The Ornstein-Uhlenbeck semigroup in finite dimension

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Abstract

We gather the main known results concerning the nondegenerate Ornstein-Uhlenbeck semigroup in finite dimension.
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Introduction

We consider the Ornstein-Uhlenbeck operator

$$\mathcal{A} = \sum_{i,j=1}^{n} q_{ij} D_{ij} + \sum_{i,j=1}^{n} b_{ij} x_j D_i = \mathrm{Tr}(QD^2) + \langle Bx, D\rangle, \qquad x \in \mathbb{R}^N. \tag{1}$$

Here $Q = (q_{ij})$ is a real symmetric and positive definite matrix and $B = (b_{ij})$ is a nonzero real matrix. We also introduce the symmetric matrices

$$Q_t = \int_0^t e^{\tau B} Q e^{\tau B^T}\, d\tau, \tag{2}$$

which share the same properties as Q. We discuss the main properties of the semigroup and the generator in the space $C_b(\mathbb{R}^N)$ of bounded continuous functions on $\mathbb{R}^N$ (domain, interpolation properties, Schauder estimates) and in the L^p spaces both endowed with the Lebesgue measure and the invariant measure γ, when it exists. In $L^p(\mathbb{R}^N)$ with the Lebesgue measure we describe the domain and the spectrum of the generator. In L^p_γ we describe the domain, the spectrum (which turns out to be completely different from the former) and some further regularity properties.

Finally, we present the hyper- (ultra-, super-) contractivity properties and the related log-Sobolev inequalities. Ornstein-Uhlenbeck operators and semigroups are of interest in several fields, from quantum mechanics, where they have been introduced by the scholars they bear the names, to stochastic analysis, control theory, partial differential equations. Evolution equations driven by Ornstein-Uhlenbeck operators are the Kolmogorov equations

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of linear stochastic ODEs, and they are one of the few examples of multidimensional linear parabolic equations for which a resolvent kernel is explicitly known. In spite of these features, as we are going to show they have generated challenging problems. An interesting discussion of the physical models and the main applications in a historical perspective can be found in [5].

We point out that we do not deal with degenerate Ornstein-Uhlenbeck operators, that can be hypoelliptic and in this case share many properties with the nondegenerate ones, and that there are several further issues connected with Ornstein-Uhlenbeck operators and semigroups, which we do not discuss, such as e.g. functional calculus, properties of maximal operators, Riesz transforms, and other questions of harmonic analysis. Our presentation is limited to the topics closer to the interests of the semigroup community.

Many of the estimates that we are going to show are dimension free, and in fact a rich extension of the theory to infinite dimensional settings (separable Hilbert and even Banach spaces) is available. We refer to the paper *Ornstein-Uhlenbeck semigroups in infinite dimension* in this volume for a survey.

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1 The integral formula for the semigroup

In this section we sketch the derivation of the explicit representation of the semigroup generated by $\mathcal{A}$ in the form (1), which is due to Kolmogorov. Formula (1.4) below can be derived also using the Fourier transform as sketched in [26].

Let us consider the following parabolic initial value problem

$$\begin{cases} u_t = \mathrm{Tr}(QD^2u) + \langle Bx, Du\rangle \\ u(0,x) = f(x) \end{cases} \tag{1.1}$$

with $f \in C_b(\mathbb{R}^N)$. Problem (1.1) is simplified getting rid of the drift term $\langle Bx, Du\rangle$ using the flow generated by B

$$\begin{cases} \dot{\xi} = B\xi \\ \xi(0) = x \end{cases} \tag{1.2}$$

whose solution is given by $\xi(t,x) = e^{tB}x$. Thus, setting $u(t,x) = v(t, e^{tB}x)$, we see that $u(t,x)$ is solution of (1.1) if and only if $v(t,x)$ is solution of the following nonautonomous parabolic problem

$$\begin{cases} v_t = \mathrm{tr}(C(t)D^2v) = A(t)v \\ v(0) = f \end{cases} \tag{1.3}$$

where $C(t) = e^{tB}Qe^{tB^T}$ and $A(t) = \mathrm{Tr}(C(t)D^2)$. If we compute formally the solution of problem (1.3) and come back we find

$$T(t)f(x) := u(t,x) = \frac{1}{(4\pi)^{N/2}(\mathrm{det}Q_t)^{1/2}} \int_{\mathbb{R}^N} e^{-\frac{\langle Q_t^{-1}y,y\rangle}{4}} f(e^{tB}x - y)dy, \tag{1.4}$$

where the matrix Q_t is defined in (2). See e.g. [4] for more details. If we set

$$g_t(y) = \frac{1}{(4\pi)^{N/2}(\mathrm{det}Q_t)^{1/2}}\, e^{-\frac{\langle Q_t^{-1}y,y\rangle}{4}}, \tag{1.5}$$

it is easily seen that $\|g_t\|_1 = 1$, and therefore $T(t)$ is conservative, namely it maps the constant function $\mathbf{1}$ into itself, and moreover for every $f \in C_b(\mathbb{R}^N)$ we have

$$T(t)f(x) = (g_t * f)(e^{tB}x), \quad t > 0, \ x \in \mathbb{R}^N. \tag{1.6}$$

Let us check the semigroup law for f in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$, so that $T(t)f \in \mathcal{S}(\mathbb{R}^N)$ by (1.4). Writing everything in terms of the Fourier transform $\mathscr{F}[f]$, we have $\mathscr{F}[g_t](\eta) = \exp\{-\langle Q_t\eta, \eta\rangle\} = \exp\{-|Q_t^{1/2}\eta|^2\}$, $\mathscr{F}[f(e^{tB}x)](\xi) = \frac{1}{|\det e^{tB}|}\mathscr{F}[f](e^{-tB^T}\xi)$ $\det e^{tB} = e^{t\,\mathrm{tr}B}$ and $Q_{t+s} = Q_s + e^{sB}Q_te^{sB^T}$ hence

$$\mathscr{F}[T(s)f](\xi) = e^{-s\ \mathrm{tr}B}\exp\{-|Q_s^{1/2}e^{-sB^T}\xi|^2\}\mathscr{F}[f](e^{-sB^T}\xi)$$

and

$$\begin{aligned}
&\mathscr{F}[T(t)T(s)f](\xi) = e^{-t\,\mathrm{tr}B}\exp\{-|Q_t^{1/2}e^{-tB^T}\xi|^2\}\mathscr{F}[T(s)f](e^{-tB^T}\xi)\\
=&e^{-t\,\mathrm{tr}B}\exp\{-|Q_t^{1/2}e^{-tB^T}\xi|^2\}e^{-s\,\mathrm{tr}B}\exp\{-|Q_s^{1/2}e^{-sB^T}e^{-tB^T}\xi|^2\}\mathscr{F}[f](e^{-tB^T}e^{-sB^T}\xi)\\
=&e^{-(t+s)\,\mathrm{tr}B}\exp\{-(|(e^{sB}Q_te^{sB^T})^{1/2}e^{-(t+s)B^T}\xi|^2 + |Q_s^{1/2}e^{-(t+s)B^T}\xi|^2)\}\mathscr{F}[f](e^{-(t+s)B^T}\xi)\\
=&\mathscr{F}[T(t+s)f](\xi).
\end{aligned}$$

Therefore $T(t+s)f = T(t)T(s)f$ if $f \in \mathcal{S}(\mathbb{R}^N)$. A comparison between (1.6) and the semigroup law in $\mathcal{S}(\mathbb{R}^N)$ gives the relation

$$g_{t+s}(y) = \int_{\mathbb{R}^n} g_t(y + e^{sB}z)g_s(z)dz,$$

whence we deduce that the semigroup law holds also in all the spaces where $T(t)$ is given by (1.6), e.g., L^p spaces with respect to the Lebesgue or the Gaussian measures, see the following Subsections, and $C_b(\mathbb{R}^N)$ as well.

Another way to deduce (1.4) is through stochastic analysis. Indeed, let us consider the ordinary SDE in $\mathbb{R}^N$

$$\begin{cases} dX_t = BX_tdt + TdW_t,\\ X_0 = x, \end{cases}$$

where W_t is a standard Brownian motion in $\mathbb{R}^N$ and T is any $N \times N$ matrix, $x \in \mathbb{R}^N$. The solution is $X_t = e^{tB}x + \int_0^t e^{(t-s)B}TdW_s$, and since $\int_0^t e^{(t-s)B}TdW_s$ is a centered Gaussian random variable with covariance $\int_0^t e^{sB}TT^*e^{sB^*}ds$, the transition semigroup P_t defined by $P_t\varphi(x) := \mathbb{E}(\varphi(X_t))$ is our $T(t)$, with $Q = (TT^*)/2$, and $u_t = \mathcal{A}u$ is the Kolmogorov equation of the above SDE, see [2, Example 6.7.6] for a simple proof in the 1-dimensional case and [3, §8.2] for a more general result.

2 Main properties of the semigroup

In this section we collect some classical results for $(T(t))_{t\geq 0}$ in spaces of continuous functions and in L^p spaces, first with respect to the Lebesgue measure and then with respect to a suitable Gaussian measure, which is invariant for the semigroup, in the case that all the eigenvalues of B have negative real part. All the functions considered here are real valued. Complex valued functions are needed only in a part of the next subsection.

2.1 The semigroup in $C_b(\mathbb{R}^N)$

As usual, $C_b(\mathbb{R}^N)$ and all its subspaces are endowed with the sup norm $\|\cdot\|_\infty$.

Since g_t is continuous, by (1.6) we see that $T(t)$ maps $L^\infty(\mathbb{R}^N)$ into $C_b(\mathbb{R}^N)$, and therefore it is a strong Feller semigroup. However, it is not strongly continuous even in $BUC(\mathbb{R}^N)$ unless $B=0$. Indeed, given any $f\in BUC(\mathbb{R}^N)$, using the fact that $\lim_{t\to 0}\int_{\{|y|>\delta\}} g_t(y)dy = 0$ for every $\delta>0$, it is easily seen that

$$\lim_{t\to 0^+}\|T(t)f - f(e^{tB}\cdot)\|_\infty = 0 \tag{2.1}$$

and therefore we have

$$\lim_{t\to 0^+}\|T(t)f - f\|_\infty = 0$$

if and only if

$$\lim_{t\to 0^+}|f(e^{tB}x)-f(x)| = 0 \text{ uniformly for } x\in\mathbb{R}^N, \tag{2.2}$$

see [12, Lemma 3.2]. For $N=1$ and $B=1$ a counterexample to strong continuity is thus provided by $f(x)=\sin x\in BUC(\mathbb{R})$. In fact, $f(e^t\cdot)$ does not converge uniformly to f as $t\to 0$. As $T(t)$ maps $C_0(\mathbb{R}^N)$ into itself and (2.2) is satisfied in $C_0(\mathbb{R}^N)$, it is strongly continuous on $C_0(\mathbb{R}^N)$.

Smoothing properties of $(T(t))_{t\geq 0}$ in $C_b(\mathbb{R}^N)$ are established as well in [12]: if $f\in C_b^1(\mathbb{R}^N)$, from (1.6) we get

$$DT(t)f(x) = e^{tB^T}(g_t * Df)(e^{tB}x) = e^{tB^T}T(t)Df(x), \tag{2.3}$$

whereas for any $f\in C_b(\mathbb{R}^N)$

$$DT(t)f(x) = e^{tB^T}(Dg_t * f)(e^{tB}x). \tag{2.4}$$

Since $Dg_t(y) = g_t(y)(-\frac{1}{2}Q_t^{-1}y)$,

$$DT(t)f(x) = -\frac{1}{2}\int_{\mathbb{R}^N} e^{tB^T}Q_t^{-1}y\, g_t(y)f(e^{tB}x-y)dy = (g_t^{(1)} * f)(e^{tB}x)$$

where $g_t^{(1)}(y) = -\frac{1}{2}e^{tB^T}Q_t^{-1}y\, g_t(y)$. We estimate $\|g_t^{(1)}\|_1$ as follows

$$\begin{aligned}\|g_t^{(1)}\|_1 &\leq \frac{1}{2}\|e^{tB^T}\|\int_{\mathbb{R}^N}|Q_t^{-1}y|\frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}}e^{-\frac{1}{4}\langle Q_t^{-1}y,y\rangle}dy\\ &= \frac{1}{2(4\pi)^{N/2}}\|e^{tB}\|\int_{\mathbb{R}^N}e^{-\frac{|z|^2}{4}}|Q_t^{-1/2}z|dz \leq c\|e^{tB}\|\|Q_t^{-1/2}\|, \qquad t>0.\end{aligned}$$

Using now the inequalities $|Q_t^{-1/2}z|\leq\frac{c}{\sqrt{t}}|z|$ for $0<t\leq 1$ (which easily follows from $Q_t/t\to I$ as $t\to 0$), and $\|Q_t\|\leq t\|Q\|\sup_{0<s<t}\|e^{sB}\|$, we obtain that there exists $c_1>0$ such that

$$\|DT(t)f\|_\infty \leq \frac{c_1}{t^{1/2}}\|f\|_\infty \qquad 0<t\leq 1.$$

If all the eigenvalues of B have negative real part there exist c_1, $\omega>0$ such that

$$\|DT(t)f\|_\infty \leq \frac{c_1 e^{-\omega t}}{t^{1/2}}\|f\|_\infty \qquad t>0.$$

Using the semigroup law and (2.3) we obtain $T(t)f \in C_b^2(\mathbb{R}^N)$, and

$$\begin{aligned} \|DD_iT(t)f\|_\infty &= \|D_iDT(t/2)T(t/2)f\|_\infty = \|D_i(e^{\frac{t}{2}B^T}T(t/2)DT(t/2)f)\|_\infty \\ &\leq \frac{c_1\|e^{\frac{t}{2}B^T}\|}{\sqrt{t}}\|DT(t/2)f\|_\infty \leq \frac{c_2}{t}\|f\|_\infty, \quad 0 < t \leq 1, \end{aligned} \tag{2.5}$$

and if all the eigenvalues of B have negative real part,

$$\|DD_iT(t)f\|_\infty \leq \frac{c_2e^{-\omega t}}{t}\|f\|_\infty \quad t > 0.$$

Iterating this argument, we conclude that if $f \in C_b(\mathbb{R}^N)$, for every $t > 0$ the function $T(t)f$ belongs to $C_b^\infty(\mathbb{R}^N)$ and for every multi-index α there exists $c = c(\alpha) > 0$ such that

$$\|D^\alpha T(t)f\|_\infty \leq \frac{c}{t^{|\alpha|/2}}\|f\|_\infty \qquad 0 < t \leq 1, \tag{2.6}$$

while the sup norm of all the derivatives of $T(t)f$ decay exponentially as $t \to \infty$ if all the eigenvalues of B have negative real part.

The semigroup $T(t)$ is neither compact, see [39, Example 5.4], nor analytic if $B \neq 0$, see [12, Lemma 3.3], in $C_b(\mathbb{R}^N)$. Theorem 4.2 of [39] and the counterexample in [12] show that it is neither compact nor analytic even in $C_0(\mathbb{R}^N)$. In fact, the counterexample in [12] shows that, in general, $T(t)$ does not map $C_0(\mathbb{R}^N)$ into the domain of the infinitesimal generator A_0 of $T(t)$ in $C_0(\mathbb{R}^N)$, and therefore $T(t)$ is not differentiable in $C_0(\mathbb{R}^N)$. Moreover, if $B \neq 0$ then $\|T(t) - T(s)\| \geq 2$ for $s \neq t$ and the semigroup is not norm-continuous.

Still, the representation formula for $T(t)$ easily yields that for every $f \in C_b(\mathbb{R}^N, \mathbb{C})$ the function $(t, x) \mapsto T(t)f(x)$ is continuous in $[0, +\infty) \times \mathbb{R}^N$, and therefore for all λ with positive real part the operators R_λ defined by

$$R_\lambda f(x) := \int_0^{+\infty} e^{-\lambda t}T(t)f(x)\,dt, \quad f \in C_b(\mathbb{R}^N; \mathbb{C}),\ x \in \mathbb{R}^N,$$

are one to one. Moreover, since $T(t)$ is a semigroup, the resolvent identity $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$ is satisfied for $\operatorname{Re}\lambda$, $\operatorname{Re}\mu > 0$. It follows that R_λ is the resolvent $(\lambda I - A_\mathbb{C})^{-1}$ of a closed operator $A_\mathbb{C} : D(A_\mathbb{C}) \subset C_b(\mathbb{R}^N; \mathbb{C}) \to C_b(\mathbb{R}^N; \mathbb{C})$, which is called *generator* of $T(t)$ in $C_b(\mathbb{R}^N; \mathbb{C})$, although it is not the infinitesimal generator in the classical sense. Here we are interested only in real valued functions; since $T(t)$ preserves real valued functions, for $\lambda > 0$ also R_λ does and we call *generator* of $T(t)$ in $C_b(\mathbb{R}^N)$ the part A of $A_\mathbb{C}$ in $C_b(\mathbb{R}^N)$.

By the general theory of strongly continuous semigroups, the restriction of R_λ to $C_0(\mathbb{R}^N)$ coincides with the resolvent of the infinitesimal generator A_0 of $T(t)$ in $C_0(\mathbb{R}^N)$.

Proposition 2.1 *We have*

$$\begin{aligned} D(A) &= \{u \in C_b(\mathbb{R}^N) \bigcap_{p>1} W^{2,p}_{\rm loc}(\mathbb{R}^N) :\ \mathcal{A}u \in C_b(\mathbb{R}^N)\}, \\ D(A_0) &= \{u \in C_0(\mathbb{R}^N) \bigcap_{p>1} W^{2,p}_{loc}(\mathbb{R}^N) :\ \mathcal{A}u \in C_0(\mathbb{R}^N)\}. \end{aligned}$$

Both statements are consequences of results from [38] about a more general class of Feller semigroups. The first one follows from Thm. 5.2(i) and Prop. 5.7, the second one from

Prop. 5.5. Hölder continuity of $T(\cdot)f$ and Schauder type theorems have been investigated in [12]. Given $f \in C_b(\mathbb{R}^N)$ and $\alpha \in (0,1)$, we have

$$\sup_{0<t\le 1} \frac{\|T(t)f-f\|_\infty}{t^\alpha} < +\infty \iff f \in (C_b(\mathbb{R}^N), D(A))_{\alpha,\infty} \tag{2.7}$$

$$\iff \begin{cases} f \in C_b^{2\alpha}(\mathbb{R}^N) \cap Y_\alpha & \alpha \neq 1/2, \\ f \in Z^1(\mathbb{R}^N) \cap Y_{1/2} & \alpha = 1/2, \end{cases}$$

where the spaces Y_α are defined by

$$Y_\alpha = \Big\{ f \in C_b(\mathbb{R}^N) : \sup_{x\in\mathbb{R}^N,\, t>0} \frac{|f(e^{tB}x)-f(x)|}{t^\alpha} < +\infty \Big\},$$

and $Z^1(\mathbb{R}^N)$ is the Zygmund space

$$Z^1(\mathbb{R}^N) = \Big\{ f \in C_b(\mathbb{R}^N) : \sup_{x\in\mathbb{R}^N,\, h\neq 0} \frac{|f(x+h)-2f(x)+f(x-h)|}{|h|} < +\infty \Big\}.$$

Schauder theorems are the following. The first part was proved in [12], the second part in [30]. We denote by $C_b^{k+\theta}(\mathbb{R}^N)$, for $k \in \mathbb{N}$ and $0 < \theta \le 1$, the space of $C_b^k(\mathbb{R}^N)$ functions with θ-Hölder continuous k-th order derivatives.

Theorem 2.1 *Let $\lambda > 0$ and $u \in D(A)$ be such that $\lambda u - \mathcal{A}u = f \in C_b^\theta(\mathbb{R}^N)$, with $\theta \in (0,1)$. Then $u \in C_b^{\theta+2}(\mathbb{R}^N)$ and all the second order derivatives of u belong to $Y_{\theta/2}$. There exists $C = C(\lambda) > 0$ independent of u such that*

$$\|u\|_{C_b^{\theta+2}(\mathbb{R}^N)} \le C\|f\|_{C_b^\theta(\mathbb{R}^N)}.$$

Let now $T > 0$, $f \in C_b^{\theta+2}(\mathbb{R}^N)$ and $g \in C_b([0,T]\times\mathbb{R}^N)$ be such that $\sup_{0\le t\le T}\|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)} < \infty$. Then the function $u(t,x) := T(t)f(x) + \int_0^t T(t-s)g(s,\cdot)(x)ds$ satisfies

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,\cdot)(x) + g(t,x), & t\in[0,T],\ x\in\mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.8}$$

and it is the unique solution of (2.8) belonging to $C_b([0,T]\times\mathbb{R}^N)$, such that $u(t,\cdot) \in C_b^{\theta+2}(\mathbb{R}^N)$ and $\sup_{0\le t\le T}\|u(t,\cdot)\|_{C_b^{\theta+2}(\mathbb{R}^N)} < \infty$, and such that the derivatives u_t, D_iu, $D_{ij}u$ are continuous in $([0,T]\times\mathbb{R}^N)$. Moreover, there is $C = C(T) > 0$ such that

$$\sup_{0\le t\le T}\|u(t,\cdot)\|_{C_b^{\theta+2}(\mathbb{R}^N)} \le C(\|f\|_{C_b^{\theta+2}(\mathbb{R}^N)} + \sup_{0\le t\le T}\|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)}).$$

Maximal regularity in the parabolic Hölder spaces $C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)$, that holds for (2.8) if $\mathcal{A}$ is replaced by any uniformly elliptic operator with coefficients in $C_b^\theta(\mathbb{R}^N)$, does not hold in the present case.

2.2 The semigroup in $L^p(\mathbb{R}^N)$

We start recalling that the semigroup $(T(t))_{t\ge 0}$ is strongly continuous on $L^p(\mathbb{R}^N)$, $1 \le p < \infty$, see [35]. One can show that a suitable realization of $\mathcal{A}$ in $L^p(\mathbb{R}^N)$ is the infinitesimal generator of $(T(t))_{t\ge 0}$. For $1 < p < \infty$ denote by $\|\cdot\|_p$ the norm in $L^p(\mathbb{R}^N)$ and define

$$D_p(\mathcal{A}) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{\rm loc}(\mathbb{R}^N) : \mathcal{A}u \in L^p(\mathbb{R}^N)\}.$$

From now on, we denote by $\mathcal{A}_p$ the realization of $\mathcal{A}$ in $L^p(\mathbb{R}^N)$ with domain $D_p(\mathcal{A})$. It is proved in [36] that if $1 < p < \infty$, the generator of $(T(t))_{t\geq 0}$ in L^p is the operator $\mathcal{A}_p$. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for $\mathcal{A}_p$. For $p = 1$ the generator is the closure $\mathcal{A}_1$ of $\mathcal{A} : C_c^\infty(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$, but in general a function u in the domain of $\mathcal{A}_1$ does not belong to $W^{2,1}_{\rm loc}(\mathbb{R}^N)$. The following more precise description of $D_p(\mathcal{A})$ is given in [40] and [41], see also [42].

Theorem 2.2 *For* $1 < p < \infty$

$$D_p(\mathcal{A}) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}(\mathbb{R}^N) : \langle Bx, Du\rangle \in L^p(\mathbb{R}^N)\}. \tag{2.1}$$

Moreover, there are positive constants c_1, c_2 *such that*

$$c_1(\|u\|_p + \|\mathcal{A}u\|_p) \leq \|u\|_{W^{2,p}(\mathbb{R}^N)} + \|\langle Bx, Du\rangle\|_p \leq c_2(\|u\|_p + \|\mathcal{A}u\|_p) \tag{2.2}$$

for every $u \in D_p(\mathcal{A})$.

The above results say that $D_p(\mathcal{A})$ is the intersection of the domains of the diffusion term $\mathrm{Tr}(QD^2u)$ and of the drift term $\langle Bx, Du\rangle$. The estimate

$$\|u\|_{W^{2,p}(\mathbb{R}^N)} \leq c_2(\|u\|_p + \|\mathcal{A}u\|_p), \qquad u \in D_p(\mathcal{A}),$$

which follows from (2.2), is the analogue of the classical Calderón-Zygmund estimate for the Laplacian, see e.g. [23, §9.4]. From (1.5) and (1.6), using Young's inequality for convolutions and the identity $\det(e^{-tB}) = e^{-t\,\mathrm{tr}B}$ we get

$$\|T(t)f\|_p \leq e^{-\frac{\mathrm{tr}B}{p}t}\|g_t\|_1\|f\|_p = e^{-\frac{\mathrm{tr}B}{p}t}\|f\|_p, \tag{2.3}$$

so that $T(t)$ is quasi-contractive and contractive if $\mathrm{tr}B$ is nonnegative. On the negative side, the semigroup $T(t)$ is never analytic in $L^p(\mathbb{R}^N)$, unless $B = 0$, as it follows from the results of Section 4 (a). An explicit counterexample for $N = 1$ is in [31].

Smoothing properties of $(T(t))_{t\geq 0}$ are established in [13] in $L^p(\mathbb{R}^N)$: if $f \in L^p(\mathbb{R}^N)$ then $T(t)f \in W^{k,p}(\mathbb{R}^N)$. Indeed, arguing as in the Subsection 2.1, Young's inequality yields

$$\|DT(t)f\|_p \leq e^{-\frac{\mathrm{trB}}{p}t}\|g_t^{(1)}\|_1 \ \|f\|_p \leq \frac{c}{\sqrt{t}}\|f\|_p \qquad 0 < t \leq 1.$$

and, by iteration,

$$\|D^\alpha T(t)f\|_p \leq \frac{c}{t^{|\alpha|/2}}\|f\|_p \qquad 0 < t \leq 1.$$

However, $\langle Bx, DT(t)f\rangle \notin L^p(\mathbb{R}^N)$, in general. Theorem 2.2 does not have a parabolic counterpart like Theorem 2.1, since the semigroup is not analytic. However the following result is proved in [31].

Proposition 2.2 *Let* $T > 0$ *and* $g \in L^p(0,T; W^{\theta,p}(\mathbb{R}^N))$ *with* $\theta \in (0,1)$ *and* $p \in (1,+\infty)$. *Then the mild solution to problem* (2.8) *with*$f \equiv 0$, *namely the function* $u(t,x) := \int_0^t T(t-s)g(s,\cdot)(x)\,ds$, *belongs to* $L^p(0,T; W^{\theta+2,p}(\mathbb{R}^N))$ *and there is* $C = C(T) > 0$ *such that*

$$\|u\|_{L^p(0,T;W^{\theta+2,p}(\mathbb{R}^N))} \leq C\|g\|_{L^p(0,T;W^{\theta,p}(\mathbb{R}^N))}.$$

2.3 The invariant measure γ and the semigroup in $L^p(\mathbb{R}^N, \gamma)$

In this section we assume that $\sigma(B) \subset \mathbb{C}_-$, i.e., the spectrum of B is contained in the open left half plane. This assumption, as proved in [14, Section 11.2.3], is equivalent to the existence of an invariant measure γ for $(T(t))_{t\geq 0}$, i.e., a probability measure γ such that

$$\int_{\mathbb{R}^N} T(t)f\, d\gamma = \int_{\mathbb{R}^N} f\, d\gamma$$

for every $t \geq 0$ and $f \in C_b(\mathbb{R}^N)$. The invariant measure, when it does exist, is usually identified by letting $t \to \infty$. As we have an explicit formula for the semigroup, we start by observing that under our hypotheses the matrix Q_t defined in (2) converges increasingly (in the sense of quadratic forms) to $Q_\infty = \int_0^\infty e^{sB} Q e^{sB^*} ds$ and that e^{tB} converges to 0 as $t \to \infty$, so that

$$T(t)f(x) \overset{t\to\infty}{\to} \frac{1}{\sqrt{(4\pi)^N \det Q_\infty}} \int_{\mathbb{R}^N} e^{-\frac{1}{4}\langle Q_\infty^{-1} y, y\rangle} f(y)\, dy \tag{2.1}$$

pointwise for every $f \in C_b(\mathbb{R}^N)$. The above computation suggests what should be the invariant measure γ, i.e., the Gaussian measure with density $g(x)$ given by

$$g(x) = \frac{1}{\sqrt{(4\pi)^N \det Q_\infty}} e^{-\frac{1}{4}\langle Q_\infty^{-1} x, x\rangle}, \quad \text{i.e.,} \quad d\gamma = g(x)\, dx. \tag{2.2}$$

Indeed, by a direct computation one can verify that $\mathcal{A}^* g = 0$ where $\mathcal{A}^* = \mathrm{Tr}(QD^2) - \langle Bx, D\rangle - \operatorname{div} B$ is the formal adjoint operator of $\mathcal{A}$. Then, if $f \in C_c^\infty(\mathbb{R}^N)$,

$$\frac{d}{dt}\int_{\mathbb{R}^N} T(t)f(x)\, d\gamma = \int_{\mathbb{R}^N} \mathcal{A}T(t)f(x)\, d\gamma = \int_{\mathbb{R}^N} T(t)f(x)\mathcal{A}^* g(x)\, dx = 0,$$

therefore

$$\int_{\mathbb{R}^N} T(t)f(x)\, d\gamma(x) = \int_{\mathbb{R}^N} f(x)\, d\gamma(x) \tag{2.3}$$

and $g(x)dx$ is an invariant measure (since $C_c^\infty(\mathbb{R}^N)$ is a core for the generator, see for example [4, Section 7], we can exploit the above computation). We define the Lebesgue and Sobolev spaces with respect to γ as follows:

$$L^p_\gamma := \Big\{ u : \mathbb{R}^N \to \mathbb{C} \text{ measurable } : \int_{\mathbb{R}^N} |u|^p\, d\gamma < \infty \Big\}, \qquad 1 \leq p < \infty,$$

$$W^{k,p}_\gamma := \Big\{ u : \mathbb{R}^N \to \mathbb{C} :\ u \in W^{k,1}_{\mathrm{loc}}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} \sum_{|\alpha|\leq k} |D^\alpha u|^p\, d\gamma < \infty \Big\}, \quad 1 \leq p < \infty,\ k \in \mathbb{N},$$

which are Banach spaces under the obvious norms

$$\|u\|_{L^p_\gamma} = \Big(\int_{\mathbb{R}^N} |u|^p d\gamma \Big)^{1/p}, \quad \|u\|_{W^{k,p}_\gamma} = \Big(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p_\gamma} \Big)^{1/p}.$$

It is proved in [14] that the embedding $W^{1,p}_\gamma \hookrightarrow L^p_\gamma$ is compact. Moreover, we observe that $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{k,p}_\gamma$, $1 \leq p < \infty$. Indeed, a simple truncation argument shows that the set of $W^{k,p}_\gamma$-functions with compact support is dense and, given $u \in W^{k,p}_\gamma$ with compact support, the usual approximating functions $\phi_\varepsilon * u$ converge to u, as $\varepsilon \to 0$, in $W^{k,p}(\mathbb{R}^N)$ and hence in $W^{k,p}_\gamma$ (here $\phi_\varepsilon(x) = \varepsilon^{-N}\phi(x/\varepsilon)$ where $\phi \in C_c^\infty(\mathbb{R}^N)$ has integral 1).

Since $|T(t)f|^p \leq T(t)(|f|^p)$ pointwise, by (2.3) $T(t)$ extends to a strongly continuous semigroup of positive contractions in $L^p_\gamma(\mathbb{R}^N)$ for every $1 \leq p < \infty$. Moreover, it is proved in [8] that $T(t)$ is *symmetric* in L^2_γ if and only if $QB^T = BQ$. We denote by $\mathcal{A}^\gamma_p$ its generator, that turns out to be a realization of $\mathcal{A}$, and we denote by $D(\mathcal{A}^\gamma_p)$ its domain. Remark that, since $Q_t < Q_\infty$ in the sense of quadratic forms, the integral in (1.4) converges for every $f \in L^p_\gamma$ and $x \in \mathbb{R}^N$, so that the extension of $T(t)$ to L^p_γ is still given by (1.4). Observe also that $D(\mathcal{A}^\gamma_p) \subset D(\mathcal{A}^\gamma_q)$ if $p \geq q$ and $\mathcal{A}^\gamma_p u = \mathcal{A}^\gamma_q u$ for $u \in D(\mathcal{A}^\gamma_p)$. A characterisation of $D(\mathcal{A}^\gamma_p)$ has been given in [40] as follows:

$$D(\mathcal{A}^\gamma_p) = W^{2,p}_\gamma. \tag{2.4}$$

This generalizes previous partial results obtained in [33] for $p = 2$ and in [11], [8] for the *symmetric* (even infinite dimensional) case.

All the results that follow in this subsection can be found in [37], unless otherwise specified. For $1 < p \leq \infty$ and for every $t > 0$, $T(t)$ maps L^p_γ into $C^\infty(\mathbb{R}^N) \cap W^{k,p}_\gamma$ for every $k \in \mathbb{N}$, see [37, Lemma 2.2]. Moreover, there exists $C = C(k,p) > 0$ such that for every $f \in L^p_\gamma$ the inequality

$$\|D^\alpha T(t) f\|_{L^p_\gamma} \leq \frac{C}{t^{|\alpha|/2}} \|f\|_{L^p_\gamma}, \qquad t \in (0,1) \tag{2.5}$$

holds for every multiindex α with $|\alpha| = k$. Observe that using (2.4) and the embedding $C_b(\mathbb{R}^N) \subset L^p_\gamma$, from Proposition 2.1 we deduce the embedding $D(A) \subset W^{2,p}_\gamma$.

Concerning the drift term, it is interesting to point out that if $1 < p < \infty$ the map $u \mapsto |x|u$ is continuous from $W^{1,p}_\gamma$ to L^p_γ. It follows, in particular, that the map $\mathcal{L}u = \langle Bx, Du\rangle$ is bounded from $W^{2,p}_\gamma$ into L^p_γ for $1 < p < \infty$: in fact

$$\|\,|x|\, Du\|_{L^p_\gamma} \leq c\|u\|_{W^{2,p}_\gamma}, \qquad u \in W^{2,p}_\gamma(\mathbb{R}^N).$$

By the compactness of the embedding $W^{1,p}_\gamma(\mathbb{R}^N) \hookrightarrow L^p_\gamma(\mathbb{R}^N)$, $1 < p < \infty$, and the fact that $T(t)$ maps L^p_γ into $W^{1,p}_\gamma$ it follows that the semigroup $T(t)$ is compact in $L^p_\gamma(\mathbb{R}^N)$, $1 < p < \infty$.

For $1 < p < \infty$ the semigroup $T(t)$ is also analytic in $L^p_\gamma(\mathbb{R}^N)$. The standard theory of analytic semigroups and the above result imply that the angle of sectoriality θ_p of $(T(t))_{t\geq 0}$ satisfies the inequality $\theta_p \leq \pi/2 - \theta$, where θ is the spectral angle of $\mathcal{A}^\gamma_p$, that in turn coincides with the spectral angle of B (the spectral angle of the generator of a contraction semigroup is the smallest angle centred at 0 and symmetric with respect to the negative real axis, which contains the spectrum). Surprisingly enough, there are situations where $\theta_2 < \pi/2 - \theta$. In these cases, the angle of sectoriality is not determined by the spectral angle of $\mathcal{A}^\gamma_p$ or, equivalently, by the spectral angle of B, see [6, Example 1].

For every $\theta \in (0, \pi]$ we define the open sector Σ_θ by

$$\Sigma_\theta := \{z \in \mathbb{C} : |\arg z| < \theta\}.$$

The following results are proved in [6, Theorem 2]. Let $1 < p < \infty$ and let $\theta_p \in (0, \frac{\pi}{2}]$ be defined by

$$\cot \theta_p = \frac{\sqrt{(p-2)^2 + p^2\kappa^2}}{2\sqrt{p-1}},$$

where $\kappa := 2\|\frac{1}{2}I + Q^{-\frac{1}{2}}Q_\infty B^T Q^{-\frac{1}{2}}\|$. Then

(i) $(T(t))_{t\geq 0}$ extends to an analytic contraction semigroup on the sector Σ_{θ_p}.

(ii) If $(T(t))_{t\geq 0}$ extends to an analytic semigroup on the sector $\Sigma_{\theta'}$ for some $\theta' \in (0, \frac{\pi}{2}]$, then $\theta' \leq \theta_p$, i.e., the angle θ_p is optimal.

In the selfadjoint case we obtain $\kappa = 0$, hence $\cot\theta_2 = 0$ and $\cot\theta_p = \dfrac{|p-2|}{2\sqrt{p-1}}$. It is worth noticing that the angle is independent of the dimension and in fact the same result holds in the infinite dimensional case as well.

The asymptotic behavior of $(T(t))_{t\geq 0}$ in L^p_γ follows from the previous considerations and the spectral results of Section 4 (b). It is clear, in fact, that $T(t)f \to Pf$ pointwise for $f \in C_b(\mathbb{R}^N)$, where Pf is the projection defined by the right hand side of (2.1). By Lebesgue theorem we obtain $\lim_{t\to\infty}\|T(t)f - Pf\|_{L^p_\gamma} = 0$ for every $f \in C_b(\mathbb{R}^N)$ and then by density for all functions $f \in L^p_\gamma$. On the other hand, 0 is an eigenvalue of $\mathcal{A}_p$ for $p > 1$, the other eigenvalues have negative real parts and the semigroup is analytic (and also comapct), see Subsection 3.2. All together, this implies that $T(t)$ converges to the projection P exponentially as $t \to \infty$ in the operator norm.

Let us show how L^q estimates for the associated parabolic problems follow from the above properties, by general theorems. We recall that an analytic semigroup $S(\cdot)$ on a Banach space X with generator L has *maximal regularity of type* L^q $(1 < q < \infty)$ if for each $f \in L^q([0,T],X)$ the function $t \mapsto u(t) = \int_0^t S(t-s)f(s)\,ds$ belongs to $W^{1,q}([0,T],X) \cap L^q([0,T],D(L))$. This means that the mild solution of the evolution equation

$$u'(t) = Lu(t) + f(t), \quad t > 0, \qquad u(0) = 0,$$

is in fact a strong solution and has the best regularity one can expect. It is known that this property does not depend on $1 < q < \infty$ and $T > 0$.

Let $X = L^p_\gamma$, $1 < p < \infty$, and denote by $\|\cdot\|_p$ the operator norm in L^p_γ and $L = \mathcal{A} - I$. Then L has maximal regularity of type L^q if its imaginary powers are bounded operators and satisfy $\|(-L)^{is}\|_p \leq Me^{a|s|}$ for some $a \in [0,\pi/2)$ and all $s \in \mathbb{R}$ thanks to the Dore–Venni theorem, see e.g. [1, Theorem II.4.10.7]. If $p = 2$, since L is maximal dissipative and invertible, then $\|(-L)^{is}\|_2 \leq Me^{\pi|s|/2}$ by a result due to Kato, [27, Theorem 5]. However we know that $e^{i\phi}L$ is maximal dissipative and invertible for some $\phi \in (0,\pi/2]$, by point (i) above. Kato's result applied to $e^{i\phi}L$ then yields $\|(-L)^{is}\|_2 \leq Me^{a'|s|}$ for $a' = \pi/2 - \phi$.

When $p \neq 2$ we first note that, since L generates a positive contraction semigroup on L^r_γ, for every $1 < r < \infty$, then $\|(-L)^{is}\|_r \leq M_\varepsilon \exp((\varepsilon + \pi/2)|s|)$ for each $\varepsilon > 0$ and $s \in \mathbb{R}$ because of the transference principle [10, §4], see also [9, Theorem 5.8]. Interpolating between the L^2 and the L^r estimates we obtain $\|(-L)^{is}\|_p \leq Me^{a|s|}$ for some $a \in [0,\pi/2)$. We have therefore proved the following result.

Proposition 2.3 *$\mathcal{A}^\gamma_p$ has maximal regularity of type L^q on L^p_γ, for all $1 < p, q < \infty$.*

3 The spectrum of the operator

In this section we describe the spectrum of the operator in the spaces $L^p(\mathbb{R}^N)$ with respect to the Lebesgue measure and in the spaces L^p_γ with respect to the invariant measure. The spectra are very different, as the operators $\mathcal{A}_p$ and $\mathcal{A}^\gamma_p$ have very different properties.

3.1 The spectrum in $L^p(\mathbb{R}^N)$

The spectrum of the Ornstein-Uhlenbeck operator $\mathcal{A}_p$ in $L^p(\mathbb{R}^N)$ has been computed in [36] under some restrictions on the spectrum of the matrix B and in [21] in full generality.

Let us consider the drift operator

$$\mathcal{L} = \sum_{i,j=1}^{N} b_{ij} x_j D_i = \langle Bx, D\rangle,$$

where B is the drift matrix in (1), and its realization $\mathcal{L}_p$ in $L^p(\mathbb{R}^N)$ $(1 \le p \le \infty)$ with domain

$$D_p(\mathcal{L}) = \{u \in L^p(\mathbb{R}^N) : \mathcal{L}u \in L^p(\mathbb{R}^N)\} \quad 1 \le p < \infty,$$

and

$$D_\infty(\mathcal{L}) = \{u \in C_0(\mathbb{R}^N) : \mathcal{L}u \in C_0(\mathbb{R}^N)\}$$

where $\mathcal{L}u$ is understood in the sense of distributions. The operator $\mathcal{L}_p$ is closed in $L^p(\mathbb{R}^N)$ and it is the generator of the C_0-group

$$S(t)f(x) = f(e^{tB}x)$$

for $f \in L^p(\mathbb{R}^N), t \in \mathbb{R}$, see (1.2). Moreover $C_c^\infty(\mathbb{R}^N)$ is a core of $\mathcal{L}_p$ and

$$\|S(t)f\|_p = e^{-\frac{\mathrm{tr(B)}}{p}t}\|f\|_p \tag{3.1}$$

for every $f \in L^p(\mathbb{R}^N)$.

In the following theorem, proved in [36], we characterize the spectrum of $\mathcal{L}_p$ for $1 \le p \le \infty$, with the agreement that $L^\infty(\mathbb{R}^N)$ stands for $C_0(\mathbb{R}^N)$. The cases of $BUC(\mathbb{R}^N)$ and $C_b(\mathbb{R}^N)$ are partially treated in [36, Cor. 6.3] and in [28, Thm. 10.2.7] under the assumption $\sigma(B) \cap i\mathbb{R} = \emptyset$.

Theorem 3.1 *1. If $\mathrm{tr}(B) \neq 0$ then $\sigma(\mathcal{L}_p) = -\mathrm{tr}(B)/p + i\,\mathbb{R}$.*

2. *If $\mathrm{tr}(B) = 0$ and B is not similar to a diagonal matrix with purely imaginary eigenvalues, then $\sigma(\mathcal{L}_p) = i\,\mathbb{R}$.*

3. *If B is similar to a diagonal matrix with purely imaginary nonzero eigenvalues $\pm i\sigma_1, \pm i\sigma_2, \dots, \pm i\sigma_k$ and possibly 0, and there are eigenvalues σ_r, σ_s such that $\sigma_r\sigma_s^{-1} \notin \mathbb{Q}$, then $\sigma(\mathcal{L}_p) = i\,\mathbb{R}$.*

4. *If B is similar to a diagonal matrix with purely imaginary nonzero eigenvalues $\pm i\sigma_1, \pm i\sigma_2, \dots, \pm i\sigma_k$ and possibly 0 and $\sigma_r\sigma_s^{-1} \in \mathbb{Q}$ for every r, s, then $(S(t))_{t\in\mathbb{R}}$ is periodic and $\sigma(\mathcal{L}_p)$ is the discrete subgroup $G = \{i(n_1\sigma_1 + \dots + n_k\sigma_k) : (n_1, \dots, n_k) \in \mathbb{Z}^k\}$.*

The following complete description of the spectrum of $\mathcal{A}$ in $L^p(\mathbb{R}^N)$ has been recently proved in [21].

Theorem 3.2 *The spectrum of $\mathcal{A}_p$ is given by*

$$\sigma(\mathcal{A}_p) = (-\infty, 0] + \sigma(\mathcal{L}_p).$$

In particular, either $\sigma(\mathcal{A}_p) = (-\infty, 0] + G$ or $\sigma(\mathcal{A}_p) = \{\gamma \in \mathbb{C} \mid \mathrm{Re}\,\gamma \le -\mathrm{tr}(B)/p\}$, according to $\sigma(\mathcal{L}_p)$ being a discrete subgroup G of $i\mathbb{R}$ or the whole line $-\mathrm{tr}(B)/p + i\mathbb{R}$.

Notice that if $B = B^T$ and $QB = BQ$ the result is easily proved, as by a linear change of variables in $\mathbb{R}^N$ the operator can be written in the form $\Delta - \langle \tilde{B}, D \rangle$ with a diagonal matrix $\tilde{B}$ and therefore it can be studied reducing to the 1-dimensional case.

The proof of the general case consists in a scaling procedure leading to a new operator $\mathcal{C}$ in the limit which is the sum of an Ornstein-Uhlenbeck operator in one or two variables and a drift operator acting in the remaining ones. Then, the scaling and the limit allow us to get rid of the upper off-diagonal blocks of the drift matrix of $\mathcal{A}$ and to separate the variables. We can recover the spectrum of $\mathcal{A}_p$ from that of the limit operator $\mathcal{C}$. The main part of the proof is thus devoted to the investigation of the spectrum of $\mathcal{C}$. Here we can assume that B has an eigenvalue with nonnegative real part, since the other case is already covered by the main result in [36]. The above splitting then reduces the problem to Ornstein-Uhlenbeck operators in $\mathbb{R}$ or in $\mathbb{R}^2$ where B has one nonnegative eigenvalue or two complex conjugate eigenvalues with nonnegative real parts. We further have to treat eigenvalues in $i\mathbb{R}$ and with positive real part separately. The detailed study of these four cases is mainly based on the construction of approximate eigenfuctions.

3.2 The spectrum in L^p_γ

In this section, following [37], we describe the spectum of the realization of $\mathcal{A}$ in L^p_γ, $1 \le p < \infty$. The following estimate is the main step to show that the eigenfunctions of $\mathcal{A}^\gamma_p$ are polynomials. We define $s(B) = \sup\{\mathrm{Re}\ \lambda :\ \lambda \in \sigma(B)\} < 0$, see Subsection 3 2.3.

Let $k \in \mathbb{N}$ and $\varepsilon > 0$ be such that $s(B) + \varepsilon < 0$. Then there exists $C = C(k, \varepsilon)$ such that for every $u \in W^{k,p}_\gamma$

$$\sum_{|\alpha|=k} \|D^\alpha T(t)u\|_p \le Ce^{tk(s(B)+\varepsilon)} \sum_{|\alpha|=k} \|D^\alpha u\|_p, \quad t \ge 0. \tag{3.1}$$

Let us first assume $1 < p < \infty$. It follows from (3.1) that all the eigenfunctions of $\mathcal{A}^\gamma_p$ in L^p_γ are polynomials. Indeed, if $u \in D(\mathcal{A}^\gamma_p)$ is an eigenfunction with eigenvalue λ, as $T(t)u = e^{\lambda t}u$, from the results recalled in Subsection 2 2.3, we know that $u \in W^{k,p}_\gamma \cap \mathcal{C}^\infty(\mathbb{R}^N)$ for every k and $D^\alpha T(t)u = e^{\lambda t} D^\alpha u$ for every multiindex α. Given $\varepsilon \in (0, |s(B)|)$, from (3.1) we get

$$e^{t\,\mathrm{Re}\,\lambda} \sum_{|\alpha|=k} \|D^\alpha u\|_p \le C(k,\varepsilon) e^{tk(s(B)+\varepsilon)} \sum_{|\alpha|=k} \|D^\alpha u\|_p$$

and hence $D^\alpha u = 0$ if $k|s(B)| \ge |\mathrm{Re}\,\lambda|$. Hence, u is a polynomial of degree less than or equal to $|\frac{\mathrm{Re}\,\gamma}{s(B)}|$. The case $r > 1$ is relevant for generalized eigenfunctions and follows by an induction argument.

The next step is the reduction to the drift term, which as before we denote by $\mathcal{L}u = \langle Bx, Du \rangle$.

Lemma 3.1 *The following statements are equivalent.*

(i) $\lambda \in \sigma(\mathcal{A}^\gamma_p)$.

(ii) *There exists a homogeneous polynomial* $u \ne 0$ *such that* $\mathcal{L}u = \lambda u$.

(iii) *There exists a homogeneous polynomial* $v \ne 0$ *such that* $v(e^{tB}x) = e^{\lambda t} v(x)$.

The proof is completely algebraic and is based on the observation that if u is a homogeneous polynomial then $\mathcal{L}u$ is a homogeneous polynomial of the same degree. After Lemma 3.1, the computation of the spectrum of $\mathcal{L}$ is based on the equality $u(e^{tB}x) = e^{\lambda t}u(x)$, valid if $\mathcal{L}u = \lambda u$, and the reduction of B to its Jordan canonical form. Putting all together, the following result follows.

Theorem 3.3 *Let $\lambda_1, \ldots, \lambda_r$ be the (distinct) eigenvalues of B . Then*

$$\sigma(\mathcal{A}_p^\gamma) = \Big\{\lambda = \sum_{j=1}^{r} n_j\lambda_j : n_j \in \mathbb{N} \cup \{0\}\Big\}, \qquad 1 < p < \infty.$$

Moreover, the linear span of the generalized eigenfunctions of $\mathcal{A}_p^\gamma$ is dense in L^p_γ.

As a byproduct, we get that the spectrum of $\mathcal{A}_p^\gamma$ is independent of $p \in (1, \infty)$, but this follows directly from the compactness of the resolvent. In [37, Section 4] it is also proved that all the eigenvalues of $\mathcal{A}_p^\gamma$ have index 1 if and only if the matrix B is diagonalizable; 0 is a simple eigenvalue and every eigenfunction is constant: in fact, if $u \in D(A_p^\gamma)$ and $A_p^\gamma u = 0$ then $T(t)u = u$. On the other hand

$$\lim_{t\to\infty} T(t)u = \int_{\mathbb{R}^N} u d\gamma$$

and therefore u is constant.

Let us come to the case $p = 1$. In this case, the spectrum of $\mathcal{A}_1^\gamma$ is the halfplane $\{\mathrm{Re}\lambda \leq 0\}$ and all complex numbers λ with $\mathrm{Re}\lambda < 0$ are eigenvalues. Moreover, all the eigenvalues associated with polynomial eigenfunctions are the same for all $p \geq 1$. In all other cases, the eigenfunctions in L^1_γ are not polynomials. As a consequence, for $p = 1$ the semigroup is neither compact or differentiable, analytic, norm-continuous, see e.g. [17, Chapter II, Section 4] for a discussion on the relations between the properties of a semigroup and the spectrum of its generator. Moreover, $T(t)$ does not map L^1_γ into $W^{1,1}_\gamma$: indeed, if this were the case, by the same argument used for $p > 1$ (notice that property (3.1) holds true for $p = 1$ as well), one would get that $T(t)$ maps L^1_γ into $W^{k,1}_\gamma$ for every $k \in \mathbb{N}$ and proceeding as in the $p > 1$ case it would follow that all the eigenfunctions are polynomials, which is false. This and more general semigroups are deeply studied in L^1 spaces in [16].

4 Hypercontractivity and Log-Sobolev inequalities

Given a semigroup $S(t)_{t\geq 0}$ defined on the scale of L^p spaces with respect to some probability measure μ, contractive in every L^p, $1 < p < \infty$, we recall the following definitions:

1. $S(t)$ is *hypercontractive* if for every $p > 1$ and $t > 0$ there is $q(t) > p$ such that $\|S(t)f\|_{q(t)} \leq \|f\|_p$;

2. $S(t)$ is *supercontractive* if for every $p > 1$ and $t > 0$ the inequality $\|S(t)f\|_q \leq \|f\|_p$ holds for every $q \geq p$;

3. $S(t)$ is *ultracontractive* if for every $p \geq 1$ and $t > 0$ there is $c_p(t) > 0$ such that $\|S(t)f\|_\infty \leq c_p(t)\|f\|_p$;

The infinitesimal counterpart of ultracontractivity are Sobolev inequalities, while the infinitesimal counterpart of hypercontractivity are the Log-Sobolev inequalities

$$\int |f|^2 \log |f| \, d\mu \le c\langle \mathcal{H}f, f\rangle_{L^2_\mu} + \|f\|^2_{L^2_\mu} \log \|f\|_{L^2_\mu} \qquad \forall\, f \in D(\mathcal{H}^\mu_2), \tag{4.1}$$

where $\mathcal{H} : D(\mathcal{H}^\mu_2) \subset L^2_\mu \to L^2_\mu$ is the generator of $S(t)$ in L^2_μ. These inequalities, already considered in [20], have been proved in [24] to be equivalent to hypercontractivity in the symmetric case, see [25] for more information and [15] for a historical account. More generally, replacing f with $|g|^{p/2}$ in (4.1) yields

$$\int_{\mathbb{R}^N} |g|^p \log |g| \, d\gamma \le c\frac{p}{2(p-1)} \int_{\mathbb{R}^N} \mathcal{H}g \ \text{sgn} g |g|^{p-1} d\gamma + \|g\|^p_{L^p_\gamma} \log \|g\|_{L^p_\gamma} \qquad g \in D(\mathcal{H}^\mu_p). \tag{4.2}$$

In this section we discuss these properties for the Ornstein-Uhlenbeck semigroup $T(t)$. The first hypercontractivity result on the Ornstein-Uhlenbeck semigroup has been proved in [43].

Theorem 4.1 *Consider the operator* $\mathcal{A} = \Delta - \langle x, D\rangle$ *and the related semigroup* $T(t)$*, and let* $1 < p < q < \infty$*. Then*

(i) if $q \le 1 + e^{2t}(p-1)$ *then* $T(t)$ *is a contraction from* L^p_γ *in* L^q_γ*;*

(ii) if $q > 1 + e^{2t}(p-1)$ *then* $T(t)$ *is not bounded from* L^p_γ *in* L^q_γ*.*

In [24], L. Gross proved the equivalence with (4.2) with $c = 1$ in the symmetric case. For general (even nonsymmetric) Ornstein-Uhlenbeck semigroups, hypercontractivity has been proved in [7] and [22] in infinite dimensional Hilbert spaces, under suitable conditions that in $\mathbb{R}^N$ are always fulfilled in the nondegenerate case. The following result hodls.

Theorem 4.2 *Consider the general operator* $\mathcal{A}$ *in* (1) *and the related semigroup* $T(t)$*, and let* $1 \le p < q < \infty$*. Then*

(i) if $q \le 1 + (p-1)\|Q_\infty^{-1/2} e^{tB} Q_\infty^{1/2}\|^{-2}$ *then* $T(t)$ *is a contraction from* L^p_γ *in* L^q_γ*;*

(ii) if $q > 1 + (p-1)\|Q_\infty^{-1/2} e^{tB} Q_\infty^{1/2}\|^{-2}$ *then* $T(t)$ *is not bounded from* L^p_γ *in* L^q_γ*.*

But, in the nonsymmetric case the equivalence with logarithmic Sobolev inequality does not hold anymore, see again [22] and also [18].

Notice that by Hölder's inequality the Ornstein-Uhlenbeck semigroup is ultracontractive in L^p spaces with respect to the Lebesgue measure, with $c_p(t) = \|g_t\|_{p'}$, with g_t in (1.5). On the negative side, Nelson's theorem 4.1 shows that $T(t)$ is neither supercontractive or ultracontractive in L^p spaces endowed with the invariant measure. This can be easily shown also by an elementary argument, which we present in the simplest case $N = 1$, $\mathcal{A} = D^2 - xD$. Notice that if a semigroup $S(t)$ is given by an integral kernel $k(t, x, y)$ then condition (iii) above for ultracontractivity is equivalent to the bound $|k(t, x, y)| \le c_1(t)$ for the integral kernel, see e.g. [25, Remark 5.5]. In the present case, $Q_t = \frac{1}{2}(1 - e^{-2t})$ and setting $y = \sqrt{1 - e^{-2t}}z$ formula (1.6) reads

$$T(t)f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(e^{-t}x + \sqrt{1 - e^{-2t}}z) e^{-z^2/2} \, dz$$

whence by the further change of variables $y = e^{-t}x + \sqrt{1-e^{-2t}}z$ we get

$$T(t)f(x) = \int_{\mathbb{R}} f(y)p(t,x,y)\, d\gamma(y),$$

where $d\gamma(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$ is the standard Gaussian measure and

$$p(t,x,y) = \frac{1}{\sqrt{1-e^{-2t}}} \exp\Big\{-\frac{1}{2e^{2t}(1-e^{-2t})}(y^2 - 2e^t xy + x^2)\Big\}.$$

Therefore, $\sup_y p(t,x,y) = e^{x^2/2}/\sqrt{1-e^{-2t}}$ and

$$\sup\Big\{\|T(t)f\|_\infty : \ \|f\|_{L^1_\gamma} \leq 1\Big\} = \sup_{x,y\in\mathbb{R}} p(t,x,y) = +\infty.$$

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