Attainable profiles for conservation laws with flux function spatially discontinuous at a single point

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Abstract

Consider a scalar conservation law with discontinuous flux

$$u_t + f(x,u)_x = 0, \qquad f(x,u) = \begin{cases} f_l(u) & \text{if } x < 0, \\ f_r(u) & \text{if } x > 0, \end{cases} \tag{1}$$

where $u = u(x,t)$ is the state variable and f_l, f_r are strictly convex maps. We study the Cauchy problem for (1) from the point of view of control theory regarding the initial datum as a control. Letting $u(x,t) \doteq \mathcal{S}_t^{AB}\overline{u}(x)$ denote the solution of the Cauchy problem for (1), with initial datum $u(\cdot,0) = \overline{u}$, that satisfy at $x = 0$ the interface entropy condition associated to a connection (A, B) (see [2]), we analyze the family of profiles that can be attained by (1) at a given time $T > 0$:

$$\mathcal{A}^{AB}(T) = \left\{ \mathcal{S}_T^{AB}\,\overline{u} : \ \overline{u} \in \mathbf{L}^\infty \right\}.$$

We provide a full characterization of $\mathcal{A}^{AB}(T)$ as a class of functions in $BV_{loc}(\mathbb{R} \setminus \{0\})$ that satisfy suitable Oleĭnik-type inequalities, and that admit one-sided limits at $x = 0$ which satisfy specific conditions related to the interface entropy criterium. Relying on this characterisation, we establish the ${\mathbf{L}^1}_{loc}$-compactness of the set of attainable profiles when the initial data $\overline{u}$ vary in a given class of uniformly bounded functions, taking values in closed convex sets. We also discuss some applications of these results to optimization problems arising in porous media flow models for oil recovery and in traffic flow.

Introduction

Consider the Cauchy problem for the scalar conservation law in one space dimension

$$u_t + f(x,u)_x = 0, \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{0.1}$$
$$u|_{t=0} = \overline{u} \qquad x \in \mathbb{R}, \tag{0.2}$$

where $u = u(x,t)$ is the state variable, and the flux $f(x,u)$ is a discontinuous function given by

$$f(x,u) = \begin{cases} f_l(u) & \text{if} \quad x < 0\,, \\ f_r(u) & \text{if} \quad x > 0\,, \end{cases} \tag{0.3}$$

with f_l and f_r smooth, strictly convex maps. The equation (0.1) is usually supplemented with appropriate coupling conditions imposed at the point of discontinuity of the flux so to guarantee uniqueness of solutions to the Cauchy problem (0.1)-(0.2). Namely, the traces

$$u_l(t) = \lim_{t \to 0-} u(x,t), \qquad u_r(t) = \lim_{t \to 0+} u(x,t), \tag{0.4}$$

of a weak distributional solution of (0.1), (0.3), must satisfy the Rankine-Hugoniot condition

$$f_l(u_l(t)) = f_r(u_r(t)) \qquad \text{for a.e.} \quad t > 0\,, \tag{0.5}$$

at the interface $x = 0$. Moreover, various type of admissibility conditions (interface entropy conditions) imposed on $u_{l,r}$ have been introduced in the literature, according with different modelling assumptions (see [8, 9]). Such conditions lead to different solutions of the Cauchy problem (0.1)-(0.2), which are appropriate for the particular physical phenomena modelled by (0.1). Alternatively, one can equivalently characterize the admissible solutions in terms of Kružkov-type (possibly singular) entropy inequalities satisfied up-to-the flux-discontinuity interface (cfr. [38]), or using extended families of entropy inequalities associated to the so called *partially adapted entropies* (see [8, 12, 18]).

Starting with the works by Isacson & Temple [33] and by Risebro and collaborators [28, 29, 41], conservation laws with discontinuous flux has been an intense subject of research in the last three decades (e.g. see [11, 15] and references therein). Solutions of (0.1), (0.3), satisfying the above mentioned admissibility criteria, can be obtained as limit of approximations constructed by regularization of the flux [13, 41, 47], by wave front-tracking [27, 29], by Godunov method [2, 40] and several other numerical schemes [18, 39, 53] or by vanishing viscosity [10, 15]. In particular, in [11] it was set up a general framework that encompasses all the notions of admissible solutions to the Cauchy problem (0.1)-(0.2) which lead to the existence of an $\mathbf{L}^1$-contractive semigroup.

In this paper we study the system (0.1)-(0.2) from the point of view of control theory, regarding the initial data u_0 as a control. Namely, we provide a characterization of the space-profile configurations that can be attained at any fixed time $T > 0$:

$$\mathcal{A}(T) = \big\{u(\cdot, T) : u \text{ is an admissible solution of (0.1),(0.3)-(0.2) with } u_0 \in \mathbf{L}^\infty(\mathbb{R})\big\}.$$

Here, u is a solution of (0.1)-(0.2) satisfying an interface entropy condition associated to a so-called *interface-connection* (A, B) [2, 18]. A connection (A, B) is a pair of states connected by a stationary weak solution of (0.1),(0.3), taking values A for $x < 0$, and B for $x > 0$, which has characteristics diverging from (or parallel to) the flux-discontinuity interface $x = 0$. Such a solution characterize the possible *undercompressive* (or *marginally undercompressive*) shock waves exhibited by admissible solutions of (0.1),(0.3) that satisfy an interface entropy condition involving the connection (A, B) (cfr.[2, 18]). The reason for choosing this type of admissibility conditions for solutions of (0.1),(0.3) is twofold. On one hand, it is consistent with the models of two-phase flows in heterogeneous porous medium [2] or of traffic flow on roads with variable surface conditions [45]. On the other hand, it allows to treat any connection (A, B) as a pair of control parameters as well.

We show that any element in $\mathcal{A}(T)$ belongs to a class of functions in $BV_{loc}(\mathbb{R} \setminus \{0\})$ (with locally bounded variation on $\mathbb{R} \setminus \{0\}$), which:

- satisfy suitable Oleĭnik-type inequalities involving the first and second derivatives of the maps f_l, f_r;
- admit one-sided limits at $x = 0$ which satisfy specific conditions related to the interface entropy criterium of the (A, B)-connection.

Viceversa, we establish an exact-time controllability result, i.e. we prove that, for any target function ω of the aforementioned class, there exist an initial datum $\overline{u}$ and a connection (A, B) that steer the system (0.1)-(0.2) to ω at a given time T. These results extend to the spatially-discontinuous setting the characterization of the attainable profiles established in [6, 7, 32] for conservation laws with convex flux depending only on the state variable. Such results are obtained exploiting, as in [6], the theory of generalized characteristics, which was developed by Dafermos [22] for conservation laws with convex flux (in the state variable) depending smoothly on the space variable. A detailed analysis of the structure of admissible solutions for a given connection (cfr. Proposition 3.1 and Remark 6) is also fundamental to derive a full characterization of the attainable profiles.

Hyperbolic partial differential equations with discontinuous coefficients arise in many different applications in physics and engineering including: two-phase flow models in porous media with changing rock types (for oil reservoir simulation) [29, 30]; slow erosion granular flow models [54]; clarifier-thickener problems of continuous sedimentation (in waste-water treatment plants) [17, 24]; population-balance models of steel ball wear in grinding mills [16]; ion etching in semiconductor industry [52]; traffic flow models with roads of varying amplitudes or surface conditions [45]; Saint Venant models of blood flow in endovascular treatments [25, 19]; radar shape-from-shading models [47]. This kind of equations appear also in the analysis of inverse problems [34, 35] or of optimal control problems [31] for conservation laws with smooth flux, where one needs to deal with the backward adjoint transport equation with discontinuous coefficients, which depend on the (possibly discontinuous) solution of the conservation law. Moreover, conservation laws with discontinuous flux arise also as a reformulation of balance laws [36] or of triangular systems of conservation laws [15, 37], in order to design efficient numerical schemes or to analyse their well-posedness. Finally, we observe that such a class of PDEs share fundamental features of conservation laws evolving on simple networks composed by a number of edges connected together by a junction [26, 27], which is a topic attracting a vast interest in the last twenty-five years for the wide range of applications [14].

Despite a large amount of literature on the theoretical and numerical aspects of conservation laws with discontinuous flux produced in the last three decades, almost no investigation of control issues for such a class of PDEs has been performed so far. The goal of the present paper is to provide a first step toward the analysis of controllability properties of these type of equations. Having in mind applications to optimization problems, we rely on the characterization of the attainable profiles to establish compactness in the $\mathbf{L}^1$-topology of the attainable set in connection with classes of uniformly bounded initial data taking values in closed convex sets. We then apply these results to two classes of optimization problems for porous media flow in oil recovery and for traffic flow, where one is interested in:

- minimising the distance from a target configuration (for both models) or the fuel consumption in a given road segment (for the latter model);
- maximising the net present value of the waterflooding process (in the first model).

We point out that a further step in the research direction pursued in this paper is the characterization of the traces of admissible solutions at the the flux-discontinuity interface as well as the analysis of the reachable set when one fixes the initial data and considers such traces as control parameters (cfr. [3] within the network setting), which is the object of the forthcoming paper [5].

The paper is organized in the following way. In Section 1 we recall the definition of interface entropy condition relative to an interface connection (A, B), and the corresponding definition of AB-entropy solution. We also review the well-posedness theory of $\mathbf{L}^1$-contractive semigroups for this particular class of entropy admissible solutions. Section 2 collects the statements of the main results on the full description of the set of attainable profiles and their topological properties. In Section 3 we establish a preliminary lemma concerning the structure of AB-entropy solutions. The proofs of the characterization of the attainable set and of its compactness is provided in Section 4 and Section 5, respectively. Finally, in Section 6 we discuss two applications arising in traffic flow models, which lead to variational problems with cost functionals depending on the profile of the solutions, where we regard as control parameters both the initial data and the connection states.

1 Preliminaries and setting of the problem

Consider the scalar conservation law (0.1) with flux given by (0.3), and assume that f_l, f_r coincide at two points of their domain which, up to a reparametrization of the unknown variable, we may suppose to be $u = 0$ and $u = 1$. Observe that, by strict convexity, f_l f_r admit a unique point of minimum which we call, respectively, θ_l and θ_r. Hence, we shall make the following standing hypotheses on the flux f in (0.3):

H1) $f_l, f_r : \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable, (uniformly) strictly convex maps

$$\min\left\{f_l''(u)\ f_r''(u)\right\} \geq c > 0 \qquad \forall\, u \in \mathbb{R};$$

H2) $f_l(0) = f_r(0), \quad f_l(1) = f_r(1);$

H3) $\theta_l \geq 0, \ \theta_r \leq 1.$

We recall that, regardless of how smooth the initial data are, nonlinear conservation laws as (0.1), (0.3) do not posses in general classical solutions globally defined in time, even when $f_l = f_r$, since they can develop discontinuities (shocks) in finite time. Hence, it is natural to consider weak solutions in the sense of distributions that, for sake of uniqueness, satisfy the classical Kružkov entropy inequalities away from the point of the flux discontinuity, and a further interface entropy condition at the flux-discontinuity interface. As observed in the introduction, for modellization and control treatment reasons, we shall employ an admissibility condition involving the so-called interface connection introduced in [2], which can be equivalently formulated in terms of an interface entropy condition or of extended entropy inequalities adapted to the particular connection taken into account (cfr. [2, 11, 18]).

Definition 1.1. (Interface Connection) Let $(A, B) \in \mathbb{R}^2$. Then (A, B) is called a connection (Fig. 1) if it satisfies:

(i) $f_l(A) = f_r(B)$; (ii) $A \leq \theta_l, \ B \geq \theta_r$.

We shall denote with $\mathscr{C}_f$ the set of pairs of connections associated to the flux $f(x, u)$ in (0.3).

Observe that condition (ii) is equivalent to: (ii)' $f_l'(A) \leq 0$ and $f_r'(B) \geq 0$; which shows that the function

$$k_{AB}(x) = \begin{cases} A & \text{if } \ x < 0\,, \\ B & \text{if } \ x > 0\,, \end{cases} \tag{1.1}$$

is a stationary undercompressive (or marginally undercompressive) weak solution of (0.1),(0.3), since its characteristics diverge from (or are parallel to) the flux-discontinuity interface $x = 0$. The function k_{AB} is used in [18] to define the adapted entropy $\eta_{AB}(x, u) = \left|u - k_{AB}(x)\right|$, which in the spirit of [12] is employed to select a unique solution of the Cauchy problem (0.1),(0.3)-(0.2), according with the following definition.

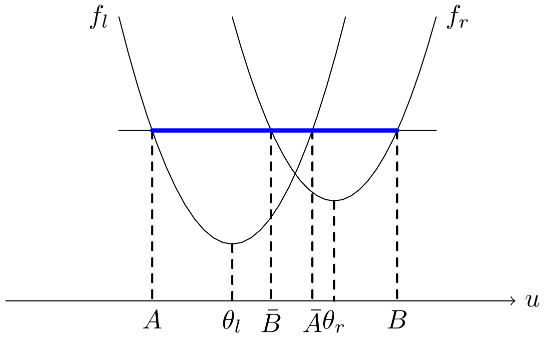

Figure 1: Example of AB connection with f_l, f_r strictly convex fluxes

Definition 1.2. **(AB-Entropy Solution)** Let (A, B) be a connection and let k_{AB} be the function defined in (1.1). A function $u \in \mathbf{L}^\infty(\mathbb{R} \times [0, +\infty))$ is said an AB-entropy solution of (0.1),(0.3)-(0.2) if the following holds:

(i) u is a weak distributional solution of (0.1),(0.3) on $\mathbb{R} \times \mathbb{R}_+$, that is, for any test function $\phi \in \mathcal{C}^1_c$ with compact support contained in $\mathbb{R} \times (0, +\infty)$, there holds

$$\int_{-\infty}^{\infty}\int_0^{\infty} \big\{u\phi_t + f(x,u)\phi_x\big\}dxdt = 0.$$

(ii) u is a Kružhkov entropy weak solution of (0.1),(0.3)-(0.2) on $(\mathbb{R} \setminus \{0\}) \times [0, +\infty)$, that is $t \to u(\cdot, t)$ is a continuous map from $[0, +\infty)$ in $\mathbf{L}^1_{loc}(\mathbb{R})$, the initial condition (0.2) is satisfied, and:

(ii.a) for any non-negative test function $\phi \in \mathcal{C}^1_c$ with compact support contained in $(-\infty, 0) \times (0, +\infty)$, there holds

$$\int_{-\infty}^{0}\int_0^{\infty} \big\{|u-k|\phi_t + \big(f_l(u) - f_l(k)\big)\operatorname{sgn}(u-k)\phi_x\big\}dxdt \ge 0 \qquad \forall\, k \in \mathbb{R};$$

(ii.b) for any non-negative test function $\phi \in \mathcal{C}^1_c$ with compact support contained in $(0, +\infty) \times (0, +\infty)$, there holds

$$\int_{0}^{+\infty}\int_0^{\infty} \big\{|u-k|\phi_t + \big(f_r(u) - f_r(k)\big)\operatorname{sgn}(u-k)\phi_x\big\}dxdt \ge 0 \qquad \forall\, k \in \mathbb{R}.$$

(iii) u satisfies a Kružhkov-type entropy inequality relative to the connection (A, B), that is, for any non-negative test function $\phi \in \mathcal{C}^1_c$ with compact support contained in $\mathbb{R} \times (0, +\infty)$, there holds

$$\int_{-\infty}^{+\infty}\int_0^{\infty} \big\{\big|u - k_{AB}(x)\big|\phi_t + \big(f(x,u) - f(x, k_{AB}(x))\big)\operatorname{sgn}(u - k_{AB}(x))\phi_x\big\}dxdt \ge 0.$$

Remark 1. If u is an AB-entropy solution, by property (ii) and because of the strict convexity of the fluxes $f_{l,r}$, it follows that $u(\cdot, t) \in BV_{loc}(\mathbb{R} \setminus \{0\})$ for any $t > 0$. Actually, it was shown in [1] that for all connections such that both $A \ne \theta_l$ and $B \ne \theta_r$, one has $u(\cdot, t) \in BV_{loc}(\mathbb{R})$ for any $t > 0$. On the other hand, when (A, B) is a critical connection, i.e. when either $A = \theta_l$ or $B = \theta_r$, the total variation of $u(\cdot, t)$ may well blow up in a neighbourhood of the flux-discontinuity interface $x = 0$, at some time $t > 0$ (see [1]). However, since u is in particular a distributional solution of $u_t + f_l(u)_x = 0$ on $(-\infty, 0) \times (0, +\infty)$, and of $u_t + f_r(u)_x = 0$ on $(0, +\infty) \times (0, +\infty)$, and since the fluxes $f_{l,r}$ are strictly convex, relying on a result in [48] (see also [56]) one deduces that $u(\cdot, t)$ still admits strong left and right traces at $x = 0$, i.e. that (after a possibly modification on a set of measure zero) for all $t > 0$ there exist the one-sided limits (0.4) (cfr [18]). Hence, since u is a distributional solution of (0.1),(0.3) on $\mathbb{R} \times (0, +\infty)$, by property (i), it follows that the Rankine-Hugoniot condition (0.5) holds. Furthermore, by the analysis in [18, Lemma 3.2] and [11, Section 4.8], it follows that, because of condition (i) of Definition 1.1 and assumption **H1)** on f_l, f_r, we can equivalently replace condition (iii) in Definition 1.2 with

(iii)' u satisfies an interface entropy condition relative to the connection (A, B), that is, the one-sided limits (0.4) satisfy

$$\begin{aligned} &f_l(u_l(t)) = f_r(u_r(t)) \ge f_l(A) = f_r(B), \\ &\big(u_l(t) \le \theta_l \quad \text{and} \quad u_r(t) \ge \theta_r\big) \quad \Longrightarrow \quad (u_l(t), u_r(t)) = (A, B) \end{aligned} \qquad \text{for a.e. } t > 0\,. \tag{1.2}$$

The first condition in (1.2) prescribes that the flux of the solution at the flux-discontinuity interface be greater or equal than the value of the flux on the (A, B) connection. Whereas the second condition in (1.2) excludes that the characteristics diverge from the flux-discontinuity interface when $(u_l(t), u_r(t)) \neq (A, B)$, i.e. the (A, B) *characteristic condition* in [18, Definition 1.4] is verified.

Remark 2. Since f_l, f_r are strictly convex maps, the Kružhkov entropy inequalities (ii.a)-(ii.b) in Definition 1.2 are equivalent to the Lax entropy condition [42, 43]

$$u(x-,t) \geq u(x+,t) \qquad \forall\, t, x > 0\,. \tag{1.3}$$

It was proved in [2, 18] (see also [11]) that AB-entropy solutions of (0.1),(0.3) with bounded initial data are unique and form an $\mathbf{L}^1$-contractive semigroup. We collect the properties of such a semigroup in the following

Theorem 1.1. (Semigroup of $\boldsymbol{AB}$-Entropy Solutions) [2, 18] *Let f be a flux as in* (0.3) *satisfying the assumptions* **H1), H2), H3)**. *Then, given a connection $(A, B) \in \mathscr{C}_f$, there exists a map*

$$\mathcal{S}^{AB} : [0, +\infty) \times \mathbf{L}^\infty(\mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R}), \qquad (t, \overline{u}) \mapsto \mathcal{S}^{AB}_t \overline{u}\,,$$

enjoying the following properties:

(i) For each $\overline{u} \in \mathbf{L}^\infty(\mathbb{R})$, the function $u(x,t) \doteq \mathcal{S}^{AB}_t \overline{u}(x)$ provides the unique AB-entropy solution of the Cauchy problem (0.1),(0.3)-(0.2).

(ii)

$$\mathcal{S}^{AB}_0 \overline{u} = \overline{u}, \qquad \mathcal{S}^{AB}_s \circ \mathcal{S}^{AB}_t \overline{u} = \mathcal{S}^{AB}_{s+t}\, \overline{u} \qquad \forall\, t, s \geq 0, \quad \forall\, \overline{u} \in \mathbf{L}^\infty(\mathbb{R})\,.$$

(iii)

$$\left\| \mathcal{S}^{AB}_t \overline{u} - \mathcal{S}^{AB}_s \overline{v} \right\|_{\mathbf{L}^1} \leq \left\| \overline{u} - \overline{v} \right\|_{\mathbf{L}^1} + L \left| t - s \right| \qquad \forall\, t, s \geq 0, \quad \forall\, \overline{u}, \overline{v} \in \mathbf{L}^\infty(\mathbb{R})\,,$$

for some positive constant $L > 0$.

In the present paper we regard as control parameters both the initial data and the connection states whose flux provides a lower bound on the flux of the solution at the flux-discontinuity interface. Then, given a set $\mathcal{U} \subset \mathbf{L}^\infty(\mathbb{R})$, and a set $\mathscr{C} \subset \mathscr{C}_f$ of connections, we consider the following *attainable sets* for (0.1),(0.3):

$$\mathcal{A}^{AB}(T, \mathcal{U}) \doteq \left\{ \mathcal{S}^{AB}_t \overline{u} : \overline{u} \in \mathcal{U} \right\}, \qquad \mathcal{A}(T, \mathcal{U}, \mathscr{C}) \doteq \bigcup_{(A,B) \in \mathscr{C}} \mathcal{A}^{AB}(T, \mathcal{U})\,, \tag{1.4}$$

which consist of all profiles that can be attained at a fixed time $T > 0$ by AB-entropy solutions of (0.1),(0.3) with initial data that varies inside $\mathcal{U}$, or by AB-entropy solutions of (0.1),(0.3) with initial data in $\mathcal{U}$ and connections $(A, B) \in \mathscr{C}$. In the case where $\mathcal{U}$ is the whole space $\mathbf{L}^\infty(\mathbb{R})$, we set

$$\mathcal{A}^{AB}(T) \doteq \mathcal{A}^{AB}\big(T, \mathbf{L}^\infty(\mathbb{R})\big), \qquad \mathcal{A}(T) \doteq \mathcal{A}\big(T, \mathbf{L}^\infty(\mathbb{R}), \mathscr{C}_f\big)\,. \tag{1.5}$$

We will provide a characterisation of the sets (1.5) in terms of certain Oleĭnik-type type estimates on the decay of positive waves, and we will establish the $\mathbf{L}^1$-compactness of (1.4) for classes $\mathcal{U}$ of initial data with values in compact convex sets, and for compact sets $\mathscr{C}$ of connections.

2 Statement of the main results

We present here the main results of the paper whose proof will be established in Sections 2.1, 2.2. Throughout the following

$$D^-\omega(x) = \liminf_{h \to 0} \frac{\omega(x+h) - \omega(x)}{h}, \qquad D^+\omega(x) = \limsup_{h \to 0} \frac{\omega(x+h) - \omega(x)}{h}, \tag{2.1}$$

will denote, respectively, the lower and the upper Dini derivative of a function ω at x. We shall also use the notations $f_{l,-}^{-1} \doteq (f_{l\,|(-\infty,\theta_l]})^{-1}$, $f_{r,-}^{-1} \doteq (f_{r\,|(-\infty,\theta_r]})^{-1}$, for the inverse of the restriction of f_l, f_r to their decreasing part, respectively, and $f_{l,+}^{-1} \doteq (f_{l\,|[\theta_l,+\infty)})^{-1}$, $f_{r,+}^{-1} \doteq (f_{r\,|[\theta_r,+\infty)})^{-1}$, for the inverse of the restriction of f_l, f_r to their increasing part, respectively. Then, we set

$$\pi_{l,\pm} \doteq f_{l,\pm}^{-1} \circ f_l\,, \qquad \pi_{r,\pm} \doteq f_{r,\pm}^{-1} \circ f_r\,, \qquad \pi^r_{l,\pm} \doteq f_{l,\pm}^{-1} \circ f_r\,, \qquad \pi^l_{r,\pm} \doteq f_{r,\pm}^{-1} \circ f_l\,. \tag{2.2}$$

and we introduce the following sets that characterize the left and right traces of an AB-entropy solution at the flux-discontinuity interface (see Remark 6):

$$\begin{aligned}
\mathcal{T}_1 &\doteq \left\{ (u_l, u_r) \in (\theta_l, +\infty) \times (\theta_r, +\infty);\ u_l \geq \pi_{l,+}(A),\ B \leq u_r \leq \pi^l_{r,+}(u_l) \right\}, \\
\mathcal{T}_2 &\doteq \left\{ (u_l, u_r) \in (-\infty, \theta_l) \times (-\infty, \theta_r);\ \pi^r_{l,-}(u_r) \leq u_l \leq A,\ u_r \leq \pi_{r,-}(B) \right\}, \\
\mathcal{T}_{3,-} &\doteq \left\{ (u_l, u_r) \in [\theta_l, +\infty) \times (-\infty, \theta_r);\ \pi_{l,+}(A) \leq u_l \leq \pi^r_{l,+}(u_r),\ u_r \leq \pi_{r,-}(B) \right\}, \\
\mathcal{T}_{3,+} &\doteq \left\{ (u_l, u_r) \in (\theta_l, +\infty) \times (-\infty, \theta_r];\ u_l \geq \pi_{l,+}(A),\ \pi^l_{r,-}(u_l) \leq u_r \leq \pi_{r,-}(B) \right\}.
\end{aligned} \tag{2.3}$$

Theorem 2.1. *Let f be a flux as in* (0.3) *satisfying the assumptions* **(H1), (H2), (H3)**, *and let $(A,B)\in\mathscr{C}_f$. Then, for any fixed $T>0$, the set $\mathcal{A}^{AB}(T)$ in* (1.5) *is given by*

$$\mathcal{A}^{AB}(T)=\mathcal{A}_1(T)\cup\mathcal{A}_2(T)\cup\mathcal{A}_3^{AB}(T)\,, \tag{2.4}$$

where $\mathcal{A}_1(T),\mathcal{A}_2(T),\mathcal{A}_3^{AB}(T)$ are sets of functions $\omega\in\mathbf{L}^\infty(\mathbb{R})$ having essential left and right limits at $x=0$, defined as follows.

$\mathcal{A}_1(T)$ *is the set of all functions ω that satisfy $\big(\omega(0-),\,\omega(0+)\big)\in\mathcal{T}_1$, and for which there exists $R>0$ such that the following conditions hold.*

$$\begin{aligned}
&\omega(x)\geq (f_l')^{-1}\big(x/T+f_l'(\omega(0-))\big)\quad\forall\, x\in(-\infty,0), && \omega(x)\geq (f_r')^{-1}\big(x/T\big)\quad\forall\, x\in(0,R),\\
&\omega(x)< (f_r')^{-1}\big(x/T\big)\quad\forall\, x\in(R,+\infty), && \omega(R-)\geq\omega(R+),
\end{aligned}\tag{2.5}$$

$$D^+\omega(x)\leq\begin{cases}1/\big(f_l''(\omega(x))\cdot T\big) & \forall\, x\in(-\infty,0)\,,\\[1ex]
\dfrac{f_r'(\omega(x))\big[f_l'\circ f_{l,+}^{-1}\circ f_r(\omega(x))\big]^2}{\big[f_l''\circ f_{l,+}^{-1}\circ f_r(\omega(x))\big]\,[f_r'(\omega(x))]^2\,\big(f_r'(\omega(x))T-x\big)+x\big[f_l'\circ f_{l,+}^{-1}\circ f_r(\omega(x))\big]^2 f_r''(\omega(x))} & \forall\, x\in(0,R),\\[1ex]
1/\big(f_r''(\omega(x))\cdot T\big) & \forall\, x\in(R,+\infty)\,.\end{cases}\tag{2.6}$$

$\mathcal{A}_2(T)$ *is the set of all functions ω that satisfy $\big(\omega(0-),\,\omega(0+)\big)\in\mathcal{T}_2$, and for which there exists $L<0$ such that the following conditions hold.*

$$\begin{aligned}
&\omega(x)> (f_l')^{-1}\big(x/T\big)\quad\forall\, x\in(-\infty,L), && \omega(x)\leq (f_l')^{-1}\big(x/T\big)\quad\forall\, x\in(L,0),\\
&\omega(x)\leq (f_r')^{-1}\big(x/T+f_r'(\omega(0+))\big)\quad\forall\, x\in(0,+\infty), && \omega(L-)\geq\omega(L+),
\end{aligned}\tag{2.7}$$

$$D^+\omega(x)\leq\begin{cases}1/\big(f_l''(\omega(x))\cdot T\big) & \forall\, x\in(-\infty,L)\,,\\[1ex]
\dfrac{f_l'(\omega(x))\,\big[f_r'\circ f_{r,-}^{-1}\circ f_l(\omega(x))\big]^2}{\big[f_r''\circ f_{r,-}^{-1}\circ f_l(\omega(x))\big]\,[f_l'(\omega(x))]^2\,\big(f_l'(\omega(x))T-x\big)+x\,\big[f_r'\circ f_{r,-}^{-1}\circ f_l(\omega(x))\big]^2\,f_l''(\omega(x))} & \forall\, x\in(L,0),\\[1ex]
\big(1/f_r''(\omega(x))\cdot T\big) & \forall\, x\in(0,+\infty)\,.\end{cases}\tag{2.8}$$

$\mathcal{A}_3^{AB}(T)$ *is the set of all functions ω for which there exist $L\leq 0\leq R$, such that*

$$\big(\omega(0-),\,\omega(0+)\big)\in\begin{cases}\mathcal{T}_{3,-}\cup\mathcal{T}_{3,+} & \textit{if}\quad L=R=0\,,\\ \big\{(A,B)\big\} & \textit{if}\quad L\leq 0\leq R\,,\end{cases}\tag{2.9}$$

and the following conditions hold.

$$\begin{gathered}
\omega(x)=A\quad\forall\, x\in(L,0),\qquad \omega(x)=B\quad\forall\, x\in(0,R),\\
\omega(L-)\geq\omega(L+)\,,\qquad \omega(R-)\geq\omega(R+)\,,\\
\omega(x)\geq\begin{cases}(f_l')^{-1}\big(x/T\big) & \textit{if}\ \ L<0,\\ (f_l')^{-1}\big(x/T+f_l'(\omega(0-))\big) & \textit{if}\ \ L=0,\end{cases}\qquad\forall\, x\in(-\infty,L),\\
\omega(x)\leq\begin{cases}(f_r')^{-1}\big(x/T\big) & \textit{if}\ \ R>0,\\ (f_r')^{-1}\big(x/T+f_r'(\omega(0+))\big) & \textit{if}\ \ R=0,\end{cases}\qquad\forall\, x\in(R,+\infty),
\end{gathered}\tag{2.10}$$

$$D^+\omega(x)\leq\begin{cases}1/\big(f_l''(\omega(x))\cdot T\big) & \forall\, x\in(-\infty,L)\,,\\ \big(1/f_r''(\omega(x))\cdot T\big) & \forall\, x\in(R,+\infty)\,.\end{cases}\tag{2.11}$$

Remark 3. Notice that, by the strict convexity assumption (H1) on f_l'',f_r'', and relying on (2.5), (2.7), we deduce that the right-hand side of (2.6), (2.8), (2.11) is always nonnegative, and it is bounded on any set bounded away from $x=0$. Therefore, any $\omega\in\mathcal{A}_1(T)\cup\mathcal{A}_2(T)\cup\mathcal{A}_3^{AB}(T)$ is an equivalence class of bounded measurable functions that haves finite total increasing variation (and hence finite total variation as well) on subsets of $\mathbb{R}$ bounded away from the origin. Moreover, by assumption any $\omega\in\mathcal{A}_1(T)\cup\mathcal{A}_2(T)\cup\mathcal{A}_3^{AB}(T)$ admits one-sided limits at $x=0$. Hence, any element of $\mathcal{A}_1(T)\cup\mathcal{A}_2(T)\cup\mathcal{A}_3^{AB}(T)$ admits one-sided limits at every point.

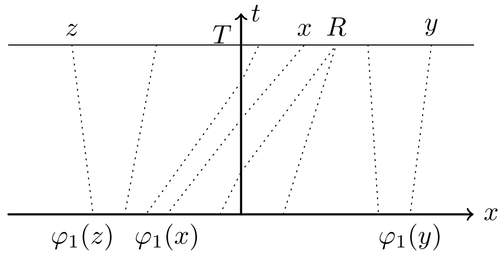

Figure 2: Characteristics's behavior for profiles in $\mathcal{A}_1(T)$

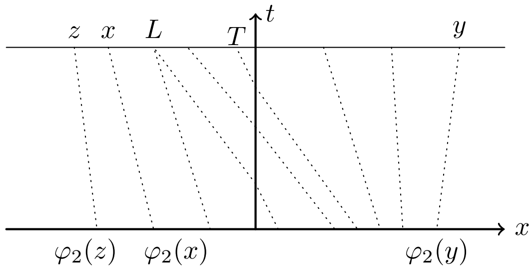

Figure 3: Characteristics's behavior for profiles in $\mathcal{A}_2(T)$

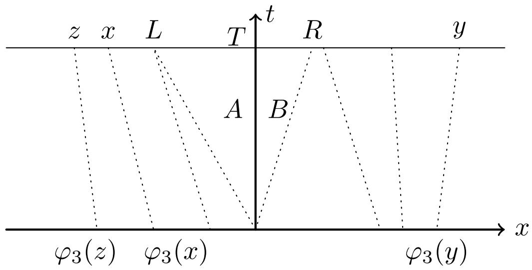

Figure 4: Characteristics's behavior for profiles in $\mathcal{A}_3^{AB}(T)$

Remark 4. The conditions (2.6), (2.8), (2.11) reflect the fact that, since the fluxes are strictly convex, positive waves of AB-entropy solutions decay in time. Such conditions are sufficient to guarantee the exact-time controllability of (0.1)-(0.2). In fact, starting at a time $T > 0$ with a profile $\omega \in \mathcal{A}_1(T) \cup \mathcal{A}_2(T) \cup \mathcal{A}_3^{AB}(T)$, because of (2.6), (2.8), (2.11) one can trace backward the (generalized) characteristics ξ_1, ξ_2 through points $x_1 < x_2$ without crossing in $\mathbb{R} \times (0,T)$, unless $\omega(x_1) = A$ and $\omega(x_2) = B$, in which cases the characteristics ξ_1, ξ_2 intersects only at $x = 0$ (see Figures 2-4). In particular, by (2.5), (2.7), the inequalities (2.6), (2.8) imply that

$$D^+\omega(x) < \frac{f_l'(\omega(x))}{x f_l''(\omega(x))} \quad \forall\, x \in (L,0)\,, \qquad D^+\omega(x) < \frac{f_r'(\omega(x))}{x f_r''(\omega(x))} \quad \forall\, x \in (0,R)\,, \tag{2.12}$$

and we recover the same type of boundary controllability condition derived in [6], if we regard the left and right traces at $x = 0$ as controls. Notice that we have in (2.12) a strict inequality since here, differently from [6], characteristics having slope with the same sign cannot intersect even at $x = 0$ (they can intersect only at $t = 0$).

The characterization of the attainable set $\mathcal{A}^{AB}(T)$ provided by Theorem 2.1 yields the $\mathbf{L^1}$-compactness of the attainable sets $\mathcal{A}^{AB}(T,\mathcal{U})$, $\mathcal{A}(T,\mathcal{U},\mathscr{C})$ in (1.4) for classes $\mathcal{U}$ of admissible controls uniformly bounded and with values in convex and closed sets, as stated in the following

Theorem 2.2. *Let $G : \mathbb{R} \hookrightarrow \mathbb{R}$ be a measurable, bounded multifunction with convex and closed values, and let $\mathscr{C} \subset \mathscr{C}_f$ be a compact set of connections. Consider the set*

$$\mathcal{U} = \{\overline{u} \in \mathbf{L}^\infty(\mathbb{R}) : \overline{u}(x) \in G(x) \ \textit{for a.e.} \ \ x \in \mathbb{R}\}\,. \tag{2.13}$$

Then, under the same assumptions of Theorem 2.1, for any fixed $T > 0$, the sets $\mathcal{A}^{AB}(T,\mathcal{U})$, $\mathcal{A}(T,\mathcal{U},\mathscr{C})$ in (1.4) are compact in the $\mathbf{L^1}_{loc}(\mathbb{R})$-topology and, letting $\mathcal{S}^{AB}_{(\cdot)}\overline{u}\,\big|_T$ denote the restriction of $\mathcal{S}^{AB}_{(\cdot)}\overline{u}$ to $\mathbb{R} \times [0,T]$, the sets

$$\mathcal{A}^{AB}(\mathcal{U}) \doteq \{\mathcal{S}^{AB}_{(\cdot)}\overline{u}\,\big|_T : \overline{u} \in \mathcal{U}\}\,, \qquad \mathcal{A}(\mathcal{U},\mathscr{C}) \doteq \bigcup_{(A,B)\in\mathscr{C}} \mathcal{A}^{AB}(\mathcal{U})\,, \tag{2.14}$$

are compact in the $\mathbf{L^1}_{loc}(\mathbb{R} \times [0,T])$-topology.

In turn, the compactness of the attainable sets yields the existence of optimal solutions for a class of minimization (maximization) problems, by considering a minimizing (maximizing) sequence for the corresponding cost functionals.

Corollary 2.3. *Let G be multivalued map as in Theorem 2.2 and assume that $G(x) = 0$ for all $x \in \mathbb{R} \setminus K$, for some bounded set $K \subset \mathbb{R}$. Given an interval $I \subset \mathbb{R}$ and $T > 0$, let $F_1 : L^1(I) \to \mathbb{R}$, $F_2 : L^1(I \times [0, T]) \to \mathbb{R}$ be lower semicontinuous functionals, and let $\mathcal{U}$ be the set of admissible controls defined in (2.13). Then, under the same assumptions of Theorem 2.1, the optimal control problems*

$$\min_{\overline{u} \in \mathcal{U}} F_1(S_T \overline{u}(\cdot)), \qquad \min_{\overline{u} \in \mathcal{U}} F_2(S_{(\cdot)} \overline{u}(\cdot)), \tag{2.15}$$

admit a solution. If we assume that F_1, F_2 are upper semicontinuous functionals, then there exists a solution of the maximization problems

$$\max_{\overline{u} \in \mathcal{U}} F_1(S_T \overline{u}(\cdot)), \qquad \max_{\overline{u} \in \mathcal{U}} F_2(S_{(\cdot)} \overline{u}(\cdot)). \tag{2.16}$$

3 Structure of AB-entropy solutions and a technical lemma

We analyze here some structural properties of AB-entropy solutions and we derive a technical lemma on the relation between upper bounds on the Dini derivative and the monotonicity of suitable maps, that will be useful for the proofs of Theorem 2.1 and Theorem 2.2.

Remark 5. By the analysis in [27, Section 3.1] it follows that the Riemann solver associated to a given connection (A, B) enjoys the following properties. Letting $u(x,t)$ be the AB-entropy solution of the Cauchy problem for (0.1),(0.3), with initial data $\overline{u}(x) = u^-$ if $x < 0$, and $\overline{u}(x) = u^+$ if $x > 0$, for any given $a \le \theta_r, b \ge \theta_l$, $a < b$, there holds

$$\{A, B, u^-, u^+\} \subseteq [a, b] \quad \Longrightarrow \quad u(x,t) \in \big[\min\{a, \pi^r_{l,-}(a)\},\ \max\{b, \pi^l_{r,+}(b)\}\big] \qquad \forall\, x \in \mathbb{R},\ t \ge 0\,. \tag{3.1}$$

Moreover, if $A, B \in [0, 1]$, by the assumptions **H1), H2)** on f_l, f_r, one has

$$\{A, B, u^-, u^+\} \subseteq [0, 1] \quad \Longrightarrow \quad u(x,t) \in [0, 1] \qquad \forall\, x \in \mathbb{R},\ t \ge 0\,. \tag{3.2}$$

Observe that, if $u(x,t)$ is a front tracking solution (cfr. [27, Section 4]) constructed with approximate Riemann solvers that satisfy (3.1), (3.2), then the same type of a-priori bounds hold. In fact, u can assume values which do not belong to the range of the initial data $\overline{u}$ only on regions adjacent to the discontinuity $x = 0$ (from the left or from the right), and such values always belong to the interval

$$\big[\inf\big\{\min\{\overline{u}(x), \pi^r_{l,-}(\overline{u}(x))\};\, x \in \mathbb{R}\big\},\ \sup\big\{\max\{\overline{u}(x), \pi^l_{r,+}(\overline{u}(x))\};\, x \in \mathbb{R}\big\}\big].$$

Hence, since a general solution of a Cauchy problems for (0.1),(0.3) can be obtained as limit of front tracking solutions (see [27, 29]), we deduce the following a-priori bounds for any $u(x,t) \doteq \mathcal{S}^{AB}_t \overline{u}(x)$, with $\overline{u} \in \mathbf{L}^\infty(\mathbb{R})$:

$$\{A, B\} \cup \{\overline{u}(x);\ x \in \mathbb{R}\} \subseteq [a, b] \quad \Longrightarrow \quad u(x,t) \in \big[\min\{a, \pi^r_{l,-}(a)\},\ \max\{b, \pi^l_{r,+}(b)\}\big] \qquad \forall\, x \in \mathbb{R},\ t \ge 0\,, \tag{3.3}$$

and

$$\{A, B\} \cup \{\overline{u}(x);\ x \in \mathbb{R}\} \subseteq [0, 1] \quad \Longrightarrow \quad u(x,t) \in [0, 1] \qquad \forall\, x \in \mathbb{R},\ t \ge 0\,. \tag{3.4}$$

Moreover, if the initial data $\overline{u}$ vanishes outside a bounded set K, then there will be some bounded set K' such that $\mathrm{supp}(u(\cdot, t)) \subset K'$ for all $t > 0$.

The classical theory of generalized characteristics for conservation laws with continuous and convex flux [22] guarantees that backward characteristics, lying in the same quarter of plane $(-\infty, 0] \times [0, +\infty)$ or $[0, +\infty) \times [0, +\infty)$, never intersect at times $t > 0$ in points $x \ne 0$. A fundamental feature of AB-entropy solutions is that backward generalized characteristics cannot intersect at times $t > 0$ even along the discontinuity interface $x = 0$, unless $(u_l(s), u_r(s)) = (A, B)$ for all $0 < s \le t$. It follows in particular that no rarefaction fan can be originated at $x = 0$ and $t > 0$. This property is the consequence of the next Proposition. We recall that a generalized characteristic $\xi(t), t \in (t', t'')$ for a conservation law $u_t + f(u)_x = 0$ is called genuine if, for almost every $t \in (t', t'')$, there holds $u(\xi(t)-, t) = u(\xi(t)+, t) = v$ for some constant v such that $f'(v) = \dot{\xi}$. Thus, genuine characteristics are segments of lines which may intersect only at their end points [22].

Proposition 3.1. *Let f be a flux as in (0.3) satisfying the assumptions* **H1), H2), H3)***, and let $u(x,t)$ be an AB-entropy solution of (0.1),(0.3)-(0.2), for some initial data $\overline{u} \in \mathbf{L}^\infty(\mathbb{R})$ and a connection $(A, B) \in \mathscr{C}_f$. Then, at any time $\overline{t} > 0$ the following hold.*

(i) If $u_l(\bar{t}+) < \theta_l$ and $u_r(\bar{t}+) > \theta_r$, then $(u_l(t\pm), u_r(t\pm)) = (A, B)$ for all $t \in (0, t^*)$, for some $t^* > \bar{t}$. Moreover, there exist exactly two forward, genuine, characteristics η', η'', starting at $(0, \bar{t}\,)$, which lie in $(-\infty, 0) \times (\bar{t}, t^*)$ and $(0, +\infty) \times (\bar{t}, t^*)$, respectively.

(ii) If $u_l(\bar{t}+) \geq \theta_l$ or $u_r(\bar{t}+) \leq \theta_r$, then there exists at most a single forward, genuine, characteristic starting at $(0, \bar{t}\,)$ and lying in $(\mathbb{R} \setminus \{0\}) \times (\bar{t}, t^*)$, for some $t^* > \bar{t}$.

Proof. We shall distinguish three cases.

Case 1. *$u_l(\bar{t}+) > \theta_l$ and $u_r(\bar{t}+) \neq \theta_r$, or $u_r(\bar{t}+) < \theta_r$ and $u_l(\bar{t}+) \neq \theta_l$.*
1a) If $u_l(\bar{t}+) > \theta_l$, $u_r(\bar{t}+) > \theta_r$ and $u(0+, \bar{t}\,) \geq u_r(\bar{t}+)$ (see Figure 3), then consider two sequences of points $\{t_n, t_n \downarrow 0\}$, and $\{(x_n, \bar{t}\,), x_n \downarrow 0\}$, of continuity for u_r and u, respectively. Tracing the backward genuine characteristics (with positive slopes) through $(0, t_n)$ and $(x_n, \bar{t}\,)$ one deduces that there exist sequences of points $\{(x'_n, \bar{t}\,), x'_n \uparrow 0\}$, and $\{t'_n, t'_n \uparrow 0\}$, such that $u(x'_n, \bar{t}\,) \to u_l(\bar{t}+)$ and $u_r(t'_n) \to u(0+, \bar{t}\,)$. Hence, there holds $u(0-, \bar{t}\,) = u_l(\bar{t}+)$, $u_r(\bar{t}-) = u(0+, \bar{t}\,)$. Now observe that, if $u_l(\bar{t}-) \neq u(0-, \bar{t}\,)$, then there should be a shock with positive slope arriving in $(0, \bar{t}\,)$ (or generated in $(0, \bar{t}\,)$) and connecting the left state $u(0-, \bar{t}\,)$ with the right state $u_l(\bar{t}-)$. Such a shock is entropy admissible for the conservation law with flux f_l and has positive slope if and only if $u_l(\bar{t}-) < u(0-, \bar{t}\,)$ and $f_l(u_l(\bar{t}-)) < f_l(u(0-, \bar{t}\,))$. Since by (1.2) one has $f_l(u_l(\bar{t}-)) = f_r(u_r(\bar{t}-))$, and because of $u(0-, \bar{t}\,) = u_l(\bar{t}+)$, from $f_l(u_l(\bar{t}-)) < f_l(u(0-, \bar{t}\,))$ it follows that $f_r(u_r(\bar{t}-)) < f_l(u_l(\bar{t}+))$. On the other hand, $\theta_r < u_r(\bar{t}+) \leq u(0+, \bar{t}\,) = u_r(\bar{t}-)$ implies $f_r(u_r(\bar{t}+)) \leq f_r(u_r(\bar{t}-))$ which is in contrast with $f_r(u_r(\bar{t}-)) < f_l(u_l(\bar{t}+))$. Therefore, $u_l(\bar{t}+) > \theta_l$, $u_r(\bar{t}+) > \theta_r$ and $u(0+, \bar{t}) \geq u_r(\bar{t}+)$, together imply that $u_l(\bar{t}-) = u(0-, \bar{t}\,) = u_l(\bar{t}+)$. Moreover, since by (1.2) one has $f_l(u_l(\bar{t}+)) = f_r(u_r(\bar{t}+))$, from $\theta_r < u_r(\bar{t}+) \leq u(0+, \bar{t}\,) = u_r(\bar{t}-)$ it follows that $f_r(u_r(\bar{t}+)) \leq f_r(u_r(\bar{t}-)) \leq f_r(u_r(\bar{t}+))$. Hence, $u_l(\bar{t}+) > \theta_l$, $u(0+, \bar{t}\,) \geq u_r(\bar{t}+) > \theta_r$ together imply that $u_l(\bar{t}\pm) = u(0-, \bar{t}\,)$ and $u_r(\bar{t}\pm) = u(0+, \bar{t}\,)$, which shows that from $(0, \bar{t}\,)$ it emerges a single forward genuine characteristic, lying on $(0, +\infty) \times (\bar{t}, t^*)$, for some $t^* > \bar{t}$, and property (ii) is verified.
1b) If $u_l(\bar{t}+) > \theta_l$ and $\theta_r \leq u(0+, \bar{t}\,) < u_r(\bar{t}+)$, then there is a shock with positive slope starting at $(0, \bar{t}\,)$ and connecting the left state $u_r(\bar{t}+)$ with the right state $u(0+, \bar{t}\,)$. Moreover, tracing the backward genuine characteristics through a sequence of points $(x_n, \bar{t}\,)$, $x_n \uparrow 0$, of continuity for u, one deduces that $u_r(\bar{t}-) = u(0+, \bar{t}\,)$. Hence, by (1.2) one has $f_l(u_l(\bar{t}-)) = f_r(u_r(\bar{t}-)) < f_r(u_r(\bar{t}+)) = f_l(u_l(\bar{t}+))$. On the other hand, by the observations in case **1a)** it follows that $u(0-, \bar{t}\,) = u_l(\bar{t}+)$, which implies $f_l(u(0-, \bar{t}\,)) > f_l(u_l(\bar{t}-))$. Thus, it must be $u(0-, \bar{t}\,) > u_l(\bar{t}-)$, and there is a shock with positive slope arriving at $(0, \bar{t}\,)$ (or generated in $(0, \bar{t}\,)$) connecting the left state $u(0-, \bar{t}\,)$ with the right state $u_l(\bar{t}-) \in \{\pi^r_{l,-}(u(0+, \bar{t}\,)), \pi^r_{l,+}(u(0+, \bar{t}\,))\}$. Therefore, if $u_l(\bar{t}+) > \theta_l$ and $\theta_r \leq u(0+, \bar{t}\,) < u_r(\bar{t}+)$, then there is no forward, genuine characteristic, emerging from $(0, \bar{t}\,)$, there is a single (forward) shock starting at $(0, \bar{t}\,)$ with positive slope, and property (ii) is verified.
1c) If $u_l(\bar{t}+) > \theta_l$ and $u(0+, \bar{t}\,) < \theta_r < u_r(\bar{t}+)$, then with similar arguments to case **1b)** one deduces that:
- there is a shock with positive slope starting at $(0, \bar{t}\,)$ and connecting the left state $u_r(\bar{t}+)$ with the right state $u(0+, \bar{t}) > \pi_{r,-}(u_r(\bar{t}+))$;
- there is a shock with positive slope arriving at $(0, \bar{t}\,)$ (or generated in $(0, \bar{t}\,)$) connecting the left state $u(0-, \bar{t}\,) = u_l(\bar{t}+)$ with the right state $u_l(\bar{t}-) \in \{\pi^r_{l,-}(u_r(\bar{t}-)), \pi^r_{l,+}(u_r(\bar{t}-))\}$;
- either $u_r(\bar{t}-) = u(0+, \bar{t}\,)$, or $u_r(\bar{t}-) > u(0+, \bar{t}\,)$, and in this latter case there is a shock with negative slope arriving at $(0, \bar{t}\,)$ (or generated in $(0, \bar{t}\,)$) that connects the left state $u_r(\bar{t}-) \in (u(0+, \bar{t}\,), \pi_{r,+}(u(0+, \bar{t}\,)))$ with the right state $u(0+, \bar{t}\,)$.
Therefore, if $u_l(\bar{t}+) > \theta_l$ and $u(0+, \bar{t}\,) < \theta_r < u_r(\bar{t}+)$, then as in case **1b)** there is no forward, genuine, characteristic emerging from $(0, \bar{t}\,)$, while there is a single (forward) shock starting at $(0, \bar{t}\,)$, which has negative slope. Hence, property (ii) is verified.
1d) If $u_l(\bar{t}+) < \theta_l$ and $u_r(\bar{t}+) < \theta_r$, then we can proceed as in cases **1a)-1c)** to conclude that: either $u_l(\bar{t}\pm) = u(0-, \bar{t}\,)$, $u_r(\bar{t}\pm) = u(0+, \bar{t}\,)$, and it emerges a single forward genuine characteristic, lying on $(-\infty, 0) \times (\bar{t}, t^*)$, for some $t^* > \bar{t}$, or $u_l(\bar{t}+) < u(0-, \bar{t}\,)$, $u_r(\bar{t}+) = u(0+, \bar{t}\,)$, and there is no forward, genuine characteristic, emerging from $(0, \bar{t}\,)$, while there is a single (forward) shock starting at $(0, \bar{t}\,)$, which has negative slope. Thus, property (ii) is verified.
1e) If $u_l(\bar{t}+) > \theta_l$ and $u_r(\bar{t}+) < \theta_r$, then with the same arguments as above we deduce that $u(0-, \bar{t}\,) = u_l(\bar{t}+)$, $u(0+, \bar{t}\,) = u_r(\bar{t}+)$, and by (1.2) one of the following subcases occurs:
- $u_l(\bar{t}+) = u_l(\bar{t}-)$, $u_r(\bar{t}+) = u_r(\bar{t}-)$, and in a neighbourhood of $\bar{t}$ the characteristics are crossing the line $x = 0$ with positive slopes on the left side, with negative slopes on the right side;
- $u_l(\bar{t}-) \leq \theta_l < u_l(\bar{t}+)$, $u_r(\bar{t}+) < u_r(\bar{t}-) \leq \theta_r$, there is a shock with positive slope arriving at $(\bar{t}, 0)$ (or generated in $(0, \bar{t}\,)$), which connects the left state $u(0-, \bar{t}\,) = u_l(\bar{t}+)$ with the right state $u_l(\bar{t}-)$, there is a shock with negative slope connecting the left state $u_r(\bar{t}-) = \pi^l_{r,-}(u_l(\bar{t}-))$ with the right state $u(0+, \bar{t}\,) = u_r(\bar{t}+)$, and in a left neighbourhood of $\bar{t}$ the characteristics are crossing the line $x = 0$ with negative slopes on both sides;
- $\theta_l < u_l(\bar{t}-) = A < u_l(\bar{t}+)$, $u_r(\bar{t}+) < u_r(\bar{t}-) = B < \theta_r$, there are two shocks with positive and negative slopes arriving at $(0, \bar{t}\,)$ as in the previous case, and in a left neighbourhood of $\bar{t}$ the characteristics are crossing

the line $x=0$ with with positive slopes on the left side, with negative slopes on the right side;
- $\theta_l < u_l(\bar{t}-) < u_l(\bar{t}+)$, $u_r(\bar{t}+) < \theta_r \le u_r(\bar{t}-)$, there are two shocks with positive and negative slopes arriving at $(0,\bar{t})$ as in the previous case, and in a left neighbourhood of $\bar{t}$ the characteristics are crossing the line $x=0$ with positive slopes on both sides.
In all subcases of **1e)** there is no forward characteristic emerging from $(0,\bar{t})$ and hence property (ii) is verified.

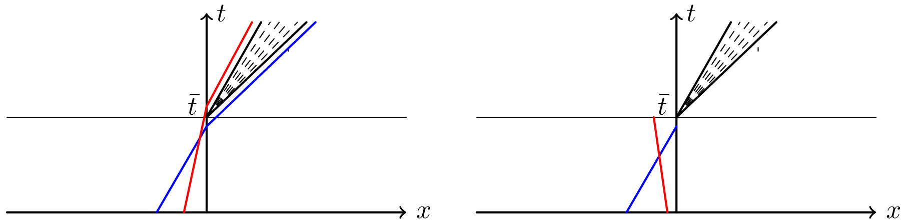

Figure 5: On the left case 1a, on the right case 2a

Case 2. $u_l(\bar{t}+) < \theta_l$ *and* $u_r(\bar{t}+) > \theta_r$.
Let $t^* > \bar{t}$ be such that $u_l(t) < \theta_l$ and $u_r(t) > \theta_r$ for all $t \in (\bar{t}, t^*)$. Then, by (1.2) this implies that $(u_l(t), u_r(t)) = (A, B)$ for all $t \in (\bar{t}, t^*)$, with $A < \theta_l$, $B > \theta_r$, and hence there holds $(u_l(\bar{t}+), u_r(\bar{t}+)) = (A, B)$.
2a) If $u(0-,\bar{t}) < A$, then tracing the backward genuine characteristics (with negative slopes) through a sequence of points $(x_n, \bar{t})$, $x_n \uparrow 0$, of continuity for u, one deduces that $u_l(\bar{t}-) = u(0-,\bar{t})$. Hence, $u_l(\bar{t}-) < A$ and $f_l(u_l(\bar{t}-)) > f(A)$. By (1.2) this implies that $u_r(\bar{t}-) < \pi_{r,-}(B)$. Observe that if $u(0+,\bar{t}) < B$, then a shock with positive slope should emerge from $(0,\bar{t})$, with left state $u_r(\bar{t}+) = B$ and right state $u(0+,\bar{t})$. But, this implies that $u(0+,\bar{t}) > \pi_{r,-}(B)$. On the other hand, from $u_r(\bar{t}-) \neq u(0+,\bar{t})$ it follows that there should be a shock with negative slope arriving in $(0,\bar{t})$ (or generated in $(0,\bar{t})$) and connecting the left state $u_r(\bar{t}-)$ with the right state $u(0+,\bar{t}) > u_r(\bar{t}-)$, which is not entropy admissible for the conservation law with flux f_r. Therefore, if $u_l(\bar{t}+) < \theta_l$, $u_r(\bar{t}+) > \theta_r$ and $u(0-,\bar{t}) < A$, then it must be $u(0+,\bar{t}) \ge B$ (see Figure 3). Hence, tracing the backward genuine characteristics (with positive slopes) through a sequence of points $(x_n, \bar{t})$, $x_n \downarrow 0$, of continuity for u, one deduces that $u_r(\bar{t}-) = u(0+,\bar{t}) \ge B$, which is in contrast with $u_r(\bar{t}-) < \pi_{r,-}(B)$. Therefore, $u_l(\bar{t}+) < \theta_l$ and $u_r(\bar{t}+) > \theta_r$ together imply $u(0-,\bar{t}) \ge A$.
2b) If $u(0-,\bar{t}) > A$, then there should be a shock with negative slope connecting the left state $u(0-,\bar{t})$ with the right state $u_l(\bar{t}+) = A$ emerging at $(0,\bar{t})$. This implies that $u(0-,\bar{t}) < \pi_{l,+}(A)$. On the other hand, if $u(0-,\bar{t}) \le \theta_l$ then tracing the backward genuine characteristics (with negative slopes) through a sequence of points $(x_n, \bar{t})$, $x_n \uparrow 0$, of continuity for u, one deduces that $u_l(\bar{t}-) = u(0-,\bar{t}) \in (A, \pi_{l,+}(A))$. This implies that $f_l(u_l(\bar{t}-)) < f_l(A)$, which is in contrast with (1.2). Hence, if $u(0-,\bar{t}) > A$, then it must be $u(0-,\bar{t}) \in (\theta_l, \pi_{l,+}(A))$. However, by (1.2) we have $u_l(\bar{t}-) \in (-\infty, A] \cup [\pi_{l,+}(A), +\infty)$, which implies $u_l(\bar{t}-) \notin (\theta_l, \pi_{l,+}(A))$. Thus, there should be a shock with positive slope arriving in $(0,\bar{t})$ (or generated in $(0,\bar{t})$) and connecting the left state $u(0-,\bar{t}) \in (\theta_l, \pi_{l,+}(A))$ with the right state $u_l(\bar{t}-) \in (-\infty, A] \cup [\pi_{l,+}(A), +\infty)$, which is not entropy admissible for the conservation law with flux f_l. Therefore, $u_l(\bar{t}+) < \theta_l$ and $u_r(\bar{t}+) > \theta_r$ together imply $u(0-,\bar{t}) = A$, and with the same arguments we deduce also that $u(0+,\bar{t}) = B$.
2c) If $u(0-,\bar{t}) = A < \theta_l$ and $u(0+,\bar{t}) = B > \theta_r$, then tracing the backward genuine characteristics through two sequences of points $(x_n, \bar{t})$, $x_n \uparrow 0$, and $(x_n, \bar{t})$, $x_n \downarrow 0$ (having negative and positive slopes, respectively), one deduces that there exists $t' < \bar{t}$ such that $u_l(t\pm) = A$ and $u_r(t\pm) = B$ for all $t \in (t', \bar{t})$. Then, set $\tau \doteq \inf\{t' < \bar{t};\ u_l(s\pm) = A < \theta_l,\ u_r(s\pm) = B > \theta_r\ \ \forall\ s \in (t', \bar{t})\}$. If $\tau > 0$, since one has $u_l(\tau+) = A$, $u_r(\tau+) = B$, repeating the above arguments of cases **2a)-2b)** one would deduce that $u_l(t\pm) = A$, $u_r(t\pm) = B$ for all $t \in (t'', \tau)$, for some $t'' < \tau$, which is in contrast with the definition of τ. Therefore it must be $\tau = 0$. On the other hand, $u_l(\bar{t}+) = A$, $u_r(\bar{t}+) = B$ clearly imply that $u_l(t\pm) = A$, $u_r(t\pm) = B$ for all $t \in (\bar{t}, t^*)$, for some $t^* > \bar{t}$. Thus, one has $u_l(t\pm) = A$, $u_r(t\pm) = B$ for all $t \in (0, t^*)$ and at any point $(0, t)$, $t \in (0, t^*)$ starts exactly two forward, genuine characteristics η', η'', which lie in $(-\infty, 0) \times (t, t^*)$ and $(0, +\infty) \times (t, t^*)$, respectively, proving property (i).
Case 3. $u_l(\bar{t}+) = \theta_l$ *or* $u_r(\bar{t}+) = \theta_r$.
Notice that, by (1.2) $u_l(\bar{t}+) = \theta_l$ implies $\theta_l = A$, while $u_r(\bar{t}+) = \theta_r$ implies $\theta_r = B$.
3a) If $u_l(\bar{t}+) = \theta_l$ and $u(0-,\bar{t}) = \theta_l$, then tracing the backward genuine characteristics through a sequence of points $(x_n, \bar{t})$, $x_n \uparrow 0$, of continuity for u, one deduces that $u_l(t) = \theta_l$ for all $t \in (0, \bar{t})$. Hence, $u_l(\bar{t}-) = \theta_l$ as well. In turn, by (1.2) this implies that $u_l(\bar{t}\pm) = A$, $u_r(\bar{t}\pm) \in \{B, \pi_{r,-}(B)\}$. Suppose that $u_r(\bar{t}+) = B$ and $u(0+,\bar{t}) < \theta_r$. Since by Definition 1.1 we have $B \ge \theta_r$, it follows that a shock with positive slope emerges from $(0,\bar{t})$, and thus $u(0+,\bar{t}) > \pi_{r,-}(B)$. However, $u_r(\bar{t}-) \in \{B, \pi_{r,-}(B)\}$ and $\pi_{r,-}(B) < u(0+,\bar{t}) < B$ imply that there should be a shock with negative slope arriving in $(0,\bar{t})$ (or generated in $(0,\bar{t})$) and connecting the left state $u_r(\bar{t}-) \in \{B, \pi_{r,-}(B)\}$ with the right state $u(0+,\bar{t}) \in (\pi_{r,-}(B), B)$, which is not entropy admissible for the conservation law with flux f_r. Therefore, if $u_r(\bar{t}+) = B$, then it must be $u(0+,\bar{t}) \ge \theta_r$. Then, tracing the backward genuine characteristics through a sequence of points $(x_n, \bar{t})$, $x_n \downarrow 0$, of continuity for u, one deduces

that $u_r(\bar{t}-) \geq \theta_r$. Since $u_r(\bar{t}-) \in \{B, \pi_{r,-}(B)\}$, this implies that $u_r(\bar{t}-) = B$. By similar arguments we deduce that, if $u_r(\bar{t}+) = \pi_{r,-}(B)$, then also $u_r(\bar{t}-) = \pi_{r,-}(B)$. Therefore, if $u_l(\bar{t}+) = \theta_l$ and $u(0-,\bar{t}) = \theta_l$, it follows that $\bar{t}$ is a point of continuity for u_l and u_r, $u_l(\bar{t}\pm) = A = \theta_l$, and $u_r(\bar{t}\pm) = B$ or $u_r(\bar{t}\pm) = \pi_{r,-}(B)$. This implies that there is no forward genuine characteristic starting from $(0,\bar{t})$ and lying on $(-\infty,0)\times(0,+\infty)$, while there is a single forward genuine characteristic starting from $(0,\bar{t})$ and lying on $(0,+\infty)\times(0,+\infty)$, which proves the property (ii).

3b) Next, assume that $u_l(\bar{t}+) = \theta_l$ and $u(0-,\bar{t}) > \theta_l$. Then, there should be a shock with negative slope connecting the left state $u(0-,\bar{t})$ with the right state θ_l emerging at $(0,\bar{t})$, which is not possible since any entropy admissible shock with right state θ_l has positive slope. Therefore, $u_l(\bar{t}+) = \theta_l$ implies that $u(0-,\bar{t}) \leq \theta_l$.

3c) Assume now that $u_l(\bar{t}+) = \theta_l > u(0-,\bar{t})$. Then, tracing the backward genuine characteristics (with negative slopes) through a sequence of points of continuity for u as above, $(x_n,\bar{t})$, $x_n \uparrow 0$, we deduce that $u_l(\bar{t}-) = u(0-,\bar{t})$. Since $\theta_l = A \neq u_l(\bar{t}-)$, by (1.2) this implies that $u_r(\bar{t}-) \leq \pi^l_{r,-}(A) = \pi_{r,-}(B)$. On the other hand, by the same observations in case **1a)** we know that $u_r(\bar{t}+) \in \{B, \pi_{r,-}(B)\}$. Moreover, with similar arguments of case **1a)** we deduce that $u_r(\bar{t}+) = B$ and $u_r(\bar{t}-) \leq \pi_{r,-}(B)$ imply $u(0+,\bar{t}) \geq \theta_r$, and $u_r(\bar{t}-) = B$. Next, assume that $u_r(\bar{t}-) \leq \pi_{r,-}(B)$, $u_r(\bar{t}+) = \pi_{r,-}(B)$. Again with similar arguments as above we deduce that $u_r(\bar{t}+) = \pi_{r,-}(B)$ implies $u(0+,\bar{t}) = \pi_{r,-}(B)$, and that there is no entropy admissible shock connecting a left state $u_r(\bar{t}-) < \pi_{r,-}(B)$ with a right state $\pi_{r,-}(B)$. Hence, if $u_r(\bar{t}-) \leq \pi_{r,-}(B)$, $u_r(\bar{t}+) = \pi_{r,-}(B)$, it must be $u_r(\bar{t}-) = \pi_{r,-}(B)$. In turn, because of (1.2) and since $u_l(\bar{t}-) < \theta_l$, this implies that $u_l(\bar{t}-) = A$, which is in contrast with $u_l(\bar{t}-) = u(0-,\bar{t}) < \theta_l = A$.

Therefore, $u_l(\bar{t}+) = \theta_l$ implies that $u(0-,\bar{t}) = \theta_l$ as well, which are the assumptions of case **1a)**, and thus property (ii) is verified. Moreover, one has $u_r(\bar{t}\pm) = u(0+,\bar{t}) \in \{B, \pi_{r,-}(B)\}$. With similar arguments we deduce that $u_r(\bar{t}+) = \theta_r$ implies $u(0+,\bar{t}) = \theta_r$, $u(0-,\bar{t}) = u_l(\bar{t}\pm) \in \{A, \pi_{l,+}(A)\}$, and then the same conclusions of the case $u_l(\bar{t}+) = \theta_l$ hold true. This completes the proof of the Proposition. □

Remark 6. By the analysis of Proposition 3.1 it follows that, if $u_l \doteq u_l(\bar{t}), u_r \doteq u_r(\bar{t})$, are the one-sided limits (0.4) at $x = 0$, and $\bar{t} > 0$, of an AB-entropy solution, then either $(u_l, u_r) = (A, B)$, or there exists a backward characteristic through $(0,\bar{t})$, defined on $[0,\bar{t}]$, and taking values in $\mathbb{R}\setminus\{0\}$ at any time $t < \bar{t}$. In this latter case, consider the minimal and maximal backward characteristics ξ_-, ξ_+ through $(0,\bar{t})$, defined on $[0,\bar{t}]$, and taking values in $\mathbb{R}$. By the proof of Proposition 3.1, and recalling the definition (2.3), we deduce that one of the following cases occurs:

1. $\xi_\pm(0) < 0$ and $(u_l, u_r) \in \mathcal{T}_1$;
2. $\xi_\pm(0) > 0$ and $(u_l, u_r) \in \mathcal{T}_2$;
3. $\xi_-(0) < \xi_+(0) = 0$, or $\xi_-(0) = 0 < \xi_+(0)$, or $\xi_-(0) < 0 < \xi_+(0)$, and $(u_l, u_r) \in \mathcal{T}_3$.

The next result shows that the upper bounds on the Dini derivative of a function $\omega \in \mathcal{A}_i(T)$, $i = 1,2,3$, given in (2.6), (2.8), (2.11), are equivalent to the monotonicity of the maps φ_i that associates to any $x \neq 0$, the starting point $\varphi_i(x)$ at time $t = 0$ of a characteristic that reaches x at time T.

Lemma 3.2. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a bounded function having right and left limits in any point. Then, the following hold.*

(i) If $\big(\omega(0-), \omega(0+)\big) \in \mathcal{T}_1$, and ω satisfies (2.5), then (2.6) holds if and only if the function

$$\varphi_1 := \begin{cases} x - f_l'(\omega(x)) \cdot T & \text{if } x < 0 \\ -f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x)) \cdot \big(T - x/f_r'(\omega(x))\big) & \text{if } 0 < x < R \\ x - f_r'(\omega(x)) \cdot T & \text{if } x > R \end{cases} \tag{3.5}$$

is nondecreasing, and the function

$$\psi_1(x) := T - x/f_r'(\omega(x)) \qquad 0 < x < R, \tag{3.6}$$

is decreasing.

(ii) If $\big(\omega(0-), \omega(0+)\big) \in \mathcal{T}_2$, and ω satisfies (2.7), then (2.8) holds if and only if the function

$$\varphi_2(x) := \begin{cases} x - f_l'(\omega(x)) \cdot T & \text{if } x < L, \\ -f_r' \circ f_{r,-}^{-1} \circ f_l(\omega(x)) \cdot \big(T - x/f_l'(\omega(x))\big) & \text{if } L < x < 0, \\ x - f_r'(\omega(x)) \cdot T & \text{if } x > 0 \end{cases} \tag{3.7}$$

is nondecreasing, and the function

$$\psi_2(x) := T - x/f_l'(\omega(x)) \qquad L < x < 0, \tag{3.8}$$

is increasing.

(iii) If ω satisfies (2.9) – (2.10)*, then the function*

$$\varphi_3(x) := \begin{cases} x - f_l'(\omega(x)) \cdot T & \text{if } x < L, \\ x - f_r'(\omega(x)) \cdot T & \text{if } x > R, \end{cases} \tag{3.9}$$

is nondecreasing if and only if (2.11) *holds.*

Proof. We prove only the statement (i), the proofs of the other two statements being entirely similar.
1. First observe that the monotonicity of φ_1, ψ_1, are equivalent to

$$D^+\varphi_1(x) \geq 0 \quad \forall\, x \in \mathbb{R}, \qquad D^+\psi_1(x) < 0 \quad \forall\, x \in (0, R)\,. \tag{3.10}$$

Next, notice that by (2.5) we have

$$f_l'(\omega(0-)) \geq 0\,, \qquad \omega(0-) \geq \theta_l\,, \qquad f_r'(\omega(x)) \cdot T - x \geq 0 \quad \forall\, x \in (0, R)\,, \tag{3.11}$$

$$R - f_r'(\omega(R+)) \cdot T \geq 0\,, \qquad f_r'(\omega(R-)) > 0\,, \qquad T - R/f_r'(\omega(R-)) \geq 0\,. \tag{3.12}$$

Moreover, $(\omega(0-), \omega(0+)) \in \mathcal{T}_1$ implies that $f_l(\omega(0-)) \geq f_r(\omega(0+))$. Hence, relying on (3.11) we deduce that

$$\omega(0-) = f_{l,+}^{-1} \circ f_l(\omega(0-)) \geq f_{l,+}^{-1} \circ f_r(\omega(0+)), \tag{3.13}$$

which in turn, together with (3.12), yields

$$\varphi_1(\omega(0-)) = -f_l'(\omega(0-)) \cdot T \leq -f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(0+)) \cdot T = \varphi_1(\omega(0+)). \tag{3.14}$$

On the other hand, since the function $f_{l,+}^{-1}$ takes values in $[\theta_l, +\infty)$ (see definition in Section 2), it follows that

$$f_l' \circ f_{l,+}^{-1} \circ f_r(v) \geq 0 \qquad \forall\, v \in \mathbb{R}\,. \tag{3.15}$$

Hence, because of (3.12), we deduce that

$$\varphi_1(R-) = -f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(R-)) \cdot \big(\mathrm{T} - R/f_r'(\omega(R-))\big) \leq 0 \leq R - f_r'(\omega(R+)) \cdot T = \varphi_1(R+). \tag{3.16}$$

Therefore, in order to establish the statement (i) it is sufficient to show that

$$D^+\varphi_1(x) \geq 0 \qquad \forall\, x \in \mathbb{R} \setminus \{0, R\} \qquad D^+\psi_1(x) < 0 \quad \forall\, x \in (0, R), \tag{3.17}$$

are verified if and only if (2.6) holds.
2. We first show that the equivalence between (2.6) and (3.17) holds at any point of discontinuity for ω. To this end observe that the maps

$$g_1(v, x) \doteq x - f_l'(v) \cdot T, \qquad g_2(v.x) \doteq \Big[-f_l' \circ f_{l,+}^{-1} \circ f_r(v)\big(\mathrm{T} - x/f_r'(v)\big)\Big]_{|\{v;\, f_r'(v)\cdot T - x \geq 0\}}, \qquad g_3(v, x) \doteq x - f_r'(v) \cdot T\,,$$

are nonincreasing in v since, by the strict convexity of the fluxes f_l, f_r, and because of (3.15), we have

$$\begin{aligned} &\partial_v\, g_1(v, x) = -f_l''(v) \cdot T < 0\,, \\ &\partial_v\, g_2(v, x) = -\frac{\big[f_l'' \circ f_{l,+}^{-1} \circ f_r(v)\big]\big[f_r'(v)\big]^2\big[f_r'(v) \cdot T - x\big] + x\big[f_l' \circ f_{l,+}^{-1} \circ f_r(v)\big]^2\big[f_r''(v)\big]}{\big[f_l' \circ f_{l,+}^{-1} \circ f_r(v)\big]\big[f_r'(v)\big]^2} \leq 0\,, \\ &\partial_v\, g_3(v, x) = -f_r''(v) \cdot T < 0\,. \end{aligned} \tag{3.18}$$

Moreover, (2.6), (3.11) and the assumption **H1)** together imply that $D^+\omega(x)$ is upper bounded since

$$D^+\omega(x) \leq \begin{cases} 1/(c \cdot T) & \text{if} \quad x < 0\,, \\ f_r'(\omega(x))/(x \cdot c) & \text{if} \quad 0 < x < R\,, \\ 1/(c \cdot T) & \text{if} \quad x > 0\,. \end{cases} \tag{3.19}$$

Hence, if x is a point of discontinuity for ω, the inequality (2.6) is verified if and only if $\omega(x-) > \omega(x+)$. On the other hand, since

$$\varphi_1(x) = \begin{cases} g_1(\omega(x), x) & \text{if} \quad x < 0\,, \\ g_2(\omega(x), x) & \text{if} \quad 0 < x < R\,, \\ g_3(\omega(x), x) & \text{if} \quad x > R\,, \end{cases}$$

by the monotonicity of the maps g_1, g_2, g_3 in v, and by the strict convexity of f_r, we have $\omega(x-) > \omega(x+)$ if and only if $\varphi_1(x-) < \varphi_1(x+)$ and $\psi_1(x-) > \psi_1(x+)$ (if $x \in (0, R)$). In turn, if x is a point of discontinuity for φ_1

and ψ_1 (if $x \in (0, R)$), then $\varphi_1(x-) < \varphi_1(x+)$, $\psi_1(x-) > \psi_1(x+)$, are verified if and only if $D^+\varphi_1(x) \geq 0$, $D^+\psi_1(x) < 0$. Thus, we conclude that in order to establish the statement (i) it is sufficient to prove that the equivalence between (2.6) and (3.17) is verified at any point of continuity for ω.

3. If $x < 0$ is a point of continuity for ω, then we get

$$D^+\varphi_1(x) = 1 + \partial_v\, g_1(\omega(x), x) \cdot D^+\omega(x) = 1 - f_l''(\omega(x)) \cdot T \cdot D^+\omega(x)\,, \tag{3.20}$$

which shows the equivalence between the first inequality in (2.6) and (3.17). With the same computation we find the equivalence between the third inequality in (2.6) and (3.17), considering a point $x > R$ of continuity for ω. Next, consider a point $0 < x < R$ where ω is continuous. Then we find

$$\begin{aligned} D^+\varphi_1(x) &= \partial_v\, g_2(\omega(x)) \cdot D^+\omega(x) + \frac{\left[f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right]}{f_r'(\omega(x))} \\ &= -\frac{\left[f_l'' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right]\left[f_r'(\omega(x))\right]^2 \left(f_r'(\omega(x)) \cdot T - x\right) + x\left[f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right]^2 f_r''(\omega(x))}{\left[f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right] \cdot \left[f_r'(\omega(x))\right]^2} \cdot D^+\omega(x) + \\ &\quad + \frac{\left[f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right]}{f_r'(\omega(x))}, \end{aligned}$$

and

$$D^+\psi_1(x) = -\frac{f_r'(\omega(x)) - x f_r''(\omega(x)) \cdot D^+\omega(x)}{\left[f_r'(\omega(x))\right]^2}\,.$$

Hence, by (3.15) we deduce that $D^+\varphi_1(x) \geq 0$ and $D^+\psi_1(x) < 0$ hold if and only if

$$\begin{aligned} &\Big[\left[f_l'' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right]\left[f_r'(\omega(x))\right]^2 \left(f_r'(\omega(x)) \cdot T - x\right) + x\left[f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right]^2 f_r''(\omega(x))\Big] \cdot D^+\omega(x) \leq \\ &\quad \leq \left[f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right]^2 \cdot f_r'(\omega(x))\,, \end{aligned} \tag{3.21}$$

and

$$x\, f_r''(\omega(x)) \cdot D^+\omega(x) < f_r'(\omega(x))\,. \tag{3.22}$$

By (3.11) and the convexity of f_r, the inequalities (3.21)-(3.22) are equivalent to the second inequality in (2.6), and the prove of the statement (i) is completed. $\square$

An immediate consequence of Lemma 3.2 is the following.

Lemma 3.3. *In the same setting and with the same notations of Theorem 2.1, the sets $\mathcal{A}_1(T), \mathcal{A}_2(T), \mathcal{A}_3^{AB}(T)$ are equivalently defined as sets of functions $\omega \in \mathbf{L}^\infty(\mathbb{R})$ having essential left and right limits at $x = 0$, that satisfy the following conditions.*

$\mathcal{A}_1(T)$ *is the set of all functions ω that satisfy $\big(\omega(0-), \omega(0+)\big) \in \mathcal{T}_1$, and for which there exists $R > 0$ such that: there holds $\omega(R-) \geq \omega(R+)$,*

$$f_l'(\omega(x)) \geq x/T + f_l'(\omega(0-)) \quad \forall\, x < 0, \qquad f_r'(\omega(x)) \geq x/T \quad \forall\, 0 < x < R, \qquad f_r'(\omega(x)) < x/T \quad \forall\, x > R, \tag{3.23}$$

the map φ_1 in (3.5) is nondecreasing, and the map ψ_1 in (3.6) is decreasing.

$\mathcal{A}_2(T)$ *is the set of all functions ω that satisfy $\big(\omega(0-), \omega(0+)\big) \in \mathcal{T}_2$, and for which there exists $L < 0$ such that: there holds $\omega(L-) \geq \omega(L+)$,*

$$f_l'(\omega(x)) > x/T \quad \forall\, x < L, \qquad f_l'(\omega(x)) \leq x/T \quad \forall\, 0 < x < L, \qquad f_r'(\omega(x)) \leq x/T + f_r'(\omega(0+)) \quad \forall\, x > 0, \tag{3.24}$$

the map φ_2 in (3.7) is nondecreasing, and the map ψ_2 in (3.8) is increasing.

$\mathcal{A}_3^{AB}(T)$ *is the set of all functions ω for which there exist $L \leq 0 \leq R$, such that:*

$$\begin{gathered} \big(\omega(0-), \omega(0+)\big) \in \begin{cases} \mathcal{T}_{3,-} \cup \mathcal{T}_{3,+} & \text{if} \quad L = R = 0\,, \\ \{(A, B)\} & \text{if} \quad L \leq 0 \leq R\,, \end{cases} \qquad \omega(L-) \geq \omega(L+)\,, \qquad \omega(R-) \geq \omega(R+)\,, \\ \omega(x) = A \quad \forall\, x \in (L, 0), \qquad \omega(x) = B \quad \forall\, x \in (0, R), \end{gathered} \tag{3.25}$$

$$\begin{gathered} f_l'(\omega(x)) \geq \begin{cases} x/T & \text{if} \;\; L < 0, \\ x/T + f_l'(\omega(0-)) & \text{if} \;\; L = 0, \end{cases} \qquad \forall\, x \in (-\infty, L), \\ f_r'(\omega(x)) \leq \begin{cases} x/T & \text{if} \;\; R < 0, \\ x/T + f_r'(\omega(0+)) & \text{if} \;\; R = 0, \end{cases} \qquad \forall\, x \in (R, +\infty), \end{gathered} \tag{3.26}$$

and the map φ_3 in (3.9) is nondecreasing.

4 Proof of Theorem 2.1

We proceed by dividing the proof into two steps: first we show that any attainable profile at time $T > 0$ of a solution to the problem (0.1),(0.3)-(0.2) satisfies all the conditions of one of the tree sets described in the statement of Lemma 3.3. Next, we prove that, for any function ω in $\mathcal{A}_1(T)$, $\mathcal{A}_2(T)$ and $\mathcal{A}_3^{AB}(T)$, there exists $\overline{u} \in \mathbf{L}^\infty(\mathbb{R})$ such that $S_T\overline{u} = \omega$.

4.1 Proof of $\mathcal{A}(\mathbf{T}) \subseteq \mathcal{A}_1(\mathbf{T}) \cup \mathcal{A}_2(\mathbf{T}) \cup \mathcal{A}_3^{\mathbf{AB}}(\mathbf{T})$.

Given $\overline{u} \in \mathbf{L}^\infty$, let $u(\cdot,t) \doteq \mathcal{S}_t^{AB}\overline{u}$, $t > 0$, we will show that $\omega \doteq \mathcal{S}_T^{AB}\overline{u}$ belongs to one of the sets $\mathcal{A}_1(T), \mathcal{A}_2(T), \mathcal{A}_3^{AB}(T)$. By Remark 1 we know that $\omega \in BV_{loc}(\mathbb{R} \setminus \{0\})$ and that ω admits one-sided limits at $x = 0$. Then, recalling Remark 6, we will distinguish the following five cases.
Case 1. $\omega(0-) = A < \theta_l$, $\omega(0+) = B > \theta_r$.
Observe that, tracing the backward characteristics through points of continuity of ω in a neighbourhood of $x = 0$, with the same arguments of the proof of Proposition 3.1 and relying on (1.2), we deduce that

$$\begin{aligned} & u_l(t) = A, \qquad u_r(t) = B \qquad \forall\, t \in (\delta_1, T)\,, \\ \omega(x) = A \qquad \forall\, x \in (-\delta_1, 0), \qquad & \omega(x) = B \qquad \forall\, x \in (0, \delta_1), \end{aligned} \tag{4.1}$$

for some there exist $\delta_1 > 0$ such that Thus, by Proposition 3.1 we deduce that

$$u_l(t) = A, \qquad u_r(t) = B \qquad \forall\, t \in (0, T)\,. \tag{4.2}$$

Next, let $R \doteq \sup\{x > 0;\, \omega(x) = B \text{ for all } y \in (0, x)\}$, $L \doteq \inf\{x < 0;\, \omega(x) = A \text{ for all } y \in (x, 0)\}$. By (4.1) one has $L < 0 < R$. Notice that $\omega(L-) \geq \omega(L+)$ and $\omega(R-) \geq \omega(R+)$ because of the Lax entropy condition (see Remark 2). Consider the maximal backward characteristic $\xi_{R,+}$ through (R, T) and assume that it crosses the axis $x = 0$ at time $t_R > 0$. Then, by (4.2) and the observations in Section 3, it follows that $\xi_{R,+}$ is a segment with positive slope $f_r'(B) = f'(\omega(R+))$. But this means that we may find $\delta_2 > 0$ such that all backward characteristics ξ_x through points (x, T), with $x \in (R, R + \delta_2)$, reach the axis $x = 0$ at times $t_x \in (\delta_2, t_R)$. This implies that $\omega(x) = u_r(t_x) = B$ for all $x \in (R, R + \delta_2)$, which is in contrast with the definition of R. Thus, the maximal backward characteristic $\xi_{R,+}$ is defined on the whole interval $[0, T]$, and there holds $\xi_{R,+}(t) \geq 0$ for all $t \in [0, T]$. With the same arguments we deduce that the minimal backward characteristic $\xi_{L,-}$ through (L, T), is defined on $[0, T]$ and there holds $\xi_{L,-}(t) \leq 0$ for all $t \in [0, T]$.

Given any $x > R$, consider the minimal and maximal backward characteristics $\xi_{x,-}, \xi_{x,+}$ through (x, T). Since $\xi_{x,\pm}, \xi_{R,+}$ are genuine characteristics for the conservation law $u_t + f_r(u)_x = 0$, it follows that they never intersect in the open quarter of plane $(0, +\infty) \times (0, +\infty)$. Hence, $\xi_{x,\pm}$ are defined on the whole interval $[0, T]$, and there holds

$$\xi_{x,-}(t) = x + f_r'(\omega(x-)) \cdot (t - T), \qquad \xi_{x,+}(t) = x + f_r'(\omega(x+)) \cdot (t - T) \qquad \forall\, t \in [0, T]\,.$$

Moreover, one has $x - f_r'(\omega(x\pm)) \cdot T = \xi_{x,\pm}(0) \geq \xi_{R,+}(0) \geq 0$, which implies $f_r'(\omega(x\pm)) \leq \frac{x}{T}$. On the other hand, recalling the definition (3.9) of φ_3, we deduce that, for every $R < x < y$, there holds $\varphi_3(x\pm) = \xi_{x,\pm}(0) \leq \xi_{y,\pm}(0) = \varphi_3(y\pm)$, which proves the nondecreasing monotonicity of φ_3 on $(R, +\infty)$. With similar arguments we deduce that $f_l'(\omega(x\pm)) \geq \frac{x}{T}$ for all $x \in (-\infty, L)$, and that φ_3 is nondecreasing also on $(-\infty, L)$. Therefore, the function ω satisfies conditions (3.25), (3.26) and φ_3 is nondecreasing on $(-\infty, L)$ and $(R, +\infty)$. Since $\varphi_3(x) \leq 0$ for all $x \in (-\infty, L)$, and $\varphi_3(x) \geq 0$ for all $x \in (R, +\infty)$, it follows that φ_3 is nonincresing on its domain and hence we have $\omega \in \mathcal{A}_3^{AB}(T)$.
Case 2. $(\omega(0-), \omega(0+)) = (A, B)$, $A = \theta_l, B > \theta_r$, or $A < \theta_l, B = \theta_r$, or $A = \theta_l, B = \theta_r$.
Assume that $A = \theta_l, B > \theta_r$, the other cases being entirely similar. Then, letting $R \doteq \sup\{x > 0;\, \omega(x) = B \text{ for all } y \in (0, x)\}$, by the same analysis of Case 1 we deduce that $R > 0$, $\omega(R-) \geq \omega(R+)$, $f_r'(\omega(x\pm)) \leq \frac{x}{T}$ for all $x > R$, and that the map φ_3 in (3.9) is nondecreasing on $(R, +\infty)$. Next, assume that there exists $x < 0$ such that $f_l'(\omega(x+)) < \frac{x}{T}$. Then, the maximal backward characteristics ξ_x starting at (x, T) crosses the axis $x = 0$ at some time $t_x > 0$. On the other hand, the maximal backward characteristics ξ_{x_n} trough a sequence of points $(x_n, T), x_n \uparrow 0$, are lines with slope $f_l'(\omega(x_n+)) \to f_l'(\omega(0-)) = 0$. Hence, there will be some n such that ξ_{x_n} intersect ξ_x in $(-\infty, 0) \times (0, +\infty)$, which is not possible. Therefore, there holds $f_l'(\omega(x\pm)) \geq \frac{x}{T}$ for all $x < 0$, and with the same arguments of Case 1 one can show that φ_3 is nondecreasing on $(-\infty, 0)$ as well, and that $\varphi_3(0-) \leq 0 < \varphi_3(R+)$. Thus, setting $L = 0$, we have shown that $\omega \in \mathcal{A}_3^{AB}(T)$.
Case 3. $(\omega(0-), \omega(0+)) \in \mathcal{T}_1$.
Notice that $(\omega(0-), \omega(0+)) \in \mathcal{T}_1$ implies $\omega(0+) > \theta_r$, and hence $f_r'(\omega(0+)) > 0$. Thus, there exist $\delta_1 > 0$ such that $f_r'(\omega(x+)) \geq \frac{x}{T}$ for all $x \in (0, \delta_1)$. Then, setting $R \doteq \sup\{x > 0;\, f_r'(\omega(x+)) \geq \frac{x}{T}\}$, one has $R > 0$ and $\omega(R-) \geq \omega(R+)$, because of the Lax entropy condition (see Remark 2). Observe that if, $f_r'(\omega(x+)) < \frac{x}{T}$ or $f_r'(\omega(x-)) < \frac{x}{T}$ for some $x \in (0, R)$, then one would deduce that the backward (minimal and maximal) characteristics $\xi_{y,\pm}$ through (y, T), $y \in (x, R)$, should cross in $(0, +\infty) \times (0, +\infty)$ the backward characteristic $\xi_{x,+}$ or $\xi_{x,+}$ through (x, T), which is not possible. Thus, there holds $f_r'(\omega(x\pm)) \geq \frac{x}{T}$ for all $x \in (0, R)$.

Next, consider the maximal backward characteristic $\xi_{R,+}$ through (R,T), and suppose that it is defined on an interval $[t_R,T]$, $t_R>0$, with $\xi_{R,+}(t_R)=0$. This means that $f_r'(\omega(R+))=\frac{R}{t_R}>\frac{R}{T}$, which implies that there exists $\delta_1>R$ such that $f_r'(\omega(x+))>\frac{x}{T}$ for all $x\in(R,\delta_1)$. But this is in contrast with the definition of R. Hence $\xi_{R,+}$ is defined on the whole interval $[0,T]$, and there holds $\xi_{R,+}(t)\geq 0$ for all $t\in[0,T]$. On the other hand, $(\omega(0-),\omega(0+))\in\mathcal{T}_1$ implies $\omega(0-)>\theta_l$, and hence the minimal backward characteristics $\xi_{0,-}$ through $(0,T)$ satisfies $\xi_{0,-}(0)=-f_l'(\omega(0-))\cdot T<0$. Then, since backward characteristics starting at points (x,T) with $x<0$ or $x>R$ cannot cross $\xi_{0,-}$ and $\xi_{R,+}$, respectively, and by the definition of R, we deduce that $f_l'(\omega(x\pm))\geq\frac{x}{T}+f_l'(\omega(0-))$ for all $x\in(-\infty,0)$ and $f_r'(\omega(x\pm))<\frac{x}{T}$ for all $x\in(R,+\infty,)$. Moreover, with the same arguments we deduce that $f_r'(\omega(x\pm))\geq\frac{x}{T}$ for all $x\in(0,R)$. Therefore, the function ω satisfies condition (3.23).

Next, with similar arguments of Case 1, we deduce that the map φ_1 defined in (3.5) is nondecreasing on the intervals $(-\infty,0)$ and $(R,+\infty)$. Regarding the monotonicity of φ_1,ψ_1 (defined in (3.6)) on $(0,R)$, first observe that, since the Lax entropy condition implies $\omega(x-)\geq\omega(x+)$, by the strict convexity of f_r it follows that $\psi_1(x-)>\psi_1(x+)$ at any point $x\in(0,R)$ of discontinuity for ω. Next, consider the maximal backward characteristic $\xi_{x,+}$ through (x,T), $0<x<R$, and the minimal backward characteristic $\xi_{y,-}$ through (y,T), $x<y<R$, given by

$$\xi_{x,+}(t)=x+f_r'(\omega(x+))\cdot(t-T)\quad t\in[t_x,T],\qquad \xi_{y,-}(t)=y+f_r'(\omega(y-))\cdot(t-T)\quad t\in[t_y,T]\,,$$

with $\xi_{x,+}(t_x)=\xi_{y,-}(t_y)=0$, $t_x\doteq\psi_1(x+)$, $t_y\doteq\psi_1(y-)$. Since $\xi_{x,+}$, $\xi_{y,-}$ cannot cross on $(0,+\infty)\times(0,+\infty)$, one has $t_x\geq t_y$. On the other hand, if $t_x=t_y$, then there would be two forward characteristics with positive slope issuing form $(0,t_x)$, which is in contrast with Proposition 3.1. Thus, it must be $\psi_1(x+)=t_x>t_y=\psi_1(y-)$, which proves the decreasing monotonicity of ψ_1.

The monotonicity of ψ_1 in particular implies $\psi_1(x\pm)>\psi_1(R-)$ for all $x\in(0,R)$. Observe that $u_r(t\pm)>\theta_r$ for all $t\in(\psi_1(R-),T)$, since any point $(0,t),t\in(\psi_1(R-),T)$ is reached by a backward characteristic (crossing $x=0$ with positive slope) issuing from a point $(x,T),x\in(0,R)$. In turn, this implies that $u_l(t\pm)>\theta_l$ for any time $t\in(\psi_1(R-),T)$ of continuity for u_l,u_r, since otherwise, by (1.2) we should have $u_l(\bar t-)=A$, $u_r(\bar t-)=B$, for some $\bar t\in(\psi_1(R-),T)$. But, by the analysis of Proposition 3.1, this implies that either

$$u_l(t)=A,\qquad u_r(t)=B\quad\ \forall\, t\in(0,\bar t\,),\qquad u_l(\bar t+)>\theta_l\,,\qquad u_r(\bar t+)<\theta_r\,,$$

or

$$u_l(t)=A,\qquad u_r(t)=B\quad\ \forall\, t\in(0,T),$$

which are in contrast with $u_r(t\pm)>\theta_r$ for all $t\in(\psi_1(R-),T)$, and with $(\omega(0-),\omega(0+))\in\mathcal{T}_1$, respectively. Therefore, we have $u_l(t\pm)>\theta_l$ for all $t\in(\psi_1(R-),T)$. Hence, by (1.2), (2.2), there holds $u_l(t)=\pi^r_{l,+}(u_r(t))$ at any time $t\in(\psi_1(R-),T)$ of continuity for u_l,u_r. Hence, in particular for $t_x\doteq\psi_1(x+)$, $t_y\doteq\psi_1(y-)$ we find

$$u_l(t_x-)=\pi^r_{l,+}(u_r(t_x-))=\pi^r_{l,+}(\omega(x+))\,,\qquad u_l(t_y+)=\pi^r_{l,+}(u_r(t_y+))=\pi^r_{l,+}(\omega(y-))\,. \tag{4.3}$$

Consider now the backward characteristics (for $u_t+f_l(u)_x=0$) $\zeta_{t_x,-}$, $\zeta_{t_y,+}$, issuing from $(0,t_x)$ and from $(0,t_y)$, respectively, given by

$$\begin{aligned}\zeta_{t_x,-}(t)&=f_l'(u_l(t_x-))\cdot(t-t_x)=f_l'\big(\pi^r_{l,+}(\omega(x+))\big)\cdot(t-t_x)\qquad t\in[0,t_x],\\ \zeta_{t_y,+}(t)&=f_l'(u_l(t_y+))\cdot(t-t_y)=f_l'\big(\pi^r_{l,+}(\omega(y-))\big)\cdot(t-t_y)\qquad t\in[0,t_y,]\,.\end{aligned}$$

By definitions (2.2), (3.5), , (3.6), we find that

$$\zeta_{t_x,-}(0)=-f_l'\big(\pi^r_{l,+}(\omega(x+))\big)\cdot\big(T-x/f_r'(\omega(x+))\big)=-f_l'\circ f_{l,+}^{-1}\circ f_r(\omega(x+))\cdot\big(T-x/f_r'(\omega(x+))\big)=\phi_1(x+)\,,$$

$$\zeta_{t_y,+}(0)=-f_l'\big(\pi^r_{l,+}(\omega(y-))\big)\cdot\big(T-y/f_r'(\omega(y-))\big)=-f_l'\circ f_{l,+}^{-1}\circ f_r(\omega(y-))\cdot\big(T-y/f_r'(\omega(y-))\big)=\phi_1(y-)\,.$$

Since $t_x>t_y$ and because backward characteristics cannot cross on $(-\infty,0)\times(0,+\infty)$, it follows that $\phi_1(x+)=\zeta_{t_x,-}(0)\leq\zeta_{t_y,+}(0)=\phi_1(y-)$, proving the nondecreasing monotonicity of φ_1. This completes the proof that $\omega\in\mathcal{A}_1(T)$ in the case $(\omega(0-),\omega(0+))\in\mathcal{T}_1$.

Case 4. $(\omega(0-),\omega(0+))\in\mathcal{T}_2$.
With entirely similar arguments to Case 3, we deduce that $\omega\in\mathcal{A}_2(T)$.

Case 5. $(\omega(0-),\omega(0+))\in\mathcal{T}_{3,-}\cup\mathcal{T}_{3,+}$.
We assume that $\omega(0-)>\theta_l$, $\omega(0+)>\theta_r$. The cases with $\omega(0-)=\theta_l$, or $\omega(0+)=\theta_r$, can be treated with entirely similar arguments, relying on the analysis of Case 2. Let $\xi_{0,-},\xi_{0,+}$ be the minimal and maximal backward characteristics through $(0,T)$. Then we have $\xi_{0,-}(0)=-f_l'(\omega(0-))\cdot T<0<\xi_{0,+}(0)=-f_r'(\omega(0+))\cdot T$. Since backward characteristics starting at points (x,T) with $x<0$ or $x>0$ cannot cross $\xi_{0,-}$ and $\xi_{0,+}$, respectively, we deduce that $f_l'(\omega(x\pm))\geq\frac{x}{T}+f_l'(\omega(0-))$ for all $x<0$ and $f_r'(\omega(x\pm))\leq\frac{x}{T}+f_r'(\omega(0+))$ for all $x>0$. Thus, setting $L=R=0$, the conditions (3.26) are satisfied. Moreover, with the same arguments of Case 1 we deduce that the map φ_3 in (3.9) is nondecreasing. Hence, we have shown that $\omega\in\mathcal{A}_3^{AB}(T)$, and this completes the proof of $\mathcal{A}(T)\subseteq\mathcal{A}_1(T)\cup\mathcal{A}_2(T)\cup\mathcal{A}_3^{AB}(T)$. $\square$

4.2 Proof of $\mathcal{A}_1(\mathbf{T}) \cup \mathcal{A}_2(\mathbf{T}) \cup \mathcal{A}_3^{\mathbf{AB}}(\mathbf{T}) \subseteq \mathcal{A}(\mathbf{T})$.

Given a function $\omega \in \mathcal{A}_1(T) \cup \mathcal{A}_2(T) \cup \mathcal{A}_3^{AB}(T)$, we will show that there exists an initial datum $\overline{u} \in \mathbf{L}^\infty(\mathbb{R})$ such that $\mathcal{S}_T^{AB}\overline{u} = \omega$. We shall analyze only two cases, the others being entirely similar.

Case 1. $\omega \in \mathcal{A}_2(T)$.

We assume that $\omega(0-) > \pi^r_{l,-}(\omega(0+))$, the case $\omega(0-) = \pi^r_{l,-}(\omega(0+))$ being entirely similar and simpler. Hence, we have $\pi^l_{r,-}(\omega(0-)) > \omega(0+)$. We will construct the initial datum $\overline{u}$ with the desired property adopting a similar procedure to [6], which consists of the following steps:

1. For every $x \neq 0$ we trace the lines $\vartheta_{x,-}$,$\vartheta_{x,+}$ through (T, x) with slope $f'_l(\omega(x-))$, $f'_l(\omega(x+))$, respectively, if $x < 0$, and $f'_r(\omega(x-))$, $f'_r(\omega(x+))$, respectively, if $x > 0$. At $x = 0$ we trace the lines $\vartheta_{0,-}$, $\vartheta_{0,+}$ through $(T, 0)$ with slope $f'_r(\pi^l_{r,-}(\omega(0-)))$, $f'_r(\omega(0+)$, respectively. Because of (2.7), ϑ^-_L and all lines $\{\vartheta_{x,\pm} : x \geq 0 \text{ or } x < L\}$ reach the x-axis without crossing the line $x = 0$ at times $t > 0$, while $\vartheta_{L,+}$ and all lines $\{\vartheta_{x,\pm} : L < x < 0\}$ cross the line $x = 0$ at a time $t \geq 0$. Then, we redefine $\vartheta_{L,+}$ and $\{\vartheta_{x,\pm} : L < x < 0\}$ as polygonal lines that, after crossing $x = 0$, continue with slope $f'_r(\pi^l_{r,-}(\omega(L+)))$ and $f'_r(\pi^l_{r,-}(\omega(x-)))$, $f'_r(\pi^l_{r,-}(\omega(x+))$, respectively. Since the curves $\vartheta_{x,\pm}$ are defined so that one has $\vartheta_{x,\pm}(0) = \varphi_2(x\pm)$ for all x, from the monotonicity of the map φ_2 in (3.7) we deduce that $\vartheta_{x,\pm}$ never intersect each other in the region $\mathbb{R} \times (0, T)$. We will treat the polygonal lines $\vartheta_{x,\pm}$, $x \in \mathbb{R}$, as (minimal and maximal) backward characteristics of the AB-entropy solution that we are constructing on $\mathbb{R} \times [0, T]$.

2. Since the solution is constant along genuine characteristics, for every $x \in (-\infty, \vartheta_{L,-}(0)) \cup (\vartheta_{0,+}(0), +\infty)$ such that $x = \vartheta_{y,\pm}(0)$ for some $y \in (-\infty, L) \cup (0, +\infty)$, we will set $\overline{u}(x) = \omega(y\pm)$, while for for every $x \in (\vartheta_{L,+}(0), \vartheta_{0,-}(0))$ such that $x = \vartheta_{y,\pm}(0)$ for some $y \in (L, 0)$, we will set $\overline{u}(x) = \pi^l_{r,-}(\omega(y\pm))$. The set of remaining x is a disjoint union of countably many open intervals, say (x^-_n, x^+_n), $n \in \mathbb{N}$, with $x^-_n = \vartheta_{y_n,-}(0), x^+_n = \vartheta_{y_n,+}(0)$, for some $y_n \in \mathbb{R}$, where $\overline{u}$ is defined so to produce a compression wave which generates a discontinuity at the point (y_n, T).

3. According with the definition of $\overline{u}$ in step 2, we define a function $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ which is constant along the lines $\vartheta_{x,\pm}$ that do not cross $x = 0$, and it is piecewise constant along the polygonal lines $\vartheta_{x,\pm}$ that intersect $x = 0$, changing value at $x = 0$ so to satisfy the interface entropy condition (1.2). Namely, we set u equal to $\omega(y\pm)$ along the line $\vartheta_{y,\pm}(t), t \in [0, T]$, when $y \in (-\infty, L) \cup (0, +\infty)$, and along the segment of polygonal $\vartheta_{y,\pm}(t), t \in [\tau_y, T]$, with $\vartheta_{y,\pm}(\tau_y) = 0$, when $y \in (L, 0)$. Instead we define u as $\pi^l_{r,-}(\omega(y\pm))$ along the segment of polygonal $\vartheta_{y,\pm}(t), t \in [0, \tau_y]$, with $\vartheta_{y,\pm}(\tau_y) = 0$, when $y \in (L, 0)$. Finally, for any $x \in (x^-_n, x^+_n)$ we let u to be equal to $\overline{u}(x)$ on the right of $x = 0$ and to be equal to $\pi^r_{l,-}(\overline{u}(x))$ on the left of $x = 0$, along a polygonal line $\eta_x(t)$, $t \in [0, T]$, which connects $(x, 0)$ with (y_n, T).

4. With the same arguments of [6] one can show that the function u constructed in step 3: is locally Lipschitz continuous on $\mathbb{R} \times [0, T]$; it is a classical solution of $u_t + f_l(u)_x$ on $(-\infty, 0) \times (0, T)$, and of $u_t + f_r(u)_x$ on $(0, +\infty) \times (0, T)$; it is continuous with respect to the the $\mathbf{L^1}_{loc}$ topology as a function from $[0, T]$ to $\mathbf{L}^\infty(\mathbb{R})$; it attains the initial data $\overline{u}$ at time $t = 0$ and the terminal profile ω at time $t = T$. Moreover, u satisfies the interface entropy condition (1.2) associated to the connection AB.

1. For each $x \neq 0, L$, consider the polygonal lines

$$\vartheta_{x,\pm}(t) := \begin{cases} x + f'_l(\omega(x\pm))(t - T) & \text{if } \ x < L, \ t \in [0, T], \\ x + f'_l(\omega(x\pm))(t - T) & \text{if } \ L < x < 0, \ t \in \big[T - x/f'_l(\omega(x\pm)),\, T\big], \\ f'_r(\pi^l_{r,-}(\omega(x\pm)))\big(t - T + x/f'_l(\omega(x\pm))\big) & \text{if } \ L < x < 0, \ t \in \big[0,\, T - x/f'_l(\omega(x\pm))\big], \\ x + f'_r(\omega(x\pm))(t - T) & \text{if } \ x > 0, \ t \in [0, T], \end{cases} \tag{4.4}$$

and, at $x = 0$, $x = L$, set

$$\begin{aligned} \vartheta_{0,-}(t) &:= f'_r(\pi^l_{r,-}(\omega(0-)))(t - T) \qquad \text{if} \qquad t \in [0, T], \\ \vartheta_{0,+}(t) &:= f'_r((\omega(0+))(t - T) \qquad \text{if} \qquad t \in [0, T], \\ \vartheta_{L,-}(t) &:= L + f'_l(\omega(L-))(t - T) \qquad \text{if} \qquad t \in [0, T], \\ \vartheta_{L,+}(t) &:= \begin{cases} x + f'_l(\omega(L+))(t - T) & \text{if } \ t \in \big[T - x/f'_l(\omega(L+)),\, T\big], \\ f'_r(\pi^l_{r,-}(\omega(L+)))\big(t - T + x/f'_l(\omega(L+))\big) & \text{if } \ t \in \big[0,\, T - x/f'_l(\omega(L+))\big]. \end{cases} \end{aligned} \tag{4.5}$$

Notice that, by definitions (2.2), (3.7), (3.8), we have $\vartheta_{x,\pm}(0) = \varphi_2(x\pm)$ for all x, and $\vartheta_{x,\pm}(\psi_2(x\pm)) = 0$ for all $x \in (L, 0)$. Then, relying on (2.7), on the nondecreasing monotonicity of φ_2, and on the increasing monotonicity of ψ_2, we deduce that the polygonal lines $\vartheta_{x,\pm}$, $x \in \mathbb{R}$, never intersect each other in the region $\mathbb{R} \times (0, T)$.

2. Consider the following partition of $\mathbb{R}$ (see Fig.6):

$$\begin{aligned}
\mathcal{I}_R &\doteq \big\{x\in\mathbb{R}:\ \ \exists\ y<z:\ \vartheta_{y,+}(0)=\vartheta_{z,-}(0)=x\big\},\\
\mathcal{I}_C &\doteq \big\{x\in\mathbb{R}:\ \ \nexists\ y\in\mathbb{R}:\ \vartheta_{y,-}(0)=x\ \text{ or }\ \vartheta_{y,+}(0)=x\big\},\\
\mathcal{I}_W &\doteq \big\{x\in\mathbb{R}:\ \ \exists!\ y:\ \vartheta_{y,-}(0)=x\ \text{ or }\ \vartheta_{y,+}(0)=x\big\}.
\end{aligned}\tag{4.6}$$

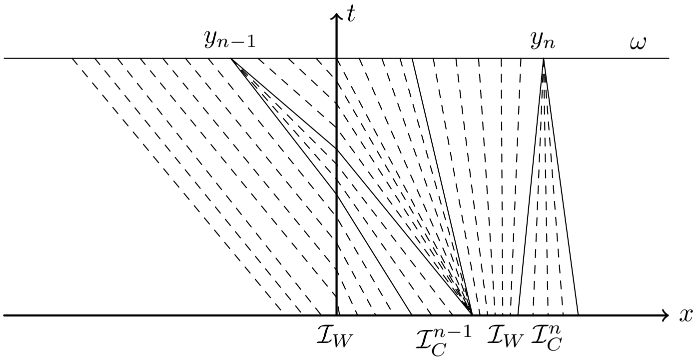

Figure 6: An example of partition of $\mathbb{R}$ associated to the profile ω

Some considerations about this partition are useful for the next. The set $\mathcal{I}_R$ consists of the centres of rarefaction waves originated at time $t=0$, the set $\mathcal{I}_C$ consists of the starting points of the compression waves that generate shocks at time T, and $\mathcal{I}_W$ collects the starting points of all other waves. With entirely similar arguments to [6], one can verify that:

- The set $\mathcal{I}_R$ contains at most countably many points;
- The set $\mathcal{I}_C$ is a disjoint union of at most countably many open intervals of the form

$$\begin{aligned}
&\mathcal{I}^n=(x_n^-,x_n^+), \qquad x_n^\pm=\vartheta_{y_n,\pm}(0), \qquad y_n\in(-\infty,L)\cup[0,+\infty),\\
&\mathcal{I}_L^n=(x_n^-,x_n^+), \qquad x_n^\pm=\vartheta_{y_n,\pm}(0), \qquad y_n\in[L,0)\,,
\end{aligned}\tag{4.7}$$

  with y_n point of discontinuity of ω. Notice that, since $\vartheta_{y_n,\pm}(0)=\varphi_2(y_n\pm)$, by the monotonicity of φ_2 and f_l, f_r', it follows that $\omega(y_n-)>\omega(y_n+)$ for all $y_n\neq 0$. Moreover, we observed at the beginning that we have $\pi^l_{r,-}(\omega(0-))>\omega(0+)$. Thus, we will construct compression waves generating a shock connecting the states $\omega(y_n-),\omega(y_n+)$ at (y_n,T), $y_n\neq 0$, and connecting the states $\pi^l_{r,-}(\omega(0-)),\omega(0+)$ at $(0,T)$.

In order to define the initial data in the sets $\mathcal{I}_L^n$, for any (x_n^-,x_n^+) with $x_n^\pm=\vartheta_{y_n,\pm}(0)$, $L<y_n<0$, setting $\alpha_n^\pm\doteq f_r'(\pi^l_{r,-}(\omega(y_n\pm)))$, consider the function

$$h_n(x,\alpha)=T-y_n/\big[f_l'\circ\pi^r_{l,-}\circ(f_r')^{-1}(\alpha)\big]+x/\alpha \qquad x\in(x_n^-,x_n^+),\quad \alpha\in[\alpha_n^+,\alpha_n^-]\,.\tag{4.8}$$

Notice that, because of the monotonicity of f_r', $\pi^l_{r,-}$, and since by (3.24) we have $\omega(y_n+)<\omega(y_n-)<\theta_l$, it follows that $\alpha_n^+<\alpha_n^-<0$. Moreover, letting $\tau_n^\pm$ be the times of intersection of $\vartheta_{y_n,\pm}$ with $x=0$, i.e., such that $\vartheta_{y_n,\pm}(\tau_n^\pm)=0$, we have

$$\tau_n^\pm=T-y_n/f_l'(\omega(y_n^\pm))=T-y_n/\big[f_l'\circ\pi^r_{l,-}\circ(f_r')^{-1}(\alpha_n^\pm)\big]=-x_n^\pm/\alpha_n^\pm\,.\tag{4.9}$$

Then, by a direct computation one finds that, for any $x\in(x_n^-,x_n^+)$, there holds

$$\begin{gathered}
h_n(x,\alpha_n^+)=(x-x_n^+)/\alpha_n^+>0, \qquad\qquad h_n(x,\alpha_n^-)=(x-x_n^-)/\alpha_n^-<0\,,\\
\partial_\alpha h_n(x,\alpha)=y_n\frac{\alpha\cdot f_l''\circ\pi^r_{l,-}\circ(f_l')^{-1}(\alpha)}{\big[f_l'\circ\pi^r_{l,-}\circ(f_r')^{-1}(\alpha)\big]^2\cdot\big[f_l'\circ\pi^r_{l,-}\circ(f_r')^{-1}(\alpha)\big]\cdot\big[f_r''\circ(f_r')^{-1}(\alpha)\big]}-\frac{x}{\alpha^2}<0 \qquad \forall\ \alpha\in(\alpha_n^+,\alpha_n^-)\,.
\end{gathered}\tag{4.10}$$

Hence, we may define a continuous, decreasing map $\alpha_n:(x_n^-,x_n^+)\to(\alpha_n^+,\alpha_n^-)$ that satisfies

$$T-y_n/\big[f_l'\circ\pi^r_{l,-}\circ(f_r')^{-1}(\alpha_n(x))\big]=-x/\alpha_n(x) \qquad x\in(x_n^-,x_n^+)\,.\tag{4.11}$$

Notice that $\lim_{x\to x_n^\pm}\alpha_n(x)=\alpha_n^\pm$. The quantity $\alpha_n(x)$ determines the slope λ_n^+ on the right of $x=0$ of a polygonal line η_x, connecting $(x,0)$ and (y_n,T), with the property that, letting $\lambda_n^-\doteq f_l'\circ\pi^r_{l,-}\circ(f_r')^{-1}(\alpha_n(x))$ be the slope of η_x on the left of $x=0$, there holds

$$(f_l')^{-1}(\lambda_n^-)=\pi^r_{l,-}\big((f_r')^{-1}(\lambda_n^+)\big)\,,\tag{4.12}$$

which guarantees that the states $u_l \doteq (f_l')^{-1}(\lambda_n^-)$, $u_r \doteq (f_l')^{-1}(\lambda_n^+)$, satisfy the interface entropy condition (1.2). In the case $(x_n^-, x_n^+) \subset \mathcal{I}_L^n$ is of the form $x_n^\pm = \vartheta_{L,\pm}(0)$, $\vartheta_{L,-}(0) < 0 < \vartheta_{L,+}(0)$, with the same arguments of above we may define a continuous, decreasing function $\alpha_n : [0, x_n^+) \to (\alpha_n^+, y_n/T]$ that satisfies the equalities in (4.11) for all $x \in [0, x_n^+)$, and there holds $\alpha_n(0) = y_n/T$. Then, we define the initial data as

$$\overline{u}(x) := \begin{cases} \omega(y\pm) & \text{if} \quad x \in \mathcal{I}_W, \quad x = \vartheta_{y,\pm}(0), \quad y \in (-\infty, L) \cup (0, +\infty), \\ \pi_{r,-}^l(\omega(0-)) & \text{if} \quad x \in \mathcal{I}_W, \quad x = \vartheta_{0,-}(0), \\ \omega(0+) & \text{if} \quad x \in \mathcal{I}_W, \quad x = \vartheta_{0,+}(0), \\ \omega(L-) & \text{if} \quad x \in \mathcal{I}_W, \quad x = \vartheta_{L,-}(0), \\ \pi_{r,-}^l(\omega(L+)) & \text{if} \quad x \in \mathcal{I}_W, \quad x = \vartheta_{L,+}(0), \\ \pi_{r,-}^l(\omega(y\pm)) & \text{if} \quad x \in \mathcal{I}_W, \quad x = \vartheta_{y,\pm}(0), \quad y \in (L, 0), \\ (f_l')^{-1}\big((y_n - x)/T\big) & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \quad y_n < L, \\ (f_r')^{-1}\big((y_n - x)/T\big) & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \quad y_n \geq 0, \\ (f_r')^{-1}(\alpha_n(x)) & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \quad L < y_n < 0, \\ (f_r')^{-1}(\alpha_n(x)) & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{L,\pm}(0), \quad x \geq 0, \\ (f_l')^{-1}\big((y_n - x)/T\big) & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{L,\pm}(0), \quad x < 0. \end{cases} \tag{4.13}$$

Notice that $\overline{u}$ is not defined on the set $\mathcal{I}_R$ which is of measure zero since it is countable. Moreover, we have

$$|\overline{u}(x)| \leq M \doteq \sup\big\{\max\{|\omega(x)|, |\pi_{r,-}^l(\omega(x))|\};\, x \in \mathbb{R}\big\}\,. \tag{4.14}$$

3. In order to define the solution u in the region of compression waves, for any $x \in \mathcal{I}_C$, consider the polygonal lines

$$\eta_x(t) := \begin{cases} x + \big((y_n - x)\,t\big)/T & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \quad y_n < L \text{ or } y_n \geq 0, \quad t \in [0, T], \\ x + \alpha_n(x)\,t & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \quad L < y_n < 0, \quad t \in [0, -\frac{x}{\alpha_n(x)}], \\ x + f_l' \circ \pi_{l,-}^r \circ (f_r')^{-1}(\alpha_n(x))\,t & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \quad L < y_n < 0, \quad t \in [-\frac{x}{\alpha_n(x)}, T], \\ x + \alpha_n(x)\,t & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{L,\pm}(0), \quad x \geq 0, \quad t \in [0, -\frac{x}{\alpha_n(x)}], \\ x + f_l' \circ \pi_{l,-}^r \circ (f_r')^{-1}(\alpha_n(x))\,t & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{L,\pm}(0), \quad x \geq 0, \quad t \in [-\frac{x}{\alpha_n(x)}, T], \\ x + \big((y_n - x)\,t\big)/T & \text{if} \quad x \in (x_n^-, x_n^+) \subseteq \mathcal{I}_C, \quad x_n^\pm = \vartheta_{L,\pm}(0), \quad x < 0, \quad t \in [0, T]. \end{cases} \tag{4.15}$$

Observe that, by construction the polygonal lines $\vartheta_{x,\pm}$, $x \in \mathbb{R}$ in (4.4)-(4.5), and $\eta_x, x \in \mathcal{I}_C$ in (4.15), never intersect each other in the region $\mathbb{R} \times (0, T)$ and there holds

$$\forall\ (x,t) \in \mathbb{R} \times (0,T) \quad \exists!\ y \in \mathbb{R} \qquad \text{s.t.} \quad x = \vartheta_{y,-}(t), \quad \text{or} \quad x = \vartheta_{y,+}(t) \quad \text{or} \quad x = \eta_y(t),\ y \in \mathcal{I}_C\,. \tag{4.16}$$

Thus, we may define on $(\mathbb{R} \setminus \{0\}) \times (0, T)$ the function:

$$u(x,t) := \begin{cases} \omega(y\pm) & \text{if} \quad \exists\ y \in (-\infty, L) \cup (0, +\infty) :\ x = \vartheta_{y,\pm}(t), \\ \omega(y\pm) & \text{if} \quad \exists\ y \in [L, 0) :\ x = \vartheta_{y,\pm}(t) < 0, \\ \pi_{r,-}^l(\omega(y\pm)) & \text{if} \quad \exists\ y \in [L, 0) :\ x = \vartheta_{y,\pm}(t) > 0, \\ \pi_{r,-}^l(\omega(0-)) & \text{if} \quad x = \vartheta_{0,-}(t), \\ \omega(0+) & \text{if} \quad x = \vartheta_{0,+}(t), \\ \overline{u}(y) & \text{if} \quad \exists\ y \in \mathcal{I}_L^n :\ x = \eta_y(t) > 0, \\ \pi_{l,-}^r(\overline{u}(y)) & \text{if} \quad \exists\ y \in \mathcal{I}_L^n :\ x = \eta_y(t) < 0, \\ \overline{u}(y) & \text{if} \quad \exists\ y \in \mathcal{I}^n :\ x = \eta_y(t)\,. \end{cases} \tag{4.17}$$

4. By construction the function u in (4.17) is continuous on $\mathbb{R} \times (0, T)$ and satisfies the interface entropy condition (1.2) at $x = 0$. Moreover, with the same type of analysis in [6] one can show that there holds

$$D_x^- u(x,t) \geq \big[f_{l,r}''(u(x,t)) \cdot (t - T)\big]^{-1} \geq \big[c \cdot (t - T)\big]^{-1} \qquad \forall (x,t) \in \mathbb{R} \times (0,T)\,. \tag{4.18}$$

On the other hand, relying on (4.14), (4.17), and on the assumption **H1)**, with the same arguments of the proof of $\mathcal{A}(T) \subseteq \mathcal{A}_1(T) \cup \mathcal{A}_2(T) \cup \mathcal{A}_3^{AB}(T)$ we derive

$$
D_x^+ u(x,t) \le \begin{cases} \big[f_l''(u(x,t))\cdot t\big]^{-1} \le \big[c\cdot t\big]^{-1} & \forall\ x<\eta_0(t)\,,\\ \dfrac{f_l'(u(x,t))}{f_l''(u(x,t))\cdot x} \le M'\big[c\cdot x\big]^{-1} & \forall\ \eta_0(t)<x<0,\\ \big[f_r''(u(x,t))\cdot t\big]^{-1} \le \big[c\cdot t\big]^{-1} & \forall\ x>0\,. \end{cases} \tag{4.19}
$$

for some constant $M'>0$. Hence u is locally Lipschitz continuous and therefore it is differentiable almost everywhere. By a direct computation one can check that u is a classical solution of $u_t+f_l(u)_x$ on $(-\infty,0)\times(0,T)$, and of $u_t+f_r(u)_x$ on $(0,+\infty)\times(0,T)$. Hence, u is an AB-entropy solution of (0.1),(0.3). Finally, with the same arguments in [6] one verifies the continuity of $t\to u(\cdot,t)$ on $[0,T]$ with respect to the $\mathbf{L^1}_{loc}$-topology, and that $u(\cdot,0)=\overline{u}$, $u(\cdot,T)=\omega$, which proves that $\omega=S_T^{AB}\,\overline{u}\in\mathcal{A}(T)$.

Case 2. $\omega\in\mathcal{A}_3^{AB}(T),\ \ L=0=R,\ \ (\omega(0-),\omega(0+))\in\mathcal{T}_{3,-}$.
Since $(\omega(0-),\omega(0+))\in\mathcal{T}_{3,-}$ it follows that $\omega(0-)\ge\theta_l$ and $f_l(\omega(0-))\le f_r(\omega(0+))$. We assume that $\omega(0-)>\theta_l$, and that $f_l(\omega(0-))<f_r(\omega(0+))$, the cases with $\omega(0-)=\theta_l$ or with $f_l(\omega(0-))=f_r(\omega(0+))$ being entirely similar. We follow the same procedure of the previous case discussing only the points where there is a difference in the construction of the initial data $\overline{u}$ and of the solution u.

1. For each $x\neq 0$, consider the lines

$$
\vartheta_{x,\pm}(t):=\begin{cases} x+f_l'(\omega(x\pm))(t-T) & \text{if}\quad x<0,\ t\in[0,T],\\ x+f_r'(\omega(x\pm))(t-T) & \text{if}\quad x>0,\ t\in[0,T], \end{cases} \tag{4.20}
$$

and, for $x=0$, set

$$
\begin{aligned} \vartheta_{0,-}(t) &:= f_l'(\omega(0-))(t-T), \qquad \vartheta_{0,+}(t):=f_r'(\omega(0+))(t-T),\\ \vartheta_{0,*}(t) &:= f_r'(\pi^l_{r,-}(\omega(0-)))(t-T), \end{aligned} \qquad \forall\ t\in[0,T]. \tag{4.21}
$$

2. Then, letting $x_0^\pm\doteq\vartheta_{0,\pm}(0)$, $x_0^*\doteq\vartheta_{0,*}(0)$, consider the partition of $\mathbb{R}\setminus\{0\}$:

$$
\begin{aligned} \mathcal{I}_R &\doteq \big\{x\in\mathbb{R}:\ \ \exists\ y<z:\ \vartheta_{y,+}(0)=\vartheta_{z,-}(0)=x\big\},\\ \mathcal{I}_C &\doteq \big\{x\in\mathbb{R}\setminus[x_0^-,x^*]:\ \ \nexists\ y\in\mathbb{R}:\ \vartheta_{y,-}(0)=x\ \ \text{or}\ \ \vartheta_{y,+}(0)=x\big\},\\ \mathcal{I}_W &\doteq \big\{x\in\mathbb{R}:\ \ \exists!\ y:\ \vartheta_{y,-}(0)=x\ \ \text{or}\ \ \vartheta_{y,+}(0)=x\big\},\\ \mathcal{I}_{0,-} &\doteq (x_0^-,0), \qquad \mathcal{I}_{0,+}\doteq(0,x_0^*)\,. \end{aligned} \tag{4.22}
$$

Here $\mathcal{I}_{0,-},\mathcal{I}_{0,+}$ are intervals where the initial data $\overline{u}$ will assume the constant value $\omega(0-)$ and $\pi^l_{r,-}(\omega(0-))$, respectively, while $\mathcal{I}_C$ is a disjoint union of at most countably many open intervals of the form

$$
\begin{aligned} \mathcal{I}^n &= (x_n^-,x_n^+), \quad x_n^\pm=\vartheta_{y_n,\pm}(0), \qquad y_n\in(-\infty,0)\cup(0,+\infty),\\ \mathcal{I}_0 &= (x_0^*,x_0^+), \quad x_0^*=\vartheta_{0,*}(0), \qquad x_0^+=\vartheta_{0,+}(0), \end{aligned} \tag{4.23}
$$

with y_n point of discontinuity of ω. Observe that $x_0^*>0$, and that $\omega(0-)>\theta_l$, $f_l(\omega(0-))<f_r(\omega(0+))$, together imply $\omega(0+)<\pi^l_{r,-}(\omega(0-))$. Hence, the states $\pi^l_{r,-}(\omega(0-)),\omega(0+)$ are connected by a shock with negative slope for the conservation law $u_t+f_r(u)_x$. Thus, we will define the initial data $\overline{u}$ on $\mathcal{I}_0$ so to produce a compression wave that generates a shock at $(0,T)$. Thus, we define

$$
\overline{u}(x):=\begin{cases} \omega(y\pm) & \text{if}\quad x\in\mathcal{I}_W,\ \ x=\vartheta_{y,\pm}(0),\ \ y\in\mathbb{R},\\ \omega(0-) & \text{if}\quad x\in(x_0^-,0),\ \ x_0^-=\vartheta_{0,-}(0),\\ \pi^l_{r,-}(\omega(0-)) & \text{if}\quad x\in(0,x_0^*),\ \ x_0^*=\vartheta_{0,*}(0),\\ (f_l')^{-1}\big((y_n-x)/T\big) & \text{if}\quad x\in(x_n^-,x_n^+)\subseteq\mathcal{I}_C,\ \ x_n^\pm=\vartheta_{y_n,\pm}(0),\ \ y_n<0,\\ (f_r')^{-1}\big((y_n-x)/T\big) & \text{if}\quad x\in(x_n^-,x_n^+)\subseteq\mathcal{I}_C,\ \ x_n^\pm=\vartheta_{y_n,\pm}(0),\ \ y_n>0,\\ (f_r')^{-1}\big(-x/T\big) & \text{if}\quad x\in(x_0^*,x_0^+)\subseteq\mathcal{I}_C,\ \ x_0^*=\vartheta_{0,*}(0),\ \ x_0^+=\vartheta_{0,+}(0). \end{cases} \tag{4.24}
$$

3. Then, setting for every $x\in\mathcal{I}_C$:

$$
\eta_x(t):=\begin{cases} x+\big((y_n-x)\,t\big)/T & \text{if}\quad x\in(x_n^-,x_n^+)\subseteq\mathcal{I}_C,\ \ x_n^\pm=\vartheta_{y_n,\pm}(0),\ \ y_n\neq0,\ \ t\in[0,T],\\ -(x\,t)/T & \text{if}\quad x\in(x_0^*,x_0^+)\subseteq\mathcal{I}_C,\ \ x_0^*=\vartheta_{0,*}(0),\ \ x_0^+=\vartheta_{0,+}(0), \end{cases} \tag{4.25}
$$

we define on $(\mathbb{R}\setminus\{0\})\times(0,T)$ the function:

$$
u(x,t):=\begin{cases} \omega(y\pm) & \text{if}\quad \exists\ y\in\mathbb{R}:\ x=\vartheta_{y,\pm}(t),\\ \omega(0-) & \text{if}\quad \vartheta_{0,-}(t)<x<0,\\ \pi^l_{r,-}(\omega(0-)) & \text{if}\quad 0<x<\vartheta_{0,*}(t),\\ \overline{u}(y) & \text{if}\quad \exists\ y\in\mathcal{I}_C:\ x=\eta_y(t)\,. \end{cases}\ . \tag{4.26}
$$

4. Observe that, since $(\omega(0-),\omega(0+))\in\mathcal{T}_{3,-}$, it follows that the pair $u_l(t)=\omega(0-), u_r(t)=\pi^l_{r,-}(\omega(0-))$ satisfies the interface entropy condition (1.2). Then, with the same arguments of the previous case, we conclude that u is an AB-entropy solution of (0.1),(0.3)-(0.2), and that $\omega=\mathcal{S}^{AB}_T\,\overline{u}$. This proves that $\omega\in\mathcal{A}(T)$, and completes the proof of Theorem 2.1. $\square$

5 Proof of Theorem 2.2

The proof is devided in three steps.

Step 1. Let $\mathcal{U}$ be as in (2.13) and let $\mathscr{C}\subset\mathscr{C}_f$ be a compact set of connections. Given $T>0$, $\{\overline{u}_n\}_n\subset\mathcal{U}$, and $(A,B)\in\mathscr{C}$, $\{(A_n,B_n)\}_n\subset\mathscr{C}$, consider the sequences

$$\big\{\mathcal{S}^{AB}_T\overline{u}_n\big\}_n,\qquad \big\{\mathcal{S}^{A_nB_n}_T\overline{u}_n\big\}_n,\qquad \big\{\mathcal{S}^{AB}_{(\cdot)}\overline{u}_n\,\big|_T\big\}_n,\qquad \big\{\mathcal{S}^{A_nB_n}_{(\cdot)}\overline{u}_n\,\big|_T\big\}_n, \tag{5.1}$$

where $u\big|_T$ denotes the restriction to $\mathbb{R}\times[0,T]$ of a map defined on $\mathbb{R}\times[0,+\infty)$. Since, G in (2.13) is bounded, $\mathscr{C}$ is compact and because of (3.3) in Remark 5, there holds

$$\big\|\mathcal{S}^{AB}_t\overline{u}_n\big\|_{\mathbf{L}^\infty(\mathbb{R})}\le C,\qquad \big\|\mathcal{S}^{A_nB_n}_t\overline{u}_n\big\|_{\mathbf{L}^\infty(\mathbb{R})}\le C\qquad \forall\, t\ge 0,\ \forall\, n\,, \tag{5.2}$$

for some constant $C>0$. Hence, the first two sequences in (5.1) are *weakly** relatively compact in $\mathbf{L}^\infty(\mathbb{R})$, the latter two are *weakly** relatively compact in $\mathbf{L}^\infty(\mathbb{R}\times[0,T])$. Thus, we can assume that

$$\overline{u}_n \ \overset{*}{\rightharpoonup}\ \overline{u}\quad \mathit{in}\quad \mathbf{L}^\infty(\mathbb{R})\,,\qquad (A_n,B_n)\ \to\ (\,\widetilde{A},\widetilde{B}\,)\,, \tag{5.3}$$

for some $\overline{u}\in\mathbf{L}^\infty(\mathbb{R})$, $(\,\widetilde{A},\widetilde{B}\,)\in\mathscr{C}$, and that

$$\mathcal{S}^{AB}_T\overline{u}_n\ \overset{*}{\rightharpoonup}\ \omega^{AB},\qquad \mathcal{S}^{A_nB_n}_T\overline{u}_n\ \overset{*}{\rightharpoonup}\ \omega^{\widetilde{A}\widetilde{B}}\quad \mathit{in}\quad \mathbf{L}^\infty(\mathbb{R}), \tag{5.4}$$

$$\mathcal{S}^{AB}_{(\cdot)}\overline{u}_n\,\big|_T\ \overset{*}{\rightharpoonup}\ u^{AB},\qquad \mathcal{S}^{A_nB_n}_{(\cdot)}\overline{u}_n\,\big|_T\ \overset{*}{\rightharpoonup}\ u^{\widetilde{A}\widetilde{B}}\quad \mathit{in}\quad \mathbf{L}^\infty(\mathbb{R}\times[0,T]), \tag{5.5}$$

for some functions ω^{AB}, $\omega^{\widetilde{A}\widetilde{B}}\in\mathbf{L}^\infty(\mathbb{R})$ and u^{AB}, $u^{\widetilde{A}\widetilde{B}}\in\mathbf{L}^\infty(\mathbb{R}\times[0,T])$. Notice that, since $\overline{u}_n(x)\in G(x)$ for almost every $x\in\mathbb{R}$, and because G is convex closed valued, by Mazur's lemma it follows from (5.3) that $\overline{u}\in\mathcal{U}$. We will show that there exist subsequences of (5.1) that converge in the $\mathbf{L^1}_{loc}$ topology to ω^{AB}, $\omega^{\widetilde{A}\widetilde{B}}$, and u^{AB}, $u^{\widetilde{A}\widetilde{B}}$, respectively, and that

$$\omega^{AB}=\mathcal{S}^{AB}_T\,\overline{u},\qquad \omega^{\widetilde{A}\widetilde{B}}=\mathcal{S}^{\widetilde{A}\widetilde{B}}_T\,\overline{u}\,,\qquad u^{AB}=\mathcal{S}^{AB}_{(\cdot)}\,\overline{u}\,\big|_T\,,\qquad u^{\widetilde{A}\widetilde{B}}=\mathcal{S}^{\widetilde{A}\widetilde{B}}_{(\cdot)}\,\overline{u}\,\big|_T\,, \tag{5.6}$$

which proves the compactness of the sets $\mathcal{A}^{AB}(T,\mathcal{U})$, $\mathcal{A}(T,\mathcal{U},\mathscr{C})$ and $\mathcal{A}^{AB}(\mathcal{U})$, $\mathcal{A}(\mathcal{U},\mathscr{C})$.

Step 2. Notice that, by Remark 3, for any $0<a<b$, there exists $C_{a,b},L_{a,b}>0$ such that, setting $I_{a,b}\doteq[-b,-a]\cup[a,b]$, one has

$$\begin{aligned}
&\mathrm{Tot.Var.}\big\{\mathcal{S}^{AB}_t\overline{u}_n:I_{a,b}\big\}\le C_{a,b},\qquad \mathrm{Tot.Var.}\big\{\mathcal{S}^{A_nB_n}_t\overline{u}_n:I_{a,b}\big\}\le C_{a,b},\qquad \forall\,t\in[a,T],\ \forall\,n,\\
&\big\|\mathcal{S}^{AB}_t\overline{u}_n-\mathcal{S}^{AB}_s\overline{u}_n\big\|_{\mathbf{L^1(I_{a,b})}}\le L_{a,b}\cdot|t-s|,\qquad \big\|\mathcal{S}^{A_nB_n}_t\overline{u}_n-\mathcal{S}^{A_nB_n}_s\overline{u}_n\big\|_{\mathbf{L^1(I_{a,b})}}\le L_{a,b}\cdot|t-s|\qquad \forall\,t,s\in[a,T],\ \forall\,n.
\end{aligned}$$

By Helly's theorem there exists subsequences $\big\{\mathcal{S}^{AB}_t\overline{u}_{n_j}\big\}_j$, $\big\{\mathcal{S}^{A_{n_j}B_{n_j}}_t\overline{u}_{n_j}\big\}_j$, which converges to some functions $w(\cdot,t)$ and $\widetilde{w}(\cdot,t)$, respectively, in $\mathbf{L^1}(I_{a,b})$ for all $t\in[0,T]$. Because of (5.5), the functions $w,\widetilde{w}$ must coincide with the restriction to $I_{a,b}\times[0,T]$ of u^{AB} and $u^{\widetilde{A},\widetilde{B}}$, respectively, and there holds

$$\mathcal{S}^{AB}_t\overline{u}_{n_j}\ \to\ u^{AB}(\cdot,t)\,,\qquad \mathcal{S}^{A_{n_j}B_{n_j}}_t\overline{u}_{n_j}\ \to\ u^{\widetilde{A}\widetilde{B}}(\cdot,t)\quad \mathit{in}\quad \mathbf{L^1}(I_{a,b})\,,\qquad \forall\,t\in[a,T]\,.$$

Then, repeating the same arguments for $I_{a_j,b_j}\doteq[-b_j,-a_j]\cup[a_j,b_j]$, with $a_j\downarrow 0$, $b_j\to+\infty$, and observing that by (5.2) one has $\|u^{AB}(\cdot,t)\|_{\mathbf{L}^\infty(\mathbb{R})}\le C$, $\|u^{\widetilde{A}\widetilde{B}}(\cdot,t)\|_{\mathbf{L}^\infty(\mathbb{R})}\le C$, for all $t\in[0,T]$, we deduce that we can select diagonal subsequences (still denoted with index j) such that

$$\mathcal{S}^{AB}_t\overline{u}_{n_j}\ \to\ u^{AB}(\cdot,t)\,,\qquad \mathcal{S}^{A_{n_j}B_{n_j}}_t\overline{u}_{n_j}\ \to\ u^{\widetilde{A}\widetilde{B}}(\cdot,t)\quad \mathit{in}\quad \mathbf{L^1}_{loc}(\mathbb{R})\,,\qquad \forall\,t\in(0,T]\,. \tag{5.7}$$

In particular, because of (5.4), (5.7), we have $u^{AB}(\cdot,T)=\omega^{AB}$, $u^{\widetilde{A}\widetilde{B}}(\cdot,T)=\omega^{\widetilde{A}\widetilde{B}}$. Therefore, in order to establish (5.6), it remains to show only that

$$u^{AB}=\mathcal{S}^{AB}_{(\cdot)}\,\overline{u}\,\big|_T\,,\qquad u^{\widetilde{A}\widetilde{B}}=\mathcal{S}^{\widetilde{A}\widetilde{B}}_{(\cdot)}\,\overline{u}\,\big|_T\,. \tag{5.8}$$

We will provide only a proof of the second equality in (5.8), the proof of the first one being entirely similar.

Step 3. First observe that, by the regularity of f_l, f_r, the convergence (5.7) implies that

$$\begin{aligned} f_l\big(\mathcal{S}_t^{A_{n_j}B_{n_j}}\overline{u}_{n_j}\big) &\to f_l\big(u^{\widetilde{A}\widetilde{B}}(\cdot,t)\big) \quad in \quad \mathbf{L^1}_{loc}((-\infty,0]) \\ f_r\big(\mathcal{S}_t^{A_{n_j}B_{n_j}}\overline{u}_{n_j}\big) &\to f_r\big(u^{\widetilde{A}\widetilde{B}}(\cdot,t)\big) \quad in \quad \mathbf{L^1}_{loc}([0,+\infty)), \end{aligned} \qquad \forall\, t \in (0,T]. \tag{5.9}$$

Therefore, since $u_{n_j}(\cdot,t) \doteq \mathcal{S}_t^{A_{n_j}B_{n_j}}\overline{u}_{n_j}$, $t \geq 0$, are in particular weak solutions of the Cauchy problem for (0.1),(0.3) with initial data $\overline{u}_{n_j}$, relying on (5.7), (5.9), and on (5.3), we find

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^{\infty}\Big\{u^{\widetilde{A}\widetilde{B}}\phi_t + f\big(x,u^{\widetilde{A}\widetilde{B}}\big)\phi_x\Big\}dxdt + \int_{-\infty}^{\infty}\overline{u}(x)\phi(x,0)dx = \\ &= \lim_{j\to\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\{u_{n_j}\phi_t + f\big(x,u_{n_j}\big)\phi_x\}dxdt + \int_{-\infty}^{\infty}\overline{u}_{n_j}(x)\phi(x,0)dx \;=\; 0, \end{aligned} \tag{5.10}$$

for any test function $\phi \in \mathcal{C}_c^1$ with compact support contained in $\mathbb{R}\times(0,+\infty)$, which shows that $u^{\widetilde{A}\widetilde{B}}$ is a weak solution of the Cauchy problem (0.1),(0.3)-(0.2). Next, setting $I_l \doteq (-\infty,0)$, $I_r \doteq (0,+\infty)$, with the same arguments we derive

$$\begin{aligned} &\int_{I_{l,r}}\int_0^{+\infty}\Big\{\big|u^{\widetilde{A}\widetilde{B}}-k\big|\phi_t + \big(f_{l,r}\big(u^{\widetilde{A}\widetilde{B}}\big)-f_{l,r}(k)\big)\,\mathrm{sgn}\,\big(u^{\widetilde{A}\widetilde{B}}-k\big)\phi_x\Big\}dxdt = \\ &\qquad \lim_{j\to\infty}\int_{I_{l,r}}\int_0^{+\infty}\Big\{\big|u_{n_j}-k\big|\phi_t + \big(f_{l,r}\big(u_{n_j}\big)-f_{l,r}(k)\big)\,\mathrm{sgn}\,\big(u_{n_j}-k\big)\phi_x\Big\}dxdt \;\geq\; 0, \end{aligned}$$

for any non negative function $\phi \in \mathcal{C}^1$ with compact support in $I_{l,r}\times(0,T]$ and for any $k\in\mathbb{R}$. Therefore, since $u^{\widetilde{A}\widetilde{B}}$ is a weak solution of the Cauchy problem (0.1),(0.3)-(0.2), that satisfies the Kružhkov entropy inequalities on $(\mathbb{R}\setminus\{0\})\times(0,T]$, invoking a result in [20] (see also [23, Coroll. 6.8.4]) we deduce that the map $t\to u^{\widetilde{A}\widetilde{B}}(\cdot,t)$ is continuous from $[0,T]$ in $\mathbf{L}^1_{loc}(\mathbb{R})$, and that the initial condition (0.2) is satisfied. This shows that $u^{\widetilde{A}\widetilde{B}}$ satisfies conditions (i)-(ii) of Definition 1.2.

Finally, observing that by definition (1.1) and because of (5.3), there holds $k_j \doteq k_{A_{n_j}\,B_{n_j}} \to k_{\widetilde{A}\,\widetilde{B}}$ in $\mathbf{L^1}_{loc}(\mathbb{R})$, we deduce that $u^{\widetilde{A}\widetilde{B}}$ satisfies also the Kružhkov-type entropy inequality associated to the $\widetilde{A}\widetilde{B}$-connection. Namely, for any non negative function $\phi\in\mathcal{C}^1$ with compact support in $\mathbb{R}\times(0,T]$, we get

$$\begin{aligned} &\int_{-\infty}^{+\infty}\int_0^{\infty}\{\big|u^{\widetilde{A}\widetilde{B}}-k_{\widetilde{A}\widetilde{B}}(x)\big|\phi_t + \big(f\big(x,u^{\widetilde{A}\widetilde{B}}\big)-f\big(x,k_{\widetilde{A}\widetilde{B}}(x)\big)\big)\,\mathrm{sgn}\,\big(u^{\widetilde{A}\widetilde{B}}-k_{\widetilde{A}\widetilde{B}}(x)\big)\phi_x\}dxdt = \\ &\qquad \lim_{j\to\infty}\int\int\{\big|u_{n_j}-k_j(x)\big|\phi_t + \big(f\big(u_{n_j}\big)-f\big(k_j(x)\big)\big)\,\mathrm{sgn}\,\big(u_{n_j}-k_j(x)\big)\phi_x\}dxdt \;\geq\; 0, \end{aligned}$$

which shows that $u^{\widetilde{A}\widetilde{B}}$ is an $\widetilde{A}\widetilde{B}$-entropy solution of the Cauchy problem (0.1),(0.3)-(0.2) on $\mathbb{R}\times[0,T]$, according with definition (1.1) . Thus, by uniqueness of $\widetilde{A}\widetilde{B}$-entropy solutions of the Cauchy problem (see Theorem 1.1), we deduce that $u^{\widetilde{A}\widetilde{B}} = \mathcal{S}_{(\cdot)}^{\widetilde{A}\widetilde{B}}\,\overline{u}\,\big|_T$, completing the proof of Theorem 2.2. □

6 Some applications in LWR traffic flow models

Starting from the seminal papers by Lighthill, Whitham [44] and Richards [50], the evolution of unidirectional traffic flow along an highway can be described at a macroscopic level with a partial differential equation (LWR model) where the dynamical variable is the traffic density $\rho(x,t)$ (the number of vehicles per unit length). The LWR model expresses the mass conservation, i.e. the conservation of the total number of vehicles, and postulates that the average traffic speed $v(x,t)$ is a function of the traffic density alone. Thus, the mean traffic flow (the number of cars crossing the point x per unit time) is given by $f(x,t) = \rho(x,t)\,v(\rho(x,t))$, and we are lead to the hyperbolic conservation law

$$\rho_t + (\rho\, v(\rho))_x = 0\,. \tag{6.1}$$

Here $\rho(x,t)$ takes values in the interval $[0,\rho_{max}]$, where ρ_{max} represents the situation in which the vehicles are bumper to bumper and thus depends only on the average length of the vehicles. The velocity $v(\rho)$ has a maximum value v_{max} (representing the limit speed) attained at $\rho = 0$, and it is strictly decreasing since in presence of larger number of cars each driver goes slower. Hence, the corresponding flux $f(\rho) = \rho\, v(\rho)$ (the so-called *fundamental diagram*) is a (uniformly) strictly concave map (see Figure 7), satisfying the assumptions

H1)′ $f_l, f_r : \mathbb{R}\to\mathbb{R}$ are twice continuously differentiable, (uniformly) strictly concave maps

$$\max\big\{f_l''(u)\ f_r''(u)\big\} \leq -c < 0 \qquad \forall\, u\in\mathbb{R},$$

and **H2)-H3)** in Section 1 (with ρ_{max} in place of 1). We refer to [26, 46, 51] for general references on macroscopic models of traffic flow.

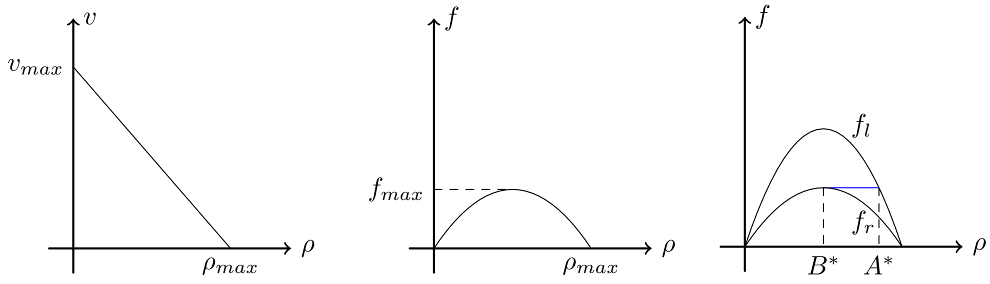

Figure 7: Velocity and flux in the LWR model, and a discontinuous flux with critical connection

The occurrence of special events (like heavy rain) that alter the road condition, or the presence along the road of obstacles (such as speed bumps, construction sites) that hinder the traffic flow, may force the vehicles to slow down or speed up in different sections of the highway. These inhomogeneities of the road are described by considering different speed-density relationships (and therefore different fundamental diagrams) on different portions of the highway. Assuming for simplicity that the change in the flow-density relation in two sections of the road of infinite length occurs at $x = 0$, we are led to a conservation law with discontinuous flux $f(x, \rho)$ as in (0.3), where the right and left fluxes are of the form $f_{l,r}(\rho) = \rho\, v_{l,r}(\rho)$. This model was considered in [45] where it was employed an admissibility criterion for the one-sided limits of the solution at $x = 0$ according with the flux maximization principle. Such a criterion is equivalent to an interface entropy condition as in (1.2) relative to a critical connection (A^*, B^*) passing trough the minimum of the two points of maximum of f_l, f_r (see Figure 7). Here, since we are considering a two-flux concave flux, we replace in the AB-interface entropy condition (1.2) the $\leq$ signs with the $\geq$ signs and viceversa. This implies that the flux of an AB-entropy solution along the discontinuity $\{x = 0\}$ must be smaller or equal to the value of the flux on the connection. We let $\mathcal{S}^*$ denote the solution operator for (0.1), (0.3) with fluxes $f_{l,r}(\rho) = \rho\, v_{l,r}(\rho)$, $\rho \in [0, \rho_{max}]$, and connection (A^*, B^*). Since our analysis will be focused on a finite section of the road, we shall assume that all the initial data have support in a bounded set $K \subset \mathbb{R}$. One can derive similar characterization of the attainable set provided by Theorem 2.1 in the two-flux concave case. Thus, the results stated in Theorem 2.2 and Corollary 2.3 continue to hold as well in the concave-concave case.

In this setting we shall consider two type of optimization problems. In the first one we treat as control parameters only the initial data. Instead, in the latter we regard as control parameters also the connection states whose flux value provides an upper limitation on the flux of the solution at $x = 0$. Such a control can be viewed as a *local point constraint control* acting at $x = 0$ (cfr. [21]). Similar problems in the context of a junction are treated in [4].

Output least square optimization with traffic density target. In order to validate the LWR models employed by transport engineers, it is fundamental to compare the experimental data with the solutions that better approximate a given observable function. A classical cost functional adopted to this pourpose is the $\mathbf{L^2}$-distance from an observation output (see for instance [35]). Thus, we are led to consider the optimization problem

$$\min_{\overline{\rho} \in \mathcal{U}} \int_{\mathbb{R}} |S^*_T \overline{\rho}(x) - l(x)|^2 dx, \tag{6.2}$$

where $\mathcal{U}$ is the admissible control set

$$\mathcal{U} \doteq \Big\{ \overline{u} \in \mathbf{L}^\infty(\mathbb{R});\ \overline{u}(x) \in G(x) \ \text{ for a.e. } \ x \in \mathbb{R} \Big\}, \tag{6.3}$$

with

$$G(x) = \begin{cases} [0, \rho_{max}] & \text{if} \quad x \in K\,, \\ \{0\} & \text{otherwise,} \end{cases} \tag{6.4}$$

and $l \in L^2(\mathbb{R})$ is a given target function. Notice that, by Remark 5, there will be some bounded set $K' \subset \mathbb{R}$ such that

$$\mathcal{S}^*_t\, \overline{s} \in \Omega \doteq \Big\{ \omega \in \mathbf{L}^\infty(\mathbb{R}; [0, c]);\ \text{supp}(\omega) \subset K' \Big\} \qquad \forall\, \overline{s} \in \mathcal{U},\ t \geq 0\,. \tag{6.5}$$

Therefore, since the map $\omega \mapsto \int_{\mathbb{R}} |\omega(x) - l(x)|^2 dx$ is clearly continuous on Ω with respect to the $\mathbf{L}^1(\mathbb{R})$ topology, we deduce the existence of a solution to problem (6.2) from the natural extension of Corollary 2.3 to the two-flux concave case.

Alternatively, in order to address road safety issues in planning design, it is important to analyse the initial density distributions and the (upper) flow limitations at the flux discontinuity interface which lead to the closest configuration to a desired density distribution. For example, one may consider two stretches of road of different capacities connected at a junction located in front of a school, where one may regulate the maximum rate at which the vehicles pass through the junction. In this case, it would be interesting to analyze the solutions of the optimization problem

$$\min_{\overline{\rho}\in\mathcal{U},\,(A,B)\in\mathscr{C}} \int_{\mathbb{R}} |S_T^{AB}\overline{\rho}(x) - l(x)|^2 dx, \tag{6.6}$$

where T is the exit time from school, $l \in L^2(\mathbb{R})$ represents a "safe" traffic distribution, $\mathcal{U}$ is the set of admissible initial data as above, and $\mathscr{C}$ is a compact set of connections. Again, relying on the analogous result of Corollary 2.3 for the two-flux concave case, we deduce the existence of a solution to (6.6).

Fuel consumption optimization. Traffic simulation is a fundamental instrument to predict the impact of road design and to examine the performance of traffic facilities under changing surface conditions. In this context, a major challenge for transport planners is to design solutions for mitigating pollution, which has huge economic impact, beside affecting people's quality of life. Various definitions to quantify the overall fuel consumption have been introduced in the literature (see [55]). We employ here the definition proposed in [49] where the fuel consumption rate of a single vehicle is expressed by a polynomial function P depending only on the average traffic speed $v(\rho)$. The overall fuel consumption rate is then obtained multiplying P by the density ρ. Thus, if we consider two stretches of road of different capacities connected at a junction where we may regulate the maximum flow rate of traffic, and we are interested in analyzing the initial density distribution that produces the minimum fuel consumption in a given interval of time $[0, T]$, we are led to the optimisation problem

$$\min_{\overline{\rho}\in\mathcal{U},\,(A,B)\in\mathscr{C}} \int_0^T \int_{\mathbb{R}} S_t^{AB}\overline{\rho}(x) P\big(v(S_t^{AB}\overline{\rho}(x))\big) dx dt\,, \tag{6.7}$$

with $\mathcal{U}$ and $\mathscr{C}$ as above. Observe that, by Remark 5, there will be some bounded set $K' \subset \mathbb{R}$ such that

$$\mathcal{S}_{(\cdot)}^{AB}\,\overline{\rho} \in \Omega \doteq \Big\{\omega \in \mathbf{L}^\infty(\mathbb{R}\times[0,T];[0,\rho_{max}]);\ \mathrm{supp}(\omega) \subset K'\Big\} \qquad \forall\,\overline{\rho} \in \mathcal{U},\ (A,B) \in \mathscr{C}\,. \tag{6.8}$$

Hence, since the map $\omega \mapsto \int_{\mathbb{T}\times[0,T]} \omega(x,t) P\big(v(\omega(x,t))\big) dx dt$ is continuous on Ω with respect to the $\mathbf{L}^1(\mathbb{R}\times[0,T])$ topology, we deduce the existence of a solution to (6.7) from the analogous result of Corollary 2.3 for the two-flux concave case.

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