Stability of optimal traffic plans in the irrigation problem

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We prove the stability of optimal traffic plans in branched transport. In particular, we show that any limit of optimal traffic plans is optimal as well. This is the Lagrangian counterpart of the recent Eulerian version proved in [CDM19a].

1 Introduction

Given two nonnegative and finite Borel measures μ^-, μ^+ on $\mathbb{R}^d$ of equal total mass, the irrigation problem consists in connecting μ^- to μ^+ with minimal cost, where in branched transport the displacement is performed on a 1-dimensional network and the transport cost for a collection of particles of total mass m travelling a distance ℓ along a common stretch is proportional to $\ell \times m^\alpha$, for a fixed parameter $\alpha \in (0,1)$. This problem may be cast in two main statical frameworks: an Eulerian one [Xia03], based on vector valued measures (more precisely normal 1-currents) called *transport paths*, and a Lagrangian one [MSM03; BCM05], based on positive measures on a set of curves (or trajectories) called *traffic plans*. We refer to the book [BCM08] for the general theory of branched transport, and to the first sections of the more recent works [Peg17b; CDM18; CDM19b] and the references therein.

In this paper, we tackle the question of the *stability* in the Lagrangian framework: if $\{\mu_n^-\}_{n\in\mathbb{N}}$ and $\{\mu_n^+\}_{n\in\mathbb{N}}$ converge respectively to μ^- and μ^+, and if $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ is a sequence of optimal traffic plans for the marginals (μ_n^-, μ_n^+), converging to a traffic plan $\mathbf{P}$, is it true that $\mathbf{P}$ is optimal for (μ^-, μ^+)? The positive answer is classically known above the critical threshold $\alpha > 1 - 1/d$ both for the Lagrangian and the Eulerian formulation. A positive answer for every $\alpha \in (0,1)$ has been recently given for the Eulerian formulation in [CDM19a]. Although the Eulerian and Lagrangian problems are essentially equivalent (see [PS06; Peg17b]), the Eulerian viewpoint carries less information than the Lagrangian one, and the Lagrangian stability is not a straightforward consequence of the Eulerian one.

Main result Denote by $\mathbf{OTP}(\mu^-,\mu^+)$ the set of optimal traffic plans with marginals (μ^-,μ^+). Modulo some technical assumptions (necessary to the validity of the statement), we prove the following result. See Theorem 2.1 for the correct statement.

Theorem 1.1 (Short statement)**.** *Let $\mu_n^\pm \overset{\star}{\rightharpoonup} \mu^\pm$ and let $\mathbf{P}_n \in \mathbf{OTP}(\mu_n^-,\mu_n^+)$ and assume that $\mathbf{P}_n$ converges to $\mathbf{P}$. Then, up to mild technical assumptions, $\mathbf{P} \in \mathbf{OTP}(\mu^-,\mu^+)$.*

Strategy of the proof Our proof relies on the general stability result proved for the Eulerian setting in [CDM19a, Theorem 1.1]. The classical way to associate to a Lagrangian traffic plan $\mathbf{P}$ an Eulerian transport path $T = T_{\mathbf{P}}$ consists in integrating (w.r.t. $\mathbf{P}$) the obvious vector measures associated to the curves supporting $\mathbf{P}$. The two suitably defined notions of transportation cost coincide on optimizers.

Taking $\mathbf{P}$, $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ as in Theorem 1.1 we consider the induced transport paths T, $\{T_n\}_{n\in\mathbb{N}}$. One can easily show that T and $\{T_n\}_{n\in\mathbb{N}}$ satisfy the hypotheses of [CDM19a, Theorem 1.1], so that T is an optimal transport path for the marginals (μ^-,μ^+). Nevertheless in principle it could happen that the cost of T as a transport path and the cost of $\mathbf{P}$ as a traffic plan do not coincide. This possibility can be attributed only to a specific phenomenon: some curves of $\mathbf{P}$ partially overlap with opposite orientations, thus producing *cancellations* at the level of vector measures. Most of our work consists in excluding the occurrence of such phenomenon.

The article closely follows the structure of the proof. After setting the notation, main definitions and preliminary results in Section 2, we argue by contradiction assuming that $\mathbf{P}$ produces cancellations at the Eulerian level. Section 3 provides existence of "many Lagrangian cycles" in $\mathbf{P}$, i.e. many pairs of distinct points (x, y) such that both the family of those trajectories crossing x after y and those crossing y after x have positive measure according to $\mathbf{P}$. From this, we deduce in Section 4 the existence of "quasi-cycles" in the $\mathbf{P}_n$'s, roughly saying that for any such pair (x, y) a certain amount of trajectories passes arbitrarily close to x and y in both orders, for n large enough. In Section 5 we show that this leads to a contradiction by constructing a better competitor for $\mathbf{P}_n$, removing portions of such trajectories, thus completing the proof of Theorem 1.1.

2 Preliminaries

In this section, we gather some definitions and basic facts that will be used throughout the paper. The notation is mostly consistent with [CDM19b].

2.1 Background, notation, and main result

We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^d$ and by $B_r(x)$, $\bar{B}_r(x)$ respectively the open and the closed ball with center x and radius r. From now on we fix, $\alpha \in (0, 1)$, $R > 0$ and $X := \bar{B}_R(0) \subseteq \mathbb{R}^d$. Except for the obvious cases, the measures that we consider are always Radon measures. Here is a list of notation used throughout the paper:

$\mathbf{1}_A$	indicator function of a set A valued in $\{0,1\}$
$d(x,A)$	$:= \inf_{y\in A}\lvert x-y\rvert$, distance between the point x and the set A
$\mathscr{M}^k(Y)$	space of finite (signed or vector) Borel measures on Y valued in $\mathbb{R}^k$
$\mathscr{M}^1_+(Y)$	set of nonnegative finite Borel measures on Y
$\mu_n \overset{\star}{\rightharpoonup} \mu$	weak-$\star$ convergence of measures in the duality between $C^0(Y,\mathbb{R}^k)$ and $\mathscr{M}^k(Y)$ when Y is compact, i.e. $\int f\,\mathrm{d}\mu_n \to \int f\,\mathrm{d}\mu$ for every $f \in C^0(Y,\mathbb{R}^k)$
$\mu \llcorner A$	$:= \mathbf{1}_A\mu$, restriction of the measure μ to the subset A
$f_\sharp\mu$	push-forward of the measure μ on Y by the map $f : Y \to Y'$, i.e. $f_\sharp\mu(A) := \mu(f^{-1}(A))$
$f\mu$	(vector) measure defined by $[f\mu](A) := \int_A f\,\mathrm{d}\mu$ for every Borel set A, when μ is a nonnegative Borel measure and f a Borel (vector-valued) map such that $\int\lvert f\rvert\,\mathrm{d}\mu < +\infty$
$\mathbb{M}(\mu)$	mass of the measure μ
$\mathbb{M}^\alpha(\mu)$	$:= \sum_{x\in Y}\lvert\mu(\{x\})\rvert^\alpha$ when $\alpha \in [0,1)$ and $\mu \in \mathscr{M}^1(Y)$ is atomic, set to $+\infty$ if μ is not atomic
$\mu \leq \nu$	means that $\mu(A) \leq \nu(A)$ for all Borel set A
$\lVert f\rVert_\infty$	$:= \sup_{x\in Y}\lvert f(x)\rvert$ supremum norm of $f : Y \to \mathbb{R}^k$
$\mathscr{H}^k$	k-dimensional Hausdorff measure
$\mathscr{H}^k_\delta$	k-dimensional Hausdorff pre-measures (see [EG15, Definition 2.1])
Lip_1	set of 1-Lipschitz curves $\gamma : \mathbb{R}_+ \to X$, endowed with the (compact and metrizable) topology of uniform convergence on compact subsets of $\mathbb{R}_+$
$\mathrm{Img}\,\gamma$	image $\gamma(I)$ of a curve $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^d$
$T_\infty(\gamma)$	$:= \inf\{t \in \mathbb{R}_+ : \gamma \text{ is constant on } [t,+\infty)\} \in [0,\infty]$, stopping time of γ
$\gamma_{\lvert[a,b]}$	restriction of $\gamma \in \mathrm{Lip}_1$ to an interval $[a,b] \subseteq \mathbb{R}_+$ defined by $t \mapsto \gamma(t+a)$ for $t \in [0,b-a]$ and $t \mapsto \gamma(b)$ for $t \geq b-a$
e_0, e_∞	evaluation maps $\gamma \mapsto \gamma(0)$ and $\gamma \mapsto \gamma(\infty) := \lim_{t\to+\infty}\gamma(t)$ when γ has finite length
$\mathrm{Tan}(x,E)$	tangent space line at point x to E when E is 1-rectifiable, i.e. it is contained in a countable union of images of Lipschitz curves up to an $\mathscr{H}^1$-null set; it is $\mathscr{H}^1$-a.e. defined on E (see [AFP00, Definition 2.86]).

A *traffic plan* $\mathbf{P}$ is a measure in $\mathscr{M}^1_+(\mathrm{Lip}_1)$ such that $\int_{\mathrm{Lip}_1} T_\infty\,\mathrm{d}\mathbf{P} < \infty$. If there exists a 1-rectifiable set E such that

$$\mathscr{H}^1(\mathrm{Img}\,\gamma \setminus E) = 0 \text{ for } \mathbf{P}\text{-almost every } \gamma \in \mathrm{Lip}_1, \tag{2.1}$$

then $\mathbf{P}$ is said rectifiable. We list the main objects that we need regarding traffic plans:

$\mathbf{TP}$	space of traffic plans
$\mathbf{TP}(\mu^-,\mu^+)$	set of traffic plans $\mathbf{P}$ such that $(e_0)_\sharp\mathbf{P}=\mu^-$, $(e_\infty)_\sharp\mathbf{P}=\mu^+$
$\mathbf{P}_n \overset{\star}{\rightharpoonup} \mathbf{P}$	weak-$\star$ convergence in $\mathscr{M}^1(\mathrm{Lip}_1)$
$\theta_{\mathbf{P}}(x)$	$:= \mathbf{P}(\{\gamma\in\mathrm{Lip}_1 : x\in\operatorname{Img}\gamma\})$, multiplicity at x w.r.t. $\mathbf{P}$
$\Theta_{\mathbf{P}}(x)$	$:= \int_{\mathrm{Lip}_1}\mathscr{H}^0(\gamma^{-1}(x))\,\mathrm{d}\mathbf{P}$, full multiplicity at x w.r.t. $\mathbf{P}$
$\mathcal{E}^\alpha(\mathbf{P})$	$:= \int_{\mathrm{Lip}_1}\int_{\mathbb{R}_+}\theta_{\mathbf{P}}(\gamma(t))^{\alpha-1}\lvert\gamma'(t)\rvert\,\mathrm{d}t\,\mathrm{d}\mathbf{P}(\gamma)$, α-energy of $\mathbf{P}$
$\mathbf{OTP}(\mu^-,\mu^+)$	set of optimal traffic plans with marginals (μ^-,μ^+), that is $\mathcal{E}^\alpha(\mathbf{P})<+\infty$ and $\mathcal{E}^\alpha(\mathbf{P})\le\mathcal{E}^\alpha(\mathbf{Q})$ for every $\mathbf{Q}\in\mathbf{TP}(\mu^-,\mu^+)$
$\Sigma_{\mathbf{P}}$	$:= \{x : \theta_{\mathbf{P}}(x)>0\}$ network associated with $\mathbf{P}$; it is 1-rectifiable (by [Peg17b, Section 2.1] or [BCM05, Lemma 6.3]), and when $\mathbf{P}$ is rectifiable then (2.1) holds with $E=\Sigma_{\mathbf{P}}$.

We can now state the correct version of Theorem 1.1.

Theorem 2.1 (Stability of optimal traffic plans in the irrigation problem)**.** *Let $\alpha\in(0,1)$, μ^-, μ^+ be mutually singular positive finite measures on $\bar{B}_R(0)\subseteq\mathbb{R}^d$, $R>0$, satisfying $\mu^-(\mathbb{R}^d)=\mu^+(\mathbb{R}^d)$. Let $\{\mu_n^-\}_{n\in\mathbb{N}}$ and $\{\mu_n^+\}_{n\in\mathbb{N}}$ be sequences of positive finite measures on $\bar{B}_R(0)$ such that $\mu_n^-(\mathbb{R}^d)=\mu_n^+(\mathbb{R}^d)$ for every $n\in\mathbb{N}$ and*

$$\mu_n^\pm \overset{\star}{\rightharpoonup} \mu^\pm,$$

and assume there exist $\mathbf{P}_n\in\mathbf{OTP}(\mu_n^-,\mu_n^+)$ satisfying

$$\sup_{n\in\mathbb{N}}\left\{\mathcal{E}^\alpha(\mathbf{P}_n)+\int_{\mathrm{Lip}_1}T_\infty(\gamma)\,\mathrm{d}\mathbf{P}_n(\gamma)\right\}<\infty,$$

and

$$\mathbf{P}_n \xrightharpoonup[n\to\infty]{\star} \mathbf{P},$$

for some $\mathbf{P}$. Then $\mathbf{P}\in\mathbf{OTP}(\mu^-,\mu^+)$.

2.2 Transport paths

A *transport path* T over X is a normal 1-current, or equivalently a vector measure on X whose distributional divergence is a signed measure. Let us summarize the classical notation for transport paths in the following table:

TP	space of transport paths
∂T	$:= -\operatorname{div}T$ where $\operatorname{div}T$ is the distributional divergence of T on $\mathbb{R}^d$
$TP(\mu^-,\mu^+)$	set of transport paths T such that $\partial T=\mu^+-\mu^-$
$T_n \overset{\star}{\rightharpoonup} T$	weak-$\star$ convergence in $\mathscr{M}^d(X)$

$[\![E,\vec{\theta}]\!]$	$:= \vec{\theta}\mathscr{H}^1 \llcorner E$ when E is 1-rectifiable and $\vec{\theta}: E \to \mathbb{R}^d$ is such that $\vec{\theta}(x) \in \mathrm{Tan}(x,E)$ for $\mathscr{H}^1$-a.e. x, $\int_E \lvert\vec{\theta}\rvert \,\mathrm{d}\mathscr{H}^1 < \infty$, and $\vec{\theta}\mathscr{H}^1 \llcorner E$ is normal ; transport paths of this form are called rectifiable
$\mathbb{M}^\alpha(T)$	defined by $\int_E \lvert\vec{\theta}\rvert^\alpha \,\mathrm{d}\mathscr{H}^1$ for $T = [\![E,\vec{\theta}]\!]$, set to $+\infty$ if T is not rectifiable
$OTP(\mu^-,\mu^+)$	set of optimal transport paths $T \in TP(\mu^-,\mu^+)$, meaning that $\mathbb{M}^\alpha(T) < +\infty$ and $\mathbb{M}^\alpha(T) \leq \mathbb{M}^\alpha(S)$ for every $S \in TP(\mu^-,\mu^+)$
I_γ	transport path induced by the curve of finite length $\gamma \in \mathrm{Lip}_1$ and defined by $\langle I_\gamma, \omega\rangle := \int_{\mathbb{R}_+} \omega(\gamma(t))\cdot\gamma'(t)\,\mathrm{d}t$ for every $\omega \in C_c^\infty(X,\mathbb{R}^d)$; its boundary is $\partial I_\gamma = \delta_{\gamma(\infty)} - \delta_{\gamma(0)}$
$T_{\mathbf{P}}$	$:= \int_{\mathrm{Lip}_1} I_\gamma \,\mathrm{d}\mathbf{P}(\gamma)$, transport path induced by $\mathbf{P}$; its boundary is $\partial T_{\mathbf{P}} = (e_\infty)_\sharp \mathbf{P} - (e_0)_\sharp \mathbf{P}$.

In the last definition, the integration should be intended in the following sense. Let I be a finite measure space and for every $t \in I$ let μ_t be a measure on $\mathbb{R}^n$, possibly real- or vector-valued, such that $t \mapsto \mu_t(E)$ is measurable for every Borel set E in $\mathbb{R}^n$; the integral $\int_I \mathbb{M}(\mu_t)\,\mathrm{d}t$ is finite. Then we denote by $\int_I \mu_t \,\mathrm{d}t$ the measure on $\mathbb{R}^n$ defined by

$$\left[\textstyle\int_I \mu_t \,\mathrm{d}t\right](E) := \int_I \mu_t(E)\,\mathrm{d}t \quad \text{for every Borel set } E \text{ in } \mathbb{R}^n. \tag{2.2}$$

When $T = T_{\mathbf{P}}$ we say that $\mathbf{P}$ *decomposes* T. Following [CDM18], a *good decomposition*, first introduced by Smirnov (see [Smi93, Section 1.2]) for normal currents, is a decomposition where neither cycles nor cancellations occur:

Definition 2.2 (Good decomposition)**.** Let T and $\mathbf{P}$ be a transport path and traffic plan such that $T = T_{\mathbf{P}}$. Then $\mathbf{P}$ is said to be a good decomposition of T if:

(A) $\mathbf{P}$ is supported on nonconstant simple curves;

(B) $\mathbb{M}(T) = \int_{\mathrm{Lip}_1} \mathbb{M}(I_\gamma)\,\mathrm{d}\mathbf{P}(\gamma)$;

(C) $\mathbb{M}(\partial T) = \int_{\mathrm{Lip}_1} \mathbb{M}(\partial I_\gamma)\,\mathrm{d}\mathbf{P}(\gamma) = 2\mathbf{P}(\mathrm{Lip}_1)$.

According to the Decomposition Theorem of Smirnov [Smi93, Theorem C] (see also [San14] for a Dacorogna-Moser approach), any acyclic transport path, hence any optimal transport path, admits a good decomposition.

2.3 On curves and rectifiability

Here we collect some basic results about 1-Lipschitz curves, 1-rectifiable sets and rectifiable traffic plans.

Definition 2.3. Let $\gamma \in \mathrm{Lip}_1$ of finite length and $\mathbf{P} \in \mathbf{TP}$ a rectifiable traffic plan. We say that:

- x is a *regular point* of γ if $x \notin \{\gamma(0), \gamma(\infty)\}$, $\mathrm{Tan}(x, \mathrm{Img}\,\gamma)$ exists, $\gamma^{-1}(x)$ is finite and for all preimage t of x, $\gamma'(t)$ exists and spans $\mathrm{Tan}(x, \mathrm{Img}\,\gamma)$. Notice that $\gamma'(t)$ might have different orientations for different preimages of x;

- x is a *regular point* of $\mathbf{P}$ if $\mathrm{Tan}(x, \Sigma_{\mathbf{P}})$ exists and if for $\mathbf{P}$-a.e. curve γ, x is a regular point of γ such that $\mathrm{Tan}(x, \Sigma_{\mathbf{P}}) = \mathrm{Tan}(x, \mathrm{Img}\,\gamma)$.

Remark 2.4. As a direct consequence of the Area formula ([EG15, Section 3.3]) and the definition of the tangent space (see [AFP00, Section 2.11] or [Mat95, pp. 212–213]), we get that $\mathscr{H}^1$-a.e. $x \in \mathrm{Img}\,\gamma$ is a regular point of γ. Using the rectifiability of $\mathbf{P}$ and Fubini's Theorem, we immediately deduce that $\mathscr{H}^1$-a.e. point $x \in \Sigma_{\mathbf{P}}$ is regular for $\mathbf{P}$. Indeed, denoting $f : \mathrm{Lip}_1 \times X \to \mathbb{R}$ such that $f(\gamma, x) = 0$ if γ has finite length and x is a regular point of γ and $f(\gamma, x) = 1$ otherwise, it holds

$$0 = \int_{\mathrm{Lip}_1} \int_X f(\gamma, x)\, \mathrm{d}\mathscr{H}^1 \llcorner \Sigma_{\mathbf{P}}\, \mathrm{d}\mathbf{P} = \int_X \int_{\mathrm{Lip}_1} f(\gamma, x)\, \mathrm{d}\mathbf{P}\, \mathrm{d}\mathscr{H}^1 \llcorner \Sigma_{\mathbf{P}}.$$

Now at each regular point x of $\gamma \in \mathrm{Lip}_1$, we define:

$$\vec{m}_\gamma(x) \coloneqq \sum_{t \in \gamma^{-1}(x)} \gamma'(t)/|\gamma'(t)|, \tag{2.3}$$

and at each regular point x of $\mathbf{P}$:

$$\vec{\theta}_{\mathbf{P}}(x) \coloneqq \int_{\mathrm{Lip}_1} \vec{m}_\gamma(x)\, \mathrm{d}\mathbf{P}(\gamma). \tag{2.4}$$

Both are well-defined $\mathscr{H}^1$-a.e. respectively on $\mathrm{Img}\,\gamma$ and $\Sigma_{\mathbf{P}}$, and set to 0 outside. Notice that by definition $\vec{m}_\gamma(x) \in \mathrm{Tan}(x, \mathrm{Img}\,\gamma)$ (with integer norm) for $\mathscr{H}^1$-a.e. $x \in \mathrm{Img}\,\gamma$ and $\vec{\theta}_{\mathbf{P}}(x) \in \mathrm{Tan}(x, \Sigma_{\mathbf{P}})$ for $\mathscr{H}^1$-a.e. $x \in \Sigma_{\mathbf{P}}$. A direct use of the Area Formula and Fubini's Theorem yields:

$$I_\gamma = [\![\mathrm{Img}\,\gamma, \vec{m}_\gamma]\!], \qquad T_{\mathbf{P}} = [\![\Sigma_{\mathbf{P}}, \vec{\theta}_{\mathbf{P}}]\!]. \tag{2.5}$$

Following [Mat95, Definition 11.9], given a 1-dimensional linear subspace $V \subseteq \mathbb{R}^d$, $x \in \mathbb{R}^d$, $s \in (0,1)$ and $r \in [0, \infty]$, we define the two-sided cone:

$$X(x, r, V, s) := \{y \in \mathbb{R}^d : d(y - x, V) \leq s|y - x|\} \cap \bar{B}_r(x),$$

and for any nonzero vector $v \in \mathbb{R}^d$, we define the one-sided cone:

$$X_\pm(x, r, v, s) := X(x, r, \mathrm{span}\, v, s) \cap \{y \in \mathbb{R}^d : \pm v \cdot (y - x) \geq 0\}.$$

Definition 2.5 (Proper crossing)**.** Consider a cone $X(x_0, r, V, s)$ and a curve $\gamma \in \mathrm{Lip}_1$ such that $\gamma(t_0) = x_0$, and $\gamma'(t_0)$ exists and spans V. We say that γ *crosses the cone properly* at time $t_0 \in (0, T_\infty(\gamma))$ if there exist $t_{\mathrm{in}} < t_0 < t_{\mathrm{out}}$ such that

(i) $\gamma([t_{\mathrm{in}}, t_0]) \subseteq X_-(x_0, r, \gamma'(t_0), s)$ and $\gamma([t_0, t_{\mathrm{out}}]) \subseteq X_+(x_0, r, \gamma'(t_0), s)$;

(ii) $\gamma(t_{\text{in}}), \gamma(t_{\text{out}}) \in \partial B_r(x_0)$;

(iii) $\gamma(s) \in B_r(x_0)$ for every $s \in (t_{\text{in}}, t_{\text{out}})$.

We say that t_{in} and t_{out} are entrance and exit times of γ inside the cone.

Proper crossing holds around regular points of any Lipschitz curve, as stated below.

Lemma 2.6. *Let $\gamma \in \mathrm{Lip}_1$ be a curve of finite length and x_0 a regular point in $\mathrm{Img}\,\gamma$. Take a preimage $t_0 \in \gamma^{-1}(x_0)$ and $s \in (0,1)$. Then*

$$r_0 := \sup\{r \geq 0 : \gamma \textit{ crosses the cone } X(\gamma(t_0), r, \operatorname{span}\gamma'(t_0), s) \textit{ properly at } t_0\} \tag{2.6}$$

is nonzero and for all $r \in (0, r_0)$, γ crosses $X(\gamma(t_0), r, \operatorname{span}\gamma'(t_0), s)$ properly at t_0.

Proof. Since γ is differentiable at t_0, as $\delta \to 0$ we have:

$$d(\gamma\,(t_0+\delta) - x_0, \operatorname{span}\gamma'(t_0)) \leq |\gamma(t_0+\delta) - x_0 - \delta\gamma'(t_0)| = o(\delta),$$

Hence for every $s \in (0,1)$ there exists $\delta_0 > 0$ such that

$$d(\gamma\,(t_0+\delta) - x_0, \operatorname{span}\gamma'(t_0)) \leq s|\gamma(t_0+\delta) - x_0|,$$

whenever $|\delta| \leq \delta_0$. Moreover, there exists $0 < \delta_1 \leq \delta_0$ such that for any $\delta \in [0, \delta_1]$, we have $\pm(\gamma(t_0 \pm \delta) - x_0) \cdot \gamma'(t_0) \geq 0$. Hence, we obtain

$$\gamma([t_0 - \delta_1, t_0]) \subseteq X_-(x_0, \infty, \gamma'(t), s) \text{ and } \gamma([t_0, t_0 + \delta_1]) \subseteq X_+(x_0, \infty, \gamma'(t), s). \tag{2.7}$$

Denote $r_0 = \min\{|\gamma(t_0 - \delta_1) - x_0|, |\gamma(t_0 + \delta_1) - x_0|\}$. By continuity of γ it follows

$$\gamma([t_0 - \delta_1, t_0]) \cap \partial B_{r_0}(x_0) \neq \emptyset \text{ and } \gamma([t_0, t_0 + \delta_1]) \cap \partial B_{r_0}(x_0) \neq \emptyset. \tag{2.8}$$

From (2.7) and (2.8), we obtain that γ crosses the cone $X(x_0, r_0, \operatorname{span}\gamma'(t_0), s)$ properly. It is clear that the same holds taking $r \leq r_0$. $\square$

2.4 Slicing traffic plans

In this section we introduce a new tool, which is the Lagrangian counterpart to the slicing of currents. We refer to [Sim83] for a complete presentation of the latter. We begin by defining a localized version of the α-energy. For any $\alpha \in [0,1]$ and any Borel set $E \subseteq \mathbb{R}^d$, we set:

$$\mathcal{E}^\alpha(\mathbf{P}, E) := \int_{\mathrm{Lip}_1} \int_{\mathbb{R}_+} \theta_{\mathbf{P}}^{\alpha-1}(\gamma(t)) \mathbf{1}_{\gamma(t)\in E} |\gamma'(t)| \,\mathrm{d}t \,\mathrm{d}\mathbf{P}(\gamma).$$

By the Area Formula and Fubini's Theorem, if $\mathbf{P}$ is rectifiable then it can be expressed as:

$$\mathcal{E}^\alpha(\mathbf{P}, E) = \int_E \theta_{\mathbf{P}}^{\alpha-1} \Theta_{\mathbf{P}} \,\mathrm{d}\mathscr{H}^1.$$

Proposition 2.7. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz map, let $\mathbf{P}$ be a traffic plan, and let $a < b$ be real numbers. Then:*

$$\int_a^b \int_{\mathrm{Lip}_1} \mathscr{H}^0((f \circ \gamma)^{-1}(\ell))\,\mathrm{d}\mathbf{P}(\gamma)\,\mathrm{d}\ell \leq \mathrm{Lip}(f)\,\mathcal{E}^1(\mathbf{P}, f^{-1}([a,b])). \tag{2.9}$$

Proof. Let us denote $g(\gamma, \ell) := \mathscr{H}^0((f \circ \gamma)^{-1}(\ell))$ for every $(\gamma, \ell) \in \mathrm{Lip}_1 \times [a, b]$. By Fubini's theorem and the Area Formula we compute

$$\begin{aligned}
\int_a^b \int_{\mathrm{Lip}_1} g(\gamma,\ell)\,\mathrm{d}\mathbf{P}(\gamma)\,\mathrm{d}\ell &= \int_{\mathrm{Lip}_1} \int_a^b g(\gamma,\ell)\,\mathrm{d}\ell\,\mathrm{d}\mathbf{P}(\gamma) \\
&= \int_{\mathrm{Lip}_1} \int_{(f\circ\gamma)^{-1}([a,b])} |(f \circ \gamma)'(t)|\,\mathrm{d}t\,\mathrm{d}\mathbf{P}(\gamma) \\
&\leq \mathrm{Lip}(f) \int_{\mathrm{Lip}_1} \int_{\mathbb{R}_+} \mathbf{1}_{\gamma(t)\in f^{-1}([a,b])} |\gamma'(t)|\,\mathrm{d}t\,\mathrm{d}\mathbf{P}(\gamma) \\
&= \mathrm{Lip}(f)\,\mathcal{E}^1(\mathbf{P}, f^{-1}([a,b])).
\end{aligned}$$

□

Proposition 2.7 allows to give the following definition:

Definition 2.8 (Slice and intensity of a slice)**.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz map, and let $\mathbf{P}$ be a traffic plan. By Proposition 2.7 the integrand in the left hand side of (2.9) is finite for a.e. $\ell \in \mathbb{R}$. For such values of ℓ we denote by $\langle\!\langle \mathbf{P}, f, \ell \rangle\!\rangle$ the finite positive measure

$$\langle\!\langle \mathbf{P}, f, \ell \rangle\!\rangle := \int_{\mathrm{Lip}_1} \sum_{t \in (f\circ\gamma)^{-1}(\ell)} \delta_{\gamma(t)}\,\mathrm{d}\mathbf{P}(\gamma),$$

where the integration is in the sense of (2.2), and we call such measure the *slice intensity* of $\mathbf{P}$ with respect to f at level ℓ. When the slice intensity is defined, we denote by $\langle \mathbf{P}, f, \ell \rangle$ the *slice* of $\mathbf{P}$ with respect to f at level ℓ, namely the real-valued measure

$$\langle \mathbf{P}, f, \ell \rangle := \int_{\mathrm{Lip}_1} \sum_{t \in (f\circ\gamma)^{-1}(\ell)} \mathrm{sign}\left((f \circ \gamma)'(t)\right) \delta_{\gamma(t)}\,\mathrm{d}\mathbf{P}(\gamma).$$

Proposition 2.9. *Let $\mathbf{P}$ be a rectifiable traffic plan, and let $a < b$ be real numbers. Then:*

$$\int_a^b \mathbb{M}^\alpha(\langle\!\langle \mathbf{P}, f, \ell \rangle\!\rangle)\,\mathrm{d}\ell \leq \mathrm{Lip}(f)\,\mathcal{E}^\alpha(\mathbf{P}, f^{-1}([a,b])).$$

Proof. The network $\Sigma_{\mathbf{P}}$ is a 1-rectifiable set. So we know (see [Sim83, Chapter 3, Remark 2.10]) that for almost every ℓ, $\Sigma_{\mathbf{P}} \cap \{f = \ell\}$ is a 0-rectifiable set, that is to say it is at most countable. In addition, it is true that for almost every ℓ:

$$\text{for } \mathbf{P}\text{-almost every } \gamma \in \mathrm{Lip}_1,\ \mathscr{H}^0(\mathrm{Img}\,\gamma \cap \{f = \ell\} \setminus \Sigma_{\mathbf{P}}) = 0, \tag{2.10}$$

i.e. $\operatorname{Img}\gamma \cap \{f=\ell\} \subseteq \Sigma_{\mathbf{P}}$. Indeed, by Fubini's Theorem and the Area Formula:

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathrm{Lip}_1} \mathscr{H}^0(\operatorname{Img}\gamma \cap \{f=\ell\} \setminus \Sigma_{\mathbf{P}})\,\mathrm{d}\mathbf{P}(\gamma)\,\mathrm{d}\ell \\
&\qquad = \int_{\mathrm{Lip}_1}\int_{\mathbb{R}}\int_{(f\circ\gamma)^{-1}(\ell)} \mathbf{1}_{\gamma(t)\notin\Sigma_{\mathbf{P}}}\,\mathrm{d}\mathscr{H}^0(t)\,\mathrm{d}\ell\,\mathrm{d}\mathbf{P}(\gamma) \\
&\qquad \le \int_{\mathrm{Lip}_1}\int_{\mathbb{R}_+} |(f\circ\gamma)'(t)|\mathbf{1}_{\gamma(t)\notin\Sigma_{\mathbf{P}}}\,\mathrm{d}t\,\mathrm{d}\mathbf{P}(\gamma) \\
&\qquad \le \mathrm{Lip}(f)\int_{\mathrm{Lip}_1}\int_{\operatorname{Img}\gamma\setminus\Sigma_{\mathbf{P}}} \mathscr{H}^0\left(\gamma^{-1}(x)\right)\mathrm{d}\mathscr{H}^1(x)\,\mathrm{d}\mathbf{P}(\gamma),
\end{aligned}$$

which equals 0 since $\mathbf{P}$ is assumed to be rectifiable.

Now let $\varphi \in C_c^0(\mathbb{R}^d)$ be nonnegative. Let ℓ be such that (2.10) is true. Then

$$\begin{aligned}
\langle \langle\!\langle \mathbf{P}, f, \ell\rangle\!\rangle, \varphi\rangle &= \int_{\mathrm{Lip}_1}\sum_{t\in(f\circ\gamma)^{-1}(\ell)} \varphi(\gamma(t))\,\mathrm{d}\mathbf{P}(\gamma) \\
&= \int_{\mathrm{Lip}_1}\sum_{x\in f^{-1}(\ell)\cap\Sigma_{\mathbf{P}}} \mathscr{H}^0\left(\gamma^{-1}(x)\right)\varphi(x)\,\mathrm{d}\mathbf{P}(\gamma) \\
&= \sum_{x\in f^{-1}(\ell)\cap\Sigma_{\mathbf{P}}} \varphi(x)\int_{\mathrm{Lip}_1} \mathscr{H}^0\left(\gamma^{-1}(x)\right)\mathrm{d}\mathbf{P}(\gamma),
\end{aligned}$$

using Fubini's Theorem for the last equality. Thus

$$\langle\!\langle \mathbf{P}, f, \ell\rangle\!\rangle = \Theta_{\mathbf{P}}\mathscr{H}^0 \llcorner (f^{-1}(\ell)\cap\Sigma_{\mathbf{P}}),$$

and $\mathbb{M}^\alpha(\langle\!\langle \mathbf{P}, f, \ell\rangle\!\rangle) = \sum_{x\in f^{-1}(\ell)\cap\Sigma_{\mathbf{P}}} \Theta_{\mathbf{P}}(x)^\alpha$. Finally, by Fubini's Theorem and the Area Formula again, we compute

$$\begin{aligned}
\int_a^b \mathbb{M}^\alpha(\langle\!\langle \mathbf{P}, f, \ell\rangle\!\rangle)\,\mathrm{d}\ell &= \int_a^b \sum_{x\in f^{-1}(\ell)\cap\Sigma_{\mathbf{P}}} \Theta_{\mathbf{P}}(x)^{\alpha-1}\Theta_{\mathbf{P}}(x)\,\mathrm{d}\ell \\
&= \int_a^b \sum_{x\in f^{-1}(\ell)\cap\Sigma_{\mathbf{P}}} \Theta_{\mathbf{P}}(x)^{\alpha-1}\int_{\mathrm{Lip}_1} \mathscr{H}^0\left(\gamma^{-1}(x)\right)\mathrm{d}\mathbf{P}(\gamma)\,\mathrm{d}\ell \\
&= \int_{\mathrm{Lip}_1}\int_a^b \sum_{t\in(f\circ\gamma)^{-1}(\ell)} \mathbf{1}_{\gamma(t)\in\Sigma_{\mathbf{P}}}\Theta_{\mathbf{P}}\left(\gamma(t)\right)^{\alpha-1}\mathrm{d}\ell\,\mathrm{d}\mathbf{P}(\gamma) \\
&= \int_{\mathrm{Lip}_1}\int_{\mathbb{R}_+} \mathbf{1}_{\gamma(t)\in\Sigma_{\mathbf{P}},(f\circ\gamma)(t)\in[a,b]}\Theta_{\mathbf{P}}\left(\gamma(t)\right)^{\alpha-1}|(f\circ\gamma)'(t)|\,\mathrm{d}t\,\mathrm{d}\mathbf{P}(\gamma) \\
&\le \mathrm{Lip}(f)\,\mathcal{E}^\alpha(\mathbf{P}, f^{-1}([a,b])). \qquad \square
\end{aligned}$$

3 From cancellations to cycles

Take $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ and $\mathbf{P}$ satisfying the hypotheses of Theorem 2.1 and set $T = T_{\mathbf{P}}$. At first glance, $\mathbf{P}$ could fail to be a good decomposition of T: in general a limit of good decompositions is not a good decomposition. In order to show that $\mathbf{P}$ is in fact a good decomposition of T, we are going to prove that, as a limit of optimal traffic plans, it cannot produce cancellations at the Eulerian level. In the following we will always assume that $\mathbf{P}_n$ and $\mathbf{P}$ are rectifiable traffic plans, which is not restrictive in view of Theorem 4.10 of [BCM08].

3.1 Cancellations

Cancellations in $\mathbf{P}$ mean "pieces of trajectories" that disappear in the induced current, due to positive amounts of curves going in opposite directions. Let us be more precise.

In general the density of the induced current $\vec{\theta}_{\mathbf{P}}$ is less or equal than the full multiplicity $\Theta_{\mathbf{P}}$, $\mathscr{H}^1$-a.e. on $\Sigma_{\mathbf{P}}$. Indeed, take a curve $\gamma \in \mathrm{Lip}_1$ such that $\mathscr{H}^1(\mathrm{Img}\,\gamma \setminus \Sigma_{\mathbf{P}}) = 0$, and a regular point $x \in \mathrm{Img}\,\gamma$ where $\mathrm{Tan}(x, \Sigma_{\mathbf{P}})$ exists. Note that by the triangle inequality:

$$|\vec{m}_\gamma(x)| \leq \#\gamma^{-1}(x), \tag{3.1}$$

hence taking a regular point x of $\mathbf{P}$, one has:

$$|\vec{\theta}_{\mathbf{P}}(x)| = \left| \int_{\mathrm{Lip}_1} \vec{m}_\gamma(x)\,\mathrm{d}\mathbf{P}(\gamma) \right| \leq \int_{\mathrm{Lip}_1} |\vec{m}_\gamma(x)|\,\mathrm{d}\mathbf{P}(\gamma) \tag{3.2}$$

$$\leq \int_{\mathrm{Lip}_1} \#\gamma^{-1}(x)\,\mathrm{d}\mathbf{P}(\gamma) = \Theta_{\mathbf{P}}(x). \tag{3.3}$$

Remark 3.1. These inequalities and their equality case are closely related to the notion of good decomposition. Indeed, notice that using Fubini's Theorem, (3.2) is an equality $\mathscr{H}^1$-a.e. if and only if

$$\mathbb{M}(T) = \int_{\mathrm{Lip}_1} \int_{\Sigma_{\mathbf{P}}} |\vec{m}_\gamma(x)|\,\mathrm{d}\mathscr{H}^1(x)\,\mathrm{d}\mathbf{P}(\gamma) = \int_{\mathrm{Lip}_1} \mathbb{M}(I_\gamma)\,\mathrm{d}\mathbf{P}(\gamma),$$

that is if (B) of Definition 2.2 holds. Moreover (A) implies equality $\mathscr{H}^1$-a.e. in (3.3) as well as $\theta_{\mathbf{P}}(x) = \Theta_{\mathbf{P}}(x)$.

We say that $\mathbf{P}$ has cancellations if we have a strict inequality:

$$|\vec{\theta}_{\mathbf{P}}(x)| < \Theta_{\mathbf{P}}(x), \tag{3.4}$$

on a subset of $\Sigma_{\mathbf{P}}$ of positive $\mathscr{H}^1$-measure. Notice that (3.4) happens if either inequality (3.2) or (3.3) is strict. Inequality (3.3) is strict if there exists a positive amount of curves γ such that equality (3.1) is strict, and since $\gamma'(t)$ belongs to $\mathrm{Tan}(x, \Sigma_{\mathbf{P}})$ for all $t \in \gamma^{-1}(x)$ by Remark 2.4, it means that γ crosses x at least twice in opposite directions: heuristically these cancellations are due to those particles flowing through the same point at least twice, with different orientations. Inequality (3.2) is strict if there are two sets of curves $A_x, B_x \in \mathrm{Lip}_1$ of positive measure such that the resultant tangent $\vec{m}_\gamma(x)$ belongs to $\mathrm{Tan}(x, \Sigma_{\mathbf{P}})$ with a certain orientation on A_x and with the opposite orientation on B_x:

heuristically these cancellations are due to the interactions between different particles flowing in opposite directions.

To capture both situations at the same time, we choose once and for all a (Borel measurable) orientation $\tau_{\Sigma_{\mathbf{P}}}(x)$ for $\mathrm{Tan}(x, \Sigma_{\mathbf{P}})$, and introduce for every $x \in \mathbb{R}^d$ the sets:

$$\Gamma^{\pm}(x) := \{\gamma \in \mathrm{Lip}_1 : \exists t \in (0, T(\gamma)) \text{ s.t. } \gamma'(t)/|\gamma'(t)| = \pm\tau_{\Sigma_{\mathbf{P}}}(x)\}, \tag{3.5}$$

as well as the corresponding multiplicities

$$\theta^{\pm}_{\mathbf{P}}(x) := \mathbf{P}\left(\Gamma^{\pm}(x)\right), \tag{3.6}$$

$$\bar{\theta}_{\mathbf{P}}(x) := \min\{\theta^{+}_{\mathbf{P}}(x), \theta^{-}_{\mathbf{P}}(x)\}. \tag{3.7}$$

We may characterize cancellations at x as stated in the following lemma:

Lemma 3.2. *Take a regular point x of $\mathbf{P}$. The following assertions are equivalent:*

1. $|\vec{\theta}_{\mathbf{P}}(x)| = \Theta_{\mathbf{P}}(x)$,
2. $\bar{\theta}_{\mathbf{P}}(x) = 0$,
3. *there exists $s \in \{-1, +1\}$ such that for $\mathbf{P}$-a.e. curve and all $t \in \gamma^{-1}(x)$,*

$$\gamma'(t) = s|\gamma'(t)|\tau_{\Sigma_{\mathbf{P}}}(x).$$

3.2 Existence of Lagrangian cycles

From the perspective of reasoning by contradiction, the goal of this section is to study properties of traffic plans producing cancellations. The theorem below guarantees the existence of “Lagrangian cycles” in $\mathbf{P}$, that is to say two families of curves with positive measures passing through two distinct points x and y in opposite order, namely $\mathbf{P}(\Gamma(x, y)), \mathbf{P}(\Gamma(y, x)) > 0$, where for any $u, v \in \mathbb{R}^d$

$$\Gamma(u, v) := \{\gamma \in \mathrm{Lip}_1 : \exists s \leq t : \gamma(s) = u, \gamma(t) = v\}.$$

These cycles are obviously obstacles to $\mathbf{P}$ being optimal, but at this point it is not yet a contradiction, as the optimality of $\mathbf{P}$ is precisely what we want to prove.

Theorem 3.3 (Existence of Lagrangian cycles)**.** *Let $\mathbf{P}$ be a traffic plan with finite energy, and assume $\mathscr{H}^1(\{\bar{\theta}_{\mathbf{P}} > 0\}) > 0$. Then there exists $F \subseteq \{\bar{\theta}_{\mathbf{P}} > 0\}$ with positive $\mathscr{H}^1$-measure such that for every $x_0 \in F$, there exists $G \subseteq F$ with positive $\mathscr{H}^1$-measure satisfying:*

$$\forall x \in G, \quad \min\{\mathbf{P}(\Gamma(x_0, x)), \mathbf{P}(\Gamma(x, x_0))\} \geq \bar{\theta}_{\mathbf{P}}(x_0)/4.$$

Before proving this theorem, we need the following lemma, which describes the geometric situation on small balls $B_r(x_0)$ around every point x_0 in a suitable subset F of $\{\bar{\theta}_{\mathbf{P}} > 0\}$: “most” of $\Sigma_{\mathbf{P}}$ lies inside F, itself contained in a cone which is crossed properly, and in both directions, by a fixed amount of curves.

Lemma 3.4. *Under the assumptions of Theorem 3.3, there exists $F \subseteq \{\bar{\theta}_{\mathbf{P}} > 0\}$ with positive $\mathscr{H}^1$-measure such that $\mathscr{H}^1$-almost every $x_0 \in F$ satisfies:*

(i) $\mathscr{H}^1(F \cap B_r(x_0)) \sim 2r$ *as* $r \to 0$*;*

(ii) $\displaystyle\int_{(\Sigma_{\mathbf{P}} \setminus F)\cap B_r(x_0)} \theta_{\mathbf{P}} \,\mathrm{d}\mathscr{H}^1 = o(r)$*;*

(iii) for all $s \in (0,1)$*, there exists* $r_1 > 0$ *such that* $F \cap B_{r_1}(x_0)$ *is contained in the cone* $X(x_0, r_1, \operatorname{span} \tau_{\Sigma_{\mathbf{P}}}(x_0), s)$*;*

(iv) for all $s \in (0,1)$*, there exists* $r_0 > 0$ *and two (not necessarily disjoint) Borel sets of curves* $\Lambda^{\pm}_{r_0} \subseteq \Gamma^{\pm}(x_0)$*, as well as Borel maps* $t^{\pm}_0 : \Lambda^{\pm}_{r_0} \to \mathbb{R}_+$ *such that:*

- $\min\{\mathbf{P}(\Lambda^+_{r_0}), \mathbf{P}(\Lambda^-_{r_0})\} \geq \bar{\theta}_{\mathbf{P}}(x_0)/2$*;*
- *every* $\gamma \in \Lambda^{\pm}_{r_0}$ *crosses the cone* $X(x_0, r_0, \operatorname{span} \tau_{\Sigma_{\mathbf{P}}}(x_0), s)$ *properly at time* $t^{\pm}_0(\gamma)$*;*
- *for any* $0 < r \leq r_0$ *there exist Borel maps* $t^{\pm}_{r,in}$*,* $t^{\pm}_{r,out}$ *from* $\Lambda^{\pm}_{r_0}$ *to* $\mathbb{R}_+$ *which are entrance and exit times in the cone* $X(x_0, r, \operatorname{span} \tau_{\Sigma_{\mathbf{P}}}(x_0), s)$ *for every curve in* $\Lambda^{\pm}_{r_0}$*.*

Proof. Since $\Sigma_{\mathbf{P}}$ is 1-rectifiable, it is contained in the union of an $\mathscr{H}^1$-negligible set and of countably many images of Lipschitz curves of finite length. Thus, there exists a Lipschitz curve $\bar{\gamma}$ such that $\mathscr{H}^1(\{\bar{\theta}_{\mathbf{P}} > 0\} \cap \operatorname{Img} \bar{\gamma}) > 0$. We set $F := \{\bar{\theta}_{\mathbf{P}} > 0\} \cap \operatorname{Img} \bar{\gamma}$. We want to show each item holds for $\mathscr{H}^1$-almost every $x_0 \in F$.

Proof of (i). We know that $\operatorname{Img} \bar{\gamma}$ is 1-rectifiable and $\mathscr{H}^1(\operatorname{Img} \bar{\gamma}) < \infty$. Then [Mat95, Theorem 17.6] implies that $\mathscr{H}^1(\operatorname{Img} \bar{\gamma} \cap B_r(x_0)) \sim 2r$ as $r \to 0$ for $\mathscr{H}^1$-almost every $x_0 \in \operatorname{Img} \bar{\gamma}$. Moreover, almost every $x_0 \in F \subseteq \operatorname{Img} \bar{\gamma}$ is a density point of the function $\mathbf{1}_F$ with respect to the Radon measure $\mathscr{H}^1 \llcorner \operatorname{Img} \bar{\gamma}$ (see [EG15, Theorem 1.32]) i.e. $\mathscr{H}^1(F \cap B_r(x_0)) \sim \mathscr{H}^1(\operatorname{Img} \bar{\gamma} \cap B_r(x_0))$, hence the result.

Proof of (ii). By the same argument, almost every $x_0 \in F \subseteq \Sigma_{\mathbf{P}}$ is a density point of the function $\mathbf{1}_F$ with respect to the Radon measure $\theta_{\mathbf{P}} \mathscr{H}^1 \llcorner \Sigma_{\mathbf{P}}$ so that

$$\int_{(\Sigma_{\mathbf{P}} \setminus F)\cap B_r(x_0)} \theta_{\mathbf{P}} \,\mathrm{d}\mathscr{H}^1 = o\left(\int_{\Sigma_{\mathbf{P}} \cap B_r(x_0)} \theta_{\mathbf{P}} \,\mathrm{d}\mathscr{H}^1\right). \tag{3.8}$$

Yet a subset of $F \subseteq \Sigma_{\mathbf{P}}$ which is negligible for $\theta_{\mathbf{P}} \mathscr{H}^1 \llcorner \Sigma_{\mathbf{P}}$ is also $\mathscr{H}^1$-negligible, so it is still true for $\mathscr{H}^1$-almost all $x_0 \in F$.

In addition, for $\mathscr{H}^1$-almost every $x_0 \in \Sigma_{\mathbf{P}}$, there exist $c = c(x_0) > 0$ and $\rho = \rho(x_0) > 0$ such that

$$\forall r \leq \rho, \quad \int_{\Sigma_{\mathbf{P}} \cap B_r(x_0)} \theta_{\mathbf{P}} \,\mathrm{d}\mathscr{H}^1 \leq cr. \tag{3.9}$$

Indeed, let us show by contraposition that the set

$$A := \left\{ x \in \Sigma_{\mathbf{P}} : \limsup_{r \to 0} \frac{1}{r} \int_{\Sigma_{\mathbf{P}} \cap B_r(x_0)} \theta_{\mathbf{P}} \,\mathrm{d}\mathscr{H}^1 = +\infty \right\}$$

is $\mathscr{H}^1$-negligible. Let $c, \varepsilon > 0$. For every $x \in A$, there exists $r(x) \in (0, \varepsilon]$ satisfying

$$\frac{1}{r(x)} \int_{\Sigma_{\mathbf{P}} \cap \bar{B}_{r(x)}(x)} \theta_{\mathbf{P}} \, \mathrm{d}\mathscr{H}^1 \geq c. \tag{3.10}$$

The family $\{\bar{B}_{r(x)}(x)\}_{x \in A}$ is a covering of A and for every $x \in A$, $r(x) \leq \varepsilon$. Then by Vitali's Covering Theorem ([Mat95, Theorem 2.1]), one may extract a (finite or countable) sequence $\{B_i\}_{i \in I} \subseteq \{\bar{B}_{r(x)}(x) : x \in A\}$ of disjoint closed balls such that $A \subseteq \bigcup_{i \in I} \hat{B}_i$, where $\hat{B}_i$ is the concentric ball to B_i with radius 5 times the radius of B_i. Therefore we get the following inequalities:

$$\mathscr{H}^1_{5\varepsilon}(A) \leq \sum_{i \in I} \operatorname{diam} \hat{B}_i \leq 5 \sum_{i \in I} \operatorname{diam} B_i \overset{(3.10)}{\leq} \frac{10}{c} \sum_{i \in I} \int_{\Sigma_{\mathbf{P}} \cap B_i} \theta_{\mathbf{P}} \, \mathrm{d}\mathscr{H}^1 \leq \frac{10}{c} \int_{\Sigma_{\mathbf{P}}} \theta_{\mathbf{P}} \, \mathrm{d}\mathscr{H}^1$$

where the last inequality is due to the fact that the balls B_i are disjoint. Since

$$\begin{aligned} \int_{\Sigma_{\mathbf{P}}} \theta_{\mathbf{P}} \, \mathrm{d}\mathscr{H}^1 &= \int_{\Sigma_{\mathbf{P}}} \left(\frac{\theta_{\mathbf{P}}}{\mathbb{M}(\mathbf{P})} \right) \mathbb{M}(\mathbf{P}) \, \mathrm{d}\mathscr{H}^1 \\ &\leq \int_{\Sigma_{\mathbf{P}}} \left(\frac{\theta_{\mathbf{P}}}{\mathbb{M}(\mathbf{P})} \right)^{\alpha} \mathbb{M}(\mathbf{P}) \, \mathrm{d}\mathscr{H}^1 = \int_{\Sigma_{\mathbf{P}}} \theta_{\mathbf{P}}^{\alpha} \mathbb{M}(\mathbf{P})^{1-\alpha} \, \mathrm{d}\mathscr{H}^1 < \infty \end{aligned}$$

and c is arbitrary, we find that for all $\varepsilon > 0$, $\mathscr{H}^1_{5\varepsilon}(A) = 0$, which yields $\mathscr{H}^1(A) = 0$.

Therefore, (3.8) and (3.9) hold for $\mathscr{H}^1$-almost every point in F, hence the result.

Proof of (iii). Recall that there is a Lipschitz curve $\bar{\gamma} \subseteq \Sigma_{\mathbf{P}}$ whose image contains F. By Remark 2.4, at $\mathscr{H}^1$-almost every point $x \in \operatorname{Img} \bar{\gamma}$, $\bar{\gamma}^{-1}(x)$ is finite and for all $t \in \bar{\gamma}^{-1}(x)$, $\operatorname{span} \bar{\gamma}'(t) = \operatorname{span} \tau_{\Sigma_{\mathbf{P}}}(x)$. Let $x_0 \in \operatorname{Img} \bar{\gamma}$ be such a point. For every $t \in \bar{\gamma}^{-1}(x_0)$, we apply Lemma 2.6 to get a radius $r_t > 0$ such that $\bar{\gamma}$ lies in the cone $X(x_0, r_t, \operatorname{span} \tau_{\Sigma_{\mathbf{P}}}(x_0), s)$ in a small interval $I_t =]t - \delta_t, t + \delta_t[$. Set $r = \min\{r_t > 0 : \bar{\gamma}(t) = x_0\} > 0$, then taking $r_1 \leq r$ small enough to make sure that $B_{r_1}(x_0) \cap \gamma(\mathbb{R}_+ \setminus \bigcup_t I_t) = \emptyset$ leads to the desired conclusion.

Proof of (iv). Let $x_0 \in F$ be a regular point for $\mathbf{P}$ and set $V := \operatorname{span} \tau_{\Sigma_{\mathbf{P}}}(x_0)$. The function

$$\begin{aligned} t_0^{\pm} : \Gamma^{\pm}(x_0) &\to \mathbb{R}_+ \\ \gamma &\mapsto \inf\{t \in \mathbb{R}_+ : \gamma(t) = x_0, \gamma'(t)/|\gamma'(t)| = \pm\tau_{\Sigma}(x_0)\} \end{aligned}$$

is well-defined and $t_0^{\pm}(\gamma) \in (0, T_\infty(\gamma))$ for $\mathbf{P}$-a.e. $\gamma \in \Gamma^{\pm}(x_0)$, as x_0 is a regular point for $\mathbf{P}$. Then fix $s \in (0,1)$. We denote by $r_0^{\pm}(\gamma)$ the (positive) radius given by (2.6) in Lemma 2.6 and we set $\Lambda_r^{\pm} := \{\gamma \in \Gamma^{\pm}(x_0) : r_0^{\pm}(\gamma) > r\}$ for any $r > 0$. Since $r_0^{\pm}(\gamma) > 0$ for every $\gamma \in \Gamma^{\pm}(x_0)$ and $\{\Lambda_{1/n}^{\pm}\}_{n \in \mathbb{N}^*}$ are a nested family of set such that $\bigcup_{n \in \mathbb{N}^*} \Lambda_{1/n}^{\pm} = \Gamma^{\pm}(x_0)$, then there exists $n \in \mathbb{N}$ such that $\mathbf{P}(\Lambda_{1/n}^{+}) \geq \mathbf{P}(\Gamma^{+}(x_0))/2 \geq \bar{\theta}_{\mathbf{P}}(x_0)/2$ and $\mathbf{P}(\Lambda_{1/n}^{-}) \geq \mathbf{P}(\Gamma^{-}(x_0))/2 \geq \bar{\theta}_{\mathbf{P}}(x_0)/2$. Therefore, setting $r_0 := 1/n$, every $\gamma \in \Lambda_{r_0}^{\pm}$ crosses the cone $X(x_0, r_0, V, s)$ properly at time $t_0^{\pm}(\gamma)$. This is also true for all the homothetic cones with radius $0 < r \leq r_0$ (see Lemma 2.6). Thus for every $\gamma \in \Lambda^{\pm}_{r_0}$ we define the entrance and exit times as:

$$\begin{aligned} t^{\pm}_{r,in}(\gamma) : \Lambda^{\pm}_{r_0} &\to \mathbb{R}_+ \\ \gamma &\mapsto \sup\{t \leq t^{\pm}_0(\gamma) : \gamma(t) \notin B_r(x_0)\}, \\ t^{\pm}_{r,out}(\gamma) : \Lambda^{\pm}_{r_0} &\to \mathbb{R}_+ \\ \gamma &\mapsto \inf\{t \geq t^{\pm}_0(\gamma) : \gamma(t) \notin B_r(x_0)\}. \end{aligned}$$

$\square$

We can now go back to the existence of Lagrangian cycles in $\mathbf{P}$:

Proof of Theorem 3.3. Let $F \subseteq \{\bar{\theta}_{\mathbf{P}} > 0\}$ given by Lemma 3.4 and let $x_0 \in F$ satisfying (i) to (iv). We fix $s \in (0,1)$ (for example $s = 1/2$) and we take $r_0 > 0$, $\Lambda^{\pm}_{r_0}$ and $t^{\pm}_{r,in}$, $t^{\pm}_{r,out}$ as in (iii) and (iv). For any $0 < r \leq r_0$, we define $\mathbf{Q}^{\pm}_r := (g_r)_\sharp(\mathbf{P} \llcorner \Lambda^{\pm}_{r_0})$ where

$$\begin{aligned} g_r : \Lambda^{\pm}_{r_0} &\to \mathrm{Lip}_1 \\ \gamma &\mapsto \gamma_{|[t^{\pm}_{r,in}(\gamma), t^{\pm}_{r,out}(\gamma)]}. \end{aligned}$$

This is obviously a traffic plan. Let us estimate its multiplicity at $x \in \mathbb{R}^d$:

$$\theta_{\mathbf{Q}^{\pm}_r}(x) = \int_{\Lambda^{\pm}_{r_0}} \mathbf{1}_{x \in \mathrm{Img}\, g_r(\gamma)} \,\mathrm{d}\mathbf{P}(\gamma) \leq \int_{\mathrm{Lip}_1} \mathbf{1}_{x \in \mathrm{Img}\,\gamma} \,\mathrm{d}\mathbf{P}(\gamma) = \theta_{\mathbf{P}}(x),$$

so that

$$\int_{(\Sigma_{\mathbf{P}} \setminus F) \cap B_r(x_0)} \theta_{\mathbf{Q}^{\pm}_r} \,\mathrm{d}\mathscr{H}^1 \leq \int_{(\Sigma_{\mathbf{P}} \setminus F) \cap B_r(x_0)} \theta_{\mathbf{P}} \,\mathrm{d}\mathscr{H}^1. \tag{3.11}$$

Furthermore, Fubini's Theorem yields:

$$\int_{\Sigma_{\mathbf{P}} \cap B_r(x_0)} \theta_{\mathbf{Q}^{\pm}_r} \,\mathrm{d}\mathscr{H}^1 = \int_{\Lambda^{\pm}_{r_0}} \mathscr{H}^1(\mathrm{Img}\, g_r(\gamma)) \,\mathrm{d}\mathbf{P}(\gamma) \geq 2r\mathbf{P}(\Lambda^{\pm}_{r_0}), \tag{3.12}$$

where the inequality comes from the fact that every curve in $\Lambda^{\pm}_{r_0}$ crosses the cone $X(x_0, r, V, s)$ properly, hence their length between the entrance and exit times is at least $2r$. Recalling (i) and (ii) of Lemma 3.4, as well as the fact that $\theta_{\mathbf{Q}^{\pm}_r} \leq \mathbf{P}(\Lambda^{\pm}_{r_0})$, (3.12) and (3.11) lead to

$$\begin{aligned} 0 &\leq \int_{F \cap B_r(x_0)} \left(\mathbf{P}(\Lambda^{\pm}_{r_0}) - \theta_{\mathbf{Q}^{\pm}_r}\right) \mathrm{d}\mathscr{H}^1 \\ &= \mathscr{H}^1(F \cap B_r(x_0))\mathbf{P}(\Lambda^{\pm}_{r_0}) - \int_{\Sigma_{\mathbf{P}} \cap B_r(x_0)} \theta_{\mathbf{Q}^{\pm}_r} \,\mathrm{d}\mathscr{H}^1 + \int_{(\Sigma_{\mathbf{P}} \setminus F) \cap B_r(x_0)} \theta_{\mathbf{Q}^{\pm}_r} \,\mathrm{d}\mathscr{H}^1 \\ &\leq \left(\mathscr{H}^1(F \cap B_r(x_0)) - 2r\right) \mathbf{P}(\Lambda^{\pm}_{r_0}) + \int_{(\Sigma_{\mathbf{P}} \setminus F) \cap B_r(x_0)} \theta_{\mathbf{P}} \,\mathrm{d}\mathscr{H}^1 \\ &= o(r). \end{aligned}$$

Thus for any $c \in (0,1)$, Markov inequality yields

$$\mathscr{H}^1\left(\left\{\mathbf{P}(\Lambda^{\pm}_{r_0}) - \theta_{\mathbf{Q}^{\pm}_r} > c\mathbf{P}(\Lambda^{\pm}_{r_0})\right\} \cap F \cap B_r(x_0)\right) = o(r),$$

and therefore

$$\frac{\mathscr{H}^1\left(\left\{\theta_{\mathbf{Q}^{\pm}_r} \geq (1-c)\mathbf{P}(\Lambda^{\pm}_{r_0})\right\} \cap F \cap B_r(x_0)\right)}{\mathscr{H}^1(F \cap B_r(x_0))} \xrightarrow{r \to 0} 1.$$

Now we take $c = 1/2$ and $r > 0$ small enough so that the quotient above (for $\pm$ being $+$ and $-$) is greater than $3/4$. We set $G := \left\{\theta_{\mathbf{Q}^+_r} \geq \mathbf{P}(\Lambda^+_{r_0})/2\right\} \cap \left\{\theta_{\mathbf{Q}^-_r} \geq \mathbf{P}(\Lambda^-_{r_0})/2\right\} \cap F \cap B_r(x_0)$. As expected, we have $\mathscr{H}^1(G \cap B_r(x_0)) \geq \mathscr{H}^1(F \cap B_r(x_0))/2$. Note that by (iii), $G \subseteq X(x_0, r, \operatorname{span} \tau_{\Sigma_\mathbf{P}}(x_0), s)$. Finally, let $x \in G$ be distinct from x_0 and assume for example that $x \in X_+(x_0, r, \tau_\Sigma(x_0), s)$. Since by (iv) every curve in $\Lambda^{\pm}_{r_0}$ crosses the cone $X(x_0, r, \operatorname{span} \tau_{\Sigma_\mathbf{P}}(x_0), s)$ properly, then every curve γ in $\Lambda^+_{r_0}$ such that $x \in \operatorname{Img} g_r(\gamma)$ goes through x_0 before going through x along the piece $g_r(\gamma)$, and vice versa for the curves in $\Lambda^-_{r_0}$, which yields:

$$\frac{\mathbf{P}(\Lambda^+_{r_0})}{2} \leq \theta_{\mathbf{Q}^+_r}(x_0) = \mathbf{P}\left(\{\gamma \in \Lambda^+_{r_0} : x \in \operatorname{Img} g_r(\gamma)\}\right) \leq \mathbf{P}\left(\Gamma(x_0, x)\right),$$
$$\frac{\mathbf{P}(\Lambda^-_{r_0})}{2} \leq \theta_{\mathbf{Q}^-_r}(x_0) = \mathbf{P}\left(\{\gamma \in \Lambda^-_{r_0} : x \in \operatorname{Img} g_r(\gamma)\}\right) \leq \mathbf{P}\left(\Gamma(x, x_0)\right).$$

Analogously, if $x \in X_-(x_0, r, \tau_\Sigma(x_0), s)$, then $\mathbf{P}(\Lambda^+_{r_0})/2 \leq \mathbf{P}(\Gamma(x, x_0))$ and $\mathbf{P}(\Lambda^-_{r_0})/2 \leq \mathbf{P}(\Gamma(x_0, x))$. We conclude recalling $\mathbf{P}(\Lambda^{\pm}_{r_0}) \geq \bar{\theta}_\mathbf{P}(x_0)/2$ from (iv). $\square$

4 Cycles and quasi-cycles in traffic plans

Consider a sequence of traffic plans $\{\mathbf{P}_n\}_{n \in \mathbb{N}}$ with bounded energy which converges to a traffic plan $\mathbf{P}$. Assume there exists $(x, y) \in X \times X$ such that $\mathbf{P}(\Gamma(x, y)) > 0$ and $\mathbf{P}(\Gamma(y, x)) > 0$. We show existence of *quasi-cycles* in the $\mathbf{P}_n$'s, namely we prove the following. Denote for any $u, v \in \mathbb{R}^d$ and $\varepsilon > 0$

$$\Gamma_\varepsilon(u, v) := \{\gamma \in \operatorname{Lip}_1 : \exists s \leq t : \gamma(s) \in B_\varepsilon(u), \gamma(t) \in B_\varepsilon(v)\},$$

there exists $\delta > 0$ such that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\min\{\mathbf{P}_n(\Gamma_\varepsilon(x, y)), \mathbf{P}_n(\Gamma_\varepsilon(y, x))\} \geq \delta, \quad \forall n \geq N.$$

These points may be well-chosen to guarantee that the energy of $\mathbf{P}_n$ vanishes somewhat uniformly in n on small balls around them. Then we estimate the energy gain obtained by removing such quasi-cycles. To simplify the construction, we will build a competing transport path rather than a traffic plan, but this is not a problem by equivalence of the two frameworks (we can always build a traffic plan with a lower or equal cost).

4.1 From cycles to quasi-cycles

We start with the lemma controlling the energy on small balls: for almost every x, the energy of T_n on small balls $B_\varepsilon(x)$ becomes arbitrarily small uniformly on a subsequence, as ε goes to 0. The lemma is proven for transport paths as justified before.

Lemma 4.1. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of transport paths such that $\sup_{n\in\mathbb{N}} \mathbb{M}^\alpha(T_n) < \infty$. Then, one has for $\mathscr{H}^1$-almost every $x \in \mathbb{R}^d$:*

$$\liminf_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner B_\varepsilon(x)) \xrightarrow{\varepsilon\to 0} 0. \tag{4.1}$$

Proof. We prove (4.1) by a simple covering argument (the same as in (ii) of Lemma 3.4 but in an $\mathscr{H}^0$ fashion). Set

$$A_p := \{x \in \mathbb{R}^d : \forall \varepsilon_0 > 0, \exists \varepsilon \le \varepsilon_0 \text{ s.t. } \liminf_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner B_\varepsilon(x)) \ge 1/p\},$$

and take k distinct points x_i in this set, as well as suitable radii $r_i > 0$ so that the balls $\bar{B}_{r_i}(x_i)$ are disjoint and for every i, $\liminf_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner B_{r_i}(x_i)) \ge 1/p$. We get

$$\frac{k}{p} \le \sum_{i=1}^{k} \liminf_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner B_{r_i}(x_i)) \le \liminf_{n\to\infty} \mathbb{M}^\alpha(T_n) \le \sup_{n\in\mathbb{N}} \mathbb{M}^\alpha(T_n).$$

So k is bounded, hence $\mathscr{H}^0(A_p) < \infty$. Therefore $\bigcup_{p\in\mathbb{N}^\star} A_p$ is at most countable. $\square$

We continue with the existence of quasi-cycles.

Lemma 4.2. *Let $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ and $\mathbf{P}$ be traffic plans such that $\mathbf{P}_n \overset{\star}{\rightharpoonup} \mathbf{P}$. Assume there exist $x, y \in \mathbb{R}^d$ and $\delta > 0$ satisfying $\min\{\mathbf{P}(\Gamma(x,y)), \mathbf{P}(\Gamma(y,x))\} \ge \delta$. Then:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \quad \textit{s.t.} \quad \forall n \ge N, \quad \min\{\mathbf{P}_n(\Gamma_\varepsilon(x,y)), \mathbf{P}_n(\Gamma_\varepsilon(y,x))\} \ge \delta/2.$$

Proof. Notice that $\Gamma_\varepsilon(x,y)$ is an open subset of Lip_1 (recall that it is endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$). Indeed, take $\gamma \in \Gamma_\varepsilon(x,y)$ and denote $s < t$ such that $\gamma(s) \in B_\varepsilon(x)$ and $\gamma(t) \in B_\varepsilon(x)$. Then any curve $\tilde{\gamma}$ such that $\|\tilde{\gamma} - \gamma\|_{\infty,[0,t]} < \min\{\varepsilon - |\gamma(s) - x|, \varepsilon - |\gamma(t) - y|\}$ belongs to $\Gamma_\varepsilon(x,y)$. Thus $\liminf_{n\to\infty} \mathbf{P}_n(\Gamma_\varepsilon(x,y)) \ge \mathbf{P}(\Gamma_\varepsilon(x,y)) \ge \mathbf{P}(\Gamma(x,y)) \ge \delta$ (by [EG15, Theorem 1.40]), hence the result. The same holds true for $\Gamma_\varepsilon(y,x)$ after exchanging x and y. $\square$

4.2 Removing quasi-cycles

Here we show that if $\mathbf{P}$ has an "ε-cycle" of mass m in the sense that

$$\min\{\mathbf{P}(\Gamma_\varepsilon(x,y)), \mathbf{P}(\Gamma_\varepsilon(y,x))\} \ge m,$$

then one may do a shortcut to reduce the α-energy of $\mathbf{P}$ up to error terms equal to the energy of $\mathbf{P}$ on the balls $B_{2\varepsilon}(x), B_{2\varepsilon}(y)$.

Proposition 4.3. *Let $\mathbf{P} \in \mathbf{TP}(\mu^-, \mu^+)$ be a traffic plan with finite energy supported on the set of simple curves. Assume that there exists $\varepsilon_0 \in (0, |y-x|/8]$ such that*

$$m := \min\{\mathbf{P}(\Gamma_{\varepsilon_0}(x,y)), \mathbf{P}(\Gamma_{\varepsilon_0}(y,x))\} > 0.$$

Then there exists $\bar{T} \in TP(\mu^-, \mu^+)$ such that

$$\mathbb{M}^\alpha(\bar{T}) \leq \mathcal{E}^\alpha(\mathbf{P}) - \alpha \mathbf{P}(\mathrm{Lip}_1)^{\alpha-1} m|y-x| + \mathcal{E}^\alpha(\mathbf{P}, B_{2\varepsilon_0}(x)) + \mathcal{E}^\alpha(\mathbf{P}, B_{2\varepsilon_0}(y)).$$

Proof. Step 1 - Choice of a suitable radius. Since $\mathbf{P}$ is rectifiable, recalling Proposition 2.9, we have:

$$\frac{1}{\varepsilon_0}\int_{\varepsilon_0}^{2\varepsilon_0} \mathbb{M}^\alpha(\langle\!\langle \mathbf{P}, d_x, \varepsilon\rangle\!\rangle) + \mathbb{M}^\alpha(\langle\!\langle \mathbf{P}, d_y, \varepsilon\rangle\!\rangle)\,\mathrm{d}\varepsilon \leq \frac{\mathcal{E}^\alpha(\mathbf{P}, \{\varepsilon_0 \leq d_x \leq 2\varepsilon_0\}) + \mathcal{E}^\alpha(\mathbf{P}, \{\varepsilon_0 \leq d_y \leq 2\varepsilon_0\})}{\varepsilon_0},$$

where $d_u : z \in \mathbb{R}^d \mapsto |z-u|$. Therefore, there exists $\varepsilon \in [\varepsilon_0, 2\varepsilon_0]$ such that

$$\begin{aligned}\mathbb{M}^\alpha(\langle\!\langle \mathbf{P}, d_x, \varepsilon\rangle\!\rangle) + \mathbb{M}^\alpha(\langle\!\langle \mathbf{P}, d_y, \varepsilon\rangle\!\rangle) &\leq \frac{\mathcal{E}^\alpha(\mathbf{P}, \{\varepsilon_0 \leq d_x \leq 2\varepsilon_0\}) + \mathcal{E}^\alpha(\mathbf{P}, \{\varepsilon_0 \leq d_y \leq 2\varepsilon_0\})}{\varepsilon_0} \\ &\leq \frac{\mathcal{E}^\alpha(\mathbf{P}, B_{2\varepsilon_0}(x)) + \mathcal{E}^\alpha(\mathbf{P}, B_{2\varepsilon_0}(y))}{\varepsilon_0}. \end{aligned} \tag{4.2}$$

Since $\varepsilon \geq \varepsilon_0$, we still have $\min\{\mathbf{P}(\Gamma_\varepsilon(x,y)), \mathbf{P}(\Gamma_\varepsilon(y,x))\} \geq m$.

Step 2 - Construction of the shortcut. Given $u \in \mathbb{R}^d$ and $\gamma \in \mathrm{Lip}_1$, we define:

$$\begin{aligned} t_u^-(\gamma) &:= \inf\{t \in [0, T_\infty(\gamma)] : \gamma(t) \in \partial B_\varepsilon(u)\} \quad \text{and} \\ t_u^+(\gamma) &:= \sup\{t \in [0, T_\infty(\gamma)] : \gamma(t) \in \partial B_\varepsilon(u)\}, \end{aligned}$$

which belong to $[0, \infty]$, accepting the abuse of notation that $\inf \emptyset = 0$ and $\sup \emptyset = 0$. For any curve $\gamma \in \Gamma_\varepsilon(x,y)$ with $T_\infty(\gamma) < \infty$, we have that $t_x^-(\gamma)$ and $t_y^+(\gamma)$ belong to $[0, T_\infty(\gamma)]$ and satisfy $t_x^-(\gamma) < t_y^+(\gamma)$, given that $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are disjoint (since $\varepsilon \leq 2\varepsilon_0 \leq |y-x|/4$). Then, for any curve $\gamma \in \Gamma_\varepsilon(x,y)$ with $T_\infty(\gamma) < \infty$, we set

$$\varphi_0^u(\gamma) := \gamma_{|[0, t_u^-(\gamma)]} \quad \text{and} \quad \varphi_\infty^u(\gamma) := \gamma_{|[t_u^+(\gamma), +\infty)}.$$

Lastly we consider arbitrary (non relabeled) measurable extensions of φ_0^u and φ_∞^u to Lip_1. We also set:

$$\begin{aligned} \Lambda_\varepsilon(x,y) &:= \Gamma_\varepsilon(x,y) \setminus \Gamma_\varepsilon(y,x), \quad \Lambda_\varepsilon(x,y,x) := \{\gamma \in \Gamma_\varepsilon(x,y) \cap \Gamma_\varepsilon(y,x) : t_x^-(\gamma) < t_y^-(\gamma)\}, \\ \Lambda_\varepsilon(y,x) &:= \Gamma_\varepsilon(y,x) \setminus \Gamma_\varepsilon(x,y), \quad \Lambda_\varepsilon(y,x,y) := \{\gamma \in \Gamma_\varepsilon(x,y) \cap \Gamma_\varepsilon(y,x) : t_y^-(\gamma) < t_x^-(\gamma)\}. \end{aligned}$$

Defining

$$m_x := \frac{\min\{\mathbf{P}(\Lambda_\varepsilon(x,y)), \mathbf{P}(\Lambda_\varepsilon(y,x))\}}{\mathbf{P}(\Lambda_\varepsilon(x,y))} \quad \text{and} \quad m_y := \frac{\min\{\mathbf{P}(\Lambda_\varepsilon(x,y)), \mathbf{P}(\Lambda_\varepsilon(y,x))\}}{\mathbf{P}(\Lambda_\varepsilon(y,x))}$$

with the convention $0/0 = 0$, we set:

$$\begin{aligned}\mathbf{Q}_1 &:= m_x\mathbf{P}\llcorner\Lambda_\varepsilon(x,y) - (\varphi_0^x)_\sharp m_x\mathbf{P}\llcorner\Lambda_\varepsilon(x,y) - (\varphi_\infty^y)_\sharp m_x\mathbf{P}\llcorner\Lambda_\varepsilon(x,y)\\ &\quad + m_y\mathbf{P}\llcorner\Lambda_\varepsilon(y,x) - (\varphi_0^y)_\sharp m_y\mathbf{P}\llcorner\Lambda_\varepsilon(y,x) - (\varphi_\infty^x)_\sharp m_y\mathbf{P}\llcorner\Lambda_\varepsilon(y,x),\\ \mathbf{Q}_2 &:= \mathbf{P}\llcorner\Lambda_\varepsilon(x,y,x) - (\varphi_0^x)_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(x,y,x) - (\varphi_\infty^x)_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(x,y,x)\\ &\quad + \mathbf{P}\llcorner\Lambda_\varepsilon(y,x,y) - (\varphi_0^y)_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(y,x,y) - (\varphi_\infty^y)_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(y,x,y).\end{aligned}$$

Furthermore, we define the traffic plan:

$$\tilde{\mathbf{P}} := \mathbf{P} - \mathbf{Q}_1 - \mathbf{Q}_2.$$

We observe $\tilde{\mathbf{P}}(\mathrm{Lip}_1) \leq 3\mathbf{P}(\mathrm{Lip}_1)$. $\tilde{\mathbf{P}}$ is supported on the set of simple curves and it is rectifiable. Moreover $\Sigma_{\tilde{\mathbf{P}}} \subseteq \Sigma_{\mathbf{P}}$.

One can easily check that

$$\begin{aligned}(e_\infty)_\sharp\mathbf{Q}_1 - (e_0)_\sharp\mathbf{Q}_1 = &\ (e_{t_y^+})_\sharp m_x\mathbf{P}\llcorner\Lambda_\varepsilon(x,y) - (e_{t_x^-})_\sharp m_x\mathbf{P}\llcorner\Lambda_\varepsilon(x,y)\\ &+ (e_{t_x^+})_\sharp m_y\mathbf{P}\llcorner\Lambda_\varepsilon(y,x) - (e_{t_y^-})_\sharp m_y\mathbf{P}\llcorner\Lambda_\varepsilon(y,x),\end{aligned}$$

and

$$\begin{aligned}(e_\infty)_\sharp\mathbf{Q}_2 - (e_0)_\sharp\mathbf{Q}_2 = &\ (e_{t_x^+})_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(x,y,x) - (e_{t_x^-})_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(x,y,x)\\ &+ (e_{t_y^+})_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(y,x,y) - (e_{t_y^-})_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(y,x,y).\end{aligned}$$

Hence $(e_\infty)_\sharp\mathbf{Q}_1 - (e_0)_\sharp\mathbf{Q}_1 + (e_\infty)_\sharp\mathbf{Q}_2 - (e_0)_\sharp\mathbf{Q}_2 = S_\varepsilon^x + S_\varepsilon^y$ where we have grouped terms in x and y, setting:

$$\begin{aligned}S_\varepsilon^x := &\ (e_{t_x^+})_\sharp m_y\mathbf{P}\llcorner\Lambda_\varepsilon(y,x) - (e_{t_x^-})_\sharp m_x\mathbf{P}\llcorner\Lambda_\varepsilon(x,y)\\ &+ (e_{t_x^+})_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(x,y,x) - (e_{t_x^-})_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(x,y,x),\\ S_\varepsilon^y := &\ (e_{t_y^+})_\sharp m_x\mathbf{P}\llcorner\Lambda_\varepsilon(x,y) - (e_{t_y^-})_\sharp m_y\mathbf{P}\llcorner\Lambda_\varepsilon(y,x)\\ &+ (e_{t_y^+})_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(y,x,y) - (e_{t_y^-})_\sharp\mathbf{P}\llcorner\Lambda_\varepsilon(y,x,y).\end{aligned}$$

Then we denote by $\tilde{T}$ the current induced by $\tilde{\mathbf{P}}$. Since the boundary of $\tilde{T}$ is equal to $(e_\infty)_\sharp\tilde{\mathbf{P}} - (e_0)_\sharp\tilde{\mathbf{P}}$, this yields:

$$\partial\tilde{T} = \mu^+ - \mu^- - (S_\varepsilon^x + S_\varepsilon^y).$$

Since $\tilde{T}$ does not irrigate the same measures as $\mathbf{P}$, we adjust it by adding cones over x and y (see [Sim83, p. 26.26]):

$$\bar{T} := \tilde{T} + x \mathbin{⨯\!\!\!\!\times} S_\varepsilon^x + y \mathbin{⨯\!\!\!\!\times} S_\varepsilon^y.$$

Since $S_\varepsilon^x(\mathbb{R}^d) = S_\varepsilon^y(\mathbb{R}^d) = 0$, then $\partial(x \mathbin{⨯\!\!\!\!\times} S_\varepsilon^x) = S_\varepsilon^x$ and $\partial(y \mathbin{⨯\!\!\!\!\times} S_\varepsilon^y) = S_\varepsilon^y$. Hence we deduce that $\bar{T} \in TP(\mu^-,\mu^+)$.

Step 3 - Energy estimate. Since $\tilde{\mathbf{P}}$ is a traffic plan supported on the set of simple curves, we can write $\mathcal{E}^\alpha(\tilde{\mathbf{P}}) = \int_{\Sigma_{\mathbf{P}}} \theta_{\tilde{\mathbf{P}}}^\alpha \,\mathrm{d}\mathscr{H}^1$ and we find a bound on its α-energy as follows:

$$
\begin{aligned}
\mathcal{E}^\alpha(\tilde{\mathbf{P}}) &= \int_{\Sigma_{\mathbf{P}}} (\theta_{\mathbf{P}} - (\theta_{\mathbf{P}} - \theta_{\tilde{\mathbf{P}}}))^\alpha \,\mathrm{d}\mathscr{H}^1 \\
&\leq \int_{\Sigma_{\mathbf{P}}} \left(\theta_{\mathbf{P}}^\alpha - \alpha(\theta_{\mathbf{P}} - \theta_{\tilde{\mathbf{P}}})\theta_{\mathbf{P}}^{\alpha-1}\right) \mathrm{d}\mathscr{H}^1 \\
&\leq \int_{\Sigma_{\mathbf{P}}} \theta_{\mathbf{P}}^\alpha \,\mathrm{d}\mathscr{H}^1 - \alpha\mathbf{P}(\mathrm{Lip}_1)^{\alpha-1} \int_{\Sigma_{\mathbf{P}}} (\theta_{\mathbf{P}} - \theta_{\tilde{\mathbf{P}}}) \,\mathrm{d}\mathscr{H}^1 \\
&= \mathcal{E}^\alpha(\mathbf{P}) - \alpha\mathbf{P}(\mathrm{Lip}_1)^{\alpha-1} \int_{\Sigma_{\mathbf{P}}} \int_{\mathrm{Lip}_1} \mathbf{1}_{x\in\mathrm{Img}\,\gamma} \,\mathrm{d}(\mathbf{P} - \tilde{\mathbf{P}})(\gamma) \,\mathrm{d}\mathscr{H}^1(x) \\
&= \mathcal{E}^\alpha(\mathbf{P}) - \alpha\mathbf{P}(\mathrm{Lip}_1)^{\alpha-1} \int_{\mathrm{Lip}_1} \mathscr{H}^1(\mathrm{Img}\,\gamma) \,\mathrm{d}(\mathbf{Q}_1 + \mathbf{Q}_2)(\gamma).
\end{aligned} \tag{4.3}
$$

The first inequality follows from the concavity of $x \mapsto x^\alpha$ on $\mathbb{R}_+$, the second one is due to the fact that $\theta_{\mathbf{P}} \leq \mathbf{P}(\mathrm{Lip}_1)$; then the equality consists in using the definition of the multiplicity and the fact that $\mathbf{P}$ is supported on simple curves; and finally, Fubini's Theorem and the rectifiability of $\mathbf{P}$ yield the last equality. In order to apply Fubini, one can first use the Hahn decomposition theorem to decompose the measure $\mathbf{Q}_1 + \mathbf{Q}_2$ in its positive and negative parts. Then one can apply Fubini on each of the two parts.

In addition, we compute:

$$
\begin{aligned}
&\int_{\mathrm{Lip}_1} \mathscr{H}^1(\mathrm{Img}\,\gamma) \,\mathrm{d}\mathbf{Q}_1(\gamma) \\
&\quad = m_x \int_{\Lambda_\varepsilon(x,y)} \mathscr{H}^1(\mathrm{Img}\,\gamma) - \mathscr{H}^1(\mathrm{Img}\,\varphi_0^x(\gamma)) - \mathscr{H}^1(\mathrm{Img}\,\varphi_\infty^y(\gamma)) \,\mathrm{d}\mathbf{P}(\gamma) \\
&\qquad + m_y \int_{\Lambda_\varepsilon(y,x)} \mathscr{H}^1(\mathrm{Img}\,\gamma) - \mathscr{H}^1(\mathrm{Img}\,\varphi_0^y(\gamma)) - \mathscr{H}^1(\mathrm{Img}\,\varphi_\infty^x(\gamma)) \,\mathrm{d}\mathbf{P}(\gamma) \\
&\quad = m_x \int_{\Lambda_\varepsilon(x,y)} \mathscr{H}^1\left(\mathrm{Img}\,\gamma_{|[t_x^-(\gamma), t_y^+(\gamma)]}\right) \mathrm{d}\mathbf{P}(\gamma) \\
&\qquad + m_y \int_{\Lambda_\varepsilon(y,x)} \mathscr{H}^1\left(\mathrm{Img}\,\gamma_{|[t_y^-(\gamma), t_x^+(\gamma)]}\right) \mathrm{d}\mathbf{P}(\gamma) \\
&\quad \geq 2(|y-x| - 2\varepsilon) \min\{\mathbf{P}(\Lambda_\varepsilon(x,y)), \mathbf{P}(\Lambda_\varepsilon(y,x))\} \\
&\quad \geq |y-x| \min\{\mathbf{P}(\Lambda_\varepsilon(x,y)), \mathbf{P}(\Lambda_\varepsilon(y,x))\}.
\end{aligned} \tag{4.4}
$$

The first equality is straightforward, the second is a consequence of the fact that $\mathbf{P}$ is supported on the set of simple curves, then the inequality comes from $\gamma(t_x^-(\gamma)) \in \bar{B}_\varepsilon(x)$ and $\gamma(t_y^+(\gamma)) \in \bar{B}_\varepsilon(y)$, and the final inequality results from the assumption on ε.

Similarly:

$$
\begin{aligned}
&\int_{\mathrm{Lip}_1} \mathscr{H}^1(\operatorname{Img}\gamma)\,\mathrm{d}\mathbf{Q}_2(\gamma)\\
&= \int_{\Lambda_\varepsilon(x,y,x)} \mathscr{H}^1(\operatorname{Img}\gamma) - \mathscr{H}^1(\operatorname{Img}\varphi_0^x(\gamma)) - \mathscr{H}^1(\operatorname{Img}\varphi_\infty^x(\gamma))\,\mathrm{d}\mathbf{P}(\gamma)\\
&\qquad + \int_{\Lambda_\varepsilon(y,x,y)} \mathscr{H}^1(\operatorname{Img}\gamma) - \mathscr{H}^1(\operatorname{Img}\varphi_0^y(\gamma)) - \mathscr{H}^1(\operatorname{Img}\varphi_\infty^y(\gamma))\,\mathrm{d}\mathbf{P}(\gamma)\\
&= \int_{\Lambda_\varepsilon(x,y,x)} \mathscr{H}^1\left(\operatorname{Img}\gamma_{|[t_x^-(\gamma),t_x^+(\gamma)]}\right)\mathrm{d}\mathbf{P}(\gamma) + \int_{\Lambda_\varepsilon(y,x,y)} \mathscr{H}^1\left(\operatorname{Img}\gamma_{|[t_y^-(\gamma),t_y^+(\gamma)]}\right)\mathrm{d}\mathbf{P}(\gamma)\\
&\geq 2(|y-x|-2\varepsilon)\left(\mathbf{P}(\Lambda_\varepsilon(x,y,x)) + \mathbf{P}(\Lambda_\varepsilon(y,x,y))\right)\\
&\geq |y-x|\mathbf{P}(\Gamma_\varepsilon(x,y)\cap\Gamma_\varepsilon(y,x)).
\end{aligned}
\tag{4.5}
$$

Combining (4.3) to (4.5) yields the following bound:

$$
\begin{aligned}
\mathcal{E}^\alpha(\tilde{\mathbf{P}}) &\leq \mathcal{E}^\alpha(\mathbf{P}) - \alpha\mathbf{P}(\mathrm{Lip}_1)^{\alpha-1}|y-x|(\min\{\mathbf{P}(\Lambda_\varepsilon(x,y)),\mathbf{P}(\Lambda_\varepsilon(y,x))\}\\
&\qquad + \mathbf{P}(\Gamma_\varepsilon(x,y)\cap\Gamma_\varepsilon(y,x)))\\
&\leq \mathcal{E}^\alpha(\mathbf{P}) - \alpha\mathbf{P}(\mathrm{Lip}_1)^{\alpha-1}|y-x|\min\{\mathbf{P}(\Gamma_\varepsilon(x,y)),\mathbf{P}(\Gamma_\varepsilon(y,x))\}\\
&= \mathcal{E}^\alpha(\mathbf{P}) - \alpha\mathbf{P}(\mathrm{Lip}_1)^{\alpha-1}|y-x|m.
\end{aligned}
$$

Moreover, by definition of $t_x^\pm$, S_ε^x is supported on the sphere centered at x with radius ε. Therefore $\mathbb{M}^\alpha(x \mathbin{⨰} S_\varepsilon^x) \leq \varepsilon\mathbb{M}^\alpha(S_\varepsilon^x)$. In addition, note that we have by definition $\mathbb{M}^\alpha(S_\varepsilon^x) \leq \mathbb{M}^\alpha(\langle\!\langle \mathbf{P}, d_x, \varepsilon\rangle\!\rangle)$. The same computations also hold at y. Thus we find that $\partial\bar{T} = \mu^+ - \mu^-$ and by subadditivity of the α-mass

$$
\begin{aligned}
\mathbb{M}^\alpha(\bar{T}) &\leq \mathbb{M}^\alpha(\tilde{T}) + \varepsilon\left(\mathbb{M}^\alpha(S_\varepsilon^x) + \mathbb{M}^\alpha(S_\varepsilon^y)\right)\\
&\leq \mathcal{E}^\alpha(\mathbf{P}) - \alpha\mathbf{P}(\mathrm{Lip}_1)^{\alpha-1}|y-x|m + \varepsilon\left(\mathbb{M}^\alpha(S_\varepsilon^x) + \mathbb{M}^\alpha(S_\varepsilon^y)\right)\\
&\leq \mathcal{E}^\alpha(\mathbf{P}) - \alpha\mathbf{P}(\mathrm{Lip}_1)^{\alpha-1}|y-x|m + \varepsilon\left(\mathbb{M}^\alpha(\langle\!\langle \mathbf{P}, d_x, \varepsilon\rangle\!\rangle) + \mathbb{M}^\alpha(\langle\!\langle \mathbf{P}, d_y, \varepsilon\rangle\!\rangle)\right)\\
&\overset{(4.2)}{\leq} \mathcal{E}^\alpha(\mathbf{P}) - \alpha\mathbf{P}(\mathrm{Lip}_1)^{\alpha-1}|y-x|m + \mathcal{E}^\alpha(\mathbf{P}, B_{2\varepsilon_0}(x)) + \mathcal{E}^\alpha(\mathbf{P}, B_{2\varepsilon_0}(y)).
\end{aligned}
$$

□

5 Proof of the main theorem

Proof of Theorem 2.1. Take $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ and $\mathbf{P}$ satisfying the hypotheses of the theorem. First, we know that $\mathbf{P} \in \mathbf{TP}(\mu^-,\mu^+)$, since the evaluation maps e_0, e_∞ are continuous on $\mathbf{TP}_C \coloneqq \{\mathbf{P} \in \mathbf{TP} : \int_{\mathrm{Lip}_1} T_\infty\,\mathrm{d}\mathbf{P} \leq C\}$ which is a closed subset of $\mathbf{TP}$, where $C \coloneqq \sup_n \int_{\mathrm{Lip}_1} T_\infty\,\mathrm{d}\mathbf{P}_n$ (see [Peg17b, Section 1]). Set $\{T_n\}_{n\in\mathbb{N}}$ and T to be the transport paths induced by $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ and $\mathbf{P}$. We first show that $\{T_n\}$ and T satisfy the hypotheses of [CDM19a, Theorem 1.1] so that T is optimal; then we prove that $\mathbf{P}$ is a good decomposition of T by showing successively (C), then (B) (which is the heart of the proof), and finally (A) of Definition 2.2. From this we conclude that $\mathbf{P}$ is optimal.

Step 1 - T is optimal. Let us prove that T_n converges weakly-$\star$ to T. Let $\omega \in C^0(X, \mathbb{R}^d)$ and fix $\varepsilon > 0$. By Markov's inequality, since all T_n's and T lie in $\mathbf{TP}_C$, there exists $t_0 \geq 0$ such that for all $n \in \mathbb{N}$, $\mathbf{P}_n(T_\infty \geq t_0) \leq \varepsilon/3\|\omega\|_\infty$ and $\mathbf{P}(T_\infty \geq t_0) \leq \varepsilon/3\|\omega\|_\infty$. The map $f_{t_0} : \gamma \mapsto \int_0^{t_0} \omega(\gamma(t)) \cdot \gamma'(t) \,\mathrm{d}t$ is continuous on Lip_1: if $\gamma_n \to \gamma$ then $\omega \circ \gamma_n \to \omega \circ \gamma$ strongly in $L^1([0, t_0])$ by the Dominated Convergence Theorem, and $\{|\gamma_n'|\}_{n\in\mathbb{N}}$ is bounded in $L^\infty([0, t_0])$ hence $f_{t_0}(\gamma_n) \to f_{t_0}(\gamma)$. Thanks to the weak-$\star$ convergence of $\mathbf{P}_n$ to $\mathbf{P}$, one has for n large enough:

$$|\langle T_n, \omega\rangle - \langle T, \omega\rangle| \leq \left| \int_{T_\infty < t_0} f_{t_0}(\gamma) \,\mathrm{d}(\mathbf{P}_n - \mathbf{P})(\gamma) \right| + \frac{2\varepsilon}{3} \leq \varepsilon,$$

thus $\langle T_n, \omega\rangle \to \langle T, \omega\rangle$ as ε is arbitrary. By equivalence of the Lagrangian and Eulerian models ([Peg17a, Theorem 2.4.1] or [PS06]), an optimal traffic plan induces an optimal transport path, hence T_n is optimal since $\mathbf{P}_n$ is, and $\mathbb{M}^\alpha(T_n) = \mathcal{E}^\alpha(\mathbf{P}_n)$ so the energy is uniformly bounded and we may apply the Eulerian stability result of [CDM19a, Theorem 1.1] to conclude that $T \in OTP(\mu^-, \mu^+)$.

Step 2 - Proof of (C). We only use the fact that μ^- and μ^+ are mutually singular. Denote by Γ the set of eventually constant curves of Lip_1 such that $e_0(\gamma) = e_\infty(\gamma)$. We obviously have $(e_0)_\sharp \mathbf{P} \llcorner \Gamma = (e_\infty)_\sharp \mathbf{P} \llcorner \Gamma$, and

$$(e_0)_\sharp \mathbf{P} \llcorner \Gamma \leq (e_0)_\sharp \mathbf{P} = \mu^-, \quad (e_\infty)_\sharp \mathbf{P} \llcorner \Gamma \leq (e_\infty)_\sharp \mathbf{P} = \mu^+.$$

Yet μ^- and μ^+ are mutually singular, hence $\mathbf{P}(\Gamma) = 0$. In particular, $\mathbf{P}$-almost every curve γ is non-constant and $e_0(\gamma) \neq e_\infty(\gamma)$, hence $\mathbb{M}(\partial I_\gamma) = 2$. The fact that μ^- and μ^+ are mutually singular allows to write $\mathbb{M}(\partial T) = \mu^-(\mathbb{R}^d) + \mu^+(\mathbb{R}^d)$. Therefore

$$\mathbb{M}(\partial T) = 2\mathbf{P}(\mathrm{Lip}_1) = \int_{\mathrm{Lip}_1} \mathbb{M}(\partial I_\gamma) \,\mathrm{d}\mathbf{P}(\gamma).$$

Step 3 - Proof of (B). It will result from the fact that $\mathscr{H}^1(\{\bar\theta_{\mathbf{P}} > 0\}) = 0$, where we recall $\bar\theta_{\mathbf{P}}$ is defined in (3.7). We argue by contradiction, assuming that $\mathscr{H}^1(\{\bar\theta_{\mathbf{P}} > 0\}) > 0$, and we wish to show that this contradicts the optimality of a T_n for some large n. Using Lemma 4.1 and Theorem 3.3, take $x_0 \in \{\bar\theta_{\mathbf{P}} > 0\}$ such that $\liminf_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner B_\varepsilon(x_0)) = o(1)$ as $\varepsilon \to 0$, together with a set G satisfying the conclusion of Theorem 3.3. In particular, $\mathscr{H}^1(G) > 0$. By monotone convergence, there exists $r > 0$ such that the set $G' := G \cap \mathbb{R}^d \setminus B_r(x_0)$ has positive $\mathscr{H}^1$-measure. Now let $0 < \tilde\varepsilon \leq r/8$ be such that:

$$\liminf_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner B_{2\tilde\varepsilon}(x_0)) \leq \alpha \mathbf{P}(\mathrm{Lip}_1)^{\alpha-1} r \frac{\bar\theta(x_0)}{128}.$$

There exists a subsequence $\{T_{\tilde n_k}\}_{k\in\mathbb{N}} \subseteq \{T_n\}_{n\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N}, \quad \mathbb{M}^\alpha(T_{\tilde n_k} \llcorner B_{2\tilde\varepsilon}(x_0)) \leq \alpha \mathbf{P}(\mathrm{Lip}_1)^{\alpha-1} r \frac{\bar\theta(x_0)}{64}. \tag{5.1}$$

Then we choose $x \in G'$ satisfying the conclusion of Lemma 4.1 for the subsequence $\{T_{\tilde n_k}\}_{k\in\mathbb{N}}$. Take $\varepsilon \leq \tilde\varepsilon$ such that for a further subsequence $\{T_{n_k}\}_{k\in\mathbb{N}}$, (5.1) holds with

$x, \varepsilon, \{T_{n_k}\}$ in place of $x_0, \tilde{\varepsilon}, \{T_{\tilde{n}_k}\}$. By monotonicity in ε, notice that (5.1) also holds with $x_0, \varepsilon, \{T_{n_k}\}$. To simplify notation, the subsequence $\{T_{n_k}\}$ will just be denoted by $\{T_n\}$. Since

$$\min\{\mathbf{P}(\Gamma(x, x_0), \mathbf{P}(\Gamma(x, x_0)\} \geq \frac{\bar{\theta}(x_0)}{4},$$

we know by Lemma 4.2 that there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\min\{\mathbf{P}_n(\Gamma_\varepsilon(x_0, x)), \mathbf{P}_n(\Gamma_\varepsilon(x, x_0))\} \geq \frac{\bar{\theta}(x_0)}{8}.$$

Moreover, notice that $\mathbf{P}_n \overset{\star}{\rightharpoonup} \mathbf{P}$ implies $\mathbf{P}_n(\mathrm{Lip}_1) \to \mathbf{P}(\mathrm{Lip}_1)$ (because Lip_1 is compact), so up to increasing N, one may assume $\mathbf{P}_n(\mathrm{Lip}_1)^{\alpha-1} \geq \mathbf{P}(\mathrm{Lip}_1)^{\alpha-1}/2$ for all $n \geq N$. Since $\varepsilon \leq r/8 \leq |x - x_0|/8$, we can apply Proposition 4.3 to $\mathbf{P}_N$. Thus there exists a transport path $\bar{T}$ connecting μ_N^- to μ_N^+ satisfying:

$$\begin{aligned}
\mathbb{M}^\alpha(\bar{T}) &\leq \mathcal{E}^\alpha(\mathbf{P}_N) - \alpha \mathbf{P}_N(\mathrm{Lip}_1)^{\alpha-1} \frac{\bar{\theta}(x_0)}{8}|x - x_0| + \alpha \mathbf{P}(\mathrm{Lip}_1)^{\alpha-1} r \frac{\bar{\theta}(x_0)}{32} \\
&\leq \mathcal{E}^\alpha(\mathbf{P}_N) - \alpha \mathbf{P}(\mathrm{Lip}_1)^{\alpha-1}|x - x_0| \frac{\bar{\theta}(x_0)}{16} + \alpha \mathbf{P}(\mathrm{Lip}_1)^{\alpha-1}|x - x_0| \frac{\bar{\theta}(x_0)}{32} \\
&\leq \mathcal{E}^\alpha(\mathbf{P}_N) - \alpha \mathbf{P}(\mathrm{Lip}_1)^{\alpha-1}|x - x_0| \frac{\bar{\theta}(x_0)}{32} \\
&< \mathcal{E}^\alpha(\mathbf{P}_N),
\end{aligned}$$

which contradicts the optimality of $\mathbf{P}_N$. Hence $\bar{\theta}_{\mathbf{P}}(x) = 0$ for $\mathscr{H}^1$-a.e. $x \in \Sigma_{\mathbf{P}}$ which by Lemma 3.2, is equivalent to $|\vec{\theta}_{\mathbf{P}}(x)| = \Theta_{\mathbf{P}}(x)$. As a consequence, since $\mathbf{P}$ is rectifiable, we have equality almost everywhere in (3.2), which is equivalent to (B) by Remark 3.1.

Step 4 - Proof of (A) This item follows from the absence of cancellations in $\mathbf{P}$ and from the fact that T is acyclic as an optimal transport path, as observed in [PS06, Theorem 10.1] (notice that [CDM19a, Theorem 1.1] is also essential here). We have already noticed in Step 2 that $\mathbf{P}$-a.e. curve is nonconstant, thus it remains to show that almost every curve is simple. Denote by Γ the set of curves that are eventually constant but not simple. For any $\gamma \in \Gamma$, there exist $s < t$ such that $\gamma(s) = \gamma(t)$ and γ is non-constant on $[s, t]$, i.e. $\gamma_{|[s,t]}$ is a nontrivial loop. Let $r : \Gamma \to \mathrm{Lip}_1$ a map that associates to each $\gamma \in \Gamma$ a nontrivial loop $\gamma_{|[s(\gamma),t(\gamma)]}$. Note that one can build r to be Borel: for example, one can check there exists a finite number of loops with maximal length and take the first one. Then for any $\gamma \in \Gamma$, $I_{r(\gamma)}$ is a cycle (in the sense of currents), that is to say $\partial I_{r(\gamma)} = 0$. Denote by S the current induced by the traffic plan $r_\sharp \mathbf{P}$: it is obviously a cycle.

If $x \in \Sigma_{\mathbf{P}}$ is a regular point for $\mathbf{P}$, since $\mathbf{P}$ has no cancellation we know by Lemma 3.2 that there exists $s \in \{-1, +1\}$ such that $\gamma'(t) = s|\gamma'(t)|\tau_{\Sigma_{\mathbf{P}}}(x)$ for every $t \in \gamma^{-1}(x)$ and $\mathbf{P}$-almost every curve γ, which implies also:

$$\begin{aligned}
|\vec{\theta}_{r_\sharp \mathbf{P}}(x)| &= \left|\int_\Gamma \vec{m}_{r(\gamma)}(x)\,\mathrm{d}\mathbf{P}(\gamma)\right| = \int_\Gamma \# r(\gamma)^{-1}(x)\,\mathrm{d}\mathbf{P}(\gamma), \\
&\leq \int_\Gamma \#\gamma^{-1}(x)\,\mathrm{d}\mathbf{P}(\gamma) = \left|\int_\Gamma \vec{m}_\gamma(x)\,\mathrm{d}\mathbf{P}(\gamma)\right| = |\vec{\theta}_{\mathbf{P}}(x)|.
\end{aligned} \tag{5.2}$$

Because they are all positively colinear, we get:

$$\left|\vec{\theta}_{\mathbf{P}}(x)\right| = |\vec{\theta}_{r_\sharp \mathbf{P}}(x)| + |\vec{\theta}_{\mathbf{P}}(x) - \vec{\theta}_{r_\sharp \mathbf{P}}(x)|, \tag{5.3}$$

and knowing that $T = [\![\Sigma_{\mathbf{P}}, \vec{\theta}_{\mathbf{P}}(x)]\!]$ and $S = [\![\Sigma_{\mathbf{P}}, \vec{\theta}_{r_\sharp \mathbf{P}}(x)]\!]$, integrating over $\Sigma_{\mathbf{P}}$ yields:

$$\mathbb{M}(T) = \mathbb{M}(S) + \mathbb{M}(T - S). \tag{5.4}$$

However, T is acyclic as an optimal transport path, thus $S = 0$. Yet by Fubini's Theorem and the Area Formula:

$$0 = \mathbb{M}(S) = \int_{\Sigma_{\mathbf{P}}} \int_{\Gamma} \# r(\gamma)^{-1}(x) \, \mathrm{d}\mathbf{P}(\gamma) \, \mathrm{d}\mathscr{H}^1(x) = \int_{\Gamma} \operatorname{length} r(\gamma) \, \mathrm{d}\mathbf{P}(\gamma),$$

from which we deduce $\mathbf{P}(\Gamma) = 0$, since $\operatorname{length} r(\gamma) > 0$ for every $\gamma \in \Gamma$.

Step 5 - $\mathbf{P}$ *is optimal.* Let us conclude. By (B), $|\vec{\theta}_{\mathbf{P}}(x)| = \int_{\mathrm{Lip}_1} |\vec{m}_\gamma(x)| \, \mathrm{d}\mathbf{P}(x)$, and by (A), $\int_{\mathrm{Lip}_1} |\vec{m}_\gamma(x)| \, \mathrm{d}\mathbf{P}(x) = \theta_{\mathbf{P}}(x) = \Theta_{\mathbf{P}}(x)$ for $\mathscr{H}^1$-a.e. x, hence:

$$\mathcal{E}^\alpha(\mathbf{P}) = \int_{\Sigma_{\mathbf{P}}} \theta_{\mathbf{P}}^{\alpha-1} \Theta_{\mathbf{P}} \, \mathrm{d}\mathscr{H}^1 = \int_{\Sigma_{\mathbf{P}}} |\vec{\theta}_{\mathbf{P}}|^\alpha \, \mathrm{d}\mathscr{H}^1 = \mathbb{M}^\alpha(T).$$

Since T is optimal, by equivalence of the Lagrangian and Eulerian models we know that the optimal costs are the same and we get that $\mathbf{P} \in \mathbf{OTP}(\mu^-, \mu^+)$. $\square$

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