9], where it is shown that a function u in $GSBD^p$ with a "small" jump set coincides with a $W^{1,p}$ function, up to a small set whose perimeter and volume are controlled by the size of the jump of u.

Dedicated to Umberto Mosco, in honour of his birthday

1. Introduction

In this note we show the existence of an extension operator for generalised special functions of bounded deformation. This result is the counterpart, in $GSBD$, of extension results for Sobolev functions (see [1, 33, 36]), and for special functions of bounded variation (see [8]).

The class of generalised special functions of bounded deformation is widely used to to describe displacements of brittle elastic bodies, in the framework of linearised elasticity.

Intuitively, given an open bounded set $\Omega \subset \mathbb{R}^n$ representing the region occupied by an elastic body, the elastic energy associated to an infinitesimal displacement $u : \Omega \to \mathbb{R}^n$, which is smooth out of a crack set $K \subset \Omega$, can be described by the *Griffith's energy*

$$E(u, \Omega, K) := \int_{\Omega \setminus K} \mathbb{C}e(u) : e(u)\, dx + \mathcal{H}^{n-1}(K). \tag{1.1}$$

In (1.1), $e(u) = (\nabla u + (\nabla u)^T)/2$ is the symmetric gradient of u, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, and $\mathbb{C}$ is the Cauchy stress tensor.

When looking for minimisers of E under given boundary conditions, however, one needs to consider a class of displacements that guarantees lower semicontinuity of the functional and compactness of minimising sequences.

The natural class for studying this problem is the space $GSBD$ of generalised special functions of bounded deformation, introduced by Gianni Dal Maso in [23] (see [15]). Roughly speaking, functions belonging to this class have the property that the symmetric part $\mathcal{E}(u)$ of the distributional gradient Du of u is a Radon measure that can be written as

$$\mathcal{E}(u) = \frac{Du + (Du)^T}{2} = e(u)\mathcal{L}^n + [u] \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u.$$

Here, $\mathcal{L}^n$ is the n-dimensional Lebesgue measure, $e(u)$ is the absolutely continuous part of $\mathcal{E}(u)$ with respect to $\mathcal{L}^n$ (which in the smooth case coincides with the whole symmetric gradient) and J_u is the *jump set* of u, which can be interpreted as the crack set K. Moreover, ν_u is the generalised normal to J_u and $[u] = u^+ - u^-$, where u^+ and u^- are the approximate limits of u on the two sides of J_u (see Section 2 for more details). In this larger space the 'weak' form of the energy (1.1) is then

$$E(u, \Omega) = \int_{\Omega} \mathbb{C}e(u) : e(u)\, dx + \mathcal{H}^{n-1}(J_u). \tag{1.2}$$

The advantage of the space $GSBD$ is that it does not require an artificial bound on the L^∞-norm of u to ensure compactness (see [31, 14]), which is the case when considering the smaller, and more classical space SBD of special functions of bounded deformation (see [3, 7]). We refer to [2, 5, 10, 11, 12, 13, 16, 19, 18, 20, 21, 22, 27, 28, 29, 30, 31, 32] for some important recent contributions in the study of this problem, and for the derivation of properties of the space $(G)SBD$.

When dealing with an energy like the one in (1.2), since the Cauchy stress typically satisfies the coercivity assumption $\mathbb{C} \gtrsim Id$ (intended as quadratic forms), it is convenient to consider the smaller class $GSBD^2(\Omega)$ of functions in $GSBD(\Omega)$ such that $e(u) \in L^2(\Omega;\mathbb{R}^{n\times n}_{\text{sym}})$ and $\mathcal{H}^{n-1}(J_u) < +\infty$. In our analysis we will in fact consider any $p > 1$, and study the energy

$$E_p(u,\Omega) = \int_\Omega |e(u)|^p\,dx + \mathcal{H}^{n-1}(J_u), \tag{1.3}$$

which is defined and finite for every $u \in GSBD^p(\Omega)$.

We are now ready to state our main result.

Theorem 1.1. *Let $n \geq 2$, let $\Omega, O \subset \mathbb{R}^n$ be open bounded sets, and assume that Ω is Lipschitz and connected. Let $p > 1$. Then, there exists an extension operator $T : GSBD^p(\Omega) \to GSBD^p(O)$ and a positive constant $c = c(n, p, \Omega, O)$ such that*

(i) $Tu = u \quad \mathcal{L}^n$*-a.e. in* $\Omega \cap O$,

(ii) $Tu \in W^{1,p}(O \setminus \overline{\Omega};\mathbb{R}^n)$,

(iii) $\displaystyle\int_O |e(Tu)|^p dx \leq c \int_\Omega |e(u)|^p dx$,

(iv) $\mathcal{H}^{n-1}(J_{Tu} \cap O) \leq c\,\mathcal{H}^{n-1}(J_u \cap \Omega)$,

for every $u \in GSBD^p(\Omega)$. The constant c is invariant under translations and dilations. If, in addition, $u \in L^\infty(\Omega;\mathbb{R}^n)$, we also have

(v) $\|Tu\|_{L^\infty(O;\mathbb{R}^n)} \leq c\left(\|u\|_{L^\infty(\Omega;\mathbb{R}^n)} + \|e(u)\|_{L^p(\Omega;\mathbb{R}^{n\times n}_{\text{sym}})}\right)$.

Theorem 1.1 shows that it is possible to extend any function $u \in GSBD^p$ from a set Ω to a set O in such a way that the Griffith's energy E_p in (1.3) increases in a controlled way, with a constant that does not depend on u. We stress here that Theorem 1.1 provides *separate* estimates on the volume and the jump part of the energy, respectively. This is important, for instance, in view of possible applications of this result to the homogenisation of functionals in $GSBD$ defined in perforated domains, in the case of Neumann boundary conditions, in the spirit of [1] and [8]. In this context, the classical way to obtain compactness is to extend to the full domain displacements that are originally only defined out of the perforations, in a way that preserves the energy bounds for the extended displacements. Alternative approaches to compactness are also possible, see e.g. [6, 26] in the SBV setting.

We observe that Theorem 1.1 does not ensure that $\mathcal{H}^{n-1}(J_{Tu} \cap (\partial\Omega \cap O)) = 0$, as it is usually the case for classical extensions obtained by reflection. In other words, the trace of Tu on $\partial\Omega \cap O$, taken from $O \setminus \overline{\Omega}$, does not necessarily coincide with the trace of u on $\partial\Omega \cap O$. For applications this is however not an issue, since the portion of $\partial\Omega$ where a load or a clamping is imposed would typically be a subset of $\partial\Omega \cap \partial O$, where the trace of u is preserved (in the case of perforated domains, for example, $\partial\Omega \cap O$ would be given by the union of the boundaries of the perforations). In any case, we show that in dimension 2 an extension preserving the trace can be obtained, see Section 4. The price to pay is that the separate estimates (*iii*) and (*iv*) in Theorem 1.1 are replaced by a bound for the functional E_q of the extension in terms of the functional E_p of the original function, with $q < p$.

The proof of Theorem 1.1 is based on a recent result proved in [9]. This states that any function in $GSBD^p$ whose jump set is sufficiently small coincides in fact with a $W^{1,p}$ function, up to a small set whose perimeter and volume are controlled by the size of the jump (see Theorem 2.1).

The paper is organised as follows. In Section 2 we recall some basic notations and in Section 3 we prove Theorem 1.1. Finally, in Section 4 we show that in dimension 2 it is possible to obtain an extension that preserves the trace.

2. Background and previous results

Let $n \in \mathbb{N}$ be fixed, with $n \geq 2$, and let $p > 1$. For every $x, y \in \mathbb{R}^n$, we denote by $x \cdot y$ the usual inner product between vectors in $\mathbb{R}^n$. We write $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times n}_{\text{sym}}$ for the set of $n \times n$ real-valued matrices, and for the set of $n \times n$ symmetric real-valued matrices, respectively. If

$x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, $|x|$ and $|A|$ stand for the Euclidean norm of x and the Frobenius norm of A, respectively, while $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ stands for the unit sphere of $\mathbb{R}^n$. For every $\xi \in \mathbb{S}^{n-1}$, the hyperplane of $\mathbb{R}^n$ containing the origin and orthogonal to ξ is denoted by

$$\Pi_\xi = \{z \in \mathbb{R}^n : \xi \cdot z = 0\}.$$

We write $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ for the n-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure of $\mathbb{R}^n$, respectively.

Let $\omega \in \mathbb{R}^n$ be a Borel set. We say that ω is a set of finite perimeter if

$$P(\omega) := \sup\left\{\int_\omega \mathrm{div}\varphi\, dx : \varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n) \text{ with } \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1\right\} < \infty,$$

where $C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ denotes the space of $\mathbb{R}^n$-valued functions of class C^1 with compact support in $\mathbb{R}^n$. If ω is a set of finite perimeter, one can define the *reduced boundary* $\partial^*\omega$ of ω. This is an $(\mathcal{H}^{n-1}, n-1)$-countably rectifiable set such that $\partial^*\omega \subset \partial\omega$ and

$$P(\omega) = \mathcal{H}^{n-1}(\partial^*\omega).$$

We now introduce the functional setting for our result. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\mathcal{M}_b(\Omega)$ ($\mathcal{M}_b^+(\Omega)$) be the space of (non-negative) bounded Radon measures in Ω. Let $u : \Omega \to \mathbb{R}^n$ be an $\mathcal{L}^n$-measurable function. We say that u belongs to the space $GBD(\Omega)$ of generalised functions of bounded deformation in Ω if there exists $\lambda_u \in \mathcal{M}_b^+(\Omega)$ such that the following is true for every $\xi \in \mathbb{S}^{n-1}$:

- For every $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} \le \tau \le \frac{1}{2}$ and $0 \le \tau' \le 1$
$$D_\xi\big(\tau(u\cdot\xi)\big) = D\big(\tau(u\cdot\xi)\big)\cdot\xi \in \mathcal{M}_b(\Omega);$$
- For every Borel set $B \subset \Omega$
$$\big|D_\xi\big(\tau(u\cdot\xi)\big)\big|(B) \le \lambda_u(B).$$

Let now $u \in GBD(\Omega)$. For every $y \in \Pi_\xi$ and $\xi \in \mathbb{S}^{n-1}$ we set

$$\Omega^\xi_y := \{t \in \mathbb{R} : y + t\xi \in \Omega\},$$

and we define the function $\hat{u}^\xi_y : \mathbb{R} \to \mathbb{R}$ as

$$\hat{u}^\xi_y(t) := u(y + t\xi)\cdot\xi \quad \text{for every } t \in \mathbb{R}.$$

We say that u belongs to the space $GSBD(\Omega)$ of generalised special functions of bounded deformation in Ω if $\hat{u}^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$. For details about the definition of the space SBV_{loc} of special functions of locally bounded variation we direct the reader to the monograph [4].

If $u \in GSBD(\Omega)$, it is possible to define (see [23]) the 'approximate symmetrised gradient' $e(u) \in L^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ of u, and the jump set J_u of u, which turns out to be $(\mathcal{H}^{n-1}, n-1)$-countably rectifiable. These agree with the standard notions of approximate symmetrised gradient and jump set for BD functions (see [3]), in case $u \in BD(\Omega)$. The space $GSBD^p(\Omega)$ is then defined as

$$GSBD^p(\Omega) := \{u \in GSBD(\Omega) : e(u) \in L^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}), \mathcal{H}^{n-1}(J_u) < +\infty\}.$$

We conclude this section with a result (see [9, Theorem 4.1]) that shows that a function in $GSBD^p$ coincides, up to a 'small' set (small both in perimeter and volume), with a Sobolev function.

Theorem 2.1. *Let $n \in \mathbb{N}$ with $n \ge 2$, $p \in (1,\infty)$, and let $\Omega \subset \mathbb{R}^n$ be a bounded and open Lipschitz set. Then, there exists a positive constant $c_1 = c_1(n,p,\Omega)$ with the following property. For every $u \in GSBD^p(\Omega)$, there exists a set of finite perimeter $\omega \subset \Omega$ and a function $v \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $u = v$ in $\Omega \setminus \omega$,*

$$\int_\Omega |e(v)|^p dx \le c_1 \int_\Omega |e(u)|^p dx, \quad \text{and} \quad \mathcal{L}^n(\omega) + \mathcal{H}^{n-1}(\partial^*\omega) \le c_1\mathcal{H}^{n-1}(J_u).$$

If in addition u is bounded, then $\|v\|_{L^\infty(\Omega;\mathbb{R}^n)} \le c_1\|u\|_{L^\infty(\Omega;\mathbb{R}^n)}$.

3. Proof of Theorem 1.1

In this section we prove our main result. We start with a technical lemma, that deals with affine functions. Similar results have been proved in [17, Lemma 4.3] and [31, Lemma 2.3].

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then, there exists $C = C(n, \Omega)$ such that for every affine function $a : \Omega \to \mathbb{R}^n$, $a(x) = Ax + b$, with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$,*

$$\|a\|_{L^\infty(\Omega;\mathbb{R}^n)} \le C\|a\|_{L^1(\Omega;\mathbb{R}^n)} \quad \text{and} \quad |A| \le C\|a\|_{L^\infty(\Omega;\mathbb{R}^n)}. \tag{3.1}$$

Proof. We start by defining the auxiliary functions $L_\Omega, F_\Omega : \mathbb{S}^{n-1} \times \mathbb{R} \to (0, \infty)$ as

$$L_\Omega(\xi, t) := \sup_{x\in\Omega+t\xi} \operatorname{dist}(x, \Pi_\xi), \qquad F_\Omega(\xi, t) := \frac{\int_{\Omega+t\xi} |\xi \cdot x|\, dx}{L_\Omega(\xi, t)}.$$

Clearly L_Ω and F_Ω are continuous.

Step 1: Estimates for L_Ω and F_Ω. Let $\bar{x} \in \Omega$ and $r > 0$ be the centre and the radius of the largest ball $B_r(\bar{x})$ contained in Ω, and let $R > 0$ be the radius of the smallest ball $B_R(0)$ containing Ω.

By the definition of L_Ω we have that, for every $\xi \in \mathbb{S}^{n-1}$ and every $t \in \mathbb{R}$,

$$r \le \sup_{x\in B_r(\bar{x})+t\xi} \operatorname{dist}(x, \Pi_\xi) \le L_\Omega(\xi, t) \le L_\Omega(\xi, 0) + |t| \le R + |t|.$$

Moreover, we claim that

$$F_\Omega(\xi, t) \ge \frac{1}{3}\mathcal{L}^n(\Omega) \quad \text{for every } \xi \in \mathbb{S}^{n-1} \text{ and } |t| > 2R. \tag{3.2}$$

Indeed, for $|t| > 2R$ we get

$$F_\Omega(\xi, t) \ge \frac{\displaystyle\int_{\Omega+t\xi} |\xi \cdot x|\, dx}{R + |t|} = \frac{\displaystyle\int_{\Omega} |t + \xi \cdot x|\, dx}{R + |t|} \ge \frac{\displaystyle\int_{\Omega} (|t| - |\xi \cdot x|)\, dx}{R + |t|} \ge \frac{(|t| - R)\mathcal{L}^n(\Omega)}{R + |t|}.$$

Since the function in the right-hand side is increasing in $|t|$, we obtain the claim (3.2) by substituting $|t| = 2R$.

Step 2: Proof of (3.1) *for scalar functions.* Let $a : \mathbb{R}^n \to \mathbb{R}$ be an affine function, namely $a(x) = A \cdot x + b$, with $A \in \mathbb{R}^n$ and $b \in \mathbb{R}$, for every $x \in \mathbb{R}^n$. We assume that $A \ne 0$ and we write $A = |A|\hat{A}$, where $\hat{A} \in \mathbb{S}^{n-1}$. Note that

$$\begin{aligned}\|a\|_{L^\infty(\Omega)} &= \sup_{x\in\Omega} |A \cdot x + b| = |A| \sup_{x\in\Omega} \left|\hat{A} \cdot x + \frac{b}{|A|}\right| = |A| \sup_{x\in\Omega} \left|\hat{A} \cdot \left(x + \frac{b}{|A|}\hat{A}\right)\right| \\ &= |A| \sup_{y\in\Omega+\frac{b}{|A|}\hat{A}} |\hat{A} \cdot y| = |A| L_\Omega(\hat{A}, b/|A|).\end{aligned}$$

Hence,

$$|A| = \frac{\|a\|_{L^\infty(\Omega)}}{L_\Omega(\hat{A}, b/|A|)} \le \frac{\|a\|_{L^\infty(\Omega)}}{r}, \tag{3.3}$$

which proves the second statement in (3.1) in the scalar case. Similarly,

$$\begin{aligned}\|a\|_{L^1(\Omega)} &= |A| \int_{\Omega+\frac{b}{|A|}\hat{A}} |\hat{A} \cdot y|\, dy = |A| L_\Omega\left(\hat{A}, \frac{b}{|A|}\right) \frac{\displaystyle\int_{\Omega+\frac{b}{|A|}\hat{A}} |\hat{A} \cdot y|\, dy}{L_\Omega\left(\hat{A}, \frac{b}{|A|}\right)} \\ &= \|a\|_{L^\infty(\Omega)} F_\Omega(\hat{A}, b/|A|).\end{aligned} \tag{3.4}$$

By the first step, and by the continuity of the function $F_\Omega(\xi, t)$ in the compact domain $\mathbb{S}^{n-1} \times [-2R, 2R]$, we conclude that

$$F_\Omega(\hat{A}, b/|A|) \ge \inf_{(\xi,t)\in\mathbb{S}^{n-1}\times\mathbb{R}} F_\Omega(\xi, t) \ge \bar{C} > 0. \tag{3.5}$$

Using (3.5) in (4.3) we prove the first statement in (3.1) in the scalar case, and conclude the proof of the second step.

Step 3: Proof of (3.1). Let now $a:\mathbb{R}^n\to\mathbb{R}^n$ be an affine function, namely $a(x)=Ax+b$, with $A\in\mathbb{R}^{n\times n}$ and $b\in\mathbb{R}^n$, for every $x\in\mathbb{R}^n$. We write $a(x)=(a_1(x),\dots,a_n(x))$, where

$$a_i(x)=A_i\cdot x+b_i \qquad \text{for } i=1,\dots,n,$$

for $A_i=A^Te_i\in\mathbb{R}^n$ and $b_i=b\cdot e_i\in\mathbb{R}$. We have that

$$\begin{aligned}\|a\|_{L^1(\Omega;\mathbb{R}^n)}&=\int_\Omega\Big(\sum_{i=1}^n a_i^2(x)\Big)^{\frac12}dx\ge\frac{1}{\sqrt{n}}\sum_{i=1}^n\int_\Omega|a_i(x)|\,dx=\frac{1}{\sqrt{n}}\sum_{i=1}^n\|a_i\|_{L^1(\Omega)}\\&\ge\frac{\overline{C}}{\sqrt{n}}\sum_{i=1}^n\|a_i\|_{L^\infty(\Omega)}\ge\frac{\overline{C}}{\sqrt{n}}\|a\|_{L^\infty(\Omega;\mathbb{R}^n)},\end{aligned}$$

so that the first statement in (3.1) is proven with $C=\frac{\sqrt{n}}{\overline{C}}$. Moreover, by (3.3),

$$|A|^2=\sum_{i=1}^n|A_i|^2\le\frac{1}{r^2}\sum_{i=1}^n\|a_i\|^2_{L^\infty(\Omega)}\le\frac{n}{r^2}\|a\|^2_{L^\infty(\Omega;\mathbb{R}^n)}.$$

□

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Let $u\in GSBD^p(\Omega)$ be fixed. By Theorem 2.1, there exist a constant $c_1=c_1(n,p,\Omega)>0$ (independent of u), a set of finite perimeter $\omega\subset\Omega$, and a function $v\in W^{1,p}(\Omega;\mathbb{R}^n)$ such that

a) $\mathcal{L}^n(\omega)+\mathcal{H}^{n-1}(\partial^*\omega)\le c_1\mathcal{H}^{n-1}(J_u)$,

b) $u=v$ in $\Omega\setminus\omega$,

c) $\displaystyle\int_\Omega|e(v)|^pdx\le c_1\int_\Omega|e(u)|^pdx.$

Thanks to the classical Korn and Korn-Poincaré inequalities applied to v (see, for instance, [34, 35, 37]), there exist two positive constants $c_2=c_2(n,p,\Omega)$ and $c_3=c_3(n,p,\Omega)$ (both independent of v), a skew-symmetric matrix $A\in\mathbb{R}^{n\times n}$, and a vector $b\in\mathbb{R}^n$ such that

$$\int_\Omega|v(x)-Ax-b|^p\,dx\le c_2\int_\Omega|\nabla v-A|^p\,dx\le c_3\int_\Omega|e(v)|^p\,dx. \tag{3.6}$$

We now define $\hat{v}\in W^{1,p}(\Omega;\mathbb{R}^n)$ as

$$\hat{v}(x):=v(x)-Ax-b,\qquad\text{for every } x\in\Omega. \tag{3.7}$$

Since $\hat{v}$ is a Sobolev function, we can apply [1, Lemma 2.6], and find that there exist a constant $c_4=c_4(n,p,\Omega,O)>0$ (independent of $\hat{v}$), and a function $\tilde{v}\in W^{1,p}(O;\mathbb{R}^n)$, such that $\tilde{v}=\hat{v}$ in $\Omega\cap O$, and

d) $\displaystyle\int_O|\nabla\tilde{v}|^p\,dx\le c_4\int_\Omega|\nabla\hat{v}|^p\,dx.$

A careful inspection of the proof of the same lemma shows that, if $\hat{v}\in L^\infty(\Omega;\mathbb{R}^n)$ (which is the case if $u\in L^\infty(\Omega;\mathbb{R}^n)$), we also have

e) $\|\tilde{v}\|_{L^\infty(O;\mathbb{R}^n)}\le c_4\,\|\hat{v}\|_{L^\infty(\Omega;\mathbb{R}^n)}.$

From d), thanks to c), (3.6), and (3.7), it follows that

$$\begin{aligned}\int_O |\nabla\tilde v|^p\,dx &\le c_4\int_\Omega |\nabla\hat v|^p\,dx = c_4\int_\Omega |\nabla v - A|^p\,dx\\ &\le \frac{c_3c_4}{c_2}\int_\Omega |e(v)|^p\,dx \le \frac{c_1c_3c_4}{c_2}\int_\Omega |e(u)|^p\,dx. \end{aligned}\tag{3.8}$$

We now introduce the function $\tilde u \in W^{1,p}(O;\mathbb{R}^n)$ defined as

$$\tilde u(x) := \tilde v(x) + Ax + b, \qquad \text{for every } x\in O. \tag{3.9}$$

Note that, thanks to (3.7), and since $\tilde v = \hat v$ in $\Omega\cap O$, we have

$$\tilde u(x) = v(x) \quad \text{for every } x\in\Omega\cap O, \quad \text{and} \quad \tilde u(x) = u(x) \quad \text{for every } x\in\Omega\setminus\omega.$$

We now set

$$(Tu)(x) := \begin{cases} u(x) & \text{if } x\in\Omega\cap O,\\ \tilde u(x) & \text{if } x\in O\setminus\Omega.\end{cases}$$

Properties (i) and (ii) follow immediately by the definition of Tu and of $\tilde u$. To show (iii) note that thanks to (3.9), recalling that A is skew-symmetric, we have

$$\begin{aligned}\int_O |e(Tu)|^p\,dx &= \int_{\Omega\cap O}|e(u)|^p\,dx + \int_{O\setminus\Omega}|e(\tilde u)|^p\,dx \le \int_\Omega |e(u)|^p\,dx + \int_O |e(\tilde v)|^p\,dx\\ &\le \int_\Omega |e(u)|^p\,dx + \int_O|\nabla\tilde v|^p\,dx \le \left(1+\frac{c_1c_3c_4}{c_2}\right)\int_\Omega |e(u)|^p\,dx,\end{aligned}$$

where we also used (3.8).

We now prove (iv). By the definition of Tu, and since $\tilde u\in W^{1,p}(O;\mathbb{R}^n)$, we have

$$\begin{aligned}\mathcal{H}^{n-1}(J_{Tu}\cap O) &= \mathcal{H}^{n-1}(J_u\cap(\Omega\cap O)) + \mathcal{H}^{n-1}(J_{\tilde u}\cap(O\setminus\overline{\Omega})) + \mathcal{H}^{n-1}(J_{Tu}\cap(O\cap\partial\Omega))\\ &= \mathcal{H}^{n-1}(J_u\cap(\Omega\cap O)) + \mathcal{H}^{n-1}(J_{Tu}\cap(O\cap\partial^*\omega))\\ &\le \mathcal{H}^{n-1}(J_u\cap\Omega) + \mathcal{H}^{n-1}(\partial^*\omega)\\ &\le (1+c_1)\mathcal{H}^{n-1}(J_u\cap\Omega),\end{aligned}$$

where we used a).

We now assume that $u\in L^\infty(\Omega;\mathbb{R}^n)$, and show that property (v) is satisfied. By the definition of Tu we have

$$\|Tu\|_{L^\infty(O;\mathbb{R}^n)} = \max\{\|u\|_{L^\infty(\Omega\cap O;\mathbb{R}^n)}, \|\tilde u\|_{L^\infty(O\setminus\Omega;\mathbb{R}^n)}\}. \tag{3.10}$$

We claim that there exists a constant $c = c(n,p,\Omega,O)>0$ such that

$$\|\tilde u\|_{L^\infty(O;\mathbb{R}^n)} \le c\left(\|e(u)\|_{L^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} + \|u\|_{L^\infty(\Omega;\mathbb{R}^n)}\right). \tag{3.11}$$

Clearly (3.11), together with (3.10), would immediately prove property (v), and hence conclude the proof of the theorem.

To prove (3.11), we start by showing that

$$\|a\|_{L^\infty(O;\mathbb{R}^n)} \le c_5\|a\|_{L^\infty(\Omega;\mathbb{R}^n)}, \tag{3.12}$$

for some positive constant $c_5 = c_5(n,\Omega,O)$, where we set $a(x) := Ax+b$, for every $x\in\mathbb{R}^n$. To this aim, let R_Ω and R_O denote the radius of the smallest open ball containing Ω and O, respectively. Then, for any $y\in\Omega$ we have

$$\begin{aligned}\|a\|_{L^\infty(O;\mathbb{R}^n)} &= \sup_{x\in O}|Ax+b| = \sup_{x\in O}|A(x-y)+Ay+b|\\ &\le |A|\sup_{x\in O}(|x|+|y|) + |Ay+b| \le |A|(R_O+R_\Omega) + \|a\|_{L^\infty(\Omega;\mathbb{R}^n)}\\ &\le (C(R_O+R_\Omega)+1)\,\|a\|_{L^\infty(\Omega;\mathbb{R}^n)},\end{aligned}$$

where $C = C(n,\Omega)$ is the positive constant given by Lemma 3.1. Hence (3.12) is proved, with $c_5 := C(R_O+R_\Omega)+1$.

Now, recalling the definition of $\tilde{u}$ in (3.9) and of $\hat{v}$ in (3.7), and using e) and (3.12), we have

$$\begin{aligned}\|\tilde{u}\|_{L^\infty(O;\mathbb{R}^n)} &= \|\tilde{v}+a\|_{L^\infty(O;\mathbb{R}^n)} \le \|\tilde{v}\|_{L^\infty(O;\mathbb{R}^n)} + \|a\|_{L^\infty(O;\mathbb{R}^n)} \\ &\le c_4\,\|\hat{v}\|_{L^\infty(\Omega;\mathbb{R}^n)} + c_5\|a\|_{L^\infty(\Omega;\mathbb{R}^n)} = c_4\,\|v-a\|_{L^\infty(\Omega;\mathbb{R}^n)} + c_5\|a\|_{L^\infty(\Omega;\mathbb{R}^n)} \\ &\le c_4\,\|v\|_{L^\infty(\Omega;\mathbb{R}^n)} + (c_4+c_5)\|a\|_{L^\infty(\Omega;\mathbb{R}^n)}. \end{aligned}\tag{3.13}$$

Note that by Lemma 3.1 we can estimate

$$\|a\|_{L^\infty(\Omega;\mathbb{R}^n)} \le C\|a\|_{L^1(\Omega;\mathbb{R}^n)} \le \overline{C}\|a\|_{L^p(\Omega;\mathbb{R}^n)},\tag{3.14}$$

where we set $\overline{C} := \overline{C}(\Omega,n,p) = C\,(\mathcal{L}^n(\Omega))^{\frac{p-1}{p}}$. Hence from (3.13) and (3.14) we have, thanks to Theorem 2.1 and (3.6),

$$\begin{aligned}\|\tilde{u}\|_{L^\infty(O;\mathbb{R}^n)} &\le c_4\,\|v\|_{L^\infty(\Omega;\mathbb{R}^n)} + \overline{C}(c_4+c_5)\|v-a\|_{L^p(\Omega;\mathbb{R}^n)} + \overline{C}(c_4+c_5)\|v\|_{L^p(\Omega;\mathbb{R}^n)} \\ &\le (c_3)^{1/p}\overline{C}(c_4+c_5)\|e(v)\|_{L^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} + \left(c_4 + C\mathcal{L}^n(\Omega)(c_4+c_5)\right)\|v\|_{L^\infty(\Omega;\mathbb{R}^n)} \\ &\le (c_1c_3)^{1/p}\overline{C}(c_4+c_5)\|e(u)\|_{L^p(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} + c_1\left(c_4 + C\mathcal{L}^n(\Omega)(c_4+c_5)\right)\|u\|_{L^\infty(\Omega;\mathbb{R}^n)}.\end{aligned}$$

This proves (3.11) and concludes the proof. ☐

4. Preserving the trace of the extension

As observed in the introduction, Theorem 1.1 does not ensure that $\mathcal{H}^{n-1}(J_{Tu}\cap(\partial\Omega\cap O)) = 0$, as it is often the case for extensions obtained by reflection (see, e.g., [8, Theorem 1.1] in $GSBV^p$, and [13, Lemma 2.8] for $GSBD^p$ functions on rectangles). One of the reasons why this is complicated is that, in the case of a general Lipschitz boundary, reflecting across a portion of the boundary can cause the derivatives of the function to 'mix'. As a consequence, the symmetrised gradient of the reflection is not controlled by the symmetric gradient of the original function, but by its full gradient. Another difficulty is due to the presence of an 'exceptional set' ω in the regularity result Theorem 2.1, where the approximate gradient of the function u is not controlled (note that the $\mathcal{L}^n$-a.e. existence in Ω of ∇u is guaranteed by [9, Corollary 5.2]).

Indeed, a direct consequence of Theorem 2.1 is that there exists an infinitesimal rigid motion a (namely an affine function a, with $e(a) = 0$), such that

$$\int_{\Omega\setminus\omega} |\nabla u - \nabla a|^p dx \le c(n,p,\Omega)\int_\Omega |e(u)|^p dx,\tag{4.1}$$

as proved in [9, Theorem 1.1]. Hence ∇u is only controlled outside ω.

We now show that if we had a full control of ∇u in Ω, then we would be able to construct an extension preserving the trace on $\partial\Omega\cap O$, by using the following extension result for special functions of bounded variation (see [8, Theorem 1.1]).

Theorem 4.1. *Let $p > 1$, let A, A' be open subsets of $\mathbb{R}^n$. Assume that A' is bounded and that A is connected and has Lipschitz boundary. Then there exists an extension operator L : $GSBV^p(A) \longrightarrow GSBV^p(A')$ and a constant $c = c(n,p,A,A') > 0$ such that*

- $Lu = u$ $\quad\mathcal{L}^n$*-a.e. in* A,
- $\displaystyle\int_{A'} |\nabla(Lu)|^p dx + \mathcal{H}^{n-1}(J_{Lu}\cap A') \le c(n,p,A,A')\left(\int_A |\nabla u|^p dx + \mathcal{H}^{n-1}(J_u\cap A)\right),$

for every $u \in GSBV^p(A)$. The constant c is invariant under translations and dilations.

If in addition $u \in L^\infty(A)$, then $Lu \in SBV^p(A')\cap L^\infty(A')$, and $\|Lu\|_{L^\infty(A')} \le \|u\|_{L^\infty(A)}$.

Remark 4.2. The result in [8] is stated and proven in $SBV^2\cap L^\infty$, but the general case of $GSBV^p$ for $p > 1$ follows immediately. In fact, a key tool of the proof in [8] is the density lower bound proved in [25] (see also [24]), which is actually valid for any $p > 1$ (see for instance [4, Theorem 7.21]).

Remark 4.3. Although it is not explicitly mentioned in the statement, a careful inspection of the proof of [8, Theorem 1.1] shows that the extension Lu can be constructed so that $\mathcal{H}^{n-1}(J_{Lu}\cap(\partial A\cap A')) = 0$.

We observe that if we had a Korn inequality of the type

$$\int_\Omega |\nabla u - \nabla a|^p dx \le c(n,p,\Omega) E_p(u,\Omega), \tag{4.2}$$

where a is some infinitesimal rigid motion, and E_p is defined in (1.3), then the function

$$v := L(u-a) + a \in GSBV^p(O;\mathbb{R}^n),$$

where L is the extension operator in Theorem 4.1, would provide an extension of u to O preserving the trace on $\partial\Omega \cap O$, and satisfying the bound $E_p(v,O) \le c(n,p,\Omega,O)E_p(u,\Omega)$.

Unfortunately, (4.2) is not true in general. Note indeed that (4.2) would imply that $GSBD^p(\Omega) \subset (GSBV^p(\Omega))^n$, while several examples show that the best possible embedding is $GSBD^p(\Omega) \subset (GSBV(\Omega))^n$, i.e., the summability of the approximate gradient is no better than L^1 in general (see for instance [17, Example 2.1], and [30, Example 2.6]).

In the two-dimensional case, however, we have a piecewise Korn inequality, proved in [30, Theorem 2.1] (see also [30, Remark 5.6]), which leads to an extension preserving the trace and satisfying a (slightly suboptimal) energy bound. We recall here the result in [30].

Theorem 4.4. *Let $p>1$, let $\Omega \subset \mathbb{R}^2$ be open, bounded with Lipschitz boundary and let $q \in [1,p)$. Then for every $u \in GSBD^p(\Omega)$ there is a Caccioppoli partition $(P_j)_{j=1}^\infty$ of Ω and corresponding infinitesimal rigid motions $(a_j)_{j=1}^\infty$ such that*

(i) $\tilde u := u - \sum_{j=1}^\infty a_j \chi_{P_j} \in SBV^q(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$,

(ii) $\sum_{j=1}^\infty \mathcal{H}^1(\partial^* P_j) \le c(p,q)(\mathcal{H}^1(J_u \cap \Omega) + \mathcal{H}^1(\partial\Omega))$,

(iii) $\|\nabla \tilde u\|_{L^q(\Omega;\mathbb{R}^{2\times 2})} \le c(\Omega)\|e(u)\|_{L^p(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})}$.

Armed with Theorem 1.1, Theorem 4.1, and Theorem 4.4, we can now construct an extension of a function $u \in GSBD^p(\Omega)$ in two steps. As a first step we construct a 'local' extension of u in a neighbourhood W of $\partial\Omega \cap O$. Then in the second step we go from a 'local' to the desired 'global' extension by means of Theorem 1.1. For the local extension, the main idea is to use the reflection given in the following theorem (see [8, Theorem 3.1]).

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and assume that $\Lambda \subset \partial\Omega$ is a bounded, relatively open, nonempty Lipschitz set, with $\Lambda \subset\subset \{x \in \partial\Omega : \partial\Omega$ has Lipschitz boundary at $x\}$. Then, there exists a bounded open set $W \subset \mathbb{R}^n$ with Lipschitz boundary, such that $\Lambda = W \cap \partial\Omega$, and a bilipschitz map $\phi : W \to W$ with $\phi_{|\Lambda} = Id$ and $\phi(W^\pm) = W^\mp$, where $W^+ := W \cap \Omega$ and $W^- := W \cap (\mathbb{R}^n \setminus \overline{\Omega})$.*

Let now $\Omega, O \subset \mathbb{R}^2$ be open bounded sets, and assume that Ω is Lipschitz and connected, as in Theorem 1.1. Let $p>1$, let $q \in [1,p)$, and let $u \in GSBD^p(\Omega)$.

Step 1: Local extension of u. Let W be the neighbourhood of $\Lambda := \partial\Omega \cap O$ given by Theorem 4.5. Since, from property (i) of Theorem 4.4, we have $\tilde u \in SBV^q(\Omega;\mathbb{R}^2)$, we can extend each component of $\tilde u$ to W by means of Theorem 4.1. We then obtain a function $L\tilde u \in GSBV^q(W;\mathbb{R}^2)$ such that $L\tilde u = \tilde u$ $\mathcal{L}^2$-a.e. in $\Omega \cap W$, $\mathcal{H}^1(J_{L\tilde u} \cap (\partial\Omega \cap W)) = 0$, and

$$\begin{aligned}\int_W |\nabla(L\tilde u)|^q\,dx + \mathcal{H}^1(J_{L\tilde u} \cap W) &\le c(q,\Omega,O)\left(\int_\Omega |\nabla \tilde u|^q\,dx + \mathcal{H}^1(J_{\tilde u} \cap \Omega)\right)\\ &\le c(p,q,\Omega,O)\left(\|e(u)\|^q_{L^p(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})} + \mathcal{H}^1(J_u \cap \Omega) + \mathcal{H}^1(\partial\Omega)\right),\end{aligned} \tag{4.3}$$

where the last inequality follows from properties (ii) and (iii) of Theorem 4.4.

Note that the function $L\tilde u$ is an extension of $\tilde u$, and not of u. To get the desired extension, we need to add to $\tilde u$ a function defined in the whole of W, that coincides with $\sum_{j=1}^\infty a_j \chi_{P_j}$ in $\Omega \cap W$, and whose discontinuity set has $\mathcal{H}^1$-negligible intersection with $\partial\Omega \cap O$.

This can be done in the following way. Let ϕ be the map given in Theorem 4.5, that keeps $\Lambda = \partial\Omega \cap O$ fixed. Then, observing that $\phi\left(\chi_{P_j \cap W}\right)$ is the characteristic function of $\phi(P_j \cap W)$ for every $j \in \mathbb{N}$, we can define a_P as

$$a_P := \begin{cases} \displaystyle\sum_{j=1}^{\infty} a_j \chi_{P_j \cap W} & \text{in } W \cap \Omega, \\ \displaystyle\sum_{j=1}^{\infty} a_j \phi\left(\chi_{P_j \cap W}\right) & \text{in } W \setminus \Omega. \end{cases}$$

Then $a_P \in SBD(W)$, and one can see that there exists a constant $c(\Omega, O)$ such that

$$\mathcal{H}^1(J_{a_P} \cap W) \leq c(\Omega, O) \sum_{j=1}^{\infty} \mathcal{H}^1(\partial^* P_j) \leq c(p, q, \Omega, O)(\mathcal{H}^1(J_u \cap \Omega) + \mathcal{H}^1(\partial\Omega)), \tag{4.4}$$

where the last inequality follows by property *(ii)* of Theorem 4.4.

Then, we have that $\tilde{v} := L\tilde{u} + a_P \in GSBD^q(W)$ is the required 'local' extension of u to W. Indeed, $\tilde{v} = u$ $\mathcal{L}^2$-a.e. in $\Omega \cap W$, $\mathcal{H}^1(J_{\tilde{v}} \cap (\partial\Omega \cap W)) = 0$, and by (4.3)

$$\begin{aligned} \|e(\tilde{v})\|^q_{L^q(W;\mathbb{R}^{2\times 2}_{\rm sym})} &= \|e(L\tilde{u})\|^q_{L^q(W;\mathbb{R}^{2\times 2}_{\rm sym})} \leq \|\nabla(L\tilde{u})\|^q_{L^q(W;\mathbb{R}^{2\times 2})} \\ &\leq c(p, q, \Omega, O)\left(\|e(u)\|^q_{L^p(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})} + \mathcal{H}^1(J_u \cap \Omega) + \mathcal{H}^1(\partial\Omega)\right). \end{aligned}$$

Moreover by (4.3) and (4.4),

$$\begin{aligned} \mathcal{H}^1(J_{\tilde{v}} \cap W) &\leq \mathcal{H}^1(J_{L\tilde{u}} \cap W) + \mathcal{H}^1(J_{a_P} \cap W) \\ &\leq \check{c}(p, q, \Omega, O)\left(\|e(u)\|^q_{L^p(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})} + \mathcal{H}^1(J_u \cap \Omega) + \mathcal{H}^1(\partial\Omega)\right). \end{aligned}$$

Therefore,

$$E_q(\tilde{v}, W) \leq \tilde{c}(p, q, \Omega, O)\left(\|e(u)\|^q_{L^p(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})} + \mathcal{H}^1(J_u \cap \Omega) + \mathcal{H}^1(\partial\Omega)\right).$$

Step 2: Global extension. We now construct a global extension of u, namely we further extend $\tilde{v}$ from W to O. To this end we apply Theorem 1.1 to the function $\tilde{v}$, and define $v := T\tilde{v} \in GSBD^q(O)$. Then v satisfies $v = u$ $\mathcal{L}^2$-a.e. in $\Omega \cap O$, $\mathcal{H}^1(J_{\tilde{v}} \cap (\partial\Omega \cap O)) = 0$, and

$$E_q(v, O) \leq c(p, q, \Omega, O) E_q(\tilde{v}, W) \leq \hat{c}(p, q, \Omega, O)\left(\|e(u)\|^q_{L^p(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})} + \mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega)\right).$$

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