Scuola Normale Superiore

Classe di Scienze Matematiche, Fisiche e Naturali

A Distributional Approach to Fractional Sobolev Spaces and Fractional Variation

Tesi di Perfezionamento in Matematica

Candidato:

Giorgio Stefani

Relatore:

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Anno Accademico 2019 – 2020

Pisa, 13 Febbraio 2020

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Abstract. In this thesis, we present the distributional approach to fractional Sobolev spaces and fractional variation developed in [**20**, **22**, **23**]. The new space $BV^\alpha(\mathbb{R}^n)$ of functions with bounded fractional variation in $\mathbb{R}^n$ of order $\alpha \in (0,1)$ is distributionally defined by exploiting suitable notions of fractional gradient and fractional divergence already existing in the literature. In analogy with the classical BV theory, we give a new notion of set E of (locally) finite fractional Caccioppoli α-perimeter and we define its fractional reduced boundary $\mathscr{F}^\alpha E$. We are able to show that $W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ continuously and, similarly, that sets with (locally) finite standard fractional α-perimeter have (locally) finite fractional Caccioppoli α-perimeter, so that our theory provides a natural extension of the known fractional framework. We first extend De Giorgi's Blow-up Theorem to sets of locally finite fractional Caccioppoli α-perimeter, proving existence of blow-ups and giving a first characterisation of these (possibly non-unique) limit sets. We then prove that the fractional α-variation converges to the standard De Giorgi's variation both pointwise and in the Γ-limit sense as $\alpha \to 1^-$ and, similarly, that the fractional β-variation converges to the fractional α-variation both pointwise and in the Γ-limit sense as $\beta \to \alpha^-$ for any given $\alpha \in (0,1)$. Finally, by exploiting some new interpolation inequalities on the fractional operators involved, we prove that the fractional α-gradient converges to the Riesz transform as $\alpha \to 0^+$ in L^p for $p \in (1,+\infty)$ and in the Hardy space and that the α-rescaled fractional α-variation converges to the integral mean of the function as $\alpha \to 0^+$.

“Quemadmodum” — inquit — “magnus luctator est, non qui omnes numeros nexusque perdidicit (quorum usus sub adversario rarus est), sed qui in uno se aut altero bene ac diligenter exercuit et eorum occasiones intentus expectat, neque enim refert quam multa sciat, si scit quantum victoriae satis est; sic in hoc studio multa delectant, pauca vincunt.”

“The great fighter” — he said — “is not he who thoroughly knows all the moves and all the catches (which are rarely used in actual fights), but he who has well and diligently trained in one or two of them and has carefully examined their possibilities, since it is not important he knows a lot, if he knows what is needed for the victory. Similarly, in this study many notions are interesting, but only few really matter.”

Lucius Annaeus Seneca, *De Beneficiis*, Liber VII, 1–4

To my mother Sonia,

the greatest fighter I have ever met.

Contents

Introduction

The definition of variation. Beyond its initial motivation related to the connection with Dirichlet's test for the convergence of Fourier series on $\mathbb{R}$ pointed out by C. Jordan [**53**], the modern definition of function of bounded variation dates back to E. De Giorgi [**30**] and G. Fichera [**40**] (see [**6**, Section 3.12] and the references therein for a historical account).

Very closely to Fichera's original idea [**40**], given an open set $\Omega \subset \mathbb{R}^n$, the *variation* of a function $f \in L^1_{\mathrm{loc}}(\Omega)$ in an open set $A \subset \Omega$ is given by

$$|Df|(A) = \left\{ \int_\Omega f \operatorname{div}\varphi \, dx : \varphi \in C^\infty_c(\Omega; \mathbb{R}^n),\ \operatorname{supp}\varphi \subset A,\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 \right\}. \tag{I.1}$$

If $|Df|(A) < +\infty$ for any open set $A \Subset \Omega$, then f has *locally finite variation* in Ω and we write $f \in BV_{\mathrm{loc}}(\Omega)$. The *space of functions of bounded variation* on Ω,

$$BV(\Omega) = \left\{ f \in L^1(\Omega) : |Df|(\Omega) < +\infty \right\}, \tag{I.2}$$

is tightly connected with Schwartz's Theory of Distributions since, thanks to Riesz's Representation Theorem, a function $f \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if its *distributional derivative* is representable by a finite n-vector valued Radon measure on Ω, $Df \in \mathscr{M}(\Omega; \mathbb{R}^n)$, i.e.,

$$\int_\Omega f \operatorname{div}\varphi \, dx = -\int_\Omega \varphi \cdot dDf \quad \text{for all } \varphi \in C^\infty_c(\Omega; \mathbb{R}^n). \tag{I.3}$$

The equality in (I.3) naturally generalises the integration-by-part formula available for functions in $C^1_c(\Omega)$ and immediately shows that the Sobolev space $W^{1,1}(\Omega)$ is continuously embedded in $BV(\Omega)$, precisely

$$f \in W^{1,1}(\Omega) \iff f \in BV(\Omega) \text{ with } Df = \nabla f \, \mathscr{L}^n. \tag{I.4}$$

Interestingly, the original definition by E. De Giorgi in [**30**] does not follow the above distributional approach but, instead, relies on the *heat flow*. Letting

$$\mathsf{p}_t(x) = \frac{1}{(4\pi t)^{n/2}} \, e^{\frac{-|x|^2}{4t}} \quad \text{for all } t > 0,\ x \in \mathbb{R}^n,$$

be the *heat kernel*, a function $f \in L^1(\mathbb{R}^n)$ belongs to $BV(\mathbb{R}^n)$ if and only if

$$I(f) = \lim_{t \to 0^+} \int_{\mathbb{R}^n} |\nabla \mathsf{P}_t f| \, dx < +\infty, \tag{I.5}$$

where $\mathsf{P}_t f = f * \mathsf{p}_t$. In this case, it actually holds that $I(f) = |Df|(\mathbb{R}^n)$. Note that the limit in (I.5) always exists since, thanks to the *semigroup property* of the heat flow,

$$|\nabla \mathsf{P}_{s+t} f| = |\nabla \mathsf{P}_s \mathsf{P}_t f| = |\mathsf{P}_s \nabla \mathsf{P}_t f| \le \mathsf{P}_s |\nabla \mathsf{P}_t f| \quad \text{for all } s, t > 0, \tag{I.6}$$

so that the function

$$(0,+\infty) \ni t \mapsto \int_{\mathbb{R}^n} |\nabla \mathsf{P}_t f|\, dx$$

is non-increasing.

A metric-measure approach to variation and $\mathrm{RCD}(K,\infty)$ **spaces.** An important feature of BV functions is that they can be approximated (in energy) with smooth functions. In fact, as proved in [**12**], a function $f \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset C^\infty(\Omega)$ such that $f_k \to f$ in $L^1(\Omega)$ as $k \to +\infty$ and

$$L = \lim_{k\to+\infty} \int_\Omega |\nabla f_k|\, dx < +\infty. \tag{I.7}$$

In this case, the least constant L in (I.7) is precisely $|Df|(\Omega)$.

As observed by M. Miranda Jr. [**66**], the approximation in (I.7) makes sense also in a metric measure space $(X,\mathrm{d},\mathfrak{m})$ once the concept of smooth function is replaced by the one of Lipschitz function. Precisely, given $f \in L^1(X,\mathfrak{m})$, one can consider the quantity

$$|\mathrm{D}f|(\Omega) = \inf\left\{\liminf_{k\to+\infty} \int_\Omega |\mathrm{D}f_k|\, dx : f_k \in \mathrm{Lip}_{\mathrm{loc}}(\Omega),\ f_k \to f \text{ in } L^1(\Omega,\mathfrak{m})\right\} \tag{I.8}$$

for any open set $\Omega \subset X$, where

$$|\mathrm{D}f|(x) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{\mathrm{d}(x,y)}, \quad x \in X,$$

is the *slope* of $f \in \mathrm{Lip}(X)$ and plays the same role of the modulus of the gradient in $\mathbb{R}^n$. Thus $f \in BV(X,\mathrm{d},\mathfrak{m})$ if and only if $|\mathrm{D}f|(X) < +\infty$ and, in this case, the map $\Omega \mapsto |\mathrm{D}f|(\Omega)$ is the restriction to open sets of a finite Borel measure, the *total variation measure* $|\mathrm{D}f| \in \mathscr{M}(X)$.

With this general metric-measure setting in mind, a natural question is whether the variation defined in (I.8) can be recovered also via De Giorgi's heat-flow approach.

Similarly to (I.8) and accordingly to the seminal paper [**21**] (see [**7**–**9**] for a general introduction), the *Dirichlet–Cheeger energy* of a function $f \in L^2(X,\mathfrak{m})$ is given by

$$\mathsf{Ch}(f) = \inf\left\{\liminf_{k\to+\infty} \int_X |\mathrm{D}f_k|^2\, d\mathfrak{m} : f_k \in \mathrm{Lip}_b(X),\ f_k \to f \text{ in } L^2(X,\mathfrak{m})\right\}.$$

Under some very general assumptions on $(X,\mathrm{d},\mathfrak{m})$, the functional $\mathsf{Ch}\colon L^2(X,\mathfrak{m}) \to [0,+\infty]$ is densely defined, lower semicontinuous and convex, so that its gradient flow $(0,+\infty) \ni t \mapsto f_t = \mathsf{P}_t f$ starting from an initial datum $f \in L^2(X,\mathfrak{m})$ in the Hilbertian space $L^2(X,\mathfrak{m})$ provides a natural definition of the *heat flow* in the metric-measure setting according to the general approach developed in [**18**]. Note that, at this level of generality, the finiteness domain $W^{1,2}(X,\mathrm{d},\mathfrak{m})$ of Ch, endowed with the natural norm $\sqrt{\|\cdot\|^2_{L^2(X,\mathfrak{m})} + \mathsf{Ch}(\cdot)}$, is a Banach space that may not be a Hilbert space and the heat flow $(\mathsf{P}_t)_{t>0}$ may not be linear, see [**9**, Remark 4.6].

In order to provide a consistent extension of (I.5) in a non-smooth space, one would like to have a suitable replacement of (I.6) in the metric-measure setting. If the ambient space X is a Riemannian manifold (M, g) with $\mathrm{d} = \mathrm{d}_{\mathrm{g}}$ and $\mathfrak{m} = \mathrm{vol}_{\mathrm{g}}$, and if the Ricci tensor satisfies $\mathrm{Ric}_{\mathrm{g}} \geq K$ for some $K \in \mathbb{R}$, then the following *Bakry–Émery inequality*

$$|\nabla_{\mathrm{g}} \mathsf{P}_t f| \leq e^{-tK}\, \mathsf{P}_t |\nabla_{\mathrm{g}} f| \tag{I.9}$$

holds for all $t > 0$ and $f \in C_c^\infty(M)$, see [**106**, Theorem 1.3] (note that $K = 0$ in the case $M = \mathbb{R}^n$). In a general non-smooth metric-measure space $(X, \mathrm{d}, \mathfrak{m})$, one thus would like to have a suitable replacement of the Bakry–Émery inequality (I.9) for some $K \in \mathbb{R}$. This requirement (equivalently) defines the so-called $\mathrm{CD}(K, \infty)$ *spaces* introduced by J. Lott and C. Villani [**57**] and K. T. Sturm [**100**, **101**]. However, at this level of generality, the heat flow $(\mathsf{P}_t)_{t>0}$ may still not be a linear functional, since the CD condition embeds Finsler manifolds, see [**70**]. In order to rule out this possibility, L. Ambrosio, N. Gigli and S. Savaré [**10**] introduced the notion of $\mathrm{RCD}(K, \infty)$ *spaces* adding the linearity of the heat flow to the $\mathrm{CD}(K, \infty)$ condition.

The RCD condition provides a pretty wide metric-measure setting in which the variation defined in (I.8) can be equivalently recovered via a suitable generalisation of De Giorgi's heat-flow approach. Indeed, if $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{RCD}(K, \infty)$ space, and if $f \in L^1(X, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$ for simplicity (the general case $f \in L^1(X, \mathfrak{m})$ can be recovered by a suitable truncation argument), then by the Bakry–Émery inequality it holds $\mathsf{P}_t f \in L^1(X, \mathfrak{m}) \cap \mathrm{Lip}_b(X)$. One can thus consider the quantity

$$I(f) = \limsup_{t \to 0^+} \int_X |\mathrm{D}\mathsf{P}_t f| \, d\mathfrak{m}$$

and prove that $f \in BV(X, \mathrm{d}, \mathfrak{m})$ if and only if $I(f) < +\infty$ with $I(f) = |Df|(X)$.

The variation in Carnot groups. Although $\mathrm{RCD}(K, \infty)$ spaces are a quite general setting, there exists a large variety of non-$\mathrm{RCD}(K, \infty)$ spaces, the so-called *Carnot groups*, that provide a natural framework in which the distributional definition of variation (I.1) can be suitably generalised and does coincide with the one given by De Giorgi's heat-flow approach (I.5).

A *Carnot group* $\mathbb{G}$ is a connected, simply connected and nilpotent Lie group whose Lie algebra $\mathfrak{g}$ of left-invariant vector fields has dimension n and admits a stratification of step κ,

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa$$

with

$$V_i = [V_1, V_{i-1}] \quad \text{for } i = 1, \dots, \kappa, \qquad [V_1, V_\kappa] = \{0\}.$$

Given an adapted basis of $\mathfrak{g}$, i.e. a basis $X_1, \dots, X_n$ such that

$$X_{h_{i-1}+1}, \dots, X_{h_i} \text{ is a basis of } V_i, \qquad i = 1, \dots, \kappa,$$

where $m_i = \dim(V_i)$ and $h_i = m_1 + \cdots + m_i$ for $i = 1, \dots, \kappa$, with $h_0 = 0$ and $h_\kappa = n$, the group $\mathbb{G}$ can be identified with the manifold $\mathbb{R}^n$ endowed with the group law determined by the Campbell–Hausdorff formula via exponential coordinates (in particular, the identity $e \in \mathbb{G}$ corresponds to $0 \in \mathbb{R}^n$ and $x^{-1} = -x$ for $x \in \mathbb{G}$). In addition, the Haar measure of the group $\mathbb{G}$ coincides with the n-dimensional Lebesgue measure $\mathscr{L}^n$.

The *horizontal tangent bundle* of the group $\mathbb{G}$ is the left-invariant sub-bundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$ such that $H_e\mathbb{G} = \{X(0) : X \in V_1\}$. Letting

$$\nabla_{\mathbb{G}} f = \sum_{j=1}^{m_1} (X_j f) X_j \quad \text{and} \quad \operatorname{div}_{\mathbb{G}} \varphi = \sum_{j=1}^{m_1} X_j \varphi_i$$

be the *horizontal gradient* of a function $f \in C^1(\mathbb{G})$ and the *horizontal divergence* of a vector field $\varphi \in C^1(\mathbb{G}; \mathbb{R}^{m_1})$ respectively, and imitating (I.1), the *horizontal variation* of a function $f \in L^1_{\mathrm{loc}}(\Omega)$ in a open set $\Omega \subset \mathbb{G}$ is given by

(I.10)

$$|D_{\mathbb{G}} f|(\Omega) = \left\{ \int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}} \varphi \, dx : \varphi \in C^\infty_c(\mathbb{G}; \mathbb{R}^m),\ \operatorname{supp} \varphi \subset \Omega,\ \|\varphi\|_{L^\infty(\mathbb{G}; \mathbb{R}^{m_1})} \le 1 \right\}.$$

The space

$$BV_{\mathbb{G}}(\Omega) = \left\{ f \in L^1(\Omega) : |D_{\mathbb{G}} f|(\Omega) < +\infty \right\}$$

of functions with bounded horizontal variation in Ω thus generalises the space in (I.2).

Once the horizontal tangent bundle $H\mathbb{G}$ is endowed with a left-invariant scalar product $\langle \cdot, \cdot \rangle_{\mathbb{G}}$ that makes $X_1, \dots, X_{m_1}$ an orthonormal basis, thanks to Chow–Rashevskii's Theorem the group $\mathbb{G}$ can be given the *Carnot–Carathéodory distance*

$$\mathrm{d_{cc}}(x, y) = \inf \left\{ \int_0^1 \sqrt{\langle \dot\gamma(t), \dot\gamma(t) \rangle_{\mathbb{G}}} \, dt : \gamma \text{ is horizontal},\ \gamma(0) = x,\ \gamma(1) = y \right\}, \quad x, y \in \mathbb{G}.$$

Here a *horizontal curve* $\gamma \colon [0,1] \to \mathbb{G}$ is a Lipschitz curve such that $\dot\gamma(t) \in H_{\gamma(t)}\mathbb{G}$ for a.e. $t \in [0,1]$.

Although the resulting metric-measure space $(\mathbb{G}, \mathrm{d_{cc}}, \mathscr{L}^n)$ is not a CD space any time $\mathbb{G}$ is non-commutative, see [**11**, Proposition 3.6], it shares many properties with RCD spaces. In fact, the *horizontal heat flow* $(\mathsf{P}^{\mathbb{G}}_t)_{t>0}$ naturally induced by the Carnot–Carathéodory metric structure is linear and, as proved in my paper in collaboration with L. Ambrosio [**11**], it does coincide with the *gradient flow* of the *relative entropy functional* in the *Wasserstein space* of probability measures on $\mathbb{G}$, see [**11**, Theorem 2.4]. Moreover, if we define

$$I_{\mathbb{G}}(f) = \limsup_{t \to 0^+} \int_{\mathbb{G}} |\nabla_{\mathbb{G}} \mathsf{P}^{\mathbb{G}}_t f| \, dx$$

for a function $f \in L^1(\mathbb{G})$, then $f \in BV_{\mathbb{G}}(\mathbb{G})$ if and only if $I_{\mathbb{G}}(f) < +\infty$, with

$$|D_{\mathbb{G}} f|(\mathbb{G}) \le I_{\mathbb{G}}(f) \le (1 + c_{\mathbb{G}}) |D_{\mathbb{G}} f|(\mathbb{G})$$

for some constant $c_{\mathbb{G}} \ge 0$ depending only on the group structure of $\mathbb{G}$, see [**17**, Theorem 2.11]. Note that, contrarily to the RCD setting, it is not known whether $c_{\mathbb{G}} = 0$.

From the variation to isoperimetric and minimal cluster problems. De Giorgi's heat-flow definition (I.5) of the variation is interesting also because it provides a natural link between the properties of the heat kernel and the isoperimetric problem in $\mathbb{R}^n$.

Given a set $E \subset \mathbb{R}^n$ of finite Lebesgue measure, De Giorgi's definition (I.5) suggests to study the properties of the function $\mathsf{P}_t \chi_E$ whenever $t > 0$. As observed in [**54**, **81**], if $B \subset \mathbb{R}^n$ is a ball, then the inequality

$$\|\mathsf{P}_t \chi_A\|_{L^2(\mathbb{R}^n)} \le \|\mathsf{P}_t \chi_B\|_{L^2(\mathbb{R}^n)},$$

for all $t > 0$ and all measurable sets $A \subset \mathbb{R}^n$ such that $|A| = |B|$, is equivalent to the *isoperimetric inequality*

$$P(B) \le P(A), \tag{I.11}$$

where

$$P(E) = |D\chi_E|(\mathbb{R}^n) \tag{I.12}$$

denotes the *Caccioppoli perimeter* of a measurable set $E \subset \mathbb{R}^n$. Inequality (I.11) was proved at this level of generality in [**32**] and constitutes one of the milestones of De Giorgi's scientific production after the development of the theory of finite perimeter sets.

If E has a sufficiently smooth (topological) boundary, then the perimeter functional in (I.12) coincides with the classical *surface measure* and, precisely, one can prove that

$$P(E) = \mathscr{H}^{n-1}(\partial E) \tag{I.13}$$

for all sets E with Lipschitz boundary, where $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional *Hausdorff measure*. One of the finest De Giorgi's intuitions [**31**] is that, for a finite perimeter set E with non-smooth boundary, the right 'boundary object' to keep the validity of (I.13) is a special subset of the topological boundary, the so-called *reduced boundary* $\mathscr{F}E$. With this notion in hand, a measurable set $E \subset \mathbb{R}^n$ has finite Caccioppoli perimeter if and only if $\mathscr{H}^{n-1}(\mathscr{F}E) < +\infty$, in which case we have

$$P(E) = \mathscr{H}^{n-1}(\mathscr{F}E). \tag{I.14}$$

Besides the Euclidean space, the isoperimetric property of (metric) balls has been proved also in the sphere and in the hyperbolic space endowed with their canonical Riemannian perimeter and volume, see [**83**]. However, isoperimetric sets may be not metric balls in general, as it happens for instance for a family of sub-Riemannian manifolds known as *Grushin spaces* [**43**, **68**], and the characterisation of their precise shape is a hard task even for relatively simple spaces, for example see the longstanding *Pansu conjecture* on the isoperimetric set in the Heisenberg groups [**67**, **71**].

Instead of looking for isoperimetric sets in particular spaces, one can approach the *isoperimetric problem* from a wider point of view by considering more general perimeter and volume functionals involving *densities*. Precisely, if $f\colon \mathbb{S}^{n-1} \to [0, +\infty]$ and $h\colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0, +\infty]$ are two L^1_{loc} and lower semi-continuous functions, then for any given measurable set $E \subset \mathbb{R}^n$ with locally finite perimeter one can consider

$$|E|_f = \int_E f(x)\,dx, \qquad P_h(E) = \int_{\mathscr{F}E} h(x, \nu_E(x))\,d\mathscr{H}^{n-1}(x). \tag{I.15}$$

Note that the canonical perimeter and volume functionals on a Riemannian manifold locally behave like (I.15), where f stands for the norm of the Riemannian metric and h for the norm of its derivative. The *isoperimetric problem with densities* (f, h),

$$\inf\{P_h(E) : |E|_f = v\} \quad \text{for a given volume } v > 0, \tag{I.16}$$

has gained a lot of attention in recent years, first in the case of the *single density* (i.e., when $h(x,\nu) = f(x)$ for all $(x,\nu) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$), and then in the more general case of the *double density* (I.16), see [**79**, **80**] and the references therein for an account on the most recent developments.

If two or more volumes are involved in the minimisation of the perimeter, then the admissible sets are called *bubble clusters*. Given $m \in \mathbb{N}$, an *m-bubble cluster* is a family of m pairwise disjoint sets $\mathcal{E} = \{E_i \subset \mathbb{R}^n : i = 1, \ldots, m\}$, such that $P_h(E_i) < +\infty$ and $|E_i|_f < +\infty$ for all $i = 1, \ldots, m$. Given a vector of volumes $v = (v_1, \ldots, v_m) \in (0, +\infty)^m$, a *minimal m-bubble cluster* is a solution of the *clustering problem*

$$\inf\left\{\mathcal{P}_h(\mathcal{E}) : \mathcal{E} = \bigcup_{i=1}^{m} E_i \subset \mathbb{R}^n,\ |E_i|_f = v_i \quad \forall i = 1, \ldots, m\right\}, \tag{I.17}$$

where $\mathcal{E} = \{E_i : i = 1, \ldots, m\}$ is an m-bubble cluster and

$$\mathcal{P}_h(\mathcal{E}) = \frac{1}{2} P_h\left(\bigcup_{i=1}^{m} E_i\right) + \frac{1}{2} \sum_{i=1}^{m} P_h(E_i)$$

is the (weighted) *cluster-perimeter* functional. In the physical case $f = h = 1$ and $n = 3$, J. Plateau [**77**] experimentally established that soap films are made of constant *mean curvature* smooth surfaces meeting in threes along an edge, the so-called *Plateau border*, at an angle of 120 degrees. These Plateau borders, in turn, meet in fours at a vertex at an angle of $\arccos(-\frac{1}{3}) \simeq 109.47$ degrees, the *tetrahedral angle*. Existence and regularity of minimisers of (I.17) in the Euclidean setting $f = h = 1$ were proved by F. J. Almgren Jr. [**4**]. Plateau's observations were rigorously confirmed by J. Taylor [**102**], while the planar case $n = 2$ was treated separately by F. Morgan [**69**].

When $m = 2$, problem (I.17) is the well-known *double bubble problem*. In the Euclidean setting, its solution is the so-called *standard double bubble* given by three $(n-1)$-dimensional spherical cups intersecting in an $(n-2)$-dimensional sphere at an angle of 120 degrees (for equal volumes, the central cup is in fact a flat disc). The first proof of this result for $n = 2$ was given in [**41**] exploiting the analysis carried out in [**69**] (a second proof appeared in [**35**]). The case $n = 3$ was established first in [**50**] for equal volumes and then in [**52**] with no restrictions. The case $n \geq 4$ was finally solved in [**82**].

When $m \geq 3$, problem (I.17) is still unsolved even in the Euclidean case $f = h = 1$ and presents several interesting open questions, see [**58**, Part IV]. In the planar Euclidean setting $n = 2$, the case $m = 3$ was solved in full generality in [**105**], while the case $m = 4$ has been completely understood only in the case of four equal volumes, see [**72**–**74**] .

The double bubble problem has been addressed in the n-dimensional sphere, in full generality for $n = 2$ in [**59**] and only with partial results for $n \geq 3$ in [**24**, **26**], on the 2-dimensional boundary of the cone in $\mathbb{R}^3$ in [**56**], and on the flat 2-torus in [**25**]. In my paper in collaboration with V. Franceschi [**45**], we address the double bubble problem in a 2-dimensional sub-Riemannian space, the so-called *Grushin plane*, in the case of equal volumes and prescribed *contact interface*.

Very little is known about the clustering problem (I.17) in the case of non-trivial densities. In my work in collaboration with V. Franceschi and A. Pratelli [**44**], we prove that Steiner's *120-degree property* is still valid for minimising planar clusters in the case $h(x, \nu) = h(x)$ for all $(x, \nu) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$, generalising the results of [**69**].

A new distributional approach to the variation in the fractional setting. Besides the validity of (I.14), an essential feature of De Giorgi's reduced boundary is the following *blow-up property*: if $x \in \mathscr{F}E$, then

$$\chi_{\frac{E-x}{r}} \to \chi_{H_{\nu_E(x)}} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \tag{I.18}$$

as $r \to 0$, where

$$H_{\nu_E(x)} = \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \geq 0\}, \qquad \nu_E(x) = \lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}.$$

The function $\nu_E \colon \mathscr{F}E \to \mathbb{S}^{n-1}$ denotes the so-called *measure-theoretic inner unit normal* of E and coincides with the usual inner unit normal of E when the boundary of E is sufficiently smooth. In other words, the blow-up property in (I.18) shows that, in a neighbourhood of a point $x \in \mathscr{F}E$, the finite perimeter set E is infinitesimally close to

$$x + H_{\nu_E(x)} = \{y \in \mathbb{R}^n : (y - x) \cdot \nu_E(x) \geq 0\}.$$

In this sense, the boundary of set of finite perimeter is very similar to a $(n-1)$-dimensional C^1-hypersurface in $\mathbb{R}^n$. Actually, starting from the blow-up property (I.18), De Giorgi [**32**] proved that the reduced boundary $\mathscr{F}E$ is in fact a $\mathscr{H}^{n-1}$*-rectifiable set*, meaning that there exist countably many $(n-1)$-dimensional C^1-hypersurfaces $(M_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ and compact sets $C_k \subset M_k$ such that

$$\mathscr{H}^{n-1}\left(\left(\bigcup_{k\in\mathbb{N}} C_k\right) \triangle \mathscr{F}E\right) = 0.$$

For a modern presentation of these results, see the recent monograph [**58**].

From this point of view, sets of (locally) finite Caccioppoli perimeter have C^1-type boundary regularity. However this type of regularity does not fit the Hölder-type boundary regularity of *fractal sets* naturally arising in Geometric Measure Theory, such as the *Smith–Volterra–Cantor set*, the *von Koch snowflake*, the *Sierpinski carpet*, the *Mandelbrot set* and the *Menger sponge* among the others, see [**37**,**38**,**76**]. For the study of fractal objects, Caccioppoli sets and, consequently, functions of bounded variation are a too restrictive class, and the definition of some sort of 'space of functions of fractional regularity' is in order.

Also because of the study of the wild nature of fractal sets, in the last decades *fractional Sobolev spaces* have been given an increasing attention (see [**34**, Section 1] for a detailed list of references in many research directions). If $p \in [1, +\infty)$ and $\alpha \in (0, 1)$, the *fractional Sobolev space* $W^{\alpha,p}(\mathbb{R}^n)$ is the space

$$W^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+p\alpha}}\, dx\, dy < +\infty \right\} \tag{I.19}$$

endowed with the norm

$$\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + [f]_{W^{\alpha,p}(\mathbb{R}^n)}, \quad f \in W^{\alpha,p}(\mathbb{R}^n).$$

In the *geometric regime* $p = 1$, somewhat imitating the classical case (I.12), the $W^{\alpha,1}$-seminorm naturally induces the *fractional* α*-perimeter*

$$P_\alpha(E) = [\chi_E]_{W^{\alpha,1}(\mathbb{R}^n)} = 2\int_{\mathbb{R}^n\setminus E}\int_E \frac{1}{|x-y|^{n+\alpha}}\, dx\, dy. \tag{I.20}$$

The notion of fractional α-perimeter can be also localised in the following way. If $\Omega \subset \mathbb{R}^n$ is an open set, then

$$P_\alpha(E;\Omega) = \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}}\,dx\,dy + 2\int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}}\,dx\,dy \tag{I.21}$$

is the *fractional α-perimeter of E relative to* Ω, see [**27**, Section 7].

Note that the fractional perimeter functional in (I.20) and (I.21) has a strong *non-local nature*, in the sense that its value depends also on points which are very far from the boundary of the set E. For this reason, it is not clear if such a perimeter measure may be linked with some kind of fractional analogue of De Giorgi's reduced boundary (which, a posteriori, cannot be expected to be a special subset of the topological boundary of E). In more general terms, the fractional Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$, differently from the standard Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, do not have an evident distributional nature, in the sense that the $W^{\alpha,p}$-seminorm does not seem to be the L^p-norm of some kind of distributionally defined gradient of fractional order.

Recently, the search for a good notion of differential operator in this fractional setting has led several authors to consider the following *fractional gradient*

$$\nabla^\alpha f(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)(f(y)-f(x))}{|y-x|^{n+\alpha+1}}\,dy, \quad x \in \mathbb{R}^n, \tag{I.22}$$

where $\mu_{n,\alpha}$ is a multiplicative normalising constant controlling the behaviour of ∇^α as $\alpha \to 1^-$. For a detailed account on the existing literature on this operator, see [**90**, Section 1]. Here we only refer to [**86**–**90**, **92**–**95**] for the articles tightly connected to the present work (see also [**78**, Section 15.2]). According to [**90**, Section 1], it is interesting to notice that [**51**] seems to be the earliest reference for the operator defined in (I.22).

A fundamental aspect of the fractional gradient in (I.22) is that it satisfies three natural 'qualitative' requirements as a fractional operator: *invariance* under translations and rotations, *homogeneity of order α* under dilations and some *continuity* properties in an appropriate functional space, e.g. Schwartz space $\mathcal{S}(\mathbb{R}^n)$. As observed by M. Šilhavý in [**92**, Theorem 2.2], these three requirements actually characterise the fractional gradient in (I.22), up to multiplicative constants (in fact, observing that $\nabla^\alpha = (-\Delta)^{\frac{\alpha}{2}} R$ on Schwartz functions, where $(-\Delta)^{\frac{\alpha}{2}}$ is the *fractional Laplacian* and R is the *Riesz transform*, this can be recovered from from [**96**, Chapter III, Proposition 2]). This characterisation shows that the definition in (I.22) is well posed not only from a *mathematical* point of view, but also from a *physical* point of view.

From its very definition, it is not difficult to see that the fractional gradient in (I.22) is well defined as an element of $L^1(\mathbb{R}^n;\mathbb{R}^n)$ for functions in $W^{\alpha,1}(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} |\nabla^\alpha f(x)|\,dx \le \mu_{n,\alpha}\,[f]_{W^{\alpha,1}(\mathbb{R}^n)}. \tag{I.23}$$

Moreover, the operator in (I.22) allows for the following fractional integration-by-part formula

$$\int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha\varphi\,dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f\,dx \tag{I.24}$$

whenever $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, where

$$\operatorname{div}^\alpha \varphi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}}\, dy, \quad x \in \mathbb{R}^n, \tag{I.25}$$

comes naturally as the *fractional divergence* of the test vector field $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, see [**92**, Section 6].

With the energetic estimate (I.23) and the integration-by-part formula (I.24) at disposal, one is now tempted to approach fractional Sobolev spaces in distributional terms. This is the new perspective introduced in my paper [**22**] in collaboration with G. E. Comi, where we combine the functional approach of [**89**, **90**] with the distributional point of view of [**92**] in order to develop a satisfactory extension of the variation and the Caccioppoli perimeter in the fractional setting.

Imitating (I.1), for a given $\alpha \in (0,1)$ the *fractional α-variation* of a function $f \in L^1(\mathbb{R}^n)$ is given by

$$|D^\alpha f|(\mathbb{R}^n) = \sup\left\{ \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1 \right\}. \tag{I.26}$$

If $|D^\alpha f|(\mathbb{R}^n) < +\infty$, then f has finite fractional α-variation and

$$BV^\alpha(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < +\infty \right\} \tag{I.27}$$

endowed with the natural norm

$$\|f\|_{BV^\alpha(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)} + |D^\alpha f|(\mathbb{R}^n), \quad f \in BV^\alpha(\mathbb{R}^n), \tag{I.28}$$

is the naturally associated *space of functions of bounded fractional α-variation* in $\mathbb{R}^n$. Note that (I.26) is well defined, since $\operatorname{div}^\alpha\varphi \in L^\infty(\mathbb{R}^n)$ for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

The space in (I.27) is actually a Banach space and its norm (I.28) is lower semicontinuous with respect to L^1-convergence. By (I.23), (I.24) and (I.26), one immediately gets that $W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ with strict continuous embedding, in perfect analogy with (I.4). Moreover, similarly to Anzellotti–Giaquinta approximation (I.7), one can prove that the sets $C^\infty(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ are dense in energy in $BV^\alpha(\mathbb{R}^n)$.

Emulating the classical definition in (I.12), it is very natural to define the fractional analogue of the Caccioppoli perimeter using the total fractional variation in (I.26). Note that this definition is well posed, since $\operatorname{div}^\alpha\varphi \in L^1(\mathbb{R}^n)$ for all $\varphi \in W^{\alpha,1}(\mathbb{R}^n;\mathbb{R}^n)$ arguing similarly as in (I.23). One can actually prove that

$$|D^\alpha \chi_E|(\Omega) \le \mu_{n,\alpha} P_\alpha(E;\Omega) \tag{I.29}$$

for all measurable sets $E \subset \mathbb{R}^n$ such that $P_\alpha(E;\Omega) < +\infty$, so that our approach naturally includes sets with finite $W^{\alpha,1}$-perimeter. Similarly to what happened for the fractional variation (I.26), fractional Caccioppoli α-perimeter is lower semicontinuous with respect to L^1_{loc}-convergence. Moreover, a *fractional isoperimetric inequality* holds, in the sense that for all $n \ge 2$ one has

$$|E|^{\frac{n-\alpha}{n}} \le c_{n,\alpha} |D^\alpha \chi_E|(\mathbb{R}^n) \quad \text{whenever } \chi_E \in BV^\alpha(\mathbb{R}^n).$$

Last but not least, a natural analogue of De Giorgi's reduced boundary, which we call *fractional reduced boundary* $\mathscr{F}^\alpha E$, is well posed for any set E with (locally) finite fractional Caccioppoli α-perimeter. In addition, one can prove that $|D^\alpha \chi_E| \ll \mathscr{H}^{n-\alpha} \llcorner$

$\mathscr{F}^\alpha E$ and that, given $x \in \mathscr{F}^\alpha E$, the family $(\frac{E-x}{r})_{r>0}$ admits limit points in the L^1_{loc}-topology and any such limit point must have constant *measure-theoretic inner unit fractional normal*.

We remark that a different approach to fractional variation was developed in [**107**]. We do not know if the fractional variation defined in (I.26) is linked to the one introduced in [**107**] and it would be very interesting to establish a connection between the two.

In order to study the relation between the fractional Sobolev space (I.19) and the fractional gradient (I.22) in the case $p > 1$, guided by the fractional integration-by-part formula (I.24), one can define the *weak fractional α-gradient* of a function $f \in L^p(\mathbb{R}^n)$, with $p \in [1, +\infty]$, as the function $\nabla^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \nabla^\alpha f \cdot \varphi \, dx$$

for all $\varphi \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$. For $\alpha \in (0,1)$ and $p \in [1, +\infty]$, one can thus define the *distributional fractional Sobolev space*

$$S^{\alpha,p}(\mathbb{R}^n) = \{ f \in L^p(\mathbb{R}^n) : \exists \, \nabla^\alpha f \in L^p(\mathbb{R}^n; \mathbb{R}^n) \}$$

naturally endowed with the norm

$$\|f\|_{S^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\, \mathbb{R}^n)}, \quad f \in S^{\alpha,p}(\mathbb{R}^n).$$

In the case $p \in (1, +\infty)$, it is known that $S^{\alpha,p}(\mathbb{R}^n) \supset L^{\alpha,p}(\mathbb{R}^n)$ with continuous embedding, where $L^{\alpha,p}(\mathbb{R}^n)$ is the *Bessel potential space* of parameters $\alpha \in (0,1)$ and $p \in (1, +\infty)$, see [**22**, Section 3.9] and the references therein. In my paper [**20**] in collaboration with E. Bruè, M. Calzi and G. E. Comi, we prove that also the inclusion $S^{\alpha,p}(\mathbb{R}^n) \subset L^{\alpha,p}(\mathbb{R}^n)$ holds continuously, so that the spaces $S^{\alpha,p}(\mathbb{R}^n)$ and $L^{\alpha,p}(\mathbb{R}^n)$ do coincide. As a consequence, one gets the following relations: $S^{\alpha+\varepsilon,p}(\mathbb{R}^n) \subset W^{\alpha,p}(\mathbb{R}^n) \subset S^{\alpha-\varepsilon,p}(\mathbb{R}^n)$ with continuous embeddings for all $\alpha \in (0,1)$, $p \in (1, +\infty)$ and $0 < \varepsilon < \min\{\alpha, 1-\alpha\}$, see [**89**, Theorem 2.2]; $S^{\alpha,2}(\mathbb{R}^n) = W^{\alpha,2}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$, see [**89**, Theorem 2.2]; $W^{\alpha,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$ with continuous embedding for all $\alpha \in (0,1)$ and $p \in (1,2]$, see [**96**, Chapter V, Section 5.3].

The fractional Sobolev space (I.19) can be understood also as an 'intermediate space' between the space $L^p(\mathbb{R}^n)$ and the standard Sobolev space $W^{1,p}(\mathbb{R}^n)$. In fact, $W^{\alpha,p}(\mathbb{R}^n)$ can be recovered as a suitable (real) *interpolation space* between the spaces $L^p(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$, see [**14, 65, 103**]. One then naturally expects that, for a sufficiently regular function f, the fractional Sobolev seminorm $[f]_{W^{\alpha,p}(\mathbb{R}^n)}$, multiplied by a suitable renormalising constant, should tend to $\|f\|_{L^p(\mathbb{R}^n)}$ as $\alpha \to 0^+$ and to $\|\nabla f\|_{L^p(\mathbb{R}^n)}$ as $\alpha \to 1^-$. Indeed, for $p \in [1, +\infty)$, it is known that

$$\lim_{\alpha \to 0^+} \alpha \, [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = A_{n,p} \, \|f\|^p_{L^p(\mathbb{R}^n)} \tag{I.30}$$

for all $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,p}(\mathbb{R}^n)$, while

$$\lim_{\alpha \to 1^-} (1-\alpha) \, [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = B_{n,p} \, \|\nabla f\|^p_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \tag{I.31}$$

for all $f \in W^{1,p}(\mathbb{R}^n)$. Here $A_{n,p}, B_{n,p} > 0$ are two constants depending only on n, p. The limit (I.30) was proved in [**61, 62**], while the limit (I.31) was established in [**15**].

As proved in [**29**], when $p = 1$ the limit (I.31) holds in the more general case of BV functions, that is,

$$\lim_{\alpha \to 1^-} (1-\alpha)\, [f]_{W^{\alpha,1}(\mathbb{R}^n)} = B_{n,1}\, |Df|(\mathbb{R}^n) \tag{I.32}$$

for all $f \in BV(\mathbb{R}^n)$. For a different approach to the limits in (I.30) and in (I.32) based on interpolation techniques, see [**65**].

Concerning the fractional perimeter P_α given in (I.21), one has some additional information besides equations (I.30) and (I.32).

On the one hand, thanks to [**75**, Theorem 1.2], the fractional α-perimeter P_α enjoys the following fractional analogue of Gustin's *Boxing Inequality* (see [**49**] and [**39**, Corollary 4.5.4]): there exists a dimensional constant $c_n > 0$ such that, for any bounded open set $E \subset \mathbb{R}^n$, one can find a covering

$$E \subset \bigcup_{k \in \mathbb{N}} B_{r_k}(x_k)$$

of open balls such that

$$\sum_{k \in \mathbb{N}} r_k^{n-\alpha} \le c_n \alpha (1-\alpha) P_\alpha(E). \tag{I.33}$$

Inequality (I.33) bridges the two limiting behaviours given by (I.30) and (I.32) and provides a useful tool for recovering Gagliardo–Nirenberg–Sobolev and Poincaré–Sobolev inequalities that remain stable as the exponent $\alpha \in (0,1)$ approaches the endpoints.

On the other hand, by [**5**, Theorem 2], the fractional α-perimeter Γ-converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to the standard De Giorgi's perimeter as $\alpha \to 1^-$, that is, if $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, then

$$\Gamma(L^1_{\mathrm{loc}})\text{-}\lim_{\alpha \to 1^-} (1-\alpha)\, P_\alpha(E;\Omega) = 2\omega_{n-1} P(E;\Omega) \tag{I.34}$$

for all measurable sets $E \subset \mathbb{R}^n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$ (it should be noted that in [**5**] the authors use a slightly different definition of the fractional α-perimeter, since they consider the functional $\mathcal{J}_\alpha(E,\Omega) = \frac{1}{2} P_\alpha(E,\Omega)$). For a complete account on Γ-*convergence*, we refer the reader to the monographs [**16**, **28**] (throughout all the thesis, with the symbol $\Gamma(X)\text{-}\lim$ we denote the Γ-convergence in the ambient metric space X). The convergence in (I.34), besides giving a Γ-convergence analogue of the limit in (I.32), is tightly connected with the study of the regularity properties of *non-local minimal surfaces*, that is, (local) minimisers of the fractional α-perimeter P_α, see [**27**, Section 7] for an account on the latest results in this directions.

In my paper [**23**] in collaboration with G. E. Comi, we study the asymptotic behaviour of the fractional α-variation (I.26) as $\alpha \to 1^-$, both in the pointwise and in the Γ-convergence sense. We provide counterparts of the limits (I.31) and (I.32) for the fractional α-variation. Indeed, we prove that, if $f \in W^{1,p}(\mathbb{R}^n)$ for some $p \in [1, +\infty)$, then $f \in S^{\alpha,p}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$ and, moreover,

$$\lim_{\alpha \to 1^-} \|\nabla^\alpha f - \nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0. \tag{I.35}$$

In the geometric regime $p = 1$, we show that if $f \in BV(\mathbb{R}^n)$ then $f \in BV^\alpha(\mathbb{R}^n)$ for all $\alpha \in (0,1)$ and, in addition,

$$D^\alpha f \rightharpoonup Df \text{ in } \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n) \text{ and } |D^\alpha f| \rightharpoonup |Df| \text{ in } \mathscr{M}(\mathbb{R}^n) \text{ as } \alpha \to 1^- \tag{I.36}$$

and

$$\lim_{\alpha\to 1^-} |D^\alpha f|(\mathbb{R}^n) = |Df|(\mathbb{R}^n). \tag{I.37}$$

We are also able to treat the case $p = +\infty$. In fact, we prove that if $f \in W^{1,\infty}(\mathbb{R}^n)$ then $f \in S^{\alpha,\infty}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$ and, moreover,

$$\nabla^\alpha f \rightharpoonup \nabla f \quad \text{in } L^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ as } \alpha \to 1^- \tag{I.38}$$

and

$$\|\nabla f\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le \liminf_{\alpha\to 1^-} \|\nabla^\alpha f\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \tag{I.39}$$

(the symbol '$\rightharpoonup$' appearing in (I.36) and (I.38) denotes the *weak*-convergence*, see below for the notation).

Some of the above results were partially announced in [**91**]. In a similar perspective, we also refer to the work [**63**], where the authors proved convergence results for non-local gradient operators on BV functions defined on bounded open sets with smooth boundary. The approach developed in [**63**] is however completely different from the asymptotic analysis we presently perform for the fractional operator defined in (I.22), since the boundedness of the domain of definition of the integral operators considered in [**63**] plays a crucial role.

Notice that the renormalising factor $(1-\alpha)^{\frac{1}{p}}$ is not needed in the limits (I.35) – (I.39), contrarily to what happened for the limits (I.31) and (I.32). In fact, this difference should not come as a surprise, since the constant $\mu_{n,\alpha}$ encoded in the definition of the operator ∇^α in (I.22) satisfies

$$\mu_{n,\alpha} \sim \frac{1-\alpha}{\omega_n} \quad \text{as } \alpha \to 1^- \tag{I.40}$$

and thus plays a similar role of the factor $(1-\alpha)^{\frac{1}{p}}$ in the limit as $\alpha \to 1^-$.

Another relevant aspect of our approach is that convergence as $\alpha \to 1^-$ holds true not only for the total energies, but also at the level of differential operators, in the strong sense when $p \in (1,+\infty)$ and in the weak* sense for $p = 1$ and $p = +\infty$. In simpler terms, the *non-local* fractional α-gradient ∇^α converges to the *local* gradient ∇ as $\alpha \to 1^-$ in the most natural way every time the limit is well defined.

We also provide a counterpart of (I.34) for the fractional α-variation as $\alpha \to 1^-$. Precisely, we prove that, if $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, then

$$\Gamma(L^1_{\mathrm{loc}})\text{-}\lim_{\alpha\to 1^-} |D^\alpha \chi_E|(\Omega) = P(E;\Omega) \tag{I.41}$$

for all measurable set $E \subset \mathbb{R}^n$. In view of (I.29), one may ask whether the $\Gamma\text{-}\limsup$ inequality in (I.41) could be deduced from the $\Gamma\text{-}\limsup$ inequality in (I.34). In fact, by employing (I.29) together with (I.34) and (I.40), one can estimate

$$\Gamma(L^1_{\mathrm{loc}})\text{-}\limsup_{\alpha\to 1^-} |D^\alpha \chi_E|(\Omega) \le \Gamma(L^1_{\mathrm{loc}})\text{-}\limsup_{\alpha\to 1^-} \mu_{n,\alpha} P_\alpha(E,\Omega) = \frac{2\omega_{n-1}}{\omega_n} P(E,\Omega).$$

However, we have $\frac{2\omega_{n-1}}{\omega_n} > 1$ for any $n \ge 2$ and thus the $\Gamma\text{-}\limsup$ inequality in (I.41) follows from the $\Gamma\text{-}\limsup$ inequality in (I.34) only in the case $n = 1$. In a similar way, one sees that the Γ - lim inf inequality in (I.41) implies the Γ - lim inf inequality in (I.34) only in the case $n = 1$.

Besides the counterpart of (I.34), our approach allows us to prove that Γ-convergence holds true also at the level of functions. Indeed, if $f \in BV(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ is an open set such that either Ω is bounded with Lipschitz boundary or $\Omega = \mathbb{R}^n$, then

$$\Gamma(L^1)\text{-}\lim_{\alpha \to 1^-} |D^\alpha f|(\Omega) = |Df|(\Omega). \tag{I.42}$$

Again, similarly as before and thanks to the asymptotic behaviour (I.40), the renormalising factor $(1-\alpha)$ is not needed in the limits (I.41) and (I.42).

As a byproduct of the techniques developed for the asymptotic study of the fractional α-variation as $\alpha \to 1^-$, we are also able to characterise the behaviour of the fractional β-variation as $\beta \to \alpha^-$, for any given $\alpha \in (0,1)$. On the one hand, if $f \in BV^\alpha(\mathbb{R}^n)$, then

$$D^\beta f \rightharpoonup D^\alpha f \text{ in } \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n) \text{ and } |D^\beta f| \rightharpoonup |D^\alpha f| \text{ in } \mathscr{M}(\mathbb{R}^n) \text{ as } \beta \to \alpha^-$$

and, moreover,

$$\lim_{\beta \to \alpha^-} |D^\beta f|(\mathbb{R}^n) = |D^\alpha f|(\mathbb{R}^n).$$

On the other hand, if $f \in BV^\alpha(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ is an open set such that either Ω is bounded and $|D^\alpha f|(\partial\Omega) = 0$ or $\Omega = \mathbb{R}^n$, then

$$\Gamma(L^1)\text{-}\lim_{\beta \to \alpha^-} |D^\beta f|(\Omega) = |D^\alpha f|(\Omega).$$

In my paper [**20**] in collaboration with E. Bruè, M. Calzi and G. E. Comi, we study the asymptotic behaviour of the fractional α-variation (I.26) as $\alpha \to 0^+$. At least for sufficiently regular functions, as $\alpha \to 0^+$ the fractional α-gradient in (I.22) is converging to the operator

$$\nabla^0 f(x) = \mu_{n,0} \int_{\mathbb{R}^n} \frac{(y-x)(f(y)-f(x))}{|y-x|^{n+1}}\, dy, \quad x \in \mathbb{R}^n, \tag{I.43}$$

where $\mu_{n,0}$ is the limit of $\mu_{n,\alpha}$ as $\alpha \to 0^+$ (in this case, there is no renormalisation factor). The operator in (I.43) is well defined for all $f \in C_c^\infty(\mathbb{R}^n)$ and, actually, coincides with the well-known vector-valued *Riesz transform* Rf, see [**47**, Section 5.1.4] and [**96**, Chapter III]. Similarly, the fractional α-divergence in (I.25) is formally converging as $\alpha \to 0^+$ to the operator

$$\operatorname{div}^0 \varphi(x) = \mu_{n,0} \int_{\mathbb{R}^n} \frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+1}}\, dy, \qquad x \in \mathbb{R}^n,$$

which is well defined for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. In perfect analogy with (I.26) above, one can thus introduce the space $BV^0(\mathbb{R}^n)$ as the space of functions $f \in L^1(\mathbb{R}^n)$ such that the quantity

$$|D^0 f|(\mathbb{R}^n) = \sup\left\{ \int_{\mathbb{R}^n} f \operatorname{div}^0 \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1 \right\}$$

is finite. Surprisingly (and differently from the fractional α-variation), it turns out that $|D^0 f| \ll \mathscr{L}^n$ for all $f \in BV^0(\mathbb{R}^n)$. More precisely, one can actually prove that

$BV^0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, in the sense that $f \in BV^0(\mathbb{R}^n)$ if and only if $f \in H^1(\mathbb{R}^n)$, with $D^0 f = Rf\mathscr{L}^n$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$. Here

$$H^1(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) : Rf \in L^1(\mathbb{R}^n;\mathbb{R}^n) \right\}$$

is the (real) *Hardy space*, see [**97**, Chapter III] for the precise definition. The Hardy space thus comes as the right target functional space for the study of the convergence of the fractional α-variation as $\alpha \to 0^+$. One can prove that, if $f \in H^1(\mathbb{R}^n) \cap \bigcup_{\alpha\in(0,1)} W^{\alpha,1}(\mathbb{R}^n)$, then

$$\lim_{\alpha\to 0^+} \|\nabla^\alpha f - Rf\|_{L^1(\mathbb{R}^n;\,\mathbb{R}^n)} = 0. \tag{I.44}$$

Of course, if $Rf \notin L^1(\mathbb{R}^n;\mathbb{R}^n)$, that is, $f \notin H^1(\mathbb{R}^n)$, then one cannot expect strong convergence in L^1 and, instead, has to consider the asymptotic behaviour of the rescaled fractional gradient $\alpha\,\nabla^\alpha f$ as $\alpha \to 0^+$ as suggested by the limit in (I.30). Precisely, if $f \in \bigcup_{\alpha\in(0,1)} W^{\alpha,1}(\mathbb{R}^n)$, then

$$\lim_{\alpha\to 0^+} \alpha \int_{\mathbb{R}^n} |\nabla^\alpha f(x)|\,dx = n\omega_n\mu_{n,0} \left| \int_{\mathbb{R}^n} f(x)\,dx \right|.$$

In the case $p \in (1,+\infty)$, since the Riesz transform (I.43) extends to a linear continuous operator $R\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, one can prove that, if $f \in \bigcup_{\alpha\in(0,1)} W^{\alpha,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha\to 0^+} \|\nabla^\alpha f - Rf\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} = 0. \tag{I.45}$$

Moreover, since the Riesz transform (I.43) also extends to a linear continuous operator $R\colon H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$, one can prove that, if $f \in \bigcup_{\alpha\in(0,1)} HS^{1,\alpha}(\mathbb{R}^n)$, then

$$\lim_{\alpha\to 0^+} \|\nabla^\alpha f - Rf\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)} = 0. \tag{I.46}$$

Here

$$HS^{\alpha,1}(\mathbb{R}^n) = \left\{ f \in H^1(\mathbb{R}^n) : \nabla^\alpha f \in H^1(\mathbb{R}^n) \right\}$$

is the fractional Hardy–Sobolev space, see [**99**]. Since one can prove that

$$H^1(\mathbb{R}^n) \cap \bigcup_{\alpha\in(0,1)} W^{\alpha,1}(\mathbb{R}^n) = \bigcup_{\alpha\in(0,1)} HS^{\alpha,1}(\mathbb{R}^n),$$

the convergence in (I.46) is actually a refinement of (I.44).

Remarkably, the limits in (I.44), (I.45) and (I.46) follow from some new *fractional interpolation inequalities*. On the one hand, by Calderón–Zygmund Theorem, if $\alpha \in (0,1]$, then there exists a constant $c_{n,\alpha} > 0$ such that

$$|D^\beta f|(\mathbb{R}^n) \le c_{n,\alpha}\, \|f\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}}\, |D^\alpha f|(\mathbb{R}^n)^{\frac{\beta}{\alpha}} \tag{I.47}$$

for all $\beta \in [0,\alpha)$ and all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$. On the other hand, by Mihlin–Hörmander Multiplier Theorem, given $p \in (1,+\infty)$, there exists a constant $c_{n,p} > 0$ such that

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \le c_{n,p}\, \|\nabla^\gamma f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}}\, \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}} \tag{I.48}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in S^{\alpha,p}(\mathbb{R}^n)$. In the particular case $\gamma = 0$, thanks to the L^p-continuity of the Riesz transform, one also has

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le c_{n,p} \|f\|_{L^p(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta}{\alpha}} \tag{I.49}$$

for all $0 \le \beta \le \alpha \le 1$ and all $f \in S^{\alpha,p}(\mathbb{R}^n)$. In a similar way, there exists a dimensional constant $c_n > 0$ such that

$$\|\nabla^\beta f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)} \le c_n \|\nabla^\gamma f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \|\nabla^\alpha f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}} \tag{I.50}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in HS^{\alpha,1}(\mathbb{R}^n)$. Again, in the particular case $\gamma = 0$, thanks to the H^1-continuity of the Riesz transform, one also has

$$\|\nabla^\beta f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)} \le c_n \|f\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} \|\nabla^\alpha f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta}{\alpha}} \tag{I.51}$$

for all $0 \le \beta \le \alpha \le 1$ and all $f \in HS^{\alpha,1}(\mathbb{R}^n)$.

The novelty of inequalities (I.47) – (I.51) lies in the fact that, differently from the classical fractional interpolation inequalities, constants do not depend on the interpolating parameter. To achieve this stability, we adopt a direct approach exploiting the precise structure of the fractional gradient (I.22) instead of relying on complex interpolation techniques.

According to [**14**, Theorem 6.4.5(6)], for all $\alpha, \vartheta \in (0,1)$ and $p \in (1,+\infty)$ we have the following complex interpolation

$$(L^p(\mathbb{R}^n), S^{\alpha,p}(\mathbb{R}^n))_{[\vartheta]} \cong S^{\vartheta\alpha,p}(\mathbb{R}^n). \tag{I.52}$$

As a consequence, (I.52) implies that, for all $0 < \beta < \alpha < 1$ and $p \in (1,+\infty)$, there exists a constant $c_{n,\alpha,\beta,p} > 0$ such that

$$\|f\|_{S^{\beta,p}(\mathbb{R}^n)} \le c_{n,\alpha,\beta,p} \|f\|_{L^p(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} \|f\|_{S^{\alpha,p}(\mathbb{R}^n)}^{\frac{\beta}{\alpha}} \tag{I.53}$$

for all $f \in S^{\alpha,p}(\mathbb{R}^n)$. Similarly, for all $\alpha, \vartheta \in (0,1)$, we have the following complex interpolation

$$(H^1(\mathbb{R}^n), HS^{\alpha,1}(\mathbb{R}^n))_{[\vartheta]} \cong HS^{\vartheta\alpha,1}(\mathbb{R}^n), \tag{I.54}$$

and thus

$$\|f\|_{HS^{\beta,1}(\mathbb{R}^n)} \le c_{n,\alpha,\beta} \|f\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} \|f\|_{HS^{\alpha,1}(\mathbb{R}^n)}^{\frac{\beta}{\alpha}} \tag{I.55}$$

for all $f \in HS^{\alpha,1}(\mathbb{R}^n)$ (the identification in (I.54) was pointed to us by M. Calzi). Inequalities (I.53) and (I.55) suggest that, in order to obtain (I.49) and (I.51) with complex interpolation methods, one essentially should prove that the identifications (I.52) and (I.54) hold uniformly with respect to the interpolating parameter. We believe that this result may be achieved but, since we do not need this level of generality for our aims, we prefer to prove (I.48) – (I.51) in a direct way.

We do not know if inequality (I.47) can be achieved also by complex interpolation methods. In fact, we do not know even if the two spaces $(H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{[\vartheta]}$ and $BV^\vartheta(\mathbb{R}^n)$ are somehow related for $\vartheta \in (0,1)$. By [**14**, Theorems 3.5.3 and 6.4.5(1)], we have the following real interpolations

$$(L^1(\mathbb{R}^n), W^{1,1}(\mathbb{R}^n))_{\vartheta,p} \cong (L^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\vartheta,p} \cong B^\vartheta_{1,p}(\mathbb{R}^n)$$

for all $\vartheta \in (0,1)$ and $p \in [1,+\infty]$, where $B^{\vartheta}_{p,q}(\mathbb{R}^n)$ denotes the Besov space (see [**14**, Section 6.2] or [**55**, Chapter 14] for the definition). By [**14**, Theorem 4.7.1], we know that

$$(H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\vartheta,1} \subsetneq (H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{[\vartheta]} \subsetneq (H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\vartheta,\infty}$$

for all $\vartheta \in (0,1)$. Since $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ continuously, on the one side we have

$$(H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\vartheta,1} \subsetneq (L^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\vartheta,1} \cong B^{\vartheta}_{1,1}(\mathbb{R}^n) \cong W^{\vartheta,1}(\mathbb{R}^n),$$

and, on the other side,

$$(H^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\vartheta,\infty} \subsetneq (L^1(\mathbb{R}^n), BV(\mathbb{R}^n))_{\vartheta,\infty} \cong B^{\vartheta}_{1,\infty}(\mathbb{R}^n),$$

for all $\vartheta \in (0,1)$. The continuous inclusion $W^{\alpha,1}(\mathbb{R}^n) \subset BV^{\alpha}(\mathbb{R}^n)$ is strict for all $\alpha \in (0,1)$ and the inclusion $BV^{\alpha}(\mathbb{R}^n) \subset B^{\alpha}_{1,\infty}(\mathbb{R}^n)$ holds continuously for all $\alpha \in (0,1)$ and is strict for all $n \geq 2$.

Organisation of the thesis. In this thesis, we present the distributional approach to fractional Sobolev spaces and fractional variation developed in [**20**, **22**, **23**]. The material is organised as follows. In Chapter 1, after having introduced the elementary properties of the fractional operators ∇^{α} and $\operatorname{div}^{\alpha}$, we define the space of functions of bounded fractional variation $BV^{\alpha}(\mathbb{R}^n)$ and we establish its fundamental features. The spaces $BV^{\alpha,p}(\mathbb{R}^n)$, $BV^{0}(\mathbb{R}^n)$ and $S^{\alpha,p}(\mathbb{R}^n)$ are also studied. In Chapter 2, we develop the theory of sets with (locally) finite fractional Caccioppoli perimeter and prove the existence of blow-ups. In Chapter 3, we deal with the asymptotic behaviour of the fractional α-variation as $\alpha \to 1^-$, studying pointwise convergence and Γ-convergence of the fractional gradient. The asymptotic behaviour of the fractional β-variation as $\beta \to \alpha^-$ is also considered. Finally, in Chapter 4, we study the asymptotic behaviour of the fractional α-variation as $\alpha \to 0^+$, proving new fractional interpolation inequalities. An application to potential estimates involving Riesz potential in Lorentz spaces is also provided.

Notation

We denote by $\mathscr{L}^n$ and $\mathscr{H}^\alpha$ the n-dimensional Lebesgue measure and the α-dimensional Hausdorff measure on $\mathbb{R}^n$ respectively, with $\alpha \geq 0$. Unless otherwise stated, a measurable set is an $\mathscr{L}^n$-measurable set. We also use the notation $|E| = \mathscr{L}^n(E)$. All functions we consider in this paper are Lebesgue measurable, unless otherwise stated. We let $B_r(x)$ be the standard open Euclidean ball with center $x \in \mathbb{R}^n$ and radius $r > 0$. We let $B_r = B_r(0)$. Recall that $\omega_n = |B_1| = \pi^{\frac{n}{2}}/\Gamma\left(\frac{n+2}{2}\right)$ and $\mathscr{H}^{n-1}(\partial B_1) = n\omega_n$, where Γ is Euler's *Gamma function*, see [**13**].

We let $\mathrm{GL}(n) \supset \mathrm{O}(n) \supset \mathrm{SO}(n)$ be the *general linear group*, the *orthogonal group* and the *special orthogonal group* respectively. We tacitly identify $\mathrm{GL}(n) \subset \mathbb{R}^{n^2}$ with the space of invertible $n \times n$ - matrices and we endow it with the usual Euclidean distance in $\mathbb{R}^{n^2}$.

For $k \in \mathbb{N}_0 \cup \{+\infty\}$ and $m \in \mathbb{N}$, we denote by $C^k_c(\Omega;\mathbb{R}^m)$ and $\mathrm{Lip}_c(\Omega;\mathbb{R}^m)$ the spaces of C^k-regular and, respectively, Lipschitz-regular, m-vector-valued functions defined on $\mathbb{R}^n$ with compact support in Ω.

For $m \in \mathbb{N}$, we denote by $\mathcal{S}(\mathbb{R}^n;\mathbb{R}^m)$ the space of m-vector-valued Schwartz functions on $\mathbb{R}^n$. For $k \in \mathbb{N}_0 \cup \{+\infty\}$ and $m \in \mathbb{N}$, we let

$$\mathcal{S}_k(\mathbb{R}^n;\mathbb{R}^m) = \left\{ f \in \mathcal{S}(\mathbb{R}^n;\mathbb{R}^m) : \int_{\mathbb{R}^n} x^{\mathsf{a}} f(x)\,dx = 0 \text{ for all } \mathsf{a} \in \mathbb{N}_0^n \text{ with } |\mathsf{a}| \leq k \right\},$$

where $x^{\mathsf{a}} = x_1^{\mathsf{a}_1} \cdot \ldots \cdot x_n^{\mathsf{a}_n}$ for all multi-indices $\mathsf{a} \in \mathbb{N}_0^n$. See [**47**, Section 2.2] for instance.

We let $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be the Fourier transform. For its precise definition and main properties, we refer to [**47**, Section 2.2.2] for instance.

For any exponent $p \in [1,+\infty]$, we denote by

$$L^p(\Omega;\mathbb{R}^m) = \left\{ u\colon \Omega \to \mathbb{R}^m : \|u\|_{L^p(\Omega;\mathbb{R}^m)} < +\infty \right\}$$

the space of m-vector-valued Lebesgue p-integrable functions on Ω. For $p \in [1,+\infty]$, we say that $(f_k)_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ *weakly converges* to $f \in L^p(\Omega;\mathbb{R}^m)$, and we write $f_k \rightharpoonup f$ in $L^p(\Omega;\mathbb{R}^m)$ as $k \to +\infty$, if

$$\lim_{k\to+\infty} \int_\Omega f_k \cdot \varphi\,dx = \int_\Omega f \cdot \varphi\,dx \tag{N.56}$$

for all $\varphi \in L^q(\Omega;\mathbb{R}^m)$, with $q \in [1,+\infty]$ the *conjugate exponent* of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$ (with the usual convention $\frac{1}{+\infty} = 0$). Note that in the case $p = +\infty$ we make a little abuse of terminology, since the limit in (N.56) actually defines the *weak*-convergence* in $L^\infty(\Omega;\mathbb{R}^m)$.

We denote by

$$W^{1,p}(\Omega;\mathbb{R}^m) = \left\{ u \in L^p(\Omega;\mathbb{R}^m) : [u]_{W^{1,p}(\Omega;\mathbb{R}^m)} = \|\nabla u\|_{L^p(\Omega;\,\mathbb{R}^{n+m})} < +\infty \right\}$$

the space of m-vector-valued Sobolev functions on Ω, see for instance [**55**, Chapter 10] for its precise definition and main properties. We also let

$$w^{1,p}(\Omega;\mathbb{R}^m) = \left\{u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m) : [u]_{W^{1,p}(\Omega;\mathbb{R}^m)} < +\infty\right\}.$$

We denote by

$$BV(\Omega;\mathbb{R}^m) = \left\{u \in L^1(\Omega;\mathbb{R}^m) : [u]_{BV(\Omega;\mathbb{R}^m)} = |Du|(\Omega) < +\infty\right\}$$

the space of m-vector-valued functions of bounded variation on Ω, see for instance [**6**, Chapter 3] or [**36**, Chapter 5] for its precise definition and main properties. We also let

$$bv(\Omega;\mathbb{R}^m) = \left\{u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m) : [u]_{BV(\Omega;\mathbb{R}^m)} < +\infty\right\}.$$

For $\alpha \in (0,1)$ and $p \in [1,+\infty)$, we denote by

$$W^{\alpha,p}(\Omega;\mathbb{R}^m) = \left\{u \in L^p(\Omega;\mathbb{R}^m) : [u]^p_{W^{\alpha,p}(\Omega;\mathbb{R}^m)} = \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy < +\infty\right\}$$

the space of m-vector-valued fractional Sobolev functions on Ω, see [**34**] for its precise definition and main properties. We also let

$$w^{\alpha,p}(\Omega;\mathbb{R}^m) = \left\{u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m) : [u]_{W^{\alpha,p}(\Omega;\mathbb{R}^m)} < +\infty\right\}.$$

For $\alpha \in (0,1)$ and $p = +\infty$, we simply let

$$W^{\alpha,\infty}(\Omega;\mathbb{R}^m) = \left\{u \in L^\infty(\Omega;\mathbb{R}^m) : \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < +\infty\right\},$$

so that $W^{\alpha,\infty}(\Omega;\mathbb{R}^m) = C^{0,\alpha}_b(\Omega;\mathbb{R}^m)$, the space of m-vector-valued bounded α-Hölder continuous functions on Ω.

Given $\alpha \in (0,n)$, we let

$$I_\alpha f(x) = 2^{-\alpha}\pi^{-\frac{n}{2}}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \qquad x \in \mathbb{R}^n, \tag{N.57}$$

be the Riesz potential of order $\alpha \in (0,n)$ of $f \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^m)$. We recall that, if $\alpha,\beta \in (0,n)$ satisfy $\alpha+\beta < n$, then we have the following *semigroup property*

$$I_\alpha(I_\beta f) = I_{\alpha+\beta} f \tag{N.58}$$

for all $f \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^m)$. In addition, if $1 < p < q < +\infty$ satisfy

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},$$

then there exists a constant $C_{n,\alpha,p} > 0$ such that the operator in (N.57) satisfies

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n;\mathbb{R}^m)} \le C_{n,\alpha,p}\|f\|_{L^p(\mathbb{R}^n;\mathbb{R}^m)} \tag{N.59}$$

for all $f \in C^\infty_c(\mathbb{R}^n;\,\mathbb{R}^m)$. As a consequence, the operator in (N.57) extends to a linear continuous operator from $L^p(\mathbb{R}^n;\mathbb{R}^m)$ to $L^q(\mathbb{R}^n;\mathbb{R}^m)$, for which we retain the same notation. For a proof of (N.58) and (N.59), we refer the reader to [**96**, Chapter V, Section 1] and to [**48**, Section 1.2.1].

Given $\alpha \in (0,1)$, we also let

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = 2^{\alpha} \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{f(x+y) - f(x)}{|y|^{n+\alpha}}\, dy, \qquad x \in \mathbb{R}^n,$$

be the fractional Laplacian (of order α) of $f \in \mathrm{Lip}_b(\mathbb{R}^n; \mathbb{R}^m)$.

For $\alpha \in (0,1)$ and $p \in (1,+\infty)$, we let

$$\begin{aligned} L^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^m) &= (\mathrm{Id} - \Delta)^{-\frac{\alpha}{2}} (L^p(\mathbb{R}^n; \mathbb{R}^m)) \\ &= \left\{ f \in \mathcal{S}'(\mathbb{R}^n; \mathbb{R}^m) : (\mathrm{Id} - \Delta)^{\frac{\alpha}{2}} f \in L^p(\mathbb{R}^n; \mathbb{R}^m) \right\} \end{aligned} \tag{N.60}$$

be the m-vector-valued Bessel potential space with norm

$$\|f\|_{L^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^m)} = \|(\mathrm{Id} - \Delta)^{\frac{\alpha}{2}} f\|_{L^p(\mathbb{R}^n; \mathbb{R}^m)}, \quad f \in L^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^m), \tag{N.61}$$

see [**2**, Sections 7.59-7.65] for its precise definition and main properties. We also refer to [**85**, Section 27.3], where the authors prove that the space in (N.60) can be equivalently defined as the space

$$\begin{aligned} L^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^m) &= L^p(\mathbb{R}^n; \mathbb{R}^m) \cap I_\alpha(L^p(\mathbb{R}^n; \mathbb{R}^m)) \\ &= \left\{ f \in L^p(\mathbb{R}^n; \mathbb{R}^m) : (-\Delta)^{\frac{\alpha}{2}} f \in L^p(\mathbb{R}^n; \mathbb{R}^m) \right\}, \end{aligned} \tag{N.62}$$

endowed with the norm

$$\|f\|_{L^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^m)} = \|f\|_{L^p(\mathbb{R}^n; \mathbb{R}^m)} + \|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^p(\mathbb{R}^n; \mathbb{R}^m)}, \quad f \in L^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^m), \tag{N.63}$$

see [**85**, Theorem 27.3] (in particular, the two norms defined in (N.61) and in (N.63) are equivalent and so, unless otherwise stated, we will use both of them with no particular distinction). We recall that $C^\infty_c(\mathbb{R}^n)$ is a dense subset of $L^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^m)$, see [**2**, Theorem 7.63(a)] and [**85**, Lemma 27.2]. Note that the space $L^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^m)$ can be defined also for any $\alpha \geq 1$ by simply using the composition properties of the Bessel potential (or of the fractional Laplacian), see [**2**, Section 7.62]. All the properties stated above remain true also for $\alpha \geq 1$ and, moreover, $L^{k,p}(\mathbb{R}^n; \mathbb{R}^m) = W^{k,p}(\mathbb{R}^n; \mathbb{R}^m)$ for all $k \in \mathbb{N}$, see [**2**, Theorem 7.63].

For $m \in \mathbb{N}$, we denote by

$$H^1(\mathbb{R}^n; \mathbb{R}^m) = \left\{ f \in L^1(\mathbb{R}^n; \mathbb{R}^m) : Rf \in L^1(\mathbb{R}^n; \mathbb{R}^{mn}) \right\}$$

the m-vector-valued (real) Hardy space endowed with the norm

$$\|f\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} = \|f\|_{L^1(\mathbb{R}^n; \mathbb{R}^m)} + \|Rf\|_{L^1(\mathbb{R}^n; \mathbb{R}^{mn})}, \quad f \in H^1(\mathbb{R}^n; \mathbb{R}^m),$$

where Rf denotes the Riesz trasform of $f \in H^1(\mathbb{R}^n; \mathbb{R}^m)$,

$$Rf_i(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon \to 0^+} \int_{\{|y| > \varepsilon\}} \frac{y\, f_i(x+y)}{|y|^{n+1}}\, dy, \quad x \in \mathbb{R}^n,\ i = 1, \dots, m, \tag{N.64}$$

see [**48**, Sections 2.1 and 2.4.4] and [**97**, Chapter III] for a more detailed exposition. We also recall that the Riesz transform (N.64) defines a continuous operator $R\colon L^p(\mathbb{R}^n; \mathbb{R}^m) \to L^p(\mathbb{R}^n; \mathbb{R}^m)$ for any given $p \in (1,+\infty)$, see [**47**, Corollary 5.2.8], and a continuous operator $R\colon H^1(\mathbb{R}^n; \mathbb{R}^m) \to H^1(\mathbb{R}^n; \mathbb{R}^m)$, see [**97**, Chapter III, Section 5.25].

We let $\mathscr{M}(\Omega;\mathbb{R}^m)$ be the space of m-vector-valued Radon measures with finite total variation. We let

$$|\mu|(\Omega) = \sup\left\{\int_\Omega \varphi \cdot d\mu : \varphi \in C_c^\infty(\Omega;\mathbb{R}^m),\ \|\varphi\|_{L^\infty(\Omega;\,\mathbb{R}^m)} \le 1\right\}$$

be the total variation of $\mu \in \mathscr{M}(\Omega;\mathbb{R}^m)$. We say that $(\mu_k)_{k\in\mathbb{N}} \subset \mathscr{M}(\Omega;\mathbb{R}^m)$ *weakly converges* to $\mu \in \mathscr{M}(\Omega;\mathbb{R}^m)$, and we write $\mu_k \rightharpoonup \mu$ in $\mathscr{M}(\Omega;\mathbb{R}^m)$ as $k \to +\infty$, if

$$\lim_{k\to+\infty} \int_\Omega \varphi \cdot d\mu_k = \int_\Omega \varphi \cdot d\mu \tag{N.65}$$

for all $\varphi \in C_c^\infty(\Omega;\mathbb{R}^m)$. Note that we make a little abuse of terminology, since the limit in (N.65) actually defines the *weak*-convergence* in $\mathscr{M}(\Omega;\mathbb{R}^m)$.

Given $\lambda \ge 0$, we let $\mathcal{L}^{1,\lambda}(\mathbb{R}^n;\mathbb{R}^m)$ be the Morrey space of measures $\mu \in \mathscr{M}(\Omega;\mathbb{R}^m)$ such that

$$\|\mu\|_{\mathcal{L}^{1,\lambda}(\mathbb{R}^n;\mathbb{R}^m)} = \sup_{x\in\mathbb{R}^n} \sup_{r>0} |\mu|(B_r(x))\, r^{-\lambda} < +\infty.$$

In the sequel, in order to avoid heavy notation, if the elements of a function space $F(\Omega;\mathbb{R}^m)$ are real-valued (i.e. $m = 1$), then we will drop the target space and simply write $F(\Omega)$.

CHAPTER 1

A distributional approach to fractional variation

1. Šilhavý's fractional calculus

1.1. Definition of ∇^α and div^α. We recall and study the non-local operators ∇^α and div^α introduced by Šilhavý in [**92**]. We refer also to [**78**, Section 15.2] and the references therein.

Let $\alpha \in (0,1)$ and set

$$\mu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)}. \tag{1.1}$$

We let

$$\nabla^\alpha f(x) = \mu_{n,\alpha} \lim_{\varepsilon\to 0} \int_{\{|z|>\varepsilon\}} \frac{z f(x+z)}{|z|^{n+\alpha+1}}\, dz \tag{1.2}$$

be the α-*gradient* of $f \in C_c^\infty(\mathbb{R}^n)$ at $x \in \mathbb{R}^n$. We also let

$$\mathrm{div}^\alpha \varphi(x) = \mu_{n,\alpha} \lim_{\varepsilon\to 0} \int_{\{|z|>\varepsilon\}} \frac{z\cdot\varphi(x+z)}{|z|^{n+\alpha+1}}\, dz \tag{1.3}$$

be the α-*divergence* of $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ at $x \in \mathbb{R}^n$. The non-local operators ∇^α and div^α are well defined in the sense that the involved integrals converge and the limits exist, see [**92**, Section 7].

Since

$$\int_{\{|z|>\varepsilon\}} \frac{z}{|z|^{n+\alpha+1}}\, dz = 0, \qquad \forall \varepsilon > 0, \tag{1.4}$$

it is immediate to check that $\nabla^\alpha c = 0$ for all $c \in \mathbb{R}$. Moreover, the cancellation in (1.4) yields

$$\nabla^\alpha f(x) = \mu_{n,\alpha} \lim_{\varepsilon\to 0} \int_{\{|y-x|>\varepsilon\}} \frac{(y-x)}{|y-x|^{n+\alpha+1}} f(y)\, dy \tag{1.5a}$$

$$= \mu_{n,\alpha} \lim_{\varepsilon\to 0} \int_{\{|x-y|>\varepsilon\}} \frac{(y-x)(f(y)-f(x))}{|y-x|^{n+\alpha+1}}\, dy \tag{1.5b}$$

$$= \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)(f(y)-f(x))}{|y-x|^{n+\alpha+1}}\, dy, \qquad \forall x \in \mathbb{R}^n, \tag{1.5c}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. Indeed, (1.5a) follows by a simple change of variables and (1.5b) is a consequence of (1.4). To prove (1.5c) it is enough to apply Lebesgue's Dominated Convergence Theorem. Indeed, we can estimate

$$\int_{\{|y-x|\le 1\}} \left| \frac{(y-x)(f(y)-f(x))}{|y-x|^{n+\alpha+1}} \right| dy \le \mathrm{Lip}(f) \int_0^1 r^{-\alpha}\, dr \tag{1.6}$$

and

$$\int_{\{|y-x|>1\}}\left|\frac{(y-x)(f(y)-f(x))}{|y-x|^{n+\alpha+1}}\right|dy \le 2\|f\|_{L^\infty(\mathbb{R}^n)}\int_1^{+\infty} r^{-(1+\alpha)}\,dr. \tag{1.7}$$

As a consequence, the operator $\nabla^\alpha f$ defined by (1.5c) is well defined for all $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and satisfies (1.2), (1.5a) and (1.5b).

By [**92**, Theorem 4.3], ∇^α is invariant by translations and rotations and is α-homogeneous. Moreover, for all $f \in C_c^\infty(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$, we have

$$(\nabla^\alpha f(\lambda\cdot))(x) = |\lambda|^\alpha \operatorname{sgn}(\lambda)(\nabla^\alpha f)(\lambda x), \qquad x \in \mathbb{R}^n. \tag{1.8}$$

Arguing similarly as above, we can write

$$\operatorname{div}^\alpha\varphi(x) = \mu_{n,\alpha}\lim_{\varepsilon\to 0}\int_{\{|x-y|>\varepsilon\}}\frac{(y-x)\cdot\varphi(y)}{|y-x|^{n+\alpha+1}}\,dy, \tag{1.9a}$$

$$= \mu_{n,\alpha}\lim_{\varepsilon\to 0}\int_{\{|x-y|>\varepsilon\}}\frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}}\,dy, \tag{1.9b}$$

$$= \mu_{n,\alpha}\int_{\mathbb{R}^n}\frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}}\,dy, \qquad \forall x \in \mathbb{R}^n, \tag{1.9c}$$

for all $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$.

Exploiting (1.5c) and (1.9c), we can extend the operators ∇^α and $\operatorname{div}^\alpha$ to functions with $w^{\alpha,1}$-regularity.

Lemma 1.1 (Extension of ∇^α and $\operatorname{div}^\alpha$ to $w^{\alpha,1}$)**.** *Let $\alpha \in (0,1)$. If $f \in w^{\alpha,1}(\mathbb{R}^n)$ and $\varphi \in w^{\alpha,1}(\mathbb{R}^n;\mathbb{R}^n)$, then the functions $\nabla^\alpha f(x)$ and $\operatorname{div}^\alpha f(x)$ given by* (1.5c) *and* (1.9c) *respectively are well defined for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. As a consequence, $\nabla^\alpha f(x)$ and $\operatorname{div}^\alpha f(x)$ satisfy* (1.2)*,* (1.5a)*,* (1.5b) *and* (1.3)*,* (1.9a)*,* (1.9b) *respectively for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.*

Proof. Let $f \in w^{\alpha,1}(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\frac{(y-x)(f(y)-f(x))}{|y-x|^{n+\alpha+1}}\right|dy\,dx \le [f]_{W^{\alpha,1}(\mathbb{R}^n)}$$

and thus the function $\nabla^\alpha f(x)$ given by (1.5c) is well defined for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$ and satisfies (1.2), (1.5a) and (1.5b) by (1.4) and by Lebesgue's Dominated Convergence Theorem. A similar argument proves the result for any $\varphi \in w^{\alpha,1}(\mathbb{R}^n;\mathbb{R}^n)$. ☐

1.2. Equivalent definition of ∇^α and $\operatorname{div}^\alpha$ via Riesz potential. Recalling the definition in (1.1), one easily sees that

$$I_{1-\alpha}f(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1}\int_{\mathbb{R}^n}\frac{f(x+y)}{|y|^{n+\alpha-1}}\,dy$$

and

$$\nabla I_{1-\alpha}f(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1}\int_{\mathbb{R}^n}\frac{\nabla_x f(x+y)}{|y|^{n+\alpha-1}}\,dy = \frac{\mu_{n,\alpha}}{n+\alpha-1}\int_{\mathbb{R}^n}\frac{\nabla_y f(x+y)}{|y|^{n+\alpha-1}}\,dy,$$

so that

$$\nabla I_{1-\alpha}f = I_{1-\alpha}\nabla f$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. A similar argument proves that

$$\mathrm{div} I_{1-\alpha}\varphi = I_{1-\alpha}\mathrm{div}\varphi$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

Thus, accordingly to the approach developed in [**51**, **86**–**90**], we can consider the operators

$$\widetilde{\nabla^\alpha} = \nabla I_{1-\alpha}\colon C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n;\mathbb{R}^n)$$

and

$$\widetilde{\mathrm{div}^\alpha} = \mathrm{div} I_{1-\alpha}\colon C_c^\infty(\mathbb{R}^n;\mathbb{R}^n) \to C^\infty(\mathbb{R}^n).$$

We can prove that these two operators coincide with the operators defined in (1.2) and (1.3). See also [**89**, Theorem 1.2].

Proposition 1.2 (Equivalence)**.** *Let* $\alpha \in (0,1)$. *We have* $\widetilde{\nabla^\alpha} = \nabla^\alpha$ *on* $\mathrm{Lip}_c(\mathbb{R}^n)$ *and* $\widetilde{\mathrm{div}^\alpha} = \mathrm{div}^\alpha$ *on* $\mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. Let $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and fix $x \in \mathbb{R}^n$. Integrating by parts, we can compute

$$\begin{aligned}\widetilde{\nabla^\alpha} f(x) &= \frac{\mu_{n,\alpha}}{n+\alpha-1}\lim_{\varepsilon\to 0}\int_{\{|y|>\varepsilon\}} \frac{\nabla_y f(x+y)}{|y|^{n+\alpha-1}}\,dy\\ &= \mu_{n,\alpha}\lim_{\varepsilon\to 0}\int_{\{|y|>\varepsilon\}} \frac{y f(y+x)}{|y|^{n+\alpha+1}}\,dy = \nabla^\alpha f(x),\end{aligned}$$

since we can estimate

$$\begin{aligned}\left|\int_{\{|y|=\varepsilon\}} \frac{f(x+y)}{|y|^{n+\alpha-1}}\frac{y}{|y|}\,d\mathscr{H}^{n-1}(y)\right| &= \left|\int_{\{|y|=\varepsilon\}} \frac{(f(x+y)-f(x))}{|y|^{n+\alpha-1}}\frac{y}{|y|}\,d\mathscr{H}^{n-1}(y)\right|\\ &\le n\omega_n \|\nabla f\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}\varepsilon^{1-\alpha}.\end{aligned}$$

The proof of $\widetilde{\mathrm{div}^\alpha}\varphi = \mathrm{div}^\alpha\varphi$ for all $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ follows similarly. □

A useful consequence of the equivalence proved in Proposition 1.2 above is the following result.

Corollary 1.3 (Representation formula for div^α and ∇^α)**.** *Let* $\alpha \in (0,1)$. *If* $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ *then* $\mathrm{div}^\alpha\varphi \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ *with*

$$\mathrm{div}^\alpha\varphi(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1}\int_{\mathbb{R}^n}\frac{\mathrm{div}\varphi(y)}{|y-x|^{n+\alpha-1}}\,dy \tag{1.10}$$

for all $x \in \mathbb{R}^n$,

$$\|\mathrm{div}^\alpha\varphi\|_{L^1(\mathbb{R}^n)} \le \mu_{n,\alpha}[\varphi]_{W^{\alpha,1}(\mathbb{R}^n;\mathbb{R}^n)} \tag{1.11}$$

and

$$\|\mathrm{div}^\alpha\varphi\|_{L^\infty(\mathbb{R}^n)} \le C_{n,\alpha,U}\|\mathrm{div}\varphi\|_{L^\infty(\mathbb{R}^n)} \tag{1.12}$$

for any bounded open set $U \subset \mathbb{R}^n$ *such that* $\mathrm{supp}(\varphi) \subset U$, *where*

$$C_{n,\alpha,U} = \frac{n\mu_{n,\alpha}}{(1-\alpha)(n+\alpha-1)}\left(\omega_n \mathrm{diam}(U)^{1-\alpha} + \left(\frac{n\omega_n}{n+\alpha-1}\right)^{\frac{n+\alpha-1}{n}}|U|^{\frac{1-\alpha}{n}}\right). \tag{1.13}$$

Analogously, if $f \in \mathrm{Lip}_c(\mathbb{R}^n)$*, then* $\nabla^\alpha f \in L^1(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n;\mathbb{R}^n)$ *with*

$$\nabla^\alpha f(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{\nabla f(y)}{|y-x|^{n+\alpha-1}}\, dy \tag{1.14}$$

for all $x \in \mathbb{R}^n$*,*

$$\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \mu_{n,\alpha} [f]_{W^{\alpha,1}(\mathbb{R}^n)} \tag{1.15}$$

and

$$\|\nabla^\alpha f\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le C_{n,\alpha,U} \|\nabla f\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \tag{1.16}$$

for any bounded open set $U \subset \mathbb{R}^n$ *such that* $\mathrm{supp}(f) \subset U$*, where* $C_{n,\alpha,U}$ *is as in* (1.13)*.*

Proof. The representation formula (1.10) follows directly from Proposition 1.2. The estimate in (1.11) is a consequence of Lemma 1.1. Finally, if $U \subset \mathbb{R}^n$ is a bounded open set such that $\mathrm{supp}(\varphi) \subset U$, then

$$\begin{aligned} |\mathrm{div}^\alpha \varphi(x)| &\le \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} |y-x|^{1-n-\alpha}\, |\mathrm{div}\varphi(y)|\, dy \\ &\le \frac{\mu_{n,\alpha} \|\mathrm{div}\varphi\|_{L^\infty(\mathbb{R}^n)}}{n+\alpha-1} \int_U |y-x|^{1-n-\alpha}\, dy \end{aligned}$$

and (1.12) follows by Lemma 1.4 below. The proof of (1.14), (1.15) and (1.16) is similar and is left to the reader. □

Lemma 1.4. *Let* $\alpha \in (0,1)$ *and let* $U \subset \mathbb{R}^n$ *be a bounded open set. For all* $x \in \mathbb{R}^n$*, we have*

$$\int_U |y-x|^{1-n-\alpha}\, dy \le \frac{n}{1-\alpha} \left(\omega_n \mathrm{diam}(U)^{1-\alpha} + \left(\frac{n\omega_n}{n+\alpha-1} \right)^{\frac{n+\alpha-1}{n}} |U|^{\frac{1-\alpha}{n}} \right). \tag{1.17}$$

Proof. For $\delta > 0$, set $U^\delta = \{x \in \mathbb{R}^n : \mathrm{dist}(x,U) < \delta\}$. Since clearly

$$x \in U^\delta \implies B_{(\mathrm{diam}(U)+\delta)}(x) \supset U,$$

for all $x \in U^\delta$ we can estimate

$$\begin{aligned} \int_U |y-x|^{1-n-\alpha}\, dy &\le \int_{B_{(\mathrm{diam}(U)+\delta)}(x)} |y-x|^{1-n-\alpha}\, dy \\ &= n\omega_n \int_0^{\mathrm{diam}(U)+\delta} r^{-\alpha}\, dr \\ &= \frac{n\omega_n}{1-\alpha} \left(\mathrm{diam}(U)+\delta\right)^{1-\alpha}. \end{aligned}$$

On the other hand, it is plain that

$$x \notin U^\delta,\ y \in U \implies |y-x| > \delta,$$

so that for all $x \notin U^\delta$ we can estimate

$$\int_U |y-x|^{1-n-\alpha}\, dy \le \delta^{1-n-\alpha} |U|.$$

Thus, for all $\delta > 0$ and $x \in \mathbb{R}^n$, we can estimate

$$\int_U |y-x|^{1-n-\alpha}\, dy \le \frac{n\omega_n}{1-\alpha} \left(\mathrm{diam}(U)+\delta\right)^{1-\alpha} + \delta^{1-n-\alpha}|U|$$

$$\leq \frac{n\omega_n}{1-\alpha}\left(\operatorname{diam}(U)^{1-\alpha}+\delta^{1-\alpha}\right)+\delta^{1-n-\alpha}|U|$$

since the function $s \mapsto s^{1-\alpha}$ is subadditive for all $s > 0$. Thus (1.17) follows minimising in $\delta > 0$ the right-hand side. □

1.3. Relation of ∇^α and $\operatorname{div}^\alpha$ with the fractional Laplacian. Following [**92**], for any $f \in C_c^\infty(\mathbb{R}^n)$ we set

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = \begin{cases} \nu_{n,\alpha} \displaystyle\int_{\mathbb{R}^n} \frac{f(x+h)}{|h|^{n+\alpha}}\,dh & \text{if } \alpha \in (-1,0), \\ f(x) & \text{if } \alpha = 0, \\ \nu_{n,\alpha} \displaystyle\int_{\mathbb{R}^n} \frac{f(x+h)-f(x)}{|h|^{n+\alpha}}\,dh & \text{if } \alpha \in (0,1), \\ \nu_{n,\alpha} \displaystyle\lim_{\varepsilon\to 0}\int_{\{|h|>\varepsilon\}} \frac{f(x+h)-f(x)}{|h|^{n+\alpha}}\,dh & \text{if } \alpha \in [1,2), \end{cases} \tag{1.18}$$

where

$$\nu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)}. \tag{1.19}$$

Note that, in the case $\alpha \in (-1,0)$, we have

$$(-\Delta)^{\frac{\alpha}{2}} = I_{-\alpha} \quad \text{on } C_c^\infty(\mathbb{R}^n).$$

We stress the fact that this definition is consistent with the previous definitions of fractional gradient and divergence in the sense that

$$-\operatorname{div}^\alpha \nabla^\beta = (-\Delta)^{\frac{\alpha+\beta}{2}}$$

for any $\alpha \in (-1,1)$ and $\beta \in (0,1)$, see [**92**, Theorem 5.3]. Thus, in particular, we get

$$-\operatorname{div}^\alpha \nabla^\alpha = (-\Delta)^\alpha$$

for any $\alpha \in (0,1)$.

1.4. Duality and Leibniz's rules. We now study the properties of the operators ∇^α and $\operatorname{div}^\alpha$. We begin with the following duality relation, see [**92**, Section 6].

Lemma 1.5 (Duality)**.** *Let $\alpha \in (0,1)$. For all $f \in \operatorname{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ it holds*

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi\,dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f\,dx. \tag{1.20}$$

Proof. Recalling Lemma 1.1 and exploiting (1.5a) and (1.9a), we can write

$$\begin{aligned} \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi\,dx &= \mu_{n,\alpha} \int_{\mathbb{R}^n} f(x) \lim_{\varepsilon\to 0} \int_{\{|x-y|>\varepsilon\}} \frac{(y-x)\cdot\varphi(y)}{|y-x|^{n+\alpha+1}}\,dy\,dx \\ &= \mu_{n,\alpha} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \int_{\{|x-y|>\varepsilon\}} f(x)\,\frac{(y-x)\cdot\varphi(y)}{|y-x|^{n+\alpha+1}}\,dy\,dx \end{aligned}$$

$$
\begin{aligned}
&= -\mu_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{\{|x-y|>\varepsilon\}} \varphi(y) \cdot \frac{(x-y)\, f(x)}{|x-y|^{n+\alpha+1}} \, dx \, dy \\
&= -\int_{\mathbb{R}^n} \varphi(y) \cdot \nabla^\alpha f(y) \, dy
\end{aligned}
$$

by the Lebesgue's Dominated Convergence Theorem and Fubini's Theorem. □

We now prove two Leibniz-type rules for the operators ∇^α and $\operatorname{div}^\alpha$, which in particular show the strong non-local nature of these two operators.

Lemma 1.6 (Leibniz's rule for ∇^α)**.** *Let $\alpha \in (0,1)$. For all $f, g \in \operatorname{Lip}_c(\mathbb{R}^n)$ it holds*

$$
\nabla^\alpha (fg) = f \nabla^\alpha g + g \nabla^\alpha f + \nabla^\alpha_{\mathrm{NL}}(f, g),
$$

where

$$
\nabla^\alpha_{\mathrm{NL}}(f,g)(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)(f(y)-f(x))(g(y)-g(x))}{|y-x|^{n+\alpha+1}} \, dy, \quad \forall x \in \mathbb{R}^n,
$$

with $\mu_{n,\alpha}$ as in (1.1)*. Moreover, it holds*

$$
\|\nabla^\alpha_{\mathrm{NL}}(f,g)\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \mu_{n,\alpha} [f]_{W^{\frac{\alpha}{p},p}(\mathbb{R}^n)} [g]_{W^{\frac{\alpha}{q},q}(\mathbb{R}^n)}
$$

with $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and similarly

$$
\|\nabla^\alpha_{\mathrm{NL}}(f,g)\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le 2\mu_{n,\alpha} \|f\|_{L^\infty(\mathbb{R}^n)} [g]_{W^{\alpha,1}(\mathbb{R}^n)}.
$$

Proof. Given $f, g \in \operatorname{Lip}_c(\mathbb{R}^n)$, by Lemma 1.1 and by (1.5c) we have

$$
\begin{aligned}
\nabla^\alpha (fg)(x) &= \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)(f(y)g(y) - f(x)g(x))}{|y-x|^{n+\alpha+1}} \, dy \\
&= \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)(f(y)g(y) - f(y)g(x) + f(y)g(x) - f(x)g(x))}{|y-x|^{n+\alpha+1}} \, dy \\
&= \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x) f(y)(g(y)-g(x))}{|y-x|^{n+\alpha+1}} \, dy + g(x) \nabla^\alpha f(x) \\
&= \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)(f(y)-f(x))(g(y)-g(x))}{|y-x|^{n+\alpha+1}} \, dy \\
&\quad + f(x) \nabla^\alpha g(x) + g(x) \nabla^\alpha f(x).
\end{aligned}
$$

We also have that

$$
\begin{aligned}
\|\nabla^\alpha_{\mathrm{NL}}(f,g)\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} &\le \mu_{n,\alpha} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)-f(x)|}{|x-y|^{\frac{n+\alpha}{p}}} \frac{|g(y)-g(x)|}{|y-x|^{\frac{n+\alpha}{q}}} \, dy \, dx, \\
&\le \mu_{n,\alpha} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)-f(x)|^p}{|x-y|^{n+\alpha}} \, dy \, dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(y)-g(x)|^q}{|x-y|^{n+\alpha}} \, dy \, dx \right)^{\frac{1}{q}}
\end{aligned}
$$

for any $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The case $p = \infty$, $q = 1$ follows similarly. □

Lemma 1.7 (Leibniz's rule for $\operatorname{div}^\alpha$)**.** *Let $\alpha \in (0,1)$. For all $f \in \operatorname{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$ it holds*

$$
\operatorname{div}^\alpha (f\varphi) = f \operatorname{div}^\alpha \varphi + \varphi \cdot \nabla^\alpha f + \operatorname{div}^\alpha_{\mathrm{NL}}(f, \varphi),
$$

where

$$\operatorname{div}^{\alpha}_{\mathrm{NL}}(f,\varphi)(x) = \mu_{n,\alpha}\int_{\mathbb{R}^n} \frac{(y-x)\cdot(\varphi(y)-\varphi(x))(f(y)-f(x))}{|y-x|^{n+\alpha+1}}\,dy, \quad \forall x\in\mathbb{R}^n, \tag{1.21}$$

with $\mu_{n,\alpha}$ *as in* (1.1)*. Moreover, it holds*

$$\|\operatorname{div}^{\alpha}_{\mathrm{NL}}(f,\varphi)\|_{L^1(\mathbb{R}^n)} \le \mu_{n,\alpha}[f]_{W^{\frac{\alpha}{p},p}(\mathbb{R}^n)}[\varphi]_{W^{\frac{\alpha}{q},q}(\mathbb{R}^n;\mathbb{R}^n)}$$

with $p,q\in(1,\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1$ *and similarly*

$$\|\operatorname{div}^{\alpha}_{\mathrm{NL}}(f,\varphi)\|_{L^1(\mathbb{R}^n)} \le 2\mu_{n,\alpha}\|f\|_{L^\infty(\mathbb{R}^n)}[\varphi]_{W^{\alpha,1}(\mathbb{R}^n;\mathbb{R}^n)},$$
$$\|\operatorname{div}^{\alpha}_{\mathrm{NL}}(f,\varphi)\|_{L^1(\mathbb{R}^n)} \le 2\mu_{n,\alpha}\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}[f]_{W^{\alpha,1}(\mathbb{R}^n)}.$$

Proof. Given $f\in\operatorname{Lip}_c(\mathbb{R}^n)$ and $\varphi\in\operatorname{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, by Lemma 1.1 and by (1.5c) we have

$$\begin{aligned}
\operatorname{div}^{\alpha}(f\varphi)(x) &= \mu_{n,\alpha}\int_{\mathbb{R}^n}\frac{(y-x)\cdot(f(y)\varphi(y)-f(x)\varphi(x))}{|y-x|^{n+\alpha+1}}\,dy\\
&= \mu_{n,\alpha}\int_{\mathbb{R}^n}\frac{(y-x)\cdot(f(y)\varphi(y)-f(y)\varphi(x)+f(y)\varphi(x)-f(x)\varphi(x))}{|y-x|^{n+\alpha+1}}\,dy\\
&= \mu_{n,\alpha}\int_{\mathbb{R}^n}\frac{(y-x)\cdot(\varphi(y)-\varphi(x))f(y)}{|y-x|^{n+\alpha+1}}\,dy + \varphi(x)\cdot\nabla^{\alpha}f(x)\\
&= \mu_{n,\alpha}\int_{\mathbb{R}^n}\frac{(y-x)\cdot(\varphi(y)-\varphi(x))(f(y)-f(x))}{|y-x|^{n+\alpha+1}}\,dy + f(x)\operatorname{div}^{\alpha}\varphi(x)+\\
&\quad + \varphi(x)\cdot\nabla^{\alpha}f(x).
\end{aligned}$$

We also have that

$$\begin{aligned}
\|\operatorname{div}^{\alpha}_{\mathrm{NL}}(f,\varphi)\|_{L^1(\mathbb{R}^n)} &\le \mu_{n,\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)-f(x)|}{|x-y|^{\frac{n+\alpha}{p}}}\frac{|\varphi(y)-\varphi(x)|}{|y-x|^{\frac{n+\alpha}{q}}}\,dy\,dx,\\
&\le \mu_{n,\alpha}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)-f(x)|^p}{|x-y|^{n+\alpha}}\,dy\,dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\varphi(y)-\varphi(x)|^q}{|x-y|^{n+\alpha}}\,dy\,dx\right)^{\frac{1}{q}}
\end{aligned}$$

for any $p,q\in(1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$. The case $p=\infty$, $q=1$ follows similarly. □

Remark 1.8 (Extension of $\nabla^{\alpha}_{\mathrm{NL}}$ and $\operatorname{div}^{\alpha}_{\mathrm{NL}}$ to fractional Sobolev spaces)**.** Thanks to the estimates in Lemma 1.6, for all $\alpha\in(0,1)$ the bilinear operator

$$\nabla^{\alpha}_{\mathrm{NL}}\colon \operatorname{Lip}_c(\mathbb{R}^n)\times\operatorname{Lip}_c(\mathbb{R}^n)\to L^1(\mathbb{R}^n;\mathbb{R}^n)$$

can be continuously extended to a bilinear operator

$$\nabla^{\alpha}_{\mathrm{NL}}\colon w^{\frac{\alpha}{p},p}(\mathbb{R}^n)\times w^{\frac{\alpha}{q},q}(\mathbb{R}^n)\to L^1(\mathbb{R}^n;\mathbb{R}^n)$$

for any $p,q\in[1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$, for which we retain the same notation (we tacitly adopt the convention $w^{\frac{\alpha}{\infty},\infty}=L^\infty$). Analogously, because of the estimates in Lemma 1.7, the bilinear operator

$$\operatorname{div}^{\alpha}_{\mathrm{NL}}\colon \operatorname{Lip}_c(\mathbb{R}^n)\times\operatorname{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)\to L^1(\mathbb{R}^n)$$

can be continuously extended to a bilinear operator

$$\operatorname{div}^{\alpha}_{\mathrm{NL}}\colon w^{\frac{\alpha}{p},p}(\mathbb{R}^n)\times w^{\frac{\alpha}{q},q}(\mathbb{R}^n;\mathbb{R}^n)\to L^1(\mathbb{R}^n)$$

for any $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, for which we retain the same notation.

2. The space BV^α

In this section we introduce and study the fractional BV space naturally induced by the operators ∇^α and $\operatorname{div}^\alpha$ defined in Section 1 following De Giorgi's distributional approach. In the presentation of the results, we will frequently refer to [**36**, Chapter 5].

2.1. Definition of $BV^\alpha(\mathbb{R}^n)$ and Structure Theorem. In analogy with the classical case (see [**36**, Definition 5.1] for instance), we start with the following definition.

Definition 1.9 ($BV^\alpha(\mathbb{R}^n)$ space)**.** Let $\alpha \in (0,1)$. A function $f \in L^1(\mathbb{R}^n)$ belongs to the space $BV^\alpha(\mathbb{R}^n)$ if

$$\sup\left\{\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \; dx : \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1\right\} < +\infty.$$

We can now state the following fundamental result relating non-local distributional gradients of BV^α functions to vector valued Radon measures.

Theorem 1.10 (Structure Theorem for BV^α functions)**.** *Let $\alpha \in (0,1)$ and $f \in L^1(\mathbb{R}^n)$. Then, $f \in BV^\alpha(\mathbb{R}^n)$ if and only if there exists a finite vector valued Radon measure $D^\alpha f \in \mathscr{M}(\mathbb{R}^n; \mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f \tag{1.22}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$. In addition, for any open set $U \subset \mathbb{R}^n$ it holds

$$|D^\alpha f|(U) = \sup\left\{\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \; dx : \varphi \in C_c^\infty(U; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(U;\mathbb{R}^n)} \le 1\right\}. \tag{1.23}$$

Proof. If $f \in L^1(\mathbb{R}^n)$ and if there exists a finite vector valued Radon measure $D^\alpha f \in \mathscr{M}(\mathbb{R}^n; \mathbb{R}^n)$ such that (1.22) holds, then $f \in BV^\alpha(\mathbb{R}^n)$ by Definition 1.9.

If $f \in BV^\alpha(\mathbb{R}^n)$, then the proof is identical to the one of [**36**, Theorem 5.1], with minor modifications. Define the linear functional $L \colon C_c^\infty(\mathbb{R}^n; \mathbb{R}^n) \to \mathbb{R}$ setting

$$L(\varphi) = -\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n).$$

Note that L is well defined thanks to Corollary 1.3. Since $f \in BV^\alpha(\mathbb{R}^n)$, we have

$$C(U) = \sup\left\{L(\varphi) : \varphi \in C_c^\infty(U; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(U;\mathbb{R}^n)} \le 1\right\} < +\infty$$

for each open set $U \subset \mathbb{R}^n$, so that

$$|L(\varphi)| \le C(U) \|\varphi\|_{L^\infty(U;\mathbb{R}^n)} \qquad \forall \varphi \in C_c^\infty(U; \mathbb{R}^n).$$

Thus, by the density of $C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ in $C_c(\mathbb{R}^n; \mathbb{R}^n)$, the functional L can be uniquely extended to a continuous linear functional $\tilde{L} \colon C_c(\mathbb{R}^n; \mathbb{R}^n) \to \mathbb{R}$ and the conclusion follows by Riesz's Representation Theorem. □

2.2. Lower semicontinuity of fractional variation. Similarly to the classical case, the *fractional variation measure* given by Theorem 1.10 in (1.23) is lower semicontinuous with respect to L^1-convergence.

Proposition 1.11 (Lower semicontinuity of fractional variation measure)**.** *Let $\alpha \in (0,1)$. If $(f_k)_{k\in\mathbb{N}} \subset BV^\alpha(\mathbb{R}^n)$ and $f_k \to f$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$, then $f \in BV^\alpha(\mathbb{R}^n)$ with*

$$|D^\alpha f|(U) \le \liminf_{k\to+\infty} |D^\alpha f_k|(U)$$

for any open set $U \subset \mathbb{R}^n$.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1$. Then $\operatorname{div}^\alpha\varphi \in L^\infty(\mathbb{R}^n)$ by Corollary 1.3 and so we can estimate

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha\varphi \, dx = \lim_{k\to+\infty} \int_{\mathbb{R}^n} f_k \operatorname{div}^\alpha\varphi \, dx = -\lim_{k\to+\infty} \int_{\mathbb{R}^n} \varphi \, dD^\alpha f_k \le \liminf_{k\to+\infty} |D^\alpha f_k|(\mathbb{R}^n).$$

This shows that

$$|D^\alpha f|(\mathbb{R}^n) \le \liminf_{k\to+\infty} |D^\alpha f_k|(\mathbb{R}^n),$$

thanks to Theorem 1.10. Finally, if U is an open set in $\mathbb{R}^n$, it is enough to take $\varphi \in C_c^\infty(U;\mathbb{R}^n)$ and to argue as above, applying (1.23). □

From Proposition 1.11 we immediately deduce the following result, whose standard proof is left to the reader.

Corollary 1.12 (BV^α is a Banach space)**.** *Let $\alpha \in (0,1)$. The linear space $BV^\alpha(\mathbb{R}^n)$ equipped with the norm*

$$\|f\|_{BV^\alpha(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)} + |D^\alpha f|(\mathbb{R}^n), \qquad f \in BV^\alpha(\mathbb{R}^n),$$

where $D^\alpha f$ is given by Theorem 1.10, is a Banach space.

2.3. Approximation by smooth functions. Here and in the following, we let $\varrho \in C_c^\infty(\mathbb{R}^n)$ be a function such that

$$\operatorname{supp}\varrho \subset B_1, \qquad \varrho \ge 0, \qquad \int_{\mathbb{R}^n} \varrho(x)\, dx = 1, \tag{1.24}$$

see [**36**, Section 4.2.1] for an example. We thus let $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^n)$ be defined as

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho\left(\frac{x}{\varepsilon}\right) \quad \forall x \in \mathbb{R}^n. \tag{1.25}$$

We call $(\varrho_\varepsilon)_{\varepsilon>0}$ a family of *standard mollifiers*. We have the following result.

Lemma 1.13 (Convolution with standard mollifiers)**.** *Let $\alpha \in (0,1)$ and let $(\varrho_\varepsilon)_{\varepsilon>0}$ as in (1.25). If $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, then*

$$\operatorname{div}^\alpha(\varrho_\varepsilon * \varphi) = \varrho_\varepsilon * \operatorname{div}^\alpha\varphi \tag{1.26}$$

for any $\varepsilon > 0$. Thus, if $f \in BV^\alpha(\mathbb{R}^n)$, then

$$D^\alpha(\varrho_\varepsilon * f) = (\varrho_\varepsilon * D^\alpha f)\mathscr{L}^n \tag{1.27}$$

for any $\varepsilon > 0$, and

$$D^\alpha(\varrho_\varepsilon * f) \rightharpoonup D^\alpha f \tag{1.28}$$

in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$ as $\varepsilon \to 0$.

Proof. Let $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Recalling (1.10), we can write

$$\mathrm{div}^\alpha\varphi = K_{n,\alpha} * \mathrm{div}\varphi,$$

where

$$K_{n,\alpha}(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1}\,|x|^{1-n-\alpha}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Since $\varrho_\varepsilon * \varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, we can compute

$$\begin{aligned}\mathrm{div}^\alpha(\varrho_\varepsilon * \varphi) &= K_{n,\alpha} * \mathrm{div}(\varrho_\varepsilon * \varphi)\\ &= K_{n,\alpha} * (\varrho_\varepsilon * \mathrm{div}\varphi)\\ &= \varrho_\varepsilon * (K_{n,\alpha} * \mathrm{div}\varphi)\\ &= \varrho_\varepsilon * \mathrm{div}^\alpha\varphi\end{aligned}$$

and (1.26) follows. Now let $f \in BV^\alpha(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. By (1.22) and (1.26), for all $\varepsilon > 0$ we can compute

$$\begin{aligned}-\int_{\mathbb{R}^n}(\varrho_\varepsilon * f)\,\mathrm{div}^\alpha\varphi\,dx &= -\int_{\mathbb{R}^n} f\,(\varrho_\varepsilon * \mathrm{div}^\alpha\varphi)\,dx\\ &= -\int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha(\varrho_\varepsilon * \varphi)\,dx\\ &= \int_{\mathbb{R}^n}(\varrho_\varepsilon * \varphi)\,dD^\alpha f\\ &= \int_{\mathbb{R}^n}\varphi \cdot (\varrho_\varepsilon * D^\alpha f)\,dx,\end{aligned}$$

proving (1.27). The convergence in (1.28) thus follows from standard properties of the mollification of Radon measures, see [**6**, Theorem 2.2] for instance. □

As an immediate application of Lemma 1.13, we can prove that a function in $BV^\alpha(\mathbb{R}^n)$ can be tested against the fractional divergence of any Lip_c-regular vector field.

Proposition 1.14 (Lip_c-regular test)**.** *Let* $\alpha \in (0,1)$. *If* $f \in BV^\alpha(\mathbb{R}^n)$, *then* (1.22) *holds for all* $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. Fix $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ and let $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^n)$ be as in (1.25). Then $\varrho_\varepsilon * \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and so, by Lemma 1.13 and (1.22), we have

$$\int_{\mathbb{R}^n}(\varrho_\varepsilon * f)\,\mathrm{div}^\alpha\varphi\,dx = \int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha(\varrho_\varepsilon * \varphi)\,dx = -\int_{\mathbb{R}^n}(\varrho_\varepsilon * \varphi)\cdot dD^\alpha f. \tag{1.29}$$

Since $\varrho_\varepsilon * \varphi \to \varphi$ uniformly and $\varrho_\varepsilon * f \to f$ in $L^1(\mathbb{R}^n)$ as $\varepsilon \to 0$, and $\mathrm{div}^\alpha\,\varphi \in L^\infty(\mathbb{R}^n)$ by Corollary 1.3, we can pass to the limit as $\varepsilon \to 0$ in (1.29) getting

$$\int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha\varphi\,dx = -\int_{\mathbb{R}^n}\varphi\cdot dD^\alpha f$$

for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$. □

As in the classical case, we can prove the density of $C^\infty(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$ in $BV^\alpha(\mathbb{R}^n)$.

Theorem 1.15 (Approximation by $C^\infty \cap BV^\alpha$ functions)**.** *Let* $\alpha \in (0,1)$. *If* $f \in BV^\alpha(\mathbb{R}^n)$, *then there exists* $(f_k)_{k\in\mathbb{N}} \subset BV^\alpha(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ *such that*

(i) $f_k \to f$ *in* $L^1(\mathbb{R}^n)$;

(ii) $|D^\alpha f_k|(\mathbb{R}^n) \to |D^\alpha f|(\mathbb{R}^n)$.

Proof. Let $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^n)$ be as in (1.25). Fix $f \in BV^\alpha(\mathbb{R}^n)$ and consider $f_\varepsilon = f * \varrho_\varepsilon$ for all $\varepsilon > 0$. Since $f_\varepsilon \to f$ in $L^1(\mathbb{R}^n)$, by Proposition 1.11 we get that

$$|D^\alpha f|(\mathbb{R}^n) \le \liminf_{\varepsilon\to 0} |D^\alpha f_\varepsilon|(\mathbb{R}^n).$$

By Lemma 1.13 we also have that

$$|D^\alpha f_\varepsilon|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |\varrho_\varepsilon * D^\alpha f| \, dx \le |D^\alpha f|(\mathbb{R}^n)$$

and the proof is complete. □

Let $(\eta_R)_{R>0} \subset C_c^\infty(\mathbb{R}^n)$ be such that

$$0 \le \eta_R \le 1, \qquad \eta_R = 1 \text{ on } B_R, \qquad \operatorname{supp}(\eta_R) \subset B_{R+1}, \qquad \operatorname{Lip}(\eta_R) \le 2. \tag{1.30}$$

We call η_R a *cut-off function*. As in the classical case, we can prove the density of $C_c^\infty(\mathbb{R}^n)$ in $BV^\alpha(\mathbb{R}^n)$.

Theorem 1.16 (Approximation by C_c^∞ functions)**.** *Let $\alpha \in (0,1)$. If $f \in BV^\alpha(\mathbb{R}^n)$, then there exists $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that*

(i) $f_k \to f$ in $L^1(\mathbb{R}^n)$;
(ii) $|D^\alpha f_k|(\mathbb{R}^n) \to |D^\alpha f|(\mathbb{R}^n)$.

Proof. Let $(\eta_R)_{R>0} \subset C_c^\infty(\mathbb{R}^n)$ be as in (1.30). Thanks to Theorem 1.15, it is enough to prove that $f\eta_R \to f$ in $BV^\alpha(\mathbb{R}^n)$ as $R \to +\infty$ for all $f \in C^\infty(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$. Clearly, $f\eta_R \to f$ in $L^1(\mathbb{R}^n)$ as $R \to +\infty$. Thus, by Proposition 1.11, we just need to prove that

$$\limsup_{R\to+\infty} |D^\alpha(f\eta_R)|(\mathbb{R}^n) \le |D^\alpha f|(\mathbb{R}^n). \tag{1.31}$$

Fix $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Then, by Lemma 1.7, we get

$$\int_{\mathbb{R}^n} f\eta_R \operatorname{div}^\alpha\varphi \, dx = \int_{\mathbb{R}^n} f \operatorname{div}^\alpha(\eta_R\varphi) \, dx - \int_{\mathbb{R}^n} f\,\varphi\cdot\nabla^\alpha\eta_R \, dx - \int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\eta_R,\varphi)\, dx.$$

Since $f \in BV^\alpha(\mathbb{R}^n)$ and $0 \le \eta_R \le 1$, we have

$$\left|\int_{\mathbb{R}^n} f \operatorname{div}^\alpha(\eta_R\varphi)\, dx\right| \le \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} |D^\alpha f|(\mathbb{R}^n).$$

Moreover, we have

$$\left|\int_{\mathbb{R}^n} f\,\varphi\cdot\nabla^\alpha\eta_R\, dx\right| \le \mu_{n,\alpha}\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y)-\eta_R(x)|}{|y-x|^{n+\alpha}}\, dy\, dx$$

and, similarly,

$$\left|\int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\eta_R,\varphi)\, dx\right| \le 2\mu_{n,\alpha}\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y)-\eta_R(x)|}{|y-x|^{n+\alpha}}\, dy\, dx.$$

Combining these three estimates, we conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f\eta_R \operatorname{div}^\alpha\varphi\, dx\right| &\le \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}|D^\alpha f|(\mathbb{R}^n)\\ &\quad + 3\mu_{n,\alpha}\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y)-\eta_R(x)|}{|y-x|^{n+\alpha}}\, dy\, dx\end{aligned}$$

and (1.31) follows by Theorem 1.10. Indeed, we have

$$\lim_{R\to+\infty}\int_{\mathbb{R}^n}|f(x)|\int_{\mathbb{R}^n}\frac{|\eta_R(y)-\eta_R(x)|}{|y-x|^{n+\alpha}}\,dy\,dx=0$$

combining (1.6), (1.7) and (1.30) with Lebesgue's Dominated Convergence Theorem. □

2.4. Gagliardo–Nirenberg–Sobolev inequality. Thanks to Theorem 1.16, we are able to prove the analogous of the Gagliardo–Nirenberg–Sobolev inequality for the space $BV^\alpha(\mathbb{R}^n)$.

Theorem 1.17 (Gagliardo–Nirenberg–Sobolev inequality)**.** *Let $\alpha\in(0,1)$ and $n\ge 2$. There exists a constant $c_{n,\alpha}>0$ such that*

$$\|f\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)}\le c_{n,\alpha}|D^\alpha f|(\mathbb{R}^n) \tag{1.32}$$

for any $f\in BV^\alpha(\mathbb{R}^n)$. As a consequence, $BV^\alpha(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q\in[1,\frac{n}{n-\alpha}]$.

Proof. By [**88**, Theorem A'], we know that (1.32) holds for any $f\in C_c^\infty(\mathbb{R}^n)$. So let $f\in BV^\alpha(\mathbb{R}^n)$ and let $(f_k)_{k\in\mathbb{N}}\subset C_c^\infty(\mathbb{R}^n)$ be as in Theorem 1.16. By Fatou's Lemma and Proposition 1.11, we thus obtain

$$\|f\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)}\le\liminf_{k\to+\infty}\|f_k\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)}\le c_{n,\alpha}\lim_{k\to+\infty}|D^\alpha f_k|(\mathbb{R}^n)=c_{n,\alpha}|D^\alpha f|(\mathbb{R}^n)$$

and the proof is complete. □

Incidentally, we observe that the continuous embedding $BV^\alpha(\mathbb{R}^n)\subset L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ for $n\ge 2$ and $\alpha\in(0,1)$ can be improved by using the main result of the recent work [**92**] (see also [**94**]). Indeed, if $n\ge 2$, $\alpha\in(0,1)$ and $f\in C_c^\infty(\mathbb{R}^n)$, then, by taking $F=\nabla^\alpha f$ in [**92**, Theorem 1.1], we have that

$$\|f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)}\le c_{n,\alpha}\|I_\alpha\nabla^\alpha f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n;\mathbb{R}^n)}\le c'_{n,\alpha}\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$$

thanks to the boundedness of the Riesz transform $R\colon L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)\to L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n;\mathbb{R}^n)$, where $c_{n,\alpha},c'_{n,\alpha}>0$ are two constants depending only on n and α, and $L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)$ is the Lorentz space of exponents $\frac{n}{n-\alpha},1$ (we refer to [**47**,**48**] for an account on Lorentz spaces and on the properties of Riesz transform). Thus, recalling Theorem 1.16, we readily deduce the continuous embedding $BV^\alpha(\mathbb{R}^n)\subset L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)$ for $n\ge 2$ and $\alpha\in(0,1)$ using Fatou's Lemma in Lorentz spaces (see [**47**, Exercise 1.4.11] for example).

Remark 1.18. We stress the fact that Theorem 1.17 does not hold for $n=1$, as will be shown in Example 1.22 below. It is worth to notice that an analogous restriction holds for [**88**, Theorem A], for which the authors provide a counterexample in the case $n=1$ (see [**88**, Counterexample 3.2]). The authors then derive [**88**, Theorem A'] as a consequence of [**88**, Theorem A], without proving the necessity of the restriction to $n\ge 2$ in this second case, as we do in Example 1.22.

2.5. Coarea inequality. In analogy with the classical case, we can prove a coarea inequality formula for functions in $BV^\alpha(\mathbb{R}^n)$.

Theorem 1.19 (Coarea inequality)**.** *Let $\alpha \in (0,1)$. If $f \in BV^\alpha(\mathbb{R}^n)$ is such that*

$$\int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|(\mathbb{R}^n)\,dt < +\infty, \tag{1.33}$$

then

$$D^\alpha f = \int_{\mathbb{R}} D^\alpha \chi_{\{f>t\}}\,dt \tag{1.34}$$

and

$$|D^\alpha f| \le \int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|\,dt. \tag{1.35}$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. By (1.33) and applying Fubini's Theorem twice, we can compute

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi \cdot\, dD^\alpha f &= -\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi(x)\,dx \\
&= -\int_{\mathbb{R}^n} \operatorname{div}^\alpha \varphi(x) \left(\int_{\mathbb{R}} \chi_{(-\infty,f(x))}(t) - \chi_{(-\infty,0)}(t)\,dt \right) dx \\
&= -\int_{\mathbb{R}} \int_{\mathbb{R}^n} \operatorname{div}^\alpha \varphi(x) \left(\chi_{\{f>t\}}(x) - \chi_{(-\infty,0)}(t) \right)\,dx\,dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \varphi \cdot\, dD^\alpha \chi_{\{f>t\}}\,dt \\
&= \int_{\mathbb{R}^n} \varphi \cdot d\left(\int_{\mathbb{R}} D^\alpha \chi_{\{f>t\}}\,dt \right)
\end{aligned}$$

proving (1.34). Thus

$$|D^\alpha f| = \left| \int_{\mathbb{R}} D^\alpha \chi_{\{f>t\}}\,dt \right| \le \int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|\,dt$$

and the proof is complete. □

2.6. A fractional version of the Fundamental Theorem of Calculus. Let $\alpha \in (0,1)$ and let $\mu_{n,-\alpha}$ be given by (1.1) (note that the expression in (1.1) makes sense for all $\alpha \in (-1,1)$). We let

$$\mathcal{T}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : D^{\mathsf{a}} f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n) \text{ for all multi-indices } \mathsf{a} \in \mathbb{N}_0^n \right\} \tag{1.36}$$

and

$$\mathcal{T}(\mathbb{R}^n;\mathbb{R}^n) = \{ \varphi \in C^\infty(\mathbb{R}^n;\mathbb{R}^n) : \varphi_i \in \mathcal{T}(\mathbb{R}^n),\ i = 1,\dots,n \}.$$

By [**92**, Section 5], the operator

$$\operatorname{div}^{-\alpha} \varphi(x) = \mu_{n,-\alpha} \int_{\mathbb{R}^n} \frac{z \cdot \varphi(x+z)}{|z|^{n+1-\alpha}}\,dz \tag{1.37}$$

is well defined for any $\varphi \in \mathcal{T}(\mathbb{R}^n;\mathbb{R}^n)$. Moreover, by [**92**, Theorem 5.3], we have the following *inversion formula*

$$-\operatorname{div}^{-\alpha} \nabla^\alpha = \mathrm{id}_{\mathcal{T}(\mathbb{R}^n)}. \tag{1.38}$$

Exploiting (1.37) and (1.38) we can prove the following fractional version of the Fundamental Theorem of Calculus. See [**89**, Theorem 2.1] for a similar approach.

Theorem 1.20 (Fractional Fundamental Theorem of Calculus)**.** *Let* $\alpha \in (0,1)$. *If* $f \in C_c^\infty(\mathbb{R}^n)$, *then*

$$f(y) - f(x) = \mu_{n,-\alpha} \int_{\mathbb{R}^n} \left(\frac{z-x}{|z-x|^{n+1-\alpha}} - \frac{z-y}{|z-y|^{n+1-\alpha}} \right) \cdot \nabla^\alpha f(z)\, dz \tag{1.39}$$

for any $x, y \in \mathbb{R}^n$.

Proof. Since clearly $C_c^\infty(\mathbb{R}^n) \subset \mathcal{T}(\mathbb{R}^n)$, we have $\nabla^\alpha f \in \mathcal{T}(\mathbb{R}^n; \mathbb{R}^n)$ by [**92**, Theorem 4.3]. Applying (1.38), we have

$$\begin{aligned} f(y) - f(x) &= (-\mathrm{div}^{-\alpha} \nabla^\alpha f)(y) - (-\mathrm{div}^{-\alpha} \nabla^\alpha f)(x) \\ &= \mu_{n,-\alpha} \int_{\mathbb{R}^n} \frac{z}{|z|^{n+1-\alpha}} \cdot \Big(\nabla^\alpha f(x+z) - \nabla^\alpha f(y+z) \Big)\, dz \end{aligned}$$

for all $x, y \in \mathbb{R}^n$. Then (1.39) follows splitting the integral and changing variables. □

An easy consequence of Theorem 1.20 is that the distributional α-divergence of the kernel appearing in (1.39) is a difference of Dirac deltas.

Proposition 1.21. *Let* $\alpha \in (0,1)$. *If* $x, y \in \mathbb{R}^n$, *then*

$$\mu_{n,-\alpha} \mathrm{div}^\alpha \left(\frac{\cdot - y}{|\cdot - y|^{n+1-\alpha}} - \frac{\cdot - x}{|\cdot - x|^{n+1-\alpha}} \right) = \delta_y - \delta_x \tag{1.40}$$

in the sense of Radon measures.

Proof. It follows immediately from (1.39). □

Example 1.22. Let $\alpha \in (0,1)$. For any $a, b \in \mathbb{R}$, with $a \neq b$, consider the function

$$f_{a,b,\alpha}(x) = |x-b|^{\alpha-1} \operatorname{sgn}(x-b) - |x-a|^{\alpha-1} \operatorname{sgn}(x-a), \quad x \in \mathbb{R} \setminus \{a, b\}.$$

We have that $f_{a,b,\alpha} \in BV^\alpha(\mathbb{R})$ with

$$D^\alpha f_{a,b,\alpha} = \frac{\delta_b - \delta_a}{\mu_{1,-\alpha}} \tag{1.41}$$

in the sense of finite Radon measures. Indeed, one can easily check that $f_{a,b,\alpha} \in L^1(\mathbb{R})$. Since $n = 1$, we have $\nabla^\alpha = \mathrm{div}^\alpha$. Thus, (1.41) follows from (1.40), proving that $f \in BV^\alpha(\mathbb{R})$. In addition, note that $f_{a,b,\alpha} \in BV^\alpha(\mathbb{R}) \setminus L^{\frac{1}{1-\alpha}}(\mathbb{R})$, since

$$|f_{a,b,\alpha}(x)|^{\frac{1}{1-\alpha}} \sim \begin{cases} |x-a|^{-1} & \text{as } x \to a, \\ |x-b|^{-1} & \text{as } x \to b. \end{cases}$$

Thus, Theorem 1.17 cannot hold for $n = 1$. By the way, note that $f_{a,b,\alpha} \in W^{\beta,1}(\mathbb{R})$ for all $\beta \in (0, \alpha)$ (this will also be a consequence of Theorem 1.30 below).

2.7. Compactness. We start with the following Hölder estimate on the L^1-norm of translations of functions in $C_c^\infty(\mathbb{R}^n)$.

Proposition 1.23 (L^1-estimate on translations)**.** *Let $\alpha \in (0,1)$. If $f \in BV(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|\,dx \le \gamma_{n,\alpha}\,|y|^\alpha\,|D^\alpha f|(\mathbb{R}^n) \tag{1.42}$$

for all $y \in \mathbb{R}^n$, where

$$\gamma_{n,\alpha} = \mu_{n,-\alpha}\int_{\mathbb{R}^n}\left|\frac{z}{|z|^{n+1-\alpha}} - \frac{z-\mathrm{e}_1}{|z-\mathrm{e}_1|^{n+1-\alpha}}\right| dz. \tag{1.43}$$

Proof. Assume $f \in C_c^\infty(\mathbb{R}^n)$. By (1.39), we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x+y)-f(x)|\,dx &\le \mu_{n,-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\frac{z}{|z|^{n+1-\alpha}} - \frac{z-y}{|z-y|^{n+1-\alpha}}\right| |\nabla^\alpha f(x+z)|\,dz\,dx\\ &= \mu_{n,-\alpha}\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\,\mathbb{R}^n)}\int_{\mathbb{R}^n}\left|\frac{z}{|z|^{n+1-\alpha}} - \frac{z-y}{|z-y|^{n+1-\alpha}}\right| dz.\end{aligned}$$

Now we notice that the integral appearing in the last term is actually a radial function of y. Indeed, let $\mathrm{R} \in \mathrm{SO}(n)$ be such that $\mathrm{R}y = |y|\nu$, for some $\nu \in \mathbb{S}^{n-1}$. Making the change of variable $z = |y|\,{}^{\mathrm{t}}\mathrm{R}w$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\frac{z}{|z|^{n+1-\alpha}} - \frac{z-y}{|z-y|^{n+1-\alpha}}\right| dz &= |y|^\alpha\int_{\mathbb{R}^n}\left|\frac{{}^{\mathrm{t}}\mathrm{R}w}{|w|^{n+1-\alpha}} - \frac{{}^{\mathrm{t}}\mathrm{R}(w-\nu)}{|w-\nu|^{n+1-\alpha}}\right| dw\\ &= |y|^\alpha\int_{\mathbb{R}^n}\left|\frac{w}{|w|^{n+1-\alpha}} - \frac{(w-\nu)}{|w-\nu|^{n+1-\alpha}}\right| dw.\end{aligned}$$

Since ν is arbitrary, we may choose $\nu = \mathrm{e}_1$. We now prove that

$$\int_{\mathbb{R}^n}\left|\frac{z}{|z|^{n+1-\alpha}} - \frac{z-\mathrm{e}_1}{|z-\mathrm{e}_1|^{n+1-\alpha}}\right| dz < +\infty.$$

To this purpose, we notice that

$$\begin{aligned}\int_{B_2}\left|\frac{z}{|z|^{n+1-\alpha}} - \frac{z-\mathrm{e}_1}{|z-\mathrm{e}_1|^{n+1-\alpha}}\right| dz &\le \int_{B_2}\frac{1}{|z|^{n-\alpha}}\,dz + \int_{B_2}\frac{1}{|z-\mathrm{e}_1|^{n-\alpha}}\,dz\\ &\le 2\int_{B_3}\frac{1}{|z|^{n-\alpha}}\,dz = 2n\omega_n\frac{3^\alpha}{\alpha}.\end{aligned}$$

On the other hand, for all $z \in \mathbb{R}^n \setminus B_2$ we have

$$\begin{aligned}\frac{z-\mathrm{e}_1}{|z-\mathrm{e}_1|^{n+1-\alpha}} - \frac{z}{|z|^{n+1-\alpha}} &= \int_0^1 \frac{d}{dt}\left(\frac{(z-t\mathrm{e}_1)}{|z-t\mathrm{e}_1|^{n+1-\alpha}}\right) dt\\ &= \int_0^1 -\frac{\mathrm{e}_1}{|z-t\mathrm{e}_1|^{n+1-\alpha}} + (n+1-\alpha)(z_1-t)\frac{(z-t\mathrm{e}_1)}{|z-t\mathrm{e}_1|^{n+3-\alpha}}\,dt\end{aligned}$$

so that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_2}\left|\frac{z}{|z|^{n+1-\alpha}} - \frac{z-\mathrm{e}_1}{|z-\mathrm{e}_1|^{n+1-\alpha}}\right| dz &\le \int_{\mathbb{R}^n\setminus B_2}\int_0^1\frac{|z-t\mathrm{e}_1| + (n-\alpha+1)|z_1-t|}{|z-t\mathrm{e}_1|^{n+2-\alpha}}\,dt\,dz\\ &\le (n-\alpha+2)\int_0^1\int_{\mathbb{R}^n\setminus B_2}\frac{1}{|z-t\mathrm{e}_1|^{n+1-\alpha}}\,dz\,dt\end{aligned}$$

$$\begin{aligned}&\le (n-\alpha+2)\int_0^1\int_{\mathbb{R}^n\setminus B_1}\frac{1}{|z|^{n+1-\alpha}}\,dz\,dt\\&=(n-\alpha+2)\frac{n\omega_n}{1-\alpha}.\end{aligned}$$

We conclude that

$$\int_{\mathbb{R}^n}\left|\frac{z}{|z|^{n+1-\alpha}}-\frac{z-\mathrm{e}_1}{|z-\mathrm{e}_1|^{n+1-\alpha}}\right|dz\le n\omega_n\left(2\frac{3^\alpha}{\alpha}+\frac{(n-\alpha+2)}{1-\alpha}\right)<+\infty.$$

Thus (1.42) follows for all $f\in C_c^\infty(\mathbb{R}^n)$. Now let $f\in BV^\alpha(\mathbb{R}^n)$ By Theorem 1.16, we can find $(f_k)_{k\in\mathbb{N}}\subset C_c^\infty(\mathbb{R}^n)$ such that $f_k\to f$ in $L^1(\mathbb{R}^n)$ and $|D^\alpha f_k|(\mathbb{R}^n)\to|D^\alpha f|(\mathbb{R}^n)$ as $k\to+\infty$. Hence, for all $y\in\mathbb{R}^n$, we get

$$\begin{aligned}\int_{\mathbb{R}^n}|f(x+y)-f(x)|\,dx&=\lim_{k\to+\infty}\int_{\mathbb{R}^n}|f_k(x+y)-f_k(x)|\,dx\\&\le\gamma_{n,\alpha}\,|y|^\alpha\lim_{k\to+\infty}|D^\alpha f_k|(\mathbb{R}^n)\\&=\gamma_{n,\alpha}\,|y|^\alpha\,|D^\alpha f|(\mathbb{R}^n)\end{aligned}$$

and the conclusion thus follows. □

Similarly to the classical case, as a consequence of the previous result we can prove the following key estimate of the L^1-distance of a function in $BV^\alpha(\mathbb{R}^n)$ and its convolution with a mollifier.

Corollary 1.24 (L^1-distance with convolution)**.** *Let $\alpha\in(0,1)$. If $f\in BV^\alpha(\mathbb{R}^n)$, then*

$$\|\varrho_\varepsilon * f-f\|_{L^1(\mathbb{R}^n)}\le\gamma_{n,\alpha}\,\varepsilon^\alpha|D^\alpha f|(\mathbb{R}^n)\tag{1.44}$$

for all $\varepsilon>0$, where $(\varrho_\varepsilon)_{\varepsilon>0}\subset C_c^\infty(\mathbb{R}^n)$ is as in (1.25) *and $\gamma_{n,\alpha}$ as in Proposition 1.23.*

Proof. By Theorem 1.16, it is enough to prove (1.44) for $f\in C_c^\infty(\mathbb{R}^n)$. By (1.42), we get

$$\begin{aligned}\|\varrho_\varepsilon * f-f\|_{L^1(\mathbb{R}^n)}&\le\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\varrho(y)|f(x-\varepsilon y)-f(x)|\,dy\,dx\\&=\int_{\mathbb{R}^n}\varrho(y)\int_{\mathbb{R}^n}|f(x-\varepsilon y)-f(x)|\,dx\,dy\\&\le\gamma_{n,\alpha}\,\varepsilon^\alpha\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}\int_{B_1}\varrho(y)|y|^\alpha\,dy\\&\le\gamma_{n,\alpha}\,\varepsilon^\alpha\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}\end{aligned}$$

and the proof is complete. □

We are now ready to prove following compactness result for the space $BV^\alpha(\mathbb{R}^n)$.

Theorem 1.25 (Compactness for $BV^\alpha(\mathbb{R}^n)$)**.** *Let $\alpha\in(0,1)$. If $(f_k)_{k\in\mathbb{N}}\subset BV^\alpha(\mathbb{R}^n)$ satisfies*

$$\sup_{k\in\mathbb{N}}\|f_k\|_{BV^\alpha(\mathbb{R}^n)}<+\infty,$$

then there exists a subsequence $(f_{k_j})_{j\in\mathbb{N}}\subset BV^\alpha(\mathbb{R}^n)$ and a function $f\in L^1(\mathbb{R}^n)$ such that

$$f_{k_j}\to f\ \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

as $j\to+\infty$.

Proof. We follow the line of the proof of [**6**, Theorem 3.23]. Let $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^n)$ be as in (1.25) and set $f_{k,\varepsilon} = \varrho_\varepsilon * f_k$. Clearly $f_{k,\varepsilon} \in C^\infty(\mathbb{R}^n)$ and

$$\|f_{k,\varepsilon}\|_{L^\infty(U)} \le \|\varrho_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|f_k\|_{L^1(\mathbb{R}^n)}, \qquad \|\nabla f_{k,\varepsilon}\|_{L^\infty(U;\mathbb{R}^n)} \le \|\nabla \varrho_\varepsilon\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}\|f_k\|_{L^1(\mathbb{R}^n)}$$

for any open set $U \Subset \mathbb{R}^n$. Thus $(f_{k,\varepsilon})_{k\in\mathbb{N}}$ is locally equibounded and locally equicontinuous for each $\varepsilon > 0$ fixed. By a diagonal argument, we can find a sequence $(k_j)_{j\in\mathbb{N}}$ such that $(f_{k_j,\varepsilon})_{j\in\mathbb{N}}$ converges in $C(U)$ for any open set $U \Subset \mathbb{R}^n$ with $\varepsilon = 1/p$ for all $p \in \mathbb{N}$. By Corollary 1.24, we thus get

$$\begin{aligned}\limsup_{h,j\to+\infty} \int_U |f_{k_h} - f_{k_j}|\,dx &= \limsup_{h,j\to+\infty} \int_U |f_{k_h,1/p} - f_{k_j,1/p}|\,dx \\ &\quad + \limsup_{h,j\to+\infty} \int_U |f_{k_h} - f_{k_h,1/p}| + |f_{k_j} - f_{k_j,1/p}|\,dx \\ &\le \frac{2\gamma_{n,\alpha}}{p^\alpha} \sup_{k\in\mathbb{N}} |D^\alpha f_k|(\mathbb{R}^n)\end{aligned}$$

for all open set $U \Subset \mathbb{R}^n$. Since $p \in \mathbb{N}$ is arbitrary and $L^1(U)$ is a Banach space, this shows that $(f_{k_j})_{j\in\mathbb{N}}$ converges in $L^1(U)$ for all open set $U \Subset \mathbb{R}^n$. Up to extract a further subsequence (which we do not relabel for simplicity), we also have that $f_{k_j}(x) \to f(x)$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. By Fatou's Lemma, we can thus infer that

$$\|f\|_{L^1(\mathbb{R}^n)} \le \liminf_{j\to+\infty} \|f_{k_j}\|_{L^1(\mathbb{R}^n)} \le \sup_{k\in\mathbb{N}} \|f_k\|_{BV^\alpha(\mathbb{R}^n)}.$$

Hence $f \in L^1(\mathbb{R}^n)$ and the proof is complete. □

Remark 1.26 (Improvement of [**89**, Theorem 2.1])**.** The argument presented above can be used to extend the validity of [**89**, Theorem 2.1] to all exponents $p \in [1, \frac{n}{\alpha})$, since our strategy does not rely on the boundedness of Riesz's transform but only on the inversion formula (1.38). We leave the details of the proof of this improvement of [**89**, Theorem 2.1] to the interested reader.

2.8. The inclusion $W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$. As in the classical case, fractional BV functions naturally include fractional Sobolev functions.

Theorem 1.27 ($W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$)**.** *Let $\alpha \in (0,1)$. If $f \in W^{\alpha,1}(\mathbb{R}^n)$ then $f \in BV^\alpha(\mathbb{R}^n)$, with*

$$|D^\alpha f|(\mathbb{R}^n) \le \mu_{n,\alpha}[f]_{W^{\alpha,1}(\mathbb{R}^n)} \tag{1.45}$$

and

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi\,dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f\,dx \tag{1.46}$$

for all $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, so that $D^\alpha f = \nabla^\alpha f\,\mathscr{L}^n$.

Moreover, if $f \in BV(\mathbb{R}^n)$, then $f \in W^{\alpha,1}(\mathbb{R}^n)$ for any $\alpha \in (0,1)$, with

$$\|f\|_{W^{\alpha,1}(\mathbb{R}^n)} \le c_{n,\alpha}\|f\|_{BV(\mathbb{R}^n)} \tag{1.47}$$

for some $c_{n,\alpha} > 0$ and

$$\nabla^\alpha f(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{dDf(y)}{|y-x|^{n+\alpha-1}} \tag{1.48}$$

for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Proof. Let $f \in W^{\alpha,1}(\mathbb{R}^n)$. For any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, by Lebesgue's Dominated Convergence Theorem, Fubini's Theorem and Lemma 1.1, and recalling (1.4), we can compute

$$\begin{aligned}
\int_{\mathbb{R}^n} f \mathrm{div}^\alpha \varphi \, dx &= \mu_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{\{|x-y|>\varepsilon\}} f(x) \frac{(y-x)\cdot\varphi(y)}{|y-x|^{n+\alpha+1}} \, dy \, dx \\
&= \mu_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{\{|x-y|>\varepsilon\}} \varphi(y) \cdot \frac{(y-x)f(x)}{|y-x|^{n+\alpha+1}} \, dx \, dy \\
&= \mu_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{\{|x-y|>\varepsilon\}} \varphi(y) \cdot \frac{(y-x)(f(x)-f(y))}{|y-x|^{n+\alpha+1}} \, dx \, dy \\
&= -\int_{\mathbb{R}^n} \varphi(y) \cdot \nabla^\alpha f(y) \, dy.
\end{aligned}$$

This proves (1.46), so that $f \in BV^\alpha(\mathbb{R}^n)$. Inequality (1.45) follows as in Lemma 1.1.

Now let $f \in BV(\mathbb{R}^n)$. We claim that $f \in W^{\alpha,1}(\mathbb{R}^n)$. Indeed, take $(f_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$ and $\|\nabla f_k\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \to |Df|(\mathbb{R}^n)$ as $k \to +\infty$ (for instance, see [**36**, Theorem 5.3]). Since $W^{1,1}(\mathbb{R}^n) \subset W^{\alpha,1}(\mathbb{R}^n)$ (the proof of this inclusion is similar to the one of [**34**, Proposition 2.2], for example), by Fatou's Lemma we get that

$$\begin{aligned}
\|f\|_{W^{\alpha,1}(\mathbb{R}^n)} &\le \liminf_{k\to+\infty} \|f_k\|_{W^{\alpha,1}(\mathbb{R}^n)} \\
&\le c_{n,\alpha} \liminf_{k\to+\infty} \|f_k\|_{W^{1,1}(\mathbb{R}^n)} \\
&= c_{n,\alpha} \lim_{k\to+\infty} (\|f_k\|_{L^1(\mathbb{R}^n)} + |Df_k|(\mathbb{R}^n)) \\
&= c_{n,\alpha} \|f\|_{BV(\mathbb{R}^n)}.
\end{aligned}$$

Since $|Df|(\mathbb{R}^n) < +\infty$, by Lemma 1.4 the function in (1.48) is well defined in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Fix $\varphi \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$. By Corollary 1.3, we can write

$$\int_{\mathbb{R}^n} f(x) \,\mathrm{div}^\alpha \varphi(x) \, dx = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) \,\frac{\mathrm{div}\varphi(y)}{|y-x|^{n+\alpha-1}} \, dy \, dx.$$

Recalling Lemma 1.4, applying Fubini's Theorem twice and integrating by parts, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) \,\frac{\mathrm{div}\varphi(y)}{|y-x|^{n+\alpha-1}} \, dy \, dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) \,\frac{\mathrm{div}_y \varphi(x+y)}{|y|^{n+\alpha-1}} \, dy \, dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) \,\frac{\mathrm{div}_x \varphi(x+y)}{|y|^{n+\alpha-1}} \, dy \, dx \\
&= \int_{\mathbb{R}^n} |y|^{1-n-\alpha} \int_{\mathbb{R}^n} f(x) \,\mathrm{div}\varphi(x+y) \, dx \, dy \\
&= -\int_{\mathbb{R}^n} |y|^{1-n-\alpha} \int_{\mathbb{R}^n} \varphi(y+x) \cdot \, dDf(x) \, dy \\
&= -\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|y-x|^{n+\alpha-1}} \, dy \cdot \, dDf(x) \\
&= -\int_{\mathbb{R}^n} \varphi(y) \cdot \int_{\mathbb{R}^n} \frac{dDf(x)}{|x-y|^{n+\alpha-1}} \, dy.
\end{aligned}$$

Thus we conclude that

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{div}^\alpha\varphi(x)\,dx = -\frac{\mu_{n,\alpha}}{n+\alpha-1}\int_{\mathbb{R}^n}\varphi(y)\cdot\int_{\mathbb{R}^n}\frac{dDf(x)}{|x-y|^{n+\alpha-1}}\,dy.$$

Recalling (1.46), this proves (1.48) and the proof is complete. □

We now want to understand in which cases inequality (1.45) is actually an equality. The key idea to the answer of this question lies in the following simple result.

Lemma 1.28. *Let $A \subset \mathbb{R}^n$ be a measurable set with $\mathscr{L}^n(A) > 0$. If $F \in L^1(A;\mathbb{R}^m)$, then*

$$\left|\int_A F(x)\,dx\right| \le \int_A |F(x)|\,dx,$$

with equality if and only if $F = f\nu$ a.e. in A for some constant direction $\nu \in \mathbb{S}^{m-1}$ and some scalar function $f \in L^1(A)$ with $f \ge 0$ a.e. in A.

Proof. The inequality is well known and it is obvious that it is an equality if $F = f\nu$ a.e. in A for some constant direction $\nu \in \mathbb{S}^{m-1}$ and some scalar function $f \in L^1(A)$ with $f \ge 0$ a.e. in A. So let us assume that

$$\left|\int_A F(x)\,dx\right| = \int_A |F(x)|\,dx.$$

If $\int_A F(x)\,dx = 0$, then also $\int_A |F(x)|\,dx = 0$. Thus $F = 0$ a.e. in A and there is nothing to prove. If $\int_A F(x)\,dx \ne 0$ instead, then we can write

$$\int_A |F(x)| - F(x)\cdot\nu\,dx = 0,$$

with

$$\nu = \frac{\int_A F(x)\,dx}{|\int_A F(x)\,dx\,|} \in \mathbb{S}^{m-1}.$$

Therefore, we obtain $|F(x)| = F(x)\cdot\nu$ for a.e. $x \in A$, so that $\frac{F(x)}{|F(x)|}\cdot\nu = 1$ for a.e. $x \in A$ such that $|F(x)| \ne 0$. This implies that $F = f\nu$ a.e. in A with $f = |F| \in L^1(A)$ and the conclusion follows. □

As an immediate consequence of Lemma 1.28, we have the following result.

Corollary 1.29. *Let $\alpha \in (0,1)$. If $f \in W^{\alpha,1}(\mathbb{R}^n)$, then*

$$\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \mu_{n,\alpha}[f]_{W^{\alpha,1}(\mathbb{R}^n)}, \tag{1.49}$$

with equality if and only if $f = 0$ a.e. in $\mathbb{R}^n$.

Proof. Inequality (1.49) is already proved in (1.45). Note that, given $f \in L^1(\mathbb{R}^n)$, $[f]_{W^{\alpha,1}(\mathbb{R}^n)} = 0$ if and only if $f = 0$ a.e. and thus, in this case, (1.49) is trivially an equality.

If (1.49) holds as an equality and f is not equivalent to the zero function, then

$$\int_{\mathbb{R}^n}\left(|\nabla^\alpha f(x)| - \mu_{n,\alpha}\int_{\mathbb{R}^n}\frac{|f(y)-f(x)|}{|y-x|^{n+\alpha}}\,dy\right)dx = 0$$

and thus

$$\left|\int_{\mathbb{R}^n}\frac{(f(y)-f(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,dy\right| = \int_{\mathbb{R}^n}\frac{|f(y)-f(x)|}{|y-x|^{n+\alpha}}\,dy \tag{1.50}$$

for all $x \in U$, for some measurable set $U \subset \mathbb{R}^n$ such that $\mathscr{L}^n(\mathbb{R}^n \setminus U) = 0$. Now let $x \in U$ be fixed. By Lemma 1.28 (applied with $A = \mathbb{R}^n$), (1.50) implies that the (non-identically zero) vector field

$$y \mapsto (f(y) - f(x))\,(y - x), \quad y \in \mathbb{R}^n,$$

has constant direction for all $y \in V_x$, for some measurable set $V_x \subset \mathbb{R}^n$ such that $\mathscr{L}^n(\mathbb{R}^n \setminus V_x) = 0$. Thus, given $y, y' \in V_x$, the two vectors $y - x$ and $y' - x$ are linearly dependent, so that the three points x, y and y' are collinear. If $n \ge 2$, then this immediately gives $\mathscr{L}^n(V_x) = 0$, a contradiction, so that (1.49) must be strict. If instead $n = 1$, then we know that

$$x \in U \implies y \mapsto (f(y) - f(x))\,(y - x) \text{ has constant sign for all } y \in V_x. \tag{1.51}$$

We claim that (1.51) implies that the function f is (equivalent to) a (non-constant) monotone function. If so, then $f \notin L^1(\mathbb{R})$, in contrast with the fact that $f \in W^{\alpha,1}(\mathbb{R})$, so that (1.49) must be strict and the proof is concluded. To prove the claim, we argue as follows. Fix $x \in U$ and assume that

$$(f(y) - f(x))\,(y - x) > 0 \tag{1.52}$$

for all $y \in V_x$ without loss of generality. Now pick $x' \in U \cap V_x$ such that $x' > x$. Then, choosing $y = x'$ in (1.52), we get $(f(x') - f(x))\,(x' - x) > 0$ and thus $f(x') > f(x)$. Similarly, if $x' \in U \cap V_x$ is such that $x' < x$, then $f(x') < f(x)$. Hence

$$\operatorname*{ess\,sup}_{z<x} f(z) \le f(x) \le \operatorname*{ess\,inf}_{z>x} f(z)$$

for all $x \in U$ (where ess sup and ess inf refer to the *essential supremum* and the *essential infimum* respectively) and thus f must be equivalent to a (non-constant) non-decreasing function. □

2.9. The inclusion $BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$ for $\beta < \alpha$. In the following result we prove that $BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$ with continuous embedding for all $0 < \beta < \alpha < 1$.

Theorem 1.30 ($BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$ for $\beta < \alpha$)**.** *Let $\alpha, \beta \in (0,1)$ with $\beta < \alpha$. Then $BV^\alpha(\mathbb{R}^n) \subset W^{\beta,1}(\mathbb{R}^n)$, with*

$$[f]_{W^{\beta,1}(\mathbb{R}^n)} \le C_{n,\alpha,\beta}\,\|f\|_{BV^\alpha(\mathbb{R}^n)}, \tag{1.53}$$

for all $f \in BV^\alpha(\mathbb{R}^n)$, where

$$C_{n,\alpha,\beta} = n\omega_n \frac{\alpha 2^{\frac{\alpha-\beta}{\beta}} \gamma_{n,\alpha}^{\beta/\alpha}}{\beta(\alpha-\beta)} \tag{1.54}$$

and $\gamma_{n,\alpha}$ is as in (1.43).

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$ and $r > 0$. By (1.42), we get

$$\begin{aligned} [f]_{W^{\beta,1}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x+y) - f(x)|}{|y|^{n+\beta}}\,dx\,dy \\ &\le \int_{\mathbb{R}^n} \frac{1}{|y|^{n+\beta}} \left(2\|f\|_{L^1(\mathbb{R}^n)}\chi_{\mathbb{R}^n\setminus B_r}(y) + \gamma_{n,\alpha}|y|^\alpha \|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}\chi_{B_r}(y)\right) dy \\ &= 2\frac{n\omega_n}{\beta} r^{-\beta}\|f\|_{L^1(\mathbb{R}^n)} + \frac{n\omega_n}{\alpha-\beta}\gamma_{n,\alpha} r^{\alpha-\beta}\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \end{aligned}$$

$$\le \left(2\frac{n\omega_n}{\beta}r^{-\beta} + \frac{n\omega_n}{\alpha-\beta}\gamma_{n,\alpha}r^{\alpha-\beta}\right)\|f\|_{BV^\alpha(\mathbb{R}^n)},$$

so that both (1.53) and (1.54) are proved by minimising in $r > 0$ for all $f \in C_c^\infty(\mathbb{R}^n)$. Now let $f \in BV^\alpha(\mathbb{R}^n)$. By Theorem 1.16, there exists $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $\|f_k\|_{BV^\alpha(\mathbb{R}^n)} \to \|f\|_{BV^\alpha(\mathbb{R}^n)}$ and $f_k \to f$ a.e. as $k \to +\infty$. Thus, by Fatou's Lemma, we get that

$$[f]_{W^{\beta,1}(\mathbb{R}^n)} \le \liminf_{k\to+\infty} [f_k]_{W^{\beta,1}(\mathbb{R}^n)} \le \lim_{k\to+\infty} C_{n,\alpha,\beta}\,\|f_k\|_{BV^\alpha(\mathbb{R}^n)} = C_{n,\alpha,\beta}\,\|f\|_{BV^\alpha(\mathbb{R}^n)}$$

and the conclusion follows. □

Note that the constant in (1.54) satisfies $\lim_{\beta\to\alpha^-} C_{n,\alpha,\beta} = +\infty$. As an immediate consequence of Theorem 1.30, we get that $BV^\alpha(\mathbb{R}^n) \subset BV^\beta(\mathbb{R}^n)$ for all $0 < \beta < \alpha < 1$.

2.10. Relation between $BV^\alpha(\mathbb{R}^n)$ and $bv(\mathbb{R}^n)$. Given $\alpha \in (0,1)$, we notice that

$$\|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^1(\mathbb{R}^n)} \le \nu_{n,\alpha}[f]_{W^{\alpha,1}(\mathbb{R}^n)} \tag{1.55}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. Thus the linear operator

$$(-\Delta)^{\frac{\alpha}{2}} \colon C_c^\infty(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

can be continuously extended to a linear operator

$$(-\Delta)^{\frac{\alpha}{2}} \colon W^{\alpha,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n), \tag{1.56}$$

for which we retain the same notation.

Given $\alpha \in (0,1)$ and $\varepsilon > 0$, for all $f \in W^{\alpha,1}(\mathbb{R}^n)$ we set

$$(-\Delta)^{\alpha/2}_\varepsilon f(x) = \nu_{n,\alpha}\int_{\{|h|>\varepsilon\}} \frac{f(x+h)-f(x)}{|h|^{n+\alpha}}\,dh.$$

By Lebesgue's Dominate Convergence Theorem, we have that

$$\lim_{\varepsilon\to 0}\|(-\Delta)^{\alpha/2}_\varepsilon f - (-\Delta)^{\frac{\alpha}{2}} f\|_{L^1(\mathbb{R}^n)} = 0$$

for all $f \in W^{\alpha,1}(\mathbb{R}^n)$. Thus, arguing as in the proof of [**84**, Lemma 2.4] (see also [**85**, Section 25.1]), for all $f \in W^{\alpha,1}(\mathbb{R}^n)$ we have

$$I_\alpha(-\Delta)^{\frac{\alpha}{2}} f = f \quad \text{in } L^1(\mathbb{R}^n). \tag{1.57}$$

Taking advantage of the identity in (1.57), we can prove the following result.

Lemma 1.31 (Relation between $BV^\alpha(\mathbb{R}^n)$ and $bv(\mathbb{R}^n)$)**.** *Let $\alpha \in (0,1)$. The following properties hold.*

(i) If $f \in BV^\alpha(\mathbb{R}^n)$, then $u = I_{1-\alpha} f \in bv(\mathbb{R}^n)$ with $Du = D^\alpha f$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$.

(ii) If $u \in BV(\mathbb{R}^n)$, then $f = (-\Delta)^{\frac{1-\alpha}{2}} u \in BV^\alpha(\mathbb{R}^n)$ with

$$\|f\|_{L^1(\mathbb{R}^n)} \le c_{n,\alpha}\|u\|_{BV(\mathbb{R}^n)} \quad \text{and} \quad D^\alpha f = Du \quad \text{in } \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n).$$

As a consequence, the operator $(-\Delta)^{\frac{1-\alpha}{2}} \colon BV(\mathbb{R}^n) \to BV^\alpha(\mathbb{R}^n)$ is continuous.

Proof. We prove the two properties separately.

Proof of (i). Let $f \in BV^\alpha(\mathbb{R}^n)$. Since $f \in L^1(\mathbb{R}^n)$, we have $I_{1-\alpha} f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. By Fubini's Theorem, for any $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = \int_{\mathbb{R}^n} f \, I_{1-\alpha} \operatorname{div} \varphi \, dx = \int_{\mathbb{R}^n} u \operatorname{div} \varphi \, dx, \tag{1.58}$$

proving that $u = I_{1-\alpha} f \in bv(\mathbb{R}^n)$ with $Du = D^\alpha f$ in $\mathscr{M}(\mathbb{R}^n; \mathbb{R}^n)$.

Proof of (ii). Let $u \in BV(\mathbb{R}^n)$. By Theorem 1.27, we know that $u \in W^{1-\alpha,1}(\mathbb{R}^n)$, so that $f = (-\Delta)^{\frac{1-\alpha}{2}} u \in L^1(\mathbb{R}^n)$ with $\|f\|_{L^1(\mathbb{R}^n)} \le c_{n,\alpha} \|u\|_{BV(\mathbb{R}^n)}$ by (1.47) and (1.55). Then, arguing as before, for any $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ we get (1.58), since we have $I_{1-\alpha} f = u$ in $L^1(\mathbb{R}^n)$ by (1.57). The proof is complete. □

Remark 1.32 (Integrability issues)**.** Note that the inclusion

$$I_{1-\alpha}(BV^\alpha(\mathbb{R}^n)) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

in Lemma 1.31 above is sharp. Indeed, by Tonelli's Theorem it is easily seen that $I_{1-\alpha}\chi_E \notin L^1(\mathbb{R}^n)$ whenever $\chi_E \in W^{\alpha,1}(\mathbb{R}^n)$. However, when $n \ge 2$, by Theorem 1.17 and by Hardy–Littlewood–Sobolev inequality (see [**96**, Chapter V, Section 1.2] for instance), the map $I_{1-\alpha} \colon BV^\alpha \to L^p(\mathbb{R}^n)$ is continuous for each $p \in \left(\frac{n}{n-1+\alpha}, \frac{n}{n-1}\right]$.

3. The space $BV^{\alpha,p}$

3.1. Definition of $BV^{\alpha,p}(\mathbb{R}^n)$. Thanks to Lemma 1.31, we can relate functions with bounded α-variation and functions with bounded variation via Riesz potential and the fractional Laplacian. We would like to prove a similar result between functions with bounded α-variation and functions with bounded β-variation, for any couple of exponents $0 < \beta < \alpha < 1$.

However, although the standard variation of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is well define, it is not clear whether the functional

$$\varphi \mapsto \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx \tag{1.59}$$

is well posed for all $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, since $\operatorname{div}^\alpha \varphi$ does not have compact support. Nevertheless, thanks to Corollary 1.3, the functional in (1.59) is well defined as soon as $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, +\infty]$. Hence, it seems natural to define the space

$$BV^{\alpha,p}(\mathbb{R}^n) = \{ f \in L^p(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < \infty \} \tag{1.60}$$

for any $\alpha \in (0,1)$ and $p \in [1, +\infty]$. In particular, $BV^{\alpha,1}(\mathbb{R}^n) = BV^\alpha(\mathbb{R}^n)$. Similarly, we let

$$BV^{1,p}(\mathbb{R}^n) = \{ f \in L^p(\mathbb{R}^n) : |Df|(\mathbb{R}^n) < +\infty \}$$

for all $p \in [1, +\infty]$. In particular, $BV^{1,1}(\mathbb{R}^n) = BV(\mathbb{R}^n)$.

3.2. Weak Gagliardo–Nirenberg–Sobolev inequality. A further justification for the definition of these new spaces comes from the following fractional version of the Gagliardo–Nirenberg–Sobolev embedding given in Theorem 1.17: if $n \ge 2$ and $\alpha \in (0,1)$, then $BV^\alpha(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$ for all $p \in \left[1, \frac{n}{n-\alpha}\right]$. Hence, thanks to (1.60), we can equivalently write

$$BV^\alpha(\mathbb{R}^n) \subset BV^{\alpha,p}(\mathbb{R}^n)$$

with continuous embedding for all $n \ge 2$, $\alpha \in (0,1)$ and $p \in \left[1, \frac{n}{n-\alpha}\right]$.

In the case $n = 1$, the space $BV^{\alpha}(\mathbb{R})$ does not embed in $L^{\frac{1}{1-\alpha}}(\mathbb{R})$ with continuity by Example 1.22. However, somehow completing the picture provided by [**92**], we can prove that the space $BV^{\alpha}(\mathbb{R})$ continuously embeds in the Lorentz space $L^{\frac{1}{1-\alpha},\infty}(\mathbb{R})$. Although this result is truly interesting only for $n = 1$, we prove it below in all dimensions for the sake of completeness.

Theorem 1.33 (Weak Gagliardo–Nirenberg–Sobolev inequality)**.** *Given $\alpha \in (0,1)$, there exists a constant $c_{n,\alpha} > 0$ such that*

$$\|f\|_{L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)} \le c_{n,\alpha}|D^{\alpha}f|(\mathbb{R}^n) \tag{1.61}$$

for all $f \in BV^{\alpha}(\mathbb{R}^n)$. As a consequence, $BV^{\alpha}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [1, \frac{n}{n-\alpha})$.

Proof. Let $f \in C_c^{\infty}(\mathbb{R}^n)$. By (1.38), we have

$$f(x) = -\operatorname{div}^{-\alpha}\nabla^{\alpha}f(x) = -\mu_{n,-\alpha}\int_{\mathbb{R}^n} \frac{(y-x)\cdot\nabla^{\alpha}f(y)}{|y-x|^{n+1-\alpha}}\,dy, \quad x \in \mathbb{R}^n,$$

so that

$$|f(x)| \le \mu_{n,-\alpha}\int_{\mathbb{R}^n} \frac{|\nabla^{\alpha}f(y)|}{|y-x|^{n-\alpha}}\,dy = \frac{\mu_{n,-\alpha}}{\mu_{n,1-\alpha}}\,(n-\alpha)\,I_{\alpha}|\nabla^{\alpha}f|(x), \quad x \in \mathbb{R}^n.$$

Since the operator $I_{\alpha}\colon L^1(\mathbb{R}^n) \to L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$ is continuous by Hardy–Littlewood–Sobolev inequality (see [**96**, Theorem 1, Chapter V] or [**47**, Theorem 1.2.3]), we can estimate

$$\|f\|_{L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)} \le \frac{n\,\mu_{n,-\alpha}}{\mu_{n,1-\alpha}}\|I_{\alpha}|\nabla^{\alpha}f|\|_{L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)} \le c_{n,\alpha}\||\nabla^{\alpha}f|\|_{L^1(\mathbb{R}^n)} = c_{n,\alpha}\,|D^{\alpha}f|(\mathbb{R}^n),$$

where $c_{n,\alpha} > 0$ is a constant depending only on n and α. Thus, inequality (1.61) follows for all $f \in C_c^{\infty}(\mathbb{R}^n)$. Now let $f \in BV^{\alpha}(\mathbb{R}^n)$. By Theorem 1.16, there exists $(f_k)_{k\in\mathbb{N}} \subset C_c^{\infty}(\mathbb{R}^n)$ such that $f_k \to f$ a.e. in $\mathbb{R}^n$ and $|D^{\alpha}f_k|(\mathbb{R}^n) \to |D^{\alpha}f|(\mathbb{R}^n)$ as $k \to +\infty$. By Fatou's Lemma in Lorentz spaces (see [**47**, Exercise 1.4.11] for example), we thus get

$$\|f\|_{L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)} \le \liminf_{k\to+\infty}\|f_k\|_{L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)} \le c_{n,\alpha}\lim_{k\to+\infty}|D^{\alpha}f_k|(\mathbb{R}^n) = c_{n,\alpha}|D^{\alpha}f|(\mathbb{R}^n)$$

and so (1.61) readily follows. Finally, thanks to [**47**, Proposition 1.1.14], we obtain the continuous embedding of $BV^{\alpha}(\mathbb{R}^n)$ in $L^q(\mathbb{R}^n)$ for all $q \in [1, \frac{n}{n-\alpha})$. □

Remark 1.34 (The embedding $BV^{\alpha}(\mathbb{R}) \subset L^{\frac{1}{1-\alpha},\infty}(\mathbb{R})$ is sharp)**.** Let $\alpha \in (0,1)$. The continuous embedding $BV^{\alpha}(\mathbb{R}) \subset L^{\frac{1}{1-\alpha},\infty}(\mathbb{R})$ is sharp at the level of Lorentz spaces, in the sense that $BV^{\alpha}(\mathbb{R}^n) \setminus L^{\frac{1}{1-\alpha},q}(\mathbb{R}) \ne \varnothing$ for any $q \in [1,+\infty)$. Indeed, if we let

$$f_{\alpha}(x) = |x-1|^{\alpha-1}\operatorname{sgn}(x-1) - |x|^{\alpha-1}\operatorname{sgn}(x), \qquad x \in \mathbb{R}\setminus\{0,1\},$$

then $f_{\alpha} \in BV^{\alpha}(\mathbb{R})$ by Example 1.22, and it is not difficult to prove that $f_{\alpha} \in L^{\frac{1}{1-\alpha},\infty}(\mathbb{R})$. However, we can find a constant $c_{\alpha} > 0$ such that

$$|f_{\alpha}(x)| \ge c_{\alpha}|x|^{\alpha-1}\chi_{\left(-\frac{1}{4},\frac{1}{4}\right)}(x) =: g_{\alpha}(x), \qquad x \in \mathbb{R}\setminus\{0,1\},$$

so that $d_{f_\alpha} \ge d_{g_\alpha}$, where d_{f_α} and d_{g_α} are the *distribution functions* of f_α and g_α. A simple calculation shows that

$$d_{g_\alpha}(s) = \begin{cases} \dfrac{1}{2} & \text{if } 0 < s \le c_\alpha 4^{1-\alpha} \\ 2\left(\dfrac{c_\alpha}{t}\right)^{\frac{1}{1-\alpha}} & \text{if } \ s > c_\alpha 4^{1-\alpha}, \end{cases}$$

so that, by [**47**, Proposition 1.4.9], we obtain

$$\begin{aligned} \|f_\alpha\|^q_{L^{\frac{1}{1-\alpha},q}(\mathbb{R})} \ge \|g_\alpha\|^q_{L^{\frac{1}{1-\alpha},q}(\mathbb{R})} &= \frac{1}{1-\alpha}\int_0^{+\infty} [d_{g_\alpha}(s)]^{q(1-\alpha)}\, s^{q-1}\, ds \\ &\ge \frac{2^{q(1-\alpha)}}{1-\alpha}\int_{c_\alpha 4^{1-\alpha}}^{+\infty} s^{-q} s^{q-1}\, ds = +\infty \end{aligned}$$

and thus $f_\alpha \notin L^{\frac{1}{1-\alpha},q}(\mathbb{R})$ for any $q \in [1, +\infty)$.

We collect the above continuous embeddings in the following statement.

Corollary 1.35 (The embedding $BV^\alpha \subset BV^{\alpha,p}$)**.** *Let $\alpha \in (0,1)$ and $p \in \left[1, \frac{n}{n-\alpha}\right)$. We have $BV^\alpha(\mathbb{R}^n) \subset BV^{\alpha,p}(\mathbb{R}^n)$ with continuous embedding. In addition, if $n \ge 2$, then also $BV^\alpha(\mathbb{R}^n) \subset BV^{\alpha,\frac{n}{n-\alpha}}(\mathbb{R}^n)$ with continuous embedding.*

3.3. Relation between BV^β and $BV^{\alpha,p}$ for $\beta < \alpha$ and $p > 1$. With Corollary 1.35 at hands, we are finally ready to investigate the relation between α-variation and β-variation for $0 < \beta < \alpha < 1$.

Lemma 1.36. *Let $0 < \beta < \alpha < 1$. The following hold.*

(i) If $f \in BV^\beta(\mathbb{R}^n)$, then $u = I_{\alpha-\beta} f \in BV^{\alpha,p}(\mathbb{R}^n)$ for any $p \in \left(\frac{n}{n-\alpha+\beta}, \frac{n}{n-\alpha}\right)$ (including $p = \frac{n}{n-\alpha}$ if $n \ge 2$), with $D^\alpha u = D^\beta f$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$.

(ii) If $u \in BV^\alpha(\mathbb{R}^n)$, then $f = (-\Delta)^{\frac{\alpha-\beta}{2}} u \in BV^\beta(\mathbb{R}^n)$ with

$$\|f\|_{L^1(\mathbb{R}^n)} \le c_{n,\alpha,\beta}\, \|u\|_{BV^\alpha(\mathbb{R}^n)} \quad \textit{and} \quad D^\beta f = D^\alpha u \quad \textit{in } \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n).$$

As a consequence, the operator $(-\Delta)^{\frac{\alpha-\beta}{2}} : BV^\alpha(\mathbb{R}^n) \to BV^\beta(\mathbb{R}^n)$ is continuous.

Proof. We begin with the following observation. Let $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and let $U \subset \mathbb{R}^n$ be a bounded open set such that $\operatorname{supp}\varphi \subset U$. By Corollary 1.3 and the *semigroup property* (N.58) of the Riesz potential, we can write

$$\operatorname{div}^\beta\varphi = I_{1-\beta}\operatorname{div}\varphi = I_{\alpha-\beta} I_{1-\alpha}\operatorname{div}\varphi = I_{\alpha-\beta}\operatorname{div}^\alpha\varphi.$$

Similarly, we also have

$$I_{\alpha-\beta}|\operatorname{div}^\alpha\varphi| = I_{\alpha-\beta}|I_{1-\alpha}\operatorname{div}\varphi| \le I_{\alpha-\beta} I_{1-\alpha}|\operatorname{div}\varphi| = I_{1-\beta}|\operatorname{div}\varphi|,$$

so that $I_{\alpha-\beta}|\operatorname{div}^\alpha\varphi| \in L^\infty(\mathbb{R}^n)$ with

$$\|I_{\alpha-\beta}|\operatorname{div}^\alpha\varphi|\|_{L^\infty(\mathbb{R}^n)} \le \|I_{1-\beta}|\operatorname{div}\varphi|\|_{L^\infty(\mathbb{R}^n)} \le C_{n,\beta,U}\|\operatorname{div}\varphi\|_{L^\infty(\mathbb{R}^n)}$$

by Lemma 1.4. We now prove the two statements separately.

Proof of (i). Let $f \in BV^\beta(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Thanks to Corollary 1.35, if $n \ge 2$, then $f \in BV^{\beta,q}(\mathbb{R}^n)$ for any $q \in [1, \frac{n}{n-\beta}]$ and so $I_{\alpha-\beta} f \in L^p(\mathbb{R}^n)$ for any $p \in \left(\frac{n}{n-\alpha+\beta}, \frac{n}{n-\alpha}\right]$ by (N.59). If instead $n = 1$, then $f \in BV^{\beta,q}(\mathbb{R})$ for any $q \in [1, \frac{1}{1-\beta})$

and so $I_{\alpha-\beta} f \in L^p(\mathbb{R})$ for any $p \in \left(\frac{1}{1-\alpha+\beta}, \frac{1}{1-\alpha}\right)$. Since $f \in L^1(\mathbb{R}^n)$ and $I_{\alpha-\beta}|\mathrm{div}^\alpha \varphi| \in L^\infty(\mathbb{R}^n)$, by Fubini's Theorem we have

$$\int_{\mathbb{R}^n} f \,\mathrm{div}^\beta \varphi \, dx = \int_{\mathbb{R}^n} f \, I_{\alpha-\beta} \mathrm{div}^\alpha \varphi \, dx = \int_{\mathbb{R}^n} u \,\mathrm{div}^\alpha \varphi \, dx, \tag{1.62}$$

proving that $u = I_{\alpha-\beta} f \in BV^{\alpha,p}(\mathbb{R}^n)$ for any $p \in \left(\frac{n}{n-\alpha+\beta}, \frac{n}{n-\alpha}\right)$ (including $p = \frac{n}{n-\alpha}$ if $n \geq 2$), with $D^\alpha u = D^\beta f$ in $\mathscr{M}(\mathbb{R}^n; \mathbb{R}^n)$.

Proof of (ii). Let $u \in BV^\alpha(\mathbb{R}^n)$. By Theorem 1.30, we know that $u \in W^{\alpha-\beta,1}(\mathbb{R}^n)$, so that $f = (-\Delta)^{\frac{\alpha-\beta}{2}} u \in L^1(\mathbb{R}^n)$ with $\|f\|_{L^1(\mathbb{R}^n)} \leq c_{n,\alpha,\beta} \|u\|_{BV^\alpha(\mathbb{R}^n)}$ by (1.56). Then, arguing as before, for any $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ we get (1.62), since we have $I_{\alpha-\beta} f = u$ in $L^1(\mathbb{R}^n)$ by (1.57). The proof is complete. □

3.4. Fractional variation of Radon measures. Let $\alpha \in (0,1)$. Exploiting Proposition 1.2 and Corollary 1.3, it is not difficult to see that, if $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ then $\mathrm{div}^\alpha \varphi \in \mathrm{Lip}_b(\mathbb{R}^n)$. Thus we can define the fractional gradient of any finite Radon measure $\mu \in \mathscr{M}(\mathbb{R}^n)$ by setting

$$\langle D^\alpha \mu, \varphi \rangle = -\int_{\mathbb{R}^n} \mathrm{div}^\alpha \varphi \, d\mu \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n),$$

see also [**92**, Section 6]. The following result shows that, in accordance with the classical BV case, BV^α functions are exactly the densities of finite Radon measures having fractional gradient equal to a finite n-vector valued Radon measure.

Proposition 1.37. *Let $\alpha \in (0,1)$. If $\mu \in \mathcal{M}(\mathbb{R}^n)$ is such that $D^\alpha \mu \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^n)$, then $\mu = f \mathscr{L}^n$ for some $f \in BV^\alpha(\mathbb{R}^n)$.*

Proof. Let $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^n)$ be a family of standard mollifiers as in (1.25). Arguing exactly as in the proof of Theorem 1.15, $\varrho_\varepsilon * \mu \in BV^\alpha(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ with $D^\alpha(\varrho_\varepsilon * \mu) = \varrho_\varepsilon * D^\alpha \mu$ for all $\varepsilon > 0$. By Theorem 1.33, we thus have

$$\|\varrho_\varepsilon * \mu\|_{L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)} \leq c_{n,\alpha} \|\varrho_\varepsilon * D^\alpha \mu\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \leq c_{n,\alpha} |D^\alpha \mu|(\mathbb{R}^n)$$

for all $\varepsilon > 0$. Hence, recalling [**47**, Propositions 1.1.6 and 1.1.14], the family $(\varrho_\varepsilon * \mu)_{\varepsilon>0}$ is uniformly bounded in $L^q(\mathbb{R}^n)$ for any given $q \in \left[1, \frac{n}{n-\alpha}\right)$. Therefore there exist a subsequence $(\varrho_{\varepsilon_k} * \mu)_{k\in\mathbb{N}}$ and a function $f \in L^q(\mathbb{R}^n)$ such that $\varrho_{\varepsilon_k} * \mu \rightharpoonup f$ in $L^q(\mathbb{R}^n)$ as $k \to +\infty$. However, $\varrho_\varepsilon * \mu \rightharpoonup \mu$ in $\mathscr{M}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$, so we must have $\mu = f \mathscr{L}^n$, which immediately implies that $f \in BV^\alpha(\mathbb{R}^n)$. □

4. The space BV^0

4.1. Definition of $BV^0(\mathbb{R}^n)$ and Structure Theorem. Somehow naturally extending the validity of Proposition 1.2 to the case $\alpha = 0$, for $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$ we define

$$\nabla^0 f = I_1 \nabla f \quad \text{and} \quad \mathrm{div}^0 \varphi = I_1 \mathrm{div} \varphi.$$

It is immediate to check that the integration-by-part formula

$$\int_{\mathbb{R}^n} f \,\mathrm{div}^0 \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^0 f \, dx \tag{1.63}$$

holds for all given $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$. Hence, in analogy with Definition 1.9, we are led to the following definition.

Definition 1.38 ($BV^0(\mathbb{R}^n)$ space)**.** A function $f \in L^1(\mathbb{R}^n)$ belongs to the space $BV^0(\mathbb{R}^n)$ if

$$\sup\left\{\int_{\mathbb{R}^n} f \operatorname{div}^0\varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1\right\} < +\infty.$$

The proof of the following result is very similar to the one of Theorem 1.10 and we omit it.

Theorem 1.39 (Structure Theorem for BV^0 functions)**.** *Let $f \in L^1(\mathbb{R}^n)$. Then, $f \in BV^0(\mathbb{R}^n)$ if and only if there exists a finite vector-valued Radon measure $D^0 f \in \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} f \operatorname{div}^0\varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot dD^0 f \tag{1.64}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. In addition, for any open set $U \subset \mathbb{R}^n$ it holds

$$|D^0 f|(U) = \sup\left\{\int_{\mathbb{R}^n} f \operatorname{div}^0\varphi \, dx : \varphi \in C_c^\infty(U;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(U;\mathbb{R}^n)} \le 1\right\}. \tag{1.65}$$

4.2. The identification $BV^0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$. We now prove that the space $BV^0(\mathbb{R}^n)$ actually coincides with the Hardy space $H^1(\mathbb{R}^n)$. Precisely, we have the following result.

Theorem 1.40 (The identification $BV^0 = H^1$)**.** *We have $BV^0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, in the sense that $f \in BV^0(\mathbb{R}^n)$ if and only if $f \in H^1(\mathbb{R}^n)$, with $D^0 f = Rf\,\mathscr{L}^n$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$.*

Proof. We prove the two inclusions separately.

Proof of $H^1(\mathbb{R}^n) \subset BV^0(\mathbb{R}^n)$. Let $f \in H^1(\mathbb{R}^n)$ and assume $f \in \mathrm{Lip}_c(\mathbb{R}^n)$. By (1.63), we immediately get that $D^0 f = Rf\,\mathscr{L}^n$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$ with $Rf = \nabla^0 f$ in $L^1(\mathbb{R}^n;\mathbb{R}^n)$, so that $f \in BV^0(\mathbb{R}^n)$. Now let $f \in H^1(\mathbb{R}^n)$. By [**97**, Chapter III, Section 5.2(b)], we can find $(f_k)_{k\in\mathbb{N}} \subset H^1(\mathbb{R}^n) \cap C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $H^1(\mathbb{R}^n)$ as $k \to +\infty$. Hence, given $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} f_k \operatorname{div}^0\varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot Rf_k \, dx$$

for all $k \in \mathbb{N}$. Passing to the limit as $k \to +\infty$, we get

$$\int_{\mathbb{R}^n} f \operatorname{div}^0\varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot Rf \, dx$$

so that $f \in BV^0(\mathbb{R}^n)$ with $D^0 f = Rf\,\mathscr{L}^n$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$ according to (1.65).

Proof of $BV^0(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$. Let $f \in BV^0(\mathbb{R}^n)$. Since $f \in L^1(\mathbb{R}^n)$, Rf is well-defined as a (vector-valued) distribution, see [**97**, Chapter III, Section 4.3]. Thanks to (1.64), we also have that $\langle Rf, \varphi\rangle = \langle D^0 f, \varphi\rangle$ for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, so that $Rf = D^0 f$ in the sense of distributions. Now let $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^n)$ be a family of standard mollifiers (see e.g. [**22**, Section 3.2]). We can thus estimate

$$\|Rf * \varrho_\varepsilon\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} = \|D^0 f * \varrho_\varepsilon\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le |D^0 f|(\mathbb{R}^n)$$

for all $\varepsilon > 0$, so that $f \in H^1(\mathbb{R}^n)$ by Proposition 3 in [**97**, Chapter III, Section 4.3], with $D^0 f = Rf\mathscr{L}^n$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$. □

4.3. Relation between $W^{\alpha,1}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$. Thanks to the identification established in Theorem 1.40, we can prove the following result.

Proposition 1.41. *Let $\alpha \in (0,1]$. The following hold.*

(i) If $f \in H^1(\mathbb{R}^n)$, then $u = I_\alpha f \in BV^{\alpha,\frac{n}{n-\alpha}}(\mathbb{R}^n)$ with $D^\alpha u = D^0 f$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$.

(ii) If $\alpha \in (0,1)$ and $u \in W^{\alpha,1}(\mathbb{R}^n)$, then $f = (-\Delta)^{\alpha/2} u \in H^1(\mathbb{R}^n)$ with

$$\|f\|_{L^1(\mathbb{R}^n)} \le \mu_{n,-\alpha} [u]_{W^{\alpha,1}(\mathbb{R}^n)} \quad \text{and} \quad Rf = \nabla^\alpha u \text{ a.e. in } \mathbb{R}^n.$$

Proof. We prove the two statements separately.

Proof of (i). Let $f \in H^1(\mathbb{R}^n)$. By Stein–Weiss inequality (see [**88**, Theorem 2] for instance), we know that $u = I_\alpha f \in L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. To prove that $|D^\alpha u|(\mathbb{R}^n) < +\infty$, we exploit Theorem 1.40 and argue similarly as in the proof of Lemma 1.31. Indeed, for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we can write

$$\int_{\mathbb{R}^n} f \operatorname{div}^0 \varphi \, dx = \int_{\mathbb{R}^n} f \, I_\alpha \operatorname{div}^\alpha \varphi \, dx = \int_{\mathbb{R}^n} u \operatorname{div}^\alpha \varphi \, dx$$

by Fubini's Theorem, since $f \in L^1(\mathbb{R}^n)$ and $I_\alpha |\operatorname{div}^\alpha \varphi| \in L^\infty(\mathbb{R}^n)$, being

$$I_\alpha |\operatorname{div}^\alpha \varphi| = I_\alpha |I_{1-\alpha} \operatorname{div} \varphi| \le I_\alpha I_{1-\alpha} |\operatorname{div} \varphi| = I_1 |\operatorname{div} \varphi| \in L^\infty(\mathbb{R}^n)$$

thanks to the semigroup property (N.58) of the Riesz potential. This proves that $D^\alpha u = D^0 f$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$.

Proof of (ii). Let $u \in W^{\alpha,1}(\mathbb{R}^n)$. Then $f = (-\Delta)^{\alpha/2} u$ satisfies

$$\|f\|_{L^1(\mathbb{R}^n)} = \mu_{n,-\alpha} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y-x|^{n+\alpha}} \, dy \right| dx \le \mu_{n,-\alpha} [u]_{W^{\alpha,1}(\mathbb{R}^n)}.$$

To prove that $f \in H^1(\mathbb{R}^n)$, we exploit Theorem 1.40 again. For all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we can write

$$\int_{\mathbb{R}^n} u \operatorname{div}^\alpha \varphi \, dx = \int_{\mathbb{R}^n} u \, (-\Delta)^{\frac{\alpha}{2}} \operatorname{div}^0 \varphi \, dx = \int_{\mathbb{R}^n} f \operatorname{div}^0 \varphi \, dx$$

by Fubini's Theorem, since $u \in L^1(\mathbb{R}^n)$ and $\operatorname{div}^0 \varphi \in \operatorname{Lip}_b(\mathbb{R}^n;\mathbb{R}^n)$, proving that $f = (-\Delta)^{\alpha/2} u \in H^1(\mathbb{R}^n)$ with $D^0 f = D^\alpha u$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$. Since $D^\alpha u = \nabla^\alpha u \, \mathscr{L}^n$ by Theorem 1.27 and $D^0 f = Rf \, \mathscr{L}^n$ by Theorem 1.40, the conclusion follows. □

5. The space $S^{\alpha,p}$

5.1. Definition of $S^{\alpha,p}(\mathbb{R}^n)$. We are now tempted to approach fractional Sobolev spaces from a distributional point of view. Recalling Corollary 1.3, we can give the following definition.

Definition 1.42 (Weak α-gradient)**.** Let $\alpha \in (0,1)$, $p \in [1,+\infty]$, $f \in L^p(\mathbb{R}^n)$. We say that $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ is a *weak α-gradient* of f, and we write $g = \nabla^\alpha f$, if

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} g \cdot \varphi \, dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

For $\alpha \in (0,1)$ and $p \in [1,+\infty]$, we can thus introduce the *distributional fractional Sobolev space* $(S^{\alpha,p}(\mathbb{R}^n), \|\cdot\|_{S^{\alpha,p}(\mathbb{R}^n)})$ letting

$$S^{\alpha,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \exists\, \nabla^\alpha f \in L^p(\mathbb{R}^n;\mathbb{R}^n)\} \tag{1.66}$$

and

$$\|f\|_{S^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}, \qquad \forall f \in S^{\alpha,p}(\mathbb{R}^n). \tag{1.67}$$

5.2. Lower semicontinuity of $S^{\alpha,p}$-energy. Similarly to the BV^α-case, the $S^{\alpha,p}$-energy is lower semicontinuous with respect to L^p-convergence. The proof of the following result is very similar to the one of Proposition 1.11 and is left to the reader.

Proposition 1.43. *Let $\alpha \in (0,1)$ and $p \in [1,+\infty)$. If $(f_k)_{k\in\mathbb{N}} \subset S^{\alpha,p}(\mathbb{R}^n)$ is such that*

$$\liminf_{k\to+\infty} \|\nabla^\alpha f_k\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} < +\infty$$

and $f_k \to f$ in $L^p(\mathbb{R}^n)$ as $k \to +\infty$, then $f \in S^{\alpha,p}(\mathbb{R}^n)$ with

$$\|\nabla^\alpha f\|_{L^p(U;\,\mathbb{R}^n)} \le \liminf_{k\to+\infty} \|\nabla^\alpha f_k\|_{L^p(U;\,\mathbb{R}^n)} \tag{1.68}$$

for any open set $U \subset \mathbb{R}^n$.

We omit the standard proof of the following result.

Proposition 1.44 ($S^{\alpha,p}$ is a Banach space)**.** *Let $\alpha \in (0,1)$ and $p \in [1,+\infty]$. The space $(S^{\alpha,p}(\mathbb{R}^n), \|\cdot\|_{S^{\alpha,p}(\mathbb{R}^n)})$ is a Banach space.*

We leave the proof of the following interpolation result to the reader.

Lemma 1.45 (Interpolation)**.** *Let $\alpha \in (0,1)$ and $p_1, p_2 \in [1,+\infty]$, with $p_1 \le p_2$. Then*

$$S^{\alpha,p_1}(\mathbb{R}^n) \cap S^{\alpha,p_2}(\mathbb{R}^n) \subset S^{\alpha,q}(\mathbb{R}^n)$$

with continuous embedding for all $q \in [p_1, p_2]$.

5.3. Approximation by smooth functions. Taking advantage of the techniques developed in the study of the space $BV^\alpha(\mathbb{R}^n)$ above, we are able to prove the following approximation result.

Theorem 1.46 (Approximation by $C^\infty \cap S^{\alpha,p}$ functions)**.** *Let $\alpha \in (0,1)$ and $p \in [1,+\infty)$. The set $C^\infty(\mathbb{R}^n) \cap S^{\alpha,p}(\mathbb{R}^n)$ is dense in $S^{\alpha,p}(\mathbb{R}^n)$.*

Proof. Let $(\varrho_\varepsilon)_{\varepsilon>0} \subset C^\infty_c(\mathbb{R}^n)$ be as in (1.25). Fix $f \in S^{\alpha,p}(\mathbb{R}^n)$ and consider $f_\varepsilon = f * \varrho_\varepsilon$ for all $\varepsilon > 0$. By Lemma 1.13, it is easy to check that $f_\varepsilon \in C^\infty(\mathbb{R}^n) \cap S^{\alpha,p}(\mathbb{R}^n)$ with $\nabla^\alpha f_\varepsilon = \varrho_\varepsilon * \nabla^\alpha f$ for all $\varepsilon > 0$, so that the conclusion follows by standard properties of the convolution. □

5.4. Approximation by test functions. Given $\alpha \in (0,1)$ and $p \in [1,+\infty]$, it is easy to see that, if $f \in C^\infty_c(\mathbb{R}^n)$, then, by Lemma 1.5, $f \in S^{\alpha,p}(\mathbb{R}^n)$ with $\nabla^\alpha f$ given as in (1.2).

In the case $p = 1$, we can prove that $C^\infty_c(\mathbb{R}^n)$ is a dense subset of $S^{\alpha,1}(\mathbb{R}^n)$ by arguing similarly as in the proof of Theorem 1.16.

Theorem 1.47 (Approximation by C^∞_c functions in $S^{\alpha,1}$)**.** *Let $\alpha \in (0,1)$. The set $C^\infty_c(\mathbb{R}^n)$ is dense in $S^{\alpha,1}(\mathbb{R}^n)$.*

Proof. Let $(\eta_R)_{R>0} \subset C_c^\infty(\mathbb{R}^n)$ be as in (1.30). Thanks to Theorem 1.46, it is enough to prove that $f\eta_R \to f$ in $S^{\alpha,1}(\mathbb{R}^n)$ as $R \to +\infty$ for all $f \in C^\infty(\mathbb{R}^n) \cap S^{\alpha,1}(\mathbb{R}^n)$. Clearly, $f\eta_R \to f$ in $L^1(\mathbb{R}^n)$ as $R \to +\infty$. We now argue as in the proof of Theorem 1.16. Fix $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Then, by Lemma 1.7, we get

$$\int_{\mathbb{R}^n} f\eta_R \operatorname{div}^\alpha \varphi \, dx = \int_{\mathbb{R}^n} f \operatorname{div}^\alpha(\eta_R\varphi)\, dx - \int_{\mathbb{R}^n} f\,\varphi \cdot \nabla^\alpha \eta_R \, dx - \int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\eta_R, \varphi)\, dx.$$

Since $f \in S^{\alpha,1}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha(\eta_R\varphi)\, dx = -\int_{\mathbb{R}^n} \eta_R \varphi \cdot \nabla^\alpha f \, dx.$$

Since $f\eta_R \in C_c^\infty(\mathbb{R}^n)$, we also have

$$\int_{\mathbb{R}^n} f\eta_R \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha(\eta_R f)\, dx.$$

Thus we can write

$$\begin{aligned}\int_{\mathbb{R}^n} (\nabla^\alpha f - \nabla^\alpha(\eta_R f)) \cdot \varphi \, dx &= \int_{\mathbb{R}^n} (1-\eta_R)\varphi \cdot \nabla^\alpha f \, dx \\ &\quad - \int_{\mathbb{R}^n} f\,\varphi \cdot \nabla^\alpha \eta_R \, dx - \int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\eta_R, \varphi)\, dx.\end{aligned}$$

Moreover, we have

$$\left| \int_{\mathbb{R}^n} f\,\varphi \cdot \nabla^\alpha \eta_R \, dx \right| \le \mu_{n,\alpha} \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y) - \eta_R(x)|}{|y-x|^{n+\alpha}}\, dy\, dx$$

and, similarly,

$$\left| \int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\eta_R, \varphi)\, dx \right| \le 2\mu_{n,\alpha} \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y) - \eta_R(x)|}{|y-x|^{n+\alpha}}\, dy\, dx.$$

Combining these two estimates, we get that

$$\begin{aligned}\left| \int_{\mathbb{R}^n} (\nabla^\alpha f - \nabla^\alpha(\eta_R f)) \cdot \varphi \, dx \right| &\le \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \int_{\mathbb{R}^n} (1-\eta_R)|\nabla^\alpha f| \, dx \\ &\quad + 3\mu_{n,\alpha} \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y) - \eta_R(x)|}{|y-x|^{n+\alpha}}\, dy\, dx.\end{aligned}$$

We thus conclude that

$$\begin{aligned}\|\nabla^\alpha f - \nabla^\alpha(\eta_R f)\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} &\le \int_{\mathbb{R}^n} (1-\eta_R)|\nabla^\alpha f| \, dx \\ &\quad + 3\mu_{n,\alpha} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y) - \eta_R(x)|}{|y-x|^{n+\alpha}}\, dy\, dx.\end{aligned}$$

Therefore $\nabla^\alpha(\eta_R f) \to \nabla^\alpha f$ in $L^1(\mathbb{R}^n;\mathbb{R}^n)$ as $R \to +\infty$. Indeed, we have

$$\lim_{R\to+\infty} \int_{\mathbb{R}^n} (1-\eta_R)|\nabla^\alpha f| \, dx = 0$$

combining (1.30) with Lebesgue's Dominated Convergence Theorem and

$$\lim_{R\to+\infty} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y) - \eta_R(x)|}{|y-x|^{n+\alpha}}\, dy\, dx = 0$$

combining (1.6), (1.7) and (1.30) with Lebesgue's Dominated Convergence Theorem. □

To prove that $C_c^\infty(\mathbb{R}^n)$ is also a dense subset of $S^{\alpha,p}(\mathbb{R}^n)$ for $p \in (1,+\infty)$, we need to adopt a different strategy. We consider the space

$$S_0^{\alpha,p}(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{S^{\alpha,p}(\mathbb{R}^n)}}$$

naturally endowed with the $S^{\alpha,p}$-norm. The space $(S_0^{\alpha,p}(\mathbb{R}^n), \|\cdot\|_{S^{\alpha,p}(\mathbb{R}^n)})$ was introduced in [**89**] (with a different, but equivalent, norm). Thanks to [**89**, Theorem 1.7], for all $\alpha \in (0,1)$ and $p \in (1,+\infty)$ we have $S_0^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$, where $L^{\alpha,p}(\mathbb{R}^n)$ is the Bessel potential space recalled in (N.60). It is known that $L^{\alpha+\varepsilon,p}(\mathbb{R}^n) \subset W^{\alpha,p}(\mathbb{R}^n) \subset L^{\alpha-\varepsilon,p}(\mathbb{R}^n)$ with continuous embeddings for all $\alpha \in (0,1)$, $p \in (1,+\infty)$ and $0 < \varepsilon < \min\{\alpha, 1-\alpha\}$, see [**89**, Theorem 2.2]. In the particular case $p = 2$, it holds that $L^{\alpha,2}(\mathbb{R}^n) = W^{\alpha,2}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$, see [**89**, Theorem 2.2]. In addition, $W^{\alpha,p}(\mathbb{R}^n) \subset L^{\alpha,p}(\mathbb{R}^n)$ with continuous embedding for all $\alpha \in (0,1)$ and $p \in (1,2]$, see [**96**, Chapter V, Section 5.3].

With this notation, the density of the set $C_c^\infty(\mathbb{R}^n)$ in $S^{\alpha,p}(\mathbb{R}^n)$ is obviously equivalent to the density of the set $S_0^{\alpha,p}(\mathbb{R}^n)$, i.e., of the set $L^{\alpha,p}(\mathbb{R}^n)$, in $S^{\alpha,p}(\mathbb{R}^n)$. We thus take advantage of some of the properties of the Bessel potential and of the fractional Laplacian. We begin with the following integration-by-part formula.

Lemma 1.48. *Let $p, q \in (1,+\infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\mathbb{R}^n)$ and $\varphi \in L^q(\mathbb{R}^n; \mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} f \operatorname{div}^0 \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^0 f \, dx. \tag{1.69}$$

Proof. Integrating by parts and applying Fubini's Theorem, formula (1.69) is easily proved for all $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$. Since the real-valued bilinear functionals

$$(f,\varphi) \mapsto \int_{\mathbb{R}^n} f \operatorname{div}^0 \varphi \, dx, \qquad (f,\varphi) \mapsto \int_{\mathbb{R}^n} \varphi \cdot \nabla^0 f \, dx,$$

are both continuous on $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n; \mathbb{R}^n)$ by Hölder's inequality and the L^p-continuity of Riesz transform, the conclusion follows by a simple approximation argument. □

Adopting the notation introduced in [**92**, Equation (1.9)], for $\alpha \in (0,1)$ and $f \in C_c^\infty(\mathbb{R}^n)$, we let

$$\mathcal{D}^\alpha f(x) = \int_{\mathbb{R}^n} \frac{|f(y+x) - f(y)|}{|y|^{n+\alpha}} \, dy$$

for all $x \in \mathbb{R}^n$. In the following result we prove that the operator $\mathcal{D}^\alpha$ naturally extends to a continuous operator from $W^{1,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R})$, see also [**60**, Lemma p. 114].

Lemma 1.49. *Let $\alpha \in (0,1)$ and $p \in [1,+\infty]$. The operator $\mathcal{D}^\alpha \colon W^{1,p}(\mathbb{R}^n) \to L^p(\mathbb{R})$ is well defined and satisfies*

$$\|\mathcal{D}^\alpha f\|_{L^p(\mathbb{R}^n)} \le \frac{2n\omega_n \nu_{n,\alpha}}{\alpha(1-\alpha)} \|f\|_{L^p(\mathbb{R}^n)}^\alpha \|\nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{1-\alpha} \tag{1.70}$$

for all $f \in W^{1,p}(\mathbb{R}^n)$.

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$ and $r > 0$. We can estimate

$$\mathcal{D}^\alpha f(x) \le \nu_{n,\alpha} \left(\int_{|y|<r} \frac{|f(y+x) - f(x)|}{|y|^{n+\alpha}} \, dy + \int_{|y| \ge r} \frac{|f(y+x) - f(x)|}{|y|^{n+\alpha}} \, dy \right)$$

for all $x \in \mathbb{R}^n$. By Minkowski's integral inequality, on the one hand we have

$$\begin{aligned}
\left\| \int_{|y|<r} \frac{|f(y+\cdot)-f(\cdot)|}{|y|^{n+\alpha}}\,dy \right\|_{L^p(\mathbb{R}^n)} &\le \int_{|y|<r} \frac{\|f(y+\cdot)-f(\cdot)\|_{L^p(\mathbb{R}^n)}}{|y|^{n+\alpha}}\,dy \\
&\le \|\nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \int_{|y|<r} \frac{dy}{|y|^{n+\alpha-1}} \\
&= \frac{n\omega_n r^{1-\alpha}}{1-\alpha} \|\nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}
\end{aligned} \tag{1.71}$$

while, on the other hand, we have

$$\begin{aligned}
\left\| \int_{|y|\ge r} \frac{|f(y+\cdot)-f(\cdot)|}{|y|^{n+\alpha}}\,dy \right\|_{L^p(\mathbb{R}^n)} &\le \int_{|y|<r} \frac{\|f(y+\cdot)\|_{L^p(\mathbb{R}^n)}+\|f\|_{L^p(\mathbb{R}^n)}}{|y|^{n+\alpha}}\,dy \\
&= 2\|f\|_{L^p(\mathbb{R}^n)} \int_{|y|\ge r} \frac{dy}{|y|^{n+\alpha}} \\
&= \frac{2n\omega_n r^{-\alpha}}{\alpha} \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Hence

$$\|\mathcal{D}^\alpha f\|_{L^p(\mathbb{R}^n)} \le 2n\omega_n \nu_{n,\alpha} \left(\frac{r^{1-\alpha}}{1-\alpha} \|\nabla f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} + \frac{r^{-\alpha}}{\alpha} \|f\|_{L^p(\mathbb{R}^n)} \right)$$

for all $r > 0$. Thus (1.70) follows by choosing $r = \frac{\|f\|_{L^p(\mathbb{R}^n)}}{\|\nabla f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}}$ for all $f \in C_c^\infty(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n)$ is a dense subset of $W^{1,p}(\mathbb{R}^n)$, we can extend $\mathcal{D}^\alpha\colon C_c^\infty(\mathbb{R}^n) \to L^p(\mathbb{R})$ to a linear bounded operator $\mathcal{D}^\alpha\colon W^{1,p}(\mathbb{R}^n) \to L^p(\mathbb{R})$ (for which we retain the same notation) still satisfying (1.70). The proof is complete. □

In the following result, we recall the following self-adjointness property the fractional Laplacian. For the reader's convenience, we give a brief proof of it below.

Lemma 1.50. *Let $\alpha \in (0,1)$ and $p, q \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$. If $f \in W^{1,p}(\mathbb{R}^n)$ and $g \in W^{1,q}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} f\,(-\Delta)^{\frac{\alpha}{2}} g\,dx = \int_{\mathbb{R}^n} g\,(-\Delta)^{\frac{\alpha}{2}} f\,dx. \tag{1.72}$$

Proof. Formula (1.72) is well known for $f, g \in \mathcal{S}(\mathbb{R}^n)$ and can be proved by exploiting Functional Calculus or by directly using the definition of $(-\Delta)^{\frac{\alpha}{2}}$ for instance. Since the real-valued functional

$$(f,g) \mapsto \int_{\mathbb{R}^n} f\,(-\Delta)^{\frac{\alpha}{2}} g\,dx$$

is bilinear and continuous on $L^p(\mathbb{R}^n) \times W^{1,q}(\mathbb{R}^n;\mathbb{R}^n)$ by Hölder's inequality and thanks to Lemma 1.49 above, the conclusion follows by a simple approximation argument. □

We are now finally ready to prove that $C_c^\infty(\mathbb{R}^n)$ is a dense subset of $S^{\alpha,p}(\mathbb{R}^n)$ for $p \in (1,+\infty)$.

Theorem 1.51 (Approximation by C_c^∞ functions in $S^{\alpha,p}$ for $p > 1$)**.** *Let $\alpha \in (0,1)$ and $p \in (1,+\infty)$. The set $C_c^\infty(\mathbb{R}^n)$ is dense in $S^{\alpha,p}(\mathbb{R}^n)$.*

Proof. We divide the proof in two steps.

Step 1. Let $f \in S^{\alpha,p}(\mathbb{R}^n)$ and assume $f \in W^{1,p}(\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n)$. Given $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we can write $\mathrm{div}^\alpha\varphi = (-\Delta)^{\frac{\alpha}{2}}\mathrm{div}^0\varphi$ with $\mathrm{div}^0\varphi \in \mathrm{Lip}_b(\mathbb{R}^n) \cap W^{1,q}(\mathbb{R}^n)$, so that

$$\int_{\mathbb{R}^n} f\,(-\Delta)^{\frac{\alpha}{2}}\mathrm{div}^0\varphi\,dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} f\,\mathrm{div}^0\varphi\,dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ by Lemma 1.50. Since $(-\Delta)^{\frac{\alpha}{2}} f \in L^p(\mathbb{R}^n)$ by Lemma 1.49, by Lemma 1.48 we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} f\,\mathrm{div}^0\varphi\,dx = -\int_{\mathbb{R}^n} \varphi\cdot\nabla^0(-\Delta)^{\frac{\alpha}{2}} f\,dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. We thus get that $\nabla^\alpha f = \nabla^0(-\Delta)^{\frac{\alpha}{2}} f$ for all $f \in S^{\alpha,p}(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n)$, so that

$$c_1\|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^p(\mathbb{R}^n)} \le [f]_{S^{\alpha,p}(\mathbb{R}^n)} \le c_2\|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in S^{\alpha,p}(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n)$, where $c_1, c_2 > 0$ are two constants depending only on $p > 1$. Thus, recalling the equivalent definition of the space $L^{\alpha,p}(\mathbb{R}^n)$ given in (N.62), we conclude that

$$S^{\alpha,p}(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n) \subset L^{\alpha,p}(\mathbb{R}^n)$$

with continuous embedding.

Step 2. Now fix $f \in S^{\alpha,p}(\mathbb{R}^n)$ and let $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^n)$ be a family of standard mollifiers as in (1.25). Setting $f_\varepsilon = f * \varrho_\varepsilon$ for all $\varepsilon > 0$, arguing as in the proof of Theorem 1.46, we have that $f_\varepsilon \to f$ in $S^{\alpha,p}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. By Young's inequality, we have that $f_\varepsilon \in S^{\alpha,p}(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n)$ for all $\varepsilon > 0$. Thus $S^{\alpha,p}(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n)$ is a dense subset of $S^{\alpha,p}(\mathbb{R}^n)$. Hence, by Step 1, we get that also $S_0^{\alpha,p}(\mathbb{R}^n)$ is a dense subset of $S^{\alpha,p}(\mathbb{R}^n)$ and the conclusion follows. □

As an immediate consequence of Theorem 1.51, we obtain the following result.

Corollary 1.52 (The identification $S^{\alpha,p} = L^{\alpha,p}$)**.** *Let $\alpha \in (0,1)$ and $p \in (1,+\infty)$. We have $S^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$.*

Thanks to the identification given by Corollary 1.52, we can prove the following result.

Proposition 1.53 (Approximation by $\mathcal{S}_0$ functions in $S^{\alpha,p}$ for $p > 1$)**.** *Let $\alpha \in (0,1)$ and $p \in (1,+\infty)$. The set $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $S^{\alpha,p}(\mathbb{R}^n)$.*

Proof. By Corollary 1.52, we equivalently need to prove that the set $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $L^{\alpha,p}(\mathbb{R}^n)$. To this aim, let us consider the functional $M\colon (\mathcal{S}(\mathbb{R}^n), \|\cdot\|_{L^p(\mathbb{R}^n)}) \to \mathbb{R}$ defined as

$$M(f) = \int_{\mathbb{R}^n} f(x)\,dx, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Clearly, the linear functional M cannot be continuous on $(\mathcal{S}(\mathbb{R}^n), \|\cdot\|_{L^p(\mathbb{R}^n)})$ and thus its kernel $\mathcal{S}_0(\mathbb{R}^n)$ must be dense $\mathcal{S}(\mathbb{R}^n)$ with respect to the L^p-norm. Since the Bessel potential

$$(\mathrm{Id}-\Delta)^{-\frac{\alpha}{2}}\colon (\mathcal{S}(\mathbb{R}^n), \|\cdot\|_{S^{\alpha,p}(\mathbb{R}^n)}) \to (\mathcal{S}(\mathbb{R}^n), \|\cdot\|_{L^p(\mathbb{R}^n)})$$

is an isomorphism, the conclusion follows. □

The following result gives an L^p-estimate on translations of functions in $S^{\alpha,p}(\mathbb{R}^n)$. Thanks to Corollary 1.52, this result can be derived from the analogous result already known for functions in $L^{\alpha,p}(\mathbb{R}^n)$. However, the estimate in (1.73) provides an explicit constant (independent of p) that may be of some interest. The proof of Proposition 1.54 below can be easily done by following the one of Proposition 1.23 and we leave it to the reader.

Proposition 1.54. *Let $\alpha \in (0,1)$ and $p \in [1,+\infty)$. If $f \in S^{\alpha,p}(\mathbb{R}^n)$, then*

$$\|f(\cdot + y) - f(\cdot)\|_{L^p(\mathbb{R}^n)} \le \gamma_{n,\alpha}\, |y|^\alpha\, \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \tag{1.73}$$

for all $y \in \mathbb{R}^n$, where $\gamma_{n,\alpha} > 0$ is as in (1.43).

As a simple consequence of Proposition 1.54, we have the following result, which is connected to the problem of finding a measurable set $E \subset \mathbb{R}^n$ such that $\chi_E \in BV^\alpha(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n)$, see Chapter 2.

Corollary 1.55. *Let $\alpha, \beta \in (0,1)$ and $p, q \in [1,+\infty)$. If $\chi_E \in S^{\beta,q}(\mathbb{R}^n)$ with $\beta q > \alpha p$, then $\chi_E \in W^{\alpha,p}(\mathbb{R}^n)$ with*

$$[\chi_E]_{W^{\alpha,p}(\mathbb{R}^n)} \le c_{\alpha,\beta,p,q,n} |E|^{\frac{1}{p} - \frac{\alpha}{\beta q}}\, \|\nabla^\beta \chi_E\|_{L^q(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\alpha}{\beta}}$$

for some constant $c_{\alpha,\beta,p,q,n} > 0$ depending only on α, β, p, q, and n.

Proof. Let $r > 0$ and write

$$\begin{aligned}
[\chi_E]_{W^{\alpha,p}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\chi_E(x+y) - \chi_E(x)|^p}{|y|^{n+\alpha p}}\, dx\, dy \\
&= \int_{\{|y|<r\}}\int_{\mathbb{R}^n} \frac{|\chi_E(x+y) - \chi_E(x)|}{|y|^{n+\alpha p}}\, dx\, dy \\
&\quad + \int_{\{|y|\ge r\}}\int_{\mathbb{R}^n} \frac{|\chi_E(x+y) - \chi_E(x)|}{|y|^{n+\alpha p}}\, dx\, dy.
\end{aligned}$$

On the one hand, we have

$$\int_{\{|y|\ge r\}} \frac{1}{|y|^{n+\alpha p}} \int_{\mathbb{R}^n} |\chi_E(x+y) - \chi_E(x)|\, dx\, dy \le 2|E| \int_{\{|y|\ge r\}} \frac{dy}{|y|^{n+\alpha p}} = 2n\omega_n |E|\, \frac{r^{-\alpha p}}{\alpha p}.$$

On the other hand, by Proposition 1.54, we can estimate

$$\begin{aligned}
&\int_{\{|y|<r\}} \frac{1}{|y|^{n+\alpha p}} \int_{\mathbb{R}^n} |\chi_E(x+y) - \chi_E(x)|\, dx\, dy \\
&\qquad = \int_{\{|y|<r\}} \frac{1}{|y|^{n+\alpha p}} \int_{\mathbb{R}^n} |\chi_E(x+y) - \chi_E(x)|^q\, dx\, dy \\
&\qquad = \int_{\{|y|<r\}} \|\chi_E(\cdot + y) - \chi_E(\cdot)\|_{L^q(\mathbb{R}^n)}^q\, \frac{dy}{|y|^{n+\alpha p}} \\
&\qquad \le \gamma_{n,\beta}^q\, \|\nabla^\beta \chi_E\|_{L^q(\mathbb{R}^n;\,\mathbb{R}^n)}^q \int_{\{|y|<r\}} \frac{dy}{|y|^{n+\alpha p - \beta q}} \\
&\qquad = \gamma_{n,\beta}^q\, \|\nabla^\beta \chi_E\|_{L^q(\mathbb{R}^n;\,\mathbb{R}^n)}^q\, \frac{r^{\beta q - \alpha p}}{\beta q - \alpha p}.
\end{aligned}$$

The conclusion thus follows by choosing $r = |E|^{1/\beta q} / \|\nabla^\beta \chi_E\|_{L^q(\mathbb{R}^n;\,\mathbb{R}^n)}^{1/\beta}$. □

5.5. Relation between $W^{\alpha,p}$ and $S^{\alpha,p}$. We can thus collect the relation between $W^{\alpha,p}(\mathbb{R}^n)$ and $S^{\alpha,p}(\mathbb{R}^n)$ in the following result.

Proposition 1.56 (Relation between $W^{\alpha,p}$ and $S^{\alpha,p}$). *The following properties hold.*

(i) If $\alpha \in (0,1)$ and $p \in [1,2]$, then $W^{\alpha,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$ with continuous embedding.

(ii) If $0 < \alpha < \beta < 1$ and $p \in (2,+\infty]$, then $W^{\beta,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$ with continuous embedding.

Proof. Property (i) follows from the discussion above for the case $p \in (1,2]$ and from Theorem 1.27 for the case $p = 1$. Property (ii) follows from the discussion above for the case $p \in (2,+\infty)$, while for the case $p = +\infty$ it is enough to observe that

$$\begin{aligned}
\|\nabla^\alpha f\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} &\le \mu_{n,\alpha} \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)-f(x)|}{|y-x|^{n+\alpha}}\,dy \\
&\le 2\mu_{n,\alpha}\|f\|_{L^\infty(\mathbb{R}^n)} \int_{\{|y|>1\}} \frac{dy}{|y|^{n+\alpha}} + \mu_{n,\alpha}[f]_{W^{\beta,\infty}(\mathbb{R}^n)} \int_{\{|y|\le 1\}} \frac{dy}{|y|^{n+\alpha-\beta}} \\
&\le c_{n,\alpha,\beta}\|f\|_{W^{\beta,\infty}(\mathbb{R}^n)}
\end{aligned}$$

for all $f \in W^{\beta,\infty}(\mathbb{R}^n)$. □

5.6. The inclusion $S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$. As in the classical case, we have $S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ with continuous embedding.

Theorem 1.57 ($S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$). *Let $\alpha \in (0,1)$. If $f \in BV^\alpha(\mathbb{R}^n)$, then $f \in S^{\alpha,1}(\mathbb{R}^n)$ if and only if $|D^\alpha f| \ll \mathscr{L}^n$, in which case*

$$D^\alpha f = \nabla^\alpha f\,\mathscr{L}^n \quad \text{in } \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n).$$

Proof. Let $f \in BV^\alpha(\mathbb{R}^n)$ and assume that $|D^\alpha f| \ll \mathscr{L}^n$. Then $D^\alpha f = g\,\mathscr{L}^n$ for some $g \in L^1(\mathbb{R}^n;\mathbb{R}^n)$. But then, by Theorem 1.10, we must have

$$\int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha\varphi\,dx = -\int_{\mathbb{R}^n} g\cdot\varphi\,dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, so that $f \in S^{\alpha,1}(\mathbb{R}^n)$ with $\nabla^\alpha f = g$. Viceversa, if $f \in S^{\alpha,1}(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha\varphi\,dx = -\int_{\mathbb{R}^n} \varphi\cdot\nabla^\alpha f\,dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, so that $f \in BV^\alpha(\mathbb{R}^n)$ with $D^\alpha f = \nabla^\alpha f\,\mathscr{L}^n$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$. □

5.7. The inclusion $S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ is strict. As a simple consequence of Lemma 1.31, we can prove that the inclusion $S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ is strict for all $\alpha \in (0,1)$ and $n \ge 1$.

Theorem 1.58 ($BV^\alpha(\mathbb{R}^n) \setminus S^{\alpha,1}(\mathbb{R}^n) \neq \varnothing$). *Let $\alpha \in (0,1)$. The inclusion $S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ is strict.*

Proof. Let $u \in BV(\mathbb{R}^n) \setminus W^{1,1}(\mathbb{R}^n)$. By Lemma 1.31, we know that $f = (-\Delta)^{\frac{1-\alpha}{2}} u \in BV^\alpha(\mathbb{R}^n)$ with $Du = D^\alpha f$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$. But then $|D^\alpha f|$ is not absolutely continuous with respect to $\mathscr{L}^n$, so that $f \notin S^{\alpha,1}(\mathbb{R}^n)$ by Theorem 1.57. □

5.8. The inclusion $W^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n)$ is strict. By Theorem 1.58, we know that the inclusion $W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ is strict. In the following result we prove that also the inclusion $W^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n)$ is strict.

Theorem 1.59 ($S^{\alpha,1}(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n) \neq \varnothing$)**.** *Let $\alpha \in (0,1)$. The inclusion $W^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n)$ is strict.*

Proof. We argue by contradiction. If $W^{\alpha,1}(\mathbb{R}^n) = S^{\alpha,1}(\mathbb{R}^n)$, then the inclusion map $W^{\alpha,1}(\mathbb{R}^n) \hookrightarrow S^{\alpha,1}(\mathbb{R}^n)$ is a linear and continuous bijection. Thus, by the Inverse Mapping Theorem, there must exist a constant $C > 0$ such that

$$[g]_{W^{\alpha,1}(\mathbb{R}^n)} \le C\|g\|_{S^{\alpha,1}(\mathbb{R}^n)} \tag{1.74}$$

for all $g \in S^{\alpha,1}(\mathbb{R}^n)$. Now let $f \in BV^\alpha(\mathbb{R}^n) \setminus S^{\alpha,1}(\mathbb{R}^n)$ be given by Theorem 1.58. By Theorem 1.16, there exists $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$ and $|D^\alpha f_k|(\mathbb{R}^n) \to |D^\alpha f|(\mathbb{R}^n)$ as $k \to +\infty$. Up to extract a subsequence (which we do not relabel for simplicity), we can assume that $f_k(x) \to f(x)$ as $k \to +\infty$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. By (1.74) and Fatou's Lemma, we have that

$$\begin{aligned}
[f]_{W^{\alpha,1}(\mathbb{R}^n)} &\le \liminf_{k\to+\infty} [f_k]_{W^{\alpha,1}(\mathbb{R}^n)} \\
&\le C \liminf_{k\to+\infty} \|f_k\|_{S^{\alpha,1}(\mathbb{R}^n)} \\
&= C \lim_{k\to+\infty} \|f_k\|_{BV^\alpha(\mathbb{R}^n)} \\
&= C\, \|f\|_{BV^\alpha(\mathbb{R}^n)} < +\infty.
\end{aligned}$$

Therefore $f \in W^{\alpha,1}(\mathbb{R}^n)$, in contradiction with Theorem 1.58. We thus must have that the inclusion map $W^{\alpha,1}(\mathbb{R}^n) \hookrightarrow S^{\alpha,1}(\mathbb{R}^n)$ cannot be surjective. ☐

5.9. The inclusion $BV^\alpha(\mathbb{R}^n) \subset B^\alpha_{1,\infty}(\mathbb{R}^n)$ is strict for all $n \ge 2$. From Proposition 1.23, one immediately deduces that the inclusion $BV^\alpha(\mathbb{R}^n) \subset B^\alpha_{1,\infty}(\mathbb{R}^n)$ holds continuously for all $\alpha \in (0,1)$, where $B^\alpha_{p,q}(\mathbb{R}^n)$ is the Besov space, see [**55**, Chapter 14]. In the following result, we show that this inclusion is actually strict whenever $n \ge 2$.

Theorem 1.60 ($B^\alpha_{1,\infty}(\mathbb{R}^n) \setminus BV^\alpha(\mathbb{R}^n) \neq \varnothing$ for $n \ge 2$)**.** *Let $\alpha \in (0,1)$ and $n \ge 2$. The inclusion $BV^\alpha(\mathbb{R}^n) \subset B^\alpha_{1,\infty}(\mathbb{R}^n)$ is strict.*

Proof. By Theorem 1.17, we just need to prove that $B^\alpha_{1,\infty}(\mathbb{R}^n) \setminus L^{\frac{n}{n-\alpha}}(\mathbb{R}^n) \neq \varnothing$. Let $\eta_1 \in C_c^\infty(\mathbb{R}^n)$ be as in (1.30) and let $f(x) = \eta_1(x)|x|^{\alpha-n}$ for all $x \in \mathbb{R}^n$. On the one side, we clearly have $f \notin L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. On the other side, for all $h \in \mathbb{R}^n$ with $|h| < 1$, we can estimate

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x+h) - f(x)|\,dx &\le \int_{\{|x|>2|h|\}} \left|\eta_1(x+h)|x+h|^{\alpha-n} - \eta_1(x)|x|^{\alpha-n}\right| dx \\
&\quad + \int_{\{|x|<3|h|\}} \eta_1(x)|x|^{\alpha-n}\,dx \\
&\le C|h| \int_{\{|x|>2|h|\}} |x|^{\alpha-n-1}\,dx + C\int_{\{|x|<3|h|\}} |x|^{\alpha-n}\,dx \\
&= C|h| \int_{2|h|}^{+\infty} r^{\alpha-2}\,dr + C\int_0^{3|h|} r^{\alpha-1}\,dr = C|h|^\alpha,
\end{aligned}$$

where $C > 0$ is a constant depending only on n and α (that may vary from line to line). Thus $f \in B^{\alpha}_{1,\infty}(\mathbb{R}^n)$ and the conclusion follows. $\square$

CHAPTER 2

A distributional approach to fractional Caccioppoli perimeter

1. Fractional Caccioppoli sets

1.1. Definition of fractional Caccioppoli sets and the Gauss–Green formula. As in the classical case (see [**6**, Definition 3.3.5] for instance), we start with the following definition.

Definition 2.1 (Fractional Caccioppoli set)**.** Let $\alpha \in (0,1)$ and let $E \subset \mathbb{R}^n$ be a measurable set. For any open set $\Omega \subset \mathbb{R}^n$, the *fractional Caccioppoli α-perimeter in Ω* is the *fractional variation* of χ_E in Ω, i.e.

$$|D^\alpha \chi_E|(\Omega) = \sup\left\{\int_E \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1\right\}.$$

We say that E is a set with *finite fractional Caccioppoli α-perimeter in Ω* if $|D^\alpha \chi_E|(\Omega) < +\infty$. We say that E is a set with *locally finite fractional Caccioppoli α-perimeter in Ω* if $|D^\alpha \chi_E|(U) < +\infty$ for any $U \Subset \Omega$.

We can now state the following fundamental result relating non-local distributional gradients of characteristic functions of fractional Caccioppoli sets and vector valued Radon measures.

Theorem 2.2 (Gauss–Green formula for fractional Caccioppoli sets)**.** *Let $\alpha \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ be an open set. A measurable set $E \subset \mathbb{R}^n$ is a set with finite fractional Caccioppoli α-perimeter in Ω if and only if $D^\alpha \chi_E \in \mathscr{M}(\Omega; \mathbb{R}^n)$ and*

$$\int_E \operatorname{div}^\alpha \varphi \, dx = -\int_\Omega \varphi \cdot dD^\alpha \chi_E \tag{2.1}$$

for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^n)$. In addition, for any open set $U \subset \Omega$ it holds

$$|D^\alpha \chi_E|(U) = \sup\left\{\int_E \operatorname{div}^\alpha \varphi \; dx : \varphi \in C_c^\infty(U; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(U;\mathbb{R}^n)} \le 1\right\}. \tag{2.2}$$

Proof. The proof is similar to the one of Theorem 1.10. If $D^\alpha \chi_E \in \mathscr{M}(\Omega; \mathbb{R}^n)$ and (2.1) holds, then E has finite fractional Caccioppoli α-perimeter in Ω by Definition 2.1.

If E is a set with finite fractional Caccioppoli α-perimeter in Ω, then define the linear functional $L \colon C_c^\infty(\Omega; \mathbb{R}^n) \to \mathbb{R}$ setting

$$L(\varphi) = -\int_E \operatorname{div}^\alpha \varphi \, dx \qquad \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^n).$$

Note that L is well defined thanks to Corollary 1.3. Since E has finite fractional Caccioppoli α-perimeter in Ω, we have

$$C(U) = \sup\left\{L(\varphi) : \varphi \in C_c^\infty(U; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(U;\mathbb{R}^n)} \le 1\right\} < +\infty$$

for each open set $U \subset \Omega$, so that

$$|L(\varphi)| \le C(U) \|\varphi\|_{L^\infty(U;\mathbb{R}^n)} \qquad \forall \varphi \in C_c^\infty(U; \mathbb{R}^n).$$

Thus, by the density of $C_c^\infty(\Omega;\mathbb{R}^n)$ in $C_c(\Omega;\mathbb{R}^n)$, the functional L can be uniquely extended to a continuous linear functional $\tilde{L}\colon C_c(\Omega;\mathbb{R}^n)\to\mathbb{R}$ and the conclusion follows by Riesz's Representation Theorem. □

1.2. Lower semicontinuity of fractional variation. As in the classical case, the variation measure of a set with finite fractional Caccioppoli α-perimeter is lower semicontinuous with respect to the local convergence in measure. We also achieve a weak convergence result.

Proposition 2.3 (Lower semicontinuity of fractional variation measure)**.** *Let $\alpha\in(0,1)$ and let $\Omega\subset\mathbb{R}^n$ be an open set. If $(E_k)_{k\in\mathbb{N}}$ is a sequence of sets with finite fractional Caccioppoli α-perimeter in Ω and $\chi_{E_k}\to\chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$D^\alpha\chi_{E_k}\rightharpoonup D^\alpha\chi_E \quad \text{in } \mathcal{M}(\Omega;\mathbb{R}^n), \tag{2.3}$$

and

$$|D^\alpha\chi_E|(\Omega)\le\liminf_{k\to+\infty}|D^\alpha\chi_{E_k}|(\Omega). \tag{2.4}$$

Proof. Up to extract a further subsequence, we can assume that $\chi_{E_k}(x)\to\chi_E(x)$ as $k\to+\infty$ for $\mathscr{L}^n$-a.e. $x\in\mathbb{R}^n$. Now let $\varphi\in C_c^\infty(\Omega;\mathbb{R}^n)$ be such that $\|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)}\le 1$. Then $\operatorname{div}^\alpha\varphi\in L^1(\mathbb{R}^n)$ by Corollary 1.3 and so, by Lebesgue's Dominated Convergence Theorem, we have

$$\int_E \operatorname{div}^\alpha\varphi\,dx=\lim_{k\to+\infty}\int_{E_k}\operatorname{div}^\alpha\varphi\,dx=-\lim_{k\to+\infty}\int_\Omega\varphi\cdot\,dD^\alpha\chi_{E_k}\le\liminf_{k\to+\infty}|D^\alpha\chi_{E_k}|(\Omega).$$

By Theorem 2.2, we get (2.4). The convergence in (2.3) easily follows. □

1.3. Fractional isoperimetric inequality. As a simple application of Theorem 1.17, we can prove the following fractional isoperimetric inequality.

Theorem 2.4 (Fractional isoperimetric inequality)**.** *Let $\alpha\in(0,1)$ and $n\ge 2$. There exists a constant $c_{n,\alpha}>0$ such that*

$$|E|^{\frac{n-\alpha}{n}}\le c_{n,\alpha}|D^\alpha\chi_E|(\mathbb{R}^n) \tag{2.5}$$

for any set $E\subset\mathbb{R}^n$ such that $|E|<+\infty$ and $|D^\alpha\chi_E|(\mathbb{R}^n)<+\infty$.

Proof. Since $\chi_E\in BV^\alpha(\mathbb{R}^n)$, the result follows directly by Theorem 1.17. □

1.4. Compactness. As an application of Theorem 1.25, we can prove the following compactness result for sets with finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$ (see for instance [**58**, Theorem 12.26] for the analogous result in the classical case).

Theorem 2.5 (Compactness for sets with finite fractional Caccioppoli α-perimeter)**.** *Let $\alpha\in(0,1)$ and $R>0$. If $(E_k)_{k\in\mathbb{N}}$ is a sequence of sets with finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$ such that*

$$\sup_{k\in\mathbb{N}}|D^\alpha\chi_{E_k}|(\mathbb{R}^n)<+\infty \quad \text{and} \quad E_k\subset B_R \quad \forall k\in\mathbb{N},$$

then there exist a subsequence $(E_{k_j})_{j\in\mathbb{N}}$ and a set $E\subset B_R$ with finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$ such that

$$\chi_{E_{k_j}}\to\chi_E \text{ in } L^1(\mathbb{R}^n)$$

as $j\to+\infty$.

Proof. Since $E_k \subset B_R$ for all $k \in \mathbb{N}$, we clearly have that $(\chi_{E_k})_{k\in\mathbb{R}^n} \subset BV^\alpha(\mathbb{R}^n)$. By Theorem 1.25, there exist a subsequence $(E_{k_j})_{j\in\mathbb{N}}$ and a function $f \in L^1(\mathbb{R}^n)$ such that $\chi_{E_{k_j}} \to f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $j \to +\infty$. Since again $E_{k_j} \subset B_R$ for all $j \in \mathbb{N}$, we have that $\chi_{E_{k_j}} \to f$ in $L^1(\mathbb{R}^n)$ as $j \to +\infty$. Up to extract a further subsequence (which we do not relabel for simplicity), we can assume that $\chi_{E_{k_j}}(x) \to f(x)$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$ as $j \to +\infty$, so that $f = \chi_E$ for some $E \subset B_R$. By Proposition 2.3 we conclude that E has finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$. □

Theorem 2.5 can be applied to prove the following compactness result for sets with locally finite fractional Caccioppoli α-perimeter.

Corollary 2.6 (Compactness for locally finite fractional Caccioppoli α-perimeter sets)**.** *Let $\alpha \in (0,1)$. If $(E_k)_{k\in\mathbb{N}}$ is a sequence of sets with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$ such that*

$$\sup_{k\in\mathbb{N}} |D^\alpha \chi_{E_k}|(B_R) < +\infty \quad \forall R > 0, \tag{2.6}$$

then there exist a subsequence $(E_{k_j})_{j\in\mathbb{N}}$ and a set E with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$ such that

$$\chi_{E_{k_j}} \to \chi_E \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

as $j \to +\infty$.

Proof. We divide the proof into two steps, essentially following the strategy presented in the proof of [**58**, Corollary 12.27].

Step 1. Let $F \subset \mathbb{R}^n$ be a set with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$. We claim that

$$|D^\alpha \chi_{F\cap B_R}|(\mathbb{R}^n) \le |D^\alpha \chi_F|(B_R) + 3\mu_{n,\alpha} P_\alpha(B_R) \quad \forall R > 0. \tag{2.7}$$

Indeed, let $R' < R$ and let $(u_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ be such that $\mathrm{supp}(u_k) \Subset B_R$ and $0 \le u_k \le 1$ for all $k \in \mathbb{N}$ and also $u_k \to \chi_{B_{R'}}$ in $W^{\alpha,1}(\mathbb{R}^n)$ as $k \to +\infty$. If $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1$, then

$$\begin{aligned}
\int_F u_k \operatorname{div}^\alpha \varphi \, dx &= \int_F \operatorname{div}^\alpha(u_k\varphi)\, dx - \int_F \varphi \cdot \nabla^\alpha u_k \, dx - \int_F \operatorname{div}^\alpha_{\mathrm{NL}}(u_k,\varphi)\, dx \\
&\le \int_F \operatorname{div}^\alpha(u_k\varphi)\, dx + 3\mu_{n,\alpha}[u_k]_{W^{\alpha,1}(\mathbb{R}^n)} \\
&\le |D^\alpha \chi_F|(B_{R'}) + 3\mu_{n,\alpha}[u_k]_{W^{\alpha,1}(\mathbb{R}^n)} \\
&\le |D^\alpha \chi_F|(B_R) + 3\mu_{n,\alpha}[u_k]_{W^{\alpha,1}(\mathbb{R}^n)}
\end{aligned}$$

by Lemma 1.7. Passing to the limit as $k \to +\infty$, we conclude that

$$\int_{F\cap B_{R'}} \operatorname{div}^\alpha \varphi \, dx \le |D^\alpha \chi_F|(B_R) + 3\mu_{n,\alpha} P_\alpha(B_{R'})$$

and thus

$$|D^\alpha \chi_{F\cap B_{R'}}|(\mathbb{R}^n) \le |D^\alpha \chi_F|(B_R) + 3\mu_{n,\alpha} P_\alpha(B_R)$$

by Theorem 2.2. Since $\chi_{F\cap B_{R'}} \to \chi_{F\cap B_R}$ in $L^1(\mathbb{R}^n)$ as $R' \to R$, the claim in (2.7) follows by Proposition 2.3.

Step 2. By (2.6) and (2.7), we can apply Theorem 2.5 to $(E_k \cap B_j)_{k\in\mathbb{N}}$ for each fixed $j \in \mathbb{N}$. By a standard diagonal argument, we find a subsequence $(E_{k_h})_{h\in\mathbb{N}}$ and a sequence $(F_j)_{j\in\mathbb{N}}$ of sets with finite fractional Caccioppoli α-perimeter such that $\chi_{E_{k_h}\cap B_j} \to \chi_{F_j}$ in $L^1(\mathbb{R}^n)$ as $h \to +\infty$ for each $j \in \mathbb{N}$. Up to null sets, we have $F_j \subset F_{j+1}$, so that $\chi_{E_{k_h}} \to \chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $E = \bigcup_{j\in\mathbb{N}} F_j$. The conclusion thus follows by Proposition 2.3. □

1.5. Fractional reduced boundary. Thanks to the scaling property of the fractional divergence, we have

$$D^\alpha \chi_{\lambda E} = \lambda^{n-\alpha} (\delta_\lambda)_\# D^\alpha \chi_E \quad \text{on } \lambda\Omega, \tag{2.8}$$

where $\delta_\lambda(x) = \lambda x$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$. Indeed, we can compute

$$\int_{\lambda E} \operatorname{div}^\alpha \varphi \, dx = \lambda^n \int_E (\operatorname{div}^\alpha \varphi) \circ \delta_\lambda \, dx = \lambda^{n-\alpha} \int_E \operatorname{div}^\alpha (\varphi \circ \delta_\lambda) \, dx$$

for all $\varphi \in C^\infty_c(\Omega; \mathbb{R}^n)$. In analogy with the classical case, we are thus led to the following definition.

Definition 2.7 (Fractional reduced boundary)**.** Let $\alpha \in (0, 1)$ and let $\Omega \subset \mathbb{R}^n$ be an open set. If $E \subset \mathbb{R}^n$ is a set with finite fractional Caccioppoli α-perimeter in Ω, then we say that a point $x \in \Omega$ belongs to the *fractional reduced boundary* of E (inside Ω), and we write $x \in \mathscr{F}^\alpha E$, if

$$x \in \operatorname{supp}(D^\alpha \chi_E) \qquad \text{and} \qquad \exists \lim_{r\to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))} \in \mathbb{S}^{n-1}.$$

We thus let

$$\nu^\alpha_E \colon \Omega \cap \mathscr{F}^\alpha E \to \mathbb{S}^{n-1}, \qquad \nu^\alpha_E(x) = \lim_{r\to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))}, \quad x \in \Omega \cap \mathscr{F}^\alpha E,$$

be the (*measure theoretic*) *inner unit fractional normal* to E (inside Ω).

As a consequence of Definition 2.7 and arguing similarly as in the proof of Proposition 1.14, if $E \subset \mathbb{R}^n$ is a set with finite fractional Caccioppoli α-perimeter in Ω, then the following Gauss–Green formula

$$\int_E \operatorname{div}^\alpha \varphi \, dx = - \int_{\Omega \cap \mathscr{F}^\alpha E} \varphi \cdot \nu^\alpha_E \, d|D^\alpha \chi_E|, \tag{2.9}$$

holds for any $\varphi \in \mathrm{Lip}_c(\Omega; \mathbb{R}^n)$.

1.6. Sets of finite fractional perimeter are fractional Caccioppoli sets. In analogy with the classical case and with the inclusion $W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$, we can show that sets with finite fractional α-perimeter have finite fractional Caccioppoli α-perimeter. Recall that the *fractional α-perimeter* of a set $E \subset \mathbb{R}$ in an open set $\Omega \subset \mathbb{R}^n$ is defined as

$$P_\alpha(E; \Omega) = \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+\alpha}} \, dx \, dy + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+\alpha}} \, dx \, dy,$$

see [**27**] for an account on this subject.

Proposition 2.8 (Sets of finite fractional perimeter are fractional Caccioppoli sets)**.** *Let $\alpha \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ be an open set. If $E \subset \mathbb{R}^n$ satisfies $P_\alpha(E;\Omega) < +\infty$, then E is a set with finite fractional Caccioppoli α-perimeter in Ω with*

$$|D^\alpha \chi_E|(\Omega) \le \mu_{n,\alpha} P_\alpha(E;\Omega) \tag{2.10}$$

and

$$\int_E \operatorname{div}^\alpha \varphi \, dx = -\int_\Omega \varphi \cdot \nabla^\alpha \chi_E \, dx \tag{2.11}$$

for all $\varphi \in \operatorname{Lip}_c(\Omega;\mathbb{R}^n)$, so that $D^\alpha \chi_E = \nu_E^\alpha |D^\alpha \chi_E| = \nabla^\alpha \chi_E \mathscr{L}^n$. Moreover, if E is such that $|E| < +\infty$ and $P(E) < +\infty$, then $\chi_E \in W^{\alpha,1}(\mathbb{R}^n)$ for any $\alpha \in (0,1)$, and

$$\nabla^\alpha \chi_E(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{\nu_E(y)}{|y-x|^{n+\alpha-1}} \, d|D\chi_E|(y) \tag{2.12}$$

for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Proof. Note that $\nabla^\alpha \chi_E \in L^1(\Omega;\mathbb{R}^n)$, because

$$\begin{aligned}
\int_\Omega |\nabla^\alpha \chi_E| \, dx &\le \mu_{n,\alpha} \int_\Omega \int_{\mathbb{R}^n} \frac{|\chi_E(y) - \chi_E(x)|}{|y-x|^{n+\alpha}} \, dy \, dx \\
&\le \mu_{n,\alpha} \int_\Omega \int_\Omega \frac{|\chi_E(y) - \chi_E(x)|}{|y-x|^{n+\alpha}} \, dy \, dx + \mu_{n,\alpha} \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(y) - \chi_E(x)|}{|y-x|^{n+\alpha}} \, dy \, dx \\
&\le \mu_{n,\alpha} P_\alpha(E;\Omega).
\end{aligned}$$

Now let $\varphi \in \operatorname{Lip}_c(\Omega;\mathbb{R}^n)$ be fixed. By Lebesgue's Dominated Convergence Theorem, by (1.4) and by Fubini's Theorem (applied for each fixed $\varepsilon > 0$), we can compute

$$\begin{aligned}
\int_E \operatorname{div}^\alpha \varphi \, dx &= \mu_{n,\alpha} \lim_{\varepsilon \to 0} \int_E \int_{\{|x-y|>\varepsilon\}} \frac{(y-x) \cdot \varphi(y)}{|y-x|^{n+\alpha+1}} \, dy \, dx \\
&= \mu_{n,\alpha} \lim_{\varepsilon \to 0} \int_\Omega \int_{\{|x-y|>\varepsilon\}} \varphi(y) \cdot \frac{(y-x)\, \chi_E(x)}{|y-x|^{n+\alpha+1}} \, dx \, dy \\
&= -\mu_{n,\alpha} \lim_{\varepsilon \to 0} \int_\Omega \int_{\{|x-y|>\varepsilon\}} \varphi(y) \cdot \frac{(y-x)(\chi_E(y) - \chi_E(x))}{|y-x|^{n+\alpha+1}} \, dx \, dy \\
&= -\int_\Omega \varphi \cdot \nabla^\alpha \chi_E \, dy.
\end{aligned}$$

Thus (2.10) and (2.11) follow by Theorem 2.2 and Definition 2.7. Finally, (2.12) follows from (1.48), since $\chi_E \in BV(\mathbb{R}^n)$. □

At the present moment, we do not know if the condition $|D^\alpha \chi_E|(\Omega) < +\infty$ implies that also $P_\alpha(E;\Omega) < +\infty$.

Remark 2.9 ($\mathscr{F}^\alpha E$ is not $\mathscr{L}^n$-negligible in general)**.** It is important to notice that, by Proposition 2.8, we have

$$P_\alpha(E;\Omega) < +\infty \implies \mathscr{L}^n(\Omega \cap \mathscr{F}^\alpha E) > 0$$

including even the case $\chi_E \in BV(\mathbb{R}^n)$. This shows a substantial difference between the standard *local* De Giorgi's perimeter measure $|D\chi_E|$ and the *non-local* fractional De Giorgi's perimeter measure $|D^\alpha \chi_E|$: the former is supported on a $\mathscr{L}^n$-negligible set contained in the topological boundary of E, while the latter, in general, can be supported on a set of positive Lebesgue measure and, for this reason, cannot be expected to be contained in the topological boundary of E.

Remark 2.10 (Fractional reduced boundary and precise representative)**.** We let

$$u^*(x) = \begin{cases} \lim\limits_{r \to 0} \dfrac{1}{|B_r(x)|} \displaystyle\int_{B_r(x)} u(y)\,dy & \text{if the limit exists and is finite,} \\ 0 & \text{otherwise,} \end{cases}$$

be the *precise representative* of a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$. Note that u^* is well defined at any Lebesgue point of u. By Proposition 2.8, if $P_\alpha(E;\Omega) < +\infty$ then $D^\alpha \chi_E = \nabla^\alpha \chi_E \mathscr{L}^n$ with $\nabla^\alpha \chi_E \in L^1(\Omega;\mathbb{R}^n)$. Therefore the set

$$\mathcal{R}^\alpha_\Omega E = \{x \in \Omega : |(\nabla^\alpha \chi_E)^*(x)| = |\nabla^\alpha \chi_E|^*(x) \neq 0\}$$

is such that

$$\mathcal{R}^\alpha_\Omega E \subset \Omega \cap \mathscr{F}^\alpha E \tag{2.13}$$

and

$$\nu^\alpha_E(x) = \frac{(\nabla^\alpha \chi_E)^*(x)}{|\nabla^\alpha \chi_E|^*(x)} \qquad \text{for all } x \in \mathcal{R}^\alpha_\Omega E.$$

The following simple example shows that the inclusion in (2.13) and the inequality in (2.10) can be strict.

Example 2.11. Let $n = 1$, $\alpha \in (0,1)$ and $a, b \in \mathbb{R}$, with $a < b$. It is easy to see that $\chi_{(a,b)} \in W^{\alpha,1}(\mathbb{R})$. By (2.12), for any $x \neq a, b$ we have that

$$\begin{aligned} \nabla^\alpha \chi_{(a,b)}(x) &= \frac{\mu_{1,\alpha}}{\alpha} \int_{\mathbb{R}} \frac{1}{|x-y|^\alpha}\, d\,(\delta_a - \delta_b)\,(y) \\ &= \frac{2^\alpha}{\alpha\sqrt{\pi}} \frac{\Gamma\left(1 + \frac{\alpha}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} \left(\frac{1}{|x-a|^\alpha} - \frac{1}{|x-b|^\alpha} \right). \end{aligned}$$

We claim that

$$\mathscr{F}^\alpha(a,b) = \mathbb{R} \setminus \left\{ \frac{a+b}{2} \right\} \tag{2.14}$$

while

$$\mathcal{R}^\alpha_{\mathbb{R}}(a,b) = \mathbb{R} \setminus \left\{ a, \frac{a+b}{2}, b \right\}, \tag{2.15}$$

so that inclusion (2.13) is strict. Finally, we also claim that

$$\|\nabla^\alpha \chi_{(a,b)}\|_{L^1(\mathbb{R})} < \mu_{1,\alpha} P_\alpha((a,b)). \tag{2.16}$$

Indeed, notice that

$$\nabla^\alpha \chi_{(a,b)}(x) \geq 0$$

if and only if $x \le \frac{a+b}{2}$, so that

$$\lim_{r\to 0} \frac{\displaystyle\int_{x-r}^{x+r} \nabla^\alpha \chi_{(a,b)}(y)\, dy}{\displaystyle\int_{x-r}^{x+r} |\nabla^\alpha \chi_{(a,b)}(y)|\, dy} = \begin{cases} 1 & \text{if } x < \dfrac{a+b}{2}, \\ -1 & \text{if } x > \dfrac{a+b}{2}. \end{cases}$$

If $x = \frac{a+b}{2}$, then

$$\int_{\frac{a+b}{2}-r}^{\frac{a+b}{2}+r} \nabla^\alpha \chi_{(a,b)}(y)\, dy = 0 \qquad \forall r > 0,$$

and claim (2.14) follows. In particular, we have

$$\nu^\alpha_{(a,b)}(x) = \begin{cases} 1 & \text{if } \ x < \dfrac{a+b}{2}, \\ -1 & \text{if } \ x > \dfrac{a+b}{2}. \end{cases}$$

On the other hand, it is clear that

$$\lim_{r\to 0} \frac{1}{2r} \int_{a-r}^{a+r} \nabla^\alpha \chi_{(a,b)}(y)\, dy = +\infty$$

and

$$\lim_{r\to 0} \frac{1}{2r} \int_{b-r}^{b+r} \nabla^\alpha \chi_{(a,b)}(y)\, dy = -\infty,$$

so that claim (2.15) follows. To prove (2.16), note that

$$P_\alpha((a,b)) = \frac{4}{\alpha(1-\alpha)}(b-a)^{1-\alpha} \tag{2.17}$$

since $P_\alpha((a,b)) = (b-a)^{1-\alpha} P_\alpha((0,1))$ by the scaling property of the fractional perimeter and

$$\begin{aligned} P_\alpha((0,1)) &= 2 \int_{\mathbb{R}\setminus(0,1)} \int_0^1 \frac{1}{|y-x|^{1+\alpha}}\, dy\, dx \\ &= \frac{2}{\alpha} \int_{\mathbb{R}\setminus(0,1)} \left[\frac{\operatorname{sgn}(x-y)}{|y-x|^\alpha} \right]_{y=0}^{y=1} dx \\ &= \frac{2}{\alpha} \int_{\mathbb{R}\setminus(0,1)} \frac{\operatorname{sgn}(x-1)}{|1-x|^\alpha} - \frac{\operatorname{sgn}(x)}{|x|^\alpha}\, dx \\ &= \frac{2}{\alpha} \int_1^\infty \frac{1}{(x-1)^\alpha} - \frac{1}{x^\alpha}\, dx + \frac{2}{\alpha} \int_{-\infty}^0 \frac{1}{(-x)^\alpha} - \frac{1}{(1-x)^\alpha}\, dx \\ &= \frac{4}{\alpha} \int_0^\infty \frac{1}{x^\alpha} - \frac{1}{(1+x)^\alpha}\, dx = \frac{4}{\alpha(1-\alpha)}. \end{aligned}$$

On the other hand, we have

$$\|\nabla^\alpha \chi_{(a,b)}\|_{L^1(\mathbb{R})} = \frac{2^{1+\alpha}\mu_{1,\alpha}}{\alpha(1-\alpha)}(b-a)^{1-\alpha}. \tag{2.18}$$

Indeed, $\|\nabla^\alpha \chi_{(a,b)}\|_{L^1(\mathbb{R})} = (b-a)^{1-\alpha}\|\nabla^\alpha \chi_{(0,1)}\|_{L^1(\mathbb{R})}$ by (2.8) and

$$\begin{aligned}
\frac{\alpha}{\mu_{1,\alpha}}\|\nabla^\alpha \chi_{(0,1)}\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} \left| \frac{1}{|x|^\alpha} - \frac{1}{|x-1|^\alpha} \right| dx \\
&= \int_1^\infty \left| \frac{1}{x^\alpha} - \frac{1}{(x-1)^\alpha} \right| dx + \int_0^1 \left| \frac{1}{x^\alpha} - \frac{1}{(1-x)^\alpha} \right| dx \\
&\quad + \int_{-\infty}^0 \left| \frac{1}{(-x)^\alpha} - \frac{1}{(1-x)^\alpha} \right| dx \\
&= \int_1^\infty \frac{1}{(x-1)^\alpha} - \frac{1}{x^\alpha}\, dx + \int_{\frac{1}{2}}^1 \frac{1}{(1-x)^\alpha} - \frac{1}{x^\alpha}\, dx \\
&\quad + \int_0^{\frac{1}{2}} \frac{1}{x^\alpha} - \frac{1}{(1-x)^\alpha}\, dx + \int_{-\infty}^0 \frac{1}{(-x)^\alpha} - \frac{1}{(1-x)^\alpha}\, dx \\
&= 2\int_0^\infty \frac{1}{x^\alpha} - \frac{1}{(1+x)^\alpha}\, dx + 2\int_0^{\frac{1}{2}} \frac{1}{x^\alpha} - \frac{1}{(1-x)^\alpha}\, dx \\
&= \frac{2}{1-\alpha}\left(1 + 2^{\alpha-1} + 2^{\alpha-1} - 1\right) = \frac{2^{1+\alpha}}{1-\alpha}.
\end{aligned}$$

Combining (2.17) and (2.18), we get (2.16).

Thanks to Example 2.11 above, we know that inequality (2.10) is strict for $E = (a,b)$ with $a, b \in \mathbb{R}$, $a < b$, and $\Omega = \mathbb{R}^n$. As we did in Section 2.8, we now want to address this problem in full generality.

Given an open set $\Omega \subset \mathbb{R}^n$ and a measurable set $E \subset \mathbb{R}^n$, we define

$$\tilde{P}_\alpha(E;\Omega) = \int_\Omega \int_\Omega \frac{|\chi_E(y) - \chi_E(x)|}{|y-x|^{n+\alpha}}\, dx\, dy + \int_{\mathbb{R}^n\setminus\Omega} \int_\Omega \frac{|\chi_E(y) - \chi_E(x)|}{|y-x|^{n+\alpha}}\, dx\, dy.$$

It is obvious to see that

$$\tilde{P}_\alpha(E;\Omega) \le P_\alpha(E;\Omega) \le 2\tilde{P}_\alpha(E;\Omega).$$

Arguing similarly as in the proof of Proposition 2.8, it is immediate to see that

$$\|\nabla^\alpha \chi_E\|_{L^1(\Omega;\mathbb{R}^n)} \le \mu_{n,\alpha}\tilde{P}_\alpha(E;\Omega), \tag{2.19}$$

an inequality stronger than that in (2.10). In analogy with Corollary 1.29, we have the following result.

Corollary 2.12. *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ be an open set and $E \subset \mathbb{R}^n$ be a measurable set such that $\tilde{P}_\alpha(E;\Omega) < +\infty$.*

(i) If $n \ge 2$, $\mathscr{L}^n(E) > 0$ and $\mathscr{L}^n(\mathbb{R}^n \setminus E) > 0$, then inequality (2.19) *is strict.*
(ii) If $n = 1$, then (2.19) *is an equality if and only if the following hold:*
  (a) for a.e. $x \in \Omega \cap E$, $\mathscr{L}^1((-\infty,x)\setminus E) = 0$ vel $\mathscr{L}^1((x,+\infty)\setminus E) = 0$;
  (b) for a.e. $x \in \Omega \setminus E$, $\mathscr{L}^1((-\infty,x)\cap E) = 0$ vel $\mathscr{L}^1((x,+\infty)\cap E) = 0$.

Proof. We prove the two statements separately.

Proof of (i). Assume $n \ge 2$. Since $\mathscr{L}^n(E) > 0$, for a given $x \in \Omega \setminus E$ the map

$$y \mapsto (y-x), \quad \text{for } y \in E,$$

does not have constant orientation. Similarly, since $\mathscr{L}^n(\mathbb{R}^n \setminus E) > 0$, for a given $x \in \Omega \cap E$ also the map

$$y \mapsto (y-x), \quad \text{for } y \in \mathbb{R}^n \setminus E,$$

does not have constant orientation. Hence, by Lemma 1.28, we must have

$$\left| \int_E \frac{y-x}{|y-x|^{n+\alpha+1}}\, dy \right| < \int_E \frac{dy}{|y-x|^{n+\alpha}}, \quad \text{for } x \in \Omega \setminus E,$$

and, similarly,

$$\left| \int_{\mathbb{R}^n \setminus E} \frac{y-x}{|y-x|^{n+\alpha+1}}\, dy \right| < \int_{\mathbb{R}^n \setminus E} \frac{dy}{|y-x|^{n+\alpha}}, \quad \text{for } x \in \Omega \cap E.$$

We thus get

$$\begin{aligned}
\|\nabla^\alpha \chi_E\|_{L^1(\Omega;\mathbb{R}^n)} &= \mu_{n,\alpha} \int_\Omega \left| \int_{\mathbb{R}^n} \frac{(\chi_E(y)-\chi_E(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\, dy \right| dx \\
&= \mu_{n,\alpha} \int_{\Omega\setminus E} \left| \int_E \frac{y-x}{|y-x|^{n+\alpha}}\, dy \right| dx + \mu_{n,\alpha} \int_{\Omega\cap E} \left| \int_{\mathbb{R}^n\setminus E} \frac{y-x}{|y-x|^{n+\alpha}}\, dy \right| dx \\
&< \mu_{n,\alpha} \int_{\Omega\setminus E} \int_E \frac{dy\, dx}{|y-x|^{n+\alpha}} + \mu_{n,\alpha} \int_{\Omega\cap E} \int_{\mathbb{R}^n\setminus E} \frac{dy\, dx}{|y-x|^{n+\alpha}} = \mu_{n,\alpha} \tilde{P}_\alpha(E;\Omega),
\end{aligned}$$

proving (i).

Proof of (ii). Let

$$f_E(y,x) = \frac{\chi_E(y)-\chi_E(x)}{|y-x|^{1+\alpha}}, \qquad \text{for } x, y \in \mathbb{R},\ y \neq x.$$

Then we can write

$$\begin{aligned}
\tilde{P}_\alpha(E;\Omega) &= \int_\Omega \int_{\mathbb{R}} |f_E(y,x)|\, dy\, dx \\
&= \int_\Omega \left(\int_{-\infty}^x |f_E(y,x)|\, dy + \int_x^{+\infty} |f_E(y,x)|\, dy \right) dx
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla^\alpha \chi_E\|_{L^1(\Omega;\mathbb{R})} &= \mu_{1,\alpha} \int_\Omega \left| \int_{\mathbb{R}} f_E(y,x)\ \mathrm{sgn}(y-x)\, dy \right| dx \\
&= \mu_{1,\alpha} \int_\Omega \left| \int_{-\infty}^x f_E(y,x)\, dy - \int_x^{+\infty} f_E(y,x)\, dy \right| dx.
\end{aligned}$$

Hence (2.19) is an equality if and only if

$$\left| \int_{-\infty}^x f_E(y,x)\, dy - \int_x^{+\infty} f_E(y,x)\, dy \right| = \int_{-\infty}^x |f_E(y,x)|\, dy + \int_x^{+\infty} |f_E(y,x)|\, dy \tag{2.20}$$

for a.e. $x \in \Omega$. Observing that

$$\begin{aligned}
\left| \int_{-\infty}^x f_E(y,x)\, dy - \int_x^{+\infty} f_E(y,x)\, dy \right| &\le \left| \int_{-\infty}^x f_E(y,x)\, dy \right| + \left| \int_x^{+\infty} f_E(y,x)\, dy \right| \\
&\le \int_{-\infty}^x |f_E(y,x)|\, dy + \int_x^{+\infty} |f_E(y,x)|\, dy
\end{aligned}$$

for a.e. $x \in \Omega$, we deduce that (2.19) is an equality if and only if

$$\left| \int_{-\infty}^{x} f_E(y,x)\,dy - \int_{x}^{+\infty} f_E(y,x)\,dy \right| = \left| \int_{-\infty}^{x} f_E(y,x)\,dy \right| + \left| \int_{x}^{+\infty} f_E(y,x)\,dy \right| \tag{2.21}$$

$$= \int_{-\infty}^{x} |f_E(y,x)|\,dy + \int_{x}^{+\infty} |f_E(y,x)|\,dy \tag{2.22}$$

for a.e. $x \in \Omega$. Now, on the one hand, squaring both sides of (2.21) and simplifying, we get that (2.19) is an equality if and only if

$$\left(\int_{-\infty}^{x} f_E(y,x)\,dy \right) \left(\int_{x}^{+\infty} f_E(y,x)\,dy \right) = 0 \tag{2.23}$$

for a.e. $x \in \Omega$. On the other hand, we can rewrite (2.22) as

$$\begin{aligned} 0 &\le \int_{-\infty}^{x} |f_E(y,x)|\,dy - \left| \int_{-\infty}^{x} f_E(y,x)\,dy \right| \\ &= \left| \int_{x}^{+\infty} f_E(y,x)\,dy \right| - \int_{x}^{+\infty} |f_E(y,x)|\,dy \le 0 \end{aligned}$$

for a.e. $x \in \Omega$, so that we must have

$$\left| \int_{-\infty}^{x} f_E(y,x)\,dy \right| = \int_{-\infty}^{x} |f_E(y,x)|\,dy$$

and

$$\left| \int_{x}^{+\infty} f_E(y,x)\,dy \right| = \int_{x}^{+\infty} |f_E(y,x)|\,dy$$

for a.e. $x \in \Omega$. Hence (2.23) can be equivalently rewritten as

$$\left(\int_{-\infty}^{x} |f_E(y,x)|\,dy \right) \left(\int_{x}^{+\infty} |f_E(y,x)|\,dy \right) = 0 \tag{2.24}$$

for a.e. $x \in \Omega$. Thus (2.19) is an equality if and only if at least one of the two integrals in the left-hand side of (2.24) is zero, and the reader can check that (ii) readily follows. □

Remark 2.13 (Half-lines in Corollary 2.12(ii))**.** In the case $n = 1$, it is worth to stress that (2.19) is always an equality when the set $E \subset \mathbb{R}$ is (equivalent to) an half-line, i.e.,

$$\|\nabla^\alpha \chi_{(a,+\infty)}\|_{L^1(\Omega;\mathbb{R})} = \mu_{1,\alpha} \tilde{P}_\alpha((a,+\infty);\Omega)$$

for any $\alpha \in (0,1)$, any $a \in \mathbb{R}$ and any open set $\Omega \subset \mathbb{R}$ such that $\tilde{P}_\alpha((a,+\infty);\Omega) < +\infty$. However, the equality cases in (2.19) are considerably richer. Indeed, on the one side,

$$\|\nabla^\alpha \chi_{(-5,-4)\cup(-1,+\infty)}\|_{L^1((0,1);\mathbb{R})} = \mu_{1,\alpha} \tilde{P}_\alpha((-5,-4)\cup(-1,+\infty);(0,1))$$

and, on the other side,

$$\|\nabla^\alpha \chi_{(-5,-4)\cup(0,+\infty)}\|_{L^1((-1,1);\mathbb{R})} < \mu_{1,\alpha} \tilde{P}_\alpha((-5,-4)\cup(0,+\infty);(-1,1))$$

for any $\alpha \in (0,1)$. We leave the simple computations to the interested reader.

2. Existence of blow-ups for fractional Caccioppoli sets

In this section we prove existence of blow-ups for sets with locally finite fractional Caccioppoli α-perimeter. We follow the approach presented in [**36**, Section 5.7].

2.1. Integration by parts on balls. We start with the following technical preliminary result.

Lemma 2.14. *Let $\alpha \in (0,1)$. For all $\varepsilon, r > 0$ and $x \in \mathbb{R}^n$ we define*

$$h_{\varepsilon,r,x}(y) = \begin{cases} 1 & \text{if } 0 \le |y-x| \le r, \\ \dfrac{r+\varepsilon-|y-x|}{\varepsilon} & \text{if } r < |y-x| < r+\varepsilon, \\ 0 & \text{if } |y-x| \ge r+\varepsilon. \end{cases}$$

Then $\nabla^\alpha h_{\varepsilon,r,x} \in L^1(\mathbb{R}^n;\mathbb{R}^n)$ with

$$\nabla^\alpha h_{\varepsilon,r,x}(y) = \frac{\mu_{n,\alpha}}{\varepsilon(n+\alpha-1)} \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|} |z-y|^{1-n-\alpha}\, dz \tag{2.25}$$

for $\mathscr{L}^n$-a.e. $y \in \mathbb{R}^n$.

Proof. Clearly $h_{\varepsilon,r,x} \in \mathrm{Lip}_c(\mathbb{R}^n)$ and

$$\nabla h_{\varepsilon,r,x}(y) = -\frac{1}{\varepsilon}\frac{y-x}{|y-x|}\, \chi_{B_{r+\varepsilon}(x)\setminus B_r(x)}(y).$$

Therefore by (1.48) we get

$$\nabla^\alpha h_{\varepsilon,r,x}(y) = -\frac{1}{\varepsilon}\frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{1}{|z-y|^{n+\alpha-1}} \frac{z-x}{|z-x|} \chi_{B_{r+\varepsilon}(x)\setminus B_r(x)}(z)\, dz$$

for $\mathscr{L}^n$-a.e. $y \in \mathbb{R}^n$. By Theorem 1.27, we get $\nabla^\alpha h_{\varepsilon,r,x} \in L^1(\mathbb{R}^n;\mathbb{R}^n)$. □

We now proceed with the following formula for integration by parts on balls, see [**36**, Lemma 5.2] for the analogous result in the classical setting.

Theorem 2.15 (Integration by parts on balls)**.** *Let $\alpha \in (0,1)$. If $E \subset \mathbb{R}^n$ is a set with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$, then*

$$\int_{E\cap B_r(x)} \mathrm{div}^\alpha \varphi\, dy + \int_E \varphi \cdot \nabla^\alpha \chi_{B_r(x)}\, dy + \int_E \mathrm{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \varphi)\, dy = -\int_{B_r(x)} \varphi \cdot dD^\alpha \chi_E \tag{2.26}$$

for all $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, $x \in \mathscr{F}^\alpha E$ and for $\mathscr{L}^1$-a.e. $r > 0$.

Proof. Fix $\varepsilon, r > 0$, $x \in \mathscr{F}^\alpha E$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ and let $h_{\varepsilon,r,x}$ be as in Lemma 2.14. On the one hand, by (2.9) we have

$$\int_E \mathrm{div}^\alpha(\varphi\, h_{\varepsilon,r,x})\, dy = -\int_{\mathscr{F}^\alpha E} (h_{\varepsilon,r,x}\, \varphi) \cdot dD^\alpha \chi_E. \tag{2.27}$$

Since $h_{\varepsilon,r,x}(y) \to \chi_{\overline{B_r(x)}}(y)$ as $\varepsilon \to 0$ for any $y \in \mathbb{R}^n$ and $|D^\alpha \chi_E|(\partial B_r(x)) = 0$ for $\mathscr{L}^1$-a.e. $r > 0$, we can compute

$$\lim_{\varepsilon\to 0} \int_{\mathscr{F}^\alpha E} (h_{\varepsilon,r,x}\, \varphi) \cdot dD^\alpha \chi_E = \int_{B_r(x)} \varphi \cdot dD^\alpha \chi_E.$$

On the other hand, by Lemma 1.1 and Lemma 1.7, we have

$$\operatorname{div}^\alpha(\varphi\, h_{\varepsilon,r,x}) = h_{\varepsilon,r,x} \operatorname{div}^\alpha\varphi + \varphi \cdot \nabla^\alpha h_{\varepsilon,r,x} + \operatorname{div}^\alpha_{\mathrm{NL}}(h_{\varepsilon,r,x}, \varphi). \tag{2.28}$$

We deal with each term of the right-hand side of (2.28) separately. For the first term, since $0 \le h_{\varepsilon,r,x} \le \chi_{B_{r+1}(x)}$ for all $\varepsilon \in (0,1)$ and $h_{\varepsilon,r,x} \to \chi_{B_r(x)}$ in $L^1(\mathbb{R}^n)$ as $\varepsilon \to 0$, by Corollary 1.3 and Lebesgue's Dominated Convergence Theorem we can compute

$$\lim_{\varepsilon\to 0} \int_E h_{\varepsilon,r,x} \operatorname{div}^\alpha\varphi \, dy = \int_{E\cap B_r(x)} \operatorname{div}^\alpha\varphi \, dy. \tag{2.29}$$

For the second term, by (2.25) we have

$$\int_E \varphi(y)\cdot\nabla^\alpha h_{\varepsilon,r,x}(y)\, dy = \frac{\mu_{n,\alpha}}{\varepsilon(n+\alpha-1)} \int_E \varphi(y) \cdot \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|} |z-y|^{1-n-\alpha}\, dz\, dy.$$

By Fubini's Theorem, we can compute

$$\begin{aligned}
&\int_E \varphi(y) \cdot \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|} |z-y|^{1-n-\alpha}\, dz\, dy \\
&\quad = \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|} \cdot \int_E \varphi(y)\, |z-y|^{1-n-\alpha}\, dy\, dz \\
&\quad = \int_r^{r+\varepsilon} \int_{\partial B_\varrho(x)} \frac{x-z}{|x-z|} \cdot \int_E \varphi(y)\, |z-y|^{1-n-\alpha}\, dy\, d\mathscr{H}^{n-1}(z)\, d\varrho.
\end{aligned}$$

By Lebesgue's Differentiation Theorem, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_E \varphi(y) \cdot \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|}\, |z-y|^{1-n-\alpha}\, dz\, dy \\
&\quad = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_r^{r+\varepsilon} \int_{\partial B_\varrho(x)} \frac{x-z}{|x-z|} \cdot \int_E \varphi(y)\, |z-y|^{1-n-\alpha}\, dy\, d\mathscr{H}^{n-1}(z)\, d\varrho \\
&\quad = \int_{\partial B_r(x)} \frac{x-z}{|x-z|} \cdot \int_E \varphi(y)\, |z-y|^{1-n-\alpha}\, dy\, d\mathscr{H}^{n-1}(z) \\
&\quad = \int_E \varphi(y) \cdot \int_{\partial B_r(x)} \frac{x-z}{|x-z|}\, |z-y|^{1-n-\alpha}\, d\mathscr{H}^{n-1}(z)\, dy \\
&\quad = \int_E \varphi(y) \cdot \int_{\mathbb{R}^n} |z-y|^{1-n-\alpha}\, dD\chi_{B_r(x)}(z)\, dy
\end{aligned}$$

for $\mathscr{L}^1$-a.e. $r > 0$. Therefore, by (1.48), we get that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_E \varphi \cdot \nabla^\alpha h_{\varepsilon,r,x}\, dy \\
&\quad = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_E \varphi(y) \cdot \int_{\mathbb{R}^n} |z-y|^{1-n-\alpha}\, dD\chi_{B_r(x)}(z)\, dy \\
&\quad = \int_E \varphi \cdot \nabla^\alpha \chi_{B_r(x)}\, dy
\end{aligned} \tag{2.30}$$

for $\mathscr{L}^1$-a.e. $r > 0$. Finally, for the third term, note that

$$\left| \frac{(z-y)\cdot(\varphi(z)-\varphi(y))(h_{\varepsilon,r,x}(z)-h_{\varepsilon,r,x}(y))}{|z-y|^{n+\alpha+1}} \right| \le 2\frac{|\varphi(z)-\varphi(y)|}{|z-y|^{n+\alpha}} \in L^1_z(\mathbb{R}^n)$$

for all $y \in \mathbb{R}^n$, so that

$$\lim_{\varepsilon\to 0} \operatorname{div}^\alpha_{\mathrm{NL}}(h_{\varepsilon,r,x}, \varphi)(y) = \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \varphi)(y)$$

for $\mathscr{L}^n$-a.e. $y \in \mathbb{R}^n$ by Lebesgue's Dominated Convergence Theorem. Since

$$|\mathrm{div}^\alpha_{\mathrm{NL}}(h_{\varepsilon,r,x},\varphi)(y)| \le 2\int_{\mathbb{R}^n} \frac{|\varphi(z)-\varphi(y)|}{|z-y|^{n+\alpha}}\,dz \in L^1_y(\mathbb{R}^n),$$

again by Lebesgue's Dominated Convergence Theorem we can compute

$$\lim_{\varepsilon\to 0}\int_E \mathrm{div}^\alpha_{\mathrm{NL}}(h_{\varepsilon,r,x},\varphi)\,dy = \int_E \mathrm{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)\,dy. \tag{2.31}$$

Combining (2.27), (2.28), (2.29), (2.30) and (2.31), we obtain (2.26). □

2.2. Decay estimates. We can now deduce the following decay estimates for the fractional De Giorgi's perimeter measure, see [**36**, Lemma 5.3] for the analogous result in the classical setting.

Theorem 2.16 (Decay estimates)**.** *Let $\alpha \in (0,1)$. There exist $A_{n,\alpha}, B_{n,\alpha} > 0$ with the following property. Let $E \subset \mathbb{R}^n$ be a set with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$. For any $x \in \mathscr{F}^\alpha E$, there exists $r_x > 0$ such that*

$$|D^\alpha\chi_E|(B_r(x)) \le A_{n,\alpha} r^{n-\alpha} \tag{2.32}$$

and

$$|D^\alpha\chi_{E\cap B_r(x)}|(\mathbb{R}^n) \le B_{n,\alpha} r^{n-\alpha} \tag{2.33}$$

for all $r \in (0, r_x)$.

Proof. We divide the proof in two steps, dealing with the two estimates separately.

Step 1: proof of (2.32). Fix $x \in \mathscr{F}^\alpha E$ and choose $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ such that $\varphi \equiv \nu^\alpha_E(x)$ in $B_1(x)$ and $\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1$. On the one hand, by Definition 2.7, there exists $r_x \in (0,1)$ such that

$$\int_{B_r(x)} \varphi\cdot dD^\alpha\chi_E \ge \frac{1}{2}|D^\alpha\chi_E|(B_r(x)) \tag{2.34}$$

for all $r \in (0, r_x)$. On the other hand, by (2.26) we have

$$\begin{aligned}\int_{B_r(x)} \varphi\cdot dD^\alpha\chi_E \le \left|\int_{E\cap B_r(x)} \mathrm{div}^\alpha\varphi\,dy\right| + \left|\int_E \varphi\cdot dD^\alpha\chi_{B_r(x)}\right| \\ + \left|\int_E \mathrm{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)\,dy\right|\end{aligned} \tag{2.35}$$

for $\mathscr{L}^1$-a.e. $r \in (0, r_x)$. We now estimate the three terms in the right-hand side separately. For the first one, since $\varphi(y) \equiv \nu^\alpha_E(x)$ in $B_r(x)$, we can estimate

$$\begin{aligned}\left|\int_{E\cap B_r(x)} \mathrm{div}^\alpha\varphi(y)\,dy\right| &\le \mu_{n,\alpha}\int_{E\cap B_r(x)}\int_{\mathbb{R}^n} \frac{|\varphi(z)-\varphi(y)|}{|z-y|^{n+\alpha}}\,dz\,dy \\ &= \mu_{n,\alpha}\int_{E\cap B_r(x)}\int_{\mathbb{R}^n\setminus B_r(x)} \frac{|\varphi(z)-\nu^\alpha_E(x)|}{|z-y|^{n+\alpha}}\,dz\,dy \\ &\le 2\mu_{n,\alpha}\int_{B_r(x)}\int_{\mathbb{R}^n\setminus B_r(x)} \frac{1}{|z-y|^{n+\alpha}}\,dz\,dy \\ &= 2\mu_{n,\alpha} P_\alpha(B_r(x))\end{aligned}$$

so that

$$\left|\int_{E\cap B_r(x)} \operatorname{div}^\alpha\varphi(y)\,dy\right| \le 2\mu_{n,\alpha}P_\alpha(B_1)\,r^{n-\alpha}. \tag{2.36}$$

For the second term, by Proposition 2.8 we can estimate

$$\left|\int_E \varphi\cdot dD^\alpha\chi_{B_r(x)}\right| \le |D^\alpha\chi_{B_r(x)}|(\mathbb{R}^n) \le \mu_{n,\alpha}P_\alpha(B_r(x)) = \mu_{n,\alpha}P_\alpha(B_1)\,r^{n-\alpha}. \tag{2.37}$$

Finally, by Lemma 1.7, we can estimate

$$\begin{aligned}\left|\int_E \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)\,dy\right| &\le \|\operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)\|_{L^1(\mathbb{R}^n)}\\ &\le 2\mu_{n,\alpha}[\chi_{B_r(x)}]_{W^{\alpha,1}(\mathbb{R}^n)}\\ &= 2\mu_{n,\alpha}P_\alpha(B_r(x))\end{aligned}$$

so that

$$\left|\int_E \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)\,dy\right| \le 2\mu_{n,\alpha}P_\alpha(B_1)\,r^{n-\alpha}. \tag{2.38}$$

Combining (2.34), (2.35), (2.36), (2.37) and (2.38), we conclude that

$$|D^\alpha\chi_E|(B_r(x)) \le 10\mu_{n,\alpha}P_\alpha(B_1)\,r^{n-\alpha} \tag{2.39}$$

for $\mathscr{L}^1$-a.e. $r\in(0,r_x)$. Hence (2.32) follows with $A_{n,\alpha} = 10\mu_{n,\alpha}P_\alpha(B_1)$ for all $r\in(0,r_x)$ by a simple continuity argument.

Step 2: proof of (2.33). Fix $x\in\mathscr{F}^\alpha E$ and $\varphi\in\mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}\le 1$. Again by (2.26) we can estimate

$$\left|\int_{E\cap B_r(x)} \operatorname{div}^\alpha\varphi\,dy\right| \le |D^\alpha\chi_E|(B_r(x)) + |D^\alpha\chi_{B_r(x)}|(\mathbb{R}^n) + \int_E |\operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)|\,dy$$

for $\mathscr{L}^1$-a.e. $r\in(0,r_x)$. Using (2.37), (2.38) and (2.39), we conclude that

$$|D^\alpha\chi_{E\cap B_r(x)}|(\mathbb{R}^n) \le 13\mu_{n,\alpha}P_\alpha(B_1)\,r^{n-\alpha}$$

for $\mathscr{L}^1$-a.e. $r\in(0,r_x)$. Hence (2.33) follows with $B_{n,\alpha} = 13\mu_{n,\alpha}P_\alpha(B_1)$ for all $r\in(0,r_x)$ by a simple continuity argument. This concludes the proof. □

As an easy consequence of Theorem 2.16, we can prove that

$$|D^\alpha\chi_E| \ll \mathscr{H}^{n-\alpha}\llcorner\mathscr{F}^\alpha E$$

for any set E with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$.

Corollary 2.17 ($|D^\alpha\chi_E| \ll \mathscr{H}^{n-\alpha}\llcorner\mathscr{F}^\alpha E$)**.** *Let $\alpha\in(0,1)$. If E is a set with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$, then*

$$|D^\alpha\chi_E| \le 2^{n-\alpha}\frac{A_{n,\alpha}}{\omega_{n-\alpha}}\mathscr{H}^{n-\alpha}\llcorner\mathscr{F}^\alpha E, \tag{2.40}$$

where $A_{n,\alpha}$ is as in (2.32).

Proof. By (2.32), we have that

$$\Theta^*_{n-\alpha}(|D^\alpha\chi_E|,x) = \limsup_{r\to 0}\frac{|D^\alpha\chi_E|(B_r(x))}{\omega_{n-\alpha}r^{n-\alpha}} \le \frac{A_{n,\alpha}}{\omega_{n-\alpha}}$$

for any $x\in\mathscr{F}^\alpha E$. Therefore, (2.40) is a simple application of [**6**, Theorem 2.56]. □

For any set E of locally finite fractional Caccioppoli α-perimeter, Corollary 2.17 enables us to obtain a lower bound on the Hausdorff dimension of $\mathscr{F}^\alpha E$.

Proposition 2.18. *Let $\alpha \in (0,1)$. If E is a set with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$, then*

$$\dim_{\mathscr{H}}(\mathscr{F}^\alpha E) \geq n - \alpha. \tag{2.41}$$

Proof. Since $|D^\alpha \chi_E|(\mathscr{F}^\alpha E) > 0$ by Definition 2.7, by Corollary 2.17 we conclude that $\mathscr{H}^{n-\alpha}(\mathscr{F}^\alpha E) > 0$, proving (2.41). □

2.3. No coarea formula in $BV^\alpha(\mathbb{R}^n)$. As another interesting consequence of Corollary 2.17, we are able to prove that assumption (1.33) in Theorem 1.19 cannot be dropped.

Corollary 2.19 (No coarea formula in $BV^\alpha(\mathbb{R}^n)$)**.** *Let $\alpha \in (0,1)$. There exist $f \in BV^\alpha(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|(\mathbb{R}^n)\,dt = +\infty. \tag{2.42}$$

Proof. Let $E \subset \mathbb{R}^n$ be such that $\chi_E \in BV(\mathbb{R}^n)$ and consider $f = (-\Delta)^{\frac{1-\alpha}{2}}\chi_E$. By Lemma 1.31, we know that $f \in BV^\alpha(\mathbb{R}^n)$ with $|D^\alpha f| = |D\chi_E| = \mathscr{H}^{n-1} \llcorner \mathscr{F}E$. If

$$\int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|(\mathbb{R}^n)\,dt < +\infty$$

then

$$|D^\alpha f| \leq \int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|\,dt$$

by Theorem 1.19. Thus $|D^\alpha f| \ll \mathscr{H}^{n-\alpha}$ by Corollary 2.17, so that $\mathscr{H}^{n-1}(\mathscr{F}E) = 0$, which is clearly absurd. □

Remark 2.20. If $f \in W^{\alpha,1}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}} |D^\alpha \chi_{\{f>t\}}|(\mathbb{R}^n)\,dt \leq \mu_{n,\alpha} \int_{\mathbb{R}} P_\alpha(\{f>t\})\,dt = \mu_{n,\alpha}[f]_{W^{\alpha,1}(\mathbb{R}^n)} < +\infty$$

by Proposition 2.8 and Tonelli's Theorem, so that (2.42) does not hold for all $f \in BV^\alpha(\mathbb{R}^n)$. We do not know if (1.35) is an equality for some functions $f \in BV^\alpha(\mathbb{R}^n)$.

2.4. Existence of blow-ups. We can now prove the existence of blow-ups for sets with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$, see [**36**, Theorem 5.13] for the analogous result in the classical setting. Here and in the following, given a set E with locally finite fractional Caccioppoli α-perimeter and $x \in \mathscr{F}^\alpha E$, we let $\mathrm{Tan}(E,x)$ be the set of all *tangent sets of E at x*, i.e. the set of all limit points in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-topology of the family $\left\{\frac{E-x}{r} : r>0\right\}$ as $r \to 0$.

Theorem 2.21 (Existence of blow-up)**.** *Let $\alpha \in (0,1)$. Let E be a set with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$. For any $x \in \mathscr{F}^\alpha E$ we have* $\mathrm{Tan}(E,x) \neq \varnothing$.

Proof. Fix $x \in \mathscr{F}^\alpha E$. Up to a translation, we can assume $x = 0$. We set $E_r = E/r = \{y \in \mathbb{R}^n : ry \in E\}$ for all $r > 0$. We divide the proof in two steps.

Step 1. For each $p \in \mathbb{N}$, we define $D_r^p = E_r \cap B_p$. By the α-homogeneity of $\operatorname{div}^\alpha$, we have

$$\int_{D_r^p} \operatorname{div}^\alpha \varphi \, dy = r^{-n} \int_{E \cap B_{rp}} (\operatorname{div}^\alpha \varphi)(r^{-1} z) \, dz = r^{\alpha - n} \int_{E \cap B_{rp}} \operatorname{div}^\alpha (\varphi(r^{-1} \cdot)) \, dz$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$. By (2.33), we thus get

$$|D^\alpha \chi_{D_r^p}|(\mathbb{R}^n) = r^{\alpha - n} |D^\alpha \chi_{E \cap B_{rp}}|(\mathbb{R}^n) \le B_{n,\alpha} p^{n-\alpha}$$

for all $r > 0$ such that $rp < r_0$. Hence, for each fixed $p \in \mathbb{N}$, we have

$$\sup_{r < r_0/p} |D^\alpha \chi_{D_r^p}|(\mathbb{R}^n) \le B_{n,\alpha} p^{n-\alpha}.$$

Step 2. Let $(r_k)_{k \in \mathbb{N}}$ be such that $r_k \to 0$ as $k \to +\infty$ and let $E_k = E_{r_k}$ and $D_k^p = D_{r_k}^p$ for simplicity. By *Step 1*, for each $p \in \mathbb{N}$ we know that

$$\sup_{r_k < r_0/p} |D^\alpha \chi_{D_k^p}|(\mathbb{R}^n) \le B_{n,\alpha} p^{n-\alpha} \quad \text{and} \quad D_k^p \subset B_p \quad \forall k \in \mathbb{N}.$$

Thanks to Theorem 2.5, by a standard diagonal argument we find a subsequence $(D_{k_j}^p)_{j \in \mathbb{N}}$ and a sequence $(F_p)_{p \in \mathbb{N}}$ of sets with finite fractional Caccioppoli α-perimeter such that $\chi_{D_{k_j}^p} \to \chi_{F_p}$ in $L^1(\mathbb{R}^n)$ as $j \to +\infty$ for each $p \in \mathbb{N}$. Up to null sets, we have $F_p \subset F_{p+1}$, so that $\chi_{E_{k_j}} \to \chi_F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where $F = \bigcup_{p \in \mathbb{N}} F_p$. We thus conclude that $F \in \operatorname{Tan}(E, x)$. □

2.5. Characterisation of blow-ups. We now give a characterisation of the blow-ups of sets with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$, see Claim #1 in the proof of [**36**, Theorem 5.13] for the result in the classical setting.

Proposition 2.22 (Characterisation of blow-ups)**.** *Let $\alpha \in (0,1)$. Let E be a set with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$ and let $x \in \mathscr{F}^\alpha E$. If $F \in \operatorname{Tan}(E, x)$, then F is a set of locally finite fractional Caccioppoli α-perimeter such that $\nu_F^\alpha(y) = \nu_E^\alpha(x)$ for $|D^\alpha \chi_F|$-a.e. $y \in \mathscr{F}^\alpha F$.*

Proof. As in the proof of Theorem 2.21, we assume $x = 0$ and we set $E_r = E/r$. By Theorem 2.21, there exists $(r_k)_{k \in \mathbb{N}}$ such that $r_k \to 0$ as $k \to +\infty$ and $\chi_{E_{r_k}} \to \chi_F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. By Proposition 1.11, it is clear that F has locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$. By (2.3), we get

$$D^\alpha \chi_{E_{r_k}} \rightharpoonup D^\alpha \chi_F \quad \text{in } \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$$

as $k \to +\infty$. Thus, for $\mathscr{L}^1$-a.e. $L > 0$, we have

$$D^\alpha \chi_{E_{r_k}}(B_L) \to D^\alpha \chi_F(B_L) \quad \text{as } k \to +\infty. \tag{2.43}$$

Since

$$D^\alpha \chi_{E_r} = r^{\alpha - n} (\delta_{\frac{1}{r}})_{\#} D^\alpha \chi_E \qquad \forall r > 0,$$

we have that

$$|D^\alpha \chi_{E_{r_k}}|(B_L) = r_k^{\alpha - n} |D^\alpha \chi_E|(B_{r_k L})$$

and

$$D^\alpha \chi_{E_{r_k}}(B_L) = r_k^{\alpha - n} D^\alpha \chi_E(B_{r_k L}).$$

Since $0 \in \mathscr{F}^\alpha E$, we thus get

$$\lim_{k\to+\infty} \frac{D^\alpha \chi_{E_{r_k}}(B_L)}{|D^\alpha \chi_{E_{r_k}}|(B_L)} = \lim_{k\to+\infty} \frac{D^\alpha \chi_E(B_{r_k L})}{|D^\alpha \chi_E|(B_{r_k L})} = \nu_E^\alpha(0). \tag{2.44}$$

Therefore, by Proposition 1.11, (2.43) and (2.44), we obtain that

$$\begin{aligned} |D^\alpha \chi_F|(B_L) &\le \liminf_{k\to+\infty} |D^\alpha \chi_{E_{r_k}}|(B_L) \\ &= \lim_{k\to+\infty} \int_{B_L} \nu_E^\alpha(0) \cdot dD^\alpha \chi_{E_{r_k}} \\ &= \int_{B_L} \nu_E^\alpha(0) \cdot dD^\alpha \chi_F \\ &= \int_{B_L} \nu_E^\alpha(0) \cdot \nu_F^\alpha \, d|D^\alpha \chi_F| \\ &\le |D^\alpha \chi_F|(B_L) \end{aligned}$$

for $\mathscr{L}^1$-a.e. $L > 0$. We thus get that $\nu_F^\alpha(y) = \nu_E^\alpha(0)$ for $|D^\alpha \chi_F|$-a.e. $y \in B_L \cap \mathscr{F}^\alpha F$ and $\mathscr{L}^1$-a.e. $L > 0$, so that the conclusion follows. □

CHAPTER 3

Asymptotic behaviour of fractional variation as $\alpha \to 1^-$

1. Truncation and approximation of BV functions

For the reader's convenience, in this starting section we state and prove two known results on BV functions and sets with locally finite perimeter.

1.1. Truncation of BV functions. Following [**6**, Section 3.6] and [**36**, Section 5.9], given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define its precise representative $f^\star \colon \mathbb{R}^n \to [0, +\infty]$ as

$$f^\star(x) = \lim_{r \to 0^+} \frac{1}{\omega_n r^n} \int_{B_r(x)} f(y)\, dy, \quad x \in \mathbb{R}^n, \tag{3.1}$$

if the limit exists, otherwise we let $f^\star(x) = 0$ by convention.

Theorem 3.1 (Truncation of BV functions)**.** *If $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$f\chi_{B_r} \in BV(\mathbb{R}^n), \ \textit{with}\ D(f\chi_{B_r}) = \chi^\star_{B_r} Df + f^\star D\chi_{B_r}, \tag{3.2}$$

for $\mathscr{L}^1$-a.e. $r > 0$. If, in addition, $f \in L^\infty(\mathbb{R}^n)$, then (3.2) *holds for all $r > 0$.*

Proof. Fix $\varphi \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$ and let $U \subset \mathbb{R}^n$ be a bounded open set such that $\mathrm{supp}(\varphi) \subset U$. Let $(\varrho_\varepsilon)_{\varepsilon > 0} \subset C^\infty_c(\mathbb{R}^n)$ be a family of standard mollifiers as in (1.25) and set $f_\varepsilon = f * \varrho_\varepsilon$ for all $\varepsilon > 0$. Note that $\mathrm{supp}\left(\varrho_\varepsilon * (\chi_{B_r}\varphi)\right) \subset U$ and $\mathrm{supp}\left(\varrho_\varepsilon * (\chi_{B_r}\mathrm{div}\varphi)\right) \subset U$ for all $\varepsilon > 0$ sufficiently small and for all $r > 0$. Given $r > 0$, by Leibniz's rule and Fubini's Theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^n} f_\varepsilon \chi_{B_r} \,\mathrm{div}\varphi \, dx &= \int_{\mathbb{R}^n} \chi_{B_r} \mathrm{div}(f_\varepsilon \varphi)\, dx - \int_{\mathbb{R}^n} \chi_{B_r} \varphi \cdot \nabla f_\varepsilon \, dx \\ &= - \int_{\mathbb{R}^n} f_\varepsilon \varphi \cdot \, dD\chi_{B_r} - \int_{\mathbb{R}^n} \varrho_\varepsilon * (\chi_{B_r}\varphi) \cdot dDf. \end{aligned} \tag{3.3}$$

Since $f_\varepsilon \to f$ a.e. in $\mathbb{R}^n$ as $\varepsilon \to 0^+$ and

$$|f|\, \varrho_\varepsilon * (\chi_{B_r} |\mathrm{div}\varphi|) \le |f| \chi_U \|\mathrm{div}\varphi\|_{L^\infty(\mathbb{R}^n)} \in L^1(\mathbb{R}^n)$$

for all $\varepsilon > 0$, by Lebesgue's Dominated Convergence Theorem we have

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} f_\varepsilon \chi_{B_r} \,\mathrm{div}\varphi \, dx = \int_{\mathbb{R}^n} f \chi_{B_r} \,\mathrm{div}\varphi \, dx$$

for all $r > 0$. Thus, since $\varrho_\varepsilon * (\chi_{B_r}\varphi) \to \chi^\star_{B_r}\varphi$ pointwise in $\mathbb{R}^n$ as $\varepsilon \to 0^+$ and

$$|\varrho_\varepsilon * (\chi_{B_r}\varphi)| \le \|\varphi\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \chi_U \in L^1(\mathbb{R}^n, |Df|)$$

for all $\varepsilon > 0$ sufficiently small, again by Lebesgue's Dominated Convergence Theorem we have

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \varrho_\varepsilon * (\chi_{B_r}\varphi) \cdot dDf = \int_{\mathbb{R}^n} \chi^\star_{B_r} \varphi \cdot dDf$$

for all $r > 0$. Now, by [**6**, Theorem 3.78 and Corollary 3.80], we know that $f_\varepsilon \to f^\star$ $\mathscr{H}^{n-1}$-a.e. in $\mathbb{R}^n$ as $\varepsilon \to 0^+$. As a consequence, given any $r > 0$, we get that $f_\varepsilon \to f^\star$ $|D\chi_{B_r}|$-a.e. in $\mathbb{R}^n$ as $\varepsilon \to 0^+$. Thus, if $f \in L^\infty(\mathbb{R}^n)$, then

$$|f_\varepsilon \varphi| \le \|f\|_{L^\infty(\mathbb{R}^n)} |\varphi| \in L^1(\mathbb{R}^n, |D\chi_{B_r}|)$$

for all $\varepsilon > 0$ and so, again by Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} f_\varepsilon \varphi \cdot dD\chi_{B_r} = \int_{\mathbb{R}^n} f^\star \varphi \cdot dD\chi_{B_r}$$

for all $r > 0$. Therefore, if $f \in L^\infty(\mathbb{R}^n)$, then we can pass to the limit as $\varepsilon \to 0^+$ in (3.3) and get

$$\int_{\mathbb{R}^n} f \chi_{B_r} \operatorname{div}\varphi \, dx = -\int_{\mathbb{R}^n} f^\star \varphi \cdot dD\chi_{B_r} - \int_{\mathbb{R}^n} \chi_{B_r}^\star \varphi \cdot dDf$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and for all $r > 0$. Since $\|f^\star\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^\infty(\mathbb{R}^n)}$, this proves (3.2) for all $r > 0$. If f is not necessarily bounded, then we argue as follows. Without loss of generality, assume that $\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1$. We can thus estimate

$$\left| \int_{\mathbb{R}^n} f_\varepsilon \varphi \cdot dD\chi_{B_r} - \int_{\mathbb{R}^n} f^\star \varphi \cdot dD\chi_{B_r} \right| \le \int_{\partial B_r} |f_\varepsilon - f^\star| \, d\mathscr{H}^{n-1}. \tag{3.4}$$

Given any $R > 0$, by Fatou's Lemma we thus get that

$$\begin{aligned}
\int_0^R \liminf_{\varepsilon \to 0^+} & \left| \int_{\mathbb{R}^n} f_\varepsilon \varphi \cdot dD\chi_{B_r} - \int_{\mathbb{R}^n} f^\star \varphi \cdot dD\chi_{B_r} \right| dr \\
&\le \int_0^R \liminf_{\varepsilon \to 0^+} \int_{\partial B_r} |f_\varepsilon - f^\star| \, d\mathscr{H}^{n-1} \, dr \\
&\le \liminf_{\varepsilon \to 0^+} \int_0^R \int_{\partial B_r} |f_\varepsilon - f^\star| \, d\mathscr{H}^{n-1} \, dr \\
&= \lim_{\varepsilon \to 0^+} \int_{B_R} |f_\varepsilon - f^\star| \, dx = 0.
\end{aligned}$$

Hence, the set

$$Z = \left\{ r > 0 : \liminf_{\varepsilon \to 0^+} \int_{\partial B_r} |f_\varepsilon - f^\star| \, d\mathscr{H}^{n-1} = 0 \right\} \tag{3.5}$$

satisfies $\mathscr{L}^1((0, +\infty) \setminus Z) = 0$ and depends neither on the choice of φ nor on the choice of the $\mathscr{L}^n$-representative of f. Now fix $r \in Z$ and let $(\varepsilon_k)_{k \in \mathbb{N}}$ be any sequence realising the lim inf in (3.5). By (3.4), we thus get

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} f_{\varepsilon_k} \varphi \cdot dD\chi_{B_r} = \int_{\mathbb{R}^n} f^\star \varphi \cdot dD\chi_{B_r}$$

uniformly for all φ satisfying $\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1$. Passing to the limit along the sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ as $k \to +\infty$ in (3.3), we get that

$$\int_{\mathbb{R}^n} f \chi_{B_r} \operatorname{div}\varphi \, dx = -\int_{\mathbb{R}^n} f^\star \varphi \cdot dD\chi_{B_r} - \int_{\mathbb{R}^n} \chi_{B_r}^\star \varphi \cdot dDf$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1$. Finally, since

$$\int_0^R \int_{\partial B_r} |f^\star| \, d\mathscr{H}^{n-1} \, dr = \int_{B_R} |f^\star| \, dx < +\infty,$$

the set

$$W = \left\{ r > 0 : \int_{\partial B_r} |f^\star| \, d\mathscr{H}^{n-1} \, dr < +\infty \right\}$$

satisfies $\mathscr{L}^1((0,+\infty) \setminus W) = 0$ and does not depend on the choice of the $\mathscr{L}^n$-representative of f. Thus (3.2) follows for all $r \in W \cap Z$ and the proof is concluded. □

1.2. Approximation by sets with polyhedral boundary. We now state and prove standard approximation results for sets with finite perimeter or, more generally, $BV_{\mathrm{loc}}(\mathbb{R}^n)$ functions, in a sufficiently regular bounded open set.

We need the following two preliminary lemmas.

Lemma 3.2. *Let $V, W \subset \mathbb{S}^{n-1}$, with V finite and W at most countable. For any $\varepsilon > 0$, there exists $\mathcal{R} \in \mathrm{SO}(n)$ with $|\mathcal{R} - \mathcal{I}| < \varepsilon$, where $\mathcal{I}$ is the identity matrix, such that $\mathcal{R}(V) \cap W = \varnothing$.*

Proof. Let $N \in \mathbb{N}$ be such that $V = \{v_i \in \mathbb{S}^{n-1} : i = 1, \dots, N\}$. We divide the proof in two steps.

Step 1. Assume that W is finite and set $A_i = \{\mathcal{R} \in \mathrm{SO}(n) : \mathcal{R}(v_i) \notin W\}$ for all $i = 1, \dots, N$. We now claim that A_i is an open and dense subset of $\mathrm{SO}(n)$ for all $i = 1, \dots, N$. Indeed, given any $i = 1, \dots, N$, since W is finite, the set $A_i^c = \mathrm{SO}(n) \setminus A_i$ is closed in $\mathrm{SO}(n)$. Moreover, we claim that $\mathrm{int}(A_i^c) = \varnothing$. Indeed, by contradiction, let us assume that $\mathrm{int}(A_i^c) \neq \varnothing$. Then there exist $\varepsilon > 0$ and $\mathcal{R} \in A_i^c$ such that any $\mathcal{S} \in \mathrm{SO}(n)$ with $|\mathcal{S} - \mathcal{R}| < \varepsilon$ satisfies $\mathcal{S} \in A_i^c$. In particular, for these $\mathcal{R} \in A_i^c$ and $\varepsilon > 0$, we have $\mathcal{R} + \frac{\varepsilon}{2^k} \frac{\mathcal{I}}{|\mathcal{I}|} \in A_i^c$ for any $k \geq 1$, which implies $\mathcal{R}(v_i) + \frac{\varepsilon}{2^k |\mathcal{I}|} v_i \in W$ for any $k \geq 1$, in contrast with the fact that W is finite. Thus, A_i is an open and dense subset of $\mathrm{SO}(n)$ for all $i = 1, \dots, N$, and so also the set

$$A^W = \bigcap_{i=1}^{N} A_i = \{\mathcal{R} \in \mathrm{SO}(n) : \mathcal{R}(v_i) \notin W \ \forall i = 1, \dots, N\}$$

is an open and dense subset of $\mathrm{SO}(n)$. The result is thus proved for any finite set W.

Step 2. Now assume that W is countable, $W = \{w_k \in \mathbb{S}^{n-1} : k \in \mathbb{N}\}$. For all $M \in \mathbb{N}$, set $W_M = \{w_k \in W : k \leq M\}$. By Step 1, we know that A^{W_M} is an open and dense subset of $\mathrm{SO}(n)$ for all $M \in \mathbb{N}$. Since $\mathrm{SO}(n) \subset \mathbb{R}^{n^2}$ is compact, by Baire's Theorem $A = \bigcap_{M \in \mathbb{N}} A^{W_M}$ is a dense subset of $\mathrm{SO}(n)$. This concludes the proof. □

Since $\det \colon \mathrm{GL}(n) \to \mathbb{R}$ is a continuous map, there exists a dimensional constant $\delta_n \in (0,1)$ such that $\det \mathcal{R} \geq \frac{1}{2}$ for all $\mathcal{R} \in \mathrm{GL}(n)$ with $|\mathcal{R} - \mathcal{I}| < \delta_n$.

Lemma 3.3. *Let $\varepsilon \in (0, \delta_n)$ and let $E \subset \mathbb{R}^n$ be a bounded set with $P(E) < +\infty$. If $\mathcal{R} \in \mathrm{SO}(n)$ satisfies $|\mathcal{R} - \mathcal{I}| < \varepsilon$, then*

$$|\mathcal{R}(E) \triangle E| \leq 2\varepsilon r_E \, P(E),$$

where $r_E = \sup\{r > 0 : |E \setminus B_r| > 0\}$.

Proof. We divide the proof in two steps.

Step 1. Let $r > 0$ and let $f \in C_c^\infty(\mathbb{R}^n)$. Setting $\mathcal{R}_t = (1-t)\mathcal{I} + t\mathcal{R}$ for all $t \in [0,1]$, we can estimate

$$\int_{B_r} |f(\mathcal{R}(x)) - f(x)| \, dx = \int_{B_r} \left| \int_0^1 \langle \nabla f(\mathcal{R}_t(x)), \mathcal{R}(x) - x \rangle \, dt \right| dx$$

$$\le |\mathcal{R} - \mathcal{I}|\, r \int_0^1 \int_{B_r} |\nabla f(\mathcal{R}_t(x))| \, dx \, dt.$$

Since $|\mathcal{R}_t - \mathcal{I}| = t|\mathcal{R} - \mathcal{I}| < t\varepsilon < \delta_n$ for all $t \in [0,1]$, $\mathcal{R}_t$ is invertible with $\det(\mathcal{R}_t^{-1}) \le 2$ for all $t \in [0,1]$. Hence we can estimate

$$\int_{B_r} |\nabla f(\mathcal{R}_t(x))| \, dx = \int_{\mathcal{R}_t(B_r)} |\nabla f(y)| \, |\det(\mathcal{R}_t^{-1})| \, dy \le 2 \int_{\mathbb{R}^n} |\nabla f(y)| \, dy,$$

so that

$$\int_{B_r} |f(\mathcal{R}(x)) - f(x)| \, dx \le 2\varepsilon r \|\nabla f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}. \tag{3.6}$$

Step 2. Since $\chi_E \in BV(\mathbb{R}^n)$, combining [**36**, Theorem 5.3] with a standard cut-off approximation argument, we find $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $f_k \to \chi_E$ pointwise a.e. in $\mathbb{R}^n$ and $|\nabla f_k|(\mathbb{R}^n) \to P(E)$ as $k \to +\infty$. Given any $r > 0$, by (3.6) in Step 1 we have

$$\int_{B_r} |f_k(\mathcal{R}(x)) - f_k(x)| \, dx \le 2\varepsilon r \|\nabla f_k\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$$

for all $k \in \mathbb{N}$. Passing to the limit as $k \to +\infty$, by Fatou's Lemma we get that

$$|(\mathcal{R}(E) \triangle E) \cap B_r| \le 2\varepsilon r \, P(E).$$

Since $E \subset B_{r_E}$ up to $\mathscr{L}^n$-negligible sets, also $\mathcal{R}(E) \subset B_{r_E}$ up to $\mathscr{L}^n$-negligible sets. Thus we can choose $r = r_E$ and the proof is complete. □

We are now ready to prove the main approximation result, see also [**5**, Proposition 15].

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $E \subset \mathbb{R}^n$ be a measurable set such that $P(E;\Omega) < +\infty$. There exists a sequence $(E_k)_{k\in\mathbb{N}}$ of bounded open sets with polyhedral boundary such that*

$$P(E_k; \partial\Omega) = 0 \tag{3.7}$$

for all $k \in \mathbb{N}$ and

$$\chi_{E_k} \to \chi_E \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{and} \quad P(E_k;\Omega) \to P(E;\Omega) \tag{3.8}$$

as $k \to +\infty$.

Proof. We divide the proof in four steps.

Step 1: cut-off. Since Ω is bounded, we find $R_0 > 0$ such that $\overline{\Omega} \subset B_{R_0}$. Let us define $R_k = R_0 + k$ and

$$C_k = \left\{ x \in \Omega^c : \operatorname{dist}(x, \partial\Omega) \le \frac{1}{k} \right\}$$

for all $k \in \mathbb{N}$. We set $E_k^1 = E \cap B_{R_k} \cap C_k^c$ for all $k \in \mathbb{N}$. Note that E_k^1 is a bounded measurable set such that

$$\chi_{E_k^1} \to \chi_E \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty$$

and

$$P(E_k^1;\Omega) = P(E;\Omega) \text{ for all } k \in \mathbb{N}.$$

Step 2: extension. Let us define

$$A_k = \left\{ x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < \frac{1}{4k} \right\}$$

for all $k \in \mathbb{N}$. Since $\chi_{E_k^1 \cap \Omega} \in BV(\Omega)$ for all $k \in \mathbb{N}$, by [**6**, Definition 3.20 and Proposition 3.21] there exists a sequence $(v_k)_{k\in\mathbb{N}} \subset BV(\mathbb{R}^n)$ such that

$$v_k = 0 \text{ a.e. in } A_k^c, \quad v_k = \chi_{E_k^1} \text{ in } \Omega, \quad |Dv_k|(\partial\Omega) = 0$$

for all $k \in \mathbb{N}$. Let us define $F_k^t = \{v_k > t\}$ for all $t \in (0,1)$. Given $k \in \mathbb{N}$, by the coarea formula [**6**, Theorem 3.40], for a.e. $t \in (0,1)$ the set F_k^t has finite perimeter in $\mathbb{R}^n$ and satisfies

$$F_k^t \subset A_k, \quad F_k^t \cap \Omega = E_k^1 \cap \Omega, \quad P(F_k^t; \partial\Omega) = 0$$

for all $k \in \mathbb{N}$. We choose any such $t_k \in (0,1)$ for each $k \in \mathbb{N}$ and define $E_k^2 = E_k^1 \cup F_k^{t_k}$ for all $k \in \mathbb{N}$. Note that E_k^2 is a bounded set with finite perimeter in $\mathbb{R}^n$ such that

$$\chi_{E_k^2} \to \chi_E \text{ in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty$$

and

$$P(E_k^2; \Omega) = P(E; \Omega) \quad \text{and} \quad P(E_k^2; \partial\Omega) = 0 \quad \text{for all } k \in \mathbb{N}.$$

Step 3: approximation. Let us define

$$D_k = \left\{ x \in \Omega^c : \operatorname{dist}(x, \partial\Omega) \in \left[\frac{1}{4k}, \frac{3}{4k} \right] \right\}$$

for all $k \in \mathbb{N}$. First arguing as in the first part of the proof of [**58**, Theorem 13.8] taking [**58**, Remark 13.13] into account, and then performing a standard diagonal argument, we find a sequence of bounded open sets $(E_k^3)_{k\in\mathbb{N}}$ with polyhedral boundary such that

$$E_k^3 \subset D_k^c \text{ for all } k \in \mathbb{N}$$

and

$$\chi_{E_k^3} \to \chi_E \text{ in } L^1_{\text{loc}}(\mathbb{R}^n), \quad P(E_k^3; \Omega) \to P(E; \Omega) \quad \text{and} \quad P(E_k^3; \partial\Omega) \to 0$$

as $k \to +\infty$. If there exists a subsequence $(E_{k_j}^3)_{j\in\mathbb{N}}$ such that $P(E_{k_j}^3; \partial\Omega) = 0$ for all $j \in \mathbb{N}$, then we can set $E_j = E_{k_j}$ for all $j \in \mathbb{N}$ and the proof is concluded. If this is not the case, then we need to proceed with the next last step.

Step 4: rotation. We now argue as in the last part of the proof of [**5**, Proposition 15]. Fix $k \in \mathbb{N}$ and assume $P(E_k^3; \partial\Omega) > 0$. Since E_k^3 has polyhedral boundary, we have $\mathscr{H}^{n-1}(\partial E_k^3 \cap \partial\Omega) > 0$ if and only if there exist $\nu \in \mathbb{S}^{n-1}$ and $U \subset \mathscr{F}\Omega$ such that $\mathscr{H}^{n-1}(U) > 0$, $\nu_\Omega(x) = \nu$ for all $x \in U$ and $U \subset \partial H$ for some half-space H satisfying $\nu_H = \nu$. Since $P(\Omega) = \mathscr{H}^{n-1}(\partial\Omega) < +\infty$, the set

$$\begin{aligned} W &= \left\{ \nu \in \mathbb{S}^{n-1} : \mathscr{H}^{n-1}\left(\{x \in \partial\Omega : \nu_\Omega(x) = \nu\}\right) > 0 \right\} \\ &= \bigcup_{h\in\mathbb{N}} \left\{ \nu \in \mathbb{S}^{n-1} : \tfrac{P(\Omega)}{h} \ge \mathscr{H}^{n-1}\left(\{x \in \partial\Omega : \nu_\Omega(x) = \nu\}\right) > \tfrac{P(\Omega)}{h+1}) \right\} \end{aligned}$$

is at most countable. Since E_k^3 has polyhedral boundary, the set

$$V_k = \left\{ \nu \in \mathbb{S}^{n-1} : \mathscr{H}^{n-1}\left(\left\{ x \in \partial E_k^3 : \nu_{E_k^3}(x) = \nu \right\} \right) > 0 \right\}$$

is finite. By Lemma 3.2, given $\varepsilon_k > 0$, there exists $\mathcal{R}_k \in \mathrm{SO}(n)$ with $|\mathcal{R}_k - \mathcal{I}| < \varepsilon_k$ such that $\mathcal{R}_k(V_k) \cap W = \varnothing$. Hence the set $E_k^4 = \mathcal{R}_k(E_k^3)$ must satisfy $P(E_k^4; \partial\Omega) = 0$. By

Lemma 3.3, we can choose $\varepsilon_k > 0$ sufficiently small in order to ensure that $|E_k^4 \triangle E_k^3| < \frac{1}{k}$. Now choose $\eta_k \in \left(0, \frac{1}{2k}\right)$ such that $P(E_k^3; Q_k) \le 2P(E_k^3; \partial\Omega)$, where

$$Q_k = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial\Omega) < \eta_k\}.$$

Since Ω is bounded, possibly choosing $\varepsilon_k > 0$ even smaller, we can also ensure that $\Omega \triangle \mathcal{R}^{-1}(\Omega) \subset Q_k$. Hence we can estimate

$$\begin{aligned}|P(E_k^4; \Omega) - P(E_k^3; \Omega)| &= |\mathscr{H}^{n-1}(\partial E_k^3 \cap \mathcal{R}^{-1}(\Omega)) - \mathscr{H}^{n-1}(\partial E_k^3 \cap \Omega)| \\ &\le \mathscr{H}^{n-1}\Big(\partial E_k^3 \cap (\Omega \triangle \mathcal{R}^{-1}(\Omega))\Big) \\ &\le \mathscr{H}^{n-1}(\partial E_k^3 \cap Q_k).\end{aligned}$$

We can thus set $E_k = E_k^4$ for all $k \in \mathbb{N}$ and the proof is complete. □

Remark 3.5 (A minor gap in the proof of [**5**, Proposition 15])**.** We warn the reader that the cut-off and the extension steps presented above were not mentioned in the proof of [**5**, Proposition 15], although they are unavoidable for the correct implementation of the rotation argument in the last step. Indeed, in general, one cannot expect the existence of a rotation $\mathcal{R} \in \mathrm{SO}(n)$ arbitrarily close to the identity map such that $P(\mathcal{R}(E); \partial\Omega) = 0$ and, at the same time, the difference between $P(\mathcal{R}(E); \Omega)$ and $P(E; \Omega)$ is small. For example, one can consider

$$\Omega = \Big\{(x_1, x_2) \in A : x_1^2 + x_2^2 < 25\Big\}$$

and

$$E = \Big\{(x_1, x_2) \in A : 1 < x_1^2 + x_2^2 < 4\Big\} \cup \Big\{(x_1, x_2) \in A^c : 9 < x_1^2 + x_2^2 < 16\Big\}$$

where $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\}$. In this case, for any rotation $\mathcal{R} \in \mathrm{SO}(2)$ arbitrarily close to the identity map, we have $P(\mathcal{R}(E); \Omega) > 2 + P(E; \Omega)$.

We conclude this section with the following result, establishing an approximation of BV_{loc} functions similar to that given in Theorem 3.4.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$. There exists $(f_k)_{k\in\mathbb{N}} \subset BV(\mathbb{R}^n)$ such that*

$$|Df_k|(\partial\Omega) = 0$$

for all $k \in \mathbb{N}$ and

$$f_k \to f \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{and} \quad |Df_k|(\Omega) \to |Df|(\Omega)$$

as $k \to +\infty$. If, in addition, $f \in L^1(\mathbb{R}^n)$, then $f_k \to f$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$.

Proof. We argue similarly as in the proof of Theorem 3.4, in two steps.

Step 1: cut-off at infinity. Since Ω is bounded, we find $R_0 > 0$ such that $\overline{\Omega} \subset B_{R_0}$. Given $(R_k)_k \subset (R_0, +\infty)$, we set $g_k = f\chi_{B_{R_k}}$ for all $k \in \mathbb{N}$. By Theorem 3.1, we have $g_k \in BV(\mathbb{R}^n)$ for a suitable choice of the sequence $(R_k)_{k\in\mathbb{N}}$, with $|Dg_k|(\Omega) = |Df|(\Omega)$ for all $k \in \mathbb{N}$ and $g_k \to f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $k \to +\infty$. If, in addition, $f \in L^1(\mathbb{R}^n)$, then $g_k \to f$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$.

Step 2: extension and cut-off near Ω. Let us define

$$A_k = \left\{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < \frac{1}{k}\right\}$$

for all $k \in \mathbb{N}$. Since $g_k \chi_\Omega \in BV(\Omega)$ with $|Dg_k|(\Omega) = |Df|(\Omega)$ for all $k \in \mathbb{N}$, by [**6**, Definition 3.20 and Proposition 3.21] there exists a sequence $(h_k)_{k\in\mathbb{N}} \subset BV(\mathbb{R}^n)$ such that

$$\operatorname{supp} h_k \subset A_{2k}, \quad h_k = g_k \text{ in } \Omega, \quad |Dh_k|(\partial\Omega) = 0$$

for all $k \in \mathbb{N}$ and

$$\lim_{k\to+\infty} \int_{A_{2k}\setminus\Omega} |h_k|\, dx = 0$$

(the latter property easily follows from the construction performed in the proof of [**6**, Proposition 3.21]). Now let $(v_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp} v_k \subset A_k^c$ and $0 \le v_k \le 1$ for all $k \in \mathbb{N}$ and $v_k \to \chi_{\Omega^c}$ pointwise in $\mathbb{R}^n$ as $k \to +\infty$. We can thus set $f_k = h_k + v_k g_k$ for all $k \in \mathbb{N}$. By [**6**, Propositon 3.2(b)], we have $v_k g_k \in BV(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, so that $f_k \in BV(\mathbb{R}^n)$ for all $k \in \mathbb{N}$. Since we can estimate

$$\begin{aligned} |f_k - f| &\le |h_k - f\chi_\Omega| + |v_k - \chi_{\Omega^c}|\,|g_k| + |g_k - f|\,\chi_{\Omega^c} \\ &= |h_k|\,\chi_{A_{2k}\setminus\Omega} + |v_k - \chi_{\Omega^c}|\,|g_k| + |g_k - f|\,\chi_{\Omega^c} \end{aligned}$$

for all $k \in \mathbb{N}$, we have $f_k \to f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $k \to +\infty$, with $f_k \to f$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$ if $f \in L^1(\mathbb{R}^n)$. By construction, we also have

$$|Df_k|(\Omega) = |Dh_k|(\Omega) \quad \text{and} \quad |Df_k|(\partial\Omega) = |Dh_k|(\partial\Omega)$$

for all $k \in \mathbb{N}$. The proof is complete. □

2. Lip_b-regular tests for fractional operators

In this section, we extend the fractional operators ∇^α and $\operatorname{div}^\alpha$ to Lip_b-regular functions and, consequently, we prove that Leibniz's rule and the integration-by-part formula still hold in this context.

2.1. Extension of ∇^α and $\operatorname{div}^\alpha$ to Lip_b-regular tests. In the following result, we extend the fractional α-divergence to Lip_b-regular vector fields.

Lemma 3.7 (Extension of $\operatorname{div}^\alpha$ to Lip_b)**.** *Let $\alpha \in (0,1)$. The operator*

$$\operatorname{div}^\alpha \colon \mathrm{Lip}_b(\mathbb{R}^n;\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$$

given by

$$\operatorname{div}^\alpha\varphi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}}\, dy, \quad x \in \mathbb{R}^n, \tag{3.9}$$

for all $\varphi \in \mathrm{Lip}_b(\mathbb{R}^n;\mathbb{R}^n)$, is well defined, with

$$\|\operatorname{div}^\alpha\varphi\|_{L^\infty(\mathbb{R}^n)} \le \frac{2^{1-\alpha} n\omega_n \mu_{n,\alpha}}{\alpha(1-\alpha)} \mathrm{Lip}(\varphi)^\alpha \|\varphi\|^{1-\alpha}_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}, \tag{3.10}$$

and satisfies

$$\begin{aligned} \operatorname{div}^\alpha\varphi(x) &= \mu_{n,\alpha} \lim_{\varepsilon\to0^+} \int_{\{|y-x|>\varepsilon\}} \frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}}\, dy \\ &= \mu_{n,\alpha} \lim_{\varepsilon\to0^+} \int_{\{|y-x|>\varepsilon\}} \frac{(y-x)\cdot\varphi(y)}{|y-x|^{n+\alpha+1}}\, dy \end{aligned} \tag{3.11}$$

for all $x \in \mathbb{R}^n$. *Moreover, if in addition* $I_{1-\alpha}|\mathrm{div}\varphi| \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *then*

$$\mathrm{div}^\alpha \varphi(x) = I_{1-\alpha}\mathrm{div}\varphi(x) \tag{3.12}$$

for a.e. $x \in \mathbb{R}^n$.

Proof. We split the proof in two steps.

Step 1: proof of (3.9), (3.10) *and* (3.11). Given $x \in \mathbb{R}^n$ and $r > 0$, we can estimate

$$\int_{\{|y-x|\le r\}} \left| \frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}} \right| dy \le n\omega_n \mathrm{Lip}(\varphi) \int_0^r \varrho^{-\alpha}\, d\varrho$$

and

$$\int_{\{|y-x|> r\}} \left| \frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}} \right| dy \le 2n\omega_n \|\varphi\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)} \int_r^{+\infty} \varrho^{-(1+\alpha)}\, d\varrho.$$

Hence the function in (3.9) is well defined for all $x \in \mathbb{R}^n$ and

$$\|\mathrm{div}^\alpha\varphi\|_{L^\infty(\mathbb{R}^n)} \le n\omega_n \left(\frac{\mathrm{Lip}(\varphi)}{1-\alpha}\, r^{1-\alpha} + \frac{2\|\varphi\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}}{\alpha}\, r^{-\alpha} \right),$$

so that (3.10) follows by optimising the right-hand side in $r > 0$. Moreover, since

$$\begin{aligned} &\left| \frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}}\, \chi_{(\varepsilon,+\infty)}(|y-x|) \right| \\ &\qquad \le \mathrm{Lip}(\varphi)\, \frac{\chi_{(0,1)}(|y-x|)}{|y-x|^{n+\alpha-1}} + 2\|\varphi\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}\, \frac{\chi_{[1,+\infty)}(|y-x|)}{|y-x|^{n+\alpha}} \in L^1_{x,y}(\mathbb{R}^n) \end{aligned}$$

and

$$\int_{\{|z|>\varepsilon\}} \frac{z}{|z|^{n+\alpha+1}}\, dy = 0$$

for all $\varepsilon > 0$, by Lebesgue's Dominated Convergence Theorem we immediately get the two equalities in (3.11) for all $x \in \mathbb{R}^n$.

Step 2: proof of (3.12). Assume that $I_{1-\alpha}|\mathrm{div}\varphi| \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then

$$\frac{|\mathrm{div}\varphi(y)|}{|y-x|^{n+\alpha-1}} \in L^1_y(\mathbb{R}^n) \tag{3.13}$$

for a.e. $x \in \mathbb{R}^n$. Hence, by Lebesgue's Dominated Convergence Theorem, we can write

$$I_{1-\alpha}\mathrm{div}\varphi(x) = \mu_{n,\alpha} \lim_{\varepsilon\to 0^+} \int_{\{|y-x|>\varepsilon\}} \frac{\mathrm{div}\varphi(y)}{|y-x|^{n+\alpha-1}}\, dy$$

for a.e. $x \in \mathbb{R}^n$. Now let $\varepsilon > 0$ be fixed and let $R > 0$. Again by (3.13) and Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{R\to+\infty} \int_{\{R>|y-x|>\varepsilon\}} \frac{\mathrm{div}\varphi(y)}{|y-x|^{n+\alpha-1}}\, dy = \int_{\{|y-x|>\varepsilon\}} \frac{\mathrm{div}\varphi(y)}{|y-x|^{n+\alpha-1}}\, dy$$

for a.e. $x \in \mathbb{R}^n$. Moreover, integrating by parts, we get

$$\begin{aligned} \int_{\{R>|y-x|>\varepsilon\}} \frac{\mathrm{div}\varphi(y)}{|y-x|^{n+\alpha-1}}\, dy &= \int_{\{R>|y|>\varepsilon\}} \frac{\mathrm{div}_y\varphi(y+x)}{|y|^{n+\alpha-1}}\, dy \\ &= \int_{\{|y|=R\}} \frac{y}{|y|}\, \frac{\varphi(y+x)}{|y|^{n+\alpha-1}}\, d\mathscr{H}^{n-1}(y) - \int_{\{|y|=\varepsilon\}} \frac{y}{|y|}\, \frac{\varphi(y+x)}{|y|^{n+\alpha-1}}\, d\mathscr{H}^{n-1}(y) \end{aligned}$$

$$+\int_{\{R>|y|>\varepsilon\}} \frac{y\cdot\varphi(y+x)}{|y|^{n+\alpha+1}}\,dy$$

for all $R>0$ and for a.e. $x\in\mathbb{R}^n$. Since $\varphi\in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$, by Lebesgue's Dominated Convergence Theorem we have

$$\lim_{R\to+\infty}\int_{\{R>|y|>\varepsilon\}} \frac{y\cdot\varphi(y+x)}{|y|^{n+\alpha+1}}\,dy=\int_{\{|y|>\varepsilon\}} \frac{y\cdot\varphi(y+x)}{|y|^{n+\alpha+1}}\,dy$$

for all $\varepsilon>0$ and all $x\in\mathbb{R}^n$. We can also estimate

$$\left|\int_{\{|y|=R\}} \frac{y}{|y|}\frac{\varphi(y+x)}{|y|^{n+\alpha-1}}\,d\mathscr{H}^{n-1}(y)\right|\le n\omega_n\|\varphi\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}R^{-\alpha}$$

for all $R>0$ and all $x\in\mathbb{R}^n$. We thus have that

$$\int_{\{|y-x|>\varepsilon\}} \frac{\mathrm{div}\varphi(y)}{|y-x|^{n+\alpha-1}}\,dy=\int_{\{|y|>\varepsilon\}} \frac{y\cdot\varphi(y+x)}{|y|^{n+\alpha+1}}\,dy-\int_{\{|y|=\varepsilon\}} \frac{y}{|y|}\frac{\varphi(y+x)}{|y|^{n+\alpha-1}}\,d\mathscr{H}^{n-1}(y)$$

for all $\varepsilon>0$ and a.e. $x\in\mathbb{R}^n$. Since also

$$\begin{aligned}\left|\int_{\{|y|=\varepsilon\}} \frac{y}{|y|}\frac{\varphi(y+x)}{|y|^{n+\alpha-1}}\,d\mathscr{H}^{n-1}(y)\right|&=\left|\int_{\{|y|=\varepsilon\}} \frac{y}{|y|}\frac{\varphi(y+x)-\varphi(x)}{|y|^{n+\alpha-1}}\,d\mathscr{H}^{n-1}(y)\right|\\&\le n\omega_n\,\mathrm{Lip}(\varphi)\,\varepsilon^{1-\alpha}\end{aligned}$$

for all $\varepsilon>0$ and $x\in\mathbb{R}^n$, we conclude that

$$\lim_{\varepsilon\to0^+}\int_{\{|y-x|>\varepsilon\}} \frac{\mathrm{div}\varphi(y)}{|y-x|^{n+\alpha-1}}\,dy=\lim_{\varepsilon\to0^+}\int_{\{|y-x|>\varepsilon\}} \frac{(y-x)\cdot\varphi(y)}{|y-x|^{n+\alpha+1}}\,dy$$

for a.e. $x\in\mathbb{R}^n$, proving (3.12). □

We can also extend the fractional α-gradient to Lip_b-regular functions. The proof is very similar to the one of Lemma 3.7 and is left to the reader.

Lemma 3.8 (Extension of ∇^α to Lip_b)**.** *Let $\alpha\in(0,1)$. The operator*

$$\nabla^\alpha\colon\ \mathrm{Lip}_b(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n;\mathbb{R}^n)$$

given by

$$\nabla^\alpha f(x)=\mu_{n,\alpha}\int_{\mathbb{R}^n}\frac{(y-x)\cdot(f(y)-f(x))}{|y-x|^{n+\alpha+1}}\,dy,\quad x\in\mathbb{R}^n,$$

for all $f\in\mathrm{Lip}_b(\mathbb{R}^n)$, is well defined, with

$$\|\nabla^\alpha f\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}\le\frac{2^{1-\alpha}n\omega_n\mu_{n,\alpha}}{\alpha(1-\alpha)}\,\mathrm{Lip}(f)^\alpha\|f\|^{1-\alpha}_{L^\infty(\mathbb{R}^n)},$$

and satisfies

$$\begin{aligned}\nabla^\alpha f(x)&=\mu_{n,\alpha}\lim_{\varepsilon\to0^+}\int_{\{|y-x|>\varepsilon\}}\frac{(y-x)\cdot(f(y)-f(x))}{|y-x|^{n+\alpha+1}}\,dy\\&=\mu_{n,\alpha}\lim_{\varepsilon\to0^+}\int_{\{|y-x|>\varepsilon\}}\frac{(y-x)\cdot f(y)}{|y-x|^{n+\alpha+1}}\,dy\end{aligned}$$

for all $x\in\mathbb{R}^n$. Moreover, if in addition $I_{1-\alpha}|\nabla f|\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then

$$\nabla^\alpha f(x)=I_{1-\alpha}\nabla f(x)$$

for a.e. $x\in\mathbb{R}^n$.

2.2. Extended Leibniz's rules for ∇^α and div^α. The following two results extend the validity of Leibniz's rules proved in Lemma 1.6 and Lemma 1.7 to Lip_b-regular functions and Lip_b-regular vector fields. The proofs are very similar to the ones given in Chapter 1 and to those of Lemma 3.7 and Lemma 3.8, and thus are left to the reader.

Lemma 3.9 (Extended Leibniz's rule for ∇^α)**.** *Let $\alpha \in (0,1)$. If $f \in \mathrm{Lip}_b(\mathbb{R}^n)$ and $\eta \in \mathrm{Lip}_c(\mathbb{R}^n)$, then*

$$\nabla^\alpha(\eta f) = \eta\, \nabla^\alpha f + f\, \nabla^\alpha \eta + \nabla^\alpha_{\mathrm{NL}}(\eta, f),$$

where

$$\nabla^\alpha_{\mathrm{NL}}(\eta, f)(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)\cdot(f(y)-f(x))(\eta(y)-\eta(x))}{|y-x|^{n+\alpha+1}}\, dy$$

for all $x \in \mathbb{R}^n$, with

$$\|\nabla^\alpha_{\mathrm{NL}}(\eta, f)\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)} \le \frac{2^{2-\alpha} n\omega_n \mu_{n,\alpha} \|f\|_{L^\infty(\mathbb{R}^n)}}{\alpha(1-\alpha)} \mathrm{Lip}(\eta)^\alpha \|\eta\|^{1-\alpha}_{L^\infty(\mathbb{R}^n)}$$

and

$$\|\nabla^\alpha_{\mathrm{NL}}(\eta, f)\|_{L^1(\mathbb{R}^n;\,\mathbb{R}^n)} \le \mu_{n,\alpha} \|f\|_{L^\infty(\mathbb{R}^n)} [\eta]_{W^{\alpha,1}(\mathbb{R}^n)}.$$

Lemma 3.10 (Extended Leibniz's rule for div^α)**.** *Let $\alpha \in (0,1)$. If $\varphi \in \mathrm{Lip}_b(\mathbb{R}^n;\mathbb{R}^n)$ and $\eta \in \mathrm{Lip}_c(\mathbb{R}^n)$, then*

$$\mathrm{div}^\alpha(\eta\varphi) = \eta\, \mathrm{div}^\alpha \varphi + \varphi \cdot \nabla^\alpha \eta + \mathrm{div}^\alpha_{\mathrm{NL}}(\eta, \varphi),$$

where

$$\mathrm{div}^\alpha_{\mathrm{NL}}(\eta, \varphi)(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)\cdot(\varphi(y)-\varphi(x))(\eta(y)-\eta(x))}{|y-x|^{n+\alpha+1}}\, dy$$

for all $x \in \mathbb{R}^n$, with

$$\|\mathrm{div}^\alpha_{\mathrm{NL}}(\eta, \varphi)\|_{L^\infty(\mathbb{R}^n)} \le \frac{2^{2-\alpha} n\omega_n \mu_{n,\alpha} \|\varphi\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}}{\alpha(1-\alpha)} \mathrm{Lip}(\eta)^\alpha \|\eta\|^{1-\alpha}_{L^\infty(\mathbb{R}^n)}$$

and

$$\|\mathrm{div}^\alpha_{\mathrm{NL}}(\eta, \varphi)\|_{L^1(\mathbb{R}^n)} \le \mu_{n,\alpha} \|\varphi\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)} [\eta]_{W^{\alpha,1}(\mathbb{R}^n)}.$$

2.3. Extended integration-by-part formulas. Thanks to Lemma 3.10, we can actually prove that a function in $BV^\alpha(\mathbb{R}^n)$ can be tested against any Lip_b-regular vector field.

Proposition 3.11 (Lip_b-regular test for BV^α functions)**.** *Let $\alpha \in (0,1)$. If $f \in BV^\alpha(\mathbb{R}^n)$, then* (1.22) *holds for all $\varphi \in \mathrm{Lip}_b(\mathbb{R}^n;\mathbb{R}^n)$.*

Proof. We argue similarly as in the proof of Theorem 1.16. Fix $\varphi \in \mathrm{Lip}_b(\mathbb{R}^n;\mathbb{R}^n)$ and let $(\eta_R)_{R>0} \subset C^\infty_c(\mathbb{R}^n)$ be a family of cut-off functions as in (1.30) On the one hand, since

$$\left| \int_{\mathbb{R}^n} f\eta_R\, \mathrm{div}^\alpha \varphi\, dx - \int_{\mathbb{R}^n} f\, \mathrm{div}^\alpha \varphi\, dx \right| \le \|\mathrm{div}^\alpha \varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f|\,(1-\eta_R)\, dx$$

for all $R > 0$, by Lebesgue's Dominated Convergence Theorem we have

$$\lim_{R\to+\infty} \int_{\mathbb{R}^n} f\eta_R\, \mathrm{div}^\alpha \varphi\, dx = \int_{\mathbb{R}^n} f\, \mathrm{div}^\alpha \varphi\, dx.$$

On the other hand, by Lemma 3.10 we can write

$$\int_{\mathbb{R}^n} f\eta_R \operatorname{div}^\alpha\varphi\,dx = \int_{\mathbb{R}^n} f\operatorname{div}^\alpha(\eta_R\varphi)\,dx - \int_{\mathbb{R}^n} f\,\varphi\cdot\nabla^\alpha\eta_R\,dx - \int_{\mathbb{R}^n} f\operatorname{div}^\alpha_{\mathrm{NL}}(\eta_R,\varphi)\,dx$$

for all $R>0$. By Proposition 1.14, we have

$$\int_{\mathbb{R}^n} f\operatorname{div}^\alpha(\eta_R\varphi)\,dx = -\int_{\mathbb{R}^n}\eta_R\varphi\cdot dD^\alpha f$$

for all $R>0$. Since

$$\left|\int_{\mathbb{R}^n}\eta_R\varphi\cdot dD^\alpha f - \int_{\mathbb{R}^n}\varphi\cdot dD^\alpha f\right| \le \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}\int_{\mathbb{R}^n}(1-\eta_R)\,d|D^\alpha f|$$

for all $R>0$, by Lebesgue's Dominated Convergence Theorem (with respect to the finite measure $|D^\alpha f|$) we have

$$\lim_{R\to+\infty}\int_{\mathbb{R}^n}\eta_R\varphi\cdot dD^\alpha f = \int_{\mathbb{R}^n}\varphi\cdot dD^\alpha f.$$

Finally, we can estimate

$$\left|\int_{\mathbb{R}^n} f\,\varphi\cdot\nabla^\alpha\eta_R\,dx\right| \le \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}\int_{\mathbb{R}^n}|f(x)|\int_{\mathbb{R}^n}\frac{|\eta_R(y)-\eta_R(x)|}{|y-x|^{n+\alpha}}\,dy\,dx$$

and, similarly,

$$\left|\int_{\mathbb{R}^n} f\operatorname{div}^\alpha_{\mathrm{NL}}(\eta_R,\varphi)\,dx\right| \le 2\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}\int_{\mathbb{R}^n}|f(x)|\int_{\mathbb{R}^n}\frac{|\eta_R(y)-\eta_R(x)|}{|y-x|^{n+\alpha}}\,dy\,dx.$$

By Lebesgue's Dominated Convergence Theorem, we thus get that

$$\lim_{R\to+\infty}\left(\int_{\mathbb{R}^n} f\,\varphi\cdot\nabla^\alpha\eta_R\,dx + \int_{\mathbb{R}^n} f\operatorname{div}^\alpha_{\mathrm{NL}}(\eta_R,\varphi)\,dx\right) = 0$$

and the conclusion follows. □

Thanks to Lemma 3.9, we can prove that a function in $\operatorname{Lip}_b(\mathbb{R}^n)$ can be tested against any Lip_c-regular vector field. The proof is very similar to the one of Proposition 3.11 and is thus left to the reader.

Proposition 3.12 (Integration by parts for Lip_b-regular functions)**.** *Let $\alpha\in(0,1)$. If $f\in\operatorname{Lip}_b(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} f\operatorname{div}^\alpha\varphi\,dx = -\int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f\,dx$$

for all $\varphi\in\operatorname{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$.

3. Estimates and representation formulas for the fractional α-gradient

3.1. Integrability properties of the fractional α-gradient. We begin with the following technical local estimate on the $W^{\alpha,1}$-seminorm of a function in BV_{loc}.

Lemma 3.13. *Let $\alpha\in(0,1)$ and let $f\in BV_{\mathrm{loc}}(\mathbb{R}^n)$. Then $f\in W^{\alpha,1}_{\mathrm{loc}}(\mathbb{R}^n)$ with*

$$[f]_{W^{\alpha,1}(B_R)} \le \frac{n\omega_n(2R)^{1-\alpha}}{1-\alpha}\,|Df|(B_{3R}) \tag{3.14}$$

for all $R>0$.

Proof. Fix $R > 0$ and let $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $f \in C^1(B_{3R})$. We can estimate

$$\begin{aligned}
[f]_{W^{\alpha,1}(B_R)} &= \int_{B_R} \int_{B_R} \frac{|f(y) - f(x)|}{|y - x|^{n+\alpha}}\, dy\, dx \\
&= \int_{B_R} \int_{B_R \cap \{|y-x|<2R\}} \frac{|f(y) - f(x)|}{|y - x|^{n+\alpha}}\, dy\, dx \\
&\le \int_{\{|h|<2R\}} \frac{1}{|h|^{n+\alpha}} \int_{B_R} |f(x+h) - f(x)|\, dx\, dh.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{B_R} |f(x+h) - f(x)|\, dx &\le \int_{B_R} \int_0^1 |\nabla f(x+th) \cdot h|\, dt\, dx \\
&\le |h| \int_0^1 \int_{B_R} |\nabla f(x+th)|\, dx\, dt \\
&\le |h| \int_{B_{R+|h|}} |\nabla f(z)|\, dz
\end{aligned}$$

for all $h \in \mathbb{R}^n$, we have

$$\begin{aligned}
[f]_{W^{\alpha,1}(B_R)} &\le \int_{\{|h|<2R\}} \frac{1}{|h|^{n+\alpha-1}} \int_{B_{R+|h|}} |\nabla f(z)|\, dz\, dh \\
&\le \int_{\{|h|<2R\}} \frac{|Df|(B_{3R})}{|h|^{n+\alpha-1}}\, dh \\
&= \frac{n\omega_n (2R)^{1-\alpha}}{1-\alpha}\, |Df|(B_{3R})
\end{aligned}$$

proving (3.14) for all $f \in BV_{\mathrm{loc}}(\mathbb{R}^n) \cap C^1(B_{3R})$. Now fix $R > 0$ and let $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$. By [**36**, Theorem 5.3], we can find a sequence $(f_k)_{k\in\mathbb{N}} \subset BV(B_{3R}) \cap C^\infty(B_{3R})$ such that $|Df_k|(B_{3R}) \to |Df|(B_{3R})$ and $f_k \to f$ a.e. in B_{3R} as $k \to +\infty$. Hence, by Fatou's Lemma, we get

$$\begin{aligned}
[f]_{W^{\alpha,1}(B_R)} &\le \liminf_{k\to+\infty} [f_k]_{W^{\alpha,1}(B_R)} \\
&\le \frac{n\omega_n (2R)^{1-\alpha}}{1-\alpha} \lim_{k\to+\infty} |Df_k|(B_{3R}) \\
&= \frac{n\omega_n (2R)^{1-\alpha}}{1-\alpha}\, |Df|(B_{3R})
\end{aligned}$$

and the proof is complete. □

In the following result, we collect several local integrability estimates involving the fractional α-gradient of a function satisfying various regularity assumptions.

Proposition 3.14. *The following statements hold.*

(i) If $f \in BV(\mathbb{R}^n)$, then $f \in BV^\alpha(\mathbb{R}^n)$ for all $\alpha \in (0,1)$ with $D^\alpha f = \nabla^\alpha f \mathscr{L}^n$ and

$$\nabla^\alpha f = I_{1-\alpha} Df \quad \text{a.e. in } \mathbb{R}^n. \tag{3.15}$$

In addition, for any bounded open set $U \subset \mathbb{R}^n$, we have

$$\|\nabla^\alpha f\|_{L^1(U;\mathbb{R}^n)} \le C_{n,\alpha,U}\, |Df|(\mathbb{R}^n) \tag{3.16}$$

for all $\alpha \in (0,1)$*, where* $C_{n,\alpha,U}$ *is as in* (1.13)*. Finally, given an open set* $A \subset \mathbb{R}^n$*, we have*

$$\|\nabla^\alpha f\|_{L^1(A;\mathbb{R}^n)} \le \frac{n\omega_n\,\mu_{n,\alpha}}{n+\alpha-1}\left(\frac{|Df|(\overline{A_r})}{1-\alpha}\,r^{1-\alpha} + \frac{n+2\alpha-1}{\alpha}\,\|f\|_{L^1(\mathbb{R}^n)}\,r^{-\alpha}\right) \tag{3.17}$$

for all $r>0$ *and* $\alpha \in (0,1)$*, where* $A_r = \{x \in \mathbb{R}^n : \operatorname{dist}(x,A) < r\}$*. In particular, we have*

$$\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \frac{n\omega_n\,\mu_{n,\alpha}(n+2\alpha-1)^{1-\alpha}}{\alpha(1-\alpha)(n+\alpha-1)}\,\|f\|_{L^1(\mathbb{R}^n)}^{1-\alpha}\,[f]_{BV(\mathbb{R}^n)}^{\alpha}. \tag{3.18}$$

(ii) If $f \in L^\infty(\mathbb{R}^n) \cap W^{\alpha,1}_{\mathrm{loc}}(\mathbb{R}^n)$*, then the fractional* α*-gradient* $D^\alpha f \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ *exists and satisfies* $D^\alpha f = \nabla^\alpha f \mathscr{L}^n$ *with* $\nabla^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ *and*

$$\begin{aligned}\|\nabla^\alpha f\|_{L^1(B_R;\mathbb{R}^n)} &\le \mu_{n,\alpha}\int_{B_R}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\,dx\,dy\\ &\le \mu_{n,\alpha}\Big([f]_{W^{\alpha,1}(B_R)} + P_\alpha(B_R)\,\|f\|_{L^\infty(\mathbb{R}^n)}\Big)\end{aligned} \tag{3.19}$$

for all $R>0$ *and* $\alpha\in(0,1)$*.*

(iii) If $f \in L^\infty(\mathbb{R}^n) \cap BV_{\mathrm{loc}}(\mathbb{R}^n)$*, then the fractional* α*-gradient* $D^\alpha f \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ *exists and satisfies* $D^\alpha f = \nabla^\alpha f \mathscr{L}^n$ *with* $\nabla^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ *and*

$$\|\nabla^\alpha f\|_{L^1(B_R;\mathbb{R}^n)} \le \mu_{n,\alpha}\left(\frac{n\omega_n(2R)^{1-\alpha}}{1-\alpha}\,|Df|(B_{3R}) + \frac{2(n\omega_n)^2R^{n-\alpha}}{\alpha\,\Gamma(1-\alpha)^{-1}}\,\|f\|_{L^\infty(\mathbb{R}^n)}\right). \tag{3.20}$$

for all $R>0$ *and* $\alpha\in(0,1)$*.*

Proof. We prove the three statements separately.

Proof of (i). Thanks to Theorem 1.27, we just need to prove (3.16) and (3.17).

We prove (3.16). By (3.15), by Tonelli's Theorem and by Lemma 1.4, we get

$$\int_U |\nabla^\alpha f|\,dx \le \int_U I_{1-\alpha}|Df|\,dx \le C_{n,\alpha,U}\,|Df|(\mathbb{R}^n),$$

where $C_{n,\alpha,U}$ is defined as in (1.13).

We now prove (3.17) in two steps.

Proof of (3.17)*, Step 1.* Assume $f \in C^\infty_c(\mathbb{R}^n)$ and fix $r>0$. We have

$$\begin{aligned}\int_A |\nabla^\alpha f|\,dx &= \int_A |I_{1-\alpha}\nabla f|\,dx\\ &\le \frac{\mu_{n,\alpha}}{n+\alpha-1}\left(\int_A\int_{\{|h|\le r\}}\frac{|\nabla f(x+h)|}{|h|^{n+\alpha-1}}\,dh\,dx + \int_A\left|\int_{\{|h|>r\}}\frac{\nabla f(x+h)}{|h|^{n+\alpha-1}}\,dh\right|dx\right).\end{aligned}$$

We estimate the two double integrals appearing in the right-hand side separately. By Tonelli's Theorem, we have

$$\begin{aligned}\int_A\int_{\{|h|\le r\}}\frac{|\nabla f(x+h)|}{|h|^{n+\alpha-1}}\,dh\,dx &= \int_{\{|h|\le r\}}\int_A|\nabla f(x+h)|\,dx\,\frac{dh}{|h|^{n+\alpha-1}}\\ &\le \|\nabla f\|_{L^1(\overline{A_r};\mathbb{R}^n)}\int_{\{|h|\le r\}}\frac{dh}{|h|^{n+\alpha-1}}\\ &= n\omega_n\,\frac{r^{1-\alpha}}{1-\alpha}\,\|\nabla f\|_{L^1(\overline{A_r};\mathbb{R}^n)}.\end{aligned}$$

Concerning the second double integral, integrating by parts we get

$$\int_{\{|h|>r\}} \frac{\nabla f(x+h)}{|h|^{n+\alpha-1}}\,dh = (n+\alpha-1)\int_{\{|h|>r\}} \frac{h f(x+h)}{|h|^{n+\alpha+1}}\,dh - \int_{\{|h|=r\}} \frac{h}{|h|}\,\frac{f(x+h)}{|h|^{n+\alpha-1}}\,d\mathscr{H}^{n-1}(h)$$

for all $x \in A$. Hence, we can estimate

$$\begin{aligned}\int_A\left|\int_{\{|h|>r\}} \frac{\nabla f(x+h)}{|h|^{n+\alpha-1}}\,dh\right|dx &\le (n+\alpha-1)\int_A\int_{\{|h|>r\}} \frac{|f(x+h)|}{|h|^{n+\alpha}}\,dh\,dx \\ &\quad + \int_A\int_{\{|h|=r\}} \frac{|f(x+h)|}{|h|^{n+\alpha-1}}\,d\mathscr{H}^{n-1}(h)\,dx \\ &\le n\omega_n \|f\|_{L^1(\mathbb{R}^n)}\, r^{-\alpha}\left(\frac{n+\alpha-1}{\alpha}+1\right) \\ &= n\omega_n\left(\frac{n+2\alpha-1}{\alpha}\right)\|f\|_{L^1(\mathbb{R}^n)}\, r^{-\alpha}.\end{aligned}$$

Thus (3.17) follows for all $f \in C_c^\infty(\mathbb{R}^n)$ and $r > 0$.

Proof of (3.17)*, Step 2.* Let $f \in BV(\mathbb{R}^n)$ and fix $r > 0$. Combining [**36**, Theorem 5.3] with a standard cut-off approximation argument, we find $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$ and $|Df_k|(\mathbb{R}^n) \to |Df|(\mathbb{R}^n)$ as $k \to +\infty$. By Step 1, we have that

$$\|\nabla^\alpha f_k\|_{L^1(A;\mathbb{R}^n)} \le \frac{n\omega_n\,\mu_{n,\alpha}}{n+\alpha-1}\left(\frac{|Df_k|(\overline{A_r})}{1-\alpha}\,r^{1-\alpha} + \frac{n+2\alpha-1}{\alpha}\,\|f_k\|_{L^1(\mathbb{R}^n)}\,r^{-\alpha}\right) \tag{3.21}$$

for all $k \in \mathbb{N}$. We claim that

$$(\nabla^\alpha f_k)\,\mathscr{L}^n \rightharpoonup (\nabla^\alpha f)\,\mathscr{L}^n \quad \text{as } k \to +\infty. \tag{3.22}$$

Indeed, if $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, then $\mathrm{div}^\alpha\varphi \in L^\infty(\mathbb{R}^n)$ by (1.12) and thus

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f_k\,dx - \int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f\,dx\right| &= \left|\int_{\mathbb{R}^n} f_k\,\mathrm{div}^\alpha\varphi\,dx - \int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha\varphi\,dx\right| \\ &\le \|\mathrm{div}^\alpha\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}\,\|f_k - f\|_{L^1(\mathbb{R}^n)}\end{aligned}$$

for all $k \in \mathbb{N}$, so that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f_k\,dx = \int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f\,dx.$$

Now fix $\varphi \in C_c^0(\mathbb{R}^n;\mathbb{R}^n)$. Let $U \subset \mathbb{R}^n$ be a bounded open set such that $\operatorname{supp}\varphi \subset U$. For each $\varepsilon > 0$ sufficiently small, pick $\psi_\varepsilon \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ such that $\|\varphi - \psi_\varepsilon\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} < \varepsilon$ and $\operatorname{supp}\psi_\varepsilon \subset U$. Then

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f_k\,dx - \int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f\,dx\right| &\le \left|\int_{\mathbb{R}^n}\psi_\varepsilon\cdot\nabla^\alpha f_k\,dx - \int_{\mathbb{R}^n}\psi_\varepsilon\cdot\nabla^\alpha f\,dx\right| \\ &\quad + \|\psi_\varepsilon - \varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}\Big(\|\nabla^\alpha f_k\|_{L^1(U;\mathbb{R}^n)} + \|\nabla^\alpha f\|_{L^1(U;\mathbb{R}^n)}\Big) \\ &\le \left|\int_{\mathbb{R}^n}\psi_\varepsilon\cdot\nabla^\alpha f_k\,dx - \int_{\mathbb{R}^n}\psi_\varepsilon\cdot\nabla^\alpha f\,dx\right| \\ &\quad + \varepsilon\,C_{n,\alpha,U}\Big(|Df_k|(\mathbb{R}^n) + |Df|(\mathbb{R}^n)\Big),\end{aligned}$$

so that

$$\lim_{k\to+\infty}\left|\int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f_k\,dx-\int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f\,dx\right|\le 2\varepsilon\, C_{n,\alpha,U}|Df|(\mathbb{R}^n).$$

Thus, (3.22) follows passing to the limit as $\varepsilon\to 0^+$. Thanks to (3.22), by [**58**, Proposition 4.29] we get that

$$\|\nabla^\alpha f\|_{L^1(A;\,\mathbb{R}^n)}\le\liminf_{k\to+\infty}\|\nabla^\alpha f_k\|_{L^1(A;\,\mathbb{R}^n)}.$$

Since

$$|Df|(U)\le\liminf_{k\to+\infty}|Df_k|(U)$$

for any open set $U\subset\mathbb{R}^n$ by [**36**, Theorem 5.2], we can estimate

$$\begin{aligned}\limsup_{k\to+\infty}|Df_k|(\overline{A_r})&\le\lim_{k\to+\infty}|Df_k|(\mathbb{R}^n)-\liminf_{k\to+\infty}|Df_k|(\mathbb{R}^n\setminus A_r)\\&\le|Df|(\mathbb{R}^n)-|Df|(\mathbb{R}^n\setminus A_r)\\&=|Df|(\overline{A_r}).\end{aligned}$$

Thus, (3.17) follows taking limits as $k\to+\infty$ in (3.21). Finally, (3.18) is easily deduced by optimising the right-hand side of (3.17) in the case $A=\mathbb{R}^n$ with respect to $r>0$.

Proof of (ii). Assume $f\in L^\infty(\mathbb{R}^n)\cap W^{\alpha,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Given $R>0$, we can estimate

$$\begin{aligned}\int_{B_R}|\nabla^\alpha f(x)|\,dx&\le\mu_{n,\alpha}\int_{B_R}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\,dx\,dy\\&=\mu_{n,\alpha}\int_{B_R}\int_{B_R}\frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\,dx\,dy+\mu_{n,\alpha}\int_{B_R}\int_{\mathbb{R}^n\setminus B_R}\frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\,dx\,dy\\&\le\mu_{n,\alpha}[f]_{W^{\alpha,1}(B_R)}+2\mu_{n,\alpha}\|f\|_{L^\infty(\mathbb{R}^n)}\int_{B_R}\int_{\mathbb{R}^n\setminus B_R}\frac{1}{|x-y|^{n+\alpha}}\,dx\,dy\\&=\mu_{n,\alpha}[f]_{W^{\alpha,1}(B_R)}+\mu_{n,\alpha}\|f\|_{L^\infty(\mathbb{R}^n)}P_\alpha(B_R)\end{aligned}$$

and (3.19) follows. To prove that $D^\alpha f=\nabla^\alpha f\mathscr{L}^n$, we argue as in the proof of Proposition 2.8. Let $\varphi\in\mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$. Since $f\in L^\infty(\mathbb{R}^n)$, we have

$$x\mapsto|f(x)|\int_{\mathbb{R}^n}\frac{|\varphi(y)-\varphi(x)|}{|y-x|^{n+\alpha}}\,dy\in L^1(\mathbb{R}^n).$$

Hence, by the definition of div^α on Lip_c-regular vector fields and by Lebesgue's Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}^n}f\,\mathrm{div}^\alpha\varphi\,dx=\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^n}f(x)\int_{\{|y-x|>\varepsilon\}}\frac{(y-x)\cdot\varphi(y)}{|y-x|^{n+\alpha+1}}\,dy\,dx.$$

Since

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\{|y-x|>\varepsilon\}}\frac{|f(x)|\,|\varphi(y)|}{|y-x|^{n+\alpha}}\,dy\,dx&\le\|f\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}|\varphi(y)|\int_{\{|y-x|>\varepsilon\}}|y-x|^{-n-\alpha}\,dx\,dy\\&\le\frac{n\omega_n}{\alpha\varepsilon^\alpha}\|f\|_{L^\infty(\mathbb{R}^n)}\|\varphi\|_{L^1(\mathbb{R}^n;\,\mathbb{R}^n)}\end{aligned}$$

for all $\varepsilon > 0$, by Fubini's Theorem we can compute

$$\begin{aligned}\int_{\mathbb{R}^n} f(x) \int_{\{|y-x|>\varepsilon\}} \frac{(y-x)\cdot\varphi(y)}{|y-x|^{n+\alpha+1}}\, dy\, dx &= -\int_{\mathbb{R}^n} \varphi(y) \int_{\{|x-y|>\varepsilon\}} \frac{(x-y)\, f(x)}{|x-y|^{n+\alpha+1}}\, dx\, dy \\ &= -\int_{\mathbb{R}^n} \varphi(y) \int_{\{|x-y|>\varepsilon\}} \frac{(x-y)\,(f(x)-f(y))}{|x-y|^{n+\alpha+1}}\, dx\, dy.\end{aligned}$$

Since

$$|\varphi(y)| \left| \int_{\{|x-y|>\varepsilon\}} \frac{(x-y)\,(f(x)-f(y))}{|x-y|^{n+\alpha+1}}\, dx \right| \le |\varphi(y)| \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\, dx$$

for all $y \in \mathbb{R}^n$ and $\varepsilon > 0$, and

$$y \mapsto \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\, dx \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

by (3.19), again by Lebesgue's Dominated Convergence Theorem we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\, \mathrm{div}^\alpha \varphi(x)\, dx &= -\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi(y) \int_{\{|x-y|>\varepsilon\}} \frac{(x-y)\,(f(x)-f(y))}{|x-y|^{n+\alpha+1}}\, dx\, dy \\ &= -\int_{\mathbb{R}^n} \varphi(y) \lim_{\varepsilon\to 0} \int_{\{|x-y|>\varepsilon\}} \frac{(x-y)\,(f(x)-f(y))}{|x-y|^{n+\alpha+1}}\, dx\, dy \\ &= -\int_{\mathbb{R}^n} \varphi(y) \cdot \nabla^\alpha f(y)\, dy\end{aligned}$$

for all $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$. Thus $D^\alpha f \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ is well defined and $D^\alpha f = \nabla^\alpha f \mathscr{L}^{n-1}$.

Proof of (iii). Assume $f \in L^\infty(\mathbb{R}^n) \cap BV_{\mathrm{loc}}(\mathbb{R}^n)$. By Lemma 3.13, we know that $f \in L^\infty(\mathbb{R}^n) \cap W^{\alpha,1}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$, so that $D^\alpha f \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ exists by (ii). Hence, inserting (3.14) in (3.19), we find

$$\|\nabla^\alpha f\|_{L^1(B_R;\mathbb{R}^n)} \le \mu_{n,\alpha} \left(\frac{n\omega_n (2R)^{1-\alpha}}{1-\alpha}\, |Df|(B_{3R}) + P_\alpha(B_1)\, R^{n-\alpha}\, \|f\|_{L^\infty(\mathbb{R}^n)} \right).$$

Since for all $x \in B_1$ we have

$$\int_{\mathbb{R}^n\setminus B_1} \frac{dy}{|y-x|^{n+\alpha}} = \int_{\mathbb{R}^n\setminus B_1(-x)} \frac{dz}{|z|^{n+\alpha}} \le \int_{\mathbb{R}^n\setminus B_{1-|x|}} \frac{dz}{|z|^{n+\alpha}} = \frac{n\omega_n}{\alpha(1-|x|)^\alpha},$$

being Γ increasing on $(0,+\infty)$ (see [**13**]), we can estimate

$$\begin{aligned}P_\alpha(B_1) = 2\int_{B_1}\int_{\mathbb{R}^n\setminus B_1} \frac{dy\, dx}{|y-x|^{n+\alpha}} &\le \frac{2n\omega_n}{\alpha} \int_{B_1} \frac{dx}{(1-|x|)^\alpha} \\ &= \frac{2(n\omega_n)^2}{\alpha} \int_0^1 \frac{t^{n-1}}{(1-t)^\alpha}\, dt = \frac{2(n\omega_n)^2}{\alpha}\, \frac{\Gamma(n)\,\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)} \\ &\le \frac{2(n\omega_n)^2}{\alpha}\, \Gamma(1-\alpha),\end{aligned}$$

so that

$$\|\nabla^\alpha f\|_{L^1(B_R;\mathbb{R}^n)} \le \mu_{n,\alpha} \left(\frac{n\omega_n (2R)^{1-\alpha}}{1-\alpha}\, |Df|_{BV(B_{3R})} + \frac{2(n\omega_n)^2 R^{n-\alpha}}{\alpha\,\Gamma(1-\alpha)^{-1}}\, \|f\|_{L^\infty(\mathbb{R}^n)} \right),$$

proving (3.20). □

Note that Proposition 3.14(i), in particular, applies to any $f \in W^{1,1}(\mathbb{R}^n)$. In the following result, we prove that a similar result holds also for any $f \in W^{1,p}(\mathbb{R}^n)$ with $p \in (1,+\infty)$.

Proposition 3.15 ($W^{1,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$ for $p \in [1,+\infty)$)**.** *Let $\alpha \in (0,1)$ and $p \in [1,+\infty)$. If $f \in W^{1,p}(\mathbb{R}^n)$, then $f \in S^{\alpha,p}(\mathbb{R}^n)$ with*

(3.23)
$$\|\nabla^\alpha f\|_{L^p(A;\,\mathbb{R}^n)} \le \frac{n\omega_n\mu_{n,\alpha}}{n+\alpha-1}\left(\frac{\|\nabla f\|_{L^p(\overline{A_r};\,\mathbb{R}^n)}}{1-\alpha}\, r^{1-\alpha} + \frac{n+2\alpha-1}{\alpha}\,\|f\|_{L^p(\mathbb{R}^n)}\, r^{-\alpha}\right)$$

for any $r>0$ and any open set $A \subset \mathbb{R}^n$, where $A_r = \{x \in \mathbb{R}^n : \operatorname{dist}(x,A) < r\}$. In particular, we have

$$\|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \le \frac{(n+2\alpha-1)^{1-\alpha}}{n+\alpha-1}\,\frac{n\omega_n\mu_{n,\alpha}}{\alpha(1-\alpha)}\,\|\nabla f\|^{\alpha}_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}\|f\|^{1-\alpha}_{L^p(\mathbb{R}^n)}. \tag{3.24}$$

In addition, if $p \in \left(1, \frac{n}{1-\alpha}\right)$ and $q = \frac{np}{n-(1-\alpha)p}$, then

$$\nabla^\alpha f = I_{1-\alpha}\nabla f \quad \text{a.e. in } \mathbb{R}^n \tag{3.25}$$

and $\nabla^\alpha f \in L^q(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. We argue similarly as in the proof of Proposition 3.14(i).

Proof of (3.23)*, Step 1.* Assume $f \in C_c^\infty(\mathbb{R}^n)$ and fix an open set $A \subset \mathbb{R}^n$ and $r>0$. Arguing as in the proof of (3.17), we can write

$$\begin{aligned}
I_{1-\alpha}\nabla f(x) &= \frac{\mu_{n,\alpha}}{n+\alpha-1}\left(\int_{\{|h|\le r\}} \frac{\nabla f(x+h)}{|h|^{n+\alpha-1}}\,dh + \int_{\{|h|>r\}} \frac{\nabla f(x+h)}{|h|^{n+\alpha-1}}\,dh\right)\\
&= \frac{\mu_{n,\alpha}}{n+\alpha-1}\left(\int_{\{|h|\le r\}} \frac{\nabla f(x+h)}{|h|^{n+\alpha-1}}\,dh + (n+\alpha-1)\int_{\{|h|>r\}} \frac{h\cdot f(x+h)}{|h|^{n+\alpha+1}}\,dh\right.\\
&\qquad\left. - \int_{\{|h|=r\}} \frac{h}{|h|}\,\frac{f(x+h)}{|h|^{n+\alpha-1}}\,d\mathscr{H}^{n-1}(h)\right)
\end{aligned}$$

for all $x \in A$. By (1.14) and Minkowski's Integral Inequality (see [**96**, Section A.1], for example), we thus have

$$\begin{aligned}
\|\nabla^\alpha f\|_{L^p(A;\,\mathbb{R}^n)} &\le \frac{\mu_{n,\alpha}}{n+\alpha-1}\left(\int_{\{|h|\le r\}} \frac{\|\nabla f(\cdot+h)\|_{L^p(A;\,\mathbb{R}^n)}}{|h|^{n+\alpha-1}}\,dh\right.\\
&\qquad + (n+\alpha-1)\int_{\{|h|>r\}} \frac{\|f(\cdot+h)\|_{L^p(A)}}{|h|^{n+\alpha}}\,dh\\
&\qquad\left. + \int_{\{|h|=r\}} \frac{\|f(\cdot+h)\|_{L^p(A)}}{|h|^{n+\alpha-1}}\,d\mathscr{H}^{n-1}(h)\right)\\
&\le \frac{\mu_{n,\alpha}}{n-\alpha+1}\left(\frac{n\omega_n}{1-\alpha}\,\|\nabla f\|_{L^p(\overline{A_r};\,\mathbb{R}^n)}\, r^{1-\alpha} + n\omega_n\frac{n+2\alpha-1}{\alpha}\,\|f\|_{L^p(\mathbb{R}^n)}\, r^{-\alpha}\right),
\end{aligned}$$

proving (3.23) for all $f \in C_c^\infty(\mathbb{R}^n)$ and $r>0$.

Proof of (3.23)*, Step 2.* Let $f \in W^{1,p}(\mathbb{R}^n)$ and fix an open set $A \subset \mathbb{R}^n$ and $r>0$. Combining [**36**, Theorem 4.2] with a standard cut-off approximation argument, we find

$(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $W^{1,p}(\mathbb{R}^n)$ as $k \to +\infty$. By Step 1, we have that
(3.26)

$$\|\nabla^\alpha f_k\|_{L^p(A;\mathbb{R}^n)} \le \frac{n\omega_n\,\mu_{n,\alpha}}{n+\alpha-1}\left(\frac{\|\nabla f_k\|_{L^p(\overline{A_r};\mathbb{R}^n)}}{1-\alpha}\,r^{1-\alpha} + \frac{n+2\alpha-1}{\alpha}\,\|f_k\|_{L^p(\mathbb{R}^n)}\,r^{-\alpha}\right)$$

for all $k \in \mathbb{N}$. Hence, choosing $A = \mathbb{R}^n$, we get that the sequence $(\nabla^\alpha f_k)_{k\in\mathbb{N}}$ is uniformly bounded in $L^p(\mathbb{R}^n;\mathbb{R}^n)$. Up to pass to a subsequence (which we do not relabel for simplicity), there exists $g \in L^p(\mathbb{R}^n;\mathbb{R}^n)$ such that $\nabla^\alpha f_k \rightharpoonup g$ in $L^p(\mathbb{R}^n;\mathbb{R}^n)$ as $k \to +\infty$. Given $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} f_k \,\mathrm{div}^\alpha\varphi\,dx = -\int_{\mathbb{R}^n} \varphi\cdot\nabla^\alpha f_k\,dx$$

for all $k \in \mathbb{N}$. Passing to the limit as $k \to +\infty$, by Corollary 1.3 we get that

$$\int_{\mathbb{R}^n} f \,\mathrm{div}^\alpha\varphi\,dx = -\int_{\mathbb{R}^n} \varphi\cdot g\,dx$$

for any $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, so that $g = \nabla^\alpha f$ and hence $f \in S^{\alpha,p}(\mathbb{R}^n)$ according to Definition 1.42. We thus have that

$$\|\nabla^\alpha f\|_{L^p(A;\mathbb{R}^n)} \le \liminf_{k\to+\infty} \|\nabla^\alpha f_k\|_{L^p(A;\mathbb{R}^n)}$$

for any open set $A \subset \mathbb{R}^n$, since

$$\int_{\mathbb{R}^n} \varphi\cdot\nabla^\alpha f\,dx = \lim_{k\to+\infty}\int_{\mathbb{R}^n} \varphi\cdot\nabla^\alpha f_k\,dx \le \|\varphi\|_{L^{\frac{p}{p-1}}(A;\mathbb{R}^n)} \liminf_{k\to+\infty} \|\nabla^\alpha f_k\|_{L^p(A;\mathbb{R}^n)}$$

for all $\varphi \in C_c^\infty(A;\mathbb{R}^n)$. Therefore, (3.23) follows by taking limits as $k \to +\infty$ in (3.26).

Proof of (3.24). Inequality (3.24) follows by applying (3.23) with $A = \mathbb{R}^n$ and minimising the right-hand side with respect to $r > 0$.

Proof of (3.25). Now assume $p \in \left(1, \frac{n}{1-\alpha}\right)$ and let $q = \frac{np}{n-(1-\alpha)p}$. Let $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be fixed. Recalling inequality (N.59), since $\varphi \in L^{\frac{q}{q-1}}(\mathbb{R}^n;\mathbb{R}^n)$ we have that

$$|\varphi|\,I_{1-\alpha}|f| \in L^1(\mathbb{R}^n), \quad |\varphi|\,I_{1-\alpha}|\nabla f| \in L^1(\mathbb{R}^n).$$

In particular, Fubini's Theorem implies that

$$f\,I_{1-\alpha}\varphi \in L^1(\mathbb{R}^n;\mathbb{R}^n), \quad I_{1-\alpha}\varphi\cdot\nabla f \in L^1(\mathbb{R}^n).$$

Since $\mathrm{div}^\alpha\varphi \in L^{\frac{p}{p-1}}(\mathbb{R}^n)$ by Corollary 1.3, we also get that

$$f\,\mathrm{div} I_{1-\alpha}\varphi = f\,\mathrm{div}^\alpha\varphi \in L^1(\mathbb{R}^n).$$

Therefore, observing that $I_{1-\alpha}\varphi \in \mathrm{Lip}_b(\mathbb{R}^n;\mathbb{R}^n)$ because

$$\nabla I_{1-\alpha}\varphi = \nabla^\alpha\varphi \in L^\infty(\mathbb{R}^n;\mathbb{R}^{n^2})$$

again by Corollary 1.3 and performing a standard cut-off approximation argument, we can integrate by parts and obtain

$$\int_{\mathbb{R}^n} \varphi\cdot I_{1-\alpha}\nabla f\,dx = \int_{\mathbb{R}^n} I_{1-\alpha}\varphi\cdot\nabla f\,dx = -\int_{\mathbb{R}^n} f\,\mathrm{div} I_{1-\alpha}\varphi\,dx = -\int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha\varphi\,dx.$$

Therefore

$$\int_{\mathbb{R}^n} \varphi\cdot I_{1-\alpha}\nabla f\,dx = -\int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha\varphi\,dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, proving (3.25). In particular, notice that $\nabla^\alpha f \in L^q(\mathbb{R}^n;\mathbb{R}^n)$ by inequality (N.59). The proof is complete. □

For the case $p = +\infty$, we have the following immediate consequence of Lemma 3.9 and Proposition 3.12.

Corollary 3.16 ($W^{1,\infty}(\mathbb{R}^n) \subset S^{\alpha,\infty}(\mathbb{R}^n)$)**.** *Let $\alpha \in (0,1)$. If $f \in W^{1,\infty}(\mathbb{R}^n)$, then $f \in S^{\alpha,\infty}(\mathbb{R}^n)$ with*

$$\|\nabla^\alpha f\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 2^{1-\alpha}\,\frac{n\omega_n\mu_{n,\alpha}}{\alpha(1-\alpha)}\,\|\nabla f\|^\alpha_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}\|f\|^{1-\alpha}_{L^\infty(\mathbb{R}^n)}. \tag{3.27}$$

3.2. Two representation formulas for the α-variation. In this section, we prove two useful representation formulas for the α-variation.

We begin with the following weak representation formula for the fractional α-variation of functions in $BV_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Here and in the following, we denote by $f^\star$ the *precise representative* of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, see (3.1) for the definition.

Proposition 3.17. *Let $\alpha \in (0,1)$ and $f \in BV_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then $\nabla^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} \varphi\cdot\nabla^\alpha f\,dx = \lim_{R\to+\infty}\int_{\mathbb{R}^n}\varphi\cdot I_{1-\alpha}(\chi^\star_{B_R}Df)\,dx \tag{3.28}$$

for all $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. By Proposition 3.14(iii), we know that $\nabla^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ for all $\alpha \in (0,1)$. By Theorem 3.1, we also know that $f\chi_{B_R} \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $D(\chi_{B_R}f) = \chi^\star_{B_R}Df + f^\star D\chi_{B_R}$ for all $R>0$. Now fix $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ and take $R>0$ such that $\operatorname{supp}\varphi \subset B_{R/2}$. By Theorem 1.27, we have that

$$\int_{\mathbb{R}^n}\chi_{B_R}f\,\mathrm{div}^\alpha\varphi\,dx = -\int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha(\chi_{B_R}f)\,dx = -\int_{\mathbb{R}^n}\varphi\cdot I_{1-\alpha}D(\chi_{B_R}f)\,dx.$$

Moreover, we can split the last integral as

$$\int_{\mathbb{R}^n}\varphi\cdot I_{1-\alpha}D(\chi_{B_R}f)\,dx = \int_{\mathbb{R}^n}\varphi\cdot I_{1-\alpha}(\chi^\star_{B_R}Df)\,dx + \int_{\mathbb{R}^n}\varphi\cdot I_{1-\alpha}(f^\star D\chi_{B_R})\,dx. \tag{3.29}$$

For all $x \in B_{R/2}$, we can estimate

$$\begin{aligned}
|I_{1-\alpha}(f^\star D\chi_{B_R})(x)| &= \left|\int_{\partial B_R}\frac{f^\star(y)}{|x-y|^{n+\alpha-1}}\,\frac{y}{|y|}\,d\mathscr{H}^{n-1}(y)\right| \\
&= \frac{1}{R^\alpha}\left|\int_{\partial B_1}\frac{f^\star(Ry)}{\left|y-\frac{x}{R}\right|^{n+\alpha-1}}\,\frac{y}{|y|}\,d\mathscr{H}^{n-1}(y)\right| \\
&\le \frac{n\omega_n}{R^\alpha\left(1-\frac{|x|}{R}\right)^{n+\alpha-1}}\,\|f\|_{L^\infty(\mathbb{R}^n)} \\
&\le \frac{2^{n+\alpha-1}n\omega_n}{R^\alpha}\,\|f\|_{L^\infty(\mathbb{R}^n)}
\end{aligned}$$

and so, since $\operatorname{supp}\varphi \subset B_{R/2}$, we get that

$$\left|\int_{\mathbb{R}^n}\varphi\cdot I_{1-\alpha}(f^\star D\chi_{B_R})\,dx\right| \le \frac{2^{n+\alpha-1}n\omega_n}{R^\alpha}\,\|\varphi\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}\,\|f\|_{L^\infty(\mathbb{R}^n)}. \tag{3.30}$$

Therefore, by (1.11), Lebesgue's Dominated Convergence Theorem, (3.29) and (3.30), we get that

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = \lim_{R \to +\infty} \int_{\mathbb{R}^n} \chi_{B_R} f \operatorname{div}^\alpha \varphi \, dx = \lim_{R \to +\infty} \int_{\mathbb{R}^n} \varphi \cdot I_{1-\alpha}(\chi^\star_{B_R} Df) \, dx$$

and the conclusion follows. □

In the following result, we show that for all functions in $bv(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ one can actually pass to the limit as $R \to +\infty$ inside the integral in the right-hand side of (3.28).

Corollary 3.18. *If either $f \in BV(\mathbb{R}^n)$ or $f \in bv(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then*

$$\nabla^\alpha f = I_{1-\alpha} Df \quad \text{a.e. in } \mathbb{R}^n. \tag{3.31}$$

Proof. If $f \in BV(\mathbb{R}^n)$, then (3.31) coincides with (3.15) and there is nothing to prove. So let us assume that $f \in bv(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Writing $Df = \nu_f |Df|$ with $\nu_f \in \mathbb{S}^{n-1}$ $|Df|$-a.e. in $\mathbb{R}^n$, for all $x \in \mathbb{R}^n$ we have

$$\lim_{R \to +\infty} \chi^\star_{B_R}(y) \, \frac{\nu_f(y)}{|y-x|^{n+\alpha-1}} = \frac{\nu_f(y)}{|y-x|^{n+\alpha-1}} \quad \text{for } |Df|\text{-a.e. } y \neq x.$$

Moreover, for a.e. $x \in \mathbb{R}^n$, we have

$$\left| \chi^\star_{B_R}(y) \, \frac{\nu_f(y)}{|y-x|^{n+\alpha-1}} \right| \le \frac{1}{|y-x|^{n+\alpha-1}} \in L^1_y(\mathbb{R}^n, |Df|) \quad \forall R > 0,$$

because $I_{1-\alpha}|Df| \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by Lemma 1.4. Therefore, by Lebesgue's Dominated Convergence Theorem (applied with respect to the finite measure $|Df|$), we get that

$$\lim_{R \to +\infty} I_{1-\alpha}(\chi^*_{B_R} Df)(x) = (I_{1-\alpha} Df)(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Now let $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$. Since

$$|\varphi \cdot I_{1-\alpha}(\chi^\star_{B_R} Df)| \le |\varphi| \, I_{1-\alpha}|Df| \in L^1(\mathbb{R}^n) \quad \forall R > 0,$$

again by Lebesgue's Dominated Convergence Theorem we get that

$$\lim_{R \to +\infty} \int_{\mathbb{R}^n} \varphi \cdot I_{1-\alpha}(\chi^*_{B_R} Df) \, dx = \int_{\mathbb{R}^n} \varphi \cdot I_{1-\alpha} Df \, dx. \tag{3.32}$$

The conclusion thus follows combining (3.28) with (3.32). □

3.3. The inclusion $BV^\alpha \subset W^{\beta,1}$ for $\beta < \alpha$: a representation formula. In Theorem 1.30, we proved that the inclusion $BV^\alpha \subset W^{\beta,1}$ is continuous for $\beta < \alpha$. In the following result we prove a useful representation formula for the fractional β-gradient of any $f \in BV^\alpha(\mathbb{R}^n)$, extending the formula obtained in Corollary 3.18.

Proposition 3.19. *Let $\alpha \in (0,1)$. If $f \in BV^\alpha(\mathbb{R}^n)$, then $f \in W^{\beta,1}(\mathbb{R}^n)$ for all $\beta \in (0,\alpha)$ with*

$$\nabla^\beta f = I_{\alpha-\beta} D^\alpha f \quad \text{a.e. in } \mathbb{R}^n. \tag{3.33}$$

In addition, for any bounded open set $U \subset \mathbb{R}^n$, we have

$$\|\nabla^\beta f\|_{L^1(U;\mathbb{R}^n)} \le C_{n,(1-\alpha+\beta),U} \, |D^\alpha f|(\mathbb{R}^n) \tag{3.34}$$

for all $\beta \in (0, \alpha)$*, where* $C_{n,\alpha,U}$ *is as in* (1.13)*. Finally, given an open set* $A \subset \mathbb{R}^n$*, we have*

(3.35)
$$\|\nabla^\beta f\|_{L^1(A;\mathbb{R}^n)} \le \frac{\mu_{n,1+\beta-\alpha}}{n+\beta-\alpha}\left(\frac{\omega_{n,1}|D^\alpha f|(\overline{A_r})}{\alpha-\beta}\, r^{\alpha-\beta} + \frac{\omega_{n,\alpha}(n+2\beta-\alpha)}{\beta}\,\|f\|_{L^1(\mathbb{R}^n)}\, r^{-\beta}\right)$$

for all $r > 0$ *and all* $\beta \in (0, \alpha)$*, where* $\omega_{n,\alpha} = \|\nabla^\alpha \chi_{B_1}\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$*,* $\omega_{n,1} = |D\chi_{B_1}|(\mathbb{R}^n) = n\omega_n$*, and, as above,* $A_r = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) < r\}$*. In particular, we have*

$$\|\nabla^\beta f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \frac{\alpha\mu_{n,1+\beta-\alpha}\,\omega_{n,1}^{\frac{\beta}{\alpha}}\,\omega_{n,\alpha}^{1-\frac{\beta}{\alpha}}\,(n+2\beta-\alpha)^{1-\frac{\beta}{\alpha}}}{\beta(n+\beta-\alpha)(\alpha-\beta)}\,\|f\|_{L^1(\mathbb{R}^n)}^{1-\frac{\beta}{\alpha}}\,|D^\alpha f|(\mathbb{R}^n)^{\frac{\beta}{\alpha}}. \tag{3.36}$$

Proof. Fix $\beta \in (0, \alpha)$. By Theorem 1.30, we already know that $f \in W^{\beta,1}(\mathbb{R}^n)$, with $D^\beta f = \nabla^\beta f \mathscr{L}^n$ according to Theorem 1.27. We thus just need to prove (3.33), (3.34) and (3.35).

We prove (3.33). Let $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Note that $I_{\alpha-\beta}\varphi \in \operatorname{Lip}_b(\mathbb{R}^n;\mathbb{R}^n)$ is such that $\operatorname{div} I_{\alpha-\beta}\varphi = I_{\alpha-\beta}\operatorname{div}\varphi$, so that

$$I_{1-\alpha}\operatorname{div} I_{\alpha-\beta}\varphi = I_{1-\alpha}I_{\alpha-\beta}\operatorname{div}\varphi = I_{1-\beta}\operatorname{div}\varphi = \operatorname{div}^\beta\varphi$$

by the semigroup property (N.58) of the Riesz potential. Moreover, in a similar way, we have

$$I_{1-\alpha}|\operatorname{div} I_{\alpha-\beta}\varphi| = I_{1-\alpha}|I_{\alpha-\beta}\operatorname{div}\varphi| \le I_{1-\alpha}I_{\alpha-\beta}|\operatorname{div}\varphi| = I_{1-\beta}|\operatorname{div}\varphi| \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

By Lemma 3.7, we thus have that $\operatorname{div}^\alpha I_{\alpha-\beta}\varphi = \operatorname{div}^\beta\varphi$. Consequently, by Proposition 3.11, we get

$$\int_{\mathbb{R}^n} f\,\operatorname{div}^\beta\varphi\,dx = \int_{\mathbb{R}^n} f\,\operatorname{div}^\alpha I_{\alpha-\beta}\varphi\,dx = -\int_{\mathbb{R}^n} I_{\alpha-\beta}\varphi\cdot dD^\alpha f.$$

Since $|D^\alpha f|(\mathbb{R}^n) < +\infty$, we have $I_{\alpha-\beta}|D^\alpha f| \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and thus, by Fubini's Theorem, we get that

$$\int_{\mathbb{R}^n} I_{\alpha-\beta}\varphi\cdot dD^\alpha f = \int_{\mathbb{R}^n} \varphi\cdot I_{\alpha-\beta}D^\alpha f\,dx.$$

We conclude that

$$\int_{\mathbb{R}^n} f\,\operatorname{div}^\beta\varphi\,dx = -\int_{\mathbb{R}^n} \varphi\cdot I_{\alpha-\beta}D^\alpha f\,dx$$

for any $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, proving (3.33).

We prove (3.34). By (3.33), by Tonelli's Theorem and by Lemma 1.4, we get

$$\int_U |\nabla^\beta f|\,dx \le \int_U I_{\alpha-\beta}|D^\alpha f|\,dx \le C_{n,(1-\alpha+\beta),U}|D^\alpha f|(\mathbb{R}^n)$$

where $C_{n,\alpha,U}$ is as in (1.13).

We now prove (3.35) in two steps. We argue similarly as in the proof of (3.17).

Proof of (3.35)*, Step 1.* Assume $f \in C_c^\infty(\mathbb{R}^n)$ and fix $r > 0$. We have

$$\begin{aligned}\int_A |\nabla^\beta f|dx &= \int_A |I_{\alpha-\beta}\nabla^\alpha f|\,dx\\ &\le \frac{\mu_{n,1+\beta-\alpha}}{n+\beta-\alpha}\left(\int_A\int_{\{|h|<r\}} \frac{|\nabla^\alpha f(x+h)|}{|h|^{n+\beta-\alpha}}dh dx + \int_A\left|\int_{\{|h|\ge r\}} \frac{\nabla^\alpha f(x+h)}{|h|^{n+\beta-\alpha}}\,dh\right|dx\right).\end{aligned}$$

We estimate the two double integrals appearing in the right-hand side separately. By Tonelli's Theorem, we have

$$\begin{aligned}\int_A \int_{\{|h|<r\}} \frac{|\nabla^\alpha f(x+h)|}{|h|^{n+\beta-\alpha}}\, dh\, dx &= \int_{\{|h|<r\}} \int_A |\nabla^\alpha f(x+h)|\, dx\, \frac{dh}{|h|^{n+\beta-\alpha}} \\ &\le |D^\alpha f|(A_r) \int_{\{|h|<r\}} \frac{dh}{|h|^{n+\beta-\alpha}} \\ &= \frac{n\omega_n\, |D^\alpha f|(A_r)}{\alpha-\beta}\, r^{\alpha-\beta}.\end{aligned}$$

Concerning the second double integral, we apply [**3**, Lemma 3.1.1(c)] to each component of the measure $D^\alpha f \in \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$ and get

$$\int_{\{|h|\ge r\}} \frac{\nabla^\alpha f(x+h)}{|h|^{n+\beta-\alpha}}\, dh = (n+\beta-\alpha) \int_r^{+\infty} \frac{D^\alpha f(B_\varrho(x))}{\varrho^{n+\beta-\alpha+1}}\, d\varrho - \frac{D^\alpha f(B_r(x))}{r^{n+\beta-\alpha}}$$

for all $x \in A$. Since

$$\begin{aligned}D^\alpha f(B_\varrho(x)) &= \int_{\mathbb{R}^n} \chi_{B_\varrho}(y)\, \nabla^\alpha f(x+y)\, dy \\ &= -\int_{\mathbb{R}^n} f(x+y)\, \nabla^\alpha \chi_{B_\varrho}(y)\, dy \\ &= -\varrho^{n-\alpha} \int_{\mathbb{R}^n} f(x+\varrho y)\, \nabla^\alpha \chi_{B_1}(y)\, dy,\end{aligned}$$

we can compute

$$\begin{aligned}&(n+\beta-\alpha) \int_r^{+\infty} \frac{D^\alpha f(B_\varrho(x))}{\varrho^{n+\beta-\alpha+1}}\, d\varrho - \frac{D^\alpha f(B_r(x))}{r^{n+\beta-\alpha}} \\ &\quad = -(n+\beta-\alpha) \int_r^{+\infty} \frac{1}{\varrho^{\beta+1}} \int_{\mathbb{R}^n} f(x+\varrho y)\, \nabla^\alpha \chi_{B_1}(y)\, dy\, d\varrho \\ &\qquad + \frac{1}{r^\beta} \int_{\mathbb{R}^n} f(x+ry)\, \nabla^\alpha \chi_{B_1}(y)\, dy \\ &\quad = \int_{\mathbb{R}^n} \left(\frac{f(x+ry)}{r^\beta} - (n+\beta-\alpha) \int_r^{+\infty} \frac{f(x+\varrho y)}{\varrho^{\beta+1}}\, d\varrho \right) \nabla^\alpha \chi_{B_1}(y)\, dy\end{aligned}$$

for all $x \in A$. Hence, we have

$$\begin{aligned}\int_A \left| \int_{\{|h|>r\}} \frac{\nabla^\alpha f(x+h)}{|h|^{n+\beta-\alpha}}\, dh \right| dx &\le \int_{\mathbb{R}^n} \left| \int_{\{|h|>r\}} \frac{\nabla^\alpha f(x+h)}{|h|^{n+\beta-\alpha}}\, dh \right| dx \\ &\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x+ry)|}{r^\beta}\, |\nabla^\alpha \chi_{B_1}(y)|\, dx\, dy \\ &\quad + (n+\beta-\alpha) \int_{\mathbb{R}^n} \int_r^{+\infty} \int_{\mathbb{R}^n} \frac{|f(x+\varrho y)|}{\varrho^{\beta+1}}\, |\nabla^\alpha \chi_{B_1}(y)|\, dx\, d\varrho\, dy \\ &= \frac{\omega_{n,\alpha}(n+2\beta-\alpha)}{\beta}\, \|f\|_{L^1(\mathbb{R}^n)}\, r^{-\beta}.\end{aligned}$$

Thus (3.17) follows for all $f \in C_c^\infty(\mathbb{R}^n)$ and $r > 0$.

Proof of (3.17), *Step 2.* Let $f \in BV^\alpha(\mathbb{R}^n)$ and fix $r > 0$. By Theorem 1.16, we find $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$ and $|D^\alpha f_k|(\mathbb{R}^n) \to |D^\alpha f|(\mathbb{R}^n)$ as

$k \to +\infty$. By Step 1, we have that

$$\|\nabla^\beta f_k\|_{L^1(A;\mathbb{R}^n)} \le \frac{\mu_{n,1+\beta-\alpha}}{n+\beta-\alpha}\left(\frac{n\omega_n|D^\alpha f_k|(\overline{A_r})}{\alpha-\beta}\, r^{\alpha-\beta} + \frac{\omega_{n,\alpha}(n+2\beta-\alpha)}{\beta}\,\|f_k\|_{L^1(\mathbb{R}^n)}\, r^{-\beta}\right) \tag{3.37}$$

for all $k \in \mathbb{N}$. We have that

$$(\nabla^\beta f_k)\,\mathscr{L}^n \rightharpoonup (\nabla^\beta f)\,\mathscr{L}^n \quad \text{as } k \to +\infty. \tag{3.38}$$

This can be proved arguing similarly as in the proof of (3.22) using (3.34). We leave the details to the reader. Thanks to (3.38), by [**58**, Proposition 4.29] we get that

$$\|\nabla^\beta f\|_{L^1(A;\mathbb{R}^n)} \le \liminf_{k\to+\infty} \|\nabla^\beta f_k\|_{L^1(A;\mathbb{R}^n)}.$$

Since

$$|D^\alpha f|(U) \le \liminf_{k\to+\infty} |D^\alpha f_k|(U)$$

for any open set $U \subset \mathbb{R}^n$ by Proposition 1.11, we can estimate

$$\begin{aligned}\limsup_{k\to+\infty} |D^\alpha f_k|(\overline{A_r}) &\le \lim_{k\to+\infty} |D^\alpha f_k|(\mathbb{R}^n) - \liminf_{k\to+\infty} |D^\alpha f_k|(\mathbb{R}^n \setminus A_r)\\ &\le |D^\alpha f|(\mathbb{R}^n) - |D^\alpha f|(\mathbb{R}^n \setminus A_r)\\ &= |D^\alpha f|(\overline{A_r}).\end{aligned}$$

Thus, (3.35) follows taking limits as $k \to +\infty$ in (3.37). Finally, (3.36) follows by considering $A = \mathbb{R}^n$ in (3.35) and optimising the right-hand side in $r > 0$. □

3.4. The inclusion $S^{\beta,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$ for $0 < \beta < \alpha < 1$. We conclude with the following result which can be derived from the theory of Bessel potential spaces. However, we state it here since our distributional approach provides explicit constants (independent of p) in the estimates that may be of some interest. The proof is very similar to the one of Proposition 3.15 and we leave it to the interested reader.

Proposition 3.20 ($S^{\beta,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$ for $0 < \beta < \alpha < 1$)**.** *Let $0 < \beta < \alpha < 1$ and $p \in (1, +\infty)$. If $f \in S^{\alpha,p}(\mathbb{R}^n)$, then $f \in S^{\beta,p}(\mathbb{R}^n)$ with*

$$\|\nabla^\beta f\|_{L^p(A;\mathbb{R}^n)} \le \frac{n\omega_n\mu_{n,1+\beta-\alpha}}{n+\beta-\alpha}\left(\frac{r^{\alpha-\beta}}{\alpha-\beta}\,\|\nabla^\alpha f\|_{L^p(\overline{A_r};\mathbb{R}^n)} + c_{n,\alpha}\,\frac{r^{-\beta}}{\beta}\,\|f\|_{L^p(\mathbb{R}^n)}\right) \tag{3.39}$$

for any $r > 0$ and any open set $A \subset \mathbb{R}^n$, where $A_r = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) < r\}$ and $c_{n,\alpha} > 0$ is a constant depending only on n and α. In particular, we have

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le c_{n,\alpha}\,\frac{\mu_{n,1+\beta-\alpha}}{\beta(\alpha-\beta)(n+\beta-\alpha)}\,\|\nabla^\alpha f\|^{\beta/\alpha}_{L^p(\mathbb{R}^n;\mathbb{R}^n)}\|f\|^{(\beta-\alpha)/\alpha}_{L^p(\mathbb{R}^n)}, \tag{3.40}$$

where $c_{n,\alpha} > 0$ is a constant depending only on n and α. In addition, if $p \in \left(1, \frac{n}{\alpha-\beta}\right)$ and $q = \frac{np}{n-(\alpha-\beta)p}$, then

$$\nabla^\beta f = I_{\alpha-\beta}\nabla^\alpha f \quad \text{a.e. in } \mathbb{R}^n \tag{3.41}$$

and $\nabla^\beta f \in L^q(\mathbb{R}^n;\mathbb{R}^n)$.

4. Asymptotic behaviour of fractional α-variation as $\alpha \to 1^-$

4.1. Convergence of ∇^α and $\operatorname{div}^\alpha$ as $\alpha \to 1^-$. We begin with the following simple result about the asymptotic behaviour of the constant $\mu_{n,\alpha}$ as $\alpha \to 1^-$.

Lemma 3.21. *Let $n \in \mathbb{N}$. We have*

$$\frac{\mu_{n,\alpha}}{1-\alpha} \le \pi^{-\frac{n}{2}} \sqrt{\frac{3}{2}} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)} =: C_n \qquad \forall \alpha \in (0,1) \tag{3.42}$$

and

$$\lim_{\alpha \to 1^-} \frac{\mu_{n,\alpha}}{1-\alpha} = \omega_n^{-1}. \tag{3.43}$$

Proof. Since $\Gamma(1) = 1$ and $\Gamma(1+x) = x\,\Gamma(x)$ for $x > 0$ (see [**13**]), we have $\Gamma(x) \sim x^{-1}$ as $x \to 0^+$. Thus as $\alpha \to 1^-$ we find

$$\mu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} \sim \pi^{-\frac{n}{2}} (1-\alpha)\, \Gamma\left(\frac{n}{2}+1\right) = \omega_n^{-1}(1-\alpha)$$

and (3.43) follows.

Since Γ is log-convex on $(0,+\infty)$ (see [**13**]), for all $x > 0$ and $a \in (0,1)$ we have

$$\Gamma(x+a) = \Gamma((1-a)x + a(x+1)) \le \Gamma(x)^{1-a}\, \Gamma(x+1)^a = x^a\, \Gamma(x).$$

For $x = \frac{n}{2}$ and $a = \frac{\alpha+1}{2}$, we can estimate

$$\Gamma\left(\frac{n+\alpha+1}{2}\right) \le \left(\frac{n}{2}\right)^{\frac{\alpha+1}{2}} \Gamma\left(\frac{n}{2}\right) \le \Gamma\left(\frac{n}{2}+1\right)$$

for all $n \ge 2$. Also, for $n = 1$, we trivially have $\Gamma\left(\frac{2+\alpha}{2}\right) \le \Gamma\left(\frac{3}{2}\right)$, because Γ is increasing on $(1,+\infty)$ (see [**13**]). For $x = 1 + \frac{1-\alpha}{2}$ and $a = \frac{\alpha}{2}$, we can estimate

$$\Gamma\left(\frac{3}{2}\right) \le \left(1 + \frac{1-\alpha}{2}\right)^{\frac{\alpha}{2}} \Gamma\left(1 + \frac{1-\alpha}{2}\right) \le \sqrt{\frac{3}{2}}\, \frac{1-\alpha}{2}\, \Gamma\left(\frac{1-\alpha}{2}\right).$$

We thus get

$$\mu_{n,\alpha}(1-\alpha)^{-1} = 2^{\alpha-1}\, \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}+1\right)} \le \pi^{-\frac{n}{2}} \sqrt{\frac{3}{2}}\, \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)}$$

and (3.42) follows. □

In the following technical result, we show that the constant $C_{n,\alpha,U}$ defined in (1.13) is uniformly bounded as $\alpha \to 1^-$ in terms of the volume and the diameter of the bounded open set $U \subset \mathbb{R}^n$.

Lemma 3.22 (Uniform upper bound on $C_{n,\alpha,U}$ as $\alpha \to 1^-$)**.** *Let $n \in \mathbb{N}$ and $\alpha \in (\frac{1}{2}, 1)$. Let $U \subset \mathbb{R}^n$ be bounded open set. If $C_{n,\alpha,U}$ is as in* (1.13)*, then*

$$C_{n,\alpha,U} \le \frac{n\omega_n C_n}{\left(n-\frac{1}{2}\right)} \left(\frac{n}{\left(n-\frac{1}{2}\right)} \max\left\{1, \frac{|U|}{\omega_n}\right\}^{\frac{1}{n}} + \max\left\{1, \sqrt{\operatorname{diam}(U)}\right\} \right) =: \kappa_{n,U}, \tag{3.44}$$

where C_n is as in (3.42)*.*

Proof. By (3.42), for all $\alpha \in (\frac{1}{2}, 1)$ we have

$$\frac{n\,\mu_{n,\alpha}}{(n+\alpha-1)(1-\alpha)} \le \frac{n\,C_n}{n+\alpha-1} \le \frac{n\,C_n}{n-\frac{1}{2}}.$$

Since $t^{1-\alpha} \le \max\left\{1, \sqrt{t}\right\}$ for any $t \ge 0$ and $\alpha \in (\frac{1}{2}, 1)$, we have

$$\omega_n(\mathrm{diam}(U))^{1-\alpha} \le \omega_n \max\left\{1, \sqrt{\mathrm{diam}(U)}\right\}$$

and

$$\begin{aligned}\left(\frac{n\omega_n}{n+\alpha-1}\right)^{\frac{n+\alpha-1}{n}} |U|^{\frac{1-\alpha}{n}} &= \frac{n\omega_n}{n+\alpha-1}\left(\frac{|U|(n+\alpha-1)}{n\omega_n}\right)^{\frac{1-\alpha}{n}} \\ &\le \frac{n\omega_n}{\left(n-\frac{1}{2}\right)} \max\left\{1, \frac{|U|}{\omega_n}\right\}^{\frac{1}{n}}.\end{aligned}$$

Combining these inequalities, we get the conclusion. □

As consequence of Corollary 1.3 and Lemma 3.22, we prove that ∇^α and div^α converge pointwise to ∇ and div respectively as $\alpha \to 1^-$.

Proposition 3.23. *If $f \in C_c^1(\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$ we have*

$$\lim_{\alpha\to 0^-} I_\alpha f(x) = f(x). \tag{3.45}$$

As a consequence, if $f \in C_c^2(\mathbb{R}^n)$ and $\varphi \in C_c^2(\mathbb{R}^n;\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$ we have

$$\lim_{\alpha\to 1^-} \nabla^\alpha f(x) = \nabla f(x), \qquad \lim_{\alpha\to 1^-} \mathrm{div}^\alpha \varphi(x) = \mathrm{div}\varphi(x). \tag{3.46}$$

Proof. Let $f \in C_c^1(\mathbb{R}^n)$ and fix $x \in \mathbb{R}^n$. Writing (1.14) in spherical coordinates, we find

$$I_\alpha f(x) = \frac{\mu_{n,1-\alpha}}{n-\alpha} \lim_{\delta\to 0} \int_{\partial B_1} \int_\delta^{+\infty} \varrho^{-1+\alpha} f(x+\varrho v)\, d\varrho\, d\mathscr{H}^{n-1}(v).$$

Since $f \in C_c^1(\mathbb{R}^n)$, for each fixed $v \in \partial B_1$ we can integrate by parts in the variable ϱ and get

$$\begin{aligned}\int_\delta^{+\infty} \varrho^{-1+\alpha} f(x+\varrho v)\, d\varrho &= \left[\frac{\varrho^\alpha}{\alpha} f(x+\varrho v)\right]_{\varrho=\delta}^{\varrho\to+\infty} - \frac{1}{\alpha}\int_\delta^{+\infty} \varrho^\alpha\, \partial_\varrho(f(x+\varrho v))\, d\varrho \\ &= -\frac{\delta^\alpha}{\alpha} f(x+\delta v) - \frac{1}{\alpha}\int_\delta^{+\infty} \varrho^\alpha\, \partial_\varrho(f(x+\varrho v))\, d\varrho.\end{aligned}$$

Clearly, we have

$$\lim_{\delta\to 0^+} \delta^\alpha \int_{\partial B_1} f(x+\delta v)\, d\mathscr{H}^{n-1}(v) = 0.$$

Thus, by Fubini's Theorem, we conclude that

$$I_\alpha f(x) = -\frac{\mu_{n,1-\alpha}}{\alpha(n-\alpha)} \int_0^\infty \int_{\partial B_1} \varrho^\alpha\, \partial_\varrho(f(x+\varrho v))\, d\mathscr{H}^{n-1}(v)\, d\varrho. \tag{3.47}$$

Since f has compact support and recalling (3.43), we can pass to the limit in (3.47) and get

$$\lim_{\alpha\to 0^+} I_\alpha f(x) = -\frac{1}{n\omega_n} \int_{\partial B_1} \int_0^\infty \partial_\varrho(f(x+\varrho v))\, d\varrho\, d\mathscr{H}^{n-1}(v) = f(x),$$

proving (3.45). The pointwise limits in (3.46) immediately follows by Corollary 1.3. □

In the following crucial result, we improve the pointwise convergence obtained in Proposition 3.23 to strong convergence in $L^p(\mathbb{R}^n)$ for all $p \in [1, +\infty]$.

Proposition 3.24. *Let $p \in [1, +\infty]$. If $f \in C_c^2(\mathbb{R}^n)$ and $\varphi \in C_c^2(\mathbb{R}^n; \mathbb{R}^n)$, then*

$$\lim_{\alpha \to 1^-} \|\nabla^\alpha f - \nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0, \qquad \lim_{\alpha \to 1^-} \|\mathrm{div}^\alpha \varphi - \mathrm{div}\varphi\|_{L^p(\mathbb{R}^n)} = 0.$$

Proof. Let $f \in C_c^2(\mathbb{R}^n)$. Since

$$\int_{B_1} \frac{dy}{|y|^{n+\alpha-1}} = n\omega_n \int_0^1 \frac{d\varrho}{\varrho^\alpha} = \frac{n\omega_n}{1-\alpha},$$

for all $x \in \mathbb{R}^n$ we can write

$$\frac{n\omega_n \mu_{n,\alpha}}{(1-\alpha)(n+\alpha-1)} \nabla f(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{B_1} \frac{\nabla f(x)}{|y|^{n+\alpha-1}}\, dy.$$

Therefore, by (1.14), we have

$$\begin{aligned} \nabla^\alpha f(x) &- \frac{n\omega_n \mu_{n,\alpha}}{(1-\alpha)(n+\alpha-1)} \nabla f(x) \\ &= \frac{\mu_{n,\alpha}}{n+\alpha-1} \left(\int_{B_1} \frac{\nabla f(x+y) - \nabla f(x)}{|y|^{n+\alpha-1}}\, dy + \int_{\mathbb{R}^n \setminus B_1} \frac{\nabla f(x+y)}{|y|^{n+\alpha-1}}\, dy \right) \end{aligned}$$

for all $x \in \mathbb{R}^n$. We now distinguish two cases.

Case 1: $p \in [1, +\infty)$. Using the elementary inequality $|v+w|^p \le 2^{p-1}(|v|^p + |w|^p)$ valid for all $v, w \in \mathbb{R}^n$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \bigg| \nabla^\alpha f(x) - &\frac{n\omega_n \mu_{n,\alpha}}{(1-\alpha)(n+\alpha-1)} \nabla f(x) \bigg|^p dx \\ &\le \frac{2^{p-1}\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \left| \int_{B_1} \frac{\nabla f(x+y) - \nabla f(x)}{|y|^{n+\alpha-1}}\, dy \right|^p dx \\ &\quad + \frac{2^{p-1}\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n \setminus B_1} \frac{\nabla f(x+y)}{|y|^{n+\alpha-1}}\, dy \right|^p dx. \end{aligned}$$

We now estimate the two double integrals appearing in the right-hand side separately.

For the first double integral, similarly as in the proof of Proposition 3.23, we pass in spherical coordinates to get

(3.48)
$$\begin{aligned} \int_{B_1} \frac{\nabla f(x+y) - \nabla f(x)}{|y|^{n+\alpha-1}}\, dy &= \int_{\partial B_1} \int_0^1 \varrho^{-\alpha} \left(\nabla f(x+\varrho v) - \nabla f(x) \right) d\varrho \, d\mathscr{H}^{n-1}(v) \\ &= \frac{1}{1-\alpha} \int_{\partial B_1} \left(\nabla f(x+v) - \nabla f(x) \right) d\mathscr{H}^{n-1}(v) \\ &\quad - \int_{\partial B_1} \int_0^1 \frac{\varrho^{1-\alpha}}{1-\alpha}\, \partial_\varrho (\nabla f(x+\varrho v))\, d\varrho \, d\mathscr{H}^{n-1}(v) \end{aligned}$$

for all $x \in \mathbb{R}^n$. Hence, by (3.43), we find

$$\lim_{\alpha \to 1^-} \frac{\mu_{n,\alpha}}{(1-\alpha)(n+\alpha-1)} \int_{\partial B_1} \left(\nabla f(x+v) - \nabla f(x) \right) d\mathscr{H}^{n-1}(v)$$

$$= \frac{1}{n\omega_n} \int_{\partial B_1} (\nabla f(x+v) - \nabla f(x)) \, d\mathscr{H}^{n-1}(v)$$

and

$$\begin{aligned}\lim_{\alpha\to 1^-} \frac{\mu_{n,\alpha}}{(1-\alpha)(n+\alpha-1)} \int_{\partial B_1} \int_0^1 \varrho^{1-\alpha}\, \partial_\varrho(\nabla f(x+\varrho v))\, d\varrho \, d\mathscr{H}^{n-1}(v) \\ = \frac{1}{n\omega_n} \int_{\partial B_1} \int_0^1 \partial_\varrho(\nabla f(x+\varrho v))\, d\varrho \, d\mathscr{H}^{n-1}(v) \\ = \frac{1}{n\omega_n} \int_{\partial B_1} (\nabla f(x+v) - \nabla f(x)) \, d\mathscr{H}^{n-1}(v)\end{aligned}$$

for all $x \in \mathbb{R}^n$. Therefore, we get

$$\lim_{\alpha\to 1^-} \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{B_1} \frac{\nabla f(x+y) - \nabla f(x)}{|y|^{n+\alpha-1}} \, dy = 0$$

for all $x \in \mathbb{R}^n$. Recalling (3.42), we also observe that

$$\frac{\mu_{n,\alpha}}{n+\alpha-1} \frac{|\nabla f(x+y) - \nabla f(x)|}{|y|^{n+\alpha-1}} \le C_n \frac{|\nabla f(x+y) - \nabla f(x)|}{|y|^n}$$

for all $\alpha \in (0,1)$, $x \in \mathbb{R}^n$ and $y \in B_1$. Moreover, letting $R > 0$ be such that $\operatorname{supp} f \subset B_R$, we can estimate

$$\int_{B_1} \frac{|\nabla f(x+y) - \nabla f(x)|}{|y|^n} \, dy \le n\omega_n \|\nabla f\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)} \chi_{B_{R+1}}(x)$$

for all $x \in \mathbb{R}^n$, so that

$$x \mapsto \left(\int_{B_1} \frac{|\nabla f(x+y) - \nabla f(x)|}{|y|^n} \, dy \right)^p \in L^1(\mathbb{R}^n).$$

In conclusion, applying Lebesgue's Dominated Convergence Theorem, we find

$$\lim_{\alpha\to 1^-} \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \left| \int_{B_1} \frac{\nabla f(x+y) - \nabla f(x)}{|y|^{n+\alpha-1}} \, dy \right|^p dx = 0.$$

For the second double integral, note that

$$\int_{\mathbb{R}^n\setminus B_1} \frac{\nabla f(x+y)}{|y|^{n+\alpha-1}} \, dy = \int_{\mathbb{R}^n\setminus B_1} \frac{\nabla (f(x+y) - f(x))}{|y|^{n+\alpha-1}} \, dy$$

for all $x \in \mathbb{R}^n$. Now let $R > 0$. Integrating by parts, we have that

$$\begin{aligned}\int_{B_R\setminus B_1} \frac{\nabla (f(x+y) - f(x))}{|y|^{n+\alpha-1}} \, dy &= (n+\alpha-1) \int_{B_R\setminus B_1} \frac{y\,(f(x+y) - f(x))}{|y|^{n+\alpha+1}} \, dy \\ &\quad + \frac{1}{R^{n+\alpha-1}} \int_{\partial B_R} (f(x+y) - f(x))\, d\mathscr{H}^{n-1}(y) \\ &\quad - \int_{\partial B_1} (f(x+y) - f(x))\, d\mathscr{H}^{n-1}(y)\end{aligned}$$

for all $x \in \mathbb{R}^n$. Since

$$\int_{\mathbb{R}^n\setminus B_R} \frac{|f(x+y) - f(x)|}{|y|^{n+\alpha}} \, dy \le \frac{2n\omega_n}{\alpha R^\alpha} \|f\|_{L^\infty(\mathbb{R}^n)}$$

and

$$\frac{1}{R^{n+\alpha-1}} \int_{\partial B_R} |f(x+y) - f(x)| \, d\mathscr{H}^{n-1}(y) \le \frac{2n\omega_n}{R^\alpha} \|f\|_{L^\infty(\mathbb{R}^n)}$$

for all $R > 0$, we conclude that

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_1} \frac{\nabla f(x+y)}{|y|^{n+\alpha-1}} \, dy &= \lim_{R \to +\infty} \int_{B_R \setminus B_1} \frac{\nabla f(x+y)}{|y|^{n+\alpha-1}} \, dy \\ &= (n+\alpha-1) \int_{\mathbb{R}^n \setminus B_1} \frac{y\,(f(x+y) - f(x))}{|y|^{n+\alpha+1}} \, dy \\ &\quad - \int_{\partial B_1} (f(x+y) - f(x)) \, d\mathscr{H}^{n-1}(y) \end{aligned} \tag{3.49}$$

for all $x \in \mathbb{R}^n$. Hence, by Minkowski's Integral Inequality (see [**96**, Section A.1], for example), we can estimate

$$\begin{aligned} \left\| \int_{\mathbb{R}^n \setminus B_1} \frac{\nabla f(\cdot + y)}{|y|^{n+\alpha-1}} \, dy \right\|_{L^p(\mathbb{R}^n;\, \mathbb{R}^n)} &\le (n+\alpha-1) \left\| \int_{\mathbb{R}^n \setminus B_1} \frac{|f(\cdot + y) - f(\cdot)|}{|y|^{n+\alpha}} \, dy \right\|_{L^p(\mathbb{R}^n)} \\ &\quad + \left\| \int_{\partial B_1} |f(\cdot + y) - f(\cdot)| \, d\mathscr{H}^{n-1}(y) \right\|_{L^p(\mathbb{R}^n)} \\ &\le \frac{n + 2\alpha - 1}{\alpha} \, 2n\omega_n \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Thus, by (3.43), we get that

$$\lim_{\alpha \to 1^-} \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n \setminus B_1} \frac{\nabla f(x+y)}{|y|^{n+\alpha-1}} \, dy \right|^p dx = 0.$$

Case 2: $p = +\infty$. We have

$$\begin{aligned} &\sup_{x \in \mathbb{R}^n} \left| \nabla^\alpha f(x) - \frac{n\omega_n \mu_{n,\alpha}}{(1-\alpha)(n+\alpha-1)} \nabla f(x) \right| \\ &\qquad \le \frac{\mu_{n,\alpha}}{n+\alpha-1} \left(\sup_{x \in \mathbb{R}^n} \left| \int_{B_1} \frac{\nabla f(x+y) - \nabla f(x)}{|y|^{n+\alpha-1}} \, dy \right| + \sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n \setminus B_1} \frac{\nabla f(x+y)}{|y|^{n+\alpha-1}} \, dy \right| \right). \end{aligned}$$

Again we estimate the two integrals appearing in the right-hand side separately. We note that

$$\begin{aligned} \int_{\partial B_1} (\nabla f(x+v) - \nabla f(x)) \, d\mathscr{H}^{n-1}(v) - \int_{\partial B_1} \int_0^1 \varrho^{1-\alpha} \, \partial_\varrho (\nabla f(x+\varrho v)) \, d\varrho \, d\mathscr{H}^{n-1}(v) \\ = \int_{\partial B_1} \int_0^1 (1 - \varrho^{1-\alpha}) \, \partial_\varrho (\nabla f(x+\varrho v)) \, d\varrho \, d\mathscr{H}^{n-1}(v), \end{aligned}$$

so that we can rewrite (3.48) as

$$\int_{B_1} \frac{\nabla f(x+y) - \nabla f(x)}{|y|^{n+\alpha-1}} \, dy = \frac{1}{1-\alpha} \int_{\partial B_1} \int_0^1 (1 - \varrho^{1-\alpha}) \, \partial_\varrho (\nabla f(x+\varrho v)) \, d\varrho \, d\mathscr{H}^{n-1}(v).$$

Hence, we can estimate

$$\sup_{x \in \mathbb{R}^n} \left| \int_{B_1} \frac{\nabla f(x+y) - \nabla f(x)}{|y|^{n+\alpha-1}} \, dy \right|$$

$$\le \frac{1}{1-\alpha} \int_{\partial B_1} \int_0^1 (1 - \varrho^{1-\alpha}) \sup_{x \in \mathbb{R}^n} |\partial_\varrho(\nabla f(x + \varrho v))| \, d\varrho \, d\mathscr{H}^{n-1}(v)$$
$$\le \frac{1}{2-\alpha} \, n\omega_n \, \|\nabla^2 f\|_{L^\infty(\mathbb{R}^n;\, \mathbb{R}^{2n})},$$

so that

$$\lim_{\alpha \to 1^-} \frac{\mu_{n,\alpha}}{n + \alpha - 1} \sup_{x \in \mathbb{R}^n} \left| \int_{B_1} \frac{\nabla f(x+y) - \nabla f(x)}{|y|^{n+\alpha-1}} \, dy \right| = 0.$$

For the second integral, by (3.49) we can estimate

$$\sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n \setminus B_1} \frac{\nabla f(x+y)}{|y|^{n+\alpha-1}} \, dy \right| dx$$
$$\le (n + \alpha - 1) \sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n \setminus B_1} \frac{|f(x+y) - f(x)|}{|y|^{n+\alpha}} \, dy \right|$$
$$+ \sup_{x \in \mathbb{R}^n} \left| \int_{\partial B_1} |f(x+y) - f(x)| \, d\mathscr{H}^{n-1}(y) \right|$$
$$\le \frac{n + 2\alpha - 1}{\alpha} \, 2n\omega_n \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Thus, by (3.43), we get that

$$\lim_{\alpha \to 1^-} \frac{\mu_{n,\alpha}}{n + \alpha - 1} \sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n \setminus B_1} \frac{\nabla f(x+y)}{|y|^{n+\alpha-1}} \, dy \right| = 0.$$

We can now conclude the proof. Again recalling (3.43), we thus find that

$$\lim_{\alpha \to 1^-} \|\nabla^\alpha f - \nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}$$
$$\le \lim_{\alpha \to 1^-} \left\| \nabla^\alpha f - \frac{n\omega_n \mu_{n,\alpha}}{(1-\alpha)(n+\alpha-1)} \nabla f \right\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}$$
$$+ \|\nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \lim_{\alpha \to 1^-} \left(\frac{n\omega_n \mu_{n,\alpha}}{(1-\alpha)(n+\alpha-1)} - 1 \right) = 0$$

for all $p \in [1, +\infty]$ and the conclusion follows. The L^p-convergence of $\operatorname{div}^\alpha \varphi$ to $\operatorname{div} \varphi$ as $\alpha \to 1^-$ for all $p \in [1, +\infty]$ follows by a similar argument and is left to the reader. □

Remark 3.25. Note that the conclusion of Proposition 3.24 still holds if instead one assumes that $f \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n; \mathbb{R}^n)$. We leave the proof of this assertion to the reader.

4.2. Weak convergence of α-variation as $\alpha \to 1^-$. In Theorem 3.27 below, we prove that the fractional α-variation weakly converges to the standard variation as $\alpha \to 1^-$ for functions either in $BV(\mathbb{R}^n)$ or in $BV_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. In the proof of Theorem 3.27, we are going to use the following technical result.

Lemma 3.26. *There exists a dimensional constant $c_n > 0$ with the following property. If $f \in L^\infty(\mathbb{R}^n) \cap BV_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$\|\nabla^\alpha f\|_{L^1(B_R;\, \mathbb{R}^n)} \le c_n \left(R^{1-\alpha} |Df|(B_{3R}) + R^{n-\alpha} \, \|f\|_{L^\infty(\mathbb{R}^n)} \right) \tag{3.50}$$

for all $R > 0$ *and* $\alpha \in (\frac{1}{2}, 1)$.

Proof. Since $\Gamma(x) \sim x^{-1}$ as $x \to 0^+$ (see [**13**]), inequality (3.50) follows immediately combining (3.20) with Lemma 3.21. □

Theorem 3.27. *If either* $f \in BV(\mathbb{R}^n)$ *or* $f \in BV_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*, then*

$$D^\alpha f \rightharpoonup Df \quad \text{as } \alpha \to 1^-.$$

Proof. We divide the proof in two steps.

Step 1. Assume $f \in BV(\mathbb{R}^n)$. By Theorem 1.27, we have

$$\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx = -\int_{\mathbb{R}^n} f \, \mathrm{div}^\alpha \varphi \, dx$$

for all $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$. Thus, given $\varphi \in C_c^2(\mathbb{R}^n; \mathbb{R}^n)$, recalling Proposition 3.23 and the estimates (1.12) and (3.44), by Lebesgue's Dominated Convergence Theorem we get that

$$\lim_{\alpha \to 1^-} \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx = -\lim_{\alpha \to 1^-} \int_{\mathbb{R}^n} f \, \mathrm{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} f \, \mathrm{div} \varphi \, dx = \int_{\mathbb{R}^n} \varphi \cdot dDf.$$

Now fix $\varphi \in C_c^0(\mathbb{R}^n; \mathbb{R}^n)$. Let $U \subset \mathbb{R}^n$ be a fixed bounded open set such that $\mathrm{supp}\, \varphi \subset U$. For each $\varepsilon > 0$ sufficiently small, pick $\psi_\varepsilon \in C_c^2(\mathbb{R}^n; \mathbb{R}^n)$ such that $\|\varphi - \psi_\varepsilon\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} < \varepsilon$ and $\mathrm{supp}\, \psi_\varepsilon \subset U$. Then, by (3.16), we can estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \varphi \cdot dDf \right| &\le \|\varphi - \psi_\varepsilon\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \left(\int_U |\nabla^\alpha f| \, dx + |Df|(\mathbb{R}^n) \right) \\ &\quad + \left| \int_{\mathbb{R}^n} \psi_\varepsilon \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \psi_\varepsilon \cdot dDf \right| \\ &\le \varepsilon (1 + C_{n,\alpha,U}) \, |Df|(\mathbb{R}^n) \\ &\quad + \left| \int_{\mathbb{R}^n} \psi_\varepsilon \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \psi_\varepsilon \cdot dDf \right| \end{aligned}$$

for all $\alpha \in (0, 1)$. Thus, by the uniform estimate (3.44) in Lemma 3.22, we get

$$\lim_{\alpha \to 1^-} \left| \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \varphi \cdot dDf \right| \le \varepsilon (1 + \kappa_{n,U}) \, |Df|(\mathbb{R}^n) \tag{3.51}$$

and the conclusion follows passing to the limit as $\varepsilon \to 0^+$.

Step 2. Assume $f \in BV_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. By Proposition 3.14(iii), we know that $D^\alpha f = \nabla^\alpha f \mathscr{L}^n$ with $\nabla^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. By Proposition 3.24, we get that

$$\lim_{\alpha \to 1^-} \left| \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \varphi \cdot dDf \right| \le \|f\|_{L^\infty(\mathbb{R}^n)} \lim_{\alpha \to 1^-} \|\mathrm{div}^\alpha \varphi - \mathrm{div} \varphi\|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} = 0$$

for all $\varphi \in C_c^2(\mathbb{R}^n; \mathbb{R}^n)$. Now fix $\varphi \in C_c^0(\mathbb{R}^n; \mathbb{R}^n)$ and choose $R \ge 1$ such that $\mathrm{supp}\, \varphi \subset B_R$. For each $\varepsilon > 0$ sufficiently small, pick $\psi_\varepsilon \in C_c^2(\mathbb{R}^n; \mathbb{R}^n)$ such that $\|\varphi - \psi_\varepsilon\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} < \varepsilon$ and $\mathrm{supp}\, \psi_\varepsilon \subset B_R$. Then, by (3.50), we can estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \varphi \cdot dDf \right| &\le \|\varphi - \psi_\varepsilon\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \left(\|\nabla^\alpha f\|_{L^1(B_R; \mathbb{R}^n)} + |Df|(B_R) \right) \\ &\quad + \left| \int_{\mathbb{R}^n} \psi_\varepsilon \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \psi_\varepsilon \cdot dDf \right| \end{aligned}$$

$$\begin{aligned} &\le \varepsilon c_n R^n \left(\|f\|_{L^\infty(\mathbb{R}^n)} + |Df|(B_{3R}) \right) \\ &\quad + \left| \int_{\mathbb{R}^n} \psi_\varepsilon \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \psi_\varepsilon \cdot dDf \right| \end{aligned}$$

for all $\alpha \in (\frac{1}{2}, 1)$. We thus get

$$\lim_{\alpha \to 1^-} \left| \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \varphi \cdot dDf \right| \le \varepsilon c_n R^n \left(\|f\|_{L^\infty(\mathbb{R}^n)} + |Df|(B_{3R}) \right) \tag{3.52}$$

and the conclusion follows passing to the limit as $\varepsilon \to 0^+$. □

We are now going to improve the weak convergence of the fractional α-variation obtained in Theorem 3.27 by establishing the weak convergence also of the total fractional α-variation as $\alpha \to 1^-$, see Theorem 3.29 below. To do so, we need the following preliminary result.

Lemma 3.28. *Let $\mu \in \mathscr{M}(\mathbb{R}^n; \mathbb{R}^n)$. We have $(I_\alpha \mu)\mathscr{L}^n \rightharpoonup \mu$ as $\alpha \to 0^+$.*

Proof. Since Riesz potential is a linear operator and thanks to Hahn–Banach Decomposition Theorem, without loss of generality we can assume that μ is a nonnegative finite Radon measure.

Let now $\varphi \in C^1_c(\mathbb{R}^n)$ and let $U \subset \mathbb{R}^n$ be a bounded open set such that $\operatorname{supp}\varphi \subset U$. We have that $\|I_\alpha|\varphi|\|_{L^\infty(\mathbb{R}^n)} \le \kappa_{n,U} \|\varphi\|_{L^\infty(\mathbb{R}^n)}$ for all $\alpha \in (0, \frac{1}{2})$ by Lemma 1.4 and Lemma 3.22. Thus, by (3.45), Fubini's Theorem and Lebesgue's Dominated Convergence Theorem, we get that

$$\lim_{\alpha \to 0^+} \int_{\mathbb{R}^n} \varphi \, I_\alpha \mu \, dx = \lim_{\alpha \to 0^+} \int_{\mathbb{R}^n} I_\alpha \varphi \, d\mu = \int_{\mathbb{R}^n} \varphi \, d\mu.$$

Now fix $\varphi \in C^0_c(\mathbb{R}^n; \mathbb{R}^n)$. Let $U \subset \mathbb{R}^n$ be a fixed bounded open set such that $\operatorname{supp}\varphi \subset U$. For each $\varepsilon > 0$ sufficiently small, pick $\psi_\varepsilon \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ such that $\|\varphi - \psi_\varepsilon\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} < \varepsilon$ and $\operatorname{supp}\psi_\varepsilon \subset U$. Then, since $\mu(\mathbb{R}^n) < +\infty$, by Lemma 1.4 and by (3.44), we can estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \varphi \, I_\alpha \mu \, dx - \int_{\mathbb{R}^n} \varphi \, d\mu \right| &\le \left| \int_{\mathbb{R}^n} \psi_\varepsilon \, I_\alpha \mu \, dx - \int_{\mathbb{R}^n} \psi_\varepsilon \, d\mu \right| + \varepsilon \|I_\alpha \mu\|_{L^1(U)} + \varepsilon \mu(U) \\ &\le \left| \int_{\mathbb{R}^n} I_\alpha \psi_\varepsilon \, d\mu - \int_{\mathbb{R}^n} \psi_\varepsilon \, d\mu \right| + \varepsilon (1 + C_{n,\alpha,U}) \, \mu(\mathbb{R}^n) \\ &\le \left| \int_{\mathbb{R}^n} I_\alpha \psi_\varepsilon \, d\mu - \int_{\mathbb{R}^n} \psi_\varepsilon \, d\mu \right| + \varepsilon (1 + \kappa_{n,U}) \, \mu(\mathbb{R}^n) \end{aligned}$$

for all $\alpha \in (0, \frac{1}{2})$, so that

$$\limsup_{\alpha \to 0^+} \left| \int_{\mathbb{R}^n} \varphi \, I_\alpha \mu \, dx - \int_{\mathbb{R}^n} \varphi \, d\mu \right| \le \varepsilon (1 + \kappa_{n,U}) \, \mu(\mathbb{R}^n).$$

The conclusion thus follows passing to the limit as $\varepsilon \to 0^+$. □

Theorem 3.29. *If either $f \in BV(\mathbb{R}^n)$ or $f \in bv(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then*

$$|D^\alpha f| \rightharpoonup |Df| \quad \textit{as } \alpha \to 1^-. \tag{3.53}$$

Moreover, if $f \in BV(\mathbb{R}^n)$*, then also*

$$\lim_{\alpha\to 1^-} |D^\alpha f|(\mathbb{R}^n) = |Df|(\mathbb{R}^n). \tag{3.54}$$

Proof. We prove (3.53) and (3.54) separately.

Proof of (3.53). By Theorem 3.27, we know that $D^\alpha f \rightharpoonup Df$ as $\alpha \to 1^-$. By [**58**, Proposition 4.29], we thus have that

$$|Df|(A) \le \liminf_{\alpha\to 1^-} |D^\alpha f|(A) \tag{3.55}$$

for any open set $A \subset \mathbb{R}^n$. Now let $K \subset \mathbb{R}^n$ be a compact set. By the representation formula (3.31) in Corollary 3.18, we can estimate

$$|D^\alpha f|(K) = \|\nabla^\alpha f\|_{L^1(K;\mathbb{R}^n)} \le \|I_{1-\alpha}|Df|\|_{L^1(K)} = (I_{1-\alpha}|Df|\,\mathscr{L}^n)(K).$$

Since $|Df|(\mathbb{R}^n) < +\infty$, by Lemma 3.28 and [**58**, Proposition 4.26] we can conclude that

$$\limsup_{\alpha\to 1^-} |D^\alpha f|(K) \le \limsup_{\alpha\to 1^-} (I_{1-\alpha}|Df|\,\mathscr{L}^n)(K) \le |Df|(K),$$

and so (3.53) follows, thanks again to [**58**, Proposition 4.26].

Proof of (3.54). Now assume $f \in BV(\mathbb{R}^n)$. By (3.17) applied with $A = \mathbb{R}^n$ and $r = 1$, we have

$$|D^\alpha f|(\mathbb{R}^n) \le \frac{n\omega_n\,\mu_{n,\alpha}}{n+\alpha-1}\left(\frac{|Df|(\mathbb{R}^n)}{1-\alpha} + \frac{n+2\alpha-1}{\alpha}\,\|f\|_{L^1(\mathbb{R}^n)}\right).$$

By (3.43), we thus get that

$$\limsup_{\alpha\to 1^-} |D^\alpha f|(\mathbb{R}^n) \le |Df|(\mathbb{R}^n). \tag{3.56}$$

Thus (3.54) follows combining (3.55) for $A = \mathbb{R}^n$ with (3.56). □

Note that Theorem 3.27 and Theorem 3.29 in particular apply to any $f \in W^{1,1}(\mathbb{R}^n)$. In the following result, by exploiting Proposition 3.15, we prove that a stronger property holds for any $f \in W^{1,p}(\mathbb{R}^n)$ with $p \in [1, +\infty)$.

Theorem 3.30. *Let* $p \in [1, +\infty)$*. If* $f \in W^{1,p}(\mathbb{R}^n)$*, then*

$$\lim_{\alpha\to 1^-} \|\nabla^\alpha f - \nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0. \tag{3.57}$$

Proof. By Proposition 3.15 we know that $f \in S^{\alpha,p}(\mathbb{R}^n)$ for any $\alpha \in (0,1)$. We now assume $\in (1, +\infty)$ and divide the proof in two steps.

Step 1. We claim that

$$\lim_{\alpha\to 1^-} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = \|\nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}. \tag{3.58}$$

Indeed, on the one hand, by Proposition 3.24, we have

$$\int_{\mathbb{R}^n} \varphi\cdot\nabla f\,dx = -\int_{\mathbb{R}^n} f\,\mathrm{div}\varphi\,dx = -\lim_{\alpha\to 1^-}\int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha\varphi\,dx = \lim_{\alpha\to 1^-}\int_{\mathbb{R}^n} \varphi\cdot\nabla^\alpha f\,dx \tag{3.59}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, so that

$$\int_{\mathbb{R}^n} \varphi\cdot\nabla f\,dx \le \|\varphi\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n;\mathbb{R}^n)} \liminf_{\alpha\to 1^-} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. We thus get that

$$\|\nabla f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le \liminf_{\alpha\to 1^-} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}. \tag{3.60}$$

On the other hand, applying (3.23) with $A = \mathbb{R}^n$ and $r = 1$, we have

$$\|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \le \frac{n\omega_n\,\mu_{n,\alpha}}{n+\alpha-1}\left(\frac{\|\nabla f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}}{1-\alpha} + \frac{n+2\alpha-1}{\alpha}\,\|f\|_{L^p(\mathbb{R}^n)}\right).$$

By (3.43), we conclude that

$$\limsup_{\alpha\to 1^-} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \le \|\nabla f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}. \tag{3.61}$$

Thus, (3.58) follows combining (3.60) and (3.61).

Step 2. We now claim that

$$\nabla^\alpha f \rightharpoonup \nabla f \quad \text{in } L^p(\mathbb{R}^n;\mathbb{R}^n) \text{ as } \alpha \to 1^-. \tag{3.62}$$

Indeed, let $\varphi \in L^{\frac{p}{p-1}}(\mathbb{R}^n;\mathbb{R}^n)$. For each $\varepsilon > 0$, let $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be such that $\|\psi_\varepsilon - \varphi\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n;\,\mathbb{R}^n)} < \varepsilon$. By (3.59) and (3.58), we can estimate

$$\begin{aligned}
\limsup_{\alpha\to 1^-}\left|\int_{\mathbb{R}^n} \varphi\cdot\nabla^\alpha f\,dx - \int_{\mathbb{R}^n}\varphi\cdot\nabla f\,dx\right| &\le \limsup_{\alpha\to 1^-}\left|\int_{\mathbb{R}^n}\psi_\varepsilon\cdot\nabla^\alpha f\,dx - \int_{\mathbb{R}^n}\psi_\varepsilon\cdot\nabla f\,dx\right| \\
&\quad + \int_{\mathbb{R}^n}|\varphi-\psi_\varepsilon|\,|\nabla^\alpha f|\,dx + \int_{\mathbb{R}^n}|\varphi-\psi_\varepsilon|\,|\nabla f|\,dx \\
&\le \varepsilon\left(\lim_{\alpha\to 1^-}\|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}\right) \\
&= 2\varepsilon\,\|\nabla f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}
\end{aligned}$$

so that (3.62) follows passing to the limit as $\varepsilon \to 0^+$.

Since $L^p(\mathbb{R}^n;\mathbb{R}^n)$ is uniformly convex (see [**19**, Section 4.3] for example), the limit in (3.57) follows from (3.58) and (3.62) by [**19**, Proposition 3.32], and the proof of the case $p \in (1,+\infty)$ is complete.

For the case $p = 1$, we argue as follows (we thank Mattia Calzi for this simple argument). Without loss of generality, it is enough to prove the limit in (3.58) for any given sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that $\alpha_k \to 1^-$ as $k \to +\infty$. By (3.54), the sequence $(\|\nabla^{\alpha_k} f\|_{L^1(\mathbb{R}^n;\,\mathbb{R}^n)})_{k\in\mathbb{N}}$ is bounded for any $f \in W^{1,1}(\mathbb{R}^n)$ and thus, by Banach–Steinhaus Theorem, the linear operators $\nabla^{\alpha_k}\colon W^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n;\mathbb{R}^n)$ are uniformly bounded (in the operator norm). The conclusion hence follows by exploiting the density of $C_c^\infty(\mathbb{R}^n)$ in $W^{1,1}(\mathbb{R}^n)$ and Proposition 3.24. □

For the case $p = +\infty$, we have the following result.

Theorem 3.31. *If $f \in W^{1,\infty}(\mathbb{R}^n)$, then*

$$\nabla^\alpha f \rightharpoonup \nabla f \quad \textit{in } L^\infty(\mathbb{R}^n;\mathbb{R}^n) \textit{ as } \alpha \to 1^- \tag{3.63}$$

and

$$\|\nabla f\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)} \le \liminf_{\alpha\to 1^-}\|\nabla^\alpha f\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}. \tag{3.64}$$

Proof. We argue similarly as in the proof of Theorem 3.30, in two steps.

Step 1: proof of (3.63). By Proposition 3.12 and Proposition 3.24, we have

$$\lim_{\alpha\to 1^-}\int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f\,dx = -\lim_{\alpha\to 1^-}\int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha\varphi\,dx = -\int_{\mathbb{R}^n} f\,\mathrm{div}\varphi\,dx = \int_{\mathbb{R}^n}\varphi\cdot\nabla f\,dx \tag{3.65}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, so that

$$\int_{\mathbb{R}^n}\varphi\cdot\nabla f\,dx \le \|\varphi\|_{L^1(\mathbb{R}^n;\,\mathbb{R}^n)}\liminf_{\alpha\to 1^-}\|\nabla^\alpha f\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. We thus get (3.64).

Step 2: proof of (3.64). Let $\varphi \in L^1(\mathbb{R}^n;\mathbb{R}^n)$. For each $\varepsilon > 0$, let $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be such that $\|\psi_\varepsilon - \varphi\|_{L^1(\mathbb{R}^n;\,\mathbb{R}^n)} < \varepsilon$. By (3.65) and (3.27), we can estimate

$$\begin{aligned}\limsup_{\alpha\to 1^-}\left|\int_{\mathbb{R}^n}\varphi\cdot\nabla^\alpha f\,dx - \int_{\mathbb{R}^n}\varphi\cdot\nabla f\,dx\right| &\le \limsup_{\alpha\to 1^-}\left|\int_{\mathbb{R}^n}\psi_\varepsilon\cdot\nabla^\alpha f\,dx - \int_{\mathbb{R}^n}\psi_\varepsilon\cdot\nabla f\,dx\right|\\ &\quad + \int_{\mathbb{R}^n}|\varphi-\psi_\varepsilon||\nabla^\alpha f|\,dx + \int_{\mathbb{R}^n}|\varphi-\psi_\varepsilon||\nabla f|\,dx\\ &\le \varepsilon\left(\limsup_{\alpha\to 1^-}\|\nabla^\alpha f\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)} + \|\nabla f\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}\right)\\ &\le \varepsilon\,(n+1)\,\|\nabla f\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}\end{aligned}$$

so that (3.62) follows passing to the limit as $\varepsilon \to 0^+$. □

Remark 3.32. We notice that Theorem 3.27 and Theorem 3.29, in the case $f = \chi_E \in BV(\mathbb{R}^n)$ with $E \subset \mathbb{R}^n$ bounded, and Theorem 3.30, were already announced in [**91**, Theorems 16 and 17].

4.3. Γ-convergence of α-variation as $\alpha \to 1^-$. In this section, we study the Γ-convergence of the fractional α-variation to the standard variation as $\alpha \to 1^-$.

We begin with the Γ - lim inf inequality.

Theorem 3.33 (Γ - lim inf inequalities as $\alpha \to 1^-$)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set.*

(i) If $(f_\alpha)_{\alpha\in(0,1)} \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $\sup_{\alpha\in(0,1)}\|f_\alpha\|_{L^\infty(\mathbb{R}^n)} < +\infty$ and $f_\alpha \to f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\alpha \to 1^-$, then

$$|Df|(\Omega) \le \liminf_{\alpha\to 1^-}|D^\alpha f_\alpha|(\Omega). \tag{3.66}$$

(ii) If $(f_\alpha)_{\alpha\in(0,1)} \subset L^1(\mathbb{R}^n)$ satisfies $f_\alpha \to f$ in $L^1(\mathbb{R}^n)$ as $\alpha \to 1^-$, then (3.66) *holds.*

Proof. We prove the two statements separately.

Proof of (i). Let $\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$ be such that $\|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1$. Since we can estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f_\alpha\,\mathrm{div}^\alpha\varphi\,dx - \int_{\mathbb{R}^n} f\,\mathrm{div}\varphi\,dx\right| &\le \int_{\mathbb{R}^n}|f_\alpha - f|\,|\mathrm{div}\varphi|\,dx + \int_{\mathbb{R}^n}|f_\alpha|\,|\mathrm{div}^\alpha\varphi - \mathrm{div}\varphi|\,dx\\ &\le \|\mathrm{div}\varphi\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}\int_{\mathrm{supp}\,\varphi}|f_\alpha - f|\,dx + \Big(\sup_{\alpha\in(0,1)}\|f_\alpha\|_{L^\infty(\mathbb{R}^n)}\Big)\,\|\mathrm{div}^\alpha\varphi - \mathrm{div}\varphi\|_{L^1(\mathbb{R}^n)},\end{aligned}$$

by Proposition 3.24 we get that

$$\int_{\mathbb{R}^n} f\,\mathrm{div}\varphi\,dx = \lim_{\alpha\to 1^-}\int_{\mathbb{R}^n} f_\alpha\,\mathrm{div}^\alpha\varphi\,dx \le \liminf_{\alpha\to 1^-}|D^\alpha f|(\Omega)$$

and the conclusion follows.

Proof of (ii). Let $\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$ be such that $\|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1$. Since we can estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f^\alpha \operatorname{div}_\alpha\varphi\,dx - \int_{\mathbb{R}^n} f\operatorname{div}\varphi\,dx\right| &\le \int_{\mathbb{R}^n}|f_\alpha - f|\,|\operatorname{div}\varphi|\,dx + \int_{\mathbb{R}^n}|f_\alpha|\,|\operatorname{div}^\alpha\varphi - \operatorname{div}\varphi|\,dx\\ &\le \|\operatorname{div}\varphi\|_{L^\infty(\mathbb{R}^n)}\|f_\alpha - f\|_{L^1(\mathbb{R}^n)} + \|\operatorname{div}^\alpha\varphi - \operatorname{div}\varphi\|_{L^\infty(\mathbb{R}^n)}\|f_\alpha\|_{L^1(\mathbb{R}^n)},\end{aligned}$$

by Proposition 3.24 we get that

$$\int_{\mathbb{R}^n} f\operatorname{div}\varphi\,dx = \lim_{\alpha\to1^-}\int_{\mathbb{R}^n} f_\alpha\operatorname{div}^\alpha\varphi\,dx \le \liminf_{\alpha\to1^-}|D^\alpha f_\alpha|(\Omega)$$

and the conclusion follows. □

We now pass to the Γ - lim sup inequality.

Theorem 3.34 (Γ - lim sup inequalities as $\alpha \to 1^-$)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set.*

(i) If $f \in BV(\mathbb{R}^n)$ and either Ω is bounded or $\Omega = \mathbb{R}^n$, then

$$\limsup_{\alpha\to1^-}|D^\alpha f|(\Omega) \le |Df|(\overline{\Omega}). \tag{3.67}$$

(ii) If $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ and Ω is bounded, then

$$\Gamma(L^1_{\mathrm{loc}})\text{ - }\limsup_{\alpha\to1^-}|D^\alpha f|(\Omega) \le |Df|(\overline{\Omega}).$$

In addition, if $f = \chi_E$, then the recovering sequences $(f_\alpha)_{\alpha\in(0,1)}$ in (i) and (ii) can be taken such that $f_\alpha = \chi_{E_\alpha}$ for some measurable sets $(E_\alpha)_{\alpha\in(0,1)}$.

Proof. Assume $f \in BV(\mathbb{R}^n)$. By Theorem 3.29, we know that $|D^\alpha f| \rightharpoonup |Df|$ as $\alpha \to 1^-$. Thus, by [**58**, Proposition 4.26], we get that

$$\limsup_{\alpha\to1^-}|D^\alpha f|(\Omega) \le \limsup_{\alpha\to1^-}|D^\alpha f|(\overline{\Omega}) \le |Df|(\overline{\Omega}) \tag{3.68}$$

for any bounded open set $\Omega \subset \mathbb{R}^n$. If $\Omega = \mathbb{R}^n$, then (3.67) follows immediately from (3.54). This concludes the proof of (i).

Now assume that $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ and Ω is bounded. Let $(R_k)_{k\in\mathbb{N}} \subset (0,+\infty)$ be a sequence such that $R_k \to +\infty$ as $k \to +\infty$ and set $f_k = f\chi_{B_{R_k}}$ for all $k \in \mathbb{N}$. By Theorem 3.1, we can choose the sequence $(R_k)_{k\in\mathbb{N}}$ such that, in addition, $f_k \in BV(\mathbb{R}^n)$ with $Df_k = \chi^\star_{B_{R_k}}Df + f^\star D\chi_{B_{R_k}}$ for all $k \in \mathbb{N}$. Consequently, $f_k \to f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $k \to +\infty$ and, moreover, since Ω is bounded, $|Df_k|(\Omega) = |Df|(\Omega)$ and $|Df_k|(\partial\Omega) = |Df|(\partial\Omega)$ for all $k \in \mathbb{N}$ sufficiently large. By (3.68), we have that

$$\limsup_{\alpha\to1^-}|D^\alpha f_k|(\Omega) \le |Df_k|(\overline{\Omega}) \tag{3.69}$$

for all $k \in \mathbb{N}$ sufficiently large. Hence, by [**16**, Proposition 1.28], by [**28**, Proposition 8.1(c)] and by (3.69), we get that

$$\begin{aligned}\Gamma(L^1_{\mathrm{loc}})\text{ - }\limsup_{\alpha\to1^-}|D^\alpha f|(\Omega) &\le \liminf_{k\to+\infty}\left(\Gamma(L^1_{\mathrm{loc}})\text{ - }\limsup_{\alpha\to1^-}|D^\alpha f_k|(\Omega)\right)\\ &\le \lim_{k\to+\infty}|Df_k|(\overline{\Omega}) = |Df|(\overline{\Omega}).\end{aligned}$$

This concludes the proof of (ii).

Finally, if $f = \chi_E$, then we can repeat the above argument *verbatim* in the metric spaces $\{\chi_F \in L^1(\mathbb{R}^n) : F \subset \mathbb{R}^n\}$ for (i) and $\{\chi_F \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : F \subset \mathbb{R}^n\}$ for (ii) endowed with their natural distances. □

Remark 3.35. Thanks to (3.67), a *recovery sequence* in Theorem 3.34(i) is the constant sequence (also in the special case $f = \chi_E$).

Combining Theorem 3.33(i) and Theorem 3.34(ii), we can prove that the fractional Caccioppoli α-perimeter Γ-converges to De Giorgi's perimeter as $\alpha \to 1^-$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We refer to [**5**] for the same result on the classical fractional perimeter.

Theorem 3.36 ($\Gamma(L^1_{\mathrm{loc}})$ - lim of perimeters as $\alpha \to 1^-$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. For every measurable set $E \subset \mathbb{R}^n$, we have*

$$\Gamma(L^1_{\mathrm{loc}})\text{ - }\lim_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) = P(E; \Omega).$$

Proof. By Theorem 3.33(i), we already know that

$$\Gamma(L^1_{\mathrm{loc}})\text{ - }\liminf_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) \geq P(E; \Omega),$$

so we just need to prove the $\Gamma(L^1_{\mathrm{loc}})$ - lim sup inequality. Without loss of generality, we can assume $P(E; \Omega) < +\infty$. Now let $(E_k)_{k \in \mathbb{N}}$ be given by Theorem 3.4. Since $\chi_{E_k} \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ and $P(E_k; \partial\Omega) = 0$ for all $k \in \mathbb{N}$, by Theorem 3.34(ii) we know that

$$\Gamma(L^1_{\mathrm{loc}})\text{ - }\limsup_{\alpha \to 1^-} |D^\alpha \chi_{E_k}|(\Omega) \leq P(E_k; \Omega)$$

for all $k \in \mathbb{N}$. Since $\chi_{E_k} \to \chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $P(E_k; \Omega) \to P(E; \Omega)$ as $k \to +\infty$, by [**16**, Proposition 1.28] we get that

$$\begin{aligned} \Gamma(L^1_{\mathrm{loc}})\text{ - }\limsup_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) &\leq \liminf_{k \to +\infty} \Big(\Gamma(L^1_{\mathrm{loc}})\text{ - }\limsup_{\alpha \to 1^-} |D^\alpha \chi_{E_k}|(\Omega) \Big) \\ &\leq \lim_{k \to +\infty} P(E_k; \Omega) = P(E; \Omega) \end{aligned}$$

and the proof is complete. □

Finally, combining Theorem 3.33(ii) and Theorem 3.34, we can prove that the fractional α-variation Γ-converges to De Giorgi's variation as $\alpha \to 1^-$ in $L^1(\mathbb{R}^n)$.

Theorem 3.37 ($\Gamma(L^1)$ - lim of variations as $\alpha \to 1^-$)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set such that either Ω is bounded with Lipschitz boundary or $\Omega = \mathbb{R}^n$. For every $f \in BV(\mathbb{R}^n)$, we have*

$$\Gamma(L^1)\text{ - }\lim_{\alpha \to 1^-} |D^\alpha f|(\Omega) = |Df|(\Omega).$$

Proof. The case $\Omega = \mathbb{R}^n$ follows immediately by [**28**, Proposition 8.1(c)] combining Theorem 3.33(ii) with Theorem 3.34(i). We can thus assume that Ω is a bounded open set with Lipschitz boundary and argue similarly as in the proof of Theorem 3.36. By Theorem 3.33(ii), we already know that

$$\Gamma(L^1)\text{ - }\liminf_{\alpha \to 1^-} |D^\alpha f|(\Omega) \geq |Df|(\Omega),$$

so we just need to prove the $\Gamma(L^1)$ - lim sup inequality. Without loss of generality, we can assume $|Df|(\Omega) < +\infty$. Now let $(f_k)_{k \in \mathbb{N}} \subset BV(\mathbb{R}^n)$ be given by Theorem 3.6.

Since $|Df_k|(\partial\Omega) = 0$ for all $k \in \mathbb{N}$, by Theorem 3.34 we know that

$$\Gamma(L^1)\text{-}\limsup_{\alpha\to1^-} |D^\alpha f_k|(\Omega) \le |Df_k|(\overline{\Omega}) = |Df_k|(\Omega)$$

for all $k \in \mathbb{N}$. Since $f_k \to f$ in $L^1(\mathbb{R}^n)$ and $|D^\alpha f_k|(\Omega) \to |D^\alpha f|(\Omega)$ as $k \to +\infty$, by [**16**, Proposition 1.28] we get that

$$\begin{aligned}\Gamma(L^1)\text{-}\limsup_{\alpha\to1^-} |D^\alpha f|(\Omega) &\le \liminf_{k\to+\infty} \Big(\Gamma(L^1)\text{-}\limsup_{\alpha\to1^-} |D^\alpha f_k|(\Omega)\Big) \\ &\le \lim_{k\to+\infty} |Df_k|(\Omega) = |Df|(\Omega)\end{aligned}$$

and the proof is complete. □

Remark 3.38. Thanks to Theorem 3.37, we can slightly improve Theorem 3.36. Indeed, if $\chi_E \in BV(\mathbb{R}^n)$, then we also have

$$\Gamma(L^1)\text{-}\lim_{\alpha\to1^-} |D^\alpha \chi_E|(\Omega) = |D\chi_E|(\Omega)$$

for any open set $\Omega \subset \mathbb{R}^n$ such that either Ω is bounded with Lipschitz boundary or $\Omega = \mathbb{R}^n$.

5. Asymptotic behaviour of fractional β-variation as $\beta \to \alpha^-$

5.1. Convergence of ∇^β and div^β as $\beta \to \alpha$. We begin with the following simple result about the L^1-convergence of the operators ∇^β and div^β as $\beta \to \alpha$ with $\alpha \in (0,1)$.

Lemma 3.39. *Let $\alpha \in (0,1)$. If $f \in W^{\alpha,1}(\mathbb{R}^n)$ and $\varphi \in W^{\alpha,1}(\mathbb{R}^n;\mathbb{R}^n)$, then*

$$\lim_{\beta\to\alpha^-} \|\nabla^\beta f - \nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} = 0, \qquad \lim_{\beta\to\alpha^-} \|\operatorname{div}^\beta\varphi - \operatorname{div}^\alpha\varphi\|_{L^1(\mathbb{R}^n)} = 0. \tag{3.70}$$

Proof. Given $\beta \in (0,\alpha)$, we can estimate

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla^\beta f(x) - \nabla^\alpha f(x)|\,dx &\le |\mu_{n,\beta} - \mu_{n,\alpha}|\,[f]_{W^{\alpha,1}(\mathbb{R}^n)} \\ &\quad + \mu_{n,\beta} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(y)-f(x)|}{|y-x|^n} \left|\frac{1}{|y-x|^\beta} - \frac{1}{|y-x|^\alpha}\right| dy\,dx.\end{aligned}$$

Since the Γ function is continuous (see [**13**]), we clearly have

$$\lim_{\beta\to\alpha^-} |\mu_{n,\beta} - \mu_{n,\alpha}|\,[f]_{W^{\alpha,1}(\mathbb{R}^n)} = 0.$$

Now write

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(y)-f(x)|}{|y-x|^n} \left|\frac{1}{|y-x|^\beta} - \frac{1}{|y-x|^\alpha}\right| dy\,dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(y)-f(x)|}{|y-x|^n} \left|\frac{1}{|y-x|^\beta} - \frac{1}{|y-x|^\alpha}\right| \chi_{(0,1)}(|y-x|)\,dy\,dx \\ &\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(y)-f(x)|}{|y-x|^n} \left|\frac{1}{|y-x|^\beta} - \frac{1}{|y-x|^\alpha}\right| \chi_{[1,+\infty)}(|y-x|)\,dy\,dx.\end{aligned}$$

On the one hand, since $f \in W^{\alpha,1}(\mathbb{R}^n)$, we have

$$\frac{|f(y)-f(x)|}{|y-x|^n} \left|\frac{1}{|y-x|^\beta} - \frac{1}{|y-x|^\alpha}\right| \chi_{(0,1)}(|y-x|)$$

$$
\begin{aligned}
&= \frac{|f(y)-f(x)|}{|y-x|^n}\left(\frac{1}{|y-x|^\alpha} - \frac{1}{|y-x|^\beta}\right)\chi_{(0,1)}(|y-x|) \\
&\le \frac{|f(y)-f(x)|}{|y-x|^{n+\alpha}}\,\chi_{(0,1)}(|y-x|) \in L^1_{x,y}(\mathbb{R}^{2n})
\end{aligned}
$$

and thus, by Lebesgue's Dominated Convergence Theorem, we get that

$$
\lim_{\beta\to\alpha^-}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)-f(x)|}{|y-x|^n}\left|\frac{1}{|y-x|^\beta}-\frac{1}{|y-x|^\alpha}\right|\chi_{(0,1)}(|y-x|)\,dy\,dx = 0.
$$

On the other hand, since one has

$$
\begin{aligned}
[f]_{W^{\beta,1}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\{|h|<1\}}\frac{|f(x+h)-f(x)|}{|h|^{n+\beta}}\,dh\,dx + \int_{\mathbb{R}^n}\int_{\{|h|\ge 1\}}\frac{|f(x+h)-f(x)|}{|h|^{n+\beta}}\,dh\,dx \\
&\le [f]_{W^{\alpha,1}(\mathbb{R}^n)} + \int_{\{|h|\ge 1\}}\frac{1}{|h|^{n+\beta}}\int_{\mathbb{R}^n}|f(x+h)|+|f(x)|\,dx\,dh \\
&= [f]_{W^{\alpha,1}(\mathbb{R}^n)} + \frac{2n\omega_n}{\beta}\,\|f\|_{L^1(\mathbb{R}^n)}
\end{aligned}
$$

for all $\beta \in (0,\alpha)$, we can estimate

$$
\begin{aligned}
&\frac{|f(y)-f(x)|}{|y-x|^n}\left|\frac{1}{|y-x|^\beta}-\frac{1}{|y-x|^\alpha}\right|\chi_{[1,+\infty)}(|y-x|) \\
&\qquad = \frac{|f(y)-f(x)|}{|y-x|^n}\left(\frac{1}{|y-x|^\beta}-\frac{1}{|y-x|^\alpha}\right)\chi_{[1,+\infty)}(|y-x|) \\
&\qquad \le \frac{|f(y)-f(x)|}{|y-x|^{n+\beta}}\,\chi_{[1,+\infty)}(|y-x|) \\
&\qquad \le \frac{|f(y)-f(x)|}{|y-x|^{n+\frac{\alpha}{2}}}\,\chi_{[1,+\infty)}(|y-x|) \in L^1_{x,y}(\mathbb{R}^{2n})
\end{aligned}
$$

for all $\beta \in \left(\frac{\alpha}{2},\alpha\right)$ and thus, by Lebesgue's Dominated Convergence Theorem, we get that

$$
\lim_{\beta\to\alpha^-}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)-f(x)|}{|y-x|^n}\left|\frac{1}{|y-x|^\beta}-\frac{1}{|y-x|^\alpha}\right|\chi_{[1,+\infty)}(|y-x|)\,dy\,dx = 0
$$

and the first limit in (3.70) follows. The second limit in (3.70) follows similarly and we leave the proof to the reader. □

Remark 3.40. Let $\alpha \in (0,1)$. If $f \in W^{\alpha+\varepsilon,1}(\mathbb{R}^n)$ and $\varphi \in W^{\alpha+\varepsilon,1}(\mathbb{R}^n)$ for some $\varepsilon \in (0, 1-\alpha)$, then, arguing as in the proof of Lemma 3.39, one can also prove that

$$
\lim_{\beta\to\alpha^+}\|\nabla^\beta f - \nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} = 0, \qquad \lim_{\beta\to\alpha^+}\|\operatorname{div}^\beta\varphi - \operatorname{div}^\alpha\varphi\|_{L^1(\mathbb{R}^n)} = 0.
$$

We leave the details of proof of this result to the interested reader.

If one deals with more regular functions, then Lemma 3.39 can be improved as follows.

Lemma 3.41. *Let $\alpha \in (0,1)$ and $p \in [1,+\infty]$. If $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, then*

$$\lim_{\beta\to\alpha^-} \|\nabla^\beta f - \nabla^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0, \qquad \lim_{\beta\to\alpha^-} \|\mathrm{div}^\beta\varphi - \mathrm{div}^\alpha\varphi\|_{L^p(\mathbb{R}^n)} = 0. \tag{3.71}$$

Proof. Since clearly $f \in W^{\alpha,1}(\mathbb{R}^n)$ for any $\alpha \in (0,1)$, the first limit in (3.71) for the case $p = 1$ follows from Lemma 3.39. Hence, we just need to prove the validity of the same limit for the case $p = +\infty$, since then the conclusion simply follows by an interpolation argument.

Let $\beta \in (0,\alpha)$ and $x \in \mathbb{R}^n$. We have

$$\begin{aligned}
|\nabla^\alpha f(x) - \nabla^\beta f(x)| &\le |\mu_{n,\beta} - \mu_{n,\alpha}| \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^{n+\alpha}}\,dy \\
&\quad + \mu_{n,\beta}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^n}\left|\frac{1}{|x-y|^\beta} - \frac{1}{|x-y|^\alpha}\right| dy \\
&= |\mu_{n,\beta} - \mu_{n,\alpha}| \int_{\mathbb{R}^n} \frac{|f(x+z)-f(x)|}{|z|^{n+\alpha}}\,dz \\
&\quad + \mu_{n,\beta}\int_{\mathbb{R}^n} \frac{|f(x+z)-f(x)|}{|z|^n}\left|\frac{1}{|z|^\beta} - \frac{1}{|z|^\alpha}\right| dz.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|f(x+z)-f(x)|}{|z|^{n+\alpha}}\,dz &\le \int_{\{|z|\le 1\}} \frac{\mathrm{Lip}(f)}{|z|^{n+\alpha-1}}\,dz + \int_{\{|z|>1\}} \frac{2\|f\|_{L^\infty(\mathbb{R}^n)}}{|z|^{n+\alpha}}\,dz \\
&\le n\omega_n\left(\frac{\mathrm{Lip}(f)}{1-\alpha} + \frac{2\|f\|_{L^\infty(\mathbb{R}^n)}}{\alpha}\right)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|f(x+z)-f(z)|}{|z|^n}\left|\frac{1}{|z|^\beta} - \frac{1}{|z|^\alpha}\right| dz &\le \int_{\{|z|\le 1\}} \frac{\mathrm{Lip}(f)}{|z|^{n-1}}\left(\frac{1}{|z|^\alpha} - \frac{1}{|z|^\beta}\right) dz \\
&\quad + \int_{\{|z|>1\}} \frac{2\|f\|_{L^\infty(\mathbb{R}^n)}}{|z|^n}\left(\frac{1}{|z|^\beta} - \frac{1}{|z|^\alpha}\right) dz \\
&\le (\alpha-\beta)n\omega_n\left(\frac{\mathrm{Lip}(f)}{(1-\alpha)(1-\beta)} + \frac{2\|f\|_{L^\infty(\mathbb{R}^n)}}{\alpha\beta}\right),
\end{aligned}$$

for all $\beta \in \left(\frac{\alpha}{2},\alpha\right)$ we obtain

$$\|\nabla^\alpha f - \nabla^\beta f\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le c_{n,\alpha} \max\left\{\mathrm{Lip}(f), \|f\|_{L^\infty(\mathbb{R}^n)}\right\}\left(|\mu_{n,\beta} - \mu_{n,\alpha}| + (\alpha-\beta)\right),$$

for some constant $c_{n,\alpha} > 0$ depending only on n and α. Thus the conclusion follows since $\mu_{n,\beta} \to \mu_{n,\alpha}$ as $\beta \to \alpha^-$. The second limit in (3.71) follows similarly and we leave the proof to the reader. □

5.2. Weak convergence of β-variation as $\beta \to \alpha^-$. In Theorem 3.42 below, we prove the weak convergence of the β-variation as $\beta \to \alpha^-$, extending the convergences obtained in Theorem 3.27 and Theorem 3.29.

Theorem 3.42. *Let* $\alpha \in (0,1)$. *If* $f \in BV^\alpha(\mathbb{R}^n)$, *then*

$$D^\beta f \rightharpoonup D^\alpha f \quad \text{and} \quad |D^\beta f| \rightharpoonup |D^\alpha f| \quad \text{as } \beta \to \alpha^-.$$

Moreover, we have

$$\lim_{\beta\to\alpha^-} |D^\beta f|(\mathbb{R}^n) = |D^\alpha f|(\mathbb{R}^n). \tag{3.72}$$

Proof. We divide the proof in three steps.

Step 1: we prove that $D^\beta f \rightharpoonup D^\alpha f$ *as* $\beta \to \alpha^-$. We argue similarly as in Step 1 of the proof of Theorem 3.27. By Proposition 3.19, we have

$$\int_{\mathbb{R}^n} \varphi \cdot \nabla^\beta f \, dx = -\int_{\mathbb{R}^n} f \operatorname{div}^\beta \varphi \, dx$$

for all $\beta \in (0,\alpha)$ and $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$. Thus, thanks to (3.71) in the case $p=\infty$, we get

$$\lim_{\beta\to\alpha^-} \int_{\mathbb{R}^n} \varphi \cdot \nabla^\beta f \, dx = -\lim_{\beta\to\alpha^-} \int_{\mathbb{R}^n} f \operatorname{div}^\beta \varphi \, dx = -\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = \int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f.$$

Now fix $\varphi \in C^0_c(\mathbb{R}^n;\mathbb{R}^n)$. Let $U \subset \mathbb{R}^n$ be a fixed bounded open set such that $\operatorname{supp}\varphi \subset U$. For each $\varepsilon > 0$ sufficiently small, pick $\psi_\varepsilon \in \operatorname{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ such that $\|\varphi - \psi_\varepsilon\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} < \varepsilon$ and $\operatorname{supp}\psi_\varepsilon \subset U$. Then, by (3.34), we can estimate

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \varphi \cdot \nabla^\beta f \, dx - \int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f \right| &\le \|\varphi - \psi_\varepsilon\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \left(\int_U |\nabla^\beta f| \, dx + |D^\alpha f|(\mathbb{R}^n) \right) \\
&\quad + \left| \int_{\mathbb{R}^n} \psi_\varepsilon \cdot \nabla^\beta f \, dx - \int_{\mathbb{R}^n} \psi_\varepsilon \cdot dD^\alpha f \right| \\
&\le \varepsilon (1 + C_{n,(1-\alpha+\beta),U}) \, |D^\alpha f|(\mathbb{R}^n) \\
&\quad + \left| \int_{\mathbb{R}^n} \psi_\varepsilon \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \psi_\varepsilon \cdot dDf \right|
\end{aligned}$$

for all $\beta \in (0,\alpha)$. Thus, by the uniform estimate (3.44) in Lemma 3.22, we get

$$\lim_{\beta\to\alpha^-} \left| \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx - \int_{\mathbb{R}^n} \varphi \cdot dDf \right| \le \varepsilon (1 + \kappa_{n,U}) \, |D^\alpha f|(\mathbb{R}^n) \tag{3.73}$$

and the conclusion follows passing to the limit as $\varepsilon \to 0^+$.

Step 2: we prove that $|D^\beta f| \rightharpoonup |D^\alpha f|$ *as* $\beta \to \alpha^-$. We argue similarly as in the first part of the proof of Theorem 3.29. Since $D^\beta f \rightharpoonup D^\alpha f$ as $\beta \to \alpha^-$ as proved in Step 1 above, by [**58**, Proposition 4.29], we have that

$$|D^\alpha f|(A) \le \liminf_{\beta\to\alpha^-} |D^\beta f|(A) \tag{3.74}$$

for any open set $A \subset \mathbb{R}^n$. Now let $K \subset \mathbb{R}^n$ be a compact set. By the representation formula (3.33) in Proposition 3.19, we can estimate

$$|D^\beta f|(K) = \|\nabla^\beta f\|_{L^1(K;\mathbb{R}^n)} \le \|I_{\alpha-\beta}|D^\alpha f|\|_{L^1(K)} = (I_{\alpha-\beta}|D^\alpha f| \, \mathscr{L}^n)(K).$$

Since $|D^\alpha f|(\mathbb{R}^n) < +\infty$, by Lemma 3.28 and [**58**, Proposition 4.26] we conclude that

$$\limsup_{\beta\to\alpha^-} |D^\beta f|(K) \le \limsup_{\beta\to\alpha^-} (I_{\alpha-\beta}|D^\alpha f| \, \mathscr{L}^n)(K) \le |D^\alpha f|(K). \tag{3.75}$$

The conclusion thus follows thanks to [**58**, Proposition 4.26].

Step 3: we prove (3.72). We argue similarly as in the proof of (3.53). By (3.35) applied with $A = \mathbb{R}^n$ and $r = 1$, we have

$$|D^\beta f|(\mathbb{R}^n) \le \frac{\mu_{n,1+\beta-\alpha}}{n+\beta-\alpha}\left(\frac{n\omega_n}{\alpha-\beta}\,|D^\alpha f|(\mathbb{R}^n) + \frac{\omega_{n,\alpha}(n+2\beta-\alpha)}{\beta}\,\|f\|_{L^1(\mathbb{R}^n)}\right).$$

By (3.43), we get that

$$\limsup_{\beta\to\alpha^-} |D^\beta f|(\mathbb{R}^n) \le |D^\alpha f|(\mathbb{R}^n). \tag{3.76}$$

Thus, (3.72) follows combining (3.74) for $A = \mathbb{R}^n$ with (3.76). □

In analogy with Theorem 3.42, from Proposition 3.20 we can extend the validity of Theorem 3.30 and deduce the following result. The proof is very similar to the one of Theorem 3.30 and is thus left to the reader.

Theorem 3.43. *Let $\alpha \in (0,1)$ and $p \in [1,+\infty)$. If $f \in S^{\alpha,p}(\mathbb{R}^n)$, then*

$$\lim_{\beta\to\alpha-} \|\nabla^\beta f - \nabla^\alpha f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} = 0. \tag{3.77}$$

5.3. Γ-convergence of β-variation as $\beta \to \alpha^-$. In this section, we study the Γ-convergence of the fractional β-variation as $\beta \to \alpha^-$, partially extending the results obtained in Section 4.3.

We begin with the $\Gamma\text{-}\liminf$ inequality.

Theorem 3.44 ($\Gamma\text{-}\liminf$ inequality for $\beta \to \alpha^-$)**.** *Let $\alpha \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ be an open set. If $(f_\beta)_{\beta\in(0,\alpha)} \subset L^1(\mathbb{R}^n)$ satisfies $f_\beta \to f$ in $L^1(\mathbb{R}^n)$ as $\beta \to \alpha^-$, then*

$$|D^\alpha f|(\Omega) \le \liminf_{\beta\to\alpha^-} |D^\beta f_\beta|(\Omega). \tag{3.78}$$

Proof. We argue similarly as in the proof of Theorem 3.33(ii). Let $\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$ be such that $\|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1$. Let $U \subset \mathbb{R}^n$ be a bounded open set such that $\operatorname{supp}\varphi \subset U$. By (1.12), we can estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f_\beta \operatorname{div}^\beta\varphi\,dx - \int_{\mathbb{R}^n} f \operatorname{div}^\alpha\varphi\,dx\right| &\le \int_{\mathbb{R}^n} |f_\beta - f|\,|\operatorname{div}^\beta\varphi|\,dx + \int_{\mathbb{R}^n} |f|\,|\operatorname{div}^\beta\varphi - \operatorname{div}^\alpha\varphi|\,dx\\ &\le C_{n,\beta,U}\|\operatorname{div}\varphi\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}\|f_\beta - f\|_{L^1(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |f|\,|\operatorname{div}^\beta\varphi - \operatorname{div}^\alpha\varphi|\,dx\end{aligned}$$

for all $\beta \in (0,\alpha)$. Since $\operatorname{div}^\beta\varphi \to \operatorname{div}^\alpha\varphi$ in $L^\infty(\mathbb{R}^n)$ as $\beta \to \alpha^-$ by (3.71), we easily obtain

$$\lim_{\beta\to\alpha^-} \int_{\mathbb{R}^n} |f|\,|\operatorname{div}^\beta\varphi - \operatorname{div}^\alpha\varphi|\,dx = 0.$$

Hence, we get

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha\varphi\,dx = \lim_{\beta\to\alpha^-} \int_{\mathbb{R}^n} f_\beta \operatorname{div}^\beta\varphi\,dx \le \liminf_{\beta\to\alpha^-} |D^\beta f_\beta|(\Omega)$$

and the conclusion follows. □

We now pass to the $\Gamma\text{-}\limsup$ inequality.

Theorem 3.45 (Γ - lim sup inequality for $\beta \to \alpha^-$). *Let $\alpha \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ be an open set. If $f \in BV^\alpha(\mathbb{R}^n)$ and either Ω is bounded or $\Omega = \mathbb{R}^n$, then*

$$\limsup_{\beta \to \alpha^-} |D^\beta f|(\Omega) \le |D^\alpha f|(\overline{\Omega}). \tag{3.79}$$

Proof. We argue similarly as in the proof of Theorem 3.34. By Theorem 3.42, we know that $|D^\beta f| \rightharpoonup |D^\alpha f|$ as $\beta \to \alpha^-$. Thus, by [**58**, Proposition 4.26] and (3.72), we get that

$$\limsup_{\beta \to \alpha^-} |D^\beta f|(\Omega) \le \limsup_{\beta \to \alpha^-} |D^\beta f|(\overline{\Omega}) \le |D^\alpha f|(\overline{\Omega}) \tag{3.80}$$

for any open set $\Omega \subset \mathbb{R}^n$ such that either Ω is bounded or $\Omega = \mathbb{R}^n$. □

Corollary 3.46 ($\Gamma(L^1)$ - lim of variations in $\mathbb{R}^n$ as $\beta \to \alpha^-$). *Let $\alpha \in (0,1)$. For every $f \in BV^\alpha(\mathbb{R}^n)$, we have*

$$\Gamma(L^1)\text{-}\lim_{\beta \to \alpha^-} |D^\beta f|(\mathbb{R}^n) = |D^\alpha f|(\mathbb{R}^n).$$

In particular, the constant sequence is a recovery sequence.

Proof. The result follows easily by combining (3.78) and (3.79) in the case $\Omega = \mathbb{R}^n$. □

Remark 3.47. We recall that, by Theorem 1.57, $f \in BV^\alpha(\mathbb{R}^n)$ satisfies $|D^\alpha f| \ll \mathscr{L}^n$ if and only if $f \in S^{\alpha,1}(\mathbb{R}^n)$. Therefore, if $f \in S^{\alpha,1}(\mathbb{R}^n)$, then $|D^\alpha f|(\partial\Omega) = 0$ for any bounded open set $\Omega \subset \mathbb{R}^n$ such that $\mathscr{L}^n(\partial\Omega) = 0$ (for instance, Ω with Lipschitz boundary). Thus, we can actually obtain the Γ-convergence of the fractional β-variation as $\beta \to \alpha^-$ on bounded open sets with Lipschitz boundary for any $f \in S^{\alpha,1}(\mathbb{R}^n)$ too. Indeed, it is enough to combine (3.78) and (3.79) and then exploit the fact that $|D^\alpha f|(\partial\Omega) = 0$ to get

$$\Gamma(L^1)\text{-}\lim_{\beta \to \alpha^-} |D^\beta f|(\Omega) = |D^\alpha f|(\Omega)$$

for any $f \in S^{\alpha,1}(\mathbb{R}^n)$.

CHAPTER 4

Asymptotic behaviour of fractional variation as $\alpha \to 0^+$

1. The space $HS^{\alpha,1}$

1.1. Definition of $HS^{\alpha,1}(\mathbb{R}^n)$. Following the classical approach of [**99**], for $\alpha \in [0,1]$ we let

$$\begin{aligned} HS^{\alpha,1}(\mathbb{R}^n) &= (I-\Delta)^{-\frac{\alpha}{2}}(H^1(\mathbb{R}^n)) \\ &= \left\{ f \in H^1(\mathbb{R}^n) : (I-\Delta)^{\frac{\alpha}{2}} f \in H^1(\mathbb{R}^n) \right\} \end{aligned}$$

be the (real) *fractional Hardy–Sobolev space* endowed with the norm

$$\|f\|_{HS^{\alpha,1}(\mathbb{R}^n)} = \|(I-\Delta)^{\frac{\alpha}{2}} f\|_{H^1(\mathbb{R}^n)}, \quad f \in HS^{\alpha,1}(\mathbb{R}^n).$$

In particular, $HS^{0,1}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ coincides with the Hardy space and $HS^{1,1}(\mathbb{R}^n)$ is the standard (real) Hardy–Sobolev space. As remarked in [**99**, p. 130], we can equivalently define

$$\begin{aligned} HS^{\alpha,1}(\mathbb{R}^n) &= H^1(\mathbb{R}^n) \cap I_\alpha(H^1(\mathbb{R}^n)) \\ &= \left\{ f \in H^1(\mathbb{R}^n) : (-\Delta)^{\frac{\alpha}{2}} f \in H^1(\mathbb{R}^n) \right\} \end{aligned}$$

endowed with the (equivalent) norm

$$\|f\|_{HS^{\alpha,1}(\mathbb{R}^n)} = \|f\|_{H^1(\mathbb{R}^n)} + \|(-\Delta)^{\frac{\alpha}{2}} f\|_{H^1(\mathbb{R}^n)}, \quad f \in HS^{\alpha,1}(\mathbb{R}^n).$$

In particular, the operator

$$(-\Delta)^{\frac{\alpha}{2}} \colon HS^{\alpha,1}(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$$

is well defined and continuous.

1.2. Approximation by test functions. For the reader's convenience we briefly prove the following density result. Here and in the following, for simplicity we let

$$L^1_0(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)\,dx = 0 \right\}$$

be the space of integrable functions with zero mean.

Lemma 4.1 (Approximation by $C^\infty_c \cap L^1_0$ functions in $HS^{\alpha,1}$)**.** *Let $\alpha \in (0,1)$. The set $C^\infty_c(\mathbb{R}^n) \cap L^1_0(\mathbb{R}^n)$ is dense in $HS^{\alpha,1}(\mathbb{R}^n)$.*

Proof. Since the set $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ by [**97**, Chapter III, Section 5.2(b)], we have that the set $(I-\Delta)^{-\frac{\alpha}{2}}(\mathcal{S}_0(\mathbb{R}^n))$ is dense in $HS^{\alpha,1}(\mathbb{R}^n)$. Since clearly $(I-\Delta)^{-\frac{\alpha}{2}}(\mathcal{S}_0(\mathbb{R}^n)) \subset \mathcal{S}_0(\mathbb{R}^n)$, we thus get that the set $\mathcal{S}_0(\mathbb{R}^n)$ is dense (and embeds continuously) in $HS^{\alpha,1}(\mathbb{R}^n)$. Since the set $C^\infty_c(\mathbb{R}^n) \cap L^1_0(\mathbb{R}^n)$ is dense in $\mathcal{S}_0(\mathbb{R}^n)$, the conclusion follows. ☐

Exploiting Lemma 4.1, for $\alpha \in (0,1)$ we can equivalently define

$$HS^{\alpha,1}(\mathbb{R}^n) = \Big\{ f \in H^1(\mathbb{R}^n) : \nabla^\alpha f \in H^1(\mathbb{R}^n;\mathbb{R}^n) \Big\}$$

endowed with the (equivalent) norm

$$\|f\|_{HS^{\alpha,1}(\mathbb{R}^n)} = \|f\|_{H^1(\mathbb{R}^n)} + \|\nabla^\alpha f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)}, \quad f \in HS^{\alpha,1}(\mathbb{R}^n).$$

Indeed, if $f \in C_c^\infty(\mathbb{R}^n) \cap L_0^1(\mathbb{R}^n)$, then we can write $\nabla^\alpha f = R(-\Delta)^{\frac{\alpha}{2}} f$, so that

$$c_n^{-1} \|(-\Delta)^{\frac{\alpha}{2}} f\|_{H^1(\mathbb{R}^n)} \le \|\nabla^\alpha f\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)} \le c_n \|(-\Delta)^{\frac{\alpha}{2}} f\|_{H^1(\mathbb{R}^n)} \tag{4.1}$$

for all $f \in C_c^\infty(\mathbb{R}^n) \cap L_0^1(\mathbb{R}^n)$ thanks to the H^1-continuity property of the Riesz transform and the fact that $R^2 = -I$ on $H^1(\mathbb{R}^n)$, where $c_n > 0$ is a dimensional constant. By Lemma 4.1, the validity of (4.1) extends to all $f \in HS^{\alpha,1}(\mathbb{R}^n)$ and the conclusion follows.

As a consequence, note that $HS^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$ with continuous embedding. We also have the following result.

Lemma 4.2. *If $0 < \beta < \alpha < 1$, then*

$$H^1(\mathbb{R}^n) \cap W^{\alpha,1}(\mathbb{R}^n) \subset HS^{\alpha,1}(\mathbb{R}^n) \tag{4.2}$$

and

$$HS^{\alpha,1}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) \cap W^{\beta,1}(\mathbb{R}^n). \tag{4.3}$$

As a consequence, we get

$$H^1(\mathbb{R}^n) \cap \bigcup_{\alpha \in (0,1)} W^{\alpha,1}(\mathbb{R}^n) = \bigcup_{\alpha \in (0,1)} HS^{\alpha,1}(\mathbb{R}^n). \tag{4.4}$$

Proof. On the one hand, by Proposition 1.41(ii), we have $(-\Delta)^{\frac{\alpha}{2}}(W^{\alpha,1}(\mathbb{R}^n)) \subset H^1(\mathbb{R}^n)$ and the inclusion (4.2) immediately follows. On the other hand, $HS^{\alpha,1}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) \cap S^{\alpha,1}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$, so that the inclusion (4.3) follows from Theorem 1.30. □

We note that the well posedness and the equivalence of the definitions of $HS^{\alpha,1}(\mathbb{R}^n)$ given above and the stated results hold for any $\alpha \ge 0$ thanks to the composition properties of the operators involved. We leave the standard verifications to the interested reader.

2. Interpolation inequalities

2.1. The case $p = 1$ via the Calderón–Zygmund Theorem. For $\alpha \in (0,1)$ and $R > 0$, we let $T_{\alpha,R} \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n;\mathbb{R}^n)$ be the linear operator defined by

$$T_{\alpha,R} f(x) = \int_{\mathbb{R}^n} f(y+x) \, \frac{y\,(1 - \eta_R(y))}{|y|^{n+\alpha+1}} \, dy, \qquad x \in \mathbb{R}^n, \tag{4.5}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Here $(\eta_R)_{R>0} \subset C_c^\infty(\mathbb{R}^n)$ is a family of cut-off functions as in (1.30). In the following result, we prove that $T_{\alpha,R}$ is a Calderón–Zygmund operator mapping $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n;\mathbb{R}^n)$.

Lemma 4.3 (Calderón–Zygmund estimate for $T_{\alpha,R}$)**.** *There is a dimensional constant $\tau_n > 0$ such that, for any given $\alpha \in (0,1)$ and $R > 0$, the operator in* (4.5) *uniquely extends to a bounded linear operator $T_{\alpha,R} \colon H^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n;\mathbb{R}^n)$ with*

$$\|T_{\alpha,R} f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \tau_n R^{-\alpha} \|f\|_{H^1(\mathbb{R}^n)}$$

for all $f \in H^1(\mathbb{R}^n)$.

Proof. We apply [**48**, Theorem 2.4.1] to the kernel

$$K_{\alpha,R}(x) = \frac{x\,(1-\eta_R(x))}{|x|^{n+\alpha+1}}, \qquad x \in \mathbb{R}^n,\ x \ne 0.$$

First of all, we have

$$|K_{\alpha,R}(x)| \le \frac{1-\eta_R(x)}{|x|^{n+\alpha}} \le \frac{2^\alpha}{R^\alpha}\,\frac{1}{|x|^n}, \qquad x \in \mathbb{R}^n,\ x \ne 0,$$

so that we can choose $A_1 = 2n\omega_n R^{-\alpha}$ in the *size estimate* (2.4.1) in [**48**]. We also have

$$|\nabla K_{\alpha,R}(x)| \le C\left(\frac{1}{R}\frac{\left|\eta'\left(\frac{|x|}{R}\right)\right|}{|x|^{n+\alpha}} + \frac{1-\eta_R(x)}{|x|^{n+\alpha+1}}\right) \le 4c_n\,\frac{2^\alpha}{R^\alpha}\,\frac{1}{|x|^{n+1}}, \qquad x \in \mathbb{R}^n,\ x \ne 0,$$

where $c_n > 0$ is a dimensional constant, so that we can choose $A_2 = c_n' R^{-\alpha}$ in the *smoothness condition* (2.4.2) in [**48**], where $c_n' > c_n$ is another dimensional constant. Finally, since clearly

$$\int_{m<|x|<M} K_{\alpha,R}(x)\,dx = 0$$

for all $m < M$, we can choose $A_3 = 0$ in the *cancellation condition* (2.4.3) in [**48**]. Since $A_1 + A_2 + A_3 = c_n'' R^{-\alpha}$ for some dimensional constant $c_n'' \ge c_n'$, the conclusion follows. □

With Lemma 4.3 at our disposal, we can prove the following result.

Theorem 4.4 ($H^1 - BV^\alpha$ interpolation inequality)**.** *Let $\alpha \in (0,1]$. There exists a constant $c_{n,\alpha} > 0$ such that*

$$[f]_{BV^\beta(\mathbb{R}^n)} \le c_{n,\alpha}\,\|f\|_{H^1(\mathbb{R}^n)}^{(\alpha-\beta)/\alpha}\,[f]_{BV^\alpha(\mathbb{R}^n)}^{\beta/\alpha} \tag{4.6}$$

for all $\beta \in [0,\alpha)$ and all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$.

Proof. Let $\alpha \in (0,1]$ be fixed. Thanks to Theorem 1.40, the case $\beta = 0$ is trivial, so we assume $\beta \in (0,\alpha)$. We divide the proof in three steps.

Step 1. Let $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$ and assume $f \in \mathrm{Lip}_b(\mathbb{R}^n)$. By Lemma 3.8, we can write

(4.7)

$$\begin{aligned}|\nabla^\beta f(x)| &= \mu_{n,\beta}\left|\int_{\mathbb{R}^n} \frac{y\cdot(f(y+x)-f(x))}{|y|^{n+\beta+1}}\,dy\right| \\ &= \mu_{n,\beta}\left|\int_{\mathbb{R}^n} \eta_R(y)\,\frac{y\cdot(f(y+x)-f(x))}{|y|^{n+\beta+1}}\,dy + \int_{\mathbb{R}^n}(1-\eta_R(y))\,\frac{y\cdot(f(y+x)-f(x))}{|y|^{n+\beta+1}}\,dy\right|\end{aligned}$$

for all $x \in \mathbb{R}^n$ and all $R > 0$. On the one hand, for $\alpha < 1$, by Proposition 1.23 we can estimate

(4.8)
$$\begin{aligned}\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \eta_R(y) \, \frac{y \cdot (f(y+x) - f(x))}{|y|^{n+\beta+1}} \, dy \right| dx &\le \int_{B_R} \frac{1}{|y|^{n+\beta}} \int_{\mathbb{R}^n} |f(y+x) - f(x)| \, dx \, dy \\ &\le \gamma_{n,\alpha} \, |D^\alpha f|(\mathbb{R}^n) \int_{B_R} \frac{dy}{|y|^{n+\beta-\alpha}} \\ &= \gamma_{n,\alpha} \, \frac{R^{\alpha-\beta}}{\alpha-\beta} \, |D^\alpha f|(\mathbb{R}^n)\end{aligned}$$

for all $R > 0$, where $\gamma_{n,\alpha} > 0$ is a constant depending only on n and α (note that the validity of Proposition 1.23 for all $f \in BV^\alpha(\mathbb{R}^n)$ follows by a simple approximation argument, thanks to Theorem 1.16). If $\alpha = 1$ instead, we simply have

$$\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \eta_R(y) \, \frac{y \cdot (f(y+x) - f(x))}{|y|^{n+\beta+1}} \, dy \right| dx \le \frac{R^{1-\beta}}{1-\beta} \, |D^\alpha f|(\mathbb{R}^n),$$

for all $R > 0$. On the other hand, by Lemma 4.3 we have

(4.9)
$$\begin{aligned}&\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (1 - \eta_R(y)) \, \frac{y \cdot (f(y+x) - f(x))}{|y|^{n+\beta+1}} \, dy \right| dx \\ &\qquad = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (1 - \eta_R(y)) \, \frac{y \cdot f(y+x)}{|y|^{n+\beta+1}} \, dy \right| dx \\ &\qquad \le \tau_n R^{-\beta} \|f\|_{H^1(\mathbb{R}^n)}\end{aligned}$$

for all $R > 0$, where $\tau_n > 0$ is the constant of Lemma 4.3. Combining the above estimates, we get

$$\begin{aligned}|D^\beta f|(\mathbb{R}^n) &\le \mu_{n,\beta} \left(\gamma_{n,\alpha} \, \frac{R^{\alpha-\beta}}{\alpha-\beta} \, [f]_{BV^\alpha(\mathbb{R}^n)} + \tau_n R^{-\beta} \, \|f\|_{H^1(\mathbb{R}^n)} \right) \\ &\le \mu_{n,\beta} \max\{\tau_n, \gamma_{n,\alpha}\} \left(\frac{R^{\alpha-\beta}}{\alpha-\beta} \, [f]_{BV^\alpha(\mathbb{R}^n)} + R^{-\beta} \, \|f\|_{H^1(\mathbb{R}^n)} \right)\end{aligned}$$

for all $R > 0$, where we have set $\gamma_{n,1} = 1$ by convention. Assuming $[f]_{BV^\alpha(\mathbb{R}^n)} \neq 0$ without loss of generality and choosing $R = \|f\|_{H^1(\mathbb{R}^n)}^{1/\alpha} \, [f]_{BV^\alpha(\mathbb{R}^n)}^{-1/\alpha}$, we get

$$|D^\beta f|(\mathbb{R}^n) \le \frac{2\mu_{n,\beta} \max\{\tau_n, \gamma_{n,\alpha}\}}{\alpha - \beta} \, \|f\|_{H^1(\mathbb{R}^n)}^{(\alpha-\beta)/\alpha} \, [f]_{BV^\alpha(\mathbb{R}^n)}^{\beta/\alpha} \tag{4.10}$$

for all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$ such that $f \in \mathrm{Lip}_b(\mathbb{R}^n)$. Using a standard approximation argument via convolution, thanks to Proposition 1.11, inequality (4.10) follows for all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$.

Step 2. If $\alpha < 1$, then, by Proposition 3.19, we know that

$$|D^\beta f|(\mathbb{R}^n) \le c_{n,\alpha} \, \frac{\mu_{n,1+\beta-\alpha}}{n+\beta-\alpha} \left(\frac{R^{\alpha-\beta}}{\alpha-\beta} \, [f]_{BV^\alpha(\mathbb{R}^n)} + \frac{R^{-\beta}}{\beta} \|f\|_{L^1(\mathbb{R}^n)} \right) \tag{4.11}$$

for all $f \in BV^\alpha(\mathbb{R}^n)$ and all $R > 0$, where $c_{n,\alpha} > 0$ is a constant depending only on n and α such that

$$c_{n,1} = \lim_{\alpha \to 1^-} c_{n,\alpha} < +\infty.$$

If $\alpha = 1$, then, by Proposition 3.14(i), inequality (4.11) holds with $\alpha = 1$ for all $f \in BV(\mathbb{R}^n)$. Assuming $\|f\|_{L^1(\mathbb{R}^n)} \neq 0$, choosing $R = [f]^{1/\alpha}_{BV^\alpha(\mathbb{R}^n)} \|f\|^{-1/\alpha}_{L^1(\mathbb{R}^n)}$ and using the inequality $\|f\|_{L^1(\mathbb{R}^n)} \le \|f\|_{H^1(\mathbb{R}^n)}$, we can estimate

$$|D^\beta f|(\mathbb{R}^n) \le \frac{c_{n,\alpha}}{\beta(\alpha-\beta)} \frac{\mu_{n,1+\beta-\alpha}}{n+\beta-\alpha} \|f\|^{(\alpha-\beta)/\alpha}_{H^1(\mathbb{R}^n)} [f]^{\beta/\alpha}_{BV^\alpha(\mathbb{R}^n)} \tag{4.12}$$

for all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$.

Step 3. Combining (4.10) and (4.12), we get

$$|D^\beta f|(\mathbb{R}^n) \le \varphi(\alpha,\beta) \|f\|^{(\alpha-\beta)/\alpha}_{H^1(\mathbb{R}^n)} [f]^{\beta/\alpha}_{BV^\alpha(\mathbb{R}^n)}$$

for all $f \in H^1(\mathbb{R}^n) \cap BV^\alpha(\mathbb{R}^n)$, where

$$\varphi(\alpha,\beta) = \min\left\{\frac{2\mu_{n,\beta}\max\{\tau_n, \gamma_{n,\alpha}\}}{\alpha-\beta}, \frac{c_{n,\alpha}}{\beta(\alpha-\beta)} \frac{\mu_{n,1+\beta-\alpha}}{n+\beta-\alpha}\right\}, \quad 0 < \beta < \alpha \le 1.$$

We observe that

$$\lim_{\beta \to \alpha^-} \varphi(\alpha,\beta) = \frac{c_{n,\alpha}}{\alpha n} \lim_{\beta \to \alpha^-} \frac{\mu_{n,1+\beta-\alpha}}{\alpha-\beta} = \frac{c_{n,\alpha}}{\alpha n \omega_n}$$

by Lemma 3.21 and that

$$\lim_{\beta \to 0^+} \varphi(\alpha,\beta) = \frac{2\mu_{n,0}\max\{\tau_n, \gamma_{n,\alpha}\}}{\alpha}.$$

The conclusion thus follows again by Lemma 3.21. □

Remark 4.5 ($H^1 - W^{\alpha,1}$ interpolation inequality)**.** Thanks to Theorem 1.27, from Theorem 4.4 one can replace the BV^α-seminorm in the right-hand side of (4.6) with the $W^{\alpha,1}$-seminorm up to multiply the constant $c_{n,\alpha}$ by $\mu_{n,\alpha}$. However, one can prove a slightly finer estimate essentially following the proof of Theorem 4.4. Indeed, for any given $f \in H^1(\mathbb{R}^n) \cap W^{\alpha,1}(\mathbb{R}^n)$ sufficiently regular, one writes $\nabla^\beta f$ as in (4.7) and estimate the second part of it as in (4.9). To estimate the first term, instead of following (4.8), one simply notes that

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} \eta_R(y)\,\frac{y\cdot(f(y+x)-f(x))}{|y|^{n+\beta+1}}\,dy\right|dx &\le \int_{\mathbb{R}^n}\int_{B_R} \frac{|f(y+x)-f(x)|}{|y|^{n+\beta}}\,dy\,dx \\ &\le R^{\alpha-\beta}\int_{\mathbb{R}^n}\int_{B_R} \frac{|f(y+x)-f(x)|}{|y|^{n+\alpha}}\,dy\,dx \\ &\le R^{\alpha-\beta}\,[f]_{W^{\alpha,1}(\mathbb{R}^n)}\end{aligned}$$

for all $R > 0$. Hence

$$|D^\beta f|(\mathbb{R}^n) \le \mu_{n,\beta}\Big(R^{\alpha-\beta}\,[f]_{W^{\alpha,1}(\mathbb{R}^n)} + \tau_n R^{-\beta}\,\|f\|_{H^1(\mathbb{R}^n)}\Big)$$

for all $R > 0$, and the desired inequality follows by optimising the right-hand side.

2.2. The cases $p > 1$ and H^1 via the Mihlin–Hörmander Multiplier Theorem. Let $0 \le \beta \le \alpha \le 1$ and consider the function

$$m_{\alpha,\beta}(\xi) = \frac{|\xi|^\beta}{1 + |\xi|^\alpha}, \qquad \xi \in \mathbb{R}^n.$$

It is not difficult to see that

$$\|m_{\alpha,\beta}\|_\star = \sup_{\mathrm{a}\in\mathbb{N}_0^n,\ |\mathrm{a}|\le\left\lfloor\frac{n}{2}\right\rfloor+1}\ \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \left|\xi^{\mathrm{a}}\, \partial_\xi^{\mathrm{a}}\, m_{\alpha,\beta}(\xi)\right| < +\infty.$$

We thus define the convolution operator $T_{m_{\alpha,\beta}} \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ with convolution kernel given by $\mathcal{F}^{-1}(m_{\alpha,\beta})$, i.e.,

$$T_{m_{\alpha,\beta}} f = \mathcal{F}(f * \mathcal{F}^{-1}(m_{\alpha,\beta})), \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{4.13}$$

In the following result, we observe that the convolution operator $T_{m_{\alpha,\beta}}$ satisfies the Mihlin–Hörmander properties uniformly with respect to the parameters $0 \le \beta \le \alpha \le 1$.

Lemma 4.6 (Mihlin–Hörmander estimates for $T_{m_{\alpha,\beta}}$)**.** *There is a dimensional constant $\sigma_n > 0$ such that the following properties hold for all given $0 \le \beta \le \alpha \le 1$.*

(i) For all given $p \in (1, +\infty)$, the operator in (4.13) uniquely extends to a bounded linear operator $T_{m_{\alpha,\beta}} \colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ with

$$\|T_{m_{\alpha,\beta}} f\|_{L^p(\mathbb{R}^n)} \le \sigma_n \max\left\{p, \frac{1}{p-1}\right\} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$.

(ii) The operator in (4.13) uniquely extends to a bounded linear operator $T_{m_{\alpha,\beta}} \colon H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ with

$$\|T_{m_{\alpha,\beta}} f\|_{H^1(\mathbb{R}^n)} \le \sigma_n \|f\|_{H^1(\mathbb{R}^n)}$$

for all $f \in H^1(\mathbb{R}^n)$.

Proof. Statements (i) and (ii) follow from the Mihlin–Hörmander Multiplier Theorem, see [**47**, Theorem 6.2.7] for the L^p-continuity and [**46**, Chapter III, Theorem 7.30] for the H^1-continuity, where

$$\sigma_n = \sup_{0\le\beta\le\alpha\le1} \|m_{\alpha,\beta}\|_\star < +\infty.$$

We leave the simple verifications to the interested reader. □

With Lemma 4.6 at our disposal, we can prove the following result.

Theorem 4.7 (Bessel and fractional Hardy–Sobolev interpolation inequalities)**.** *The following statements hold.*

(i) Given $p \in (1, +\infty)$, there exists a constant $c_{n,p} > 0$ such that

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \le c_{n,p}\, \|\nabla^\gamma f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}}\, \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}} \tag{4.14}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in S^{\alpha,p}(\mathbb{R}^n)$. In the case $\gamma = 0$, we also have

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \le c_{n,p}\, \|f\|_{L^p(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}}\, \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\beta}{\alpha}} \tag{4.15}$$

for all $0 \le \beta \le \alpha \le 1$ and all $f \in S^{\alpha,p}(\mathbb{R}^n)$.

(ii) There exists a dimensional constant $c_n > 0$ such that

$$\|\nabla^\beta f\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)} \le c_n \,\|\nabla^\gamma f\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \,\|\nabla^\alpha f\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}} \tag{4.16}$$

for all $0 \le \gamma \le \beta \le \alpha \le 1$ and all $f \in HS^{\alpha,1}(\mathbb{R}^n)$. In the case $\gamma = 0$, we also have

$$\|\nabla^\beta f\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)} \le c_n \,\|f\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} \,\|\nabla^\alpha f\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\beta}{\alpha}} \tag{4.17}$$

for all $0 \le \beta \le \alpha \le 1$ and all $f \in HS^{\alpha,1}(\mathbb{R}^n)$.

Proof. Without loss of generality, we can directly assume that $0 \le \gamma < \beta < \alpha \le 1$. We prove the two statements separately.

Proof of (i). Given $f \in S^{\alpha,p}(\mathbb{R}^n)$, we can write

$$(-\Delta)^{\frac{\beta}{2}} f = \mathcal{F}^{-1}(m_{\alpha,\beta}) * \Big((I + (-\Delta)^{\frac{\alpha}{2}})f\Big) = T_{m_{\alpha,\beta}}\Big((I + (-\Delta)^{\frac{\alpha}{2}})f\Big),$$

so that

$$\begin{aligned}
\|(-\Delta)^{\frac{\beta}{2}} f\|_{L^p(\mathbb{R}^n)} &= \Big\| T_{m_{\alpha,\beta}}\Big((I + (-\Delta)^{\frac{\alpha}{2}})f\Big)\Big\|_{L^p(\mathbb{R}^n)} \\
&\le \sigma_n \max\left\{p, \frac{1}{p-1}\right\} \|f + (-\Delta)^{\frac{\alpha}{2}} f\|_{L^p(\mathbb{R}^n)} \\
&\le \sigma_n \max\left\{p, \frac{1}{p-1}\right\} \Big(\|f\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^p(\mathbb{R}^n)}\Big) \qquad (4.18)
\end{aligned}$$

thanks to Lemma 4.6(i). Now let $f \in C_c^\infty(\mathbb{R}^n)$. Since

$$(-\Delta)^{\frac{\alpha}{2}} \nabla^\gamma f = R\,(-\Delta)^{\frac{\alpha+\gamma}{2}} f \in L^p(\mathbb{R}^n;\mathbb{R}^n)$$

because $f \in L^{\alpha+\gamma,p}(\mathbb{R}^n)$ and by the L^p-continuity property of the Riesz transform, we get that $\nabla^\gamma f \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^n)$ according to the definition given in (N.62) and the identification established in Corollary 1.52. By applying (4.18) to the components of the function $\nabla^\gamma f \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^n)$ with exponents $\alpha - \gamma$ and $\beta - \gamma$ in place of α and β respectively, we get

$$\begin{aligned}
\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} &= \|(-\Delta)^{\frac{\beta-\gamma}{2}} \nabla^\gamma f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \\
&\le \sigma_n \max\left\{p, \frac{1}{p-1}\right\} \Big(\|\nabla^\gamma f\|_{L^p(\mathbb{R}^n)} + \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n)}\Big)
\end{aligned}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. By performing a dilation and by optimising the right-hand side, we find that

$$\|\nabla^\beta f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \le c_{n,p} \,\|\nabla^\gamma f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \,\|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$, where $c_{n,p} > 0$ is a constant depending only on n and p. Thanks to Theorem 1.51, Proposition 1.43 and Proposition 3.20, inequality (4.14) follows by performing a standard approximation argument.

In the case $\gamma = 0$, inequality (4.15) follows from (4.14) by the L^p-continuity of the Riesz transform. This concludes the proof of (i).

Proof of (ii). Given $f \in HS^{\alpha,1}(\mathbb{R}^n)$, arguing as above, we can write

$$(-\Delta)^{\frac{\beta}{2}} f = \mathcal{F}^{-1}(m_{\alpha,\beta}) * \Big((I + (-\Delta)^{\frac{\alpha}{2}}) f\Big) = T_{m_{\alpha,\beta}}\Big((I + (-\Delta)^{\frac{\alpha}{2}}) f\Big),$$

so that

$$\|(-\Delta)^{\frac{\beta}{2}} f\|_{H^1(\mathbb{R}^n)} \le \sigma_n \left(\|f\|_{H^1(\mathbb{R}^n)} + \|(-\Delta)^{\frac{\alpha}{2}} f\|_{H^1(\mathbb{R}^n)}\right) \tag{4.19}$$

thanks to Lemma 4.6(ii). Now let $f \in C_c^\infty(\mathbb{R}^n)$. Note that $\nabla^\gamma f \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, because $\nabla^\gamma f \in L^1(\mathbb{R}^n; \mathbb{R}^n)$ and

$$\operatorname{div}^0 \nabla^\gamma f = \operatorname{div}^0 R (-\Delta)^{\frac{\gamma}{2}} f = (-\Delta)^{\frac{\gamma}{2}} f \in H^1(\mathbb{R}^n)$$

by Proposition 1.41(ii). Moreover,

$$(-\Delta)^{\frac{\alpha}{2}} \nabla^\gamma f = R\,(-\Delta)^{\frac{\alpha+\gamma}{2}} f \in H^1(\mathbb{R}^n; \mathbb{R}^n)$$

because $f \in HS^{\alpha+\gamma,1}(\mathbb{R}^n)$ and by the H^1-continuity property of the Riesz transform. Thus $\nabla^\gamma f \in HS^{\alpha,1}(\mathbb{R}^n; \mathbb{R}^n)$. By applying (4.19) to the components of the function $\nabla^\gamma f \in HS^{\alpha,1}(\mathbb{R}^n; \mathbb{R}^n)$ with exponents $\alpha - \gamma$ and $\beta - \gamma$ in place of α and β respectively, by arguing as above we get

$$\begin{aligned}\|(-\Delta)^{\frac{\beta}{2}} f\|_{H^1(\mathbb{R}^n)} &= \|(-\Delta)^{\frac{\beta-\gamma}{2}} \nabla^\gamma f\|_{H^1(\mathbb{R}^n; \mathbb{R}^n)} \\ &\le c_n \left(\|\nabla^\gamma f\|_{H^1(\mathbb{R}^n; \mathbb{R}^n)} + \|\nabla^\alpha f\|_{H^1(\mathbb{R}^n; \mathbb{R}^n)}\right)\end{aligned}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. By performing a dilation and by optimising the right-hand side, we find that

$$\|\nabla^\beta f\|_{H^1(\mathbb{R}^n; \mathbb{R}^n)} \le \sigma_n \, \|\nabla^\gamma f\|_{H^1(\mathbb{R}^n; \mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha-\gamma}} \, \|\nabla^\alpha f\|_{H^1(\mathbb{R}^n; \mathbb{R}^n)}^{\frac{\beta-\gamma}{\alpha-\gamma}}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$, where $c_n > 0$ is a dimensional constant. Thanks to Lemma 4.1, inequality (4.16) follows by performing a standard approximation argument.

In the case $\gamma = 0$, inequality (4.17) follows from (4.14) by the H^1-continuity of the Riesz transform. This concludes the proof of (ii). □

3. Asymptotic behaviour of fractional α-variation as $\alpha \to 0^+$

In this section, we study the asymptotic behaviour of ∇^α as $\alpha \to 0^+$.

3.1. Pointwise convergence of ∇^α as $\alpha \to 0^+$. We start with the pointwise convergence of ∇^α to ∇^0 as $\alpha \to 0^+$ for sufficiently regular functions.

Lemma 4.8 (Pointwise convergence of ∇^α as $\alpha \to 0^+$)**.** *Let $\alpha \in (0,1]$ and $p \in (1,+\infty)$. For $\beta \in [0,\alpha)$, the operator*

$$\nabla^\beta \colon C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \to L^\infty_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$$

defined as

$$\nabla^\beta f(x) = \mu_{n,\beta} \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \frac{y \cdot f(y+x)}{|y|^{n+\beta+1}} \, dy, \qquad x \in \mathbb{R}^n, \tag{4.20}$$

for all $f \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, is well defined and satisfies

$$\|\nabla^\beta f\|_{L^\infty(B_R; \mathbb{R}^n)} \le c_{n,p} \mu_{n,\beta} \left(\frac{r^{\alpha-\beta}}{\alpha-\beta} \, [f]_{C^{0,\alpha}(B_{R+r})} + r^{-\frac{n}{p}-\beta} \, \|f\|_{L^p(\mathbb{R}^n)}\right) \tag{4.21}$$

for all $r, R > 0$, where $c_{n,p} > 0$ is a constant depending only on n and p. In addition, it holds

$$\lim_{\beta \to 0^+} \nabla^\beta f(x) = \nabla^0 f(x) \tag{4.22}$$

for all $x \in \mathbb{R}^n$.

Proof. Given $f \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we can estimate

$$\begin{aligned}\left| \int_{|y|>\varepsilon} \frac{y \cdot f(y+x)}{|y|^{n+\beta+1}}\, dy \right| &\le \int_{\varepsilon<|y|<r} \frac{|f(y+x) - f(x)|}{|y|^{n+\beta}}\, dy + \int_{|y|\ge r} \frac{|f(y+x)|}{|y|^{n+\beta}} \\ &\le [f]_{C^{0,\alpha}(B_r(x))} \int_{|y|<r} \frac{dy}{|y|^{n+\beta-\alpha}} + \|f\|_{L^p(\mathbb{R}^n)} \left(\int_{|y|\ge r} \frac{dy}{|y|^{(n+\beta)q}} \right)^{\frac{1}{q}} \\ &\le \frac{n\omega_n r^{\alpha-\beta}}{\alpha-\beta} [f]_{C^{0,\alpha}(B_r(x))} + \left(\frac{n\omega_n r^{n-(n+\beta)q}}{(n+\beta)q-n} \right)^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

for all $r > \varepsilon > 0$ and $\beta \in [0, \alpha)$, where $q = \frac{p}{p-1}$. Thus the limit in (4.20) is well posed and (4.21) follows. In addition, since $\mu_{n,\beta} \to \mu_{n,0}$ as $\beta \to 0^+$ and

$$\begin{aligned}|\nabla^\beta f(x) - \nabla^0 f(x)| \le \left| 1 - \frac{\mu_{n,\beta}}{\mu_{n,0}} \right| |\nabla^0 f(x)| + \mu_{n,\beta} [f]_{C^{0,\alpha}(B_1(x))} \int_{|y|<1} \left(\frac{1}{|y|^\beta} - 1 \right) \frac{dy}{|y|^{n-\alpha}} \\ + \mu_{n,\beta} \int_{|y|>1} \left(1 - \frac{1}{|y|^\beta} \right) \frac{|f(y+x)|}{|y|^n}\, dy\end{aligned}$$

for all $\beta \in (0, \alpha)$ and $x \in \mathbb{R}^n$, the limit in (4.22) follows by the Monotone Convergence Theorem and Lebesgue's Dominated Convergence Theorem. □

As an immediate consequence of Lemma 4.8, we can show that the fractional α-variation is lower semicontinuous as $\alpha \to 0^+$.

Corollary 4.9 (Lower semicontinuity of BV^α-seminorm as $\alpha \to 0^+$)**.** *If $f \in L^1(\mathbb{R}^n)$, then*

$$|D^0 f|(\mathbb{R}^n) \le \liminf_{\alpha \to 0^+} |D^\alpha f|(\mathbb{R}^n). \tag{4.23}$$

Proof. Given $\varphi \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1$, thanks to Lemma 4.8 and Corollary 1.3, we have

$$\int_{\mathbb{R}^n} f \operatorname{div}^0 \varphi\, dx = \lim_{\alpha \to 0^+} \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi\, dx \le \liminf_{\alpha \to 0^+} |D^\alpha f|(\mathbb{R}^n),$$

by Lebesgue's Dominated Convergence Theorem, so that (4.23) follows by (1.65). □

3.2. Strong and energy convergence of ∇^α as $\alpha \to 0^+$. We now study the strong and the energy convergence of ∇^α as $\alpha \to 0^+$.

For the strong convergence, we have the following result. See Section 3.3 for the proof.

Theorem 4.10 (Strong convergence of ∇^α as $\alpha \to 0^+$)**.** *The following hold.*

(i) If $f \in \bigcup_{\alpha \in (0,1)} HS^{\alpha,1}(\mathbb{R}^n)$, then

$$\lim_{\alpha \to 0^+} \|\nabla^\alpha f - Rf\|_{H^1(\mathbb{R}^n;\mathbb{R}^n)} = 0. \tag{4.24}$$

(ii) If $p \in (1,+\infty)$ and $f \in \bigcup_{\alpha\in(0,1)} S^{\alpha,p}(\mathbb{R}^n)$, then

$$\lim_{\alpha\to 0^+} \|\nabla^\alpha f - Rf\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0. \tag{4.25}$$

For the convergence of the (rescaled) energy, we instead have the following result. See Section 3.4 for the proof.

Theorem 4.11 (Energy convergence of ∇^α as $\alpha \to 0^+$)**.** *If $f \in \bigcup_{\alpha\in(0,1)} W^{\alpha,1}(\mathbb{R}^n)$, then*

$$\lim_{\alpha\to 0^+} \alpha \int_{\mathbb{R}^n} |\nabla^\alpha f|\,dx = n\omega_n \mu_{n,0} \left|\int_{\mathbb{R}^n} f\,dx\right|.$$

3.3. Proof of Theorem 4.10. Before the proof of Theorem 4.10, we need to recall the following well-known result, see the first part of the proof of [**42**, Lemma 1.60]. For the reader's convenience and to keep the paper the most self-contained as possible, we briefly recall its simple proof.

Lemma 4.12. *Let $m \in \mathbb{N}_0$. If $f \in \mathcal{S}_m(\mathbb{R}^n)$, then $f = \operatorname{div} g$ for some $g \in \mathcal{S}_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ (with $g \in \mathcal{S}(\mathbb{R}^n;\mathbb{R}^n)$ in the case $m = 0$).*

Proof. Applying Fourier transform everywhere, the problem can be equivalently restated as follows: if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\partial^{\mathsf{a}}\varphi(0) = 0$ for all $\mathsf{a} \in \mathbb{N}_0^n$ such that $|\mathsf{a}| \le m$, then $\varphi = \sum_1^n \xi_i \psi_i(\xi)$ for some $\psi_1, \dots, \psi_n \in \mathcal{S}(\mathbb{R}^n)$ with $\partial^{\mathsf{a}}\psi_i(0) = 0$ for all $i = 1, \dots, n$ and all $\mathsf{a} \in \mathbb{N}_0^n$ such that $|\mathsf{a}| \le m - 1$. This can be achieved as follows. Fixed any $\zeta \in C_c^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\zeta \subset B_2 \quad \text{and} \quad \zeta = 1 \text{ on } B_1,$$

we can define

$$\psi_i(\xi) = \zeta(\xi) \int_0^1 \partial_i\varphi(t\xi)\,dt + \frac{1-\zeta(\xi)}{|\xi|^2}\,\xi_i\,\varphi(\xi), \qquad \xi \in \mathbb{R}^n,$$

for all $i = 1, \dots, n$. It is now easy to prove that such ψ_i's satisfy the required properties and we leave the simple calculations to the reader. □

Thanks to Lemma 4.12, we can prove the following L^p-convergence result of the fractional α-Laplacian and the fractional α-gradient as $\alpha \to 0^+$ for functions in $\mathcal{S}_0(\mathbb{R}^n)$.

Lemma 4.13. *Let $p \in [1,+\infty]$. If $f \in \mathcal{S}_0(\mathbb{R}^n)$, then*

$$\lim_{\alpha\to 0^+} \|(-\Delta)^{\frac{\alpha}{2}} f - f\|_{L^p(\mathbb{R}^n)} = 0. \tag{4.26}$$

As a consequence, if $p \in (1,+\infty)$ and $f \in \mathcal{S}_0(\mathbb{R}^n)$, then

$$\lim_{\alpha\to 0^+} \|\nabla^\alpha f - Rf\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = 0. \tag{4.27}$$

Proof. Let $f \in \mathcal{S}_0(\mathbb{R}^n)$ be fixed. If $p \in (1,+\infty)$, then

$$\|\nabla^\alpha f - Rf\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} = \|R(-\Delta)^{\frac{\alpha}{2}} f - Rf\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le c_{n,p}\|(-\Delta)^{\frac{\alpha}{2}} f - f\|_{L^p(\mathbb{R}^n)}$$

by the L^p-continuity of the Riesz transform, so that (4.27) follows from (4.26). To prove (4.26), given $x \in \mathbb{R}^n$ write

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = \nu_{n,\alpha} \int_{\{|h|>1\}} \frac{f(x+h)-f(x)}{|h|^{n+\alpha}}\,dh + \nu_{n,\alpha} \int_{\{|h|\le 1\}} \frac{f(x+h)-f(x)}{|h|^{n+\alpha}}\,dh,$$

where as usual

$$\nu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)}$$

for $\alpha \in (0,1)$. One easily sees that

$$\lim_{\alpha\to 0^+} \frac{\nu_{n,\alpha}}{\alpha} = -\frac{1}{n\omega_n}. \tag{4.28}$$

On the one hand, as in (1.71), we can estimate

$$\left\| \nu_{n,\alpha} \int_{\{|h|>1\}} \frac{f(\cdot + h) - f(\cdot)}{|h|^{n+\alpha}} \, dh \right\|_{L^p(\mathbb{R}^n)} \le \frac{n\omega_n \nu_{n,\alpha}}{1-\alpha} \|\nabla f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)},$$

so that

$$\lim_{\alpha\to 0^+} \left\| \nu_{n,\alpha} \int_{\{|h|>1\}} \frac{f(\cdot + h) - f(\cdot)}{|h|^{n+\alpha}} \, dh \right\|_{L^p(\mathbb{R}^n)} = 0$$

by (4.28) for all $p \in [1,+\infty]$. On the other hand, by Lemma 4.12 there exists $g \in \mathcal{S}(\mathbb{R}^n;\mathbb{R}^n)$ such that $f = \mathrm{div} g$ and thus we can write

$$\begin{aligned} \nu_{n,\alpha} \int_{\{|h|\le 1\}} \frac{f(x+h) - f(x)}{|h|^{n+\alpha}} \, dh &= \nu_{n,\alpha} \int_{\{|h|\le 1\}} \frac{f(x+h)}{|h|^{n+\alpha}} \, dh - \frac{n\omega_n \nu_{n,\alpha}}{\alpha} f(x) \\ &= \nu_{n,\alpha} \int_{\{|h|\le 1\}} \frac{\mathrm{div} g(x+h)}{|h|^{n+\alpha}} \, dh - \frac{n\omega_n \nu_{n,\alpha}}{\alpha} f(x). \end{aligned}$$

Integrating by parts, the reader can easily verify that

$$\lim_{\alpha\to 0^+} \left\| \nu_{n,\alpha} \int_{\{|h|\le 1\}} \frac{\mathrm{div} g(\cdot + h)}{|h|^{n+\alpha}} \, dh \right\|_{L^p(\mathbb{R}^n)} = 0$$

for all $p \in [1,+\infty]$. Hence we get

$$\lim_{\alpha\to 0^+} \|(-\Delta)^{\frac{\alpha}{2}} f - f\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} \lim_{\alpha\to 0^+} \left(1 + \frac{n\omega_n \nu_{n,\alpha}}{\alpha}\right) = 0$$

for all $p \in [1,+\infty]$. The proof is complete. □

We can now prove Theorem 4.10.

Proof of Theorem 4.10. We prove the two statements separately.

Proof of (i). Let $f \in HS^{\alpha,1}(\mathbb{R}^n)$. By Lemma 4.1, there exists $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}_0(\mathbb{R}^n)$ such that $f_k \to f$ in $HS^{\alpha,1}(\mathbb{R}^n)$ as $k \to +\infty$. If $\beta \in (0,\alpha)$, then we can estimate

$$\begin{aligned} \|\nabla^\beta f - Rf\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)} &\le \|\nabla^\beta f_k - Rf_k\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)} + \|\nabla^\beta f - \nabla^\beta f_k\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)} \\ &\quad + \|Rf - Rf_k\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)} \\ &\le \|\nabla^\beta f_k - Rf_k\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)} + c_n \|f - f_k\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} \|\nabla^\alpha f - \nabla^\alpha f_k\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)}^{\frac{\beta}{\alpha}} \\ &\quad + c_n' \|f - f_k\|_{H^1(\mathbb{R}^n)} \end{aligned}$$

for all $k \in \mathbb{N}$ by (4.17) in Theorem 4.7(ii) and the H^1-continuity of the Riesz transform, where $c_n, c_n' > 0$ are dimensional constants. Thus

$$\limsup_{\beta\to 0^+} \|\nabla^\beta f - Rf\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)} \le \limsup_{\beta\to 0^+} \|\nabla^\beta f_k - Rf_k\|_{H^1(\mathbb{R}^n;\,\mathbb{R}^n)} + c_n'' \|f - f_k\|_{H^1(\mathbb{R}^n)}$$

$$= c_n'' \|f - f_k\|_{H^1(\mathbb{R}^n)}$$

for all $k \in \mathbb{N}$ by (4.27) in Lemma 4.13, where $c_n'' = c_n + c_n'$. Hence (4.24) follows by passing to the limit as $k \to +\infty$ and the proof of (i) is complete.

Proof of (ii). We argue similarly as in the proof of (i). Let $f \in S^{\alpha,p}(\mathbb{R}^n)$. By Proposition 1.53, there exists $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}_0(\mathbb{R}^n)$ such that $f_k \to f$ in $S^{\alpha,p}(\mathbb{R}^n)$ as $k \to +\infty$. If $\beta \in (0, \alpha)$, then we can estimate

$$\begin{aligned}
\|\nabla^\beta f - Rf\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} &\le \|\nabla^\beta f_k - Rf_k\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} + \|\nabla^\beta f - \nabla^\beta f_k\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \\
&\quad + \|Rf - Rf_k\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \\
&\le \|\nabla^\beta f_k - Rf_k\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} + c_{n,p} \|f - f_k\|_{H^1(\mathbb{R}^n)}^{\frac{\alpha-\beta}{\alpha}} \|\nabla^\alpha f - \nabla^\alpha f_k\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta}{\alpha}} \\
&\quad + c_{n,p}' \|f - f_k\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

for all $k \in \mathbb{N}$ by (4.15) in Theorem 4.7(i) and the L^p-continuity of the Riesz transform, where the constants $c_{n,p}, c_{n,p}' > 0$ depend only on n and p. Thus

$$\begin{aligned}
\limsup_{\beta\to 0^+} \|\nabla^\beta f - Rf\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} &\le \limsup_{\beta\to 0^+} \|\nabla^\beta f_k - Rf_k\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} + c_{n,p}'' \|f - f_k\|_{L^p(\mathbb{R}^n)} \\
&= c_{n,p}'' \|f - f_k\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

for all $k \in \mathbb{N}$ by (4.26) in Lemma 4.13, where $c_{n,p}'' = c_{n,p} + c_{n,p}'$. Hence (4.25) follows by passing to the limit as $k \to +\infty$ and the proof of (ii) is complete. □

3.4. Proof of Theorem 4.11. We now pass to the proof of Theorem 4.11. We need some preliminaries. We begin with the following result.

Lemma 4.14. *Let $f \in L^1_c(\mathbb{R}^n)$ and let $R > 0$ be such that* $\operatorname{supp} f \subset B_R$. *If $\varepsilon > R$, then*

$$\lim_{\alpha\to 0^+} \alpha \mu_{n,\alpha} \int_{\mathbb{R}^n} \left| \int_{\{|y|>\varepsilon\}} \frac{y \cdot f(y+x)}{|y|^{n+\alpha+1}} \, dy \right| dx = n\omega_n \mu_{n,0} \left| \int_{\mathbb{R}^n} f \, dx \right|.$$

Proof. Since $\mu_{n,\alpha} \to \mu_{n,0}$ as $\alpha \to 0^+$, we just need to prove that

$$\lim_{\alpha\to 0^+} \alpha \int_{\mathbb{R}^n} \left| \int_{\{|y|>\varepsilon\}} \frac{y \cdot f(y+x)}{|y|^{n+\alpha+1}} \, dy \right| dx = n\omega_n \left| \int_{\mathbb{R}^n} f \, dx \right|. \tag{4.29}$$

We now divide the proof in two steps.

Step 1. We claim that

$$\lim_{\alpha\to 0^+} \alpha \int_{\mathbb{R}^n} \left| \int_{\{|y|>\varepsilon\}} \frac{x \cdot f(y+x)}{|x|^{n+\alpha+1}} \, dy \right| dx = n\omega_n \left| \int_{\mathbb{R}^n} f \, dx \right|. \tag{4.30}$$

Indeed, since $\operatorname{supp} f \subset B_R$, we have that

$$\int_{\{|y|>\varepsilon\}} \frac{x \cdot f(y+x)}{|x|^{n+\alpha+1}} \, dy = 0 \quad \text{for all } x \in \mathbb{R}^n \text{ such that } |x+y| \ge R.$$

Since $|y| > \varepsilon$, we have that

$$|x| \le \varepsilon - R \implies |x+y| \ge R \tag{4.31}$$

and thus we can write

$$\begin{aligned}\alpha \int_{\mathbb{R}^n} \left| \int_{|y|>\varepsilon} \frac{x \cdot f(y+x)}{|x|^{n+\alpha+1}} \, dy \right| dx &= \alpha \int_{\mathbb{R}^n} \frac{1}{|x|^{n+\alpha}} \left| \int_{|y|>\varepsilon} f(y+x) \, dy \right| dx \\ &= \alpha \int_{|x|>\varepsilon-R} \frac{1}{|x|^{n+\alpha}} \left| \int_{|y|>\varepsilon} f(y+x) \, dy \right| dx.\end{aligned}$$

Now, on the one hand, we have

$$\alpha \int_{\varepsilon-R<|x|\le\varepsilon+R} \frac{1}{|x|^{n+\alpha}} \left| \int_{|y|>\varepsilon} f(y+x) \, dy \right| dx \le \alpha n \omega_n \|f\|_{L^1(\mathbb{R}^n)} \int_{\varepsilon-R}^{\varepsilon+R} \frac{dr}{r^{\alpha+1}} \tag{4.32}$$

for all $\alpha \in (0,1)$. On the other hand, since

$$|x| > \varepsilon + R \implies B_R \subset B_\varepsilon(x)^c,$$

we have

$$\begin{aligned}\alpha \int_{|x|>\varepsilon+R} \frac{1}{|x|^{n+\alpha}} \left| \int_{|y|>\varepsilon} f(y+x) \, dy \right| dx &= \alpha \int_{|x|>\varepsilon+R} \frac{1}{|x|^{n+\alpha}} \left| \int_{\mathbb{R}^n} f \, dz \right| dx \\ &= \frac{n\omega_n}{(\varepsilon+R)^\alpha} \left| \int_{\mathbb{R}^n} f \, dz \right|\end{aligned} \tag{4.33}$$

for all $\alpha \in (0,1)$. Hence claim (4.30) follows by first combining (4.32) and (4.33) and then passing to the limit as $\alpha \to 0^+$.

Step 2. We claim that

$$\left| \frac{y}{|y|^{n+\alpha+1}} + \frac{x}{|x|^{n+\alpha+1}} \right| \le (n+3) \frac{|x+y|}{|y|^{n+\alpha+1}} \left(\frac{\varepsilon}{\varepsilon - R} \right)^{n+\alpha+1} \tag{4.34}$$

for all $x, y \in \mathbb{R}^n$ such that $|x| > \varepsilon - R$, $|y| > \varepsilon$ and $|y+x| < R$. Indeed, setting $F(z) = \frac{z}{|z|^{n+\alpha+1}}$ for all $z \in \mathbb{R}^n \setminus \{0\}$, we can estimate

$$\begin{aligned}\left| \frac{y}{|y|^{n+\alpha+1}} + \frac{x}{|x|^{n+\alpha+1}} \right| &= |F(y) - F(-x)| \le |y+x| \sup_{t\in[0,1]} |DF|((1-t)y - tx) \\ &\le (n+\alpha+2) \, |y+x| \sup_{t\in[0,1]} \frac{1}{|(1-t)y - tx|^{n+\alpha+1}}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{|(1-t)y - tx|^{n+\alpha+1}} &\le \frac{1}{||y| - t|y+x||^{n+\alpha+1}} \\ &\le \frac{1}{(|y| - R)^{n+\alpha+1}} \\ &\le \frac{1}{|y|^{n+\alpha+1}} \left(\frac{|y|}{|y| - R} \right)^{n+\alpha+1} \\ &\le \frac{1}{|y|^{n+\alpha+1}} \left(\frac{\varepsilon}{\varepsilon - R} \right)^{n+\alpha+1}\end{aligned}$$

for all $t \in [0,1]$, claim (4.34) immediately follows. Now, recalling (4.31), we can estimate

$$\begin{aligned}
&\left|\alpha \int_{\mathbb{R}^n} \left| \int_{|y|>\varepsilon} \frac{y \cdot f(y+x)}{|y|^{n+\alpha+1}}\, dy \right| dx - \alpha \int_{\mathbb{R}^n} \left| \int_{|y|>\varepsilon} \frac{x \cdot f(y+x)}{|x|^{n+\alpha+1}}\, dy \right| dx \right| \\
&\quad \le \alpha \int_{\mathbb{R}^n} \int_{|y|>\varepsilon} |f(y+x)| \left| \frac{y}{|y|^{n+\alpha+1}} + \frac{x}{|x|^{n+\alpha+1}} \right| dy\, dx \\
&\quad = \alpha \int_{|x|>\varepsilon - R} \int_{|y|>\varepsilon} |f(y+x)| \left| \frac{y}{|y|^{n+\alpha+1}} + \frac{x}{|x|^{n+\alpha+1}} \right| dy\, dx \\
&\quad \le \alpha(n+3) \left(\frac{\varepsilon - R}{\varepsilon - 2R} \right)^{n+\alpha+1} \int_{|x|>\varepsilon - R} \int_{|y|>\varepsilon} |f(y+x)| \, \frac{|y+x|}{|y|^{n+\alpha+1}}\, dy\, dx
\end{aligned}$$

for all $\alpha \in (0,1)$ thanks to (4.34). Since

$$\alpha \int_{|y|>\varepsilon} \frac{1}{|y|^{n+\alpha+1}} \int_{|x|>\varepsilon - R} |f(y+x)|\, |y+x| \, dx\, dy \le \alpha n \omega_n R \, \|f\|_{L^1(\mathbb{R}^n)} \int_{r>\varepsilon} \frac{dr}{r^{\alpha+2}},$$

we conclude that

$$\limsup_{\alpha \to 0^+} \left|\alpha \int_{\mathbb{R}^n} \left| \int_{|y|>\varepsilon} \frac{y \cdot f(y+x)}{|y|^{n+\alpha+1}}\, dy \right| dx - \alpha \int_{\mathbb{R}^n} \left| \int_{|y|>\varepsilon} \frac{x \cdot f(y+x)}{|x|^{n+\alpha+1}}\, dy \right| dx \right| = 0. \tag{4.35}$$

Thus (4.29) follows by combining (4.30) with (4.35) and the proof is complete. □

Thanks to Lemma 4.14, we can prove the following result.

Lemma 4.15. *Let $f \in L^1(\mathbb{R}^n)$ and $\eta > 0$. There exists $\varepsilon > 0$ such that*

$$\limsup_{\alpha \to 0^+} \left| \alpha \mu_{n,\alpha} \int_{\mathbb{R}^n} \left| \int_{|y|>\varepsilon} \frac{y \cdot f(y+x)}{|y|^{n+\alpha+1}}\, dy \right| dx - n\omega_n \mu_{n,0} \left| \int_{\mathbb{R}^n} f\, dx \right| \right| < \eta.$$

Proof. Let $\eta' > 0$ be such that $\eta = 2n\omega_n \mu_{n,0}\, \eta'$. Since $f \in L^1(\mathbb{R}^n)$, we can find $R > 0$ such that $\int_{B_R^c} |f|\, dx < \eta'$. Let $g = f\chi_{B_R} \in L^1_c(\mathbb{R}^n)$ and $\varepsilon > R$. Then

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \left| \int_{|y|>\varepsilon} \frac{y \cdot f(y+x)}{|y|^{n+\alpha+1}}\, dy \right| dx - \int_{\mathbb{R}^n} \left| \int_{|y|>\varepsilon} \frac{y \cdot g(y+x)}{|y|^{n+\alpha+1}}\, dy \right| dx \right| \\
&\quad \le \int_{|y|>\varepsilon} \frac{1}{|y|^{n+\alpha}}\, dy \int_{\mathbb{R}^n} |f(y+x) - g(y+x)|\, dx \\
&\quad = \frac{n\omega_n \|f-g\|_{L^1(\mathbb{R}^n)}}{\alpha \varepsilon^\alpha} < \frac{n\omega_n}{\alpha\varepsilon^\alpha}\, \eta'.
\end{aligned}$$

Since clearly

$$\left| \left| \int_{\mathbb{R}^n} f\, dx \right| - \left| \int_{\mathbb{R}^n} g\, dx \right| \right| \le \|f-g\|_{L^1(\mathbb{R}^n)} < \eta',$$

by Lemma 4.14 we conclude that

$$\begin{aligned}
&\limsup_{\alpha \to 0^+} \left| \alpha\, \mu_{n,\alpha} \int_{\mathbb{R}^n} \left| \int_{|y|>\varepsilon} \frac{y \cdot f(y+x)}{|y|^{n+\alpha+1}}\, dy \right| dx - n\omega_n\, \mu_{n,0} \left| \int_{\mathbb{R}^n} f\, dx \right| \right| \\
&\quad < \limsup_{\alpha \to 0^+} \left| \alpha\, \mu_{n,\alpha} \int_{\mathbb{R}^n} \left| \int_{|y|>\varepsilon} \frac{y \cdot g(y+x)}{|y|^{n+\alpha+1}}\, dy \right| dx - n\omega_n\, \mu_{n,0} \left| \int_{\mathbb{R}^n} g\, dx \right| \right|
\end{aligned}$$

$$+\left(n\omega_n\mu_{n,0}+n\omega_n\lim_{\alpha\to0^+}\mu_{n,\alpha}\varepsilon^{-\alpha}\right)\eta'$$
$$=2n\omega_n\mu_{n,0}\,\eta'=\eta$$

and the proof is complete. □

We are now ready to prove Theorem 4.11.

Proof of Theorem 4.11. Assume $f\in W^{\beta,1}(\mathbb{R}^n)$ for some $\beta\in(0,1)$ and fix $\eta>0$. By Lemma 4.15, there exists $\varepsilon>0$ such that

$$\limsup_{\alpha\to0^+}\left|\alpha\,\mu_{n,\alpha}\int_{\mathbb{R}^n}\left|\int_{|y|>\varepsilon}\frac{y\cdot f(y+x)}{|y|^{n+\alpha+1}}\,dy\right|dx-n\omega_n\mu_{n,0}\left|\int_{\mathbb{R}^n}f\,dx\right|\right|<\eta. \tag{4.36}$$

Since for all $\alpha\in(0,\beta)$ we can estimate

$$\begin{aligned}
&\left|\alpha\int_{\mathbb{R}^n}|\nabla^\alpha f|\,dx-n\omega_n\mu_{n,0}\left|\int_{\mathbb{R}^n}f\,dx\right|\right|\\
&\quad\le\left|\alpha\,\mu_{n,\alpha}\int_{\mathbb{R}^n}\left|\int_{|y|>\varepsilon}\frac{y\cdot f(y+x)}{|y|^{n+\alpha+1}}\,dy\right|dx-n\omega_n\mu_{n,0}\left|\int_{\mathbb{R}^n}f\,dx\right|\right|\\
&\qquad+\alpha\,\mu_{n,\alpha}\int_{\mathbb{R}^n}\int_{|y|\le\varepsilon}\frac{|f(y+x)-f(x)|}{|y|^{n+\alpha}}\,dy\,dx\\
&\quad\le\left|\alpha\,\mu_{n,\alpha}\int_{\mathbb{R}^n}\left|\int_{|y|>\varepsilon}\frac{y\cdot f(y+x)}{|y|^{n+\alpha+1}}\,dy\right|dx-n\omega_n\mu_{n,0}\left|\int_{\mathbb{R}^n}f\,dx\right|\right|+\alpha\,\mu_{n,\alpha}\,\varepsilon^{\beta-\alpha}[f]_{W^{\beta,1}(\mathbb{R}^n)},
\end{aligned}$$

by (4.36) we have

$$\limsup_{\alpha\to0^+}\left|\alpha\int_{\mathbb{R}^n}|\nabla^\alpha f|\,dx-n\omega_n\mu_{n,0}\left|\int_{\mathbb{R}^n}f\,dx\right|\right|<\eta$$

and the conclusion follows passing to the limit as $\eta\to0^+$. □

4. An application to potential estimates in Lorentz spaces

4.1. A fractional version of Meyers–Ziemer's trace inequality. The following result is a fractional generalisation of Meyers–Ziemer trace inequality, see [**64**], in the spirit of [**95**, Problem 7.1].

Theorem 4.16 (Fractional Meyers–Ziemer trace inequality)**.** *Let $\alpha\in(0,1)$. There exists a dimensional constant $c_n>0$ such that*

$$\int_{\mathbb{R}^n}|(I_{1-\alpha}f)^\star|\,d\mu\le c_n\|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)}|D^\alpha f|(\mathbb{R}^n) \tag{4.37}$$

for all $f\in BV^\alpha(\mathbb{R}^n)$ and all $\mu\in\mathcal{L}^{1,n-1}(\mathbb{R}^n)$.

Proof. Assume $f\in C_c^\infty(\mathbb{R}^n)$. We have $u=|I_{1-\alpha}f|\in\mathrm{Lip}_b(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n}|\nabla u|\,dx=\int_{\mathbb{R}^n}|\nabla^\alpha f|\,dx\le\mu_{n,\alpha}[f]_{W^{\alpha,1}(\mathbb{R}^n)}<+\infty.$$

Thus, $E_t=\{x\in\mathbb{R}^n:u(x)>t\}$ is an open set with finite perimeter for a.e. $t>0$. Since

$$\frac{|E_t\cap B_r(x)|}{|B_r(x)|}\le\frac{\min\{|E_t|,|B_r(x)|\}}{|B_r(x)|}$$

and

$$|E_t| = |\{x \in \mathbb{R}^n : |I_{1-\alpha} f| > t\}| \le c_{n,\alpha} \left(\frac{\|f\|_{L^1(\mathbb{R}^n)}}{t} \right)^{\frac{n}{n-1+\alpha}} < +\infty$$

by Hardy–Littlewood–Sobolev inequality, for all $x \in E_t$ the function $r \mapsto \frac{|E_t \cap B_r(x)|}{|B_r(x)|}$ is continuous, equals 1 for small $r > 0$ (since E_t is open) and tends to zero as $r \to +\infty$. Thus, arguing as in [**95**, Section 6], we can estimate

$$\mu(E_t) \le c_n |D\chi_{E_t}|(\mathbb{R}^n)$$

for a.e. $t > 0$. Therefore, applying the coarea formula to the function u, we find

$$\int_{\mathbb{R}^n} u \, d\mu = \int_{\mathbb{R}} \mu(E_t) \, dt \le c_n \int_{\mathbb{R}} |D\chi_{E_t}|(\mathbb{R}^n) \, dt = c_n \int_{\mathbb{R}^n} |\nabla u| \, dx = c_n |D^\alpha f|(\mathbb{R}^n),$$

proving (4.37) for all $f \in C_c^\infty(\mathbb{R}^n)$.

Now let $f \in BV^\alpha(\mathbb{R}^n)$. By Lemma 1.31(i) , $I_{1-\alpha} f \in bv(\mathbb{R}^n)$ with $DI_{1-\alpha} f = D^\alpha f$ in $\mathscr{M}(\mathbb{R}^n; \mathbb{R}^n)$. Let $(\varrho_\varepsilon)_{\varepsilon > 0} \subset C_c^\infty(\mathbb{R}^n)$ be a family of standard mollifies as in (1.25). By Remark 1.32, we have $\varrho_\varepsilon * I_{1-\alpha} f \in \mathrm{Lip}_b(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} |\nabla(\varrho_\varepsilon * I_{1-\alpha} f)| \, dx \le |D^\alpha f|(\mathbb{R}^n)$$

for all $\varepsilon > 0$. Since $\mu \ll \mathscr{H}^{n-1}$ and $(\varrho_\varepsilon * I_{1-\alpha} f)(x) \to (I_{1-\alpha} f)^\star(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$ as $\varepsilon \to 0^+$, the conclusion follows by Fatou's Lemma. □

As a simple consequence of Theorem 4.16 and the asymptotic analysis of the fractional operators, we get the following result.

Corollary 4.17 (Meyer–Ziemer trace inequalities)**.** *There exists a dimensional constant $c_n > 0$ with the following properties. Let $\mu \in \mathcal{L}^{1,n-1}(\mathbb{R}^n)$.*

(i) If $f \in BV(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |f^\star| \, d\mu \le c_n \|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)} |Df|(\mathbb{R}^n). \tag{4.38}$$

(ii) If $f \in H^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |(I_1 f)^\star| \, d\mu \le c_n \|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)} \|Rf\|_{L^1(\mathbb{R}^n)}. \tag{4.39}$$

Proof. We prove the two statements separately.

Proof of (i). Assume $f \in C_c^\infty(\mathbb{R}^n)$. By Proposition 3.23, we know that $I_{1-\alpha} f(x) \to f(x)$ for all $x \in \mathbb{R}^n$ as $\alpha \to 1-$. Hence, by Fatou's Lemma, Theorem 4.16 and Theorem 3.29, we get

$$\begin{aligned} \int_{\mathbb{R}^n} |f| \, d\mu &\le \liminf_{\alpha \to 1^-} \int_{\mathbb{R}^n} |I_{1-\alpha} f| \, d\mu \\ &\le c_n \|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)} \lim_{\alpha \to 1^-} |D^\alpha f|(\mathbb{R}^n) \\ &= c_n \|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)} |Df|(\mathbb{R}^n), \end{aligned}$$

proving (4.38) for all $f \in C_c^\infty(\mathbb{R}^n)$.

Now let $f \in BV(\mathbb{R}^n)$. There exists $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f^\star$ $\mathscr{H}^{n-1}$-a.e. in $\mathbb{R}^n$ and $|Df_k|(\mathbb{R}^n) \to |Df|(\mathbb{R}^n)$ as $k \to +\infty$. Since $\mu \ll \mathscr{H}^{n-1}$, again by Fatou's Lemma we get

$$\begin{aligned}\int_{\mathbb{R}^n} |f^\star|\, d\mu &\le \liminf_{k\to+\infty} \int_{\mathbb{R}^n} |f_k|\, d\mu \\ &\le c_n \|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)} \lim_{k\to+\infty} |Df_k|(\mathbb{R}^n) \\ &= c_n \|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)} |Df|(\mathbb{R}^n)\end{aligned}$$

and the conclusion follows.

Proof of (ii). Assume $f \in \mathrm{Lip}_b(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$. By the Dominated Convergence Theorem, we easily get that $I_{1-\alpha} f(x) \to I_1 f(x)$ for all $x \in \mathbb{R}^n$ as $\alpha \to 0^+$. Hence, by Fatou's Lemma, Theorem 4.16 and Theorem 4.10, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |I_1 f|\, d\mu &\le \liminf_{\alpha\to 0^+} \int_{\mathbb{R}^n} |I_{1-\alpha} f|\, d\mu \\ &\le c_n \|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)} \lim_{\alpha\to 0^+} |D^\alpha f|(\mathbb{R}^n) \\ &= c_n \|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)} \|Rf\|_{L^1(\mathbb{R}^n)},\end{aligned}$$

proving (4.39) for all $f \in \mathrm{Lip}_b(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$.

Now let $f \in H^1(\mathbb{R}^n)$ and define $f_\varepsilon = f * \varrho_\varepsilon$ for all $\varepsilon > 0$. Then $f_\varepsilon \in \mathrm{Lip}_b(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ and $f_\varepsilon \to f$ in $H^1(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Since $I_1 f \in bv(\mathbb{R}^n)$ by Proposition 1.41(i) and $I_1 f_\varepsilon = (I_1 f) * \varrho_\varepsilon$ for all $\varepsilon > 0$, we know that $I_1 f_\varepsilon(x) \to (I_1 f)^\star(x)$ as $\varepsilon \to 0^+$ for $\mathscr{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$. Since $\mu \ll \mathscr{H}^{n-1}$, again by Fatou's Lemma we get

$$\begin{aligned}\int_{\mathbb{R}^n} |(I_1 f)^*|\, d\mu &\le \liminf_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} |I_1 f_\varepsilon|\, d\mu \\ &\le c_n \|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)} \lim_{\varepsilon\to 0^+} \|Rf_\varepsilon\|_{L^1(\mathbb{R}^n)} \\ &= c_n \|\mu\|_{\mathcal{L}^{1,n-1}(\mathbb{R}^n)} \|Rf\|_{L^1(\mathbb{R}^n)}\end{aligned}$$

and the conclusion follows. □

We incidentally note that Corollary 4.17(i) can be proved exactly in the same way of Theorem 4.16, see [**95**, Section 6]. However, we decided to present (4.38) and (4.39) as consequences of (4.37) motivated by [**95**, Problem 7.1]. Also, note that Corollary 4.17(ii) positively answers [**95**, Problem 7.1] for the extremal case $\alpha = 1$.

4.2. Potential estimates. In the following result we prove the equivalence between three inequalities involving Riesz's potential, see [**93**, **94**].

Theorem 4.18 (Potential estimates)**.** *Let $n \ge 2$ and $\alpha \in (0,1)$. There exists a constant $c_{n,\alpha} > 0$ such that the following inequalities are equivalent (and hold with the same constant).*

(i) For all $f \in H^1(\mathbb{R}^n)$ it holds

$$\|I_\alpha f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha} \|Rf\|_{L^1(\mathbb{R}^n)} \tag{4.40}$$

(ii) For all $f \in BV^\alpha(\mathbb{R}^n)$ it holds

$$\|f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha} |D^\alpha f|(\mathbb{R}^n) \tag{4.41}$$

(iii) Given $\beta \in (0, \alpha)$, for all $f \in BV^\beta(\mathbb{R}^n)$ it holds

$$\|I_{\alpha-\beta} f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha} |D^\beta f|(\mathbb{R}^n). \tag{4.42}$$

Proof. Inequalities (4.40) and (4.41) were proved in [**93**] (see also [**94**]) and their equivalence was briefly explained in [**93**, Introduction]. For the reader's convenience, we prove all the implications.

Proof of (i) $\implies$ (ii). Let $f \in C_c^\infty(\mathbb{R}^n)$. By Proposition 1.41(ii), we know that $u = (-\Delta)^{\frac{\alpha}{2}} f \in H^1(\mathbb{R}^n)$ with $Ru = \nabla^\alpha f$ in $L^1(\mathbb{R}^n)$. Hence, by (4.40), we get that

$$\|f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} = \|I_\alpha u\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha} \|Ru\|_{L^1(\mathbb{R}^n)} = c_{n,\alpha} |D^\alpha f|(\mathbb{R}^n),$$

proving (4.42) for all $f \in C_c^\infty(\mathbb{R}^n)$.

Now let $f \in BV^\alpha(\mathbb{R}^n)$. By Theorem 1.16, we can find $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ a.e. in $\mathbb{R}^n$ and $|D^\alpha f_k|(\mathbb{R}^n) \to |D^\alpha f|(\mathbb{R}^n)$ as $k \to +\infty$. Thus, by Fatou's Lemma, we can estimate

$$\|f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le \liminf_{k\to+\infty} \|f_k\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha} \lim_{k\to+\infty} |D^\alpha f_k|(\mathbb{R}^n) = c_{n,\alpha} |D^\alpha f|(\mathbb{R}^n)$$

and the conclusion follows.

Proof of (ii) $\implies$ (i). Let $f \in C_c^\infty(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$. By Lemma 4.12, there exists $g \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ such that $f = \operatorname{div} g$, so that $u = I_\alpha f = \operatorname{div}^\alpha g \in L^1(\mathbb{R}^n)$. Hence, by Proposition 1.41(i), we get that $u \in BV^\alpha(\mathbb{R}^n)$ with $D^\alpha u = Rf$. Thus, by (4.41), we get

$$\|I_\alpha f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} = \|u\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha} \|Ru\|_{L^1(\mathbb{R}^n)} = c_{n,\alpha} |D^\alpha f|(\mathbb{R}^n),$$

proving (4.41) for all $f \in C_c^\infty(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$.

Now let $f \in H^1(\mathbb{R}^n)$. By [**97**, Chapter III, Section 5.2(b)], we can find $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$ and $\|Rf_k\|_{L^1(\mathbb{R}^n)} \to \|Rf\|_{L^1(\mathbb{R}^n)}$ as $k \to +\infty$. Since $I_\alpha f_k \to I_\alpha f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, possibly passing to a subsequence, by Fatou's Lemma we can estimate

$$\|I_\alpha f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le \liminf_{k\to+\infty} \|I_\alpha f_k\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha} \lim_{k\to+\infty} \|Rf_k\|_{L^1(\mathbb{R}^n)} = c_{n,\alpha} \|Rf\|_{L^1(\mathbb{R}^n)}$$

and the conclusion follows.

Proof of (i) $\implies$ (iii). Fix $\beta \in (0, \alpha)$ and $f \in C_c^\infty(\mathbb{R}^n)$. By Proposition 1.41(ii), we know that $u = (-\Delta)^{\frac{\beta}{2}} f \in H^1(\mathbb{R}^n)$ with $Ru = \nabla^\beta f$ in $L^1(\mathbb{R}^n)$. Hence, by (4.40), we get that

$$\|I_{\alpha-\beta} f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} = \|I_\alpha u\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha} \|Ru\|_{L^1(\mathbb{R}^n)} = c_{n,\alpha} |D^\beta f|(\mathbb{R}^n),$$

proving (4.42) for all $f \in C_c^\infty(\mathbb{R}^n)$.

Now let $f \in BV^\beta(\mathbb{R}^n)$. By Theorem 1.16, we can find $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$ and $|D^\beta f_k|(\mathbb{R}^n) \to |D^\beta f|(\mathbb{R}^n)$ as $k \to +\infty$. Since $I_{\alpha-\beta} f_k \to I_{\alpha-\beta} f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, possibly up to pass to a subsequence, by Fatou's Lemma we can estimate

$$\|I_{\alpha-\beta} f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le \liminf_{k\to+\infty} \|I_{\alpha-\beta} f_k\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha} \lim_{k\to+\infty} |D^\beta f_k|(\mathbb{R}^n) = c_{n,\alpha} |D^\beta f|(\mathbb{R}^n)$$

and the conclusion follows.

Proof of (iii) $\implies$ (i). Let $f \in C_c^\infty(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$, so that $f \in BV^\beta(\mathbb{R}^n)$ for all $\beta \in (0, \alpha)$. Since $I_{\alpha-\beta} f \to I_\alpha f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\beta \to 0^+$, there exists $(\beta_k)_{k\in\mathbb{N}} \subset (0, \alpha)$ with $\beta_k \to 0^+$ as $k \to +\infty$ such that $I_{\alpha-\beta_k} f \to I_\alpha f$ a.e. in $\mathbb{R}^n$. Thus, by Fatou's Lemma, inequality (4.42) and Theorem 4.10, we get

$$\begin{aligned}\|I_\alpha f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} &\le \liminf_{k\to+\infty} \|I_{\alpha-\beta_k} f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \\ &\le c_{n,\alpha} \lim_{k\to+\infty} \|D^{\beta_k} f\|_{L^1(\mathbb{R}^n)} = c_{n,\alpha} \|Rf\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)},\end{aligned}$$

proving (4.40) for all $f \in C_c^\infty(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$. The conclusion thus follows as in the last part of the proof of the implication (ii) $\implies$ (i).

Proof of (iii) $\implies$ (ii). Let $f \in C_c^\infty(\mathbb{R}^n)$, so that $f \in BV^\beta(\mathbb{R}^n)$ for all $\beta \in (0, \alpha)$. Since $I_{\alpha-\beta} f(x) \to f(x)$ for all $x \in \mathbb{R}^n$ as $\beta \to \alpha^-$ by Proposition 3.23, by Fatou's Lemma, inequality (4.42) and Theorem 3.42 we get

$$\|f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le \liminf_{\beta\to\alpha^-} \|I_{\alpha-\beta} f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha} \lim_{\beta\to\alpha^-} |D^\beta f|(\mathbb{R}^n) = c_{n,\alpha} |D^\alpha f|(\mathbb{R}^n),$$

proving (4.40) for all $f \in C_c^\infty(\mathbb{R}^n)$. The conclusion thus follows as in the last part of the proof of the implication (i) $\implies$ (ii). □

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