Existence and Uniqueness of the Motion by Curvature of Regular Networks

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Abstract

We prove existence and uniqueness of the motion by curvature of networks with triple junctions in $\mathbb{R}^d$ when the initial datum is of class $W_p^{2-2/p}$ and the unit tangent vectors to the concurring curves form angles of 120 degrees. Moreover we investigate the regularisation effect due to the parabolic nature of the system. An application of the well-posedness is a new proof and a generalization of the long–time behaviour result [39, Theorem 3.18].

Our study is motivated by an open question proposed in [38]: does there exist a unique solution of the motion by curvature of networks with initial datum being a regular network of class C^2? We give a positive answer.

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1 Introduction

The mean curvature flow of surfaces in $\mathbb{R}^d$, and in Riemannian manifolds in general, is one of the most significant examples of geometric evolution equations. This evolution can be understood as the gradient flow of the area functional: a time–dependent surface evolves with normal velocity equal to its mean curvature at any point and time.

Since the 80s the curve shortening flow (mean curvature flow of one–dimensional objects) has been widely studied by many authors both for closed curves [16, 17, 18, 23] and for curves with fixed end–points [26, 45, 46]. Also initial curves forming an angle or a cusp has been studied, and in this case the singularity disappears immediately [4, 5, 3]. When more than two curves meet at a junction, the description of the motion cannot be reduced to the case of a single curve and the problem presents additional interesting features. The simplest example of motion by mean curvature of a set which is essentially singular is the motion by curvature of *networks* that are finite unions of curves that meet at junctions.

To find a good definition of the network flow in the framework of classical PDE is tricky. Because of the variational nature of the flow, it is natural to expect that configurations with multipoints of order greater than three or 3–points with angles different from 120 degrees,

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being unstable for the length functional, should be present only in the initial network or that they should appear only at some discrete set of times, during the flow. In this paper we consider as initial data *regular* networks, that is, networks at least of class C^1 that possess only triple junctions with angles of 120 degrees.
The motion by curvature of regular networks can be expressed as a boundary value problem (see Definition 2.18). Consider a time–dependent parametrisation of the evolving network $\mathcal{N}_t = (\gamma_t^1, \dots, \gamma_t^m)$ with $\gamma_t^i : [0,T] \times [0,1] \to \mathbb{R}^d$. Suppose that $\mathcal{N}_t$ has q triple junctions $\mathcal{O}^1, \dots, \mathcal{O}^q$ parametrised by

$$\gamma_0^{j_1}(t, y_1) = \gamma_0^{j_2}(t, y_2) = \gamma_0^{j_3}(t, y_3) = \mathcal{O}^j(t) \quad \text{with } y_1, y_2, y_3 \in \{0, 1\} \tag{1.1}$$

for $j \in \{1, \dots, q\}$. Then the evolution of each curve is described at each point and time by the second order quasilinear PDE

$$V^i(t, x) = \boldsymbol{k}^i(t, x)\,, \tag{1.2}$$

where V is the normal velocity and $\boldsymbol{k}$ is the curvature. Apart from the concurrency condition at the junction (1.1), another condition appears in the system:

$$\tau^{j_1}(t, y_1) + \tau_0^{j_2}(t, y_2) + \tau^{j_3}(t, y_3) = 0\,, \tag{1.3}$$

with $\tau(y_i) = (-1)^{y_i} \frac{\gamma_x(y_i)}{|\gamma_x(y_i)|}$, for $j \in \{1, \dots, q\}$. This second condition says that the curves form angles of 120 degrees (at all times).

Since conditions (1.1), (1.2) and (1.3) are purely geometric, additional solutions can be constructed simply by re–parametrisation. Hence uniqueness has to be understood in a purely geometric sense, namely, up to reparametrisations. Moreover there is a tangential degree of freedom in the definition of the main equation: the motion by curvature of networks is described by a parabolic system of degenerate PDEs where only the normal movements of the curves are prescribed. One can take advantage of this property, by specifying a suitable tangential component of the velocity and turn the problem into a system of non–degenerate second order quasilinear PDEs, the so–called Special Flow (Definition 2.22).

This approach has been proven successful to show existence of solutions. Indeed, the first attempt to find strong solutions to the network flow was by Bronsard and Reitich [9], who provided local existence and uniqueness of solutions to the Special Flow in $\mathbb{R}^2$ for admissible initial regular networks of class $C^{2+\alpha}$ with the sum of the curvature at the junctions equal to zero. Clearly it is preferable to remove this additional regularity condition on the initial datum. When the initial datum is a regular network of class C^2 without any restriction on the curvature at the junctions, existence has been established in [38].
Uniqueness for solution to the Special Flow and geometric uniqueness for the original problem (1.1), (1.2), (1.3) remained open.

Another point that we want to address is the parabolic regularisation of the flow. Bronsard and Reitich [9] proved that (under suitable conditions) the regularity of the initial datum is preserved in time. However, it is natural to ask whether the regularity of the evolving network increases, that is, whether the flow is smooth for positive times.

We state our first result:

Theorem 1.1 (Existence, uniqueness and smoothness of the motion by curvature)**.** *Let $p \in (3, \infty)$ and $\mathcal{N}_0$ be a regular network in $\mathbb{R}^d$ of class $W_p^{2-2/p}$. Then there exists a maximal solution*

$(\mathcal{N}(t))_{t\in[0,T_{\max})}$ *to the motion by curvature with initial datum* $\mathcal{N}_0$ *in the maximal time interval* $[0,T_{\max})$ *which is geometrically unique and parametrised by curves of class*

$$W_p^1((0,T_{\max}),L_p(0,1))\cap L_p((0,T_{\max}),W_p^2(0,1))\,.$$

Furthermore, up to re–parametrisation, the maps $\gamma^i : [0,T_{max})\times[0,1]\to\mathbb{R}^d$ *are smooth for all positive times.*

Theorem 3.7 improves the result by Bronsard and Reitich passing from initial data in $C^{2+\alpha}$ to $W_p^{2-2/p}$. Moreover it shows *geometric uniqueness* of solutions. Combining Theorem 1.1 with [38, Theorem 6.8] we get *a fortiori* uniqueness for initial regular networks of class C^2. This answers in the positive a question asked in [38]. Finally, it also shows that the flow is smooth for positive times.

Once the wellposedness of the flow is settled, we investigate what happens at the maximal time of existence. The study of the long time behaviour of the evolving networks moving in the plane was undertaken in [39], completed in [37] for trees composed of three curves and extended to more general cases in [27, 38, 41]. A key element of the analysis are integral estimates which are quite intricate, due to the presence of the triple junctions.

Our short time existence result allows us to give a new prove of the following:

Theorem 1.2 (Long time behaviour). *Let* $p\in(3,\infty)$, $\mathcal{N}_0$ *an admissible initial network of class* $W_p^{2-2/p}$ *and* $(\mathcal{N}(t))_{t\in[0,T_{\max})}$ *be a maximal solution to the motion by curvature with initial datum* $\mathcal{N}_0$ *in* $[0,T_{\max})$ *with* $T_{\max}>0$. *Then at least one of the following happens:*

$i)$ $T_{\max}=\infty$;

$ii)$ *the inferior limit as* $t\nearrow T_{\max}$ *of the length of at least one curve of the network* $\mathcal{N}(t)$ *is zero;*

$iii)$ *the superior limit as* $t\nearrow T_{\max}$ *of the* L^2*–norm of the curvature of the network is* $+\infty$.

Theorem 1.2 was first proven in [39, Theorem 3.18] only for smooth initial data and in dimension $d=2$. Their proof is based on bounding the L^2–norm of all derivatives of the curvature with the L^2–norm of the curvature itself, see [39, pages 257–273]. Our proof is almost effortless compared to the one in [39] due to the fact that, thanks to our improved short time existence result, the estimates can be completely avoided.

Furthermore, we stress the fact that our results are valid for every dimension d.

Here is the strategy that allows us to prove Theorem 1.1 and Theorem 1.2. We consider the Special Flow and linearise it around the initial datum. Then we prove existence and uniqueness for the linearised problem in Section 3.1. Wellposedness of the linear system follows by Solonnikov's theory [44] provided that the system is parabolic and that the complementary conditions hold. Solutions to the Special Flow are obtained by a contraction argument in Section 3.2. Notice that the choice of the solution space $W_p^1((0,T_{\max}),L_p(0,1))\cap L_p((0,T_{\max}),W_p^2(0,1))$ is crucial to define the boundary conditions pointwise and to use the theory of [44] to solve the associated linear system. Moreover this regularity is needed in the contraction estimates because of the quasilinear nature of the equations. Clearly the solution to the Special Flow induces a solution to the motion by curvature of networks, so we get existence.

To get geometric uniqueness one has to prove that two solutions differ only by a reparametrisation as we show in Section 3.3. Existence and uniqueness of maximal solutions can then be deduced by standard arguments.

In Section 4 we prove that the flow is smooth for all positive times (Theorem 4.8). The idea of the proof is based on the so called parameter trick due to Angenent [3]. Although this strategy has been generalized to several situations [35, 36, 42], it should be pointed out that our system is not among the cases treated above because of the fully non-linear boundary condition

$$\sum_{i=1}^{3}(-1)^y \frac{\gamma^i_x(y)}{|\gamma^i_x(y)|} = 0 \quad \text{with } y \in \{0,1\}\,.$$

In [22] a strategy has been developed to prove smoothness for positive times of the surface diffusion flow for triple junction clusters with the same non–linear boundary condition. We follow that approach and adapt the arguments to our setting to complete the proof of Theorem 1.1.

Thanks to Theorem 1.1 and the quantification of the existence time of solutions to the Special Flow in terms of the initial values as given in Theorem 3.14 we are then also able to prove Theorem 1.2 by contradiction.

In this last part of the introduction we describe the existing literature related with the motion by curvature of networks.

First of all it is worth to mention that the problem of the motion by curvature of networks has been generalized to the anisotropic setting: [31], [6].

In this paper we describe the evolution till the first singularity and do not investigate what happens afterwards. Classical solutions "with restarting" have been considered in [27, 34, 38]. Although uniqueness fails in this context, there exists only finitely many solutions.
Moreover, apart from classical solutions (with or without restarting) defined in the framework of classical PDE, there are several generalised (weak) notions of the flow, see for instance [2, 8, 14, 28, 32, 47, 24]. In principle the class of admissible solutions could be much larger, so one may wonder if these weak solutions still resemble classical ones. A possible answer comes from showing the regularity of weak solutions. An important progress in this direction has been made by Kim and Tonegawa [28, 29] for an improved notion of Brakke's flow: the evolving varifold is coupled with a finite number of time–dependent mutually disjoint open sets. In this setting, when the initial datum is a closed 1–rectifiable set in $\mathbb{R}^2$ with (locally) finite measure, then for almost every time the support of the evolving varifolds consists of embedded W^2_2 curves whose endpoints meet at junctions forming angles of $0, 60$ or 120 degrees.
Another way to better understand weak solutions are the so–called weak strong uniqueness results. The first result in this direction is due to Fischer, Hensel, Laux and Simon [15] proving uniqueness of their "BV solutions" (see also [25]). If there exists a classical solution to the evolution of networks that does not undergo topological changes, then the BV solutions coincide with the classical solutions and in particular uniqueness holds. To this aim they develop a gradient–flow analogue of the notion of calibrations (for calibrations for minimal networks we refer to [33, 40, 10, 11, 7]). Just like the existence of a calibration guarantees that one has reached a global minimum in the energy landscape, the existence of a gradient flow calibration ensures that the route of steepest descent in the energy landscape is unique and stable.

However another question remains completely open: could there be more solutions in the sense of [15], [28] or [47] than the classical solutions "with restarting" that by their nature go through singularities and topological changes?

We finally describe the structure of the paper. In Section 2 we define the motion by curvature of networks and we introduce the solution space together with useful properties. Section 3 is devoted to prove existence of solutions to the motion by curvature and their geometric uniqueness. Then in Section 4 we explore the regularisation effect of the flow resulting in the proof of Theorem 1.1. We conclude with the proof of Theorem 1.2 in Section 5 giving a description of the behaviour of solutions at their maximal time of existence.

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2 Solutions to the Motion by Curvature of Networks

2.1 Preliminaries on function spaces

This paper is devoted to show well-posedness of a second order evolution equation. One natural solution space is given by

$$W_p^{1,2}\left((0,T)\times(0,1);\mathbb{R}^d\right) := W_p^1\left((0,T);L_p((0,1);\mathbb{R}^d)\right)\cap L_p\left((0,T);W_p^2\left((0,1);\mathbb{R}^d\right)\right)$$

where T represents the time of existence and $d\in\mathbb{N}$ is any natural number. This space should be understood as the intersection of two *Bochner spaces* that are Sobolev spaces defined on a measure space with values in a Banach space. We give a brief summary in the case that the measure space is an interval. A detailed introduction on Bochner spaces can be found in [50].

Let $I\subset\mathbb{R}$ be an open interval and X be a Banach space. A function $f:I\to X$ is called *strongly measurable* if it is pointwise limit a.e. of a sequence of piecewise constant functions. If $f:I\to X$ is strongly measurable, then $\|f\|_X : I\to\mathbb{R}$ is Lebesgue measurable. This justifies the following definition.

Definition 2.1 (L_p–spaces)**.** Let $I\subset\mathbb{R}$ be an open interval and X be a Banach space. For $1\leq p\leq\infty$, we define the *L_p–space*

$$L_p(I;X) := \left\{f:I\to X \text{ strongly measurable } : \|f\|_{L_p(I;X)}<\infty\right\},$$

where $\|f\|_{L_p(I;X)} := \|\|f(\cdot)\|_X\|_{L_p(I;\mathbb{R})}$. Furthermore, we let

$$L_{1,loc}(I;X) := \left\{f:I\to X \text{ strongly measurable} :\ \text{for all } K\subset I \text{ compact, } f_{|K}\in L_1(K;X)\right\}.$$

Let $I\subset\mathbb{R}$ be an open interval, X be a Banach space, $f\in L_{1,loc}(I;X)$ and $k\in\mathbb{N}_0$. The k-th distributional derivative $\partial_x^k f$ of f is the functional on $C_0^\infty(I;\mathbb{R})$ given by

$$\langle\phi\,,\,\partial_x^k f\rangle := (-1)^k\int_I f(x)\partial_x^k\phi(x)\mathrm{d}x\,.$$

The distribution $\partial_x^k f$ is called regular if it is (represented by) a function in $L^{1,loc}$

Definition 2.2 (Sobolev spaces). Let $m \in \mathbb{N}$, $I \subset \mathbb{R}$ be an open interval and X be a Banach space. For $1 \le p \le \infty$ the *Sobolev space of order* $m \in \mathbb{N}$ is defined as

$$W_p^m(I;X) := \left\{ f \in L_p(I;X) : \partial_x^k f \in L_p(I;X) \text{ for all } 1 \le k \le m \right\} .$$

The space $W_p^m(I;X)$ is a Banach space with the norm

$$\|f\|_{W_p^m(I;X)} := \begin{cases} \left(\sum_{0 \le k \le m} \|\partial_x^k f\|_{L_p(I;X)}^p \right)^{1/p} , & 1 \le p < \infty , \\ \max_{0 \le k \le m} \|\partial_x^k f\|_{L_\infty(I;X)} , & p = \infty . \end{cases} \tag{2.1}$$

Elements in the solution space

$$\boldsymbol{E}_T := W_p^1\left((0,T); L_p((0,1);(\mathbb{R}^d)^m)\right) \cap L_p\left((0,T); W_p^2\left((0,1);(\mathbb{R}^d)^m\right)\right)$$

are thus functions $f \in L_p\left((0,T); L_p\left((0,1);(\mathbb{R}^d)^m\right)\right)$ possessing one distributional derivative with respect to time $\partial_t f \in L_p\left((0,T); L_p\left((0,1);(\mathbb{R}^d)^m\right)\right)$. Furthermore, for almost every $t \in (0,T)$, the function $f(t)$ lies in $W_p^2\left((0,1);(\mathbb{R}^d)^m\right)$ and thus has two spacial derivatives $\partial_x(f(t))$, $\partial_x^2(f(t)) \in L_p\left((0,1);(\mathbb{R}^d)^m\right)$. One easily sees that the functions $t \mapsto \partial_x^k(f(t))$ for $k \in \{1,2\}$ lie in $L_p\left((0,T); L_p\left((0,1);(\mathbb{R}^d)^m\right)\right)$.
The space $\boldsymbol{E}_T$ is often denoted by $W_p^{1,2}\left((0,T)\times(0,1);(\mathbb{R}^d)^m\right)$. We also use the notation $\|\cdot\|_{\boldsymbol{E}_T} := \|\cdot\|_{W_p^{1,2}}$ where $\|\cdot\|_{W_p^{1,2}}$ is defined in (2.1).

Definition 2.3 (Sobolev–Slobodeckij spaces). Given $d \in \mathbb{N}$, $p \in [1,\infty)$ and $\theta \in (0,1)$ the *Slobodeckij semi-norm* of an element $f \in L_p\left((0,1);\mathbb{R}^d\right)$ is defined as

$$[f]_{\theta,p} := \left(\int_0^1 \int_0^1 \frac{|f(x)-f(y)|^p}{|x-y|^{\theta p+1}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p} .$$

Let $s \in (0,\infty)$ be non–integer. The *Sobolev–Slobodeckij space* $W_p^s\left((0,1);\mathbb{R}^d\right)$ is defined by

$$W_p^s\left((0,1);\mathbb{R}^d\right) := \left\{ f \in W_p^{\lfloor s \rfloor}\left((0,1);\mathbb{R}^d\right) : \left[\partial_x^{\lfloor s \rfloor} f\right]_{s-\lfloor s \rfloor,p} < \infty \right\} .$$

Theorem 2.4. *Let T be positive, $p \in (3,\infty)$ and $\alpha \in (0, 1-3/p]$. We have continuous embeddings*

$$W_p^{1,2}\left((0,T)\times(0,1)\right) \hookrightarrow C\left([0,T]; W_p^{2-2/p}\left((0,1)\right)\right) \hookrightarrow C\left([0,T]; C^{1+\alpha}\left([0,1]\right)\right) .$$

Proof. The first embedding follows from [12, Lemma 4.4], the second is an immediate consequence of the Sobolev Embedding Theorem [48, Theorem 4.6.1.(e)]. □

Similarly, we can specify the spaces of the boundary values.

Lemma 2.5. *Let T be positive, $d \in \mathbb{N}$ and $p \in [1,\infty)$. Then the operator*

$$W_p^{1,2}\left((0,T)\times(0,1);\mathbb{R}\right) \to W_p^{1/2-1/2p}\left((0,T);\mathbb{R}^d\right) ,$$
$$f \mapsto (f_x)_{|x=0}$$

is linear and continuous.

Proof. This follows from [44, Theorem 5.1]. □

Another important feature of Sobolev Slobodeckij spaces is their Banach algebra property.

Proposition 2.6. *Let $I \subset \mathbb{R}$ be a bounded open interval, $p \in [1, \infty)$ and $s \in (0,1)$ with $s - \frac{1}{p} > 0$. Then for $f, g \in W_p^s(I;\mathbb{R})$ the product fg lies in $W_p^s(I;\mathbb{R})$ and satisfies*

$$\|fg\|_{W_p^s(I;\mathbb{R})} \leq C(s,p)\left(\|f\|_{C(\bar{I})}\|g\|_{W_p^s(I;\mathbb{R})} + \|g\|_{C(\bar{I})}\|f\|_{W_p^s(I;\mathbb{R})}\right).$$

Furthermore, given a smooth function $F : \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, and a function $f \in W_p^s(I;\mathbb{R}^d)$, the function $t \mapsto F(f(t))$ lies in $W_p^s(I;\mathbb{R})$.

Proof. As $W_p^s((0,1);\mathbb{R}) \hookrightarrow C(\bar{I};\mathbb{R})$ due to the Sobolev Embedding Theorem [48, Theorem 4.6.1.(e)], we obtain for $f, g \in W_p^s(I;\mathbb{R})$ the estimate

$$\|fg\|_{L_p(I;\mathbb{R})} \leq \|f\|_{C(\bar{I})}\|g\|_{L_p(I;\mathbb{R})} \leq C(s,p)\|f\|_{W_p^s(I;\mathbb{R})}\|g\|_{W_p^s(I;\mathbb{R})}$$

and

$$\begin{aligned}
[fg]_{s,p}^p &= \int_I\int_I \frac{|(fg)(x)-(fg)(y)|^p}{|x-y|^{sp+1}}\mathrm{d}x\,\mathrm{d}y \\
&\leq \int_I\int_I \frac{|g(x)|^p|f(x)-f(y)|^p + |f(y)|^p|g(x)-g(y)|^p}{|x-y|^{sp+1}}\mathrm{d}x\,\mathrm{d}y \\
&\leq \|g\|_{C(\bar{I})}^p[f]_{s,p}^p + \|f\|_{C(\bar{I})}[g]_{s,p}^p \leq C(s,p)\|f\|_{W_p^s(I;\mathbb{R})}\|g\|_{W_p^s(I;\mathbb{R})}.
\end{aligned}$$

Let $F : \mathbb{R}^d \to \mathbb{R}$ be smooth and $f \in W_p^s(I;\mathbb{R}^d)$. As f lies in $C(\bar{I};\mathbb{R}^d)$, there exists $R > 0$ such that $f(\bar{I}) \subset \overline{B_R(0)}$. Thus we obtain

$$\|F(f)\|_{L_p(I;\mathbb{R})}^p = \int_I |F(f(x))|^p \mathrm{d}x \leq \max_{z \in \overline{B_R(0)}} |F(z)|^p|I|$$

where $|I|$ denotes the length of the interval I. Using

$$\begin{aligned}
|F(f(x)) - F(f(y))| &= \left|\int_0^1 (DF)(\xi f(x) + (1-\xi)f(y))\,\mathrm{d}\xi\,(f(x)-f(y))\right| \\
&\leq \max_{z\in\overline{B_R(0)}} |DF(z)||f(x)-f(y)|
\end{aligned}$$

we obtain

$$[F(f)]_{s,p}^p = \int_I\int_I \frac{|F(f(x))-F(f(y))|^p}{|x-y|^{sp+1}}\mathrm{d}x\,\mathrm{d}y \leq [f]_{s,p}^p \max_{z\in\overline{B_R(0)}} |DF(z)|^p.$$

□

To show well-posedness of evolution equations it is important to have embeddings with constants independent of the time interval one is working with. To this end one needs to change the norm on the solution space. In the following, we collect the results that are needed in our specific case.

Lemma 2.7. *Let $p \in (3, \infty)$. For every $T > 0$,*

$$\|\!|g|\!\|_{W_p^{1,2}((0,T)\times(0,1))} := \|g\|_{W_p^{1,2}((0,T)\times(0,1))} + \|g(0)\|_{W_p^{2-2/p}((0,1))}$$

defines a norm on $W_p^{1,2}((0,T)\times(0,1))$ that is equivalent to the usual one.

Proof. This is a consequence of Theorem 2.4. □

Lemma 2.8 (Extension Operator, I). *Let T_0 be positive, $T \in (0, T_0)$ and $p \in (3, \infty)$. There exists a linear operator*

$$\boldsymbol{E} : W_p^{1,2}((0,T)\times(0,1)) \to W_p^{1,2}((0,T_0)\times(0,1))$$

such that for all $g \in W_p^{1,2}((0,T)\times(0,1))$, $(\boldsymbol{E}g)_{|(0,T)} = g$ and

$$\|\boldsymbol{E}g\|_{W_p^{1,2}((0,T_0)\times(0,1))} \le C\left(\|g\|_{W_p^{1,2}((0,T)\times(0,1))} + \|g(0)\|_{W_p^{2-2/p}(0,1)}\right) = C\|\!|g|\!\|_{W_p^{1,2}((0,T)\times(0,1))}$$

with a constant $C = C(p, T_0)$ depending only on p and T_0.

Proof. In the case that $g(0) = 0$, the function g can be extended to $(0, \infty)$ by reflecting it with respect to the axis $t = T$. The general statement can be deduced from this case by solving a linear parabolic equation of second order and using results on maximal regularity as given in [42, Proposition 3.4.3]. □

Given $d \in \mathbb{N}$ we obtain an extension operator on the space $W_p^{1,2}\left((0,T)\times(0,1);\mathbb{R}^d\right)$ by applying $\boldsymbol{E}$ to every component.

Lemma 2.9. *Let $p \in (1, \infty)$ and $\alpha > \frac{1}{p}$. For every positive T,*

$$\|\!|b|\!\|_{W_p^{\alpha}((0,T);\mathbb{R})} := \|b\|_{W_p^{\alpha}((0,T);\mathbb{R})} + |b(0)|$$

defines a norm on $W_p^{\alpha}((0,T);\mathbb{R})$ that is equivalent to the usual one.

Proof. This is an immediate consequence of the Sobolev Embedding Theorem [48, Theorem 4.6.1.(e)]. □

Lemma 2.10 (Extension Operator, II). *Let T be positive, $p \in (1, \infty)$ and $\alpha > \frac{1}{p}$. There exists a linear operator*

$$E : W_p^{\alpha}((0,T);\mathbb{R}) \to W_p^{\alpha}((0,\infty);\mathbb{R})$$

such that for all $b \in W_p^{\alpha}((0,T);\mathbb{R})$, $(Eb)_{|(0,T)} = b$ and

$$\|Eb\|_{W_p^{\alpha}((0,\infty);\mathbb{R})} \le C_p\left(\|b\|_{W_p^{\alpha}((0,T);\mathbb{R})} + |b(0)|\right) = C_p\|\!|b|\!\|_{W_p^{\alpha}((0,T);\mathbb{R})}$$

with a constant C_p depending only on p.

Proof. In the case $b(0) = 0$ the operator obtained by reflecting the function with respect to the axis $t = T$ has the desired properties. The general statement can be deduced from this case using surjectivity of the temporal trace $_{|t=0} : W_p^{\alpha}((0,\infty);\mathbb{R}) \to \mathbb{R}$. □

Theorem 2.11 (Uniform embedding I). *Let $p \in (3, \infty)$ and T_0 be positive. There exist constants $C(p)$ and $C(T_0, p)$ such that for all $T \in (0, T_0]$ and all $g \in W_p^{1,2}((0,T)\times(0,1))$,*

$$\|g\|_{C([0,T];C^1([0,1]))} \le C(p)\,\|g\|_{C\left([0,T];W_p^{2-2/p}((0,1))\right)} \le C(T_0, p)\,\|\!|g|\!\|_{W_p^{1,2}((0,T)\times(0,1))}.$$

Proof. Let $T \in (0, T_0]$ be arbitrary, $g \in W_p^{1,2}((0,T)\times(0,1))$ and $\boldsymbol{E}g$ the extension according to Lemma 2.8. Then $\boldsymbol{E}g$ lies in $W_p^{1,2}((0,T_0)\times(0,1))$ and Theorem 2.4 and Lemma 2.8 imply

$$\begin{aligned}\|g\|_{C\left([0,T];W_p^{2-2/p}((0,1))\right)} &\le \|\boldsymbol{E}g\|_{C\left([0,T_0];W_p^{2-2/p}((0,1))\right)} \le C(T_0,p)\,\|\boldsymbol{E}g\|_{W_p^{1,2}((0,T_0)\times(0,1))} \\ &\le C(T_0,p)\,\|g\|_{W_p^{1,2}((0,T)\times(0,1))}.\end{aligned}$$

□

Theorem 2.12 (Uniform embedding II). *Let $p \in (3,\infty)$, $\theta \in \left(\frac{1+1/p}{2-2/p}, 1\right)$, $\delta \in (0, 1-1/p)$ and T_0 be positive. There exists a constant $C(T_0,p,\theta,\delta) > 0$ such that for all $T \in (0,T_0]$ there holds the embedding*

$$W_p^{1,2}((0,T)\times(0,1)) \hookrightarrow C^{(1-\theta)(1-1/p-\delta)}\left([0,T]; C^1([0,1])\right)$$

and all $g \in W_p^{1,2}((0,T)\times(0,1))$ satisfy the uniform estimate

$$\|g\|_{C^{(1-\theta)(1-1/p-\delta)}([0,T];C^1([0,1]))} \le C(T_0,p,\theta,\delta)\,\|g\|_{W_p^{1,2}((0,T)\times(0,1))}.$$

Proof. By [43, Corollary 26] there holds for any $\delta \in (0, 1-1/p)$ the continuous embedding

$$W_p^{1,2}((0,T_0)\times(0,1)) \hookrightarrow C^{1-1/p-\delta}([0,T_0]; L_p((0,1)))$$

with operator norm depending on T_0. Furthermore, Theorem 2.4 gives

$$W_p^{1,2}((0,T_0)\times(0,1)) \hookrightarrow C\left([0,T_0]; W_p^{2-2/p}((0,1))\right).$$

The results in [48] yield that the real interpolation space satisfies

$$W_p^{\theta(2-2/p)}((0,1)) = \left(L_p((0,1)), W_p^{2-2/p}((0,1))\right)_{\theta,p}$$

with equivalent norms. In particular, for all $f \in W_p^{\theta(2-2/p)}((0,1))$ there holds the estimate

$$\|f\|_{W_p^{\theta(2-2/p)}((0,1))} \le C\,\|f\|_{L_p((0,1))}^{1-\theta}\,\|f\|_{W_p^{2-2/p}((0,1))}^{\theta}.$$

A direct computation using the above estimate shows that for all $\alpha \in (0,1)$,

$$C\left([0,T_0]; W_p^{2-2/p}((0,1))\right) \cap C^{\alpha}([0,T_0]; L_p((0,1))) \hookrightarrow C^{(1-\theta)\alpha}\left([0,T_0]; W_p^{\theta(2-2/p)}((0,1))\right)$$

which yields for all $\delta \in (0, 1-1/p)$ the continuous embedding

$$W_p^{1,2}((0,T_0)\times(0,1)) \hookrightarrow C^{(1-\theta)(1-1/p-\delta)}\left([0,T_0]; W_p^{\theta(2-2/p)}((0,1))\right).$$

Due to $\theta(2-2/p) - \frac{1}{p} > 1$ the Sobolev Embedding Theorem yields

$$W_p^{1,2}((0,T_0)\times(0,1)) \hookrightarrow C^{(1-\theta)(1-1/p-\delta)}\left([0,T_0]; C^1([0,1])\right).$$

The claim now follows using the extension operator $\boldsymbol{E}$ constructed in Lemma 2.8 with similar arguments as in the proof of Theorem 2.11. □

2.2 Motion by curvature of networks

Let $d \in \mathbb{N}, d \geq 2$. Consider a curve $\gamma : [0,1] \to \mathbb{R}^d$ of class C^1. A curve is said to be *regular* if $|\gamma_x(x)| \neq 0$ for every $x \in [0,1]$. Let us denote with s the arclength parameter. We remind that $\partial_s = \frac{\partial_x}{|\gamma_x|}$. If a curve γ is of class C^1 and regular, its unit tangent vector is given by $\tau = \gamma_s = \frac{\gamma_x}{|\gamma_x|}$. The *curvature vector* of a regular C^2–curve γ is defined by

$$\boldsymbol{\kappa} := \gamma_{ss} = \tau_s = \frac{\gamma_{xx}}{|\gamma_x|^2} - \frac{\langle \gamma_{xx}, \gamma_x \rangle \gamma_x}{|\gamma_x|^4} .$$

The *curvature* is given by $\kappa = |\tau_s|$.

Definition 2.13. A *network* $\mathcal{N}$ is a connected set in $\mathbb{R}^d$ consisting of a finite union of regular curves $\mathcal{N}^i$ that meet at their endpoints in junctions. Each curve $\mathcal{N}^i$ admits a regular C^1–parametrisation, namely a map $\gamma^i : [0,1] \to \mathbb{R}^d$ of class C^1 with $|\gamma^i_x| \neq 0$ on $[0,1]$ and $\gamma^i([0,1]) = \mathcal{N}^i$.

Although a network is a *set* by definition, we will mainly deal with its parametrisations. It is then natural to speak about the regularity of these maps.

Definition 2.14. Let $k \in \mathbb{N}$, $k \geq 2$, and $1 \leq p \leq \infty$ with $p > \frac{1}{k-1}$. A network $\mathcal{N}$ is of class C^k (or W^k_p, respectively) if it admits a regular parametrisation of class C^k (or W^k_p, respectively).

In this paper we restrict to the class of *regular networks*.

Definition 2.15. A network is called *regular* if its curves meet at triple junctions forming equal angles.

Notice that this notion is geometric in the sense that it does not depend on the choice of the parametrisations of the curves of the network $\mathcal{N}$.

Definition 2.16 (Geometrically admissible initial datum)**.** A network $\mathcal{N}_0 = \cup_{i=1}^m \sigma^i([0,1])$ is a *geometrically admissible initial datum* for the motion by curvature if it is regular and each of its curves can be parametrised by a regular curve $\sigma^i \in W_p^{2-2/p}([0,1], \mathbb{R}^d)$ with $p \in (3, \infty)$.

Remark 2.17. For $p \in (3, \infty)$ the Sobolev Embedding Theorem [48, Theorem 4.6.1.(e)] implies

$$W_p^{2-2/p}\left((0,1); \mathbb{R}^d\right) \hookrightarrow C^{1+\alpha}\left([0,1]; \mathbb{R}^d\right)$$

for $\alpha \in (0, 1 - 3/p)$. In particular, any admissible initial network is of class C^1 and the angle condition at the boundary is well-defined.

We define now the *motion by curvature* of regular networks: a time dependent family of regular networks evolves with normal velocity V^i equal to the curvature vector at any point and any time, namely

$$V^i = \boldsymbol{\kappa}^i .$$

To be more precise, given a time dependent family of curves γ^i, we denote by $\boldsymbol{P}^i : \mathbb{R}^d \to \mathbb{R}^d$ the projection onto the normal space to γ^i, namely $\boldsymbol{P}^i := \mathrm{Id} - \gamma^i_s \otimes \gamma^i_s$. The motion equation reads as

$$\boldsymbol{P}^i \gamma^i_t = \boldsymbol{\kappa}^i .$$

To write the precise system of equations that describe the motion by curvature of a time dependent family of network it is convenient to describe more in detail the structure/topology of the initial datum, and thus the structure/topology of the evolving network.
Let $m, \ell, q \in \mathbb{N}$ and suppose that we consider a regular network $\mathcal{N}_0$ composed of m curves with ℓ endpoints $P^1, \dots, P^\ell$ and with q triple junctions $\mathcal{O}^1, \dots, \mathcal{O}^q$. We parametrise the curves of the network in such a way that if P^i is an endpoint of order one of a curve $\mathcal{N}_0^i$ and σ^i is its parametrisation, then $\sigma^i(1) = P^i$. Consider now one of the triple junctions, say $\mathcal{O}^j$, where the curves $\mathcal{N}_0^{j_1}, \mathcal{N}_0^{j_2}$ and $\mathcal{N}_0^{j_3}$ meet (with j_1, j_2, j_3 not all equal). If $\sigma^{j_1}, \sigma^{j_2}$ and σ^{j_3} are the parametrisations of $\mathcal{N}_0^{j_1}, \mathcal{N}_0^{j_2}$ and $\mathcal{N}_0^{j_3}$ we cannot impose that $\sigma^{j_1}(0) = \sigma^{j_2}(0) = \sigma^{j_3}(0) = \mathcal{O}^j$ whatever j is, because both endpoints of a curve can be part of a triple junction (see for instance the networks composed of five or two curves in the picture below).

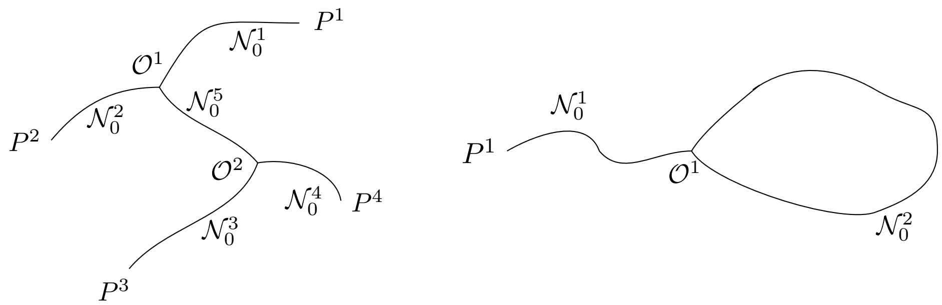

Figure 1: A network composed of five curves and another composed of two.

We will instead have that

$$\sigma^{j_1}(y_1) = \sigma^{j_2}(y_2) = \sigma^{j_3}(y_3) = \mathcal{O}^j \text{ with } j \in \{1, \dots, q\}, y_1, y_2, y_3 \in \{0, 1\}\,.$$

The fact that y_1, y_2, y_3 could be either 0 or 1 affects how the angle condition reads, that it

$$(-1)^{y_1}\tau_0^{j_1}(y_1) + (-1)^{y_2}\tau_0^{j_2}(y_2) + (-1)^{y_3}\tau_0^{j_3}(y_3) = 0\,,$$

where $\tau_0 = \sigma_s$.

Definition 2.18 (Solutions to the motion by curvature)**.** Let $m, \ell, q \in \mathbb{N}$, $p \in (3, \infty)$ and $T > 0$. Let $\mathcal{N}_0$ be a geometrically admissible initial datum composed of m curves, possibly with endpoints $P^1, \dots, P^\ell$ and with triple junctions $\mathcal{O}^1, \dots, \mathcal{O}^q$, parametrised as described above. A time dependent family of networks $(\mathcal{N}(t))$ is a *solution to the motion by curvature in* $[0, T]$ *with initial datum* $\mathcal{N}_0$ if there exists a collection of time dependent parametrisations

$$\gamma_n^i \in W_p^1(I_n; L_p((0,1);\mathbb{R}^d)) \cap L_p(I_n; W_p^2((0,1);\mathbb{R}^d))\,,$$

with $n \in \{0, \dots, N\}$ for some $N \in \mathbb{N}$, $I_n := (a_n, b_n) \subset \mathbb{R}$, $a_n \leq a_{n+1}$, $b_n \leq b_{n+1}$, $a_n < b_n$ and $\bigcup_n (a_n, b_n) = (0, T)$ such that for all $n \in \{0, \dots, N\}$ and $t \in I_n$, $\gamma_n(t) = \big(\gamma^1(t), \dots, \gamma^m(t)\big)$ is a regular parametrisation of $\mathcal{N}(t)$. Moreover, each γ_n needs to satisfy the following system:

$$\begin{cases} \boldsymbol{P}^i\gamma^i_t(t,x) = \boldsymbol{\kappa}^i(t,x) & \text{motion by curvature,}\\ \gamma^k(t,1) = P^k & \text{fixed endpoints,}\\ \gamma^{j_1}(t,y_1) = \gamma^{j_2}(t,y_2) = \gamma^{j_3}(t,y_3) & \text{concurrency condition,}\\ (-1)^{y_1}\tau^{j_1}(t,y_1) + (-1)^{y_2}\tau^{j_2}(t,y_2) + (-1)^{y_3}\tau^{j_3}(t,y_3) = 0 & \text{angle condition,} \end{cases} \tag{2.2}$$

for almost every $t \in I_n, x \in (0,1)$, for all $i \in \{1,\dots,m\}$, $k \in \{1,\dots,\ell\}$, $j \in \{1,\dots,q\}$. Finally, we ask that $\gamma_n(a_n,[0,1])$ parametrises $\mathcal{N}_0$ when $a_n = 0$.

Remark 2.19. In the motion by curvature equation only the normal component of the velocity γ^i_t is prescribed. This does not mean that there is no tangential motion. Indeed, a non–trivial tangential velocity is generally needed to allow for motion of the triple junctions.

Remark 2.20. We are interested in finding a time–dependent family of networks $(\mathcal{N}(t))$ solving the motion by curvature. Our notion of solution allows the network to be parametrised by different sets of functions in different (but overlapping) time intervals. Namely a solution can be parametrised by $\gamma = (\gamma^1,\dots,\gamma^m)$ with $\gamma^i : (a_0,b_0)\times[0,1] \to \mathbb{R}^d$ and $\eta = (\eta^1,\dots,\eta^m)$ with $\eta^i : (a_1,b_1)\times[0,1] \to \mathbb{R}^d$ if $a_0 \le a_1 < b_0 \le b_1$ and $\gamma^i((a_1,b_0)\times[0,1]) = \eta^i((a_1,b_0)\times[0,1])$. Requiring that the family of networks $(\mathcal{N}(t))$ is parametrised by *one map* $\gamma(t) = (\gamma^1(t),\dots,\gamma^m(t))$ in the whole time interval of existence $[0,T]$ as in [38] gives a slightly stronger definition of the motion by curvature in comparison to Definition 2.18. This difference does not affect the proof of the short time existence result, but in principle using our definition the *maximal* time interval of existence could be longer.

The first step to find solutions to the motion by curvature is to turn system (2.2) into a system of quasilinear parabolic PDEs by choosing a suitable tangential velocity T. We choose T such that

$$\gamma^i_t(t,x) = \boldsymbol{P}^i\gamma^i_t(t,x) + \left\langle \gamma^i_t(t,x)\,,\tau^i(t,x)\right\rangle\tau^i(t,x) = \boldsymbol{\kappa}^i(t,x) + T^i(t,x)\tau^i(t,x) = \frac{\gamma^i_{xx}(t,x)}{|\gamma^i_x(t,x)|^2}\,.$$

Since the expression of the curvature reads as

$$\boldsymbol{\kappa}^i(t,x) = \frac{\gamma^i_{xx}(t,x)}{|\gamma^i_x(t,x)|^2} - \left\langle \frac{\gamma^i_{xx}(t,x)}{|\gamma^i_x(t,x)|^2}\,,\tau^i(t,x)\right\rangle\tau^i(t,x)$$

we choose

$$T^i(t,x) = \left\langle \frac{\gamma^i_{xx}(t,x)}{|\gamma^i_x(t,x)|^2}\,,\tau^i(t,x)\right\rangle\,.$$

The equation $\gamma^i_t = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2}$ is called *Special Flow*.

Definition 2.21 (Admissible initial parametrisation)**.** Let $p \in (3,\infty)$. An *admissible initial parametrisation* for a network $\mathcal{N}_0$ composed of m curves, possibly with endpoints $P^1,\dots,P^\ell$ and with q triple junctions $\mathcal{O}^1,\dots,\mathcal{O}^q$ is a tuple

$$\sigma = (\sigma^1,\dots,\sigma^m)$$

where $\bigcup_i \sigma^i([0,1]) = \mathcal{N}_0$, with σ^i regular and of class $W^{2-2/p}_p((0,1),\mathbb{R}^d)$. The endpoints are parametrised by $\sigma^k(t) = P^k$ with $k \in \{1,\dots,\ell\}$, the triple junctions by $\sigma^{j_1}(y_1) = \sigma^{j_2}(y_2) = \sigma^{j_3}(y_3)$ with $j \in \{1,\dots,q\}, y_1,y_2,y_3 \in \{0,1\}$. Moreover at the junctions it holds $(-1)^{y_1}\tau^{j_1}_0(y_1) + (-1)^{y_2}\tau^{j_2}_0(y_2) + (-1)^{y_3}\tau^{j_3}_0(y_3) = 0$.

Notice that it follows by the very definition that a geometrically admissible network admits an admissible parametrisation.

Definition 2.22 (Solution of the Special Flow)**.** Let $T > 0$ and $p \in (3, \infty)$. Consider an admissible initial parametrisation $\sigma = (\sigma^1, \dots, \sigma^m)$ for a network $\mathcal{N}_0$ composed of m curves in $\mathbb{R}^d$ with ℓ endpoints $P^1, \dots, P^\ell \in \mathbb{R}^d$ parametrised by $\gamma^k(1) = P^k$ and q triple junctions $\mathcal{O}^1, \dots, \mathcal{O}^q$ parametrised by $\sigma^{j_1}(y_1) = \sigma^{j_2}(y_2) = \sigma^{j_3}(y_3)$. Then we say that $\gamma = (\gamma^1, \dots, \gamma^m)$ is a *solution of the Special Flow in the time interval* $[0, T]$ *with initial datum* σ if

$$\gamma = \left(\gamma^1, \dots, \gamma^m\right) \in \boldsymbol{E}_T = W^1_p((0,T); L_p((0,1); (\mathbb{R}^d)^m)) \cap L_p((0,T); W^2_p((0,1); (\mathbb{R}^d)^m)) \,,$$

$|\gamma^i_x(t,x)| \neq 0$ for all $(t,x) \in [0,T] \times [0,1]$ and the following system is satisfied for almost every $x \in (0,1)$, $t \in (0,T)$, for every $i \in \{1, \dots, m\}$, $k \in \{1, \dots, \ell\}$, $j \in \{1, \dots, q\}$:

$$\begin{cases} \gamma^i_t(t,x) = \frac{\gamma^i_{xx}(t,x)}{|\gamma^i_x(t,x)|^2} & \text{Special Flow,} \\ \gamma^k(t,1) = P^k & \text{fixed endpoints,} \\ \gamma^{j_1}(t,y_1) = \gamma^{j_2}(t,y_2) = \gamma^{j_3}(t,y_3) & \text{concurrency condition,} \\ (-1)^{y_1} \frac{\gamma^{j_1}_x(t,y_1)}{\left|\gamma^{j_1}_x(t,y_i)\right|} + (-1)^{y_2} \frac{\gamma^{i_2}_x(t,y_2)}{\left|\gamma^{i_2}_x(t,y_2)\right|} + (-1)^{y_3} \frac{\gamma^{j_3}_x(t,y_3)}{\left|\gamma^{j_3}_x(t,y_3)\right|} = 0 & \text{angle condition,} \\ \gamma^i(0,x) = \sigma^i(x) & \text{initial datum.} \end{cases} \tag{2.3}$$

Remark 2.23. Both in [9] and in [39] the authors define the motion by curvature introducing directly the Special Flow. This is not restrictive to get a short time existence result because a solution of the Special Flow as defined in Definition 2.22 induces a solution of the motion by curvature in the sense of Definition 2.18, as shown in Theorem 3.16 below. However, we will see that it is not easy to deduce *geometric uniqueness* of solutions to the motion by curvature from uniqueness of solutions to the Special Flow.

For the sake of presentation, we will often restrict to the motion by curvature of a *Triod* and we give the proofs in full details for this simple configuration. The adaptation to more general situations is easy, nevertheless we will carefully explain how to deal with it in the Appendix. To fix the precise notation we write now Definition 2.18, Definition 2.21, Definition 2.22 in the specific case of Triods.

Definition 2.24. A *Triod* $\mathbb{T} = \bigcup_{i=1}^3 \gamma^i([0,1])$ is a network composed of three regular C^1–curves $\gamma^i : [0,1] \to \mathbb{R}^d$ that intersect each other at the triple junction $O := \gamma^1(0) = \gamma^2(0) = \gamma^3(0)$. The other three endpoints of the curves $\gamma^i(1)$ with $i \in \{1,2,3\}$ coincide with three points $P^i \in \mathbb{R}^d$, that is, $P^i := \gamma^i(1)$. The Triod is called *regular* if it is a regular network.

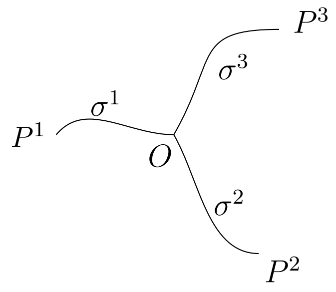

Figure 2: A regular Triod in $\mathbb{R}^2$.

Definition 2.25 (Solutions to the motion by curvature of a Triod)**.** Let $p \in (3, \infty)$ and $T > 0$. Let $\mathbb{T}_0$ be a geometrically admissible initial Triod with endpoints P^1, P^2, P^3. A time dependent family of Triods $(\mathbb{T}(t))$ is a *solution to the motion by curvature in* $[0, T]$ *with initial datum* $\mathbb{T}_0$ if there exists a collection of time dependent parametrisations

$$\gamma_n^i \in W_p^1(I_j; L_p((0,1); \mathbb{R}^d)) \cap L_p(I_j; W_p^2((0,1); \mathbb{R}^d)),$$

with $n \in \{0, \ldots, N\}$ for some $N \in \mathbb{N}$, $I_n := (a_n, b_n) \subset \mathbb{R}$, $a_n \leq a_{n+1}$, $b_n \leq b_{n+1}$, $a_n < b_n$ and $\bigcup_n (a_n, b_n) = (0, T)$ such that for all $n \in \{0, \ldots, N\}$ and $t \in I_n$, $\gamma_n(t) = \big(\gamma^1(t), \gamma^2(t), \gamma^3(t)\big)$ is a regular parametrisation of $\mathbb{T}(t)$. Moreover, each γ_n needs to satisfy the following system:

$$\begin{cases} \boldsymbol{P}^i \gamma_t^i(t,x) = \boldsymbol{\kappa}^i(t,x) & \text{motion by curvature,} \\ \gamma^i(t,1) = P^i & \text{fixed endpoints,} \\ \gamma^1(t,0) = \gamma^2(t,0) = \gamma^3(t,0) & \text{concurrency condition,} \\ \sum_{i=1}^3 \tau^i(t,0) = 0 & \text{angle condition,} \end{cases} \tag{2.4}$$

for almost every $t \in I_n, x \in (0,1)$ and for $i \in \{1,2,3\}$. Finally, we ask that $\gamma_n(a_n, [0,1]) = \mathbb{T}_0$ whenever $a_n = 0$.

Definition 2.26 (Admissible initial parametrisation)**.** Let $p \in (3, \infty)$. An *admissible initial parametrisation* for a Triod $\mathbb{T}_0$ is a triple $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ where $\bigcup_i \sigma^i([0,1]) = \mathbb{T}_0$, $\sigma^1(0) = \sigma^2(0) = \sigma^3(0)$ and $\sum_{i=1}^3 \frac{\sigma_x^i(0)}{|\sigma_x^i(0)|} = 0$ with σ^i regular and of class $W_p^{2-2/p}((0,1), \mathbb{R}^d)$.

Definition 2.27 (Solution of the Special Flow)**.** Let $T > 0$ and $p \in (3, \infty)$. Consider an admissible initial parametrisation $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ for a Triod $\mathbb{T}_0$ in $\mathbb{R}^d$ with $\sigma^i(1) = P^i \in \mathbb{R}^d$. Then we say that $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ is a *solution of the Special Flow in the time interval* $[0, T]$ *with initial datum* σ if

$$\gamma = \big(\gamma^1, \gamma^2, \gamma^3\big) \in \boldsymbol{E}_T = W_p^1((0,T); L_p((0,1); (\mathbb{R}^d)^3)) \cap L_p((0,T); W_p^2((0,1); (\mathbb{R}^d)^3)),$$

$|\gamma_x^i(t,x)| \neq 0$ for all $(t,x) \in [0,T] \times [0,1]$ and the following system is satisfied for $i \in \{1,2,3\}$ and for almost every $x \in (0,1)$, $t \in (0,T)$:

$$\begin{cases} \gamma_t^i(t,x) = \frac{\gamma_{xx}^i(t,x)}{|\gamma_x^i(t,x)|^2} & \text{Special Flow,} \\ \gamma^i(t,1) = P^i & \text{fixed endpoints,} \\ \gamma^1(t,0) = \gamma^2(t,0) = \gamma^3(t,0) & \text{concurrency condition,} \\ \sum_{i=1}^3 \frac{\gamma_x^i(t,0)}{|\gamma_x^i(t,0)|} = 0 & \text{angle condition,} \\ \gamma^i(0,x) = \sigma^i(x) & \text{initial datum.} \end{cases} \tag{2.5}$$

3 Existence and Uniqueness of the Motion by Curvature

3.1 Existence and uniqueness of the linearised Special Flow

For the moment we restrict to Triods. We refer to the Appendix for the generalizations needed in the case of more general networks.
We fix an admissible initial parametrisation $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ of a triod. Linearising the main equation of system (2.5) and the angle condition at $x = 0$ around the initial datum and considering the principal part of the respective linearisation we obtain

$$\gamma_t^i(t,x) - \frac{1}{|\sigma_x^i(x)|^2}\,\gamma_{xx}^i(t,x) = \left(\frac{1}{|\gamma_x^i(t,x)|^2} - \frac{1}{|\sigma_x^i(x)|^2}\right)\gamma_{xx}^i(t,x) \tag{3.1}$$

and

$$-\sum_{i=1}^{3}\left(\frac{\gamma_x^i}{|\sigma_x^i|} - \frac{\sigma_x^i\left\langle \gamma_x^i, \sigma_x^i\right\rangle}{|\sigma_x^i|^3}\right) = \sum_{i=1}^{3}\left(\left(\frac{1}{|\gamma_x^i|} - \frac{1}{|\sigma_x^i|}\right)\gamma_x^i + \frac{\sigma_x^i\left\langle \gamma_x^i, \sigma_x^i\right\rangle}{|\sigma_x^i|^3}\right), \tag{3.2}$$

where in (3.2) we have omitted the dependence of σ_x^i and γ_x^i on 0 and $(t,0)$, respectively. The concurrency and the fixed endpoints conditions are already linear and affine. We obtain the following linearised system for a general right hand side (f, η, b, ψ).

$$\begin{cases} \gamma_t^i(t,x) - \frac{1}{|\sigma_x^i(x)|^2}\,\gamma_{xx}^i(t,x) &= f^i(t,x)\,, & t\in(0,T)\,, x\in(0,1)\,, i\in\{1,2,3\}\,,\\ \gamma(t,1) &= \eta(t)\,, & t\in[0,T]\,,\\ \gamma^1(t,0) - \gamma^2(t,0) &= 0\,, & t\in[0,T]\,,\\ \gamma^2(t,0) - \gamma^3(t,0) &= 0\,, & t\in[0,T]\,,\\ -\sum_{i=1}^3\left(\frac{\gamma_x^i(t,0)}{|\sigma_x^i(0)|} - \frac{\sigma_x^i(0)\left\langle\gamma_x^i(t,0),\sigma_x^i(0)\right\rangle}{|\sigma_x^i(0)|^3}\right) &= b(t)\,, & t\in[0,T]\,,\\ \gamma(0,x) &= \psi(x)\,, & x\in[0,1]\,. \end{cases} \tag{3.3}$$

Definition 3.1 (Linear compatibility conditions)**.** Let $p \in (3,\infty)$. A function $\psi = (\psi^1, \psi^2, \psi^3)$ of class $W_p^{2-2/p}\left((0,1);(\mathbb{R}^d)^3\right)$ satisfies the *linear compatibility conditions* for system (3.3) with respect to given functions $\eta \in W_p^{1-1/2p}((0,T);(\mathbb{R}^d)^3)$, $b \in W_p^{1/2-1/2p}((0,T);\mathbb{R}^d)$ if for $i,j \in \{1,2,3\}$ it holds $\psi^i(0) = \psi^j(0)$, $\psi^i(1) = \eta^i(0)$ and

$$-\sum_{i=1}^{3}\left(\frac{\psi_x^i(0)}{|\sigma_x^i(0)|} - \frac{\sigma_x^i(0)\left\langle\psi_x^i(0),\sigma_x^i(0)\right\rangle}{|\sigma_x^i(0)|^3}\right) = b(0)\,.$$

We want to show that system (3.3) admits a unique solution $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ in $\boldsymbol{E}_T$. The result follows from the classical theory for linear parabolic systems by Solonnikov [44] provided that the system is parabolic and that the *complementary conditions* hold (see [44, p. 11]). Both the parabolicity and the complementary (initial and boundary) conditions have been proven in [9] when the ambient space is $\mathbb{R}^2$. Parabolicity does not depend on the dimension of the ambient space. We underline the fact that to prove the complementary conditions we follow a different and simpler strategy with respect to [9]. Our proof is based on the fact that the complementary conditions at the boundary follow from the *Lopatinskii–Shapiro condition* (see for instance [13, pages 11–15]).

Definition 3.2 (Lopatinskii–Shapiro condition)**.** Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ be arbitrary. The Lopatinskii–Shapiro condition for system (3.3) is satisfied at the triple junction if every solution $\varrho = (\varrho^1, \varrho^2, \varrho^3) \in C^2([0,\infty), (\mathbb{C}^2)^3)$ to

$$\begin{cases} \lambda \varrho^i(x) - \frac{1}{|\sigma_x^i(0)|^2} \varrho_{xx}^i(x) &= 0\,, \quad x \in [0,\infty)\,,\ i \in \{1,2,3\}\,, \\ \varrho^1(0) - \varrho^2(0) &= 0\,, \\ \varrho^2(0) - \varrho^3(0) &= 0\,, \\ \sum_{i=1}^3 \left(\frac{\varrho_x^i(0)}{|\sigma_x^i(0)|} - \frac{\sigma_x^i(0)\langle \varrho_x^i(0), \sigma_x^i(0)\rangle}{|\sigma_x^i(0)|^3} \right) &= 0 \end{cases} \tag{3.4}$$

which satisfies $\lim_{x\to\infty}|\varrho^i(x)| = 0$ is the trivial solution.

Similarly, the Lopatinskii–Shapiro condition for system (3.3) is satisfied at the fixed endpoints if every solution $\varrho = (\varrho^1, \varrho^2, \varrho^3) \in C^2([0,\infty), (\mathbb{C}^2)^3)$ to

$$\begin{cases} \lambda \varrho^i(x) - \frac{1}{|\sigma_x^i(0)|^2} \varrho_{xx}^i(x) &= 0\,, \quad x \in [0,\infty)\,,\ i \in \{1,2,3\}\,, \\ \varrho^i(0) &= 0\,, \quad i \in \{1,2,3\} \end{cases}$$

which satisfies $\lim_{x\to\infty}|\varrho^i(x)| = 0$ is the trivial solution.

Lemma 3.3. *The Lopatinskii–Shapiro condition is satisfied.*

Proof. We first check the condition at the triple junction. Let ϱ be a solution to (3.4) satisfying $\lim_{x\to\infty}|\varrho^i(x)| = 0$. Due to the specific exponential representation of solutions to the linear system (3.4), one observes that also the derivatives of ϱ^i up to order two decay to zero as x tends to infinity. We multiply $\lambda\varrho^i(x) - \frac{1}{|\sigma_x^i(0)|^2}\varrho_{xx}^i(x) = 0$ by $|\sigma_x^i(0)|\boldsymbol{P}^i\overline{\varrho}^i(x)$ with $\boldsymbol{P}^i := \mathrm{Id} - \sigma_s^i(0) \otimes \sigma_s^i(0)$, then we integrate and sum. Note that in $\boldsymbol{P}^i$ we only want to project the real part of a function. So, $\boldsymbol{P}^i$ is the identity on the complex part and as consequence we get that

$$\overline{\boldsymbol{P}^i\varrho^i} = \boldsymbol{P}^i\overline{\varrho^i}\,, \quad \overline{\boldsymbol{P}^i\varrho_x^i} = \boldsymbol{P}^i\overline{\varrho_x^i}\,,$$

and with the fact that $\sigma_s^i(0) \cdot \boldsymbol{P}^i\overline{\rho}^i = 0 = \sigma_s^i(0) \cdot \boldsymbol{P}^i\overline{\rho}_x^i$ it follows that

$$\varrho^i \cdot \boldsymbol{P}^i\overline{\varrho}^i = \boldsymbol{P}^i\varrho^i \cdot \overline{\boldsymbol{P}^i\varrho^i} = |\boldsymbol{P}^i\varrho^i|^2\,, \quad \varrho_x^i \cdot \boldsymbol{P}^i\overline{\varrho}_x^i = \boldsymbol{P}^i\varrho_x^i \cdot \overline{\boldsymbol{P}^i\varrho_x^i} = |\boldsymbol{P}^i\varrho_x^i|^2\,.$$

Using that the boundary conditions can be written as $\varrho^1(0) = \varrho^2(0) = \varrho^3(0)$ and

$$\sum_{i=1}^3 \boldsymbol{P}^i \left(\frac{\varrho_x^i(0)}{|\sigma_x^i(0)|} \right) = \sum_{i=1}^3 \frac{\varrho_x^i(0)}{|\sigma_x^i(0)|} - \frac{\sigma_x^i(0)\,\langle \varrho_x^i(0), \sigma_x^i(0)\rangle}{|\sigma_x^i(0)|^3} = 0\,,$$

we obtain

$$
\begin{aligned}
0 &= \sum_{i=1}^{3} \int_0^\infty \lambda |\sigma_x^i(0)| |\boldsymbol{P}^i(\varrho^i(x))|^2 - \frac{1}{|\sigma_x^i(0)|} \left\langle \varrho_{xx}^i(x), \boldsymbol{P}^i \overline{\varrho}^i(x) \right\rangle \,\mathrm{d}x \\
&= \sum_{i=1}^{3} \int_0^\infty \lambda |\sigma_x^i(0)| |\boldsymbol{P}^i(\varrho^i(x))|^2 + \frac{|\boldsymbol{P}^i(\varrho_x^i(x))|^2}{|\sigma_x^i(0)|} \,\mathrm{d}x - \sum_{i=1}^{3} \frac{1}{|\sigma_x^i(0)|} \left\langle \boldsymbol{P}^i \varrho_x^i(0), \overline{\varrho}^i(0) \right\rangle \\
&= \sum_{i=1}^{3} \int_0^\infty \lambda |\sigma_x^i(0)| |\boldsymbol{P}^i(\varrho^i(x))|^2 + \frac{|\boldsymbol{P}^i(\varrho_x^i(x))|^2}{|\sigma_x^i(0)|} \,\mathrm{d}x - \left\langle \overline{\varrho}^1(0), \sum_{i=1}^{3} \boldsymbol{P}^i \left(\frac{\varrho_x^i(0)}{|\sigma_x^i(0)|} \right) \right\rangle \\
&= \sum_{i=1}^{3} \int_0^\infty \lambda |\sigma_x^i(0)| |\boldsymbol{P}^i(\varrho^i(x))|^2 + \frac{|\boldsymbol{P}^i(\varrho_x^i(x))|^2}{|\sigma_x^i(0)|} \,\mathrm{d}x \,.
\end{aligned}
$$

As a consequence we get that $\boldsymbol{P}^i(\varrho^i(x)) = 0$ for all $x \in [0, \infty)$ and $i \in \{1, 2, 3\}$ and in particular $\boldsymbol{P}^i(\varrho^1(0)) = 0$ for all $i \in \{1, 2, 3\}$. As the orthogonal complements of $\sigma_x^i(0)$ with $i \in \{1, 2, 3\}$ span all $\mathbb{R}^d$, we conclude that $\varrho^i(0) = 0$ for all $i \in \{1, 2, 3\}$. Repeating the argument and testing the motion equation by $|\sigma_x^i(0)| \langle \overline{\varrho}^i(x), \sigma_s^i(0) \rangle \sigma_s^i(0)$ we can conclude that $\varrho^i(x) = 0$ for every $x \in [0, \infty)$. Indeed, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{3} \lambda |\sigma_x^i(0)| \int_0^\infty |\left\langle \varrho^i(x), \sigma_s^i(0) \right\rangle|^2 \,\mathrm{d}x + \sum_{i=1}^{3} \frac{1}{|\sigma_x^i(0)|} \int_0^\infty |\left\langle \varrho_x^i(x), \sigma_s^i(0) \right\rangle|^2 \,\mathrm{d}x \\
&+ \sum_{i=1}^{3} \frac{1}{|\sigma_x^i(0)|} \left\langle \overline{\varrho}^i(0), \sigma_s^i(0) \right\rangle \left\langle \varrho_x^i(0), \sigma_s^i(0) \right\rangle = 0 \,. \qquad (3.5)
\end{aligned}
$$

This time the boundary condition vanishes since we get $\varrho^i(0) = 0$ from the previous step. Taking again the real part of (3.5) we can conclude that $\left\langle \varrho^i(x), \sigma_s^i(0) \right\rangle = 0$ for all $x \in [0, \infty)$. Hence $\varrho^i(x) = 0$ for every $x \in [0, \infty)$ as desired.
The condition at the fixed endpoints follows in exactly the same way using the boundary condition $\varrho^i(0) = 0$. ☐

Given $T > 0$ we introduce the spaces

$$
\begin{aligned}
\mathbb{E}_T &:= \left\{ \gamma \in \boldsymbol{E}_T \,, \gamma^1(t,0) = \gamma^2(t,0) = \gamma^3(t,0) \text{ for } i \in \{1,2,3\}, t \in [0,T] \right\}, \\
\mathbb{F}_T &:= \Big\{ (f, \eta, 0, b, \psi) \text{ with } f \in L_p((0,T); L_p((0,1); (\mathbb{R}^d)^3)),\ \eta \in W_p^{1-1/2p}((0,T); (\mathbb{R}^d)^3)\,, \\
&\qquad 0 \in W_p^{1-1/2p}((0,T); \mathbb{R}^{2n})\,, b \in W_p^{1/2-1/2p}((0,T); \mathbb{R}^d)\,,\ \psi \in W_p^{2-2/p}((0,1); (\mathbb{R}^d)^3) \\
&\qquad \text{such that the linear compatibility conditions in Definition 3.1 hold} \Big\}.
\end{aligned}
$$

Theorem 3.4. *Let $p \in (3, \infty)$. For every $T > 0$ system (3.3) has a unique solution $\gamma \in \mathbb{E}_T$ provided that $f \in L_p((0,T); L_p((0,1); (\mathbb{R}^d)^3))$, $\eta \in W_p^{1-1/2p}((0,T); (\mathbb{R}^d)^3)$ $b \in W_p^{1/2-1/2p}((0,T); \mathbb{R}^d)$ and $\psi \in W_p^{2-2/p}((0,1); (\mathbb{R}^d)^3)$ fulfil the linear compatibility conditions given in Definition 3.1. Moreover, there exists a constant $C = C(T) > 0$ such that the following estimate holds:*

$$
\|\gamma\|_{\boldsymbol{E}_T} \le C \left(\|f\|_{L_p((0,T);L_p((0,1)))} + \|\eta\|_{W_p^{1-1/2p}((0,T))} + \|b\|_{W_p^{1/2-1/2p}((0,T))} + \|\psi\|_{W_p^{2-2/p}((0,1))} \right) .
$$

Proof. This follows from [44, Theorem 5.4]. ☐

As explained in the Appendix, repeating the previous arguments and applying again [44, Theorem 5.4] one gets the more general:

Theorem 3.5. *Let $p \in (3,\infty)$. For every $T > 0$ system (A.3) has a unique solution $\gamma \in \mathbb{E}_T$ provided that $f \in L_p((0,T); L_p((0,1);(\mathbb{R}^d)^m))$, $\eta \in W_p^{1-1/2p}((0,T);(\mathbb{R}^d)^\ell)$ $b \in W_p^{1/2-1/2p}((0,T);(\mathbb{R}^d)^q)$ and $\psi \in W_p^{2-2/p}((0,1);(\mathbb{R}^d)^m)$ fulfil the linear compatibility conditions given in Definition A.1. Moreover, there exists a constant $C = C(T) > 0$ such that the following estimate holds:*

$$\|\gamma\|_{\boldsymbol{E}_T} \leq C\left(\|f\|_{L_p((0,T);L_p((0,1)))} + \|\eta\|_{W_p^{1-1/2p}((0,T))} + \|b\|_{W_p^{1/2-1/2p}((0,T))} + \|\psi\|_{W_p^{2-2/p}((0,1))}\right).$$

Theorem 3.5 implies in particular that the linear operator $L_T : \mathbb{E}_T \to \mathbb{F}_T$ defined by

$$L_T(\gamma) = \begin{pmatrix} \left(\gamma_t^i - \frac{\gamma_{xx}^i}{|\sigma_x^i|^2}\right)_{i\in\{1,2,3\}} \\ \gamma_{|x=1} \\ \left(\gamma^1_{|x=0} - \gamma^2_{|x=0}, \gamma^2_{|x=0} - \gamma^3_{|x=0}\right) \\ -\sum_{i=1}^3 \left(\frac{\gamma_x^i}{|\sigma_x^i|} - \frac{\sigma_x^i\langle\gamma_x^i,\sigma_x^i\rangle}{|\sigma_x^i|^3}\right)_{|x=0} \\ \gamma_{|t=0} \end{pmatrix}$$

is a continuous isomorphism.
Corollary 2.7 and Lemma 2.9 imply that for every positive T the spaces $\mathbb{E}_T$ and $\mathbb{F}_T$ endowed with the norms

$$|||\gamma|||_{\boldsymbol{E}_T} := |||\gamma|||_{W_p^{1,2}\left((0,T)\times(0,1);(\mathbb{R}^d)^3\right)} = \|\gamma\|_{W_p^{1,2}\left((0,T)\times(0,1);(\mathbb{R}^d)^3\right)} + \|\gamma(0)\|_{W_p^{2-2/p}\left((0,1);(\mathbb{R}^d)^3\right)}$$

and

$$\begin{aligned} |||(f,\eta,0,b,\psi)|||_{\mathbb{F}_T} &:= \|f\|_{L_p\left((0,T);L_p((0,1);(\mathbb{R}^d)^3)\right)} + |||\eta|||_{W_p^{1-1/2p}\left((0,T);(\mathbb{R}^d)^3\right)} \\ &\quad + |||b|||_{W_p^{1/2-1/2p}\left((0,T);\mathbb{R}^d\right)} + \|\psi\|_{W_p^{2-2/p}\left((0,1);(\mathbb{R}^d)^3\right)}, \end{aligned}$$

respectively, are Banach spaces. Given a linear operator $A : \mathbb{F}_T \to \mathbb{E}_T$ we let

$$|||A|||_{\mathcal{L}(\mathbb{F}_T,\mathbb{E}_T)} := \sup\{|||A(f,\eta,0,b,\psi)|||_{\boldsymbol{E}_T} : (f,\eta,0,b,\psi) \in \mathbb{F}_T, |||(f,\eta,0,b,\psi)|||_{\mathbb{F}_T} \leq 1\}.$$

Lemma 3.6. *Let $p \in (3,\infty)$. For all $T_0 > 0$ there exists a constant $c(T_0,p)$ such that*

$$\sup_{T\in(0,T_0]} |||L_T^{-1}|||_{\mathcal{L}(\mathbb{F}_T,\mathbb{E}_T)} \leq c(T_0,p).$$

Proof. Let $T \in (0,T_0]$ be arbitrary, $(f,\eta,0,b,\psi) \in \mathbb{F}_T$ and $E_{T_0}b := (Eb)_{|(0,T_0)}$, $E_{T_0}\eta := (E\eta)_{|(0,T_0)}$ where E is the extension operator defined in Lemma 2.10. Extending f by 0 to $E_{T_0}f \in L_p\left((0,T_0); L_p\left((0,1);(\mathbb{R}^d)^3\right)\right)$ we observe that $(E_{T_0}f, E_{T_0}\eta, 0, E_{T_0}b, \psi)$ lies in $\mathbb{F}_{T_0}$. As L_T and L_{T_0} are isomorphisms, there exist unique $\gamma \in \mathbb{E}_T$ and $\widetilde{\gamma} \in \mathbb{E}_{T_0}$ such that $L_T\gamma = (f,\eta,0,b,\psi)$ and $L_{T_0}\widetilde{\gamma} = (E_{T_0}f, E_{T_0}\eta, 0, E_{T_0}b, \psi)$ satisfying

$$L_T\gamma = (f,\eta,0,b,\psi) = (E_{T_0}f, E_{T_0}\eta, 0, E_{T_0}b, \psi)_{|(0,T)} = (L_{T_0}\widetilde{\gamma})_{|(0,T)} = L_T\left(\widetilde{\gamma}_{|(0,T)}\right)$$

and thus $\gamma = \widetilde{\gamma}_{|(0,T)}$. Using Theorem 3.5, Lemma 2.10 and the equivalence of norms on $\boldsymbol{E}_{T_0}$ this implies

$$\begin{aligned}&\left|\!\left|\!\left|L_T^{-1}(f,\eta,0,b,\psi)\right|\!\right|\!\right|_{\boldsymbol{E}_T} = \left|\!\left|\!\left|\left(L_{T_0}^{-1}(E_{T_0}f,E_{T_0}\eta,0,E_{T_0}b,\psi)\right)_{|(0,T)}\right|\!\right|\!\right|_{\boldsymbol{E}_T}\\&\leq \left|\!\left|\!\left|L_{T_0}^{-1}(E_{T_0}f,E_{T_0}\eta,0,E_{T_0}b,\psi)\right|\!\right|\!\right|_{\boldsymbol{E}_{T_0}} \leq c(T_0,p)\left\|L_{T_0}^{-1}(E_{T_0}f,E_{T_0}\eta,0,E_{T_0}b,\psi)\right\|_{\boldsymbol{E}_{T_0}}\\&\leq c(T_0,p)\left\|(E_{T_0}f,E_{T_0}\eta,0,E_{T_0}b,\psi)\right\|_{\mathbb{F}_{T_0}} \leq c(T_0,p)\left|\!\left|\!\left|(f,\eta,0,b,\psi)\right|\!\right|\!\right|_{\mathbb{F}_T}.\end{aligned}$$

□

3.2 Existence and uniqueness of the Special Flow

Given M positive we introduce the notation

$$\overline{B_M} := \left\{\gamma \in \boldsymbol{E}_T : |\!|\!|\gamma|\!|\!|_{\boldsymbol{E}_T} \leq M\right\}.$$

This section is devoted to the proof of the following:

Theorem 3.7. *Let $p \in (3,\infty)$ and let $\sigma = (\sigma^1,\dots,\sigma^m)$ be an admissible initial parametrisation. There exists a positive radius M and a positive time T such that the system (2.3) has a unique solution $\mathcal{E}\sigma$ in $\boldsymbol{E}_T \cap \overline{B_M}$.*

We prove the theorem for a Triod, in particular we have an admissible parametrisation $\sigma = (\sigma^1,\sigma^2,\sigma^3)$ and we consider system (2.5). See the Appendix for the generalization to networks with more complicated structure.

Given an admissible initial parametrisation σ and $T > 0$ we consider the complete metric spaces

$$\begin{aligned}&\mathbb{E}_T^\sigma := \{\gamma \in \mathbb{E}_T \text{ such that } \gamma_{|t=0} = \sigma \text{ and } \gamma_{|x=1} = \sigma(1)\},\\&\mathbb{F}_T^\sigma := \mathbb{F}_T \cap \left(L_p\left((0,T);L_p\left((0,1);(\mathbb{R}^d)^3\right)\right) \times \{\sigma(1)\} \times \{0\} \times W_p^{1/2-1/2p}\left((0,T);\mathbb{R}^d\right) \times \{\sigma\}\right).\end{aligned}$$

Lemma 3.8. *Let $p \in (3,\infty)$, $T > 0$ and $\sigma = (\sigma^1,\sigma^2,\sigma^3)$ be an admissible initial parametrisation. Then the space $\mathbb{E}_T^\sigma$ is non-empty.*

Proof. As σ is an admissible initial parametrisation, one easily checks that $f \equiv 0$, $\eta \equiv \sigma(1)$, $b \equiv 0$ and $\psi \equiv \sigma$ is an admissible right hand side for system (3.3). In other words, $(0,\sigma(1),0,0,\sigma) \in \mathbb{F}_T$ and hence Theorem 3.5 yields the existence of $\varrho \in \mathbb{E}_T$ with $L_T\varrho = (0,\sigma(1),0,0,\sigma)$. In particular, $\varrho_{|t=0} = \sigma$ and $\varrho_{|x=1} = \sigma(1)$ which gives $\varrho \in \mathbb{E}_T^\sigma$. □

Lemma 3.9. *Let $p \in (3,\infty)$ and*

$$\boldsymbol{c} := \frac{1}{2}\min_{i\in\{1,2,3\},x\in[0,1]} |\sigma_x^i(x)|.$$

Given $T_0 > 0$ and $M > 0$ there exists a time $\widetilde{T}(\boldsymbol{c},M) \in (0,T_0]$ such that for all $\gamma \in \mathbb{E}_T^\sigma \cap \overline{B_M}$ with $T \in [0,\widetilde{T}(\boldsymbol{c},M)]$ it holds

$$\inf_{x\in[0,1],t\in[0,T],i\in\{1,2,3\}} \left|\gamma_x^i(t,x)\right| \geq \boldsymbol{c}.$$

In particular, the curves $\gamma^i(t)$ are regular for all $t \in [0,T]$.

Proof. Let $p \in (3, \infty)$, $\theta \in \left(\frac{1+1/p}{2-2/p}, 1\right)$ and $\delta \in (0, 1-1/p)$. By Theorem 2.12 there exists a constant $C(T_0, p, \theta, \delta) > 0$ such that for all $T \in (0, T_0]$ and all $\gamma \in \mathbb{E}_T^\sigma \cap \overline{B_M}$ with $\alpha := (1-\theta)(1-1/p-\delta)$ it holds

$$\|\gamma\|_{C^\alpha\left([0,T];C^1([0,1];(\mathbb{R}^d)^3)\right)} \leq C(T_0, p, \theta, \delta)\|\gamma\|_{\boldsymbol{E}_T} \leq C(T_0, p, \theta, \delta)M \,,$$

which implies in particular for all $t \in [0, T]$,

$$\|\gamma(t) - \sigma\|_{C^1([0,1];(\mathbb{R}^d)^3)} \leq T^\alpha C(T_0, p, \theta, \delta)M \,.$$

We let $\widetilde{T}(\boldsymbol{c}, M)$ be so small that $\widetilde{T}(\boldsymbol{c}, M)^\alpha C(T_0, p, \theta, \delta)M \leq \boldsymbol{c}$. Then it follows for all $\gamma \in \mathbb{E}_T^\sigma$ with $T \in (0, \widetilde{T}(\boldsymbol{c}, M))$,

$$\inf_{t\in[0,T],x\in[0,1]} |\gamma_x^i(t,x)| \geq \inf_{x\in[0,1]} |\sigma_x^i(x)| - \sup_{t\in[0,T],x\in[0,1]} |\gamma_x^i(t,x) - \gamma_x^i(0,x)| \geq \boldsymbol{c} \,.$$

□

Let us now define the operator N_T that encodes the non–linearity of our problem. The map $N_T : \mathbb{E}_T^\sigma \to \mathbb{F}_T^\sigma$ is given by $\gamma \mapsto \left(N_T^1(\gamma), \gamma_{|x=1}, 0, N_T^2(\gamma), \gamma_{|t=0}\right)$ where the two components N_T^1, N_T^2 are defined as

$$N_T^1 : \begin{cases} \mathbb{E}_T^\sigma & \to L_p((0,T); L_p((0,1); (\mathbb{R}^d)^3)) \,, \\ \gamma & \mapsto f(\gamma) \,, \end{cases}$$

$$N_T^2 : \begin{cases} \mathbb{E}_T^\sigma & \to W_p^{1/2-1/2p}((0,T); \mathbb{R}^d) \,, \\ \gamma & \mapsto b(\gamma) \end{cases}$$

with

$$f(\gamma)^i(t,x) := \left(\frac{1}{|\gamma_x^i(t,x)|^2} - \frac{1}{|\sigma_x^i(x)|^2}\right)\gamma_{xx}^i(t,x) \,,$$

$$b(\gamma)(t) := \sum_{i=1}^3 \left(\left(\frac{1}{|\gamma_x^i(t,0)|} - \frac{1}{|\sigma_x^i(0)|}\right)\gamma_x^i(t,0) + \frac{\sigma_x^i(0)\left\langle \gamma_x^i(t,0), \sigma_x^i(0)\right\rangle}{|\sigma_x^i(0)|^3}\right)$$

defined by the right hand side of (3.1) and (3.2), respectively.

Proposition 3.10. *Let $p \in (3, \infty)$ and M be positive. Then for all $T \in (0, \widetilde{T}(\boldsymbol{c}, M)]$ the map*

$$N_T : \mathbb{E}_T^\sigma \cap \overline{B_M} \to \mathbb{F}_T^\sigma \,, \quad N_T(\gamma) := \left(N_T^1(\gamma), \gamma_{|x=1}, 0, N_T^2(\gamma), \gamma_{|t=0}\right)$$

is well-defined.

Proof. Let $T \in (0, \widetilde{T}(\boldsymbol{c}, M)]$ and $\gamma \in \mathbb{E}_T^\sigma \cap \overline{B_M}$ be given. Lemma 3.9 implies

$$\begin{aligned}
&\left\| \left(\frac{1}{|\gamma_x^i|^2} - \frac{1}{|\sigma_x^i|^2}\right)\gamma_{xx}^i \right\|_{L_p((0,T);L_p((0,1);\mathbb{R}^d))}^p = \int_0^T\int_0^1 \left|\frac{1}{|\gamma_x^i|^2} - \frac{1}{|\sigma_x^i|^2}\right|^p |\gamma_{xx}^i|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C\left(\sup_{x\in[0,1],t\in[0,T]} \frac{1}{|\gamma_x^i|^{2p}} + \sup_{x\in[0,1]} \frac{1}{|\sigma_x^i|^{2p}}\right)\int_0^T\int_0^1 |\gamma_{xx}^i|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C(\boldsymbol{c})\|\gamma_{xx}^i\|_{L_p((0,T);L_p((0,1);\mathbb{R}^d))}^p \leq C(\boldsymbol{c}, M) < \infty \,.
\end{aligned}$$

We now show that $N_T^2(\gamma)$ lies in $W_p^{1/2-1/2p}\left((0,T);\mathbb{R}^d\right)$. Let $h:\mathbb{R}^d\to\mathbb{R}^d$ be a smooth function such that $h(p)=\frac{p}{|p|}$ for all $p\in\mathbb{R}^d\setminus B_{c/2}(0)$. Then one observes that for all $t\in[0,T]$

$$b(\gamma)(t)=\sum_{i=1}^{3} h(\gamma_x^i(t))-(Dh)\,(\sigma_x^i)\gamma_x^i(t) \tag{3.6}$$

where we omitted the evaluation in $x=0$ to ease notation. Each term in the sum can be expressed as

$$\begin{aligned} h(\gamma_x^i(t))-(Dh)\,(\sigma_x^i)\gamma_x^i(t) &= \int_0^1 (Dh)(\xi\gamma_x^i(t)+(1-\xi)\sigma_x^i)\mathrm{d}\xi\,(\gamma_x^i(t)-\sigma_x^i)\\ &\quad-(Dh)(\sigma_x^i)\left(\gamma_x^i(t)-\sigma_x^i\right)+h(\sigma_x^i)-Dh(\sigma_x^i)\sigma_x^i\,. \end{aligned}$$

All terms that are constant in t are smooth in t and by Lemma 2.5 we have

$$t\mapsto\gamma_x^i(t,0)\in W_p^{1/2-1/2p}\left((0,T);\mathbb{R}^d\right)\,.$$

As $W_p^{1/2-1/2p}\left((0,T);\mathbb{R}\right)$ is a Banach algebra according to Proposition 2.6, it only remains to show

$$t\mapsto\int_0^1 (Dh)(\xi\gamma_x^i(t,0)+(1-\xi)\sigma_x^i(0))\mathrm{d}\xi\in W_p^{1/2-1/2p}\left((0,T);\mathbb{R}^{n\times n}\right)$$

which follows from the second assertion in Proposition 2.6. Observe that $\gamma_{|x=1}=\sigma(1)$ and $\gamma_{|t=0}=\sigma$ by definition of $\mathbb{E}_T^\sigma$. As

$$N_T^2(\gamma)_{|t=0}=\sum_{i=1}^{3}\frac{\sigma_x^i(0)}{|\sigma_x^i(0)|}=0=-\sum_{i=1}^{3}\left(\frac{\sigma_x^i(0)}{|\sigma_x^i(0)|}-\frac{\sigma_x^i(0)\left\langle\sigma_x^i(0),\sigma_x^i(0)\right\rangle}{|\sigma_x^i(0)|^3}\right)$$

and as $\sigma^i(0)=\sigma^j(0)$, $\sigma^i(1)=\gamma^i(0,1)$, we may conclude that

$$\left(N_T^1(\gamma),\gamma_{|x=1},0,N_T^2(\gamma),\gamma_{|t=0}\right)=\left(N_T^1(\gamma),\sigma(1),0,N_T^2(\gamma),\sigma\right)\in\mathbb{F}_T^\sigma.$$

□

Corollary 3.11. *Let $p\in(3,\infty)$ and M be positive. Then for all $T\in(0,\widetilde{T}(\boldsymbol{c},M)]$ the map*

$$K_T:\mathbb{E}_T^\sigma\cap\overline{B_M}\to\mathbb{E}_T^\sigma\,,\quad K_T:=L_T^{-1}N_T$$

is well-defined.

Proof. Let $T\in(0,\widetilde{T}(\boldsymbol{c},M)]$ and $\gamma\in\mathbb{E}_T^\sigma\cap\overline{B_M}$. By the previous proof we have

$$N_T(\gamma)=\left(N_T^1(\gamma),\gamma_{|x=1},0,N_T^2(\gamma),\gamma_{|t=0}\right)\in\mathbb{F}_T^\sigma\subset\mathbb{F}_T$$

and thus in particular

$$K_T(\gamma)=L_T^{-1}(N_T(\gamma))\in\mathbb{E}_T\,.$$

To verify that $K_T(\gamma)$ lies in $\mathbb{E}_T^\sigma$ we observe that

$$\begin{aligned} K_T(\gamma)_{|t=0}&=N_T(\gamma)_5=\gamma_{|t=0}=\sigma\,,\\ K_T(\gamma)_{|x=1}&=N_T(\gamma)_2=\gamma_{|x=1}=\sigma(1)\,. \end{aligned}$$

□

Proposition 3.12. *Let $p \in (3,\infty)$ and M be positive. There exists $T(\boldsymbol{c},M) \in (0,\widetilde{T}(\boldsymbol{c},M)]$ such that for all $T \in (0,T(\boldsymbol{c},M)]$ the map $K_T : \mathbb{E}_T^\sigma \cap \overline{B_M} \to \mathbb{E}_T^\sigma$ is a contraction.*

Proof. Let $T \in (0,\widetilde{T}(\boldsymbol{c},M)]$ and $\gamma,\widetilde{\gamma} \in \mathbb{E}_T^\sigma \cap \overline{B_M}$ be fixed. We begin by estimating

$$\|N_T^1(\gamma) - N_T^1(\widetilde{\gamma})\|_{L_p\left((0,T);L_p\left((0,1);(\mathbb{R}^d)^3\right)\right)} = \|f(\gamma) - f(\widetilde{\gamma})\|_{L_p\left((0,T);L_p\left((0,1);(\mathbb{R}^d)^3\right)\right)}\,.$$

The i-th component of $f(\gamma) - f(\widetilde{\gamma})$ is given by

$$\begin{aligned}
&\left(\frac{1}{|\gamma_x^i|^2} - \frac{1}{|\sigma_x^i|^2}\right)(\gamma_{xx}^i - \widetilde{\gamma}_{xx}^i) + \left(\frac{1}{|\gamma_x^i|^2} - \frac{1}{|\widetilde{\gamma}_x^i|^2}\right)\widetilde{\gamma}_{xx}^i \\
&= \left(\frac{1}{|\gamma_x^i|^2|\sigma_x^i|} + \frac{1}{|\gamma_x^i||\sigma_x^i|^2}\right)\left(|\sigma_x^i| - |\gamma_x^i|\right)(\gamma_{xx}^i - \widetilde{\gamma}_{xx}^i) \\
&\quad + \left(\frac{1}{|\gamma_x^i|^2|\widetilde{\gamma}_x^i|} + \frac{1}{|\gamma_x^i||\widetilde{\gamma}_x^i|^2}\right)\left(|\widetilde{\gamma}_x^i| - |\gamma_x^i|\right)\widetilde{\gamma}_{xx}^i\,.
\end{aligned}$$

Lemma 3.9 implies

$$\sup_{t\in[0,T],x\in[0,1]}\left|\frac{1}{|\gamma_x^i|^2|\sigma_x^i|} + \frac{1}{|\gamma_x^i||\sigma_x^i|^2}\right| \leq C(\boldsymbol{c}) < \infty\,,$$

and

$$\sup_{t\in[0,T],x\in[0,1]}\left|\frac{1}{|\gamma_x^i|^2|\widetilde{\gamma}_x^i|} + \frac{1}{|\gamma_x^i||\widetilde{\gamma}_x^i|^2}\right| \leq C(\boldsymbol{c}) < \infty\,.$$

Hence we obtain

$$\begin{aligned}
&\left\|f(\gamma)^i - f(\widetilde{\gamma})^i\right\|_{L_p(0,T;L_p\left((0,1);(\mathbb{R}^d)^3\right))} \\
&\leq C(\boldsymbol{c})\left(\left\|\left(|\sigma_x^i| - |\gamma_x^i|\right)(\gamma_{xx}^i - \widetilde{\gamma}_{xx}^i)\right\|_{L_p\left((0,T);L_p(0,1);\mathbb{R}^d\right)} + \left\|\left(|\widetilde{\gamma}_x^i| - |\gamma_x^i|\right)\widetilde{\gamma}_{xx}^i\right\|_{L_p\left((0,T);L_p(0,1);\mathbb{R}^d\right)}\right) \\
&\leq C(\boldsymbol{c})\Bigg(\sup_{t\in[0,T],x\in[0,1]}\left||\sigma_x^i(x)| - |\gamma_x^i(t,x)|\right|\left\|\gamma_{xx}^i - \widetilde{\gamma}_{xx}^i\right\|_{L_p\left((0,T);L_p(0,1);\mathbb{R}^d\right)} \\
&\qquad + \sup_{t\in[0,T],x\in[0,1]}\left||\widetilde{\gamma}_x^i(t,x)| - |\gamma_x^i(t,x)|\right|\left\|\widetilde{\gamma}_{xx}^i\right\|_{L_p\left((0,T);L_p((0,1);\mathbb{R}^d)\right)}\Bigg) \\
&\leq C(\boldsymbol{c})\sup_{t\in[0,T],x\in[0,1]}\left|\sigma_x^i(x) - \gamma_x^i(t,x)\right| \|\!|\gamma - \widetilde{\gamma}|\!\|_{\boldsymbol{E}_T} \\
&\quad + C(\boldsymbol{c})\sup_{t\in[0,T],x\in[0,1]}\left|\widetilde{\gamma}_x^i(t,x) - \gamma_x^i(t,x)\right| \|\!|\widetilde{\gamma}|\!\|_{\boldsymbol{E}_T}\,.
\end{aligned}$$

Let $\theta \in \left(\frac{1+1/p}{2-2/p},1\right)$, $\delta \in (0,1-1/p)$ be fixed and define $\alpha := (1-\theta)(1-1/p-\delta)$. Theorem 2.12 implies

$$\begin{aligned}
&\sup_{t\in[0,T],x\in[0,1]}\left|\sigma_x^i(x) - \gamma_x^i(t,x)\right| = \sup_{t\in[0,T]}\left\|\gamma_x^i(0) - \gamma_x^i(t)\right\|_{C([0,1];\mathbb{R}^d)} \\
&\leq \sup_{t\in[0,T]}\left\|\gamma^i(t) - \gamma^i(0)\right\|_{C^1([0,1];\mathbb{R}^d)} \leq \sup_{t\in[0,T]} t^\alpha\left\|\gamma^i\right\|_{C^\alpha\left([0,T];C^1([0,1];\mathbb{R}^d)\right)} \\
&\leq T^\alpha\left\|\gamma^i\right\|_{C^\alpha\left([0,T];C^1([0,1];\mathbb{R}^d)\right)} \leq T^\alpha C(T_0,p,\theta,\delta)\|\!|\gamma|\!\|_{\boldsymbol{E}_T} \leq C(M)T^\alpha\,.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}\sup_{t\in[0,T],x\in[0,1]}\left|\widetilde{\gamma}^i_x(t,x)-\gamma^i_x(t,x)\right| &= \sup_{t\in[0,T],x\in[0,1]}\left|\left(\widetilde{\gamma}^i_x-\gamma^i_x\right)(t,x)-\left(\widetilde{\gamma}^i_x-\gamma^i_x\right)(0,x)\right|\\ &\leq \sup_{t\in[0,T]}\left\|\left(\widetilde{\gamma}^i-\gamma^i\right)(t)-\left(\widetilde{\gamma}^i-\gamma^i\right)(0)\right\|_{C^1([0,1];\mathbb{R}^d)}\\ &\leq \sup_{t\in[0,T]} t^\alpha\left\|\widetilde{\gamma}^i-\gamma^i\right\|_{C^\alpha\left([0,T];C^1([0,1];\mathbb{R}^d)\right)} \leq CT^\alpha \|\widetilde{\gamma}-\gamma\|_{\boldsymbol{E}_T}\,.\end{aligned}$$

This allows us to conclude

$$\|f(\gamma)-f(\widetilde{\gamma})\|_{L_p\left((0,T);L_p((0,1);(\mathbb{R}^d)^3)\right)} \leq C(\boldsymbol{c},M)T^\alpha \|\gamma-\widetilde{\gamma}\|_{\boldsymbol{E}_T}\,.$$

We proceed by estimating

$$\left\|N^2_T(\gamma)-N^2_T(\widetilde{\gamma})\right\|_{W_p^{1/2-1/2p}((0,T);\mathbb{R}^d)} = \|b(\gamma)-b(\widetilde{\gamma})\|_{W_p^{1/2-1/2p}((0,T);\mathbb{R}^d)}\,.$$

Let $T\in(0,\widetilde{T}(\boldsymbol{c},M)]$ be fixed and $h:\mathbb{R}^d\to\mathbb{R}^d$ be a smooth function such that $h(p)=\frac{p}{|p|}$ on $\mathbb{R}^d\setminus B_{\boldsymbol{c}/2}(0)$. As for all $t\in[0,T]$ and $\eta\in\mathbb{E}^\sigma_T\cap\overline{B_M}$,

$$|\eta^i_x(t,0)|\geq \boldsymbol{c}\,,$$

we may conclude that for all $\gamma,\widetilde{\gamma}\in\mathbb{E}^\sigma_T\cap\overline{B_M}$, the function

$$t\mapsto g^i(t):=\int_0^1 (Dh)\left(\xi\gamma^i_x(t,0)+(1-\xi)\widetilde{\gamma}^i_x(t,0)\right)\mathrm{d}\xi$$

lies in $W_p^{1/2-1/2p}(0,T;\mathbb{R}^{n\times n})$. To ease notation we let $s:=1/2-1/2p$. Observe that $g^i(0)=(Dh)(\sigma^i_x(0))$ and thus using identity (3.6) we obtain

$$b(\gamma)(t)-b(\widetilde{\gamma})(t)=\sum_{i=1}^3\left(g^i(t)-g^i(0)\right)\left(\gamma^i_x(t,0)-\widetilde{\gamma}^i_x(t,0)\right)\,.$$

Using the product estimate in Proposition 2.6 we obtain

$$\begin{aligned}\|b(\gamma)-b(\widetilde{\gamma})\|_{W^s_p((0,T);\mathbb{R}^d)} &\leq \sum_{i=1}^3\left\|\left(g^i-g^i(0)\right)\left(\gamma^i_x(\cdot,0)-\widetilde{\gamma}^i_x(\cdot,0)\right)\right\|_{W^s_p((0,T);\mathbb{R}^d)}\\ &\leq \sum_{i=1}^3\left\|g^i-g^i(0)\right\|_{C([0,T];\mathbb{R}^{n\times n})}\left\|\gamma^i_x(\cdot,0)-\widetilde{\gamma}^i_x(\cdot,0)\right\|_{W^s_p(0,T;\mathbb{R}^d)}\\ &\quad+\left\|g^i-g^i(0)\right\|_{W^s_p(0,T;\mathbb{R}^{n\times n})}\left\|\gamma^i_x(\cdot,0)-\widetilde{\gamma}^i_x(\cdot,0)\right\|_{C([0,T];\mathbb{R}^d)}\,.\end{aligned}$$

As $s-\frac{1}{p}>0$ due to $p\in(3,\infty)$ there exists $\beta\in(0,1)$ such that

$$W^s_p\left(0,T;\mathbb{R}^d\right)\hookrightarrow C^\beta\left([0,T];\mathbb{R}^d\right)$$

with embedding constant $C(s,p)$. This implies in particular

$$\sup_{t\in[0,T]}|g^i(t)-g^i(0)|\leq T^\beta\left\|g^i\right\|_{C^\beta([0,T];\mathbb{R}^{n\times n})}\leq T^\beta C(s,p)\left\|g^i\right\|_{W^s_p((0,T);\mathbb{R}^{n\times n})}\,.$$

Reading carefully through the estimates in Proposition 2.6 we observe that

$$\left\| g^i \right\|_{W_p^s((0,T);\mathbb{R}^{n\times n})} \leq C(T_0, \boldsymbol{c}, M)\,.$$

Furthermore, given $\theta \in \left(\frac{1+1/p}{2-2/p}, 1\right)$ and $\delta \in (0, 1-1/p)$, Theorem 2.12 implies with $\alpha := (1-\theta)\,(1-1/p-\delta) > 0$ the estimate

$$\begin{aligned}
\sup_{t\in[0,T]} \left|\gamma_x^i(t,0) - \widetilde{\gamma}_x^i(t,0)\right| &= \sup_{t\in[0,T]} \left|\left(\gamma_x^i - \widetilde{\gamma}_x^i\right)(t,0) - (\gamma_x^i - \widetilde{\gamma}_x^i)(0,0)\right| \\
&\leq \sup_{t\in[0,T]} \left\|\left(\gamma^i - \widetilde{\gamma}^i\right)(t) - \left(\gamma^i - \widetilde{\gamma}^i\right)(0)\right\|_{C^1\left([0,1];\mathbb{R}^d\right)} \\
&\leq T^\alpha \left\|\gamma^i - \widetilde{\gamma}^i\right\|_{C^\alpha\left([0,T];C^1\left([0,1],\mathbb{R}^d\right)\right)} \leq T^\alpha \|\!|\gamma - \widetilde{\gamma}|\!\|_{\boldsymbol{E}_T}\,.
\end{aligned}$$

This allows us to conclude

$$\|b(\gamma) - b(\widetilde{\gamma})\|_{W_p^{1/2-1/2p}\left(0,T;\mathbb{R}^d\right)} \leq C(s,p,T_0,\boldsymbol{c},M)T^\alpha \|\!|\gamma - \widetilde{\gamma}|\!\|_{\boldsymbol{E}_T}\,.$$

Finally, Lemma 3.6 implies for all $T \in (0, \widetilde{T}(\boldsymbol{c}, M)]$,

$$\begin{aligned}
&\|\!|K_T(\gamma) - K_T(\widetilde{\gamma})|\!\|_{\boldsymbol{E}_T} = \left|\!\left|\!\left| L_T^{-1}\left(N_T(\gamma) - N_T(\widetilde{\gamma})\right)\right|\!\right|\!\right|_{\boldsymbol{E}_T} \leq c(T_0,p)\|\!|N_T(\gamma) - N_T(\widetilde{\gamma})|\!\|_{\mathbb{F}_T} \\
&= c(T_0,p)\left(\|f(\gamma) - f(\widetilde{\gamma})\|_{L_p\left((0,T);L_p\left((0,1);(\mathbb{R}^d)^3\right)\right)} + \|b(\gamma) - b(\widetilde{\gamma})\|_{W_p^{1/2-1/2p}\left(0,T;\mathbb{R}^d\right)}\right) \\
&\leq C(T_0,p,\boldsymbol{c},M)T^{\min\{\alpha,\beta\}}\|\!|\gamma - \widetilde{\gamma}|\!\|_{\boldsymbol{E}_T}\,.
\end{aligned}$$

This completes the proof. □

To conclude the existence of a solution with the Banach Fixed Point Theorem we have to show that there exists a radius $M > 0$ such that K_T is a self-mapping of $\mathbb{E}_T^\sigma \cap \overline{B_M}$.

Proposition 3.13. *Let $p \in (3,\infty)$. There exists a positive radius M depending on $\boldsymbol{c}$ and the norm of σ in $W_p^{2-2/p}\left((0,1);(\mathbb{R}^d)^3\right)$ and a positive time $\widehat{T}(\boldsymbol{c}, M)$ such that for all $T \in (0, \widehat{T}(\boldsymbol{c}, M)]$ the map*

$$K_T : \mathbb{E}_T^\sigma \cap \overline{B_M} \to \mathbb{E}_T^\sigma \cap \overline{B_M}$$

is well-defined.

Proof. We let $T_0 = 1$ and define

$$M := 2\max\left\{\sup_{T\in(0,1]} \left|\!\left|\!\left| L_T^{-1}\right|\!\right|\!\right|_{\mathcal{L}(\mathbb{F}_T,\mathbb{E}_T)}, 1\right\} \max\left\{\|\!|\mathcal{L}\sigma|\!\|_{\boldsymbol{E}_1}, \left|\!\left|\!\left|\left(N_1^1(\mathcal{L}\sigma), \sigma(1), 0, N_1^2(\mathcal{L}\sigma), \sigma\right)\right|\!\right|\!\right|_{\mathbb{F}_1}\right\}$$

where $\mathcal{L}\sigma := L_1^{-1}\left(0, \sigma(1), 0, 0, \sigma\right)$ denotes the extension defined in Lemma 3.8 with $T = 1$. In particular, $\mathcal{L}\sigma$ lies in $\mathbb{E}_T^\sigma \cap \overline{B_M}$ for all $T \in (0,1]$. Moreover, for all $T \in (0,1]$ we have

$$\|\!|K_T\left(\mathcal{L}\sigma\right)|\!\|_{\boldsymbol{E}_T} \leq \sup_{T\in(0,1]} \left|\!\left|\!\left| L_T^{-1}\right|\!\right|\!\right|_{\mathcal{L}(\mathbb{F}_T,\mathbb{E}_T)} \left|\!\left|\!\left|\left(N_1^1(\mathcal{L}\sigma), \sigma(1), 0, N_1^2(\mathcal{L}\sigma), \sigma\right)\right|\!\right|\!\right|_{\mathbb{F}_T} \leq M/2\,.$$

Let $T\left(\boldsymbol{c}, M\right)$ be the time as in Proposition 3.12. Given $T \in (0, T\left(\boldsymbol{c}, M\right)]$ and $\gamma \in \mathbb{E}_T^\sigma \cap \overline{B_M}$ we observe that for some $\beta \in (0,1)$,

$$\|\!|K_T\left(\gamma\right) - K_T\left(\mathcal{L}\sigma\right)|\!\|_{\boldsymbol{E}_T} \leq C\left(\boldsymbol{c}, M\right)T^\beta \|\!|\gamma - \mathcal{L}\sigma|\!\|_{\boldsymbol{E}_T} \leq C\left(\boldsymbol{c}, M\right)T^\beta 2M\,.$$

We choose a time $\widehat{T}(\boldsymbol{c}, M) \in (0, T(\boldsymbol{c}, M)]$ so small that for all $T \in (0, \widehat{T}(\boldsymbol{c}, M)]$ it holds $C(\boldsymbol{c}, M) T^{\beta} 2M \leq M/2$. Finally, we conclude for all $T \in (0, \widehat{T}(\boldsymbol{c}, M)]$ and $\gamma \in \mathbb{E}_T^{\sigma} \cap \overline{B_M}$,

$$\|\!|K_T(\gamma)|\!\|_{\mathbb{E}_T} \leq \|\!|K_T(\gamma) - K_T(\mathcal{L}\sigma)|\!\|_{\mathbb{E}_T} + \|\!|K_T(\mathcal{L}\sigma)|\!\|_{\mathbb{E}_T} \leq M/2 + M/2 = M\,.$$

□

Theorem 3.14. *Let $p \in (3, \infty)$ and σ be an admissible initial parametrisation of a Triod. There exists a positive time $\widetilde{T}(\sigma)$ depending on $\min_{i\in\{1,2,3\}, x\in[0,1]} |\sigma_x^i(x)|$ and $\|\sigma\|_{W_p^{2-2/p}((0,1);(\mathbb{R}^d)^3)}$ such that for all $\boldsymbol{T} \in (0, \widetilde{T}(\sigma)]$ the system (2.3) has a solution $\mathcal{E}\sigma$ in*

$$\boldsymbol{E_T} = W_p^1\left((0, \boldsymbol{T}); L_p\left((0,1); (\mathbb{R}^d)^3\right)\right) \cap L_p\left((0, \boldsymbol{T}); W_p^2\left((0,1); (\mathbb{R}^d)^3\right)\right)$$

which is unique in $\boldsymbol{E_T} \cap \overline{B_M}$ with

$$M := 2 \max\left\{\sup_{T\in(0,1]} \|\!|L_T^{-1}|\!\|_{\mathcal{L}(\mathbb{F}_T, \mathbb{E}_T)}, 1\right\} \max\left\{\|\!|\mathcal{L}\sigma|\!\|_{\boldsymbol{E}_1}, \|\!|\left(N_1^1(\mathcal{L}\sigma), \sigma(1), 0, N_1^2(\mathcal{L}\sigma), \sigma\right)|\!\|_{\mathbb{F}_1}\right\}$$

where $\mathcal{L}\sigma := L_1^{-1}(0, \sigma(1), 0, 0, \sigma)$ denotes the extension defined in Lemma 3.8 with $T = 1$.

Proof. Let M and $\widehat{T}(\boldsymbol{c}, M)$ be as in Proposition 3.13 and let $\boldsymbol{T} \in (0, \widehat{T}(\boldsymbol{c}, M)]$. The fixed points of the mapping $K_{\boldsymbol{T}}$ in $\mathbb{E}_{\boldsymbol{T}}^{\sigma} \cap \overline{B_M}$ are precisely the solutions of the system (2.3) in the space $\boldsymbol{E_T} \cap \overline{B_M}$. As $K_{\boldsymbol{T}}$ is a contraction of the complete metric space $\mathbb{E}_{\boldsymbol{T}}^{\sigma} \cap \overline{B_M}$, existence and uniqueness of a solution follow from the Contraction Mapping Principle. □

Remark 3.15. If we replace $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ admissible initial parametrisation of a Triod with $\sigma = (\sigma^1, \dots, \sigma^m)$ admissible initial parametrisation of a network composed of m curves, then the time $\widetilde{T}(\sigma)$ depends on $\min_{i\in\{1,\dots,m\}, x\in[0,1]} |\sigma_x^i(x)|$ and $\|\sigma\|_{W_p^{2-2/p}((0,1);(\mathbb{R}^d)^m)}$.

Proof of Theorem 3.7. This follows from Theorem 3.14 where the appropriate time $\boldsymbol{T}$ and radius M are specified. □

3.3 Existence and uniqueness of solutions to the motion by curvature

Now that we obtained existence and uniqueness of solutions to the Special Flow (2.3) we can come back to our geometric problem.

Theorem 3.16 (Local existence of the motion by curvature)**.** *Let $p \in (3, \infty)$ and $\mathcal{N}_0$ be a geometrically admissible initial network. Then there exists $T > 0$ such that there exists a solution to the motion by curvature in $[0, T]$ with initial datum $\mathcal{N}_0$ as defined in Definition 2.18 which can be described by one parametrisation in the whole time interval $[0, T]$.*

Proof. By Definition 2.16 the geometrically admissible initial datum $\mathcal{N}_0$ admits a parametrisation $\sigma = (\sigma^1, \dots, \sigma^m)$ that is an admissible initial parametrisation for the Special Flow. Theorem 3.7 implies that there exists $\boldsymbol{T} > 0$ and a solution $\mathcal{E}\sigma \in \boldsymbol{E_T}$ to the Special Flow (2.3) in $[0, \boldsymbol{T}]$ with $(\mathcal{E}\sigma)^i(0) = \sigma^i$. Then, by Definition 2.18, $\mathcal{N} = \bigcup_{i=1}^m (\mathcal{E}\sigma)^i([0, \boldsymbol{T}] \times [0,1])$ is a solution to the motion by curvature in $[0, \boldsymbol{T}]$ with initial datum $\mathbb{T}_0$. □

Lemma 3.17 (A composition property). *Let $p \in (3,\infty)$, T be positive and*

$$f, g \in L_p\left((0,T); W_p^2\left((0,1)\right)\right) \cap W_p^1\left((0,T); L_p((0,1))\right)$$

be such that for every $t \in [0,T]$ the map $g(t,\cdot) : [0,1] \to [0,1]$ is a C^1–diffeomorphism. Then the map $h(t,x) := f(t, g(t,x))$ lies in $L_p\left((0,T); W_p^2\left((0,1)\right)\right) \cap W_p^1\left((0,T); L_p((0,1))\right)$ and all derivatives can be calculated by the chain rule.

Proof. This can be shown with similar arguments as in [21, Lemma 5.3] using the embedding in Theorem 2.4. □

Theorem 3.18 (Local uniqueness of the motion by curvature). *Let $p \in (3,\infty)$, $T, \widetilde{T} > 0$, $\mathcal{N}_0$ be a geometrically admissible initial network and $(\mathcal{N}(t))$, $(\widetilde{\mathcal{N}}(t))$ be two solutions to the motion by curvature with initial datum $\mathcal{N}_0$ in $[0,T]$ and $[0,\widetilde{T}]$, respectively, as defined in Definition 2.18. Then there exists a positive time $\widehat{T} \leq \min\{T, \widetilde{T}\}$ such that $\mathcal{N}(t) = \widetilde{\mathcal{N}}(t)$ for all $t \in [0,\widehat{T}]$.*

Proof. For the sake of notation we restrict to the case of Triods. Let $\mathbb{T}_0$ be a geometrically admissible initial Triod with regular parametrisation $\sigma \in W_p^{2-2/p}\left((0,1); (\mathbb{R}^d)^3\right)$. Then σ is an admissible initial value for the Special Flow (2.3) and Theorem 3.7 yields that there exists $\boldsymbol{T} > 0$ and a solution $\mathcal{E}\sigma = ((\mathcal{E}\sigma)^1, (\mathcal{E}\sigma)^2, (\mathcal{E}\sigma)^3) \in \boldsymbol{E_T}$ of (2.3) with initial datum σ which is unique in $\boldsymbol{E_T} \cap \overline{B_M}$ with M as in Theorem 3.14. In particular, $\mathbb{T}(t) := (\mathcal{E}\sigma)(t,[0,1])$ defines a solution to the motion by curvature (2.4) in $[0,\boldsymbol{T}]$ with initial datum $\mathbb{T}_0$. Suppose that there is another solution $(\widetilde{\mathbb{T}}(t))$ to the motion by curvature in the sense of Definition 2.25 with initial datum $\mathbb{T}_0$ in a time interval $[0,\widetilde{T}]$ for some positive $\widetilde{T}$. By possibly decreasing the time of existence $\widetilde{T}$ we may assume that there exists one parametrisation $\widetilde{\gamma} \in \boldsymbol{E}_{\widetilde{T}}$ for the evolution $(\widetilde{\mathbb{T}}(t))$ in the whole time interval $[0,\widetilde{T}]$.
We show that there exists a family of time dependent diffeomorphisms $\psi^i(t) : [0,1] \to [0,1]$ with $t \in [0,\widehat{T}]$ for some $\widehat{T} \leq \min\{\widetilde{T}, \boldsymbol{T}\}$ such that the equality

$$\widetilde{\gamma}^i(t, \psi^i(t,x)) = (\mathcal{E}\sigma)^i(t,x)$$

holds in the space $\boldsymbol{E}_{\widehat{T}}$. In order to make use of the uniqueness assertion in Theorem 3.7 we construct the reparametrisations $\psi = \left(\psi^1, \psi^2, \psi^3\right)$ in such a way that the functions $(t,x) \mapsto \widetilde{\gamma}^i(t, \psi^i(t,x))$ are a solution to the Special Flow in $\boldsymbol{E}_{\widehat{T}}$ with initial datum σ.
One easily shows that there exist unique diffeomorphisms $\psi_0^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, of regularity $\psi_0^i \in W_p^{2-2/p}((0,1);\mathbb{R})$ such that $\psi_0^i(0) = 0$, $\psi_0^i(1) = 1$ and $\widetilde{\gamma}^i(0, \psi_0^i(x)) = \sigma^i(x)$. Taking into account the special tangential velocity in (2.3) (formal) differentiation shows that the reparametrisations ψ^i need to satisfy the following boundary value problem:

$$\begin{cases} \psi_t^i(t,x) &= \frac{\psi_{xx}^i(t,x)}{|\widetilde{\gamma}_x^i(t,\psi^i(t,x))|^2 \psi_x^i(t,x)^2} - \frac{\langle \widetilde{\gamma}_t^i(t,\psi^i(t,x)), \widetilde{\gamma}_x^i(t,\psi^i(t,x))\rangle}{|\widetilde{\gamma}_x^i(t,\psi^i(t,x))|^2} \\ &\quad + \frac{1}{|\widetilde{\gamma}_x^i(t,\psi^i(t,x))|} \left\langle \frac{\widetilde{\gamma}_{xx}^i\left(t,\psi^i(t,x)\right)}{|\widetilde{\gamma}_x^i(t,\psi^i(t,x))|^2}, \frac{\widetilde{\gamma}_x^i(t,\psi^i(t,x))}{|\widetilde{\gamma}_x^i(t,\psi^i(t,x))|} \right\rangle, \\ \psi^i(t,0) &= 0\,, \\ \psi^i(t,1) &= 1\,, \\ \psi^i(0,x) &= \psi_0^i(x)\,. \end{cases} \tag{3.7}$$

Lemma 3.19 yields that there exists a solution

$$\psi = \left(\psi^1, \psi^2, \psi^3\right) \in W_p^1((0,\widehat{T}); L_p((0,1);\mathbb{R}^3)) \cap L_p((0,\widehat{T}); W_p^2((0,1);\mathbb{R}^3))$$

to system (3.7) for some $\widehat{T} \leq \min\{\widetilde{T}, \boldsymbol{T}\}$ such that $\psi^i(t) : [0,1] \to [0,1]$ is a diffeomorphism for every $t \in [0, \widehat{T}]$. Then Lemma 3.17 implies that the composition $(t,x) \mapsto \widetilde{\gamma}^i(t, \psi^i(t,x))$ lies in $\boldsymbol{E}_{\widehat{T}}$ and by construction, it is a solution to the Special Flow. We may now argue as in the proof of [21, Theorem 5.4] to obtain that $(t,x) \mapsto (\mathcal{E}\sigma)^i(t,x)$ and $(t,x) \mapsto \widetilde{\gamma}^i(t, \psi^i(t,x))$ coincide in $\boldsymbol{E}_{\widehat{T}}$. In particular, the networks $\mathbb{T}(t)$ and $\widetilde{\mathbb{T}}(t)$ coincide for all $t \in [0, \widehat{T}]$. ☐

Lemma 3.19. *Let $p \in (3, \infty)$, $\psi_0 = (\psi_0^1, \psi_0^2, \psi_0^3) \in W_p^{2-2/p}\left((0,1); \mathbb{R}^3\right)$ with $\psi_0^i : [0,1] \to [0,1]$ a diffeomorphism with $\psi_0^i(0) = 0$, $\psi_0^i(1) = 1$, $\widetilde{T} > 0$ and $\widetilde{\gamma} \in \boldsymbol{E}_{\widetilde{T}}$ be such that $\widetilde{\gamma}_x^i(x,t) \geq c$ for some $c > 0$, for all $t \in [0, \widetilde{T}]$, all $x \in [0,1]$ and $i \in \{1,2,3\}$. Then there exists a time $\widehat{T} \in (0, \widetilde{T}]$ and a solution*

$$\psi = \left(\psi^1, \psi^2, \psi^3\right) \in W_p^1((0, \widehat{T}); L_p((0,1); \mathbb{R}^3)) \cap L_p((0, \widehat{T}); W_p^2((0,1); \mathbb{R}^3))$$

to system (3.7) such that $\psi^i(t) : [0,1] \to [0,1]$ is a diffeomorphism for every $t \in [0, \widehat{T}]$.

Proof. We observe that the right hand side of the motion equation in system (3.7) contains terms of the form $f^i(t, \psi^i(t,x))$ with $f^i \in L_p\left((0,T); L_p((0,1))\right)$. To remove this dependence it is convenient to consider the associated problem for the inverse diffeomorphisms $\xi = (\xi^1, \xi^2, \xi^3)$ given by $\xi^i(t) := \psi^i(t)^{-1}$. Indeed suppose that $\psi \in W_p^{1,2}((0, \widetilde{T}) \times (0,1); \mathbb{R}^3)$ is a solution to (3.7) with $\psi^i(t) : [0,1] \to [0,1]$ a C^1–diffeomorphism. Similar arguments as in [21, Lemma 5.3] show that also ξ is of class $W_p^{1,2}((0, \widetilde{T}) \times (0,1); \mathbb{R}^3)$. Moreover, the differentiation rules

$$\begin{aligned}
\xi_y^i(t,y) &= \psi_x^i(t, \xi^i(t,y))^{-1}\,, \\
\xi_{yy}^i(t,y) &= -\xi_y^i(t,y)^3 \psi_{xx}^i(t, \xi^i(t,y))
\end{aligned}$$

yield the evolution equation

$$\begin{aligned}
\xi_t^i(t,y) =& -\psi_t^i(t, \xi^i(t,y)) \xi_y^i(t,y) \\
=& -\frac{\psi_{xx}^i\left(t, \xi^i(t,y)\right)}{\left|\widetilde{\gamma}_x^i\left(t,y\right)\right|^2} \xi_y^i(t,y)^3 + \frac{\left\langle \widetilde{\gamma}_t^i(t,y), \widetilde{\gamma}_x^i(t,y) \right\rangle}{|\widetilde{\gamma}_x^i(t,y)|^2} \xi_y^i(t,y) \\
& - \frac{\xi_y^i(t,y)}{\left|\widetilde{\gamma}_x^i\left(t,y\right)\right|} \left\langle \frac{\widetilde{\gamma}_{xx}^i\left(t,y\right)}{\left|\widetilde{\gamma}_x^i\left(t,y\right)\right|^2}, \frac{\widetilde{\gamma}_x^i(t,y)}{\left|\widetilde{\gamma}_x^i\left(t,y\right)\right|} \right\rangle,
\end{aligned}$$

and in conclusion the following system for ξ:

$$\begin{cases}
\xi_t^i(t,y) &= \frac{\xi_{yy}^i(t,y)}{|\widetilde{\gamma}_x^i(t,y)|^2} + \frac{\left\langle \widetilde{\gamma}_t^i(t,y), \widetilde{\gamma}_x^i(t,y)\right\rangle}{|\widetilde{\gamma}_x^i(t,y)|^2} \xi_y^i(t,y) - \frac{\xi_y^i(t,y)}{|\widetilde{\gamma}_x^i(t,y)|} \left\langle \frac{\widetilde{\gamma}_{xx}^i(t,y)}{|\widetilde{\gamma}_x^i(t,y)|^2}, \frac{\widetilde{\gamma}_x^i(t,y)}{|\widetilde{\gamma}_x^i(t,y)|} \right\rangle, \\
\xi^i(t,0) &= 0\,, \\
\xi^i(t,1) &= 1\,, \\
\xi^i(0,y) &= (\psi_0^i)^{-1}(y)
\end{cases} \tag{3.8}$$

for all $t \in [0, \widetilde{T}]$, $y \in [0,1]$. We observe that the boundary value problem (3.8) has a very similar structure as the Special Flow. Analogous arguments as in the proof of Theorem 3.7 allow us to conclude that there exists a solution $\xi \in W_p^{1,2}((0, \widehat{T}) \times (0,1); (\mathbb{R}^2)^3)$ to (3.8) with $\widehat{T} \in (0, \widetilde{T}]$ such that for $t \in [0, \widehat{T}]$ the map $\xi^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism.

Indeed the resulting system for ξ^i has a very similar structure as Problem (2.3) we studied before: one linearises system (3.8) and apply the linear theory developed by Solonnikov [44] to get well–posedness. Contraction estimates similar to our previous one allows to conclude the existence and uniqueness of solution with a fixed point argument. Reversing the above argumentation yields that the inverse functions $\psi^i(t) := \xi^i(t)^{-1}$ solve (3.7) and possess the desired properties. □

Theorem 3.20 (Geometric uniqueness of the motion by curvature)**.** *Let $p \in (3, \infty)$, $\mathcal{N}_0$ be a geometrically admissible initial network and T be positive. Solutions to the motion by curvature in $[0, T]$ with initial datum $\mathcal{N}_0$ are geometrically unique in the sense that given any two solutions $(\mathcal{N}(t))$ and $(\widetilde{\mathcal{N}}(t))$ to the motion by curvature in the time interval $[0, T]$ with initial datum $\mathcal{N}_0$ the networks $\mathcal{N}(t)$ and $\widetilde{\mathcal{N}}(t)$ coincide for all $t \in [0, T]$.*

Proof. Let $(\mathcal{N}(t))$ and $(\widetilde{\mathcal{N}}(t))$ be two solutions to the motion by curvature in $[0, T]$ with initial datum $\mathcal{N}_0$. Suppose by contradiction that the set

$$\mathcal{S} := \left\{ t \in [0, T] : \mathcal{N}(t) \neq \widetilde{\mathcal{N}}(t) \right\}$$

is non-empty and let $t^* := \inf \mathcal{S}$. As $\mathcal{S}$ is an open subset of $[0, T]$, we have $t^* \in [0, T)$ and $\mathcal{N}(t^*) = \widetilde{\mathcal{N}}(t^*)$. The Triod $\mathcal{N}(t^*)$ is geometrically admissible and both $t \mapsto \mathcal{N}(t^* + t)$ and $t \mapsto \widetilde{\mathcal{N}}(t^* + t)$ are solutions to the motion by curvature in the time interval $[0, T - t^*]$ with initial datum $\mathcal{N}(t^*)$. Theorem 3.18 yields that there exists a time $\widehat{T} \in (0, T - t^*]$ such that for all $t \in [0, \widehat{T}]$, $\mathcal{N}(t^* + t) = \widetilde{\mathcal{N}}(t^* + t)$ which contradicts the definition of t^*. □

Definition 3.21 (Maximal solutions to the motion by curvature)**.** Let $p \in (3, \infty)$ and $\mathcal{N}_0$ be a geometrically admissible initial network. A time–dependent family of networks $(\mathcal{N}(t))_{t\in[0,T)}$ with $T \in (0, \infty) \cup \{\infty\}$ is a *maximal solution to the motion by curvature* in $[0, T)$ with initial datum $\mathcal{N}_0$ if it is a solution (in the sense of Definition 2.18) in $[0, \hat{T}]$ for all $\hat{T} < T$ and if there does not exist a solution $(\widetilde{\mathcal{N}}(\tau))$ to the motion by curvature in the sense of Definition 2.18 in $[0, \widetilde{T}]$ with $\widetilde{T} \geq T$ and such that $\mathcal{N} = \widetilde{\mathcal{N}}$ in $[0, T)$. In this case the time T is called *maximal time of existence* and is denoted by T_{max}.

Proposition 3.22 (Existence and uniqueness of maximal solutions)**.** *Let $p \in (3, \infty)$ and $\mathcal{N}_0$ be a geometrically admissible initial network. There exists a maximal solution to the motion by curvature with initial datum $\mathcal{N}_0$ which is geometrically unique.*

Proof. Given an admissible network $\mathcal{N}_0$ we let

$$T_{max} := \sup \{T > 0 : \text{ there exists a solution } (\mathcal{N}^T(t)) \text{ to the motion by curvature in } [0, T] \\ \text{with initial datum } \mathbb{T}_0\} \; .$$

Theorem 3.16 yields $T_{max} \in (0, \infty) \cup \{\infty\}$. Given any $t \in [0, T_{max})$ we may consider a solution $\mathcal{N}^T$ with $T \in (t, T_{max})$ to the motion by curvature in $[0, T]$ with initial datum $\mathcal{N}_0$ and set

$$\mathcal{N}(t) := \mathcal{N}^T(t) \, .$$

We note that $\mathcal{N}$ is well-defined on $[0, T_{max})$ as any two solutions $\mathcal{N}^{T_1}$ and $\mathcal{N}^{T_2}$ with $T_1, T_2 \in [0, T_{max})$ to the motion by curvature with initial datum $\mathcal{N}_0$ coincide on their common interval of existence by Theorem 3.20. One easily verifies that $(\mathcal{N}(t))_{t\in[0,T_{max})}$ satisfies the properties of a maximal solution stated in Definition 3.21. Indeed, if there existed a solution $\widetilde{\mathcal{N}}(\tau)$ to the motion by curvature in $[0,\widetilde{T}]$ for $\widetilde{T} \geq T_{max}$, Theorem 3.16 would imply the existence of a solution with initial datum $\widetilde{\mathcal{N}}(\widetilde{T})$ in a time interval $[0,\delta]$, $\delta > 0$. This would yield the existence of a solution in the time interval $[0,\widetilde{T}+\delta]$ with initial datum $\mathcal{N}_0$ contradicting the definition of T_{max}. The uniqueness assertion follows from Theorem 3.20. □

4 Smoothness of the Special Flow

This section is devoted to prove that solutions to the Special Flow are smooth for positive times. Heuristically, this regularisation effect is due to the parabolic nature of the problem. The basic idea of the proof is based on the so called parameter trick which is due to Angenent [3] and has been generalized to several situations [35, 36, 42]. However, due to the fully non-linear boundary condition

$$\sum_{i=1}^{3} \frac{\gamma_x^i(t,0)}{|\gamma_x^i(t,0)|} = 0$$

the Special Flow is not treated in the above mentioned results. An adaptation of the parameter trick that allows to treat fully non-linear boundary terms is presented in [22]. We follow [22, Chapter 6.6] modifying the arguments for the application in our Sobolev setting. In the following we let $\mathcal{E}\sigma \in \boldsymbol{E}_T$ be a solution to the Special Flow on $[0,T]$, $T>0$, with initial datum $\sigma \in W_p^{2-2/p}\left((0,1);(\mathbb{R}^d)^3\right)$.

The key idea to apply Angenent's parameter trick lies in an implicit function type argument which itself relies on the invertibility of the linearisation of the Special Flow in the solution $\mathcal{E}\sigma$. Thus, the linear analysis from Subsection 3.1 will not be enough to apply this method. So before we can actually start we have to generalise Theorem 3.5.

Definition 4.1. We consider the full linearisation of system (2.3) around $\mathcal{E}\sigma$ which gives

$$\begin{cases} \gamma_t^i(t,x) - \frac{1}{|(\mathcal{E}\sigma)_x^i(t,x)|^2}\,\gamma_{xx}^i(t,x) + 2\frac{(\mathcal{E}\sigma)_{xx}^i(t,x)\left\langle \gamma_x^i(t,x),(\mathcal{E}\sigma)_x^i(t,x)\right\rangle}{|(\mathcal{E}\sigma)_x^i(t,x)|^4} &= f^i(t,x)\,,\\ \gamma(t,1) &= \eta(t)\,,\\ \gamma^1(t,0)-\gamma^2(t,0) &= 0\,,\\ \gamma^2(t,0)-\gamma^3(t,0) &= 0\,,\\ -\sum_{i=1}^3\left(\frac{\gamma_x^i(t,0)}{|(\mathcal{E}\sigma)_x^i(t,0)|} - \frac{(\mathcal{E}\sigma)_x^i(t,0)\left\langle\gamma_x^i(t,0),(\mathcal{E}\sigma)_x^i(t,0)\right\rangle}{|(\mathcal{E}\sigma)_x^i(t,0)|^3}\right) &= b(t)\,,\\ \gamma(0,x) &= \psi(x)\,. \end{cases} \tag{4.1}$$

Here ψ is an admissible initial value with respect to the given right hand side η and b. For $\gamma \in \boldsymbol{E}_T$ we define $\mathcal{A}_{T,\mathcal{E}}(\gamma) \in L_p\left((0,T);L_p((0,1);(\mathbb{R}^d)^3)\right)$ by

$$(\mathcal{A}_{T,\mathcal{E}}(\gamma))^i := \frac{1}{|(\mathcal{E}\sigma)_x^i(t,x)|^2}\,\gamma_{xx}^i(t,x) - 2\frac{(\mathcal{E}\sigma)_{xx}^i(t,x)\left\langle\gamma_x^i(t,x),(\mathcal{E}\sigma)_x^i(t,x)\right\rangle}{|(\mathcal{E}\sigma)_x^i(t,x)|^4}\,.$$

Definition 4.2 (The linearised boundary operator)**.** Let $T>0$ and

$$\mathcal{B}_{T,\mathcal{E}} : \boldsymbol{E}_T = W_p^{1,2}\left((0,T)\times(0,1);(\mathbb{R}^d)^3\right) \to W_p^{1-1/2p}\left((0,T);(\mathbb{R}^d)^5\right) \times W_p^{1/2-1/2p}\left((0,T);\mathbb{R}^d\right)$$

be the linearised boundary operator induced by the linearisation in $\mathcal{E}\sigma$, i.e.,

$$\mathcal{B}_{T,\mathcal{E}}(\gamma) = \begin{pmatrix} \gamma(\cdot,1) \\ \gamma^1(\cdot,0) - \gamma^2(\cdot,0) \\ \gamma^2(\cdot,0) - \gamma^3(\cdot,0) \\ -\sum_{i=1}^3 \left(\frac{\gamma^i_x(\cdot,0)}{|(\mathcal{E}\sigma)^i_x(\cdot,0)|} - \frac{(\mathcal{E}\sigma)^i_x(\cdot,0)\left\langle \gamma^i_x(\cdot,0),(\mathcal{E}\sigma)^i_x(\cdot,0)\right\rangle}{|(\mathcal{E}\sigma)^i_x(\cdot,0)|^3} \right) \end{pmatrix}.$$

Moreover we let

$$\boldsymbol{X}_T := \ker(\mathcal{B}_{T,\mathcal{E}}) \,.$$

As $\mathcal{B}_{T,\mathcal{E}}$ is continuous, the space $\boldsymbol{X}_T$ is a closed subspace of $\boldsymbol{E}_T$ and thus a Banach space.

Remark 4.3 (Existence analysis for (4.1)). Note that the compatibility conditions in Definition 3.1 for system (3.3) are precisely the same as the ones for (4.1) due to the fact that $\mathcal{B}_{T,\mathcal{E}}\big|_{t=0}$ equals the original linearisation. Also, with the same arguments as in the proof of Lemma 3.3 we can derive the Lopatinskii-Shapiro conditions for $\mathcal{B}_{T,\mathcal{E}}$. Therefore, the result from Theorem 3.5 holds also for problem (4.1). For $\gamma \in \boldsymbol{E}_T$ we write

$$L_{T,\mathcal{E}}(\gamma) := \begin{pmatrix} \gamma_t - A_{T,\mathcal{E}}(\gamma) \\ \mathcal{B}_{T,\mathcal{E}}(\gamma) \\ \gamma_{|t=0} \end{pmatrix} \,.$$

With the previous considerations we have the basics to start the work on the parameter trick. As a first step we have to construct a parametrisation of the non-linear boundary conditions over the linear boundary conditions. We need to do this as we cannot have the non-linear boundary operator to be part of the operator used in the parameter trick due to technical reasons with the compatibility conditions.
In the following lemma we construct a partition of the solution space $\boldsymbol{E}_T = \boldsymbol{X}_T \oplus \boldsymbol{Z}_T$.

Lemma 4.4. *Let $T > 0$. There exists a closed subspace $\boldsymbol{Z}_T$ of $\boldsymbol{E}_T$ such that $\boldsymbol{E}_T = \boldsymbol{X}_T \oplus \boldsymbol{Z}_T$.*

Proof. Firstly, we consider the space

$$\overline{Z}^1_T := \left\{ \mathfrak{b} \in W_p^{1-1/2p}\left((0,T);(\mathbb{R}^d)^5\right) \times W_p^{1/2-1/2p}\left((0,T);\mathbb{R}^d\right) : \mathfrak{b}_{|t=0} = 0 \right\}.$$

We notice that $f = 0$, $\mathfrak{b} \in \overline{Z}^1_T$, $\psi = 0$ is a suitable right hand side for problem (4.1). Hence for every $\mathfrak{b} \in \overline{Z}^1_T$ there exists a unique solution $L^{-1}_{T,\mathcal{E}}(0,\mathfrak{b},0) \in \boldsymbol{E}_T$ to (4.1) and the space $Z^1_T := L^{-1}_{T,\mathcal{E}}\left((0,\overline{Z}^1_T,0)\right)$ is a closed subspace of $\boldsymbol{E}_T$.
Next we define the space

$$\overline{Z}^2 := (\mathbb{R}^d)^5 \times \mathbb{R}^d \,.$$

Given $\tilde{b} \in \overline{Z}^2$ the elliptic system $\tilde{L}\eta = (0, \tilde{b})$ defined by

$$\begin{cases} -\frac{1}{|\sigma_x^i(x)|^2}\,\eta_{xx}^i(x) &= 0\,, \qquad\qquad x \in (0,1)\,, i \in \{1,2,3\}\,, \\ \eta^1(1) &= \tilde{b}^1\,, \\ \eta^2(1) &= \tilde{b}^2\,, \\ \eta^3(1) &= \tilde{b}^3\,, \\ \eta^1(0) - \eta^2(0) &= \tilde{b}^4\,, \\ \eta^2(0) - \eta^3(0) &= \tilde{b}^5\,, \\ -\sum_{i=1}^3 \left(\frac{\eta_x^i(0)}{|\sigma_x^i(0)|} - \frac{\sigma_x^i(0)\langle \eta_x^i(0), \sigma_x^i(0)\rangle}{|\sigma_x^i(0)|^3} \right) &= \tilde{b}^6\,, \end{cases} \tag{4.2}$$

has a unique solution $\eta \in W_p^2\left((0,1);(\mathbb{R}^d)^3\right)$ which we denote by $\tilde{L}^{-1}(0,\tilde{b})$. This is guaranteed due to the results in [1] and the fact that the boundary operator fulfils the Lopatinskii-Shapiro conditions according to Lemma 3.3. The space $\tilde{L}^{-1}(0,\overline{Z}^2)$ is a closed subspace of $W_p^2\left((0,1);(\mathbb{R}^d)^3\right)$ due to continuity of the solution operator which is guaranteed by the energy estimates in [1]. Extending every function in $\tilde{L}^{-1}(0,\overline{Z}^2)$ constantly in time we can view $\tilde{L}^{-1}(0,\overline{Z}^2)$ as a closed subspace of $\boldsymbol{E}_T$. This space will be denoted by Z_T^2. It is straightforward to check that $Z_T^1 \cap Z_T^2 = \{0\}$ which allows us to define $\boldsymbol{Z}_T$ as the subspace of $\boldsymbol{E}_T$ given by

$$\boldsymbol{Z}_T := Z_T^1 \oplus Z_T^2\,.$$

Note that $\boldsymbol{Z}_T$ is indeed a closed subspace which one sees as follows. Suppose that

$$(z_n)_{n\in\mathbb{N}} = (z_n^1 + z_n^2)_{n\in\mathbb{N}} \subset \boldsymbol{Z}_T$$

is a convergent sequence in $\boldsymbol{E}_T$.
Due to $\boldsymbol{E}_T \hookrightarrow C([0,T]; C^{1+\alpha}([0,1];(\mathbb{R}^n)^3))$ for $\alpha \in (0, 1-3/p]$ according to Theorem 2.4 we may conclude that the sequence $(z_n\big|_{t=0})_{n\in\mathbb{N}} = (z_n^2\big|_{t=0})_{n\in\mathbb{N}}$ converges in $C^{1+\alpha}([0,1];(\mathbb{R}^n)^3)$. In particular, this yields the convergence of the boundary data needed for the elliptic system we used to construct z_n^2. Continuity of the elliptic solution operator then implies that $(z_n^2\big|_{t=0})_{n\in\mathbb{N}}$ converges in $W_p^2((0,1);(\mathbb{R}^n)^3)$. Due to its constant extension in time we see that $(z_n^2)_{n\in\mathbb{N}}$ converges in $\boldsymbol{E}_T$ to a limit z^2 which is also in Z_T^2 being a closed subspace of $\boldsymbol{E}_T$. Then $(z_n^1)_{n\in\mathbb{N}} = (z_n)_{n\in\mathbb{N}} - (z_n^2)_{n\in\mathbb{N}}$ converges in $\boldsymbol{E}_T$ as sum of two convergent sequences to an element z^1 of the closed space Z_T^1. We conclude that $(z_n)_{n\in\mathbb{N}}$ converges to $z^1 + z^2 \in \boldsymbol{Z}_T$ which shows that $\boldsymbol{Z}_T$ is closed.
It remains to prove that $\boldsymbol{X}_T \cap \boldsymbol{Z}_T = \{0\}$ and $\boldsymbol{E}_T = \boldsymbol{X}_T + \boldsymbol{Z}_T$. To this end let $\gamma \in \boldsymbol{X}_T \cap \boldsymbol{Z}_T$. By definition of $\boldsymbol{X}_T$ we have $\mathcal{B}_{T,\mathcal{E}}(\gamma) = 0$ which implies in particular $\mathcal{B}_{T,\mathcal{E}}(\gamma)_{|t=0} = 0$. As γ lies in $\boldsymbol{Z}_T$, there exist $z_1 \in Z_T^1$, $z_2 \in Z_T^2$ with $\gamma = z_1 + z_2$. Using that $\mathcal{B}_{T,\mathcal{E}}(z_1)$ lies in $\overline{Z}_T^1$, we observe

$$0 = \mathcal{B}_{T,\mathcal{E}}(z_1+z_2)_{|t=0} = \mathcal{B}_{T,\mathcal{E}}(z_1)_{|t=0} + \mathcal{B}_{T,\mathcal{E}}(z_2)_{|t=0} = \mathcal{B}_{T,\mathcal{E}}(z_2)_{|t=0}\,.$$

Due to the uniqueness of the elliptic system (4.2) this shows $(z_2)_{|t=0} = 0$. By definition of Z_T^2 we obtain $z_2 = 0$. This implies $0 = \mathcal{B}_{T,\mathcal{E}}(\gamma) = \mathcal{B}_{T,\mathcal{E}}(z_1)$ which gives $z_1 = L_{T,\mathcal{E}}^{-1}(0,0,0) = 0$.
To prove that $\boldsymbol{E}_T = \boldsymbol{X}_T + \boldsymbol{Z}_T$ we let $\gamma \in \boldsymbol{E}_T$. We define

$$z_2 := \tilde{L}^{-1}(0, \mathcal{B}_{T,\mathcal{E}}(\gamma)_{|t=0}) \in Z_T^2$$

viewing z_2 as an element of $\boldsymbol{E}_T$ by extending it constantly in time. By definition of the boundary operator in the elliptic system (4.2) and due to $(\mathcal{E}\sigma)_{|t=0} = \sigma$ we have

$$\mathcal{B}_{T,\mathcal{E}}(z_2)_{|t=0} = \mathcal{B}_{T,\mathcal{E}}(\gamma)_{|t=0}\,.$$

In particular, $\mathcal{B}_{T,\mathcal{E}}(\gamma) - \mathcal{B}_{T,\mathcal{E}}(z_2)$ lies in $\overline{Z}_T^1$ and we may define

$$z_1 := L_{T,\mathcal{E}}^{-1}\left(0, \mathcal{B}_{T,\mathcal{E}}(\gamma) - \mathcal{B}_{T,\mathcal{E}}(z_2), 0\right) \in Z_T^1\,.$$

Now it remains to show that $\gamma - z_1 - z_2$ lies in $\boldsymbol{X}_T$ which is equivalent to $\mathcal{B}_{T,\mathcal{E}}(\gamma - z_1 - z_2) = 0$ which follows by construction. □

Lemma 4.5 (Parametrisation of the nonlinear boundary conditions)**.** *Let $T > 0$. There exists a neighbourhood U of 0 in $\boldsymbol{X}_T$, a function $\varrho : U \to \boldsymbol{Z}_T$ and a neighbourhood V of $\mathcal{E}\sigma$ in $\boldsymbol{E}_T$ such that*

$$\{\mathcal{E}\sigma + \boldsymbol{u} + \varrho(\boldsymbol{u}) \;:\; \boldsymbol{u} \in U\} = \{\gamma \in V : \mathcal{G}(\gamma) = 0\}$$

where $\mathcal{G}$ denotes the operator

$$\gamma \mapsto \mathcal{G}(\gamma) := \begin{pmatrix} \gamma^1(\cdot,1) - \sigma^1(1) \\ \gamma^2(\cdot,1) - \sigma^2(1) \\ \gamma^3(\cdot,1) - \sigma^3(1) \\ \gamma^1(\cdot,0) - \gamma^2(\cdot,0) \\ \gamma^2(\cdot,0) - \gamma^3(\cdot,0) \\ \sum_{i=1}^3 \frac{\gamma_x^i(\cdot,0)}{|\gamma_x^i(\cdot,0)|} \end{pmatrix}.$$

Furthermore, it holds that $(D\varrho)_{|0} \equiv 0$.

Proof. We let

$$\boldsymbol{Y}_T := W_p^{1-1/2p}\left((0,T);(\mathbb{R}^d)^5\right) \times W_p^{1/2-1/2p}\left((0,T);\mathbb{R}^d\right)$$

and consider the operator

$$\begin{aligned} F : \boldsymbol{X}_T \oplus \boldsymbol{Z}_T &\to \boldsymbol{Y}_T\,, \\ (\boldsymbol{x},\boldsymbol{z}) &\mapsto \mathcal{G}\left(\mathcal{E}\sigma + \boldsymbol{x} + \boldsymbol{z}\right)\,. \end{aligned}$$

By definition of $\mathcal{E}\sigma$ we have that $F(0,0) = 0$. We observe that $(\partial_2 F(0,0))\boldsymbol{z} = \mathcal{B}_{T,\mathcal{E}}(\boldsymbol{z})$. To apply the implicit function theorem we have to show that

$$\mathcal{B}_{T,\mathcal{E}} : \boldsymbol{Z}_T \to \boldsymbol{Y}_T$$

is an isomorphism. The map is injective as $\ker \mathcal{B}_{T,\mathcal{E}} \cap \boldsymbol{Z}_T = \boldsymbol{X}_T \cap \boldsymbol{Z}_T = \{0\}$. Given $\boldsymbol{b} \in \boldsymbol{Y}_T$ we let $z_2 := \tilde{L}^{-1}(0, \boldsymbol{b}_{|t=0}) \in Z_T^2$ and $z_1 := L_{T,\mathcal{E}}^{-1}(0, \boldsymbol{b} - \mathcal{B}_{T,\mathcal{E}}(z_2)) \in Z_T^1$ and observe that $z_1 + z_2 \in \boldsymbol{Z}_T$ satisfies

$$\mathcal{B}_{T,\mathcal{E}}(z_1 + z_2) = \mathcal{B}_{T,\mathcal{E}}(z_1) + \mathcal{B}_{T,\mathcal{E}}(z_2) = \boldsymbol{b} - \mathcal{B}_{T,\mathcal{E}}(z_2) + \mathcal{B}_{T,\mathcal{E}}(z_2) = \boldsymbol{b}\,.$$

The implicit function theorem implies that there exist neighbourhoods U and W of 0 in $\boldsymbol{X}_T$ and $\boldsymbol{Z}_T$, respectively, and a function $\varrho : U \to W$ with $\varrho(0) = 0$ such that for a neighbourhood $\tilde{V}$ of 0 in $\boldsymbol{E}_T$, it holds

$$\{\boldsymbol{u} + \varrho(\boldsymbol{u}) : \boldsymbol{u} \in U\} = \{\boldsymbol{x} + \boldsymbol{z} \in \boldsymbol{E}_T : F(\boldsymbol{x},\boldsymbol{z}) = 0\} \cap \tilde{V}\,.$$

To show that $(D\varrho)_{|0} = 0$ we let $\boldsymbol{u} \in \boldsymbol{X}_T$ be arbitrary. Due to $(D\varrho)_{|0} : \boldsymbol{X}_T \to \boldsymbol{Z}_T$ we obtain $(D\varrho)_{|0}\boldsymbol{u} \in \boldsymbol{Z}_T$. Hence it is enough to show that $(D\varrho)_{|0}\boldsymbol{u}$ lies also in $\boldsymbol{X}_T$. To this end we differentiate the identity

$$0 = F(\delta \boldsymbol{u}, \varrho(\delta \boldsymbol{u})) = \mathcal{G}\left(\mathcal{E}\sigma + \delta \boldsymbol{u} + \varrho(\delta \boldsymbol{u})\right)$$

with respect to δ and obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}\delta}\mathcal{G}\left(\mathcal{E}\sigma + \delta \boldsymbol{u} + \varrho(\delta \boldsymbol{u})\right)_{|\delta=0} = (D\mathcal{G})\left(\mathcal{E}\sigma\right)(\boldsymbol{u} + (D\varrho)_{|0}\boldsymbol{u}) = \mathcal{B}_{T,\mathcal{E}}(\boldsymbol{u} + (D\varrho)_{|0}\boldsymbol{u})\,.$$

This implies $\boldsymbol{u} + (D\varrho)_{|0}\boldsymbol{u} \in \ker \mathcal{B}_{T,\mathcal{E}} = \boldsymbol{X}_T$ and thus $(D\varrho)_{|0}\boldsymbol{u} \in \boldsymbol{X}_T$. □

With this result we can finally start the proof of the parabolic smoothing. We will first derive higher time regularity of the solution (this is actually the classical parameter trick argument by Angenent), and will then get from this higher regularity in space using the parabolic problem and finally start a bootstrap procedure.

Proposition 4.6 (Higher time regularity of solutions to the Special Flow)**.**
Let $\mathcal{E}\sigma \in \boldsymbol{E}_T$ be a solution to the Special Flow in $[0,T]$ with $T > 0$ and initial value $\sigma \in W_p^{2-2/p}\left((0,1);(\mathbb{R}^d)^3\right)$. Then we have for all $\tilde{t} \in (0,T]$ the increased time regularity

$$\partial_t(\mathcal{E}\sigma) \in \boldsymbol{E}_T\big|_{[\tilde{t},T]}\,. \tag{4.3}$$

Proof. We consider the space

$$\begin{aligned}\boldsymbol{I} := \Bigg\{\psi \in W_p^{2-2/p}\left((0,1);(\mathbb{R}^d)^3\right) : \psi(1) = 0\,, \psi^1(0) = \psi^2(0) = \psi^3(0)\,,\\ \sum_{i=1}^{3}\left(\frac{\psi_x^i(0)}{|\sigma_x^i(0)|} - \frac{\sigma_x^i(0)\left\langle \psi_x^i(0), \sigma_x^i(0)\right\rangle}{|\sigma_x^i(0)|^3}\right) = 0\Bigg\}\,.\end{aligned}$$

We let U, V and ϱ be as in the previous Lemma and define $\overline{\varrho}(\boldsymbol{u}) := \mathcal{E}\sigma + \boldsymbol{u} + \varrho(\boldsymbol{u})$. For some small $\varepsilon \in (0,1)$ we consider the map

$$\begin{aligned}G : (1-\varepsilon, 1+\varepsilon) \times \boldsymbol{I} \times \boldsymbol{X}_T \to \boldsymbol{I} \times L_p\left((0,T)\times(0,1);(\mathbb{R}^d)^3\right)\,,\\ (\lambda,\psi,\boldsymbol{u}) \mapsto \left(\boldsymbol{u}_{|t=0} - \psi, \partial_t\overline{\varrho}(\boldsymbol{u}) - \lambda\frac{\overline{\varrho}(\boldsymbol{u})_{xx}}{|\overline{\varrho}(\boldsymbol{u})_x|^2}\right)\,.\end{aligned}$$

Notice that $G(1,0,0) = 0$. Due to $(D\varrho)_{|0} = 0$ the Fréchet derivative

$$\partial_3 G(1,0,0) : \boldsymbol{X}_T \to \boldsymbol{I} \times L_p\left((0,T)\times(0,1);(\mathbb{R}^d)^3\right)$$

is given by

$$\partial_3 G(1,0,0)\boldsymbol{u} = \left(\boldsymbol{u}_{|t=0}, \partial_t \boldsymbol{u} - \mathcal{A}_{T,\mathcal{E}}(\boldsymbol{u})\right)\,.$$

As explained in Remark 4.3 we have that $(DG)_{|(1,0,0)}(0,0,\cdot)$ is an isomorphism. Hence the implicit function theorem implies the existence of neighbourhoods $\mathcal{U}$ of $(1,0)$ in $(1-\varepsilon,1+\varepsilon)\times\boldsymbol{I}$ and $\mathcal{V}$ of 0 in $\boldsymbol{X}_T$ and a function $\zeta : \mathcal{U} \to \mathcal{V}$ with $\zeta((1,0)) = 0$ and

$$\{(\lambda,\psi,\boldsymbol{u}) \in \mathcal{U}\times\mathcal{V} : G(\lambda,\psi,\boldsymbol{u}) = 0\} = \{(\lambda,\psi,\zeta(\lambda,\psi)) : (\lambda,\psi) \in \mathcal{U}\}\,.$$

Consider now the map $P : \boldsymbol{E}_T \to \boldsymbol{X}_T$ given by $P(\gamma) := P_{\boldsymbol{X}_T}(\gamma - \mathcal{E}\sigma)$ with $P_{\boldsymbol{X}_T}(\eta) = \boldsymbol{u}$ for the unique partition $\eta = \boldsymbol{u} + \overline{\boldsymbol{u}} \in \boldsymbol{X}_T \oplus \boldsymbol{Z}_T$. Clearly, we have that $\overline{\varrho}(P(\gamma)) = \gamma$ for all γ in the neighbourhood V constructed in Lemma 4.5. Given λ close to 1 we consider the time-scaled function

$$(\mathcal{E}\sigma)_\lambda(t,x) := (\mathcal{E}\sigma)(\lambda t, x)\,.$$

By definition this satisfies for $\psi := P((\mathcal{E}\sigma)_\lambda)\big|_{t=0}$

$$G(\lambda, \psi, P((\mathcal{E}\sigma)_\lambda)) = 0\,.$$

By uniqueness we conclude that

$$P((\mathcal{E}\sigma)_\lambda) = \zeta(\lambda, \psi)$$

and therefore

$$(\mathcal{E}\sigma)_\lambda = \bar{\varrho}(\zeta(\lambda, \psi))\,.$$

As both ζ and $\bar{\varrho}$ are smooth, this shows that $(\mathcal{E}\sigma)_\lambda$ is a smooth function in λ with values in $\boldsymbol{E}_T$. This implies now

$$t\partial_t(\mathcal{E}\sigma) = \partial_\lambda((\mathcal{E}\sigma)_\lambda)\big|_{\lambda=1} \in \boldsymbol{E}_T$$

from which we directly conclude (4.3). □

Next, we want to derive higher regularity in space for our solution. But this follows almost immediately from the associated ODE we have at a fixed time.

Corollary 4.7 (Higher space regularity of solutions to the Special Flow)**.**
Let $\mathcal{E}\sigma \in \boldsymbol{E}_T$ be a solution to the Special Flow in $[0,T]$ with $T > 0$ and initial value $\sigma \in W_p^{2-2/p}\left((0,1);(\mathbb{R}^d)^3\right)$. Given $\tilde{t} \in (0,T]$ we have for all $t \in [\tilde{t}, T]$ that

$$(\mathcal{E}\sigma)(t) \in C^3([0,1];(\mathbb{R}^n)^3)\,.$$

Additionally, all derivatives in space up to order two are smooth in time.

Proof. Considering $\partial_t((\mathcal{E}\sigma)^i)(t)$ as given functions $\mathfrak{f}^i \in C^1([0,1];\mathbb{R}^n)$ we see that $(\mathcal{E}\sigma)^i(t,\cdot)$ solves

$$\frac{(\mathcal{E}\sigma)^i_{xx}(t,\cdot)}{|(\mathcal{E}\sigma)^i_x(t,\cdot)|^2} = \mathfrak{f}^i.$$

As we already constructed $\mathcal{E}\sigma$, we may include independent boundary conditions at $x = 0$ for the values of $\mathcal{E}\sigma$ and $\partial_x\mathcal{E}\sigma$. On this problem one may again apply the implicit function theorem together with standard well-posedness results for ODEs to get that $\mathcal{E}\sigma$ is indeed in C^2 and depends smoothly on the data. Then the smoothness of the space derivatives in time follows from the smoothness of $\partial_t\mathcal{E}\sigma$ and the smooth dependence of the data. □

With the two previous results we are now able to start a bootstrap procedure.

Theorem 4.8 (Smoothness of solutions to the Special Flow)**.**
Let $\mathcal{E}\sigma \in \boldsymbol{E}_T$ be a solution to the Special Flow in $[0,T]$ with $T > 0$ and initial value $\sigma \in W_p^{2-2/p}\left((0,1);(\mathbb{R}^d)^3\right)$. Then $\mathcal{E}\sigma$ is smooth on $[\tilde{t}, T] \times [0,1]$ for all $\tilde{t} \in (0,T)$.

Proof. Due to Corollary 4.7 we can use $(\mathcal{E}\sigma)(t)$ for almost all $t > 0$ as initial data for a regularity result in parabolic Hölder space, cf. [20] for such a result for the Willmore flow. As we checked that the Lopatinskii-Shapiro conditions are still valid in higher co-dimensions, the analysis works as in the planar case. Additionally, the needed compatibility conditions due to the zero order boundary conditions are guaranteed by the fact that $\partial_t(\mathcal{E}\sigma)$ lies in $C([\tilde{t}, T]; C([0,1]; (\mathbb{R}^n)^3)$. With this new maximal regularity result, which is the key argument in the proof of Proposition 4.6, we can repeat the whole procedure to derive $C^{3+\alpha,(3+\alpha)/2}$-regularity. Note that in this situation of higher regularity we have to include compatibility conditions in $\boldsymbol{X}_T$. But this makes the construction of $\boldsymbol{Z}_T$ in Lemma 4.4 very difficult. Thus, a modification is necessary, moving the boundary conditions in the operator itself. For details we refer to [19, Section 4]. This starts now the bootstrapping yielding the desired smoothness result. Note that in every step the needed compatibility conditions are guaranteed by the fact that our flow already has the regularity related to these compatibility conditions (see for instance [39, Theorem 3.1]). □

In analogy to [21] we may now use smoothness of the Special Flow to prove Theorem 1.1.

Proof of Theorem 1.1. The existence of maximal solutions and their geometric uniqueness are shown in Proposition 3.22. Using smoothness of the Special Flow shown in Theorem 4.8 one may argue analogously to [21, Section 5.2, Section 7.2] to show that parametrising each curve $\mathbb{T}^i(t)$ with constant speed equal to the length of $\mathbb{T}^i(t)$ yields a global parametrisation $\gamma : [0, T_{max}) \times [0,1] \to (\mathbb{R}^d)^m$ of the evolution that is smooth for positive times. □

5 Long Time Behaviour of the Motion by Curvature

Proof of Theorem 1.2. Let $p \in (3,\infty)$ and $\mathcal{N}(t)$ the maximal solution. Thanks to uniqueness and regularity we can consider $p \in (3,6)$. Let $\varepsilon \in (0, T_{\max}/1000)$ be fixed. Suppose that $T_{\max}$ is finite and that the two assertions $i)$ and $ii)$ are not fulfilled. Let $\gamma = (\gamma^1, \ldots, \gamma^m) : [0, T_{max}) \times [0,1] \to (\mathbb{R}^d)^m$ be the parametrisation of the evolution such that each curve $\mathcal{N}^i(t)$ is parametrised with constant speed equal to its length $L(\mathcal{N}^i(t))$. As γ is smooth on $[\varepsilon, T]$ for all positive ε and all $T \in (\varepsilon, T_{\max})$, hypothesis $ii)$ yields

$$\boldsymbol{\kappa}^i \in L^\infty\left((\varepsilon, T_{\max}); L^2((0,1);\mathbb{R}^d)\right).$$

As $\boldsymbol{E}_T$ embeds continuously into $C\left([0,T]; C^1([0,1]; (\mathbb{R}^d)^m)\right)$, hypothesis $i)$ implies that the lengths $L(\mathbb{T}^i)$ of all three curves composing the network are uniformly bounded away from zero in $[0, T_{\max})$. Moreover, thanks to the gradient flow structure of the motion by curvature the single lengths of the networks satisfy $L(\mathcal{N}^i(t)) \le L(\mathcal{N}_0)$ for all $t \in [0, T_{\max})$. In particular, we obtain for all $t \in [0, T_{max})$, $x \in [0,1]$,

$$0 < \mathfrak{c} \le |\gamma^i_x(t,x)| = L(\mathcal{N}^i(t)) \le C < \infty\,. \tag{5.1}$$

With our choice of parametrisation the curvature can be expressed as $\boldsymbol{\kappa}^i = \gamma^i_{xx}/L(\mathcal{N}^i)^2$ from which we can infer for all $t \in [0, T_{max})$,

$$\int_0^1 |\gamma^i_{xx}|^2\,\mathrm{d}x = \left(L(\mathcal{N}^i)\right)^3 \int_{\mathbb{T}} |\boldsymbol{\kappa}^i|^2\,\mathrm{d}s \le C < \infty\,.$$

As the endpoints $P^1, \dots, P^\ell$ are fixed and as the single lengths $L(\mathcal{N}^i(t))$ are uniformly bounded from above in $[0, T_{max})$, there exists a constant $R > 0$ such that for every $t \in [0, T_{\max})$ it holds $\mathcal{N}(t) \subset B_R(0)$. With the above arguments we conclude

$$\gamma^i \in L^\infty((\varepsilon, T_{\max}); W_2^2((0,1); \mathbb{R}^d)) .$$

The Sobolev Embedding Theorem implies for all $p \in (3, 6]$ the estimate

$$\sup_{t \in (\varepsilon, T_{max})} \left\| \gamma^i(t) \right\|_{W_p^{2-2/p}((0,1);\mathbb{R}^d)} \leq \boldsymbol{C} \tag{5.2}$$

for a uniform constant $\boldsymbol{C} > 0$. We note further that for all $\delta \in (0, T_{max}/4)$ the parametrisation $\gamma(T_{max} - \delta)$ is an admissible initial value for the Special Flow (2.3). Due to (5.1) and (5.2) Theorem 3.14 yields that there exists a uniform time $\boldsymbol{T}$ of existence of solutions to the Special Flow (2.3) for all initial values $\gamma(T_{max} - \delta)$ depending on $\boldsymbol{C}$ and $\mathfrak{c}$. Let $\delta := \min\{\boldsymbol{T}/2, T_{max}/4\}$. Then Theorem 3.14 implies the existence of a solution $\eta = (\eta^1, \dots, \eta^\ell)$ with η^i regular and

$$\eta^i \in W_p^1\left((T_{max} - \delta, T_{max} + \delta); L_p\left((0,1); \mathbb{R}^d\right)\right) \cap L_p\left((T_{max} - \delta, T_{max} + \delta); W_p^2\left((0,1); \mathbb{R}^d\right)\right)$$

to system (2.3) with $\eta\,(T_{max} - \delta) = \gamma\,(T_{max} - \delta)$. The two parametrisations γ and η defined on $(0, T_{max} - \frac{\delta}{3})$ and $\left(T_{max} - \frac{\delta}{2}, T_{max} + \delta\right)$, respectively, define a solution $(\widetilde{\mathcal{N}}(t))$ to the motion by curvature on the time interval $(0, T_{max} + \delta]$ with initial datum $\mathcal{N}_0$ coinciding with $\mathcal{N}$ on $(0, T_{max})$. This contradicts the maximality of T_{max}. □

A Appendix

We explain here how to pass from the analysis of the evolution of a single Triod to networks with more complicated topologies. Naturally it is not the first time that this generalisation has been consider and there is more than one way to deal with it. We will follow the method outlined in [30], that is extensively based on the work for linear system done in [49].

We consider an initial network composed of m curves, with ℓ endpoints $\gamma^k(t,1) = P^k$ with $P^k \in \mathbb{R}^n$, $k \in \{1, \dots, \ell\}$ and with q triple junctions $\sigma^{j_1}(y_1) = \sigma^{j_2}(y_2) = \sigma^{j_3}(y_3) = \mathcal{O}^j$ with $j \in \{1, \dots, q\}, y_1, y_2, y_3 \in \{0, 1\}$.

Let us start from Section 3.1. The motion equations of the linearised Special Flow will not differ to much from the version for three curves. Formula (3.1) holds for each curve of the network:

$$\gamma_t^i(t,x) - \frac{1}{|\sigma_x^i(x)|^2}\,\gamma_{xx}^i(t,x) = \left(\frac{1}{|\gamma_x^i(t,x)|^2} - \frac{1}{|\sigma_x^i(x)|^2}\right)\gamma_{xx}^i(t,x)\,. \tag{A.1}$$

Then one has to write formula (3.2) at each triple junction, γ^{j_i} is evaluated at (t, y_i) with $y_i \in \{0,1\}$, taking care of the fact that if $y_i = 1$ there is a change of sign with respect to (3.2). So for $j \in \{1, \dots, q\}$

$$\begin{aligned} &-\sum_{i=1}^{3}\left((-1)^{y_i}\left(\frac{\gamma_x^{j_i}}{|\sigma_x^{j_i}|} - \frac{\sigma_x^{j_i}\left\langle \gamma_x^{j_i}, \sigma_x^{j_i}\right\rangle}{|\sigma_x^{j_i}|^3}\right)\right) \\ &= \sum_{i=1}^{3}\left((-1)^{y_i}\left(\left(\frac{1}{|\gamma_x^{j_i}|} - \frac{1}{|\sigma_x^{j_i}|}\right)\gamma_x^{j_i} + \frac{\sigma_x^{j_i}\left\langle \gamma_x^{j_i}, \sigma_x^{j_i}\right\rangle}{|\sigma_x^{j_i}|^3}\right)\right)\,, \end{aligned} \tag{A.2}$$

where we have omitted the dependence of t and on $y_1, y_2, y_3 \in \{0,1\}$.
In the system (3.3), instead of three evolution equations, m equations should appear, together with the compatibility condition and the linearised angle condition for each junction. Indeed in place of (3.3) one gets the following

$$\begin{cases} \gamma^i_t(t,x) - \frac{1}{|\sigma^i_x(x)|^2}\,\gamma^i_{xx}(t,x) = f^i(t,x)\,, & t\in(0,T)\,, x\in(0,1)\,,\\ \gamma^k(t,1) = \eta^k(t)\,, & t\in[0,T]\,, k\in\{1,\dots,\ell\}\,,\\ \gamma^{j_1}(t,y_1) - \gamma^{j_2}(t,y_2) = 0\,, & t\in[0,T]\,, j\in\{1,\dots,q\}\,,\\ \gamma^{j_2}(t,y_2) - \gamma^{j_3}(t,y_3) = 0\,, & t\in[0,T]\,, j\in\{1,\dots,q\}\,,\\ -\sum_{i=1}^3(-1)^{y_i}\left(\frac{\gamma^{j_i}_x(t,y_i)}{|\sigma^{j_i}_x(y_i)|} - \frac{\sigma^{j_i}_x(y_i)\left\langle\gamma^{j_i}_x(t,y_i),\sigma^{j_i}_x(y_i)\right\rangle}{|\sigma^{j_i}_x(y_i)|^3}\right) = b^j(t)\,, & t\in[0,T]\,, j\in\{1,\dots,q\}\,,\\ \gamma(0,x) = \psi(x)\,, & x\in[0,1]\,. \end{cases} \tag{A.3}$$

for $i\in\{1,\dots,m\}$ and for a general right hand side (f,η,b,ψ) with $\eta = (\eta^1,\dots,\eta^\ell)$, $b = (b^1,\dots,b^q)$.

One needs to adapt also Definition 3.1.

Definition A.1. Let $p\in(3,\infty)$. A function $\psi = (\psi^1,\dots,\psi^m)$ of class $W^{2-2/p}_p\left((0,1);(\mathbb{R}^d)^m\right)$ satisfies the *linear compatibility conditions* for system (A.3) with respect to given functions $\eta\in W^{1-1/2p}_p((0,T);(\mathbb{R}^d)^k)$, $b\in W^{1/2-1/2p}_p((0,T);(\mathbb{R}^d)^q)$ if for $k\in\{1,\dots,\ell\}, j\in\{1,\dots,q\}$ it holds $\psi^k(1) = \eta^k(0)$ $\psi^{j_1}(0) = \psi^{j_2}(0) = \psi^{j_3}(0)$, and

$$-\sum_{i=1}^3(-1)^{y_i}\left(\frac{\psi^{j_i}_x(y_i)}{|\sigma^{j_i}_x(y_i)|} - \frac{\sigma^{j_i}_x(y_i)\left\langle\psi^{j_i}_x(y_i),\sigma^{j_i}_x(y_i)\right\rangle}{|\sigma^{j_i}_x(y_i)|^3}\right) = b^j(0)\,.$$

At this point one wants to apply Solonnikov's theory [44] to get Theorem 3.5. As usual, the difficulty concerned the boundary conditions. Theorem [44, Theorem 5.4] requires the fulfilment of the complementary conditions at the boundary: basically the two matrices $\mathcal{B}(0,t,\partial_x,\partial_t)$ and $\mathcal{B}(1,t,\partial_x,\partial_t)$ must be invertible. However, we have parametrized the curves in such a way that the conditions at $x=0$ and $x=1$ are entangled and we cannot write two separate invertible matrices. One has to write a new system, equivalent to (A.3), that has a suitable structure to directly use Solonnikov's theory [44]. Namely, one has to arrange that a given triple junction is the image of either $x=0$ under the three curves or $x=1$ under the three curves. It is necessary to break some curves imposing artificial Cauchy conditions at the intermediate breaking points, as explained in [30, Section 5]. In [49] the author carry on with full details this procedure. The great advantage is that Von Below not only gets to separate matrices, both each one is a block matrix and to show their invertibility it is enough to prove that the determinant of each block is different from zero. Every block describes one single triple junction, and the invertibility of the block is equivalent to the Lopatinskii-Shapiro condition, that we have already shown in Lemma 3.3. Hence, thanks to [49] we have existence and uniqueness and suitable estimates for the new system and then for system (A.3) as well. So Theorem 3.5 is valid.

Then one will have new spaces $\mathbb{E}_T$ and $\mathbb{F}_T$ properly defined and as a consequence of Theorem 3.5 the operator $L_T(\gamma) : \mathbb{E}_T \to \mathbb{F}_T$

$$L_T(\gamma) = \begin{pmatrix} \left(\gamma^i_t - \frac{\gamma^i_{xx}}{|\sigma^i_x|^2}\right)_{i\in\{1,\dots,m\}} \\ \left(\gamma^k_{|x=1}\right)_{k\in\{1,\dots,\ell\}} \\ \left(\gamma^{j_1}_{|x=0} - \gamma^{j_2}_{|x=0}, \gamma^{j_2}_{|x=0} - \gamma^{j_3}_{|x=0}\right)_{j\in\{1,\dots,q\}} \\ \left(-\sum_{i=1}^3 \left(\frac{\gamma^{j_i}_x}{|\sigma^{j_i}_x|} - \frac{\sigma^{j_i}_x\left\langle \gamma^{j_i}_x, \sigma^{j_i}_x\right\rangle}{|\sigma^{j_i}_x|^3}\right)_{|x=0}\right)_{j\in\{1,\dots,q\}} \\ \gamma_{|t=0} \end{pmatrix}$$

that is still a continuous isomorphism. So Section 3.1 does not need any other alteration and one gets also Lemma 3.6.

It is straightforward to adapt the arguments of Section 3.2 to the case of general networks. Indeed, this is just a matter of suitably redefining the operators and spaces appearing in the proofs. With a careful look one realizes that no additional estimates are needed.
In particular the constant $\boldsymbol{c}$ in Lemma 3.9 becomes $\boldsymbol{c} := \frac{1}{2}\min_{i\in\{1,\dots,m\},x\in[0,1]} |\sigma^i_x(x)|$ and the proof does not undergo changes. The two components N^1_T, N^2_T of the operator N_T are defined as

$$N^1_T : \begin{cases} \mathbb{E}^\sigma_T & \to L_p((0,T); L_p((0,1); (\mathbb{R}^d)^m))\,, \\ \gamma & \mapsto f(\gamma)\,, \end{cases}$$
$$N^2_T : \begin{cases} \mathbb{E}^\sigma_T & \to W_p^{1/2-1/2p}((0,T); (\mathbb{R}^d)^q)\,, \\ \gamma & \mapsto b(\gamma) \end{cases}$$

with

$$f(\gamma)^i(t,x) := \left(\frac{1}{|\gamma^i_x(t,x)|^2} - \frac{1}{|\sigma^i_x(x)|^2}\right)\gamma^i_{xx}(t,x)\,,$$
$$b(\gamma)^j(t) := \sum_{i=1}^3 (-1)^{y_j}\left(\left(\frac{1}{|\gamma^{j_i}_x(t,y_j)|} - \frac{1}{|\sigma^{j_i}_x(y_j)|}\right)\gamma^{j_i}_x(t,y_j) + \frac{\sigma^{j_i}_x(y_j)\left\langle \gamma^{j_i}_x(t,y_j), \sigma^{j_i}_x(y_j)\right\rangle}{|\sigma^{j_i}_x(y_j)|^3}\right)\,.$$

this time defined by the right hand side of (A.1) and (A.2), respectively.
As the estimates concerning N^1_T are done for each $i \in \{1,\dots,m\}$ instead of $i \in \{1,2,3\}$ and the estimate related to N^2_T can be done component–wise, for $j \in \{1,\dots,q\}$, it is possible to obtain again Proposition 3.10 and Proposition 3.12. Instead Corollary 3.11 and Proposition 3.13 are more abstract and do not need to be modified.

In the same spirit one also adapts the whole Section 3.3. Indeed, the only case in which we restricted our analysis to Triods is the proof of Theorem 3.18 which is entirely based on the resolution with a very similar structure to the Special Flow.

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