

ADVECTION DIFFUSION EQUATIONS WITH SOBOLEV VELOCITY FIELD

ELIA BRUÉ AND QUOC-HUNG NGUYEN

ABSTRACT. In this note we study advection-diffusion equations associated to incompressible $W^{1,p}$ velocity fields with $p > 2$. We present new estimates on the energy dissipation rate and we discuss applications to the study of upper bounds on the enhanced dissipation rate, lower bounds on the L^2 norm of the density, and quantitative vanishing viscosity estimates. The key tools employed in our argument are a propagation of regularity result, coming from the study of transport equations, and a new result connecting the energy dissipation rate to regularity estimates for transport equations. Eventually we provide examples which underline the sharpness of our estimates.

Key words: Advection-diffusion equation with Sobolev vector field; transport equation; log-Sobolev space; Bressan's mixing conjecture.

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INTRODUCTION AND MAIN RESULT

Let \mathbb{T}^d be the torus of dimension $d \geq 2$ and $T \in (0, +\infty]$. Given a divergence-free velocity field $b \in L^1([0, T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ with $p > 1$, and an initial datum $u_0 \in L^\infty(\mathbb{T}^d)$ we study the

School of Mathematics, Institute for Advanced Study, 1 Einstein Dr. Princeton NJ 05840 U.S.A., elia.brue@math.ias.edu.

ShanghaiTech University, 393 Middle Huaxia Road, Pudong, Shanghai, 201210, China, qhnguyen@shanghaitech.edu.cn.

Cauchy problem associated to the advection-diffusion equation

$$\begin{cases} \partial_t u^\nu + b \cdot \nabla u^\nu - \nu \Delta u^\nu = 0 & \text{on } \mathbb{T}^d \times (0, T] \\ u^\nu(0, x) = u_0(x), \end{cases} \quad (E_\nu)$$

and the linear transport equation

$$\begin{cases} \partial_t u^0 + b \cdot \nabla u^0 = 0 & \text{on } \mathbb{T}^d \times (0, T] \\ u^0(0, x) = u_0(x). \end{cases} \quad (E_0)$$

Above, $\nu > 0$ is a constant molecular diffusivity. In order to ease notation we often write $u_t^\nu(x)$ and $b_t(x)$ in place of, respectively, $u^\nu(t, x)$ and $b(t, x)$.

Solutions to (E_ν) and (E_0) are understood in the distributional sense, are mean free, and belong to the natural classes

$$u^\nu \in L^\infty([0, T] \times \mathbb{T}^d) \cap C([0, T], L^2(\mathbb{T}^d)) \cap L^2([0, T], W^{1,2}(\mathbb{T}^d)), \quad (0.1)$$

and $u^0 \in C([0, T], (L^\infty(\mathbb{T}^d), w^*))$, where $(L^\infty(\mathbb{T}^d), w^*)$ denotes the space of bounded functions endowed with the weak-star topology.

Existence and uniqueness of solutions to (E_0) are guaranteed by the DiPerna-Lions theory [DPL89, A04] (see also [AC14]). Regarding the advection-diffusion equation, standard energy estimates ensure that (E_ν) posses a unique solution in (0.1) which satisfies the energy balance

$$\|u_t^\nu\|_{L^2}^2 - \|u_0\|_{L^2}^2 = -2\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds \quad \text{for every } t \in [0, T]. \quad (0.2)$$

Motivated by recent developments in the mathematical understanding of the *dissipation enhancement by mixing* [CKRZ08, BCZ17, CZDE18, FI19, DEIJ2019, CZDO19], in this note we study quantitative properties of solutions to (E_ν) at low regularity, i.e. in the setting of Sobolev divergence-free velocity fields. This framework is quite natural in view of possible applications to problems coming from fluid dynamics and conservation laws, where very often the setting of smooth vector fields is too restrictive.

For transport problems, a theory in weaker regularity settings has been developed in the last decades and it is nowadays clear that nonuniqueness results [MSz18, MSz19, MS19, BCDL20] and new loss of regularity phenomena [ACM14, ACM16, ACM18, J16, BN18c] may occur. These phenomena affect also advection-diffusion problems leading to challenging open questions.

Enhanced dissipation and mixing. Enhanced dissipation is the notion that solutions to (E_ν) dissipate the energy $\|u_t^\nu\|_{L^2}$ faster than $e^{-\nu t}$, the rate at which the heat equation dissipates energy. More rigorously, we give the following definition (Cf. [CZDR19, Definition 1]).

Definition 0.1. Let $r : (0, \nu_0) \rightarrow (0, 1)$ be an increasing function satisfying

$$\lim_{\nu \rightarrow 0} \frac{\nu}{r(\nu)} = 0.$$

We say that a divergence-free vector field b is diffusion enhancing on a subspace $H \subset L^2(\mathbb{T}^d)$ with rate $r(\nu)$, if for any $\nu \in (0, \nu_0)$ there exists $t_\nu > 0$ such that

$$\|u_t^\nu\|_{L^2}^2 \leq C e^{-r(\nu)t} \|u_0\|_{L^2}^2 \quad \text{for every } t \geq t_\nu, \text{ and } u_0 \in H. \quad (0.3)$$

The constant $C > 0$ above depends only on b .

It is nowadays well known that mixing by the flow of b is responsible for enhancing diffusion. [CKRZ08, CZDE18, FI19].

Definition 0.2. Let $\rho : (0, \infty) \rightarrow [0, \infty)$ be a decreasing function satisfying $\lim_{t \rightarrow +\infty} \rho(t) = 0$. We say that a time dependent divergence-free velocity field b on \mathbb{T}^d mixes with rate ρ if for any $t_0 > 0$, and $u_{t_0} \in W^{1,2}$, with $\int u_{t_0} dx = 0$, denoting by $u : [t_0, \infty) \rightarrow \mathbb{R}$ the solution to (E_ν) starting from u_{t_0} at time $t = t_0$, one has

$$\|u_t\|_{H^{-1}} \leq \rho(t - t_0) \|u_{t_0}\|_{W^{1,2}} \quad \text{for any } t \geq t_0.$$

In [CZDE18, FI19] it has been estimated the diffusion enhancing rate $r(\nu)$ in terms of the mixing rate $\rho(t)$, when the drift is Lipschitz regular uniformly in time, i.e. $b \in L_t^\infty W_x^{1,\infty}$.

Let us recall that, for smooth velocity fields, a simple Gronwall argument gives

$$\|u_t\|_{H^{-1}} \geq e^{-t\|\nabla b\|_{L^\infty}} \frac{\|u_0\|_{L^2}^2}{\|\nabla u_0\|_{L^2}} \quad \text{for all } t \geq 0 \text{ and } u_0 \in W^{1,2}(\mathbb{T}^d) \quad (0.4)$$

ensuring that the mixing rate cannot be faster than exponential. In this meaningful case, i.e. $\rho(t) := Me^{-\mu t}$ for some constants $M > 0$ and $\mu > 0$, the diffusion enhancing rate obtained in [CZDE18, Theorem 2.5] is

$$r(\nu) = C \log(1/\nu)^{-2} \quad \text{with } C = C(M, \mu, \|\nabla b\|_{L^\infty}). \quad (0.5)$$

As far as we know it is not known whether a velocity fields having a diffusion enhancing rate slower than $r(\nu) = O(\log(1/\nu)^{-2})$ does exist. However, relying on an old result by Poon [Poon96, MD18]

$$\|u_t^\nu\|_{L^2}^2 \geq \|u_0\|_{L^2}^2 \exp \left\{ -\nu \frac{\|\nabla u_0\|_{L^2}^2}{\|u_0\|_{L^2}^2} \int_0^t \exp \left\{ 2 \int_0^s \|\nabla b_r\|_{L^\infty} dr \right\} ds \right\} \quad \forall t > 0 \quad \nu \in (0, 1) \quad (0.6)$$

it is straightforward to see that $r(\nu) \leq O(\log(1/\nu)^{-1})$, regardless of the mixing rate. We remark in passing that it is unknown whether there exists a smooth velocity field realizing a double exponential dissipation of $\|u_t^\nu\|_{L^2}$ for some $u_0 \in L^2(\mathbb{T}^d)$, a positive result in this direction has been obtained in [IKX14], dealing with a discrete model for (E_ν) .

One of the most interesting problems in this field consists in closing the gap between the universal upper bound $r(\nu) \leq O(\log(1/\nu)^{-1})$ and the best known lower bound (0.5) in the case of exponential mixing. Recently there has been a lot of interest in finding sharp upper and lower bounds on the diffusion rate, under various assumptions on the drift. Let us mention the work of Coti Zelati and Drivas [CZDR19] where a class of meaningful examples, such as shear flows and circular flows have been studied.

Out of the smooth setting it is even unknown whether a double exponential lower bound on the L^2 norm, as in (0.6), holds. The main difficulty here is that energy methods are not suitable to attack the problem due to a possible loss of regularity for transport equations [ACM14, ACM16, ACM18, J16, BN18c, BN19]. We refer the reader to [DEIJ2019, Section 1.3] for a discussion on this topic.

Bressan's mixing conjecture. In the non smooth setting is still unknown whether the mixing rate for passive scalars has a universal lower bound. This is related to the famous Bressan's mixing conjecture [B03] that can be formulated as follows.

Conjecture 0.3. Given a divergence-free velocity field $b \in L^\infty([0, \infty), W^{1,1}(\mathbb{T}^d, \mathbb{R}^d))$ there exist $c > 0$ and $C > 0$ depending only on the initial datum u_0 such that

$$\rho(t) \geq C \exp \left\{ -ct \|\nabla b\|_{L_t^\infty L_x^1} \right\} \quad \text{for every } t \geq 0.$$

Where ρ is the mixing rate according to Definition 0.2.

We have already pointed out (see for instance (0.4)) that Bressan's conjecture follows from a standard Gronwall estimate when the velocity field is Lipschitz, uniformly in time. A positive result has been obtained also for $b \in L^\infty([0, \infty), W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ with $p > 1$ in [CDL08] (see also [IKX14]), while the case $p = 1$ seems to require some new ideas. On the other hand, in the last years beautiful examples of mixing velocity fields under various constraint have been provided, see for instance [ACM16, EZ, YZ17].

In view of the enhanced dissipation estimates, the problems of finding lower bounds on the energy $\|u_t^\nu\|_{L^2}^2$ and on the diffusion enhancing rate $r(\nu)$ have natural connections with the challenging Bressan's mixing conjecture [B03]. We refer to [DEIJ2019, section 1,3] for a detailed discussion.

Energy dissipation rate in the Sobolev setting. Aiming at better understanding enhanced dissipation and energy's lower bounds, the key quantity to study is the energy dissipation rate

$$2\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds = \|u_0\|_{L^2}^2 - \|u_t^\nu\|_{L^2}^2.$$

Notice that, when the divergence-free velocity field b has the property that (E_0) admits a unique solution that conserves the L^2 norm, it must hold

$$\lim_{\nu \downarrow 0} 2\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds = 0. \quad (0.7)$$

It can be easily checked by observing that, up to extracting a subsequence, $u_t^\nu \rightarrow u_t^0$ weakly in L^2 and by using the fact that the L^2 norm is lower semicontinuous with respect to weak convergence.

In particular, if the drift is either Sobolev or BV the DiPerna-Lions-Ambrosio theory [DPL89, A04, AC14] guarantees (0.7) (see also the recent paper [QN18] for a quantitative analysis in BV and the study of velocity fields which can be represented as singular integral of functions in BV). One of the main achievement of this work is the correct estimate of the rate of convergence of (0.7). Before stating the result and its consequences let us recall that, in view of (0.6) it is easily seen that in the Lipschitz setting (i.e. $b \in L_t^\infty W_x^{1,\infty}$) any solution to (E_ν) with $u_0 \in W^{1,2}$ satisfies

$$\nu \int_0^1 \|\nabla u_s^\nu\|_{L^2}^2 ds \leq C\nu \quad \text{for } \nu \in (0, 1). \quad (0.8)$$

Hence the energy dissipation rate is $O(\nu)$ for $\nu \rightarrow 0$. On the other hand, if one relaxes the regularity assumption on the velocity field the situation may change dramatically. For instance, in [DEIJ2019] a divergence-free vector field was constructed

$$b \in C^\infty([0, 1] \times \mathbb{T}^d) \cap L^1([0, 1], C^\alpha(\mathbb{T}^d)) \cap L^\infty([0, 1] \times \mathbb{T}^d)$$

such that

$$\limsup_{\nu \downarrow 0} \nu \int_0^1 \|\nabla u_s^\nu\|_{L^2}^2 ds \geq c > 0,$$

for a broad family of initial data $u_0 \in W^{2,2}(\mathbb{T}^d)$. Notice that this implies the existence of passive scalars advected by b with non constant L^2 norm.

Let us also mention a recent result [JY20, Theorem 1.1] which provides solutions to the 3D Navier-Stokes equation with energy dissipation slower than (0.8).

In the Sobolev setting we have the following logarithmic rate.

Theorem 0.4. *Let $b \in L^\infty([0, T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ be a divergence-free vector field for some $p > 2$. Any solution u^ν to (E_ν) with $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$ satisfies*

$$\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds \leq C(\|u_0\|_{W^{1,2}}^2 + \|u_0\|_{L^\infty}^2) \left[\nu t + \frac{t^p \|\nabla b\|_{L_t^\infty L_x^p}^p + 1}{\log\left(\frac{1}{\nu t} + 2\right)^{p-1}} \right] \quad \forall t \geq 0, \quad (0.9)$$

where $C = C(p, d)$. In particular, for any $t > 0$, we have

$$\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds \leq C \log(1/\nu)^{-p+1} \quad \text{for every } \nu \in (0, 1/5). \quad (0.10)$$

Here $C = C(p, d, t, \|u_0\|_{W^{1,2}}^2 + \|u_0\|_{L^\infty}^2, \|\nabla b\|_{L_t^\infty L_x^p}) > 0$.

The next result shows that the logarithmic rate is ‘‘almost’’ sharp.

Theorem 0.5. *Let $d \geq 2$ and $p > 2$ be fixed. There exist a divergence-free velocity field $b \in L^\infty([0, 1], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ and $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$ such that*

$$\limsup_{\nu \rightarrow 0} \log(1/\nu)^r \nu \int_0^1 \|\nabla u_s^\nu\|_{L^2}^2 ds = +\infty \quad (0.11)$$

for any $r > p \frac{(p-1)}{p-2}$. Here u^ν denotes the solution to (E_ν) .

We strongly believe that both results Theorem 0.4 and Theorem 0.5 can be sharpened as follows.

Conjecture 0.6. The rate in (0.10) can be sharpened to

$$\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds \leq C \log(1/\nu)^{-p} \quad \text{for every } \nu \in (0, 1/5),$$

and (0.11) holds for any $r > p$.

Idea of the proof of Theorem 0.4. A crucial ingredient of proof is a new propagation of regularity result (Theorem 1.1) for solutions to (E_ν) . The main novelty is that the constants appearing in the regularity estimate do not depend on the diffusivity parameter $\nu > 0$. It extends known result for transport equations [BBJ19, LF16, BN18c, BN19] to the advection-diffusion problem. To prove Theorem 1.1 we took advantage of the *Lagrangian representation* for solutions to (E_ν) , via Feynman-Kac's formula [K97]:

$$u^\nu(t, x) = \mathbb{E}[u_0 \circ X_{t,0}(x)].$$

Here $X_{t,0}$ denotes the solution to the backward stochastic differential equation

$$dX_{t,s} = b(s, X_{t,s}) ds + \sqrt{2\nu} dW_s \quad \text{with} \quad X_{t,t}(x) = x, \quad (\text{SDE})$$

where W_s is an \mathbb{T}^d valued Brownian motion adapted to the backwards filtration (i.e. satisfying $W_t = 0$) in the probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

The core of the proof of Theorem 1.1 consists in estimating the rate of change of *log-Sobolev* norms of $X_{t,s}$, by exploiting the Sobolev regularity of the drift b . We refer the reader to section 1 for a detailed outline of the argument.

In order to explain the connection between propagation of regularity results and estimates on the energy dissipation rate we recall that, in the simple case $\nabla b \in L^\infty$, solutions to (E_0) and (E_ν) propagate the Sobolev regularity of the initial data for any $1 \leq p \leq \infty$ according to

$$\|\nabla u_t^\nu\|_{L^p} \leq \|\nabla u_0\|_{L^p} e^{ct\|\nabla b\|_{L^\infty}}, \quad (0.12)$$

where $c > 0$ does not depend on ν . This can be checked either by means of energy estimates or by studying the regularity of the stochastic flow map $X_{t,s}$. Having such a strong regularity result at hand, the upper bound on the energy dissipation rate (0.8) immediately follows:

$$\nu \int_0^1 \|\nabla u_s^\nu\|_{L^2}^2 ds \leq \nu \int_0^1 \|\nabla u_0\|_{L^2}^2 e^{2ct\|\nabla b\|_{L^\infty}} ds \leq C(t, \|\nabla b\|_{L^\infty}) \|\nabla u_0\|_{L^2}^2 \nu.$$

The main issue of working with Sobolev vector fields is that regularity estimates like (0.12) are known to be false for the inviscid problem [ACM14, ACM16, ACM18]. On the other hand, the regularity theory for (E_0) in the framework of Sobolev velocity fields [LF16, BN18c] provides us with propagation of regularity results on *Sobolev spaces of logarithmic order*, see section 1. Unfortunately the latter are too weak to be suitable to bound directly the energy dissipation rate.

To get around this problem we use an interpolation argument which combines the *log-Sobolev* estimate of Theorem 1.1 with a new a priori estimate on $\nu^2 \int_0^t \|\Delta u_s^\nu\|_{L^2}^2 ds$, given in terms of the energy dissipation rate (cf. Proposition 2.3).

Idea of the proof of Theorem 0.5. To prove the existence of solutions with “slow dissipation rate” we exploit the existence of rough solutions to the transport equation (see Proposition 1.3).

The main idea is that quantitative bounds on the energy dissipation rate imply regularity results for transport equations. This has been made quantitative in Proposition 2.1 by showing the implication

$$\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds \leq C \log(1/\nu)^q \implies u_t^0 \in H^{\log, r} \quad \text{for any } 0 < r < q \frac{p-2}{p-1},$$

when $b \in L_t^\infty W_x^{1,p}$. Here $H^{\log, r}$ denotes a Sobolev space of functions with “derivative of logarithmic order” introduced in section 1. Although the logarithmic regularity is very mild in [BN18c] we have built solutions to (E_0) , associated to $W^{1,p}$ velocity fields, that do not propagate the $H^{\log, r}$ regularity for $r > p$. This clearly leads to the sought conclusion.

Applications. An immediate consequence of Theorem 0.5 is that the double exponential lower bound as in Poon's estimate (0.6) does not hold in the Sobolev setting since it forces

$$\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds \leq C\nu \quad \text{for all } \nu \in (0, 1).$$

In view of Theorem 0.4 and Conjecture 0.6, it is natural to conjecture the following variant of Poon's estimate:

Conjecture 0.7. Fix $p \in [1, +\infty)$. Let $b \in C^\infty([0, T] \times \mathbb{T}^d)$ be divergence-free and $u_0 \in W^{1,2}(\mathbb{T}^d)$. Then, any solution u_t^ν to (E_ν) satisfies

$$\|u_t\|_{L^2}^2 \geq \|u_0\|_{L^2}^2 \exp \left\{ -\log(1/\nu)^{-p} C_1 \int_0^t \exp \left\{ C_2 \int_0^s \|\nabla b_r\|_{L^p} dr \right\} ds \right\}, \quad (0.13)$$

for any $\nu \in (0, 1)$ and $t > 0$. Here $C_1 = C_1(u_0, p, d) > 0$, and $C_2 = C_2(p, d) > 0$.

In the case $p = 1$ Conjecture 0.7 has been already presented and thoroughly discussed in [DEIJ2019, Conjecture 1.7]. We refer the reader to subsection 3.2 for the discussion of Proposition 3.2, a positive result towards Conjecture 0.7.

An other interesting consequence of Theorem 0.4 is the following upper bound on the enhanced dissipation rate in the setting of $W^{1,p}$ divergence-free vector fields.

Proposition 0.8. Let $b \in L^\infty([0, +\infty), W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ be a divergence-free vector field for some $p > 2$. Given $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$, if there exists $r : (0, \nu_0) \rightarrow (0, +\infty)$ for some $0 < \nu_0 < 1$, which satisfies

$$\|u_t^\nu\|_{L^2}^2 \leq e^{-r(\nu)t} \|u_0\|_{L^2}^2 \quad \text{for any } t > 1/\nu_0 \text{ and } \nu \in (0, \nu_0), \quad (0.14)$$

then

$$\limsup_{\nu \downarrow 0} \frac{r(\nu)}{\log(1/\nu)^{-\frac{p-1}{p}}} < \infty. \quad (0.15)$$

In other words the upper bound $r(\nu) \leq O(\log(1/\nu)^{-\frac{p-1}{p}})$ holds in the Sobolev setting. Notice that it is little worse than $O(\log(1/\nu)^{-1})$, the one available for smooth vector fields.

After finishing this note we got to know that a better upper bound on the enhanced dissipation rate has been proven in [S20, Theorem 2] by Seis. The approach of Seis is different from our, it relies on the quantitative analysis of solutions by means of weak norms and techniques coming from the optimal transport theory (see also [S17]).

The last application of Theorem 0.4 is a quantitative estimate on the rate of convergence in the vanishing viscosity limit.

Theorem 0.9. Let $b \in L^\infty([0, +\infty), W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ be a divergence-free vector field for some $p > 2$. Given $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$ we consider u^0, u^ν , respectively, solutions to (E_ν) and (E_0) . Then it holds

$$\sup_{s \in [0, t]} \|u_s^\nu - u_s^0\|_{L^2}^2 \leq Ct \left[\nu + t\nu^{\frac{p-2}{p-1}} + \frac{t^{p-1} + 1}{\log\left(\frac{1}{\nu t} + 2\right)^{p-2}} \right] \quad \text{for every } \nu > 0 \text{ and } t > 0, \quad (0.16)$$

where $C = C_0(1 + \|\nabla b\|_{L_t^\infty L_x^p}^p)(\|u_0\|_{W^{1/2}}^2 + \|u_0\|_{L^\infty}^2)$.

As far as we know (0.16) is the first quantitative vanishing viscosity estimate in terms of strong norms in the framework of Sobolev velocity fields. Previous results, such as [S18, Theorem 2] have dealt with weak norms.

It is worth noticing that Theorem 0.9 is almost optimal, we refer to subsection 3.3 for a discussion on this.

Organization of the paper. The rest of the paper is devoted to the proof of the outlined results. More specifically in section 1 we present the propagation of regularity result (Theorem 1.1) while section 2 is devoted to the proof of existence of “slow dissipating solutions” (Theorem 0.5). In section 3 we show the logarithmic estimate on the energy dissipation rate (Theorem 0.4) and its corollaries. Precisely, in subsection 3.2 we present the proof of Proposition 0.8 and we discuss a positive result in the direction of Conjecture 0.7. Eventually we show Theorem 0.9 in subsection 3.3.

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1. REGULARITY RESULT

In this section, we present a propagation of regularity result for solutions to (E_ν) , that will play a central role in the sequel. Here and in the rest of the paper we tacitly identify any $f : \mathbb{T}^d \rightarrow \mathbb{R}$ with a 1-periodic function on \mathbb{R}^d .

Let us begin by introducing a class of functional spaces. For any $\alpha \in (0, +\infty)$ we define

$$[u]_{H^{\log, \alpha}}^2 := \int_{B_{1/3}} \int_{\mathbb{T}^d} \frac{|u(x+h) - u(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-\alpha}} dx dh \quad (1.1)$$

and the related log-Sobolev class

$$H^{\log, \alpha} := \{ u \in L^2(\mathbb{T}^d) : \|u\|_{H^{\log, \alpha}}^2 := \|u\|_{L^2}^2 + [u]_{H^{\log, \alpha}}^2 < \infty \}. \quad (1.2)$$

The following characterisation of $H^{\log, \alpha}$ will play a role in the rest of the paper ¹

$$\|u\|_{H^{\log, \alpha}}^2 \sim_{\alpha, d} \sum_{k \in \mathbb{Z}^d} \log(2 + |k|)^\alpha |\hat{u}(k)|^2, \quad (1.3)$$

where $\hat{u}(k) := \int u(x) e^{-ix \cdot k} dx$. We refer to [BN18c] for a proof of (1.3) in the case in which the ambient space is \mathbb{R}^d .

Here and in the following we adopt the notation $a \wedge b$ to indicate $\min\{a, b\}$. The main result of the section is the following.

Theorem 1.1. *Let $b \in L^1([0, T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ be a divergence-free vector field for some $p > 1$. Then, any solution $u \in L^\infty([0, T] \times \mathbb{T}^d)$ to (E_ν) satisfies*

$$\begin{aligned} & \int_{B_{\frac{1}{10}}} \int_{\mathbb{T}^d} \frac{1 \wedge |u_t(x+h) - u_t(x)|^q}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} dx dh \\ & \lesssim_{p,q,d} \left(\int_0^t \|\nabla b_s\|_{L^p} ds \right)^p + \int_{B_{3/4}} \int_{\mathbb{T}^d} \frac{1 \wedge |u_0(x+h) - u_0(x)|^q}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} dx dh \end{aligned} \quad (1.4)$$

for any $0 < q < \infty$.

In particular, choosing $q = 2$ we get

Corollary 1.2. *Under the assumptions of Theorem 1.1 one has*

$$[u_t]_{H^{\log, p}} \lesssim_{p,d} \left(\int_0^t \|\nabla b_s\|_{L^p} ds \right)^{p/2} \|u_0\|_{L^\infty} + \|u_0\|_{H^{\log, p}} \quad \text{for any } t \in [0, T]. \quad (1.5)$$

It is worth remarking that (1.5) does not depend on $\nu > 0$, hence the inequality holds even in the case $\nu = 0$, i.e. for solution of the transport equation (E_0) (Cf. [BN18c, LF16]).

Moreover, the following example borrowed from [BN18c, Theorem 3.2] shows that Corollary 1.2 is sharp, in the sense that $H^{\log, p}$ cannot be replaced with a $H^{\log, q}$ for $q > p$.

¹Here and in the sequel we use the notation $A \sim_c B$ to mean that $C^{-1}A \leq B \leq CA$ where C depends only on c . Similar notation will be adopted for \lesssim_c and \gtrsim_c .

Proposition 1.3. *Let $p \geq 1$. There exist a divergence-free vector field $b \in L^\infty([0, +\infty); W^{1,p}(\mathbb{R}^d))$ and $u_0 \in L^\infty(\mathbb{R}^d) \cap W^{1,d}(\mathbb{R}^d)$ supported, respectively, in $B_1 \times [0, +\infty)$ and B_1 , such that the solution $u \in L^\infty([0, +\infty) \times \mathbb{R}^d)$ to (E_0) satisfies*

$$u_t \notin H^{\log,q} \quad \text{for any } t > 0 \text{ whenever } q > p.$$

The remaining part of this section is devoted to the proof of Theorem 1.1. The argument is a refinement of the one presented in [BN18c] and has its roots in the very influential paper [CDL08]. In a nutshell, it goes as follows. First, by employing the Lusin-Lipschitz inequality for Sobolev maps (1.10) and Gronwall's lemma, one studies regularity properties of the backwards stochastic flow (Cf. Proposition 1.5) associated to b . Next, one translates the Lagrangian regularity result into an Eulerian one by using Feynman-Kac formula (1.7) and Lusin-type characterisations of $H^{\log,p}$ functions (Cf. Proposition 1.6).

Remark 1.4. In what follows it is technically convenient to assume that $b \in L^1([0, T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ is pointwise defined, with respect to the space variable, according to

$$b(t, x) := \begin{cases} \lim_{r \downarrow 0} \frac{1}{\omega_d r^d} \int_{B_r(x)} b(t, y) \, dy & \text{whenever it exists,} \\ 0 & \text{otherwise.} \end{cases} \quad (1.6)$$

1.1. Stochastic representation and Lagrangian estimate. As we already mentioned in the introduction, for any $t \in (0, \infty)$ we consider the following backward stochastic differential equation

$$dX_{t,s} = b(s, X_{t,s}) \, ds + \sqrt{2\nu} \, dW_s \quad \text{with} \quad X_{t,t}(x) = x, \quad (\text{SDE})$$

where W_s is an \mathbb{T}^d valued Brownian motion adapted to the backwards filtration (i.e. satisfying $W_t = 0$) in the probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Then, the Feynman-Kac formula [K97] expresses the solution of (E_ν) as

$$u^\nu(t, x) = \mathbb{E}[u_0 \circ X_{t,0}(x)]. \quad (1.7)$$

Exploiting the Sobolev regularity of b one gets a following Lusin type estimate for the stochastic flow map $X_{s,t}$ that does not depend on ν .

Proposition 1.5. *Let $b \in L^1([0, T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ be a divergence-free vector field, for some $p > 1$. Fix $t \in (0, T)$. Then, there exists a nonnegative random function $g_t(\omega, x) = g_t(x)$ for $\omega \in \Omega$ and $x \in \mathbb{T}^d$, which for \mathcal{P} -a.e. ω satisfies the inequalities*

$$\|g_t\|_{L^p(\mathbb{T}^d)} \lesssim_{p,d} \int_0^t \|\nabla b_s\|_{L^p(\mathbb{T}^d)} \, ds, \quad (1.8)$$

$$e^{-g_t(x) - g_t(y)} \leq \frac{|X_{t,s}(x) - X_{t,s}(y)|}{|x - y|} \leq e^{g_t(x) + g_t(y)} \quad \text{for any } 0 \leq s \leq t, x, y \in \mathbb{T}^d. \quad (1.9)$$

Here $X_{t,s}$ is a realization of the solution to (SDE).

Proof. Let us introduce the local Hardy-littelwood maximal function

$$Mf(x) := \sup_{0 < r < 3} \frac{1}{\omega_d r^d} \int_{B_r(x)} |f(y)| \, dy,$$

for $f \in L^1(\mathbb{T}^d)$, and set

$$g_t(x) := \int_0^t M|\nabla b_s|(X_{t,s}(x)) \, ds \quad \text{for any } x \in \mathbb{T}^d,$$

and notice that (1.8) is a simple consequence of the Minkoski's inequality and the fact that $X_{t,s}$ is measure preserving.

The inequality (1.9) follows from the Gronwall's lemma, along with the observation that, \mathcal{P} -a.e., for any $x, y \in \mathbb{T}^d$, the map $s \rightarrow |X_{t,s}(x) - X_{t,s}(y)|$ is absolutely continuous and satisfies

$$\begin{aligned} \frac{d}{ds} |X_{t,s}(x) - X_{t,s}(y)| &\leq |b(s, X_{t,s}(x)) - b(s, X_{t,s}(y))| \\ &\leq C_d |X_{t,s}(x) - X_{t,s}(y)| (M|\nabla b_s|(X_{t,s}(x)) + M|\nabla b_s|(X_{t,s}(y))), \end{aligned}$$

for a.e. $s \in (0, t)$. Above we have used the the Lusin-Lipschitz inequality for Sobolev functions $f \in W_{\text{loc}}^{1,1}(\mathbb{T}^d)$, pointwise defined according to (1.6):

$$|f(x) - f(y)| \leq C_d |x - y| (M|\nabla f|(x) + M|\nabla f|(y)) \quad \text{for any } x, y \in \mathbb{T}^d. \quad (1.10)$$

□

1.2. Lusin type characterisation of $H^{\log,p}$ functions and proof of Theorem 1.1. Let us begin by presenting a refined version of [BN18c, Theorem 1.11].

Proposition 1.6. *Let $q > 0$ and $p > 0$. For any $u \in L_{\text{loc}}^1(\mathbb{R}^d)$ it holds*

$$1 \wedge |u(y) - u(x)|^q \lesssim_{p,q,d} \log(1/r)^{-p} (G(r, x) + G(r, y)),$$

for any $x, y \in \mathbb{R}^d$ with $2|x - y| \leq r < \frac{1}{10}$, where

$$G(r, z) := \int_{r \leq |h| \leq r^{1/2}} \frac{1 \wedge |u(z+h) - u(z)|^q}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} dh \quad \text{for any } z \in \mathbb{R}^d.$$

Proof. First observe that, for any $x, y \in \mathbb{R}^d$ and $s \geq 2|x - y|$ one has

$$\begin{aligned} & 1 \wedge |u(x) - u(y)|^q \\ & \lesssim_{d,q} \int_{B_{3s}(0) \setminus B_s(0)} 1 \wedge |u(x+h) - u(x)|^q dh + \int_{B_{3s}(0) \setminus B_s(0)} 1 \wedge |u(y+h) - u(y)|^q dh, \end{aligned} \quad (1.11)$$

see [BN18c, Lemma 1.12] for a simple proof. Next, we integrate both sides of (1.11) with respect to the variable s against a suitable kernel, getting

$$\begin{aligned} & 1 \wedge |u(x) - u(y)|^q \int_r^{r^{1/2}} \frac{1}{s \log(1/s)^{1-p}} ds \\ & \lesssim_{d,p} \int_r^{r^{1/2}} \int_{B_{3r}(0) \setminus B_r(0)} 1 \wedge |u(x+h) - u(x)|^q dh \frac{ds}{s \log(1/s)^{1-p}} \\ & \quad + \int_r^{r^{1/2}} \int_{B_{3s}(0) \setminus B_s(0)} 1 \wedge |u(y+h) - u(y)|^q dh \frac{ds}{s \log(1/s)^{1-p}}. \end{aligned}$$

Observe that

$$\int_r^{r^{1/2}} \int_{B_{3s}(0) \setminus B_s(0)} 1 \wedge |u(x+h) - u(x)|^q dh \frac{ds}{s \log(1/s)^{1-p}} \lesssim_{d,p} G(r, x),$$

and

$$\int_r^{r^{1/2}} \frac{1}{s \log(1/s)^{1-p}} ds = \frac{1}{p} \left(\log\left(\frac{1}{r}\right)^p - \log\left(\frac{3}{r^{1/2}}\right)^p \right) \gtrsim_p \log\left(\frac{1}{r}\right)^p,$$

where we have used $r < \frac{1}{10}$. The proof is complete. □

Proof of Theorem 1.1. Let us begin by noticing that our conclusion follows from the \mathcal{P} -a.e. inequality

$$\begin{aligned} & \int_{B_{1/5}} \int_{\mathbb{T}^d} \frac{1 \wedge |u_0(X_{t,0}(x+h)) - u_0(X_{t,0}(x))|^q}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} dx dh \\ & \lesssim_{p,q,d} \left(\int_0^t \|\nabla b_s\|_{L^p} ds \right)^p + \int_{B_{3/4}} \int_{\mathbb{T}^d} \frac{1 \wedge |u_0(x+h) - u_0(x)|^q}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} dx dh, \end{aligned} \quad (1.12)$$

by taking the expectation and using (1.7).

Let us then prove (1.12). Fix $t \in (0, T)$ and g given by Proposition 1.5, in order to keep notation short we drop the dependence of g on ω and t . For \mathcal{P} -a.e. ω we have

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{|h| < \frac{1}{10}} \frac{1 \wedge |u_0(X_{t,0}(x+h)) - u_0(X_{t,0}(x))|^q}{|h|^d \log(1/|h|)^{1-p}} dh dx \\ & \leq \int_{\mathbb{T}^d} \int_{|h| < \frac{1}{10}} \mathbf{1}_{|h|^{1/2} \exp\{g(x+h)+g(x)\} \geq 1} \frac{1}{|h|^d \log(1/|h|)^{1-p}} dh dx \\ & \quad + \int_{\mathbb{T}^d} \int_{|h| < \frac{1}{10}} \mathbf{1}_{|h|^{1/2} \exp\{g(x+h)+g(x)\} < 1} \frac{1 \wedge |u_0(X_{t,0}(x+h)) - u_0(X_{t,0}(x))|^q}{|h|^d \log(1/|h|)^{1-p}} dh dx \\ & =: I + II. \end{aligned}$$

Let us estimate I by means of (1.8):

$$\begin{aligned} I & \leq \int_{\mathbb{T}^d} \int_{|h| < \frac{1}{10}} \mathbf{1}_{|h|^{1/2} \exp\{g(x+h)+g(x)\} \geq 1} \frac{1}{|h|^d \log(1/|h|)^{1-p}} dh dx \\ & \leq 2 \int_{\mathbb{T}^d} \int_{|h| < \frac{1}{10}} \mathbf{1}_{|h|^{1/2} e^{2g(x)} \geq 2} \frac{1}{|h|^d \log(1/|h|)^{1-p}} dh dx \\ & = 2 \int_{|h| < \frac{1}{10}} \mathcal{L}^d \left(\left\{ g \geq \frac{1}{4} \log(4/|h|) \right\} \right) \frac{1}{|h|^d \log(1/|h|)^{1-p}} dh \\ & \lesssim_d \int_{r < \frac{1}{10}} \mathcal{L}^d \left(\left\{ g \geq \frac{1}{4} \log(4/r) \right\} \right) \log(1/r)^{p-1} \frac{dr}{r} \\ & \leq 4 \int_{\log(40)/4}^{\infty} \mathcal{L}^d(\{g \geq \lambda\}) (4\lambda - 2 \log(2))^{p-1} d\lambda \\ & \lesssim_p \|g\|_{L^p}^p \lesssim_{p,d} \left(\int_0^T \|\nabla b_s\|_{L^p} ds \right)^p. \end{aligned}$$

Let us now estimate II . Let G be given by Proposition 1.6 and associated to u_0 , we have

$$\begin{aligned} & 1 \wedge |u_0(X_{t,0}(x+h)) - u_0(X_{t,0}(x))|^q \\ & \lesssim \log(1/r)^{-p} (G(r, X_{t,0}(x+h)) + G(r, X_{t,0}(x))), \end{aligned} \quad (1.13)$$

with

$$r := \frac{1}{20} \wedge |X_{t,0}(x+h) - X_{t,0}(x)|.$$

Note that, by Proposition 1.5 we have

$$\frac{1}{20} \wedge [|h| \exp\{-g(x+h) - g(x)\}] \leq r \leq \frac{1}{20} \wedge [|h| \exp\{g(x+h) + g(x)\}]. \quad (1.14)$$

Let us fix $h \in B_{\frac{1}{10}}(0)$. For any $x \in \mathbb{T}^d$ such that

$$|h|^{1/2} \exp\{g(x+h) + g(x)\} < 1,$$

it follows from (1.13) and (1.14) that $|h|^{3/2} \leq r \leq |h|^{1/2}$, and

$$\begin{aligned} & 1 \wedge |u_0(X_{t,0}(x+h)) - u_0(X_{t,0}(x))|^q \\ & \lesssim \log\left(\frac{1}{|h|}\right)^{-p} (H(|h|, X_{t,0}(x+h)) + H(|h|, X_{t,0}(x))), \end{aligned}$$

where

$$H(r, z) := \int_{r^{3/2} \leq |h| \leq r^{1/4}} \frac{1 \wedge |u_0(z+h) - u_0(z)|^q}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} dh \quad \text{for any } z \in \mathbb{T}^d.$$

This implies,

$$\begin{aligned}
II &\lesssim_p \int_{\mathbb{T}^d} \int_{|h| < \frac{1}{10}} \frac{H(|h|, X_{t,0}(x+h))}{|h|^d \log(1/|h|)} dh dx + \int_{\mathbb{T}^d} \int_{|h| < \frac{1}{10}} \frac{H(|h|, X_{t,0}(x))}{|h|^d \log(1/|h|)} dh dx \\
&= 2 \int_{\mathbb{T}^d} \int_{|h| < \frac{1}{10}} \frac{H(|h|, X_{t,0}(x))}{|h|^d \log(1/|h|)} dh dx \\
&\simeq_{p,d} \int_{\mathbb{T}^d} \int_0^{\frac{1}{10}} \frac{1}{r \log(1/r)} \int_{r^{3/2} \leq |h| \leq r^{1/4}} \frac{1 \wedge |u_0(x+h) - u_0(x)|^q}{|h|^d \log(1/|h|)^{1-p}} dh dr dx \\
&\lesssim_{p,d} \int_{\mathbb{T}^d} \int_{|h| < 3/4} \left(\int_{|h|^4}^{|h|^{2/3}} \frac{1}{r \log(1/r)} dr \right) \frac{1 \wedge |u_0(x+h) - u_0(x)|^q}{|h|^d \log(1/|h|)^{1-p}} dh dx \\
&\simeq_{p,d} \int_{\mathbb{T}^d} \int_{|h| < 3/4} \frac{1 \wedge |u_0(x+h) - u_0(x)|^q}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} dh dx,
\end{aligned}$$

here we have used the fact that

$$\int_{|h|^4}^{|h|^{2/3}} \frac{1}{r \log(1/r)} dr = \log(\log(1/|h|^4)) - \log(\log(1/|h|^{2/3})) = \log(6).$$

The proof is over. \square

Remark 1.7. Notice that (1.12) is stronger than the regularity estimate in (1.1), indeed when we take the expectation we are losing information. We believe that a more precise analysis, which do not lose this information, could lead to the following improved version of (1.2):

$$[u_t]_{H^{\log,p}}^2 + \nu \int_0^t [\nabla u_s]_{H^{\log,p}}^2 ds \lesssim_{d,p} \left(\int_0^t \|\nabla b_s\|_{L^p} \right)^p \|u_0\|_{L^\infty}^2 + \|u_0\|_{H^{\log,p}}^2 \quad \forall t \in [0, T]. \quad (1.15)$$

Unfortunately we are not able to show this estimate by means of our approach. However it is worth stressing that if (0.7) were true then it would lead to significant improvements of Theorem 0.4, Theorem 0.5 and their applications.

2. PROOF OF THEOREM 0.5: EXISTENCE OF SLOW DISSIPATING SOLUTIONS

The core of the argument in the proof of Theorem 0.5 is the following.

Proposition 2.1. *Let $b \in L^1([0, T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ be a divergence-free vector field, for some $p > 2$. Let u^ν and u^0 solve, respectively, (E_ν) and (E_0) . For any $t \in [0, T]$, if there exists $q > 0$ such that*

$$\limsup_{\nu \downarrow 0} \log(1/\nu)^q \nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds < \infty, \quad (2.1)$$

then $u_t \in H^{\log,r}$ (see (1.1)) for any $0 < r < q \frac{p-2}{p-1}$.

Remark 2.2. By exploiting the ideas developed in the proof of Proposition 2.1 (Cf. Remark 2.4) one can prove the following variant: if there exists $\theta \in (0, 1]$ such that

$$\limsup_{\nu \downarrow 0} \nu^{1-\theta} \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds < \infty, \quad (2.2)$$

then $u_t \in H^r(\mathbb{T}^d)$ for any $0 < r < \theta \frac{p-2}{2(p-1)}$. Here $H^r(\mathbb{T}^d) := \{u \in L^2(\mathbb{T}^d) : [u]_{H^r} < \infty\}$ denotes the fractional Sobolev space defined by means of the Gagliardo semi-norm

$$[u]_{H^r}^2 := \int_{B_2} \int_{\mathbb{T}^d} \frac{|u(x+h) - u(x)|^2}{|h|^{d+2r}} dx dh.$$

Proof of Theorem 0.5 given Proposition 2.1. We argue by contradiction. If the conclusion were false then the assumptions of Proposition 2.1 are satisfied for some $q > p \frac{p-1}{p-2}$, therefore there exists $r > p$ such that $u_t^0 \in H^{\log,r}$. This is not possible in general in view of Proposition 1.3. \square

2.1. Interpolation estimate. In this subsection we present an estimate on $\nu^2 \int_0^t \|\Delta u_s^\nu\|_{L^2}^2 ds$, which plays a central role in Proposition 2.1 and Theorem 0.4.

Proposition 2.3. *Let $\gamma \in (2, +\infty]$ be fixed. Assume $b \in L^\infty([0, T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ for some $p > \frac{2\gamma}{\gamma-2}$. Any solution $u^\nu \in L^\infty([0, T], W^{1,2}(\mathbb{T}^d) \cap L^\gamma)$ to (E_ν) satisfies*

$$\begin{aligned} & \nu \|\nabla u_t^\nu\|_{L^2}^2 + \nu^2 \int_0^t \|\Delta u_s^\nu\|_{L^2}^2 ds \\ & \leq \nu \|\nabla u_0\|_{L^2}^2 + C_{d,p,\gamma} \|u_0\|_{L^\gamma}^{2(1-\beta)} \|\nabla b\|_{L_t^\infty L_x^p}^{2-\beta} t^{1-\beta} \left(\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds \right)^\beta, \end{aligned} \quad (2.3)$$

where

$$\beta = 1 - \frac{1}{p-1-\frac{2p}{\gamma}} \in (0, 1).$$

In the sequel we will use (2.3) just in the case $\gamma = \infty$.

Proof. It is enough to prove the result for $\|\nabla b\|_{L_t^\infty L_x^p} = 1$, the general case follows by a simple scaling argument. Testing (E_ν) against Δu_t we get

$$\begin{aligned} \|\nabla u_t^\nu\|_{L^2}^2 + 2\nu \int_0^t \|\Delta u_s^\nu\|_{L^2}^2 ds & \leq \|\nabla u_0\|_{L^2}^2 + \int_0^t \|\nabla u_s^\nu\|_{L^{2p'}}^2 ds \\ & \leq \|\nabla u_0\|_{L^2}^2 + \int_0^t \|\nabla u_s^\nu\|_{L^2}^{2\alpha} \|\nabla u_s^\nu\|_{L^{2q}}^{2(1-\alpha)} ds, \end{aligned}$$

with

$$p' = \frac{p}{p-1}, \quad \frac{1}{p'} = \alpha + \frac{1-\alpha}{q}, \quad \alpha \in (0, 1). \quad (2.4)$$

By using the Gagliardo–Nirenberg interpolation inequality we deduce

$$\|\nabla u^\nu\|_{L^{2q}} \leq C_{d,q,\gamma} \|\Delta u^\nu\|_{L^2}^{1/2} \|u^\nu\|_{L^\gamma}^{1/2} \quad \text{for} \quad \frac{1}{q} = \frac{1}{2} + \frac{1}{\gamma}, \quad (2.5)$$

hence

$$\begin{aligned} \|\nabla u_t^\nu\|_{L^2}^2 + 2\nu \int_0^t \|\Delta u_s^\nu\|_{L^2}^2 ds & \leq \|\nabla u_0\|_{L^2}^2 + C_{d,q,\gamma} \|u_0\|_{L^\gamma}^{1-\alpha} \int_0^t \|\nabla u_s^\nu\|_{L^2}^{2\alpha} \|\Delta u_s^\nu\|_{L^2}^{1-\alpha} ds \\ & \leq \|\nabla u_0\|_{L^2}^2 + C_{d,q,\gamma,\alpha} \|u_0\|_{L^\gamma}^{2\frac{1-\alpha}{1+\alpha}} \nu^{-\frac{1-\alpha}{1+\alpha}} \int_0^t \|\nabla u_s^\nu\|_{L^2}^{\frac{4\alpha}{1+\alpha}} ds \\ & \quad + \nu \int_0^t \|\Delta u_s^\nu\|_{L^2}^2 ds, \end{aligned}$$

which amounts to

$$\begin{aligned} \|\nabla u_t^\nu\|_{L^2}^2 + \nu \int_0^t \|\Delta u_s^\nu\|_{L^2}^2 ds & \leq \|\nabla u_0\|_{L^2}^2 + C_{d,q,\gamma,\alpha} \|u_0\|_{L^\gamma}^{2\frac{1-\alpha}{1+\alpha}} \nu^{-\frac{1-\alpha}{1+\alpha}} \int_0^t \|\nabla u_s^\nu\|_{L^2}^{\frac{4\alpha}{1+\alpha}} ds \\ & \leq \|\nabla u_0\|_{L^2}^2 + C_{d,q,\gamma,\alpha} \|u_0\|_{L^\gamma}^{2\frac{1-\alpha}{1+\alpha}} t\nu^{-1} \left(t^{-1}\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds \right)^{\frac{2\alpha}{1+\alpha}}. \end{aligned}$$

In order to conclude the proof we just need to combine (2.4) and (2.5) to find the expression of α and q in terms of p and γ . \square

2.2. Proof of Proposition 2.1. Fix $t \in (0, T)$ and a convolution kernel $\rho_\varepsilon(x) := \varepsilon^{-d} \rho(x\varepsilon^{-1})$ where $\rho \in C_c^\infty(B_{1/2}(0))$ satisfies $\int_{\mathbb{T}^d} \rho = 1$ and $\varepsilon > 0$. For any $f \in L^1(\mathbb{T}^d)$ we denote by

$$f * \rho_\varepsilon(x) := \int_{\mathbb{R}^d} u(x-y) \rho_\varepsilon(y) dy$$

its convolution against ρ_ε , which is continuous and 1-periodic. Then, for any $\nu > 0$, it holds

$$\begin{aligned} \left\| u_t^0 * \rho_\varepsilon - u_t^0 \right\|_{L^2} &\leq \left\| u_t^0 * \rho_\varepsilon - u_t^\nu * \rho_\varepsilon \right\|_{L^2} + \left\| u_t^\nu * \rho_\varepsilon - u_t^\nu \right\|_{L^2} + \left\| u_t^\nu - u_t^0 \right\|_{L^2} \\ &\leq 2 \left\| u_t^\nu - u_t^0 \right\|_{L^2} + \left\| u_t^\nu * \rho_\varepsilon - u_t^\nu \right\|_{L^2} \\ &\lesssim \left\| u_t^\nu - u_t^0 \right\|_{L^2} + \varepsilon \left\| \nabla u_t^\nu \right\|_{L^2}. \end{aligned} \quad (2.6)$$

From (3.12) and Proposition 2.3 (with $\gamma = \infty$) we get

$$\left\| u_t^\nu - u_t^0 \right\|_{L^2}^2 \lesssim_p t\nu \left\| \nabla u_0 \right\|_{L^2}^2 + t^{\frac{p}{p-1}} \left\| u_0 \right\|_{L^2}^{\frac{2}{p-1}} \left\| \nabla b \right\|_{L_t^\infty L_x^p}^{\frac{p}{p-1}} \left(\nu \int_0^t \left\| \nabla u_s^\nu \right\|_{L^2}^2 ds \right)^{\frac{p-2}{p-1}}, \quad (2.7)$$

while Proposition 2.3 and (0.2) yield

$$\varepsilon^2 \left\| \nabla u_t^\nu \right\|_{L^2}^2 \lesssim_p \varepsilon^2 \left\| \nabla u_0 \right\|_{L^2}^2 + \varepsilon^2 \nu^{-1} t^{\frac{1}{p-1}} \left\| u_0 \right\|_{L^2}^{\frac{2}{p-1}} \left\| \nabla b \right\|_{L_t^\infty L_x^p}^{\frac{p}{p-1}} \left(\nu \int_0^t \left\| \nabla u_s^\nu \right\|_{L^2}^2 ds \right)^{\frac{p-2}{p-1}}. \quad (2.8)$$

By combining (2.6), (2.8), (2.7), assuming without loss of generality $\|u_0\|_{W^{1,2}} + \|u_0\|_{L^\infty} \leq 1$, and choosing $\varepsilon = \nu$ one gets

$$\left\| u_t^0 * \rho_\nu - u_t^0 \right\|_{L^2}^2 \lesssim_p \nu(t+1) + t^{\frac{p}{p-1}} \left\| \nabla b \right\|_{L_t^\infty L_x^p}^{\frac{p}{p-1}} \left(\nu \int_0^t \left\| \nabla u_s^\nu \right\|_{L^2}^2 ds \right)^{\frac{p-2}{p-1}} \quad \text{for every } \nu \in (0, 1). \quad (2.9)$$

Thanks to (2.1) there exists $\nu_0 \in (0, 1)$ such that $\nu \int_0^t \left\| \nabla u_s \right\|_{L^2}^2 ds \leq C \log(1/\nu)^{-q}$ for any $\nu \in (0, \nu_0)$, hence

$$\left\| u_t^0 * \rho_\nu - u_t^0 \right\|_{L^2}^2 \lesssim_{t,p} \log(1/\nu)^{-q \frac{p-2}{p-1}} \quad \text{for any } 0 < \nu < \nu_0. \quad (2.10)$$

We claim that (2.10) implies $u_t^0 \in H^{\log, r}$ for every $0 < r < q \frac{p-2}{p-1}$. To this end we note that

$$\sum_{k \in \mathbb{Z}} |\widehat{u_t^0}(k)|^2 \int_0^{\nu_0} \frac{|\hat{\rho}(\nu k) - 1|^2}{\log(1/\nu)^{1-r}} \frac{d\nu}{\nu} = \int_0^{\nu_0} \int_{\mathbb{T}^d} |u_t^0 * \rho_\nu - u_t^0|^2 \frac{1}{\log(1/\nu)^{1-r}} dx \frac{d\nu}{\nu} \lesssim \frac{1}{q \frac{p-2}{p-1} - r}$$

for any $0 < r < q \frac{p-2}{p-1}$, where $\hat{\rho}$ denotes the Fourier transform of ρ in \mathbb{R}^d . Moreover it is not hard to check that

$$C_{\nu_0} + \int_0^{\nu_0} |\hat{\rho}(\nu k) - 1|^2 \frac{1}{\log(1/\nu)^{1-r}} \frac{d\nu}{\nu} \gtrsim_{\nu_0, d} \log(2 + |k|)^r, \quad (2.11)$$

Thus (1.3) yields

$$\left\| u_t^0 \right\|_{H^{\log, r}}^2 \lesssim_{r, d} \sum_{k \in \mathbb{Z}^d} \log(2 + |k|)^r |\widehat{u_t^0}(k)|^2 < \infty.$$

The proof is over.

Remark 2.4. Under the assumption (2.2), the estimate (2.9) gives

$$\left\| u_t^0 * \rho_\nu - u_t^0 \right\|_{L^2}^2 \lesssim_{t,p} \nu^{\theta \frac{p-2}{p-1}} \quad \text{for any } 0 < \nu < \nu_0,$$

for some $\nu_0 > 0$. Hence $u_t^0 \in H^r(\mathbb{T}^d)$ for any $0 < r < \theta \frac{p-2}{2(p-1)}$.

3. LOGARITHMIC ESTIMATE ON THE DISSIPATION RATE AND CONSEQUENCES

In this section we prove Theorem 0.4 and we draw a series of consequences.

3.1. Proof of Theorem 0.4: logarithmic bound on the dissipation rate. Since (E_ν) is linear we can assume without loss of generality that

$$\|u_0\|_{W^{1,2}} + \|u_0\|_{L^2} \leq 1.$$

Observe that, from (1.3), we deduce

$$[u_0]_{H^{\log,p}}^2 \lesssim_{d,p} \sum_{k \in \mathbb{Z}^d} \log(2 + |k|)^p |\hat{u}_0(k)|^2 \lesssim \sum_{k \in \mathbb{Z}^d} (1 + |k|^2) |\hat{u}_0(k)|^2 \leq \|u_0\|_{W^{1,2}}^2 \leq 1.$$

We apply (2.3) with $\gamma = \infty$ and $\beta = \frac{p-2}{p-1}$ obtaining

$$\begin{aligned} \nu \|\nabla u_t^\nu\|_{L^2}^2 + \nu^2 \int_0^t \|\Delta u_s^\nu\|_{L^2}^2 ds \\ \leq \nu \|\nabla u_0\|_{L^2}^2 + C_p \|u_0\|_{L^\infty}^{\frac{2}{p-1}} \|\nabla b\|_{L_t^\infty L_x^p}^{\frac{p}{p-1}} t^{\frac{1}{p-1}} \left(\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds \right)^{\frac{p-2}{p-1}} \\ \leq \nu + C_{p,d} \|\nabla b\|_{L_t^\infty L_x^p}^{\frac{p}{p-1}} t^{\frac{1}{p-1}} \left(\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds \right)^{\frac{p-2}{p-1}}. \end{aligned} \quad (3.1)$$

Let us now set

$$D_\nu(t) := \nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 ds, \quad (3.2)$$

and fix $\lambda > 0$. Exploiting (1.3) and (3.1) we obtain

$$\begin{aligned} D_\nu(t) &= \nu \int_0^t \sum_{k \in \mathbb{Z}^d} |k|^2 |\widehat{u}_t^\nu(k)|^2 \\ &\lesssim \frac{\nu \lambda^2}{\log(\lambda + 2)^p} \int_0^t \sum_{|k| < \lambda} \log(2 + |k|)^p |\widehat{u}_s^\nu(k)|^2 ds + \frac{\nu}{\lambda^2} \int_0^t \|\Delta u_s^\nu\|_{L^2}^2 ds \\ &\leq \frac{\nu \lambda^2}{\log(\lambda + 2)^p} \int_0^t \|u_s\|_{H^{\log,p}}^2 ds + \frac{1}{\lambda^2} + \frac{1}{\nu \lambda^2} \|\nabla b\|_{L_t^\infty L_x^p}^{\frac{p}{p-1}} t^{\frac{1}{p-1}} D_\nu(t)^{\frac{p-2}{p-1}} \end{aligned}$$

for any $t \in (0, T)$. By means of the Young inequality we can estimate

$$\frac{1}{\nu \lambda^2} \|\nabla b\|_{L_t^\infty L_x^p}^{\frac{p}{p-1}} t^{\frac{1}{p-1}} D_\nu(t)^{\frac{p-2}{p-1}} \leq \frac{C_p}{\nu^{p-1} \lambda^{2(p-1)}} \|\nabla b\|_{L_t^\infty L_x^p}^p t + \frac{1}{2} D_\nu(t),$$

while Corollary 1.2 gives

$$\int_0^t \|u_s\|_{H^{\log,p}}^2 ds \lesssim_{p,d} t^{p+1} \|\nabla b\|_{L_t^\infty L_x^p}^p + t.$$

Putting all together we end up with

$$D_\nu(t) \lesssim_{p,d} \frac{\nu \lambda^2 t}{\log(\lambda + 2)^p} \left(t^p \|\nabla b\|_{L_t^\infty L_x^p}^p + 1 \right) + \frac{1}{\lambda^2} + \frac{1}{\nu^{p-1} \lambda^{2(p-1)}} t \|\nabla b\|_{L_t^\infty L_x^p}^p. \quad (3.3)$$

Choosing

$$\lambda = (\nu t)^{-\frac{1}{2}} \log \left(\frac{1}{\nu t} + e \right)^{1/2}, \quad (3.4)$$

and using the elementary inequality

$$\frac{\log \left(\frac{1}{\nu t} + e \right)}{\log \left(\frac{\log((\nu t)^{-1} + e)^{1/2}}{\sqrt{\nu t}} + 2 \right)^p} \leq \frac{\log \left(\frac{1}{\nu t} + e \right)}{\log \left(\frac{1}{\sqrt{\nu t}} + 2 \right)^p} \leq 2^p \frac{1}{\log \left(\frac{1}{\nu t} + 4 \right)^{p-1}},$$

one gets (0.9).

3.2. Lower bound on L^2 norms. Let us now present two consequences of Theorem 0.4.

The first conclusion is Proposition 3.1 below. It provides an upper bound on the enhanced diffusion rate $r(\nu)$, we refer to the introduction for a detailed discussion.

Proposition 3.1. *Let $b \in L^\infty([0, +\infty), W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ be a divergence-free vector field for some $p > 2$. Given $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$, if there exists $r : (0, \nu_0) \rightarrow (0, +\infty)$ for some $0 < \nu_0 < 1$, which satisfies*

$$\|u_t^\nu\|_{L^2}^2 \leq e^{-r(\nu)t} \|u_0\|_{L^2}^2 \quad \text{for any } t > 1/\nu_0 \text{ and } \nu \in (0, \nu_0), \quad (3.5)$$

then

$$\limsup_{\nu \downarrow 0} \frac{r(\nu)}{\log(1/\nu)^{-\frac{p-1}{p}}} \leq C, \quad (3.6)$$

where $C = C(p, d, \|u_0\|_{W^{1,2}}, \|u_0\|_{L^\infty}, \|\nabla b\|_{L_t^\infty L_x^p})$.

Proof. Let $C = C(p, d)$ as in the statement of Theorem 0.4 and set

$$K := C(\|u_0\|_{W^{1,2}}^2 + \|u_0\|_{L^\infty}^2).$$

Fix $\alpha > 0$ to be chosen later. Given $\nu > 0$ small enough we set $t := \alpha \log(1/\nu)^{\frac{p-1}{p}} > 1/\nu_0$. From (0.2) and Theorem 0.4 we get

$$\begin{aligned} \|u_t\|_{L^2}^2 &= \|u_0\|_{L^2}^2 - \nu \int_0^t \|\nabla u_s\|_{L^2}^2 ds \\ &\geq \|u_0\|_{L^2}^2 \left[1 - \frac{K}{\|u_0\|_{L^2}^2} \left(\nu t + \frac{t^p \|\nabla b\|_{L_t^\infty L_x^p}^p + 1}{\log\left(\frac{1}{\nu t} + 2\right)^{p-1}} \right) \right] \\ &= \|u_0\|_{L^2}^2 \left[1 - \frac{K}{\|u_0\|_{L^2}^2} \left(\alpha \nu \log(1/\nu)^{\frac{p-1}{p}} + \alpha^p \frac{\log(1/\nu)^{p-1} \|\nabla b\|_{L_t^\infty L_x^p}^p + 1}{\log\left(\frac{1}{\alpha \nu} \log(1/\nu)^{-\frac{p-1}{p}} + 2\right)^{p-1}} \right) \right] \\ &= \|u_0\|_{L^2}^2 \left(1 - \frac{K \alpha^p \|\nabla b\|_{L_t^\infty L_x^p}^p}{\|u_0\|_{L^2}^2} + o(1) \right) \end{aligned}$$

where $o(1) \rightarrow 0$ for $\nu \rightarrow 0$, and K is as in Theorem 0.4. We deduce

$$\liminf_{\nu \downarrow 0} \exp \left\{ -\alpha \frac{r(\nu)}{\log(1/\nu)^{-\frac{p-1}{p}}} \right\} \geq 1 - \frac{K \alpha^p \|\nabla b\|_{L_t^\infty L_x^p}^p}{\|u_0\|_{L^2}^2},$$

and choosing α such that

$$\frac{K \alpha^p \|\nabla b\|_{L_t^\infty L_x^p}^p}{\|u_0\|_{L^2}^2} = \frac{1}{2},$$

we easily get (0.15). □

A second consequence of Theorem 0.4 is a step toward Conjecture 0.7.

Proposition 3.2. *Let $b \in L^\infty([0, +\infty), W^{1,p}(\mathbb{T}^d, \mathbb{R}^d) \cap L^\infty)$ be a divergence-free vector field for some $p > 2$. Let u^ν solve (E_ν) with $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$. Then, for any $\alpha \in [0, p-1)$ there exist $\nu_0 = \nu_0(u_0, \|b\|_{L_t^\infty W_x^{1,p} \cap L^\infty}, \alpha, p, d) \in (0, 1)$ and $C = C(u_0, \|b\|_{L_t^\infty W_x^{1,p} \cap L^\infty}, p, d) > 0$ such that*

$$\|u_t^\nu\|_{L^2}^2 \geq \|u_0\|_{L^2}^2 \exp \left\{ -\log(1/\nu)^{-\alpha} \exp \left\{ e^{Ct^{\frac{p}{p-1}-\alpha}} \right\} \right\}, \quad (3.7)$$

for every $1 < t < +\infty$ and $\nu \in (0, \nu_0)$.

Proof. Let $C = C(p, d)$ as in the statement of Theorem 0.4 and define

$$K := C(\|u_0\|_{W^{1,2}}^2 + \|u_0\|_{L^\infty}^2).$$

Set

$$t_\nu := \left(\frac{\|u_0\|_{L^2}^2}{2K} \right)^{1/p} \frac{1}{\|\nabla b\|_{L_t^\infty L_x^p}} \log(1/\nu)^{\frac{p-1-\alpha}{p}}.$$

Let us begin by considering the case $0 < t \leq t_\nu$, arguing as in the proof of Proposition 0.8, we get

$$\begin{aligned} \|u_t\|_{L^2}^2 &= \|u_0\|_{L^2}^2 - \nu \int_0^t \|\nabla u_s\|_{L^2}^2 ds \geq \|u_0\|_{L^2}^2 - \nu \int_0^{t_\nu} \|\nabla u_s\|_{L^2}^2 ds \\ &\geq \|u_0\|_{L^2}^2 \left[1 - \frac{K}{\|u_0\|_{L^2}^2} \left(\nu t_\nu + \frac{t_\nu^p \|\nabla b\|_{L_t^\infty L_x^p}^p + 1}{\log\left(\frac{1}{\nu t_\nu} + 2\right)^{p-1}} \right) \right] \\ &= \|u_0\|_{L^2}^2 \left(1 - \frac{1}{2} \log(1/\nu)^{-\alpha} + o(1) \right) \end{aligned}$$

where $o(1) \rightarrow 0$ for $\nu \rightarrow 0$. Therefore, we can find $\nu_0 = \nu_0(u_0, b, p, d) \in (0, 1)$ such that, for any $\nu \in (0, \nu_0)$ it holds

$$\|u_t^\nu\|_{L^2}^2 \geq e^{-\log(1/\nu)^{-\alpha}} \|u_0\|_{L^2}^2 \quad \text{for every } t \in (0, t_\nu). \quad (3.8)$$

Observe that (3.8) implies (3.7) for any $t \in (0, t_\nu)$.

Let us now consider the case $t > t_\nu$, for $\nu \in (0, \nu_0)$. From [MD18] we know that

$$\|u_t^\nu\|_{L^2}^2 \geq \|u_0\|_{L^2}^2 \exp \left\{ -\nu^2 \frac{\|\nabla u_0\|_{L^2}^2}{\|u_0\|_{L^2}^2} \left(e^{t\nu^{-1}\|b\|_{L^\infty}} - 1 \right) \right\}, \quad (3.9)$$

it is easily seen that, for $t \geq t_\nu$ one has

$$\nu^2 \frac{\|\nabla u_0\|_{L^2}^2}{\|u_0\|_{L^2}^2} \left(e^{t\nu^{-1}\|b\|_{L^\infty}} - 1 \right) \leq \frac{\|\nabla u_0\|_{L^2}^2}{\|u_0\|_{L^2}^2} \exp \left\{ \|b\|_{L^\infty} e^{Ct\frac{p}{p-1-\alpha}} \right\}, \quad (3.10)$$

where $C = C(u_0, \|b\|_{L_t^\infty W_x^{1,p} \cap L^\infty}, p, d) > 0$, hence (3.7) is satisfied provided $\nu_0 > 0$ is small enough. \square

3.3. Vanishing viscosity limit. Another interesting consequence of Theorem 0.4 regards the vanishing viscosity limit $\nu \rightarrow 0$. More precisely we aim at estimating the L^2 distance between u^ν and u^0 which, respectively, solve (E_ν) and (E_0) . To this end the key estimate to take into account is

$$\nu^2 \int_0^t \|\Delta u_s\|_{L^2}^2 ds \leq C \left[\nu + t\nu^{\frac{p-2}{p-1}} + \frac{t^{p-1} + 1}{\log\left(\frac{1}{\nu t} + 2\right)^{p-2}} \right] \quad \text{for every } \nu > 0 \text{ and } t > 0, \quad (3.11)$$

where $C = (1 + \|\nabla b\|_{L_t^\infty L_x^p}^p)(\|u_0\|_{W^{1,2}}^2 + \|u_0\|_{L^\infty}^2)$. Notice that (3.11) easily follows by combining (2.3) and (0.4).

The connection between (3.11) and the vanishing viscosity estimate is given by

$$\sup_{s \in [0, t]} \|u_s^\nu - u_s^0\|_{L^2}^2 \leq t\nu^2 \int_0^t \|\Delta u_s^\nu\|_{L^2}^2 ds, \quad (3.12)$$

that comes from

$$\frac{d}{dt} \|u_t^\nu - u_t^0\|_{L^2}^2 \leq 2\nu \left| \int_{\mathbb{T}^d} (u_t^\nu - u_t^0) \Delta u_t^\nu dx \right| \leq 2\nu \|u_t^\nu - u_0\|_{L^2} \|\Delta u_t^\nu\|_{L^2},$$

by applying the Hölder inequality. What we have proven is Theorem 0.9 that we state again below for the reader's convenience.

Theorem 3.3. *Let $b \in L^\infty([0, +\infty), W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ be a divergence-free vector field for some $p > 2$. Given $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$ we consider u^ν, u^ν , respectively, solutions to (E_ν) and (E_0) . Then it holds*

$$\sup_{s \in [0, t]} \|u_s^\nu - u_s^0\|_{L^2}^2 \leq Ct \left[\nu + t\nu^{\frac{p-2}{p-1}} + \frac{t^{p-1} + 1}{\log\left(\frac{1}{\nu t} + 2\right)^{p-2}} \right] \quad \text{for every } \nu > 0 \text{ and } t > 0,$$

where $C = (1 + \|\nabla b\|_{L_t^\infty L_x^p}^p)(\|u_0\|_{W^{1,2}}^2 + \|u_0\|_{L^\infty}^2)$.

Relying on ideas developed in subsection 2.2 we are able to prove that the bound

$$\sup_{s \in [0, t]} \|u_s^\nu - u_s^0\|_{L^2}^2 \leq O(\log(1/\nu)^{2-p}) \quad \text{for } \nu \rightarrow 0$$

is almost optimal. More precisely, we show that for any $C > 0$ one can find $b \in L^1([0, T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ and $u_0 \in W^{1,2}(\mathbb{R}^d) \cap L^\infty$ such that, for every $r > p$ it holds

$$\limsup_{\nu \downarrow 0} \log(1/\nu)^r \|u_t^\nu - u_t^0\|_{L^2}^2 = \infty.$$

This easily follows from Proposition 3.4 below and the example in Proposition 1.3.

Proposition 3.4. *Fix $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$. Let u_t^ν and u_t^0 solve, respectively, (E_ν) (E_0) with $b \in L^1([0, T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$, for some $p > 2$. If there exist $t \in (0, T)$, $\nu_0 \in (0, 1)$, $C > 0$ and $r > 0$ such that*

$$\|u_t^\nu - u_t^0\|_{L^2}^2 \leq C \log(1/\nu)^{-r}, \quad \text{for every } 0 < \nu < \nu_0, \quad (3.13)$$

then

$$u_t^0 \in H^{\log, r_1} \quad \text{for any } 0 < r_1 < r.$$

Proof. We can assume without loss of generality that

$$\|\nabla b\|_{L_t^\infty L_x^p} + \|u_0\|_{W^{1,2}} + \|u_0\|_{L^\infty} \leq 1.$$

Fix $\nu \in (0, \nu_0)$ and $\varepsilon \in (0, 1)$. By (2.6) and our assumptions we have

$$\begin{aligned} \|u_t^0 * \rho_\varepsilon - u_t^0\|_{L^2} &\leq 2 \|u_t^\nu - u_t^0\|_{L^2} + \|u_t^\nu * \rho_\varepsilon - u_t^\nu\|_{L^2} \\ &\lesssim C \log(1/\nu)^{-r/2} + \varepsilon \|\nabla u_t^\nu\|_{L^2}, \end{aligned}$$

that along with Proposition 2.3, gives

$$\|u_t^0 * \rho_\varepsilon - u_t^0\|_{L^2} \lesssim_{p,d,\gamma} C \log(1/\nu)^{-r/2} + \varepsilon \nu^{-1/2} t^{(1-\beta)/2}. \quad (3.14)$$

In particular, choosing $\varepsilon = \nu$, there exists $C' = C'(t, C, p, d, \gamma)$ such that

$$\|u_t^0 * \rho_\nu - u_t^0\|_{L^2} \leq C' \log(1/\nu)^{-r/2} \quad \text{for every } 0 < \nu < \nu_0. \quad (3.15)$$

As we have already shown in subsection 2.2, the inequality (3.15) implies $u_t^\nu \in H^{\log, r_1}$ for any $0 \leq r_1 < r$. \square

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