LOWER SEMICONTINUITY FOR FUNCTIONALS DEFINED ON PIECEWISE RIGID FUNCTIONS AND ON $GSBD$

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Abstract. In this work, we provide a characterization result for lower semicontinuity of surface energies defined on piecewise rigid functions, i.e., functions which are piecewise affine on a Caccioppoli partition where the derivative in each component is a skew symmetric matrix. This characterization is achieved by means of an integral condition, called *BD-ellipticity*, which is in the spirit of BV-ellipticity defined by Ambrosio and Braides [5]. By specific examples we show that this novel concept is in fact stronger compared to its BV analog. We further provide a sufficient condition implying BD-ellipticity which we call *symmetric joint convexity*. This notion can be checked explicitly for certain classes of surface energies which are relevant for applications, e.g., for variational fracture models. Finally, we give a direct proof that surface energies with symmetric jointly convex integrands are lower semicontinuous also on the larger space of $GSBD^p$ functions.

1. Introduction

The minimization of surface energies for configurations which represent partitions of the domain into regions of finite perimeter appears in many problems in materials science, physics, computer science, and other fields (see, for instance, [33, Introduction] and the references therein). In the framework of the calculus of variations, these energies are often given in the form of integral functionals defined on *Caccioppoli partitions* or *piecewise constant functions* on such partitions, see [7, Section 4.4] for their definition. After the seminal work by Almgren [1], Ambrosio and Braides [4, 5] developed a thorough analysis concerning integral representation, compactness, Γ-convergence, and relaxation for this class of functionals. They also formulated a general theory of lower semicontinuity in this setting, which we will discuss in detail below. This approach was further developed by subsequent contributions over the last years, see, e.g., [7, Section 5.3] or [14, 15, 16]. Let us also mention some recent advances dealing with density and continuity results [8, 36], witnessing that the study of this class of functionals is of ongoing interest.

Background: We now briefly discuss the framework for lower semicontinuity devised in [5] since it will be relevant for the purpose of our paper. There, integral functionals of the form

$$u \mapsto \int_{J_u \cap \Omega} f(u^+, u^-, \nu_u)\, \mathrm{d}\mathcal{H}^{d-1} \tag{1.1}$$

are considered. Above, $u = \sum_{k\in\mathbb{N}} b_k \chi_{P_k}$ is a piecewise constant function where the sets P_k partition a d-dimensional reference configuration Ω into subsets of finite perimeter. Thus, the jump set J_u of u, locally oriented by a normal unit vector ν_u, consists of the interfaces between two different P_k's where u jumps from the value u^+ to u^-. The constants b_k are taken from a prescribed finite subset T of $\mathbb{R}^m$, so that without restriction one can assume the integrand $f\colon \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{d-1} \to [0, +\infty)$ to be continuous and bounded. (Here, $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$.) In [5], it is shown with

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a localization technique that lower semicontinuity of energies of the kind (1.1) can be equivalently reformulated in terms of an integral condition, named BV-ellipticity. The latter plays a similar role as Morrey quasi-convexity [35] for integral functionals on Sobolev spaces. It requires that, for all $(i,j,\nu) \in T \times T \times \mathbb{S}^{d-1}$ with $i \neq j$, we have

$$\int_{J_v \cap Q_1^\nu} f(v^+, v^-, \nu_v)\, \mathrm{d}\mathcal{H}^{d-1} \geq f(i,j,\nu)$$

for all piecewise constant functions v (with values in T) which take the values i and j, respectively, on the upper and the lower part of the boundary of Q_1^ν which is the unit cube in $\mathbb{R}^d$ oriented by ν. This condition, however, is not easy to handle because it is given by an integral inequality. To overcome this difficulty, a sufficient condition for semicontinuity has been introduced which can be easily verified in many practical cases: it is called *regular biconvexity* or *joint convexity* (this latter expression is used in the reference book [7]) and amounts to require that

$$f(i,j,\nu) = \sup_{h \in \mathbb{N}} \langle V_h(i) - V_h(j), \nu \rangle, \tag{1.2}$$

where $(V_h)_h \subset C_0(\mathbb{R}^m, \mathbb{R}^d)$ is a countable collection of continuous functions vanishing at infinity.

Understanding the properties of functionals on Caccioppoli partitions has also proved to be a fundamental step in the analysis of free-discontinuity problems [7, 29] defined on *(generalized) special functions of bounded variation* ($(G)SBV$) (see [7, Section 4]). The study of lower semicontinuity conditions for surface energies of the form (1.1), but considered in the larger space $GSBV$, is indeed one of the relevant issues that can be reduced to corresponding problems on partitions, see [2, 3]. Since piecewise constant functions are a subset of $GSBV$, it is clear that BV-ellipticity still provides a necessary condition for lower semicontinuity in a suitable weak topology, essentially the one where Ambrosio's compactness theorem [7, Theorems 4.7-4.8] holds. Remarkably, using an approximation argument of SBV functions with piecewise constant ones (which essentially relies on the BV coarea formula), [2, Theorem 3.3] shows that BV-ellipticity actually provides also a sufficient condition for lower semicontinuity along sequences which are uniformly bounded both in L^∞ and in the weak topology of SBV. If L^∞ bounds are not available and one still wants to allow for possibly unbounded integrands, lower semicontinuity results in $GSBV$ can be provided only under additional structural assumptions, see [2, Theorem 3.7], since the traces u^+ and u^- are not necessarily integrable on J_u in this case. In the proof, the possibility of using Lipschitz truncations to approximate $GSBV$ with bounded SBV functions plays a relevant role, a tool which is not available in our setting described below.

Setting of the paper: In the present paper, we are interested in analogous problems for functionals defined on *piecewise rigid functions*, denoted by $PR(\Omega)$, i.e., functions which are piecewise affine on a Caccioppoli partition where the derivative in each component is constant and lies in the set of skew symmetric matrices $\mathbb{R}^{d\times d}_{\mathrm{skew}}$. Functions in this space are vector-valued and take the form

$$u(x) = \sum\nolimits_{k\in\mathbb{N}} (Q_k\, x + b_k)\chi_{P_k}(x),$$

where $(P_k)_{k\in\mathbb{N}}$ is a Caccioppoli partition of Ω, $Q_k \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$, and $b_k \in \mathbb{R}^d$ for all $k \in \mathbb{N}$. Due to a remarkable *piecewise rigidity* result in [18], the set of these functions coincides with the (seemingly larger) set of functions $u \in GSBD$ with approximate symmetrized gradient $e(u) = 0$ almost everywhere. Here, $(G)SBD$ is the space of *(generalized) special functions of bounded deformation*, introduced in [6, 27]. Actually, our primary motivation comes exactly from the study of free-discontinuity problems defined on the space $GSBD^p$, see [27], which has obtained steadily increasing attention over the last years, cf., e.g., [17, 19, 20, 21, 22, 23, 24, 25, 30, 31, 32, 33, 34]. (Here, the exponent p refers to summability of the approximate symmetrized gradient.) In these

problems, only a control on the symmetrized gradient of the admissible configurations is available. Hence, a larger space than piecewise constant functions must be taken into account in order to provide lower semicontinuity conditions for surface integrands. It is quite natural to expect (and indeed our results in Sections 4.6 and 5 will justify this point of view) that the understanding of energies defined on piecewise rigid functions is a significant ingredient of such a research program.

The results of this paper (see description below) also complement the ones we obtained in a first paper on this topic [33], where integral representation and Γ-convergence for functionals defined on piecewise rigid functions have been investigated. The proof strategy there was based on the global method for relaxation developed in [11, 12], but some highly nontrivial issues had to be faced. In particular, a key ingredient for the results in [33] (and actually also for the ones in the present paper) is a construction for *joining* two functions $u, v \in PR(\Omega)$, which is usually called the *fundamental estimate*. In the space $PR(\Omega)$, this cannot be achieved by means of a cut-off construction of the form $w := u\varphi + (1-\varphi)v$ for some smooth φ with $0 \le \varphi \le 1$, since in general w is not in $PR(\Omega)$. In the case of piecewise constant functions, this issue was solved in [4] by using the coarea formula in BV, see [4, Lemma 4.4], a tool which is not available in spaces of functions with bounded deformation. However, the issue can be successfully overcome: a statement of the fundamental estimate in $PR(\Omega)$ is given in Lemma 2.3, while we refer the interested reader to [33, Introduction, Lemmas 4.1 and 4.4] for an overview of the proof strategy, and the detailed proof, respectively.

Results of the paper: We now come to the description of our results. We essentially follow the program of [5]. On the space $PR(\Omega)$, we consider functionals of the form

$$u \mapsto \int_{J_u \cap \Omega} f(u^+, u^-, \nu_u)\, \mathrm{d}\mathcal{H}^{d-1} \tag{1.3}$$

for densities $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0, +\infty)$. We introduce an integral notion which we call *BD-ellipticity*: it requires that for all $(i, j, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$ with $i \neq j$ we have

$$\int_{J_v \cap Q_1^\nu} f(v^+, v^-, \nu_v)\, \mathrm{d}\mathcal{H}^{d-1} \ge f(i, j, \nu)$$

for all $v \in PR(\Omega)$ which take the values i and j, respectively, on the upper and the lower part of the boundary of the unit cube oriented by ν, again denoted by Q_1^ν. The fundamental estimate in Lemma 2.3 and a localization procedure allows us to show that this condition is equivalent to lower semicontinuity of the functionals (1.3) along sequences which converge in measure and whose jump sets have uniformly bounded $\mathcal{H}^{d-1}$-measure, provided that f is uniformly continuous and bounded (Theorem 2.2). If this latter requirement is dropped, lower semicontinuity is still guaranteed on sequences which are bounded in L^∞, see Corollary 2.5. However, this uniform bound is quite a restrictive assumption in the variational modeling of fracture where the interest in functionals of the kind (1.3) originates. Hence, it is relevant to know that we may get rid of L^∞-bounds also in some cases where f is unbounded, as we point out in Corollary 2.6. There, we prove that, if f is the supremum of BD-elliptic, uniformly continuous, and bounded integrands, it is itself BD-elliptic and the associated functional is lower semicontinuous.

In case f is not BD-elliptic, we also study the relaxation of functionals of the form (1.3), see Theorem 2.8 for details. After providing an abstract framework for lower semicontinuity, the rest of the paper is devoted to investigate more closely related notions. On the one hand, our objective is to provide sufficient conditions and to find relevant explicit examples of functionals fulfilling our assumptions. On the other hand, we compare the notion of BD-ellipticity with its BV analog: as one may expect, we show that it is actually more restrictive.

To provide a sufficient notion for lower semicontinuity, we introduce a suitable subclass of jointly convex integrands, which we call *symmetric jointly convex* functions: they are still of the form

$$f(i,j,\nu) = \sup_{h\in\mathbb{N}} \langle V_h(i) - V_h(j), \nu\rangle\,, \tag{1.4}$$

cf. (1.2), but, along with boundedness and uniform continuity, we additionally require the vector fields $V_h\colon \mathbb{R}^d \to \mathbb{R}^d$ to be *conservative*. The role of this condition, which for smooth fields V_h implies that the differentials DV_h are symmetric matrices, is apparent in the proof of Theorem 3.4 as it implies that the distributional divergence of the composite functions $V_h(u)$, with $u \in PR(\Omega)$, is concentrated on the jump set J_u. This allows to reproduce the arguments in [5], see also [7, Theorem 5.20], and to prove that symmetric joint convexity is a sufficient condition both for BD-ellipticity and for the lower semicontinuity of the associated integral functionals (1.3).

With symmetric joint convexity at our disposal, we can give explicit examples of functionals complying with our setting, see Section 4. For instance, in Subsection 4.1 we show that the functionals frequently used to describe *cohesive fracture* energies, namely

$$u \mapsto \int_{J_u} |u^+ - u^-|\,\mathrm{d}\mathcal{H}^{d-1}\,, \qquad u \mapsto \int_{J_u} \min\{|u^+ - u^-|, M\}\,\mathrm{d}\mathcal{H}^{d-1},$$

are lower semicontinuous. The key observation to this purpose is that for all (i,j,ν) it holds

$$|i-j|\,|\nu| = \sup\langle Bi - Bj, \nu\rangle,$$

where the supremum is taken over all symmetric matrices B having *operator norm* at most 1, see Lemma 4.3. Note that the vector field $x \mapsto Bx$ is conservative by symmetry of the matrix B. Then, by decomposing the action of B into one-dimensional, orthogonal eigenspaces and truncating the resulting functions of one variable, we can approximate $x \mapsto Bx$ from below with bounded, uniformly continuous, and conservative vector fields V_h. This shows that the integrands are symmetric jointly convex, which yields the desired lower semicontinuity. The results in Subsection 4.1 also apply to more general surface integrands of the form $g(|i-j|)|\nu|$, for increasing subadditive functions g. (Actually, an additional restriction has to be imposed, cf. the statement of Theorem 4.1.) Notice that these integrands are *isotropic*, in contrast to similar ones considered in [5] which (as we will discuss later) may instead fail to be BD-elliptic.

A slight modification of the arguments in Subsection 4.1 also allows to prove the symmetric joint convexity of the surface integrands introduced in [28], one of the very rare examples of lower semicontinuous energies on BD to date. In particular, the functionals

$$u \mapsto \int_{J_u} |(u^+ - u^-)\odot \nu_u|\,\mathrm{d}\mathcal{H}^{d-1}\,, \qquad u \mapsto \int_{J_u} G_M\big((u^+ - u^-)\odot\nu_u\big)\,\mathrm{d}\mathcal{H}^{d-1} \tag{1.5}$$

are lower semicontinuous in $PR(\Omega)$, where $|\cdot|$ is the Frobenius norm, $\odot$ denotes the symmetrized tensor product, and the second integrand is a suitable truncation of the first one, explicitly calculated in [28, Section 6].

After further examples in Subsections 4.3–4.5, in Subsection 4.6 we instead address the comparison between BD- and BV-ellipticity. In particular, we show with a counterexample (Example 4.12) that anisotropic integrands of the form $|i-j|\psi(\nu)$, where ψ denotes a norm different from the Euclidean one, are in general not BD-elliptic[1]. As these functions are known to be BV-elliptic, the associated functionals are lower semicontinuous in the SBV-weak topology considered in [2], but not in the analogous topology in the space of special functions of bounded deformation. A

[1]The counterexample can be extended straightforwardly to the case $\theta(|i-j|)\psi(\nu)$ for a suitable bounded θ if one does not want to cope with unbounded integrands.

similar counterexample can also be provided for the case of integrands which are anisotropic in the jump direction, see Example 4.13.

Whereas BD-ellipticity provides a complete theoretical framework for lower semicontinuity of surface energies in $PR(\Omega)$ and can be used for providing counterexamples in the larger space $GSBD^p$, a complete characterization of lower semicontinuity in $GSBD^p$ is still missing. Indeed, unlike the BV case, we cannot reconduct in general the problem to the analogous one for piecewise rigid functions, essentially due to the lack of a "coarea-like" formula in our setting. However, our results can be succesfully exploited to tackle the well-posedness of miminum problems for energies in $GSBD^p$, provided we assume the surface integrands to be symmetric jointly convex. In fact, we can give a direct proof that functionals of the form (1.3), with f symmetric jointly convex, are lower semicontinuous along sequences of $GSBD^p$ functions which converge in measure, whose jump sets have uniformly bounded $\mathcal{H}^{d-1}$-measure, and whose symmetrized gradient have equibounded L^p norm, see Theorem 5.1. The proof makes use of an integration-by-parts formula, which in its turn relies on the fact that the vector fields in (1.4) are conservative. The latter ensures that the Lebesgue part of the distributional divergence of the composite functions $V_h(u)$, with $u \in GSBD(\Omega)$, only depends on V_h, u, and the symmetrized gradient $e(u)$, see Lemma 5.3. Eventually, this allows us to successfully adapt the localization procedure of [2, Theorem 3.6] to our setting.

As a final remark, we point out that, if we combine the above-mentioned results of Sections 4 and 5 with compactness in $GSBD$ [27, Theorem 11.3], we obtain the well-posedness of some variational problems of relevant applied interest, such as

$$
\begin{aligned}
&u \mapsto \int_\Omega |e(u)|^p \,\mathrm{d}x + \int_{J_u} (1 + |u^+ - u^-|) \,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega \Psi(|u|) \,\mathrm{d}x\,,\\
&u \mapsto \int_\Omega |e(u)|^p \,\mathrm{d}x + \int_{J_u} \big(1 + \min\{|u^+ - u^-|, M\}\big) \,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega \Psi(|u|) \,\mathrm{d}x\\
&u \mapsto \int_\Omega |e(u)|^p \,\mathrm{d}x + \int_{J_u} (1 + |(u^+ - u^-) \odot \nu_u|) \,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega \Psi(|u|) \,\mathrm{d}x\\
&u \mapsto \int_\Omega |e(u)|^p \,\mathrm{d}x + \int_{J_u} \big(1 + G_M((u^+ - u^-) \odot \nu_u)\big) \,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega \Psi(|u|) \,\mathrm{d}x\,,
\end{aligned}
$$

where G_M is the suitable truncation of the Frobenius norm of $(u^+ - u^-) \odot \nu_u$ introduced in [28], see (1.5) above, and $\Psi : [0, +\infty) \to [0, +\infty)$ is a coercive function needed for applying the compactness theorem. We consider this as being a major outcome of our results. Let us mention that we did not address in this paper the possibility of working in the larger space $GSBD^p_\infty$ introduced recently in [25], building on a recent compactness result by Chambolle and Crismale [21]. This would allow us to drop the additional term $\int_\Omega \Psi(|u|) \,\mathrm{d}x$, in favor of Dirichlet boundary conditions. In any case, this is a further interesting issue which we plan to address in the future.

Organization of the paper and notation: The paper is organized as follows. In Section 2 we introduce our setting, define BD-ellipticity, and prove the lower semicontinuity of BD-elliptic functionals in $PR(\Omega)$. Here, we also address the problem of relaxation. Section 3 is devoted to the notion of symmetric joint convexity. There, we prove BD-ellipticity and lower semicontinuity in $PR(\Omega)$ of the associated energies. In Section 4, we discuss the aforementioned relevant examples of functionals which comply with our assumptions, as well as the comparison between the notions of BV- and BD-ellipticity. Finally, in Section 5 we prove that surface energies associated to symmetric jointly convex integrands are lower semicontinuous in $GSBD^p$.

We close the introduction by fixing notations. Throughout the paper, $\Omega \subset \mathbb{R}^d$ is open, bounded with Lipschitz boundary. Let $\mathcal{A}(\Omega)$ be the family of open subsets of Ω, and let $\mathcal{A}_0(\Omega) \subset \mathcal{A}(\Omega)$ be

the subset of sets with regular boundary. The notations $\mathcal{L}^d$ and $\mathcal{H}^{d-1}$ are used for the Lebesgue measure, and the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$, respectively. For an $\mathcal{L}^d$-measurable set $E \subset \mathbb{R}^d$, the symbol χ_E denotes its characteristic function. For $A, B \in \mathcal{A}(\Omega)$ with $\overline{B} \subset A$, we write $B \subset\subset A$. By $A \triangle B = (A \setminus B) \cup (B \setminus A)$ we denote the symmetric difference of two sets. The symbol $B_R(x)$ denotes a ball of radius R centered at x.

Components of vectors $\mu \in \mathbb{R}^d$ are generally indicated by μ_k, $k = 1, \dots, d$. We write $\langle \mu, \mu' \rangle$ for the scalar product of two vectors $\mu, \mu' \in \mathbb{R}^d$. The space of symmetric and skew-symmetric matrices is denoted by $\mathbb{R}^{d\times d}_{\rm sym}$ and $\mathbb{R}^{d\times d}_{\rm skew}$, respectively, while the identity in $\mathbb{R}^{d\times d}$ is indicated by Id. The Frobenius norm of a matrix $A \in \mathbb{R}^{d\times d}$ is indicated by $|A|$, and $\|A\|$ denotes the operator norm. The scalar product of two matrices $A, B \in \mathbb{R}^{d\times d}$ is indicated by $A : B$. The symbol $\mathbb{S}^{d-1}$ stands for the unit sphere in $\mathbb{R}^d$. For $\nu \in \mathbb{S}^{d-1}$, we denote by $Q^\nu_\rho \subset \mathbb{R}^d$ the d-dimensional cube, centered in the origin, with sidelength $\rho > 0$, and two faces orthogonal to ν.

2. BD-ellipticity and lower semicontinuity

In this section, we consider functionals defined on piecewise rigid functions, and we characterize lower semicontinuity in terms of an integral condition that we call *BD-ellipticity*. Afterwards, we also address the problem of relaxation.

2.1. **Definitions.**

In this subsection, we collect the basic definitions.

Function spaces: First, we define the space of *piecewise rigid functions* by

$$PR(\Omega) := \big\{ u\colon \Omega \to \mathbb{R}^d \ \mathcal{L}^d\text{-measurable:} \ \ u(x) = \sum\nolimits_{k\in\mathbb{N}} (Q_k\, x + b_k)\chi_{P_k}(x) \quad \forall x \in \Omega, \\ \text{where } Q_k \in \mathbb{R}^{d\times d}_{\rm skew},\ b_k \in \mathbb{R}^d, \text{ and } (P_k)_k \text{ is a Caccioppoli partition of } \Omega \big\}. \tag{2.1}$$

We will sometimes use the shorthand $a_{Q,b}(x) := Qx + b$ with $Q \in \mathbb{R}^{d\times d}_{\rm skew}$ and $b \in \mathbb{R}^d$. It follows from the properties of Caccioppoli partitions, see [7, Section 4.4], that for each $u \in PR(\Omega)$ we have that $\mathcal{H}^{d-1}(J_u \setminus \bigcup_k \partial^* P_k) = 0$ and thus $\mathcal{H}^{d-1}(J_u) < +\infty$. We also note that the representation in (2.1) can always be chosen in such a way that also $\mathcal{H}^{d-1}(J_u \triangle (\bigcup_k \partial^* P_k \setminus \partial\Omega)) = 0$ holds, cf. [33, Equation (3.2)]. In the following, we say that a sequence $(u_h)_h$ converges to u in $PR(\Omega)$ if $\sup_h \mathcal{H}^{d-1}(J_{u_h}) < +\infty$ and $u_h \to u$ in measure on Ω.

If $u \in PR(\Omega)$ has the form $u = \sum_{k\in\mathbb{N}} b_k \chi_{P_k}$, i.e., $Q_k = 0$ for all $k \in \mathbb{N}$ in representation (2.1), then $u \in PC(\Omega)$, where $PC(\Omega) \subset PR(\Omega)$ denotes the subspace of *piecewise constant functions*. We note that there hold the inclusions $PC(\Omega) \subset GSBV(\Omega)$ and $PR(\Omega) \subset GSBD(\Omega)$, see [7, Section 4.5] and [27], respectively, for definitions and properties of the latter function spaces.

BV- and BD-ellipticity: Given $\mathcal{L}^d$-measurable functions $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \mapsto [0, +\infty)$, we consider integral functionals $\mathcal{F}\colon PR(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty)$ of the form

$$\mathcal{F}(u, A) = \int_{J_u \cap A} f(u^+, u^-, \nu_u)\, \mathrm{d}\mathcal{H}^{d-1} \qquad \forall\, u \in PR(\Omega), \qquad \forall\, A \in \mathcal{A}(\Omega), \tag{2.2}$$

where u^+, u^- represent the approximate one-sided traces of u on J_u, ν_u denotes a unit normal to the jump (i.e., a normal to the interface), and f represents an interfacial energy density. (In the following, we will sometimes also write $[u] := u^+ - u^-$.) We often write $\mathcal{F}(u)$ instead of $\mathcal{F}(u, \Omega)$ if no confusion arises. We assume that the functions f satisfy the symmetry condition

$$f(i, j, \nu) = f(j, i, -\nu) \qquad \text{for all } (i, j, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}. \tag{2.3}$$

We first restrict the functionals onto the subspace $PC(\Omega)$ and recall the notion of BV-ellipticity introduced in [5]. Fix $(i, j, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$ with $i \neq j$, and define the function $u_{i,j,\nu}\colon Q^\nu_1 \to \mathbb{R}^d$

by

$$u_{i,j,\nu}(x) := \begin{cases} i & \langle x, \nu\rangle > 0, \\ j & \langle x, \nu\rangle \le 0. \end{cases} \tag{2.4}$$

Then, we say that f is BV-elliptic if

$$\int_{J_v} f(v^+, v^-, \nu_v)\, \mathrm{d}\mathcal{H}^{d-1} \ge f(i, j, \nu)$$

for any $v \in PC(Q_1^\nu)$ such that $\{u_{i,j,\nu} \neq v\} \subset\subset Q_1^\nu$ and for any triple (i, j, ν) in the domain of f with $i \neq j$. This notion is necessary and sufficient for lower semicontinuity in the space $PC(\Omega)$, whenever f is continuous and bounded, see [7, Theorem 5.14]. It plays the analogous role of quasiconvexity for integral functionals defined on Sobolev spaces. In particular, we recall from [7, Theorem 5.11] the following two necessary conditions for lower semicontinuity for continuous densities:

i) (*subadditivity*) for any $\rho \in \mathbb{S}^{d-1}$ we have

$$f(i, j, \rho) \le f(i, k, \rho) + f(k, j, \rho) \quad \forall\, i, j, k \in \mathbb{R}^d;$$

ii) (*convexity*) for any $i, j \in \mathbb{R}^d$, the function $\rho \mapsto f(i, j, \rho)$ is convex in $\mathbb{R}^d$.

(We point out that, strictly speaking, [7, Theorem 5.11, Theorem 5.14] have been shown only when i and j are chosen from a countable, bounded subset of $\mathbb{R}^d$. An inspection of the proofs, however, shows that the results can be generalized to the whole $\mathbb{R}^d$.)

We now introduce a similar notion for functionals defined on $PR(\Omega)$ which we call *BD-ellipticity*.

Definition 2.1. Let $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \mapsto [0, +\infty)$ be an $\mathcal{L}^d$-measurable function. We say that f is *BD-elliptic* if

$$\int_{J_v} f(v^+, v^-, \nu_v)\, \mathrm{d}\mathcal{H}^{d-1} \ge f(i, j, \nu) \tag{2.5}$$

for any $v \in PR(Q_1^\nu)$ such that $\{u_{i,j,\nu} \neq v\} \subset\subset Q_1^\nu$ and for any triple (i, j, ν) in the domain of f with $i \neq j$.

We have chosen this name in analogy to BV-ellipticity to highlight that $PR(\Omega)$ is related to the space BD, whereas the space $PC(\Omega)$ is related to the theory of BV-functions. We also remark that inequality (2.5) needs to hold only for $i \neq j$, as the values $f(i, i, \nu)$ clearly do not matter for the functionals in (2.2).

We observe that every BD-elliptic function is of course also BV-elliptic since $PC(\Omega) \subset PR(\Omega)$. In particular, the two properties stated above (subadditivity and convexity) are necessary for BD-ellipticity. The reverse implication does not hold, i.e., BV- and BD-ellipticity are really different notions. In fact, functions $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0, +\infty)$ of the form

$$f(i, j, \nu) = \theta(i, j)\, \psi(\nu) \tag{2.6}$$

are BV-elliptic if $\theta\colon \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ is a pseudo-distance (i.e., positive, symmetric obeying the triangle inequality) and $\psi\colon \mathbb{R}^d \to [0, +\infty)$ is even, positively 1-homogeneous, and convex. We refer to [7, Example 5.23] for details. On the other hand, as we will detail below in Examples 4.12–4.13, these functions are in general not BD-elliptic if they are *anisotropic*, i.e., if $\theta(i, j)$ oscillates on $\{(i, j)\colon |i - j| = const.\}$ or $\psi(\nu)$ oscillates for $\nu \in \mathbb{S}^{d-1}$.

2.2. **Characterization of lower semicontinuity.** Recall that BV-ellipticity has been identified as a necessary and sufficient condition for lower semicontinuity of functionals defined on $PC(\Omega)$. In this subsection, we establish a corresponding result in $PR(\Omega)$ in terms of BD-ellipticity. To this end, we need to assume a slightly stronger continuity condition of the integrands f, namely uniform continuity in the first two variables: there exists an increasing modulus of continuity $\sigma : [0,+\infty) \to [0,+\infty)$ with $\sigma(0) = 0$ such that for any $(i_1, j_1), (i_2, j_2) \in \mathbb{R}^d \times \mathbb{R}^d$ we have

$$|f(i_1, j_1, \nu) - f(i_2, j_2, \nu)| \le \sigma\big(|i_1 - i_2| + |j_1 - j_2|\big). \tag{2.7}$$

Theorem 2.2 (Characterization of lower semicontinuity)**.** *Let $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ be a bounded, continuous function satisfying* (2.3) *and* (2.7)*. Then, $\mathcal{F}$ defined in (2.2) is lower semicontinuous in $PR(\Omega)$ if and only if f is BD-elliptic.*

In order to prove the above result, we need the following fundamental estimate slightly adapted for our purposes, see [33, Lemma 4.5].

Lemma 2.3. *Let $\eta > 0$ and $A', A, B \in \mathcal{A}_0(\Omega)$ with $A' \subset\subset A$. Let $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ be a measurable, bounded function satisfying* (2.3)*,* (2.7)*, and* $\inf f > 0$*. Let $\psi\colon [0,+\infty) \to [0,+\infty)$ be continuous and strictly increasing with $\psi(0) = 0$. Then, there exist a function $\Psi\colon PR(B) \to (0,+\infty]$ and a lower semicontinuous function $\Lambda : PR(A) \times PR(B) \to [0,+\infty]$ satisfying*

$$\Lambda(z_1, z_2) \to 0 \quad \text{whenever} \quad \int_{(A\setminus A')\cap B} \psi(|z_1 - z_2|) \to 0 \tag{2.8}$$

such that for all $u \in PR(A)$ and $v \in PR(B)$ satisfying the condition

$$\Lambda(u, v) \le \Psi(v) \tag{2.9}$$

there exists a function $w \in PR(A' \cup B)$ such that

$$\begin{aligned}
&\text{(i)} \quad \mathcal{F}(w, A' \cup B) \le \mathcal{F}(u, A) + \mathcal{F}(v, B) + \big(M + \mathcal{F}(u, A) + \mathcal{F}(v, B)\big)\big(2\eta + M'\sigma(\Theta(u, v))\big),\\
&\text{(ii)} \quad \|\min\{|w - u|, |w - v|\}\|_{L^\infty(A'\cup B)} \le \Theta(u, v),\\
&\text{(iii)} \quad w = v \text{ on } B \setminus A.
\end{aligned} \tag{2.10}$$

Here, $\mathcal{F}$ is of the form (2.2)*, σ is given in* (2.7)*, and $M, M' > 0$ as well as $\Theta\colon PR(A) \times PR(B) \to [0,+\infty]$ are independent of u and v. Moreover, M is also independent of η and Θ is a lower semicontinuous function satisfying*

$$\Theta(z_1, z_2) \to 0 \quad \text{whenever} \quad \int_{(A\setminus A')\cap B} \psi(|z_1 - z_2|) \to 0\,. \tag{2.11}$$

In the above result, we follow the convention that $u \in PR(A)$ and $v \in PR(B)$ are extended by $u = 0$ and $v = 0$ outside of A and B, respectively. Condition (2.9) is necessary to ensure (2.10)(iii). As detailed in [33, Lemma 4.1], (2.9) can be removed at the expense of dropping also (2.10)(iii).

Proof. We briefly explain how the result follows from [33, Lemma 4.5]. The functional $\mathcal{F}$ satisfies (H_1) since f is measurable and (H_3) holds due to the integral representation (2.2). Property (H_4) follows from the fact that f is bounded and satisfies $\inf f > 0$, where we set $\alpha = \inf f > 0$ and $\beta = \sup f < \infty$. Property (2.7) implies (H_5'). Condition (2.9) is equivalent to [33, Equation (4.6)] for given $\delta > 0$, $M_1 \ge 0$, and $\Psi(v) := \frac{1}{M_1}\Phi(A', A' \cup B; v|_{B\setminus\overline{A'}}, \delta) \in (0,+\infty]$. Then, [33, Equation (4.7)] implies (2.10), where we set $M := \mathcal{H}^{d-1}(\partial A' \cup \partial A \cup \partial B)$ and $M' := M_2$. □

Proof of Theorem 2.2. Our proof is in the spirit of [7, Theorem 5.14] with the essential difference that in the implication "BD-ellipticity implies lower semicontinuity" we replace the lemma of

joining two functions, see [7, Lemma 5.15] or [4, Lemma 4.4], by our fundamental estimate stated in Lemma 2.3. We show the two directions separately.

Step 1: Lower semicontinuity implies BD-ellipticity. The argument is very similar to the one used in [7, Theorem 5.14], and we therefore only sketch it. Up to a rescaling and a translation of Q_1^ν, we may assume that $Q_1^\nu \subset \Omega$. Consider $i, j \in \mathbb{R}^d$ with $i \neq j$, $\nu \in \mathbb{S}^{d-1}$, and some $v \in PR(Q_1^\nu)$ with $\{u_{i,j,\nu} \neq v\} \subset\subset Q_1^\nu$. For each $h \in \mathbb{N}$, we define $u_h \in PR(\Omega)$ by $u_h = u_{i,j,\nu}$ on $\Omega \setminus Q_1^\nu$ and inside Q_1^ν we set

$$u_h(x) = \begin{cases} i & \text{if } \langle x, \nu\rangle > 1/h, \\ v(h(x - x_n)) & \text{if } 0 < \langle x, \nu\rangle < 1/h \text{ and } x \in Q_n, \\ j & \text{if } \langle x, \nu\rangle < 0, \end{cases}$$

where $(Q_n)_n$ denotes a partition of the set $\{x \in Q_1^\nu \colon 0 < \langle x, \nu\rangle < 1/h\}$ consisting of h^{d-1} cubes with sidelength $1/h$, and x_n indicates the center of Q_n. As $\{u_{i,j,\nu} \neq v\} \subset\subset Q_1^\nu$, we find by a scaling argument

$$\begin{aligned} \mathcal{F}(u_h) &\leq \sum_{n=1}^{h^{d-1}} \int_{J_{u_h} \cap Q_n} f(u_h^+, u_h^-, \nu_{u_h}) \, \mathrm{d}\mathcal{H}^{d-1} + \|f\|_\infty \mathcal{H}^{d-1}(J_{u_h} \cap \partial Q_1^\nu) + \mathcal{H}^{d-1}(\Gamma) f(i,j,\nu) \\ &\leq \int_{J_v \cap Q_1^\nu} f(v^+, v^-, \nu_v) \, \mathrm{d}\mathcal{H}^{d-1} + C\|f\|_\infty / h + \mathcal{H}^{d-1}(\Gamma) f(i,j,\nu), \end{aligned}$$

where $\Gamma := \{x \colon \langle x, \nu\rangle = 0\} \cap (\Omega \setminus Q_1^\nu)$ and $C > 0$ is a universal constant. Since $\mathcal{L}^d(\{u_h \neq u_{i,j,\nu}\}) \leq 1/h$, we find $u_h \to u_{i,j,\nu}$ in measure on Ω. Therefore, by the lower semicontinuiy of $\mathcal{F}$ we conclude

$$\begin{aligned} \int_{J_v \cap Q_1^\nu} f(v^+, v^-, \nu_v) \, \mathrm{d}\mathcal{H}^{d-1} &\geq \liminf_{h\to\infty} \mathcal{F}(u_h) - \mathcal{H}^{d-1}(\Gamma) f(i,j,\nu) \geq \mathcal{F}(u_{i,j,\nu}) - \mathcal{H}^{d-1}(\Gamma) f(i,j,\nu) \\ &= \big(\mathcal{H}^{d-1}(J_{u_{i,j,\nu}} \cap \Omega) - \mathcal{H}^{d-1}(\Gamma)\big) f(i,j,\nu) = f(i,j,\nu). \end{aligned}$$

This shows that f is BD-elliptic.

Step 2: BD-ellipticity implies lower semicontinuity. We detail this step only in the special case $\Omega = Q_1^\nu$ for the special limiting function $u_{i,j,\nu}$ for some $i, j \in \mathbb{R}^d$ with $i \neq j$ and $\nu \in \mathbb{S}^{d-1}$. The general case follows by standard covering and blow up arguments. We refer to Step 2 and Step 3 in the proof of [7, Theorem 5.14] for details.

Let $(u_h)_h \subset PR(Q_1^\nu)$ be a sequence converging to $u_{i,j,\nu}$ in $PR(Q_1^\nu)$. In particular, we have $\sup_h \mathcal{H}^{d-1}(J_{u_h}) < +\infty$, and the boundedness of f then implies

$$\sup_h \mathcal{F}(u_h, Q_1^\nu) < +\infty. \tag{2.12}$$

We first suppose that $\inf f > 0$ and explain the small adaptions for $\inf f = 0$ at the end of the proof. We want to construct a sequence $(w_h)_h \subset PR(Q_1^\nu)$ such that $\{u_{i,j,\nu} \neq w_h\} \subset\subset Q_1^\nu$ and such that the energy of w_h is asymptotically controlled by the one of u_h. Our strategy relies on Lemma 2.3.

To this end, we first fix $\eta > 0$, $\rho > 0$, and define the sets $B, A, A' \subset \mathcal{A}_0(Q_1^\nu)$ by $A' = Q_{1-2\rho}^\nu$, $A = Q_{1-\rho}^\nu$, and $B = Q_1^\nu \setminus Q_{1-3\rho}^\nu$. Note that $A' \cup B = Q_1^\nu$. In order to apply Lemma 2.3 for $u = u_h|_A$ and $v = u_{i,j,\nu}|_B$, we need to check (2.9). As $u_h \to u_{i,j,\nu}$ in measure on Q_1^ν, we clearly get

$$\int_{Q_1^\nu} \psi(|u_h - u_{i,j,\nu}|) \to 0 \tag{2.13}$$

as $h \to \infty$, where $\psi\colon [0,+\infty) \to [0,+\infty)$ is defined by $\psi(t) := t/(1+t)$ for $t \geq 0$. Therefore, $\Lambda(u_h|_A, u_{i,j,\nu}|_B) \to 0$ by (2.8). Consequently, there holds $\Lambda(u_h|_A, u_{i,j,\nu}|_B) \leq \Psi(u_{i,j,\nu}|_B)$ for all h sufficiently large and thus (2.9) holds.

We apply Lemma 2.3 for $u = u_h|_A$ and $v = u_{i,j,\nu}|_B$, and obtain $w_h \in PR(Q_1^\nu)$ such that by (2.10) there holds $w_h = u_{i,j,\nu}$ on $Q_1^\nu \setminus Q_{1-\rho}^\nu$ and

$$\mathcal{F}(w_h, Q_1^\nu) \leq \mathcal{F}\big(u_h, Q_{1-\rho}^\nu\big) + \mathcal{F}\big(u_{i,j,\nu}, Q_1^\nu \setminus Q_{1-3\rho}^\nu\big) + I_1^{h,\eta} + I_2^{h,\eta}, \tag{2.14}$$

where for shorthand we have set

$$I_1^{h,\eta} = 2\eta\left(M + \mathcal{F}(u_h, A) + \mathcal{F}(u_{i,j,\nu}, B)\right),$$
$$I_2^{h,\eta} = M'\sigma\big(\Theta(u_h|_A, u_{i,j,\nu}|_B)\big)\left(M + \mathcal{F}(u_h, A) + \mathcal{F}(u_{i,j,\nu}, B)\right).$$

Since f is nonnegative and BD-elliptic, and there holds $\{u_{i,j,\nu} \neq w_h\} \subset\subset Q_1^\nu$, we get by (2.14)

$$\mathcal{F}(u_{i,j,\nu}, Q_1^\nu) \leq \mathcal{F}(w_h, Q_1^\nu) \leq \mathcal{F}\big(u_h, Q_1^\nu\big) + \mathcal{F}\big(u_{i,j,\nu}, Q_1^\nu \setminus Q_{1-3\rho}^\nu\big) + I_1^{h,\eta} + I_2^{h,\eta}. \tag{2.15}$$

By $\sigma(0) = 0$, (2.11), (2.12), and (2.13) we obtain $\lim_{h\to\infty} I_2^{h,\eta} = 0$. This along with (2.15) implies

$$\mathcal{F}(u_{i,j,\nu}, Q_1^\nu) \leq \liminf_{h\to\infty} \mathcal{F}(u_h, Q_1^\nu) + \mathcal{F}\big(u_{i,j,\nu}, Q_1^\nu \setminus Q_{1-3\rho}^\nu\big) + \sup_h I_1^{h,\eta}.$$

By (2.12) and the fact that M is independent of η we get $\lim_{\eta\to 0}(\sup_h I_1^{h,\eta}) = 0$. Thus, passing to the limits $\eta, \rho \to 0$, we obtain the desired estimate

$$\mathcal{F}(u_{i,j,\nu}, Q_1^\nu) \leq \liminf_{h\to\infty} \mathcal{F}(u_h, Q_1^\nu).$$

This concludes the proof in the case $\inf f > 0$. If $\inf f = 0$ instead, we consider densities $f_\varepsilon = f + \varepsilon$ for arbitrary $\varepsilon > 0$. As f is BD-elliptic and the constant function with value ε is BD-elliptic (see e.g. Proposition 4.10 below), we see that also f_ε is BD-elliptic. Then, the functional $\mathcal{F}_\varepsilon$ with density f_ε is lower semicontinuous and we obtain

$$\mathcal{F}(u) \leq \mathcal{F}_\varepsilon(u) \leq \liminf_{h\to\infty} \mathcal{F}_\varepsilon(u_h) \leq \liminf_{h\to\infty} \mathcal{F}(u_h) + \varepsilon \sup_h \mathcal{H}^{d-1}(J_{u_h}).$$

We conclude the proof by passing to $\varepsilon \to 0$ and using (2.12). □

Remark 2.4. For later purposes, we note that in Step 1 of the proof we only used the boundedness of f but not its continuity. In other words, lower semicontinuity in $PR(\Omega)$ implies BD-ellipticity for bounded, measurable functions $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ satisfying (2.3).

We now drop the condition that f is a bounded function, and obtain the following two corollaries.

Corollary 2.5 (Lower semicontinuity for unbounded functions)**.** *If f is a continuous, BD-elliptic function satisfying* (2.3)*, then $\mathcal{F}$ defined in (2.2) is lower semicontinuous along sequences converging in $PR(\Omega)$ which are bounded in $L^\infty(\Omega;\mathbb{R}^d)$.*

Proof. Given $(u_h)_h$ with $M := \sup_{h\in\mathbb{N}} \|u_h\|_\infty < +\infty$, we choose a bounded, continuous function $\tilde{f}\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that $f(i,j,\nu) = \tilde{f}(i,j,\nu)$ whenever $|i|, |j| \leq M$ and $\nu \in \mathbb{S}^{d-1}$. By uniform continuity on compact sets, this can be achieved in such a way that $\tilde{f}$ satisfies also (2.7). The statement now follows from Theorem 2.2 noting that the sequence of energies $(\mathcal{F}(u_h))_h$ remains unchanged when f is replaced by $\tilde{f}$ in (2.2). □

Corollary 2.6 (Supremum of bounded BD-elliptic functions)**.** *Let $f_h\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$, $h \in \mathbb{N}$, be a sequence of continuous, bounded, and BD-elliptic functions satisfying* (2.3) *and* (2.7)*. Suppose $f(i,j,\nu) := \sup_{h\in\mathbb{N}} f_h(i,j,\nu) < +\infty$ for all $i,j \in \mathbb{R}^d$, $i \neq j$, and $\nu \in \mathbb{S}^{d-1}$. Then, f is BD-elliptic and the corresponding functional $\mathcal{F}$ defined in (2.2) is lower semicontinuous on $PR(\Omega)$.*

To prove the above corollary, let us recall the following lemma (see, e.g., [13, Lemma 15.2]).

Lemma 2.7. *Let Ω be an open subset of $\mathbb{R}^d$. Let Λ be a set function defined on $\mathcal{A}(\Omega)$, which is supperadditive on open sets with disjoint compact closure, i.e., $\Lambda(U\cup V)\geq\Lambda(U)+\Lambda(V)$ whenever U, $V\subset\subset\Omega$ and $\overline{U}\cap\overline{V}=\emptyset$. Let λ be a positive measure on Ω, and let $(\varphi_h)_h$ be a sequence of nonnegative Borel functions on Ω such that $\Lambda(A)\geq\int_A\varphi_h\,\mathrm{d}\lambda$ for every $A\in\mathcal{A}(\Omega)$ and $h\in\mathbb{N}$. Then, $\int_A\sup_h\varphi_h\mathrm{d}\lambda\leq\Lambda(A)$ for every $A\in\mathcal{A}(\Omega)$.*

Proof of Corollary 2.6. We first prove BD-ellipticity and then lower semicontinuity.

Step 1: BD-ellipticity. We first show that f is BD-elliptic. Fix a triple $(i,j,\nu)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}$ with $i\neq j$. Let $u\in PR(Q_1^\nu)$ be such that $\{u\neq u_{i,j,\nu}\}\subset\subset Q_1^\nu$, where $u_{i,j,\nu}$ is defined in (2.4). Then, since each f_h is BD-elliptic, we get

$$\begin{aligned}\mathcal{F}(u,Q_1^\nu)&=\int_{J_u}f(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}=\int_{J_u}\sup_{h\in\mathbb{N}}f_h(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}\\&\geq\sup_{h\in\mathbb{N}}\int_{J_u}f_h(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}\geq\sup_{h\in\mathbb{N}}f_h(i,j,\nu)=f(i,j,\nu).\end{aligned}$$

Step 2: Lower semicontinuity. Consider a sequence $(u_n)_n\subset PR(\Omega)$ and $u\in PR(\Omega)$ such that $u_n\to u$ in $PR(\Omega)$ as $n\to\infty$. Our goal is to show

$$\liminf_{n\to\infty}\mathcal{F}(u_n,\Omega)\geq\mathcal{F}(u,\Omega).\tag{2.16}$$

In view of Theorem 2.2, the functional with integrand f_h is lower semicontinuous for every $h\in\mathbb{N}$. Therefore, we get for every $U\in\mathcal{A}(\Omega)$

$$\begin{aligned}\liminf_{n\to\infty}\int_{J_{u_n}\cap U}f(u_n^+,u_n^-,\nu_{u_n})\,\mathrm{d}\mathcal{H}^{d-1}&\geq\liminf_{n\to\infty}\int_{J_{u_n}\cap U}f_h(u_n^+,u_n^-,\nu_{u_n})\,\mathrm{d}\mathcal{H}^{d-1}\\&\geq\int_{J_u\cap U}f_h(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}\tag{2.17}$$

We define the superadditive function $\Lambda\colon\mathcal{A}(\Omega)\to[0,+\infty)$ by

$$\Lambda(U):=\liminf_{n\to\infty}\int_{J_{u_n}\cap U}f(u_n^+,u_n^-,\nu_{u_n})\,\mathrm{d}\mathcal{H}^{d-1}$$

for each $U\in\mathcal{A}(\Omega)$. Thus, by (2.17) we obtain

$$\Lambda(U)\geq\int_{J_u\cap U}f_h(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}$$

for all $U\in\mathcal{A}(\Omega)$ and all $h\in\mathbb{N}$. By applying Lemma 2.7 we get that

$$\Lambda(U)\geq\int_{J_u\cap U}\sup_{h\in\mathbb{N}}f_h(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}=\int_{J_u\cap U}f(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}$$

for all $U\in\mathcal{A}(\Omega)$. For $U=\Omega$, we obtain (2.16). This concludes the proof. □

2.3. **Relaxation.** In this subsection, we address the relaxation of integral functionals of the form (2.2). For simplicity, we restrict our study to the class of translational invariant integrands, i.e., functions $f\colon\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}\mapsto[0,+\infty)$ satisfying

$$f(i,j,\nu)=f(i+t,j+t,\nu)\qquad\text{for all }(i,j,\nu)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}\text{ and }t\in\mathbb{R}^d.\tag{2.18}$$

In other words, differently from what considered so far, functionals of the form (2.2) depend on the two vectors $([u],\nu_u)$ rather than on the more general triple (u^+,u^-,ν_u). (For consistency, we keep the notation (u^+,u^-,ν_u) in the following.) This assumption is due to the fact that we will use

an integral representation result [33] which has been proved in this slightly more specific setting only. In [33], however, translational invariance is assumed just to simplify the exposition, and a generalization to the general situation of (2.2) would in principle be possible. We note that, under (2.18), the continuity condition (2.7) can be reduced to

$$|f(\xi,0,\nu)-f(\tau,0,\nu)|\le\sigma\big(|\xi-\tau|\big)\qquad\text{for all }\xi,\tau\in\mathbb{R}^d,\nu\in\mathbb{S}^{d-1}. \tag{2.19}$$

Before we come to the main result of this subsection, we introduce a further notation: for every $u\in PR(\Omega)$ and $A\in\mathcal{A}(\Omega)$ we define

$$\mathbf{m}_{\mathcal{F}}(u,A)=\inf_{v\in PR(\Omega)}\big\{\mathcal{F}(v,A)\colon\ \{v\neq u\}\subset\subset A\big\}.$$

Theorem 2.8. *Let $f:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}\to[0,+\infty)$ be a bounded, continuous function satisfying* (2.3), (2.18), (2.19), *and* $\inf f>0$. *Let $\mathcal{F}$ be defined in (2.2). Then, the relaxed functional defined as*

$$\bar{\mathcal{F}}(u,A):=\inf\left\{\liminf_{h\to\infty}\mathcal{F}(u_h,A)\colon\ u_h\to u\text{ in }PR(A)\right\} \tag{2.20}$$

admits an integral representation, namely

$$\bar{\mathcal{F}}(u,A)=\int_{J_u\cap A}\varphi(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}\qquad\forall\,u\in PR(\Omega),\qquad\forall\,A\in\mathcal{A}(\Omega). \tag{2.21}$$

Here, the function $\varphi\colon\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}\to[0,+\infty)$ is the greatest BD-elliptic function with $\varphi\le f$ and is characterized by

$$\varphi(i,j,\nu)=\mathbf{m}_{\mathcal{F}}\big(u_{i,j,\nu},Q_1^\nu\big)\qquad\text{for all }(i,j,\nu)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1},\ i\neq j. \tag{2.22}$$

Moreover, φ is bounded, continuous and satisfies (2.3), (2.18), *and* (2.19).

The key ingredient is the following Γ-convergence and integral representation result, see [33, Theorem 2.3], which is slightly adapted for our purposes. For an exhaustive treatment of Γ-convergence we refer to [9, 26]. In particular, we recall that for a constant sequence of functionals the Γ-limit is given by the lower semicontinuous envelope, cf. [26, Remark 4.5].

Lemma 2.9 (Γ-convergence and integral representation)**.** *Let $(\mathcal{F}_n)_n$ be a sequence of functionals of the form* (2.2) *for continuous densities $f_n\colon\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}\to[0,+\infty)$ satisfying $0<\alpha\le\inf f_n\le\sup f_n\le\beta<+\infty$,* (2.3), (2.18), *and* (2.19) *for the same function σ. Then, there exists $\bar{\mathcal{F}}\colon PR(\Omega)\times\mathcal{A}(\Omega)\to[0,+\infty)$ and a subsequence (not relabeled) such that*

$$\bar{\mathcal{F}}(\cdot,A)=\Gamma\text{-}\lim_{n\to\infty}\mathcal{F}_n(\cdot,A)\qquad\text{with respect to convergence in measure on }A \tag{2.23}$$

for all $A\in\mathcal{A}_0(\Omega)$. Moreover, if there holds

$$\limsup_{n\to\infty}\mathbf{m}_{\mathcal{F}_n}(u,B_\varepsilon(x_0))\le\mathbf{m}_{\bar{\mathcal{F}}}(u,B_\varepsilon(x_0))\le\sup_{0<\varepsilon'<\varepsilon}\liminf_{n\to\infty}\mathbf{m}_{\mathcal{F}_n}(u,B_{\varepsilon'}(x_0)) \tag{2.24}$$

for all $u\in PR(\Omega)$ and each ball $B_\varepsilon(x_0)\subset\Omega$, then $\bar{\mathcal{F}}$ admits an integral representation of the form (2.2) *for a density f which satisfies $\alpha\le\inf f\le\sup f\le\beta$,* (2.3), (2.18), *and* (2.19).

Proof. We briefly explain how the result follows from [33, Theorem 2.3]. The functionals $\mathcal{F}_n$ satisfy (H_1) since f_n are measurable and (H_3) holds due to the integral representation (2.2). Property (H_4) follows from $\alpha\le\inf f_n\le\sup f_n\le\beta$. Property (2.19) implies (H_5).

Then, by [33, Theorem 2.3] we find a limiting functional $\bar{\mathcal{F}}$ satisfying (H_1)–(H_5) such that (2.23) holds and $\bar{\mathcal{F}}$ admits an integral representation. It remains to show that the corresponding density f satisfies $\alpha\le\inf f\le\sup f\le\beta$, (2.3), (2.18), and (2.19). In fact, (2.3) and (2.18) are obvious by [33, Equation (2.7)] and (H_4) implies $\alpha\le\inf f\le\sup f\le\beta$. Finally, by ($\mathrm{H}_5$) we get (2.19). □

Proof of Theorem 2.8. We divide the proof into three steps.

Step 1: Integral representation. In this step, we prove that (2.21) holds true. We start by applying Lemma 2.9 on the constant sequence of functionals $\mathcal{F}_n = \mathcal{F}$ for all $n \in \mathbb{N}$. (Note that Lemma 2.9 is applicable as f is bounded, $\inf f > 0$, as well as f satisfies (2.3), (2.18), and (2.19).) As $\inf f > 0$, convergence of measure is equivalent to convergence in $PR(\Omega)$ for sequences of bounded energy. Therefore, we get that the functional $\bar{\mathcal{F}}$ defined in (2.20) coincides with the Γ-limit given in (2.23), cf. [26, Remark 4.5]. Now, to show that $\bar{\mathcal{F}}$ admits an integral representation, it remains to check that (2.24) holds true. To this end, it suffices to prove that

$$\mathbf{m}_{\mathcal{F}}(u, A) = \mathbf{m}_{\bar{\mathcal{F}}}(u, A) \qquad \forall\, u \in PR(\Omega), \qquad \forall\, A \in \mathcal{A}_0(\Omega). \tag{2.25}$$

Observe that inequality "$\geq$" follows directly by definition of $\mathbf{m}_{\mathcal{F}}$ and by the fact that $\bar{\mathcal{F}} \leq \mathcal{F}$, see (2.20). The other inequality is a direct consequence of [33, Lemma 6.3]. (This result essentially relies on the fundamental estimate Lemma 2.3.) Thus, (2.25) holds true. Then, Lemma 2.9 yields that $\bar{\mathcal{F}}$ admits an integral representation. The corresponding integrand is denoted by φ in the following. From Lemma 2.9 we also get that φ is bounded and satisfies (2.3), (2.18), and (2.19).

Step 2: BD-ellipticity and representation (2.22). As Γ-limit, the functional $\bar{\mathcal{F}}$ is lower semicontinuous on $PR(\Omega)$. Since φ is also bounded and satisfies (2.3) by Step 1, we get that φ is BD-elliptic, see Remark 2.4. Now, since φ is BD-elliptic, (2.5) implies

$$\varphi(i, j, \nu) = \mathbf{m}_{\bar{\mathcal{F}}}\big(u_{i,j,\nu}, Q_1^\nu\big) \qquad \text{for all } (i, j, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1},\ i \neq j.$$

This along with (2.25) yields (2.22) and concludes Step 2 of the proof.

Step 3: Further properties of φ. To conclude the proof, it remains to show that φ is continuous and that it is the greatest BD-elliptic function below f. In view of (2.18)–(2.19), for the continuity it suffices to check that $\nu \mapsto \varphi(i, j, \nu)$ is continuous for fixed $i, j \in \mathbb{R}^d$. As φ is BD-elliptic, the mapping $\nu \mapsto \varphi(i, j, \nu)$ is convex, see Subsection 2.1. In particular, the mapping is also continuous, as desired.

Finally, we show that φ is the greatest BD-elliptic function with $\varphi \leq f$. First, $\varphi \leq f$ clearly follows from (2.22) by using $u_{i,j,\nu}$ as a competitor. On the other hand, let $\bar{\varphi}$ be another BD-elliptic function satisfying $\bar{\varphi} \leq f$. Let us prove that $\bar{\varphi} \leq \varphi$. Denoting by $\mathcal{F}_{\bar{\varphi}}$ the functional in (2.2) with density $\bar{\varphi}$, we find

$$\bar{\varphi}(i, j, \nu) = \mathbf{m}_{\mathcal{F}_{\bar{\varphi}}}\big(u_{i,j,\nu}, Q_1^\nu\big) \qquad \text{for all } (i, j, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1},\ i \neq j$$

since $\bar{\varphi}$ is BD-elliptic. Then, $\bar{\varphi} \leq f$ and (2.22) imply

$$\bar{\varphi}(i, j, \nu) = \mathbf{m}_{\mathcal{F}_{\bar{\varphi}}}\big(u_{i,j,\nu}, Q_1^\nu\big) \leq \mathbf{m}_{\mathcal{F}}\big(u_{i,j,\nu}, Q_1^\nu\big) = \varphi(i, j, \nu).$$

This shows indeed that φ is the greatest BD-elliptic function with $\varphi \leq f$ on $\{i \neq j\}$. (The values on the diagonal $\{i = j\}$ are irrelevant.) □

3. A sufficient condition for lower semicontinuity: symmetric joint convexity

Whereas Theorem 2.2 provides a characterization of lower semicontinuity in $PR(\Omega)$ for functionals defined in (2.2), the drawback is that it is in general a difficult task to check whether an integrand f is BD-elliptic or not. Therefore, we seek for a sufficient condition that (a) implies BD-ellipticity and lower semicontinuity, as well as that (b) can be checked in practice for concrete examples. For BV-ellipticity, this role is played by *jointly convex functions*. In the setting of piecewise rigid functions, we introduce a corresponding notion that we call *symmetric joint convexity*. In this section, we prove sufficiency for lower semicontinuity. We defer important examples of symmetric jointly convex functions to Section 4 below.

We recall that a vector field $g \in C(\mathbb{R}^d;\mathbb{R}^d)$ is *conservative* if there exists a potential $G \in C^1(\mathbb{R}^d)$ such that $\nabla G = g$.

Definition 3.1 (Symmetric joint convexity)**.** We say that $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$ is a *symmetric jointly convex function* if

$$f(i,j,\nu) = \sup_{h\in\mathbb{N}} \langle g_h(i) - g_h(j),\, \nu\rangle \qquad \text{for all } (i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \quad \text{with } i \neq j, \tag{3.1}$$

where $g_h\colon \mathbb{R}^d \to \mathbb{R}^d$ is a uniformly continuous, bounded, and conservative vector field for every $h \in \mathbb{N}$.

The notion is related to the class of *jointly convex functions*, see [7, Definition 5.17], which constitutes an important class of BV-elliptic functions. The essential difference in our definition is that we require the vector fields to be *conservative*. This additional property is instrumental to deal with functions for which only the symmetric part of the gradient can be controlled. We point out that the definition directly implies that f as in (3.1) satisfies (2.3). Before we proceed with the main statement of this section, we remark that the functions $(g_h)_h$ can be approximated by more regular functions.

Remark 3.2. We will sometimes approximate functions of the kind (3.1) with the supremum of more regular fields, which belong to $C^1(\mathbb{R}^d;\mathbb{R}^d)\cap W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$. In fact, each uniformly continuous, bounded, and conservative vector field can be approximated uniformly by a conservative vector field in $C^1(\mathbb{R}^d;\mathbb{R}^d)\cap W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$. (This follows by approximating the corresponding potential.) Therefore, for a given f as in (3.1) and each $\varepsilon > 0$ we can find a sequence $(g_h^\varepsilon)_h \subset C^1(\mathbb{R}^d;\mathbb{R}^d)\cap W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ such that

$$f(i,j,\nu) - \varepsilon \le \sup_{h\in\mathbb{N}} \langle g_h^\varepsilon(i) - g_h^\varepsilon(j),\, \nu\rangle \le f(i,j,\nu) + \varepsilon \quad \text{for all } (i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \text{ with } i \neq j. \tag{3.2}$$

Let us also recall that conservative C^1-vector fields are *curl-free*, where the curl is defined by $\operatorname{curl}(g) = (\partial_i g_j - \partial_j g_i)_{i,j=1,\dots,d}$ for $g \in C^1(\mathbb{R}^d;\mathbb{R}^d)$.

Remark 3.3. It follows from the definition that the class of symmetric jointly convex functions is closed under finite sum and countable supremum, provided the latter is pointwise finite.

The main result of this section addresses the relation of symmetric joint convexity and BD-ellipticity, as well as lower semicontinuity of the corresponding functionals.

Theorem 3.4 (Symmetric joint convexity implies BD-ellipticity)**.** *Any symmetric jointly convex function $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$ is BD-elliptic, and the corresponding functional $\mathcal{F}$ defined in (2.2) is lower semicontinuous on $PR(\Omega)$.*

Proof. We divide the proof into two steps: first, we prove the statement if f is bounded, continuous, and satisfies (2.7), then we come to the general case.

Step 1. Assume, in addition, that f is bounded, continuous on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$ and satisfies (2.7). Fix a triple $(i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$, $i \neq j$. Let $u \in PR(Q_1^\nu)$ be such that $\{u \neq u_{i,j,\nu}\} \subset\subset Q_1^\nu$, where $u_{i,j,\nu}$ is the function defined in (2.4). In view of (2.1), we can write $u = \sum_{k\in\mathbb{N}} a_{Q_k,b_k}\chi_{P_k}$, where $P_1 = \{u = i\}$ and $P_2 = \{u = j\}$. Fix any $\varepsilon > 0$ and define $v = \sum_{k=1}^K a_{Q_k,b_k}\chi_{P_k}$ such that $\mathcal{H}^{d-1}(\bigcup_{k\ge K+1} \partial^* P_k) \le \varepsilon$ and $\{v \neq u_{i,j,\nu}\} \subset\subset Q_1^\nu$. Then, we clearly have $v \in BV(Q_1^\nu;\mathbb{R}^d)\cap L^\infty(Q_1^\nu;\mathbb{R}^d)$, and

$$\int_{J_v} f(v^+,v^-,\nu_v)\,\mathrm{d}\mathcal{H}^{d-1} \le \int_{J_u} f(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} + \varepsilon\|f\|_\infty. \tag{3.3}$$

Now, it suffices to prove

$$\int_{J_v} f(v^+, v^-, \nu_v)\,\mathrm{d}\mathcal{H}^{d-1} \geq f(i,j,\nu). \tag{3.4}$$

In fact, (3.3) and the arbitrariness of ε then show that f is BD-elliptic.

To see (3.4), we first fix $g \in C^1(\mathbb{R}^d;\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$, and use the chain rule in BV (see [7, Theorem 3.96]) to obtain $g(v) \in BV(Q_1^\nu;\mathbb{R}^d)$ with

$$Dg(v) = \nabla g(v)\nabla v\mathcal{L}^d + \big(g(v^+) - g(v^-)\big) \otimes \nu_v \mathcal{H}^{d-1} \llcorner J_v, \tag{3.5}$$

where $Dg(v) = (D_k g_l(v))_{k,l=1,\dots,d}$ denotes the distributional derivative. Since $g(v) - g(u_{i,j,\nu})$ has compact support in Q_1^ν, there holds $Dg(v)(Q_1^\nu) = Dg(u_{i,j,\nu})(Q_1^\nu)$. In particular,

$$\mathrm{tr}\big(Dg(v)(Q_1^\nu)\big) = \mathrm{tr}\big(Dg(u_{i,j,\nu})(Q_1^\nu)\big), \tag{3.6}$$

where "tr" stands for the trace, i.e., $\mathrm{tr}\,(Dg(v)(Q_1^\nu)) = \sum_{k=1}^d D_k g_k(v)(Q_1^\nu)$. Now assume that g is also conservative, i.e., curl-free. We get by (3.5) that

$$\mathrm{tr}\big(Dg(v)(Q_1^\nu)\big) = \int_{Q_1^\nu} \nabla g(v) : (\nabla v)^T\,\mathrm{d}\mathcal{L}^d + \int_{J_v} \big\langle g(v^+) - g(v^-), \nu_v\big\rangle\,\mathrm{d}\mathcal{H}^{d-1}.$$

Since g is curl-free and thus $\nabla g(v)$ is a symmetric matrix, whereas ∇v is a skew symmetric matrix pointwise a.e., we then get

$$\mathrm{tr}\big(Dg(v)(Q_1^\nu)\big) = \int_{J_v} \big\langle g(v^+) - g(v^-), \nu_v\big\rangle\,d\mathcal{H}^{d-1}.$$

In a similar fashion, we obtain $\mathrm{tr}\,(Dg(u_{i,j,\nu})(Q_1^\nu)) = \langle g(i) - g(j), \nu\rangle$. Therefore, by (3.6) we derive

$$\int_{J_v} \big\langle g(v^+) - g(v^-), \nu_v\big\rangle\,\mathrm{d}\mathcal{H}^{d-1} = \mathrm{tr}\big(Dg(v)(Q_1^\nu)\big) = \mathrm{tr}\big(Dg(u_{i,j,\nu})(Q_1^\nu)\big) = \langle g(i) - g(j), \nu\rangle\,.$$

Let $g_h^\varepsilon \in C^1(\mathbb{R}^d;\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ be curl-free for every $h \in \mathbb{N}$ as in Remark 3.2. Then, taking the supremum on both sides of the above relation for $g = g_h^\varepsilon$ and using (3.2) we get

$$\begin{aligned}\int_{J_v} f(v^+, v^-, \nu_v)\,d\mathcal{H}^{d-1} + \varepsilon\mathcal{H}^{d-1}(J_v) &\geq \sup_{h\in\mathbb{N}} \int_{J_v} \big\langle g_h^\varepsilon(v^+) - g_h^\varepsilon(v^-), \nu_v\big\rangle\,d\mathcal{H}^{d-1} \\ &= \sup_{h\in\mathbb{N}} \langle g_h^\varepsilon(i) - g_h^\varepsilon(j), \nu\rangle \geq f(i,j,\nu) - \varepsilon.\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$, this shows (3.4). By using also Theorem 2.2 we get that $\mathcal{F}$ defined in (2.2) is lower semicontinuous in $PR(\Omega)$. This concludes the proof of Step 1.

Step 2. We now address the general case. For each $M \in \mathbb{N}$, set $\hat{g}_h = g_h$ if $h \leq M$, and $\hat{g}_h = 0$ else. Consider $f_M(i,j,\nu) := \sup_{h\in\mathbb{N}} \langle \hat{g}_h(i) - \hat{g}_h(j), \nu\rangle$ (so that in particular, $f_M \geq 0$). Clearly, Step 1 can be applied to the functions f_M as each function g_h in (3.1) is bounded and uniformly continuous. Since $f = \sup_M f_M$, the conclusion follows from Corollary 2.6. □

We close this section by providing a prototypical class of symmetric jointly convex functions.

Example 3.5 (Prototype of symmetric jointly convex functions)**.** Given any orthonormal basis $\{\xi_1,\dots,\xi_d\}$ of $\mathbb{R}^d$ and bounded, uniformly continuous functions $h_k \in C(\mathbb{R})$, $k = 1,\dots,d$, consider the function

$$g(w) := \sum_{k=1}^d h_k\big(\langle w, \xi_k\rangle\big)\,\xi_k \quad \text{ for all } w \in \mathbb{R}^d.$$

Then, clearly $g \in C(\mathbb{R}^d;\mathbb{R}^d)$ is bounded, uniformly continuous, and conservative with potential

$$G(w) = \sum_{k=1}^{d} H_k\big(\langle w, \xi_k\rangle\big) \quad \text{for all } w \in \mathbb{R}^d,$$

where H_k denotes a primitive of h_k. Then, functions f as in (3.1) with functions g_h of the above form are symmetric jointly convex. We will exploit this several times in the examples in Section 4.

4. Examples of BD-elliptic functionals

In this section, we present various examples of functions that are BD-elliptic. We start with some classes of symmetric jointly convex functions, including in particular the density $(i,j,\nu) \mapsto |i-j||\nu|$. Afterwards, we consider so-called *biconvex functions*, and then functions which either only depend on the normal or have a "mild" dependence on the traces i and j. Finally, we provide examples of functions that are BV-elliptic but not BD-elliptic.

4.1. Subadditive isotropic integrands.

In this subsection, we show that certain subadditive isotropic integrands are BD-elliptic. This result constitutes one of our main results since the class of considered functions contains in particular the density $(i,j,\nu) \mapsto |i-j||\nu|$.

Theorem 4.1 (Subadditive, isotropic integrands)**.** *Let $g : [0,+\infty) \to [0,+\infty)$ be an increasing function satisfying*

$$\frac{g(t)}{t} \quad \text{is nonincreasing on } (0,+\infty)\,. \tag{4.1}$$

Then, $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$ defined as

$$f(i,j,\nu) := g(|i-j|)|\nu| \tag{4.2}$$

is symmetric jointly convex and thus BD-elliptic. In particular, the function

$$(i,j,\nu) \mapsto |i-j||\nu|$$

is symmetric jointly convex.

We remark that (4.1) particularly implies that g is *subadditive*. This condition is satisfied, for instance, if g is concave (as we have $g(0) \geq 0$). By choosing $g \equiv 1$, we re-derive the well-known fact that the Hausdorff-measure $\mathcal{H}^{d-1}$ is lower semicontinuous on $PR(\Omega)$, see [27, Theorem 11.3] or [33, Lemma 3.3].

The notion of *isotropy* refers to the fact that $f(i,j,\nu) = f(i,j,R\nu)$ and $f(i,j,\nu) = f(Ri,Rj,\nu)$ for all proper rotations $R \in SO(d)$. Note that this class is much smaller than the corresponding class of BV-elliptic functions considered in (2.6) where f can be anisotropic as long as θ is a pseudo-distance and ψ is even, positively 1-homogeneous, and convex. In fact, as we will show in Subsection 4.6 below, for certain anisotropies it turns out that the functions in (2.6) are BV-elliptic, but not BD-elliptic.

The proof of Theorem 4.1 follows directly from the following lemma.

Lemma 4.2. *For each $M, a \geq 0$ the function $\theta_{M,a}\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by*

$$\theta_{M,a}(i,j,\nu) := \min\{a|i-j|, M\}\,|\nu| \qquad \text{for all } (i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$$

is symmetric jointly convex.

Proof of Theorem 4.1. Let f as in (4.2) be given. For each $t > 0$, $t \in \mathbb{Q}$, we choose $M_t \geq 0$ and $a_t \geq 0$ such that $M_t = g(t)$ and $a_t t = g(t)$. Then the monotonicity of g and (4.1) imply $\min\{a_t z, M_t\} \leq g(z)$ for all $z > 0$ and $\min\{a_t t, M_t\} = g(t)$. This yields

$$f(i,j,v) = \sup_{t>0, t\in\mathbb{Q}} \theta_{M_t,a_t}(i,j,\nu) \qquad \text{for all } (i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \quad \text{with } i \neq j.$$

Consequently, as each θ_{M_t,a_t} is symmetric jointly convex, also f is symmetric jointly convex, see Remark 3.3. □

The remainder of this subsection is devoted to the proof of Lemma 4.2. Let us start with a technical lemma that each element of $\mathbb{S}^{d-1}$ can be mapped to any other element of $\mathbb{S}^{d-1}$ by a symmetric matrix. In the following, $\|\cdot\|$ denotes the operator norm of a matrix $B \in \mathbb{R}^{d\times d}$, i.e., $\|B\| := \max_{\nu\in\mathbb{S}^{d-1}} |B\nu|$.

Lemma 4.3. *For all $u, v \in \mathbb{S}^{d-1}$, there exists a symmetric matrix $B \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that $\|B\| = 1$ and $Bu = v$.*

Proof. First, observe that the result is trivial if $u = \pm v$ by choosing $B = \pm\mathrm{Id}$. We start by proving the statement for $d = 2$ (Step 1) and then address the general case (Step 2).

Step 1. We prove the statement for $d = 2$. Let $u = (\cos\alpha, \sin\alpha), v = (\cos\beta, \sin\beta) \in \mathbb{S}^1$ with $\alpha, \beta \in [0, 2\pi)$. We let $\gamma = \alpha + \beta$ and introduce the following matrix which is a composition of a rotation and a reflection:

$$B := \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix}.$$

Note that B is symmetric. By using the angle sum identities $\cos(\gamma - \alpha) = \cos\alpha\cos\gamma + \sin\alpha\sin\gamma$ and $\sin(\gamma - \alpha) = \cos\alpha\sin\gamma - \sin\alpha\cos\gamma$ we get $Bu = v$. Moreover, as $|Bw| = |w|$ for all $w \in \mathbb{R}^2$, we find that $\|B\| = 1$. This concludes the first step.

Step 2. Let $u, v \in \mathbb{S}^{d-1}$ with $u \neq \pm v$. Let us consider the two-dimensional plane $\Pi_{u,v}$ in $\mathbb{R}^d$ spanned by the two vectors u and v. Fix an orthonormal basis $\xi_1, \xi_2 \in \mathbb{R}^d$ of $\Pi_{u,v}$ and note that $u = \cos(\alpha)\xi_1 + \sin(\alpha)\xi_2$ and $v = \cos(\beta)\xi_1 + \sin(\beta)\xi_2$ for some $\alpha, \beta \in [0, 2\pi)$. We define the matrix

$$B = \cos(\gamma)\big(\xi_1 \otimes \xi_1 - \xi_2 \otimes \xi_2\big) + \sin(\gamma)\big(\xi_1 \otimes \xi_2 + \xi_2 \otimes \xi_1\big),$$

where $\gamma = \alpha + \beta$. As in Step 1, we can check that $B \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\|B\| = 1$, and $Bu = v$. This concludes the proof. □

In the following, the symmetry of the matrices given in Lemma 4.3 will be crucial as it allows us to diagonalize the matrices and to represent the function f in (4.2) in terms of functions similar to the prototype introduced in Example 3.5. We are now in a position to prove Lemma 4.2.

Proof of Lemma 4.2. Without restriction we assume that $M, a > 0$. We start by defining the functions g_h (Step 1) and then show equality in (3.1) (Steps 2–3).

Step 1: Definition of the functions g_h. We start by introducing the class of bounded, uniformly continuous, and conservative vector fields $(g_h)_{h\in\mathbb{N}} \in C(\mathbb{R}^d; \mathbb{R}^d)$. Given $M, a > 0$, define $\eta_M \colon \mathbb{R} \to [0, +\infty)$ by $\eta_M(t) := \min\{|t|, M\}$ for $t \in \mathbb{R}$, i.e.,

$$\theta_{M,a}(i,j,\nu) = \eta_M(a|i-j|)|\nu| \qquad \text{for } (i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d. \tag{4.3}$$

It is elementary to check that η_M is an even, uniformly continuous, and subadditive function. Consider a symmetric matrix B with $\|B\| = 1$, as provided by Lemma 4.3. Let $\{\xi_1, \dots, \xi_d\}$ be an orthonormal basis of $\mathbb{R}^d$ made of eigenvectors of B and let $\{\lambda_1, \dots, \lambda_d\}$ be the set of corresponding

eigenvalues. Note that $|\lambda_k| \le 1$ for $k = 1, \dots, d$ since $\|B\| = 1$. Finally, let us fix $\mu \in \mathbb{S}^{d-1}$ and $c \in \mathbb{R}^d$. We define $g_{B,\mu,c} \colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$g_{B,\mu,c}(w) := \sum_{k=1}^{d} \lambda_k \mu_k \, \eta_M \left(a \left(\frac{\langle w, \xi_k \rangle}{\mu_k} - c_k \right) \right) \xi_k \quad \text{for all } w \in \mathbb{R}^d. \tag{4.4}$$

Here and in the following, an addend is interpreted to be zero whenever $\mu_k = 0$. Clearly, each $g_{B,\mu,c}$ is bounded and uniformly continuous. It is elementary to check that $g_{B,\mu,c}$ is conservative with potential

$$\sum_{k=1}^{d} \frac{\lambda_k \mu_k^2}{a} \, \Theta_M \left(a \left(\frac{\langle w, \xi_k \rangle}{\mu_k} - c_k \right) \right),$$

where Θ_M denotes a primitive of η_M, cf. the prototypes discussed in Example 3.5.

We denote by $(u_k, v_k)_k$ a countable dense set in $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ and denote by $(B^k)_k$ the symmetric matrices from Lemma 4.3 satisfying $B^k u_k = v_k$. Moreover, let $(\mu^l)_l$ be a countable, dense set in $\mathbb{S}^{d-1}$ and let $(c^n)_n$ be a countable, dense set in $\mathbb{R}^d$. To shorten the notation, we label the countable set of functions $(g_{B^k,\mu^l,c^n})_{k,l,n}$ by $(g_h)_{h\in\mathbb{N}}$.

Let us now show that $\theta_{M,a}$ is symmetric jointly convex, namely, for every $(i, j, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$, $i \neq j$, there holds

$$\theta_{M,a}(i, j, \nu) = \sup_{h\in\mathbb{N}} \langle g_h(i) - g_h(j), \nu \rangle \, . \tag{4.5}$$

We split the proof into two inequalities. Before we enter into the details, let us briefly explain the rough ideas behind the parameters B, μ, and c: we will choose B, μ, c in an optimal way in order to obtain one inequality, see (4.6)–(4.8). In particular, we can choose B such that $g_{B,\mu,c}(i) - g_{B,\mu,c}(j)$ and ν are aligned. Moreover, c can be selected such that $g_{B,\mu,c}(i) - g_{B,\mu,c}(j) = g_{B,\mu,0}(i-j)$. (This is inspired by [7, Example 5.23].) Finally, μ will allow us to deal with the nonlinearity of η_M.

Step 2: Proof of "$\le$". We consider the symmetric matrix B given by Lemma 4.3 such that

$$\frac{B(i-j)}{|i-j|} = \frac{\nu}{|\nu|}. \tag{4.6}$$

Moreover, we choose $\mu = (\mu_1, \mu_2, \dots, \mu_d) \in \mathbb{S}^{d-1}$ by

$$\mu_k = \frac{\langle i - j, \xi_k \rangle}{|i-j|} \quad \text{for } k = 1, \dots, d. \tag{4.7}$$

By $c = (c_1, c_2, \dots, c_d) \in \mathbb{R}^d$ we denote the vector with

$$c_k = \langle j, \xi_k \rangle / \mu_k, \tag{4.8}$$

whenever $\mu_k \neq 0$ and $c_k = 0$ else. For brevity, we write $g = g_{B,\mu,c}$. In view of (4.4) and by the choices of μ and c, we get

$$\begin{aligned} \langle g(i) - g(j), \nu \rangle &= \left\langle \sum_{k=1}^{d} \lambda_k \mu_k \left[\eta_M \left(a \frac{\langle i, \xi_k \rangle}{\mu_k} - a c_k \right) - \eta_M \left(a \frac{\langle j, \xi_k \rangle}{\mu_k} - a c_k \right) \right] \xi_k, \nu \right\rangle \\ &= \left\langle \sum_{k=1}^{d} \lambda_k \mu_k \, \eta_M \left(a \frac{\langle i - j, \xi_k \rangle}{\mu_k} \right) \xi_k, \nu \right\rangle = \left\langle \sum_{k=1}^{d} \lambda_k \frac{\langle i - j, \xi_k \rangle}{|i-j|} \eta_M \left(a |i-j| \right) \xi_k, \nu \right\rangle . \end{aligned}$$

Since $(\xi_k)_k$ is an orthonormal basis of $\mathbb{R}^d$ made of eigenvectors of B and $(\lambda_k)_k$ are the corresponding eigenvalues, we get by (4.3) and (4.6)

$$\begin{aligned}\langle g(i)-g(j),\nu\rangle &= \frac{\eta_M\left(a|i-j|\right)}{|i-j|}\left\langle \sum_{k=1}^d \lambda_k\langle i-j,\xi_k\rangle\,\xi_k,\nu\right\rangle = \frac{\eta_M\left(a|i-j|\right)}{|i-j|}\,\langle B(i-j),\nu\rangle\\ &= \eta_M(a|i-j|)|\nu| = \theta_{M,a}(i,j,\nu).\end{aligned}$$

By the density of $(u_k,v_k)_k$, $(\mu^l)_l$, and $(c^n)_n$ we get that the function g considered above can be approximated by $(g_h)_{h\in\mathbb{N}}$. Thus, we obtain inequality "$\leq$" in (4.5).

Step 3: Proof of "$\geq$". Fix any $g = g_{B,\mu,c}$ as above. For brevity, we define

$$b_k := \eta_M\left(a\frac{\langle i,\xi_k\rangle}{\mu_k} - ac_k\right) - \eta_M\left(a\frac{\langle j,\xi_k\rangle}{\mu_k} - ac_k\right)$$

for $k=1,\dots,d$. Since η_M is nonnegative, subadditive, and even, we get

$$|b_k| \leq \eta_M\left(a\frac{\langle i-j,\xi_k\rangle}{\mu_k}\right). \tag{4.9}$$

Since $|\lambda_k|\leq 1$ for $k=1,\dots,d$ (recall $\|B\|=1$) and $(\xi_k)_k$ forms an orthonormal basis, we get for every $h\in\mathbb{N}$ by (4.4) and the Cauchy-Schwarz inequality

$$|\langle g(i)-g(j),\nu\rangle| = \left|\left\langle \sum_{k=1}^d \lambda_k\mu_k b_k\xi_k,\nu\right\rangle\right| \leq \left(\sum_{k=1}^d(\mu_k b_k)^2\right)^{1/2}|\nu|.$$

We now distinguish two cases: if $a|i-j|\leq M$, we deduce from (4.3), (4.9), and the fact that $\eta_M(t)\leq|t|$ for $t\in\mathbb{R}$ that

$$|\langle g(i)-g(j),\nu\rangle| \leq \left(\sum_{k=1}^d \mu_k^2\left(a\frac{\langle i-j,\xi_k\rangle}{\mu_k}\right)^2\right)^{1/2}|\nu| = a|i-j||\nu| = \theta_{M,a}(i,j,\nu).$$

Otherwise, if $a|i-j|>M$, in view of (4.3), (4.9), we find by using $\|\eta_M\|_\infty\leq M$ and $\mu\in\mathbb{S}^{d-1}$ that

$$|\langle g(i)-g(j),\nu\rangle| \leq \left(\sum_{k=1}^d M^2\mu_k^2\right)^{1/2}|\nu| = M|\nu| = \theta_{M,a}(i,j,\nu).$$

Taking the supremum over all $(g_h)_{h\in\mathbb{N}}$ we obtain inequality "$\geq$" in (4.5). This concludes the proof. □

4.2. **A further class of symmetric jointly functions.** In this subsection, we revisit a class of functions considered in a more general context in [28], where the authors prove that the associated energy functionals, see (2.2), are lower semicontinuous. Given even, continuous, and subadditive function $\theta_k\in C(\mathbb{R};[0,+\infty))$, $k=1,\dots,d$, with $\theta_k(0)=0$, we define the function $f\colon \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\to[0,+\infty)$ by

$$f(i,j,\nu) := \sup_{(\xi_1,\dots,\xi_k)}\left(\sum_{k=1}^d \theta_k\big(\langle i-j,\xi_k\rangle\big)^2\,|\langle\nu,\xi_k\rangle|^2\right)^{1/2}, \tag{4.10}$$

where the supremum is taken over all orthonormal bases $(\xi_k)_{k=1}^d$ of $\mathbb{R}^d$.

We prove that functions of this form are symmetric jointly convex which provides an alternative (and in our opinion simpler) approach to the lower semicontinuity of the functional in (2.2) for f as

above, when restricted to $PR(\Omega)$. Let us also mention that in contrast to the examples considered in Subsection 4.1, the functions in (4.10) may in general be anisotropic.

Proposition 4.4. *Let $\theta_k \in C(\mathbb{R};[0,+\infty))$ be even, continuous, and subadditive functions with $\theta_k(0)=0$ for $k=1\dots,d$. Then, the function $f\colon \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\to[0,+\infty)$ defined in (4.10) is symmetric jointly convex.*

Proof. We start by noticing that each θ_k is uniformly continuous. In fact, suppose by contradiction that this was false. Then, there exists $\delta>0$ and a sequence of pairs $(x_n,y_n)\in\mathbb{R}^2$ such that $|x_n-y_n|\to 0$ as $n\to\infty$ and $\theta_k(x_n)\le\theta_k(y_n)-\delta$ for all $n\in\mathbb{N}$. But as θ_k is subadditive and even, we get for n large enough that $\theta_k(y_n)\le\theta_k(x_n)+\theta(|x_n-y_n|)\le\theta_k(x_n)+\delta/2$, where the last step follows from $|x_n-y_n|\to 0$ and $\theta_k(0)=0$. This is a contradiction.

We also observe that it is not restrictive to assume that each θ_k is bounded. In fact, otherwise we consider the truncations θ_k^M defined by $\theta_k^M(t):=\min\{\theta_k(t),M\}$ for $t\in\mathbb{R}$ which are again even, uniformly continuous, and subadditive. Then, by $\sup_{M>0}\theta_k^M=\theta_k$ and Remark 3.3 it clearly suffices to prove that f in (4.10) with θ_k^M in place of θ_k is symmetric jointly convex. For simplicity, we assume in the following that each θ_k is bounded.

By definition of f and Remark 3.3, it is sufficient to show that the function $\bar f\colon\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\to[0,+\infty)$ defined by

$$\bar f(i,j,\nu):=\left(\sum_{k=1}^d\theta_k\big(\langle i-j,\xi_k\rangle\big)^2\,|\langle\nu,\xi_k\rangle|^2\right)^{1/2}$$

is symmetric jointly convex, where $(\xi_k)_k$ is any orthonormal basis of $\mathbb{R}^d$. To this end, for each $p\in\mathbb{Q}^d$ with $|p|\le 1$, each $q\in\mathbb{Q}^d$, and each $\sigma\in\{-1,1\}^d$, we define the conservative vector field $g_{p,q,\sigma}\colon\mathbb{R}^d\to\mathbb{R}^d$ by

$$g_{p,q,\sigma}(w):=\sum_{k=1}^d\sigma_k\langle p,\xi_k\rangle\,\theta_k\big(\langle w-q,\xi_k\rangle\big)\,\xi_k\qquad\text{for all } w\in\mathbb{R}^d,$$

cf. Example 3.5. Clearly, $g_{p,q,\sigma}$ is bounded and uniformly continuous. Our goal is to prove that

$$\bar f(i,j,\nu)=\sup_{p,q,\sigma}\langle g_{p,q,\sigma}(i)-g_{p,q,\sigma}(j),\nu\rangle\qquad\text{for all } (i,j,\nu)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d,\ i\neq j.\tag{4.11}$$

We show the two inequalities separately.

Step 1: Proof of "$\ge$". Fix $(i,j,\nu)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$ with $i\neq j$. Consider $p\in\mathbb{Q}^d$ with $|p|\le 1$, $q\in\mathbb{Q}^d$, and $\sigma\in\{-1,1\}^d$. We get

$$\big|\,\langle g_{p,q,\sigma}(i)-g_{p,q,\sigma}(j),\nu\rangle\,\big|\le\sum_{k=1}^d\big|\theta_k\big(\langle i-q,\xi_k\rangle\big)-\theta_k\big(\langle j-q,\xi_k\rangle\big)\big|\,|\langle p,\xi_k\rangle||\langle\nu,\xi_k\rangle|.\tag{4.12}$$

Since θ_k is nonnegative, subadditive, and even, we obtain

$$\big|\theta_k\big(\langle i-q,\xi_k\rangle\big)-\theta_k\big(\langle j-q,\xi_k\rangle\big)\big|\le\theta_k\big(\langle i-j,\xi_k\rangle\big)\qquad\text{for all } k=1,\dots,d.$$

Then, by (4.12), the Cauchy-Schwarz inequality, and the fact that $(\xi_k)_k$ is an orthonormal basis we get

$$\big|\,\langle g_{p,q,\sigma}(i)-g_{p,q,\sigma}(j),\nu\rangle\,\big|\le\sum_{k=1}^d\theta_k\big(\langle i-j,\xi_k\rangle\big)|\langle\nu,\xi_k\rangle||\langle p,\xi_k\rangle|\le\left(\sum_{k=1}^d\theta_k\big(\langle i-j,\xi_k\rangle\big)^2|\langle\nu,\xi_k\rangle|^2\right)^{1/2}|p|.$$

By recalling $|p|\le 1$ and by passing to the supremum over (p,q,σ) in their domain, we obtain inequality "$\ge$" in (4.11).

Step 2: Proof of "$\leq$". Fix $(i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ with $i \neq j$. Let us set $\sigma_k = \operatorname{sign}\langle \nu, \xi_k \rangle$ for $k = 1, \dots, d$. Moreover, consider a sequence of vectors $(q_h)_{h\in\mathbb{N}} \subset \mathbb{Q}^d$ such that $\lim_{h\to\infty} q_h = j$. Then, for each $p \in \mathbb{Q}^d$ with $|p| \leq 1$, we get by $\theta_k(0) = 0$ that

$$\lim_{h\to\infty} \langle g_{p,q_h,\sigma}(i) - g_{p,q_h,\sigma}(j), \nu \rangle = \sum_{k=1}^{d} \langle p, \xi_k \rangle \, \theta_k\big(\langle i-j, \xi_k \rangle\big) |\langle \nu, \xi_k \rangle| = \langle p, \mu \rangle, \tag{4.13}$$

where the vector $\mu \in \mathbb{R}^d$ is defined by

$$\mu := \sum_{k=1}^{d} \theta_k\big(\langle i-j, \xi_k \rangle\big) |\langle \nu, \xi_k \rangle| \xi_k.$$

We choose $(p_m)_{m\in\mathbb{N}} \subset \mathbb{Q}^d$, $|p_m| \leq 1$, such that $\lim_{m\to\infty} p_m = \mu/|\mu|$. Then, by (4.13) we conclude

$$\begin{aligned} \sup_{p,q,\sigma} \langle g_{p,q,\sigma}(i) - g_{p,q,\sigma}(j), \nu \rangle &\geq \lim_{m\to\infty} \lim_{h\to\infty} \langle g_{p_m,q_h,\sigma}(i) - g_{p_m,q_h,\sigma}(j), \nu \rangle = |\mu| \\ &= \left(\sum_{k=1}^{d} \theta_k\big(\langle i-j, \xi_k \rangle\big)^2 |\langle \nu, \xi_k \rangle|^2 \right)^{1/2} = \bar{f}(i,j,\nu). \end{aligned}$$

This proves inequality "$\leq$" in (4.11). □

Remark 4.5. Among the integrands of the form (4.10), we may mention $f(i,j,\nu) = |(i-j) \odot \nu|$, where the symbol $|\cdot|$ denotes the Frobenius norm, and $\odot$ denotes the symmetric tensor product $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$ for $a, b \in \mathbb{R}^d$. This is obtained for $\theta_k(t) = |t|$, $k = 1, \dots, d$, in (4.10). If one instead chooses $\theta_k(t) = \min\{|t|, M\}$ for a fixed $M > 0$, the resulting f is a bounded integrand satisfying $f(i,j,\nu) = |(i-j) \odot \nu|$ when $|i-j| \leq M$, $f(i,j,\nu) = M|\nu|$ when $|i-j| \geq \sqrt{2}M$, with a smooth transition in the annulus between the radii M and $\sqrt{2}M$ (see [28, Section 6]).

4.3. **Symmetric biconvex functions.** In this subsection, we introduce and study *symmetric biconvex functions*.

Definition 4.6 (Symmetric biconvexity)**.** We say that $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ is a *symmetric biconvex function* if there exists a convex and positively 1-homogeneous function $\theta : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, +\infty)$ such that

$$f(i,j,\nu) = \theta\big((i-j) \odot \nu\big) \qquad \text{for all } (i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d.$$

Here, $\odot$ denotes the symmetric tensor product $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$ for $a, b \in \mathbb{R}^d$. This notion is related to biconvexity defined in [5, Section 2.2]. For θ being the Frobenius norm, the corresponding integrand is symmetric jointly convex (and hence BD-elliptic), as discussed in Remark 4.5. In particular, the functional

$$u \mapsto \int_{J_u} |(u^+ - u^-) \odot \nu_u| \, \mathrm{d}\mathcal{H}^{d-1}$$

is lower semicontinuous on $PR(\Omega)$, see Theorem 3.4. In the general case, the situation is more complicated. We may indeed prove that

- symmetric biconvex functions are symmetric jointly convex when restricted to compact subsets, in a sense made precise by Proposition 4.7 below.
- symmetric biconvex functions with $\{\theta = 0\} = \{0\}$ are BD-elliptic, cf. Proposition 4.8.

Both results only allow us, in general, to deduce that for biconvex functions the functional $\mathcal{F}$ defined in (2.2) is lower semicontinuous in $PR(\Omega)$ along uniformly bounded sequences, see Corollary 2.5. This can also be inferred by the results in [10], where lower semicontinuity in the space SBD of integral functionals corresponding to symmetric biconvex functions has already been addressed.

We now state and prove the announced results. We first address the relation of symmetric biconvex functions to symmetric jointly convex functions.

Proposition 4.7 (Biconvexity and joint convexity)**.** *Let $f\colon \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to [0,+\infty)$ be a symmetric biconvex function. Then, there exists a sequence $(f_M)_{M\in\mathbb{N}}$ of symmetric jointly convex functions such that $f(i,j,\nu) = f_M(i,j,\nu)$ for all $(i,j,\nu) \in B_M(0)\times B_M(0)\times\mathbb{R}^d$.*

Proof. By assumption, we have

$$f(i,j,\nu) = \theta\big((i-j)\odot\nu\big),$$

with $\theta\colon \mathbb{R}^{d\times d}_{\rm sym} \to [0,+\infty)$ convex and positively 1-homogeneous. It is a well known fact that

$$\theta(F) = \sup_{Z\in W_\theta} F : Z \qquad \text{for all } F\in\mathbb{R}^{d\times d}_{\rm sym},$$

where $W_\theta \subset \mathbb{R}^{d\times d}$ is a bounded set depending on θ. (Here, the symbol : denotes the scalar product for matrices in $\mathbb{R}^{d\times d}$.) Let us consider a countable, dense set of matrices $(Z_h)_{h\in\mathbb{N}}$ in W_θ. Since $F : Z^{\rm skew} = 0$ whenever $F\in\mathbb{R}^{d\times d}_{\rm sym}$ and $Z^{\rm skew}\in\mathbb{R}^{d\times d}_{\rm skew}$, we get

$$\theta\big((i-j)\odot\nu\big) = \sup_{h\in\mathbb{N}} \big((i-j)\odot\nu\big) : Z_h = \sup_{h\in\mathbb{N}} \big((i-j)\odot\nu\big) : Z_h^{\rm sym} = \sup_{h\in\mathbb{N}} \langle Z_h^{\rm sym}\, i - Z_h^{\rm sym}\, j, \nu\rangle\,, \tag{4.14}$$

where $Z_h^{\rm sym} := \frac{1}{2}(Z_h^T + Z_h)$. Consequently,

$$\theta\big((i-j)\odot\nu\big) = \sup_{h\in\mathbb{N}} \langle g_h(i) - g_h(j), \nu\rangle, \tag{4.15}$$

where g_h is defined by $g_h(x) := Z_h^{\rm sym}x$ for $x\in\mathbb{R}^d$. We define a truncation of each g_h as follows. For $M>0$, we consider the function $\tau_M\colon\mathbb{R}\to\mathbb{R}$ given by $\tau_M(t) = t$ for $|t|\le M$ and $\tau_M(t) = \mathrm{sgn}(t)M$ else. Let $\{\xi_1,\dots,\xi_d\}$ be an orthonormal basis of $\mathbb{R}^d$ made of eigenvectors of $Z_h^{\rm sym}$ and let $\{\lambda_1,\dots,\lambda_d\}$ be the set of corresponding eigenvalues. Then, we introduce

$$g_h^M(w) := \sum_{k=1}^d \lambda_k \tau_M\big(\langle w,\xi_k\rangle\big)\xi_k \qquad \text{for all } w\in\mathbb{R}^d.$$

We observe that each g_h^M is bounded, uniformly continuous, and conservative, cf. the prototype in Example 3.5. Then, we define the symmetric jointly convex function

$$f_M(i,j,\nu) := \sup_{h\in\mathbb{N}} \langle g_h^M(i) - g_h^M(j), \nu\rangle\,.$$

By the definition of τ_M and (4.15) we get $f(i,j,\nu) = f_M(i,j,\nu)$ whenever $i,j\in B_M(0)$. □

We remark that, in general, it appears to be difficult to approximate the functions g_h defined after (4.14) from below by conservative and bounded vector fields on the *entire* $\mathbb{R}^d$. We now show that certain symmetric biconvex functions are BD-elliptic.

Proposition 4.8 (Biconvexity implies BD-ellipticity)**.** *Let $f : \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to [0,+\infty)$ be a symmetric biconvex function such that the associated θ satisfies $\{\theta=0\} = \{0\}$. Then, f is BD-elliptic.*

In the proof, we will use the following technical property.

Lemma 4.9. *Let $B \subset \mathbb{R}^d$ open and bounded. Suppose $u \in PR(B)$ with $\{u \neq u_0\} \subset\subset B$ for some $u_0 \in L^1(B)$ such that $\int_{J_u} |[u]| \, \mathrm{d}\mathcal{H}^{d-1} < +\infty$. Then $u \in SBD(B)$.*

The proof of the lemma relies on standard slicing techniques for functions of bounded deformation. We provide the proof in Appendix A for convenience of the reader. We proceed with the proof of Proposition 4.8.

Proof of Proposition 4.8. Let us fix $(i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$ with $i \neq j$. Consider $v \in PR(Q_1^\nu)$ with $\{v \neq u_{i,j,\nu}\} \subset\subset Q_1^\nu$, where $u_{i,j,\nu}$ is the function defined in (2.4). Without restriction, we may assume that $\int_{J_v} \theta\big([v] \odot \nu_v\big) \, \mathrm{d}\mathcal{H}^{d-1} < \infty$. Since θ is positively 1-homogeneous with $\{\theta = 0\} = \{0\}$, we get $\theta(F) \geq c|F|$ for all $F \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ for some $c > 0$. This implies

$$\int_{J_v} |[v]| \, \mathrm{d}\mathcal{H}^{d-1} \leq \sqrt{2} \int_{J_v} \big|[v] \odot \nu_v\big| \, \mathrm{d}\mathcal{H}^{d-1} \leq \sqrt{2}/c \int_{J_v} \theta\big([v] \odot \nu_v\big) \, \mathrm{d}\mathcal{H}^{d-1} < +\infty.$$

Therefore, by Lemma 4.9 we get $v \in SBD(Q_1^\nu)$. In particular, the symmetric distributional derivative Ev of v is a finite Radon measure, and is given by

$$Ev(B) = \int_{J_v \cap B} \big([v] \odot \nu_v\big) \, \mathrm{d}\mathcal{H}^{d-1}$$

for all Borel sets $B \subset Q_1^\nu$. Since $\{v \neq u_{i,j,\nu}\} \subset\subset Q_1^\nu$, we have $Ev(Q_1^\nu) = Eu_{i,j,\nu}(Q_1^\nu)$. This along with Jensen's inequality and the fact that θ is positively 1-homogeneous and convex yields

$$\begin{aligned}
\int_{J_v} f(v^+, v^-, \nu_v) \, \mathrm{d}\mathcal{H}^{d-1} &= \int_{J_v} \theta\big([v] \odot \nu_v\big) \, \mathrm{d}\mathcal{H}^{d-1} \geq \theta\left(\int_{J_v} \big([v] \odot \nu_v\big) \, \mathrm{d}\mathcal{H}^{d-1} \right) \\
&= \theta\left(\int_{J_{u_{i,j,\nu}}} \big([u_{i,j,\nu}] \odot \nu\big) \, \mathrm{d}\mathcal{H}^{d-1} \right) = \mathcal{H}^{d-1}\big(J_{u_{i,j,\nu}} \cap Q_1^\nu\big) \, \theta\big((i-j) \odot \nu\big) \\
&= f(i,j,\nu).
\end{aligned}$$

This shows that f is BD-elliptic and concludes the proof. □

4.4. **Independence of the traces at the jump.** We now proceed with examples which are possibly not related to symmetric jointly convex functions. In this subsection, we consider functions which are independent of the traces at the jump set and only dependent on the normal, i.e.,

$$f(i,j,\nu) = \psi(\nu) \qquad \text{for all } (i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}, \tag{4.16}$$

for $\psi \colon \mathbb{R}^d \to [0,+\infty)$ even, positively 1-homogeneous, and convex. In this setting, it turns out that the notions of BV-ellipticity and BD-ellipticity coincide. Recall that convexity of ψ is a necessary condition for BV-ellipticity, see [7, Theorem 5.11, Theorem 5.14], and thus also necessary for BD-ellipticity.

Proposition 4.10. *A function $f \colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ of the form (4.16) is BD-elliptic if $\psi \colon \mathbb{R}^d \to [0,+\infty)$ is even, positively 1-homogeneous, and convex.*

Proof. Fix $(i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$, $i \neq j$, and consider $v \in PR(Q_1^\nu)$ with $\{v \neq u_{i,j,\nu}\} \subset\subset Q_1^\nu$, where $u_{i,j,\nu}$ is defined in (2.4). In view of (2.1), it is elementary to see that there exists $u \in PC(Q_1^\nu)$ with $\{u \neq u_{i,j,\nu}\} \subset\subset Q_1^\nu$ such that $\mathcal{H}^{d-1}(J_u \triangle J_v) = 0$. This along with the fact that f is BV-elliptic (see [7, Example 5.23]) yields

$$\int_{J_v} f(v^+, v^-, \nu_v) \, \mathrm{d}\mathcal{H}^{d-1} = \int_{J_u} f(u^+, u^-, \nu_u) \, \mathrm{d}\mathcal{H}^{d-1} \geq f(i,j,\nu).$$

This concludes the proof. □

By choosing $\psi \equiv 1$ on $\mathbb{S}^{d-1}$, we re-derive the well-known fact that the Hausdorff-measure $\mathcal{H}^{d-1}$ is lower semicontinuous on $PR(\Omega)$. Moreover, we briefly remark that lower semicontinuity of functionals with integrands of the form (4.16) has already been addressed in [25, Corollary 5.5] in the setting of $GSBD^p$ functions.

4.5. **Functions with mild dependence on the traces.** In this subsection, we consider another class of BD-elliptic functions

$$f(i,j,\nu) = g(i-j) \qquad \text{for all } (i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1},$$

where $g\colon \mathbb{R}^d \to [0,+\infty)$ is a bounded, even function with

$$\sup g \le 2 \inf g. \tag{4.17}$$

Due to (4.17), we say that f has only a *mild dependence* on the traces at the jump.

Proposition 4.11. *Under* (4.17)*, the function f is BD-elliptic.*

Proof. Fix $(i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$ with $i \neq j$. Consider $u \in PR(Q_1^\nu)$ such that $\{u \neq u_{i,j,\nu}\} \subset\subset Q_1^\nu$, where $u_{i,j,\nu}$ is the function defined in (2.4). In view of (2.1), we can write $u = \sum_{k\in\mathbb{N}} a_{Q_k,b_k}\chi_{P_k}$, where $P_1 = \{u = i\}$ and $P_2 = \{u = j\}$, and $\mathcal{H}^{d-1}(J_u \triangle (\bigcup_{k\ge 1} \partial^* P_k \setminus \partial Q_1^\nu)) = 0$. We define

$$\Gamma_1 = \partial^* P_1 \cap \partial^* P_2, \qquad \Gamma_2 = \bigcup_{k\ge 3} \partial^* P_k \setminus \Gamma_1.$$

The local structure of Caccioppoli partitions (see [7, Theorem 4.17]) and the fact that $\partial^* P_k \cap \partial Q_1^\nu = \emptyset$ for $k \ge 3$ imply that $J_u = \Gamma_1 \cup \Gamma_2$ up to a set of $\mathcal{H}^{d-1}$-negligible measure. We now introduce some more notation. We define $\Pi_\nu = \{x \in \mathbb{R}^d \colon \langle x,\nu\rangle = 0\}$ and, for $B \subset \mathbb{R}^d$, we let $B_y^\nu = \{t \in \mathbb{R} \colon y + t\nu \in B\}$ for each $y \in \Pi_\nu$. We decompose the set $\Pi_\nu \cap Q_1^\nu$ into the sets

$$T_1 = \big\{y \in \Pi_\nu \cap Q_1^\nu \colon (\Gamma_1)_y^\nu \neq \emptyset\big\}, \qquad T_2 = (\Pi_\nu \cap Q_1^\nu) \setminus T_1.$$

As $\{u \neq u_{i,j,\nu}\} \subset\subset Q_1^\nu$, we find that each line $y + \mathbb{R}\nu$, $y \in \Pi_\nu \cap Q_1^\nu$, intersects P_1 and P_2 on a set of positive Lebesgue measure. Thus, for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi_\nu \cap Q_1^\nu$, by slicing properties [7, Theorem 3.108] for the BV functions χ_{P_1} and χ_{P_2}, we get $\mathcal{H}^0\big((y+\mathbb{R}\nu)\cap \partial^* P_k\big) \ge 1$ for $k = 1,2$. Then, the local structure of Caccioppoli partitions (see [7, Theorem 4.17]) implies that for $\mathcal{H}^{d-1}$-a.e. $y \in T_2$ there exist other components P_k, P_l for some $k,l \ge 3$ (possibly $k = l$) such that the line $y + \mathbb{R}\nu$ intersects $\partial^* P_1 \cap \partial^* P_k$ and $\partial^* P_2 \cap \partial^* P_l$. Thus, there holds

$$\mathcal{H}^0\big((\Gamma_2)_y^\nu\big) \ge 2 \qquad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in T_2. \tag{4.18}$$

As $u = i$ on P_1 and $u = j$ on P_2, we obtain

$$\mathcal{F}(u) = \int_{\Gamma_1} g([u])\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_2} g([u])\,\mathrm{d}\mathcal{H}^{d-1} \ge g(i-j)\mathcal{H}^{d-1}(\Gamma_1) + \inf g\,\mathcal{H}^{d-1}(\Gamma_2).$$

By ν_{Γ_1} and ν_{Γ_2} we denote unit normals to the rectifiable sets Γ_1 and Γ_2, respectively. By the area formula (cf. e.g. [37, (12.4) in Section 12]) and by (4.18) there holds

$$\begin{aligned}
\mathcal{F}(u) &\ge g(i-j)\int_{\Gamma_1} |\langle \nu, \nu_{\Gamma_1}\rangle|\,\mathrm{d}\mathcal{H}^{d-1} + \inf g \int_{\Gamma_2} |\langle \nu, \nu_{\Gamma_2}\rangle|\,\mathrm{d}\mathcal{H}^{d-1} \\
&\ge g(i-j)\int_{T_1} \mathcal{H}^0\big((\Gamma_1)_y^\nu\big)\,\mathrm{d}\mathcal{H}^{d-1}(y) + \inf g \int_{T_2} \mathcal{H}^0\big((\Gamma_2)_y^\nu\big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&\ge g(i-j)\,\mathcal{H}^{d-1}(T_1) + 2\inf g\,\mathcal{H}^{d-1}(T_2).
\end{aligned}$$

By (4.17) and the fact that $\mathcal{H}^{d-1}(T_1) + \mathcal{H}^{d-1}(T_2) = 1$, we conclude $\mathcal{F}(u) \ge g(i-j) = f(i,j,\nu)$. □

4.6. **BV-elliptic, but not BD-elliptic functions.** In this subsection, we provide two examples of BV-elliptic functions which are not BD-elliptic.

Example 4.12 (Anisotropy in jump normal)**.** Consider functions $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ of the form

$$f(i,j,\nu) := |i-j|\,\psi(\nu),$$

where $\psi : \mathbb{R}^d \to [0,+\infty)$ is convex, even, and positively 1-homogeneous. Recall from (2.6) that densities of this form are BV-elliptic. We show that f is in general not BD-elliptic if ψ is anisotropic. To see this, we let $d = 2$ for simplicity and suppose that $\nu = e_2$, where $\{e_1, e_2\}$ denotes the standard orthonormal basis of $\mathbb{R}^2$. Assume that $\psi(e_1) = \varepsilon$ and $\psi(e_2) = 1$ for some $\varepsilon > 0$ small to be specified later. For notational simplicity, we consider functions defined on Q_6^ν. Let $i = 0$, $j = (2\lambda, 2\lambda)$ for $\lambda > 0$, and let $u \in PR(Q_6^{e_2})$ be defined by

$$u(x) := \begin{cases} u_{i,j,e_2}(x) & \text{on } Q_6^{e_2} \setminus Q_2^{e_2}, \\ a_{Q,b}(x) & \text{on } Q_2^{e_2}, \end{cases}$$

where $Q = \lambda(e_1 \otimes e_2 - e_2 \otimes e_1) \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $b = (\lambda, \lambda) \in \mathbb{R}^2$. The affine function is chosen in such a way that the set of discontinuities of the scalar functions $u_1 = \langle u, e_1\rangle$ and $u_2 = \langle u, e_2\rangle$ is the one represented in Figure 1. We define

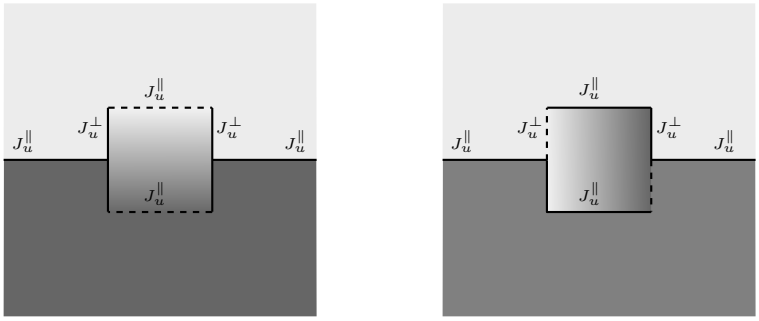

FIGURE 1. The lines (not dashed) in the above pictures are a pictorial representation of the set of discontinuities of the function u_1 (on the left) and of the function u_2 (on the right). The dashed bold lines correspond to points where the function is not discontinuous.

$$J_u^{\parallel} := \{x \in J_u\colon\ |\langle \nu_u(x), e_2\rangle| = 1\}, \qquad J_u^{\perp} := \{x \in J_u\colon\ \langle \nu_u(x), e_2\rangle = 0\}. \tag{4.19}$$

Up to a set of negligible $\mathcal{H}^1$-measure, we can write $J_u^{\parallel}$ as the union of four pairwise disjoint sets Γ_k with $\mathcal{H}^1(\Gamma_k) = 2$ for $k = 1, \dots, 4$, see Figure 1. Then, by $\psi(e_1) = \varepsilon$ and $\psi(e_2) = 1$ we get

$$\mathcal{F}(u, Q_6^{e_2}) = \int_{J_u} |[u]|\psi(\nu_u)\,\mathrm{d}\mathcal{H}^1 = \int_{J_u^{\perp}} |[u]|\psi(e_1)\,\mathrm{d}\mathcal{H}^1 + \int_{J_u^{\parallel}} |[u]|\psi(e_2)\,\mathrm{d}\mathcal{H}^1 \le C\lambda\varepsilon + \int_{J_u^{\parallel}} |[u]|\,\mathrm{d}\mathcal{H}^1$$

for a universal $C > 0$. Since $|[u]| = 2\sqrt{2}\lambda$ on $\Gamma_1 \cup \Gamma_4$, as well as $a_{Q,b}(t,-1) = \lambda(1-t)e_2$ and $a_{Q,b}(t,1) = 2\lambda e_1 + \lambda(1-t)e_2$ for $t \in (-1,1)$, a direct computation shows

$$\begin{aligned} \int_{J_u^{\parallel}} |[u]|\,\mathrm{d}\mathcal{H}^1 &= \int_{\Gamma_1\cup\Gamma_4} |[u]|\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma_2\cup\Gamma_3} |[u]|\,\mathrm{d}\mathcal{H}^1 = 4\cdot 2\sqrt{2}\lambda + \int_{\Gamma_2\cup\Gamma_3} |[u]|\,\mathrm{d}\mathcal{H}^1 \\ &= 8\sqrt{2}\lambda + \int_{-1}^{1} \lambda(1-t)\,\mathrm{d}t + \int_{-1}^{1} \lambda(1+t)\,\mathrm{d}t = (8\sqrt{2}+4)\lambda. \end{aligned}$$

Thus, $\mathcal{F}(u, Q_6^{e_2}) \le C\lambda\varepsilon + (8\sqrt{2}+4)\lambda < 12\sqrt{2}\lambda$ for ε small enough. Observing that $\mathcal{F}(u_{i,j,e_2}, Q_6^{e_2}) = 6 \cdot 2\sqrt{2}\lambda = 12\sqrt{2}\lambda$, we find $\mathcal{F}(u, Q_6^{e_2}) < \mathcal{F}(u_{i,j,e_2}, Q_6^{e_2})$. This shows that f is not BD-elliptic and concludes the example.

Example 4.13 (Anisotropy in jump direction)**.** Consider functions $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0, +\infty)$ of the form

$$f(i, j, \nu) := \psi(i - j),$$

where $\psi : \mathbb{R}^d \to [0, +\infty)$ is a norm on $\mathbb{R}^d$. Recall from (2.6) that densities of this form a BV-elliptic. We show that f is in general not BD-elliptic if ψ is anisotropic. We again consider $d = 2$, $\nu = e_2$, and define $\psi(x_1, x_2) = \sqrt{x_1^2 + \varepsilon x_2^2}$ for $(x_1, x_2) \in \mathbb{R}^2$ for some $\varepsilon > 0$ small to be specified later. Let $i = 0$ and $j = (2\lambda, 2\lambda)$ for $\lambda > 0$. For $\delta = \varepsilon^{1/4} > 0$, we let $u \in PR(Q_6^{e_2})$ be given by

$$u(x) := \begin{cases} u_{i,j,e_2}(x) & \text{on } Q_6^{e_2} \setminus R_\delta, \\ a_{Q,b}(x) & \text{on } R_\delta, \end{cases}$$

where $R_\delta := (-1, 1) \times (-\delta, \delta)$, $Q = \lambda/\delta(e_1 \otimes e_2 - e_2 \otimes e_1) \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $b = (\lambda, \lambda/\delta) \in \mathbb{R}^2$. We define $J_u^{\parallel}$ and $J_u^{\perp}$ as in (4.19) and again note that, up to a set of negligible $\mathcal{H}^1$-measure, $J_u^{\parallel}$ consists of four pairwise disjoint sets Γ_k with $\mathcal{H}^1(\Gamma_k) = 2$ for $k = 1, \dots, 4$. Since $\|\langle a_{Q,b}, e_1\rangle\|_{L^\infty(R_\delta)} \le C\lambda$ and $\|\langle a_{Q,b}, e_2\rangle\|_{L^\infty(R_\delta)} \le C\lambda/\delta$ for a universal $C > 0$, we get $|[u_1]| \le C\lambda$ and $|[u_2]| \le C\lambda/\delta$ on $J_u^{\perp}$, where $u_k := \langle u, e_k\rangle$, $k = 1, 2$. Therefore, we obtain

$$\begin{aligned} \mathcal{F}(u, Q_6^{e_2}) &= \int_{J_u} \psi([u])\,\mathrm{d}\mathcal{H}^1 = \int_{J_u^{\perp}} \psi([u])\,\mathrm{d}\mathcal{H}^1 + \int_{J_u^{\parallel}} \psi([u])\,\mathrm{d}\mathcal{H}^1 \\ &\le C\mathcal{H}^1(J_u^{\perp})\sqrt{\lambda^2 + \varepsilon(\lambda/\delta)^2} + \int_{J_u^{\parallel}} \psi([u])\,\mathrm{d}\mathcal{H}^1 \le C\lambda\delta\sqrt{1 + \varepsilon/\delta^2} + \int_{J_u^{\parallel}} \psi([u])\,\mathrm{d}\mathcal{H}^1, \end{aligned} \tag{4.20}$$

where we have also used that $\mathcal{H}^1(J_u^{\perp}) \le 4\delta$. Since $\psi([u]) = 2\sqrt{1+\varepsilon}\lambda$ on $\Gamma_1 \cup \Gamma_4$, as well as $a_{Q,b}(t, -\delta) = \frac{\lambda}{\delta}(1 - t)e_2$ and $a_{Q,b}(t, \delta) = 2\lambda e_1 + \frac{\lambda}{\delta}(1 - t)e_2$ for $t \in (-1, 1)$, we compute

$$\begin{aligned} \int_{J_u^{\parallel}} \psi([u])\,\mathrm{d}\mathcal{H}^1 &= \int_{\Gamma_1\cup\Gamma_4} \psi([u])\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma_2\cup\Gamma_3} \psi([u])\,\mathrm{d}\mathcal{H}^1 = 4 \cdot 2\sqrt{1+\varepsilon}\lambda + \int_{\Gamma_2\cup\Gamma_3} \psi([u])\,\mathrm{d}\mathcal{H}^1 \\ &= 8\sqrt{1+\varepsilon}\lambda + \int_{-1}^{1} \sqrt{\varepsilon}\frac{\lambda}{\delta}(1 - t)\,\mathrm{d}t + \int_{-1}^{1} \sqrt{\varepsilon}\Big|2\lambda - \frac{\lambda}{\delta}(1 - t)\Big|\,\mathrm{d}t \le 8\lambda + C\lambda\sqrt{\varepsilon}/\delta. \end{aligned}$$

In view of (4.20) and $\delta = \varepsilon^{1/4}$, we thus get $\mathcal{F}(u, Q_6^{e_2}) \le 8\lambda + C\lambda\varepsilon^{1/4}$. By choosing ε sufficiently small, we find $\mathcal{F}(u, Q_6^{e_2}) < 6 \cdot 2\lambda\sqrt{1+\varepsilon} = \mathcal{F}(u_{i,j,e_2}, Q_6^{e_2})$. This shows that f is not BD-elliptic and concludes the example.

5. Lower semicontinuity in $GSBD^p$ for symmetric jointly convex functions

This section is devoted to a lower semicontinuity result for surface integrals in $GSBD^p(\Omega)$, $p > 1$, where the integrands are symmetric jointly convex functions, see Definition 3.1. We also discuss well-posedness of certain minimization problems. We refer to [27] for the definition and the properties of this function space.

Theorem 5.1 (Lower semicontinuity of surface integrals in $GSBD^p$)**.** *Let $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ be a symmetric jointly convex function. Then, for every sequence $(u_k)_k \subset GSBD^p(\Omega)$, $p > 1$, converging in measure to $u \in GSBD^p(\Omega)$, and satisfying the condition*

$$\sup_{k\in\mathbb{N}} \big(\|e(u_k)\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_k})\big) < +\infty, \tag{5.1}$$

we have that

$$\int_{J_u} f(u^+, u^-, \nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \le \liminf_{k\to\infty} \int_{J_{u_k}} f(u_k^+, u_k^-, \nu_{u_k})\,\mathrm{d}\mathcal{H}^{d-1}. \tag{5.2}$$

For various examples of symmetric jointly convex integrands we refer the reader to Section 4. Restricting to the space of SBD-functions, the above result also holds for symmetric biconvex functions introduced in Subsection 4.3, see Remark 5.5 below for details. As a consequence of the above result, we get that the following minimization problems are well-posed.

Theorem 5.2 (Existence of minimizers)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $c > 0$. Let $W\colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$ be convex with $W(F) \ge c|F|^p$ for all $F \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ for some $p > 1$. Let $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [c,+\infty)$ be symmetric jointly convex and let $\Psi : [0,+\infty) \to [0,+\infty)$ be continuous such that $\lim_{t\to\infty} \Psi(t) = +\infty$. Then, the functional*

$$\mathcal{E}(u) := \int_\Omega W\big(e(u)\big)\,\mathrm{d}x + \int_{J_u} f(u^+, u^-, \nu)\,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega \Psi(|u|)\,\mathrm{d}x \qquad \text{for all } u \in GSBD^p(\Omega)$$

has a minimizer in $GSBD^p(\Omega)$.

Proof of Theorem 5.2. Let $(u_k)_k \subset GSBD^p(\Omega)$ be a minimizing sequence. Then, by the growth condition of W and the fact that $f \ge c > 0$, we obtain

$$\sup_{k\in\mathbb{N}} \big(\|e(u_k)\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_k})\big) < +\infty. \tag{5.3}$$

This along with the fact that $\sup_{k\in\mathbb{N}} \int_\Omega \Psi(|u_k|)\mathrm{d}x < +\infty$ allows us to apply [27, Theorem 11.3]: we find $u \in GSBD^p(\Omega)$ such that $u_k \to u$ a.e. on Ω and $e(u_k) \rightharpoonup e(u)$ weakly in $L^p(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. By the convexity of W and Fatou's lemma we obtain

$$\int_\Omega W\big(e(u)\big)\,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega W\big(e(u_k)\big)\,\mathrm{d}x, \qquad \int_\Omega \Psi(|u|)\,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega \Psi(|u_k|)\,\mathrm{d}x.$$

By (5.3) and Theorem 5.1 we also get that the surface term is lower semicontinuous. We thus conclude that u is a minimizer. □

The remainder of the section is devoted to the proof of Theorem 5.1. The proof will rely on the following integration by parts formula.

Lemma 5.3 (Integration by parts in $GSBD$)**.** *Let $G \in C^1(\mathbb{R}^d;\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ be conservative and let $u \in GSBD^p(\Omega)$, $1 < p < \infty$. Then, for all $A \in \mathcal{A}(\Omega)$ and $\varphi \in C^1_c(A)$ there holds*

$$\int_{A\cap J_u} \big\langle G(u^+) - G(u^-), \nu_u\big\rangle\, \varphi\,\mathrm{d}\mathcal{H}^{d-1} + \int_A \big((\nabla G(u) : e(u)\big)\,\varphi\,\mathrm{d}x = -\int_A \langle G(u), \nabla\varphi\rangle\,\mathrm{d}x. \tag{5.4}$$

In order to prove this formula, we will combine the corresponding formula in SBV (see [2, Lemma 3.5]) with an approximation result for $GSBD^p$ functions stated in [20, Theorem 1.1]. A slightly simplified statement of the latter result is the following.

Theorem 5.4 (Density in $GSBD^p$)**.** *Let $u \in GSBD^p(\Omega)$, $p > 1$. Then, there exists a sequence of functions $(u_k)_k \subset SBV^p(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ such that each J_{u_k} is closed in Ω and included in*

a finite union of closed connected pieces of C^1 hypersufaces, $u_k \in W^{1,\infty}(\Omega \setminus J_{u_k};\mathbb{R}^d)$, and

$$\begin{aligned} &\text{(i)} \quad u_k \to u \text{ a.e. on } \Omega,\\ &\text{(ii)} \quad \|e(u_k)-e(u)\|_{L^p(\Omega)} \to 0,\\ &\text{(iii)} \quad \mathcal{H}^{d-1}(J_{u_k} \triangle J_u) \to 0,\\ &\text{(iv)} \quad \int_{J_{u_k}\cup J_u} \tau(|u_k^\pm - u^\pm|)\,\mathrm{d}\mathcal{H}^{d-1} \to 0, \end{aligned} \tag{5.5}$$

for some $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} \le \tau \le \frac{1}{2}$, $0 \le \tau' \le 1$, and $\{\tau = 0\} = \{0\}$.

Proof of Lemma 5.3. Given $u \in GSBD^p(\Omega)$ with $p > 1$, we let $(u_k)_k \subset SBV^p(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ be the approximation sequence provided by Theorem 5.4. Then, by [2, Lemma 3.5], for all $A \in \mathcal{A}(\Omega)$, for every $\varphi \in C_c^1(A)$, and for all $k \in \mathbb{N}$ we have that

$$\int_{A\cap J_{u_k}} \big\langle G(u_k^+) - G(u_k^-), \nu_{u_k}\big\rangle\, \varphi\, \mathrm{d}\mathcal{H}^{d-1} + \int_A \big((\nabla G(u_k))^T : \nabla u_k\big)\, \varphi\, \mathrm{d}x = -\int_A \langle G(u_k), \nabla\varphi\rangle\, \mathrm{d}x. \tag{5.6}$$

Our goal is to pass to the limit $k \to \infty$ in each of the three terms separately.

Step 1. As $G \in C(\mathbb{R}^d;\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $\nabla\varphi \in L^\infty(A)$, we get by (5.5)(i) and dominated convergence that

$$\lim_{k\to\infty} \int_A \langle G(u_k), \nabla\varphi\rangle\, \mathrm{d}x = \int_A \langle G(u), \nabla\varphi\rangle\, \mathrm{d}x. \tag{5.7}$$

Step 2. We now show that

$$\lim_{k\to\infty} \int_A \big((\nabla G(u_k))^T : \nabla u_k\big)\, \varphi\, \mathrm{d}x = \int_A \big(\nabla G(u) : e(u)\big)\, \varphi\, \mathrm{d}x. \tag{5.8}$$

In fact, we first note that $(\nabla G(u_k))^T : \nabla u_k = \nabla G(u_k) : e(u_k)$ a.e. due to the fact that G is a conservative vector field and thus $\nabla G\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$. As $\nabla G \in C(\mathbb{R}^d;\mathbb{R}^{d\times d}) \cap L^\infty(\mathbb{R}^d;\mathbb{R}^{d\times d})$, we get $\nabla G(u_k) \to \nabla G(u)$ in $L^q(\Omega;\mathbb{R}^{d\times d})$ for any $q \in [1,\infty)$ by (5.5)(i) and dominated convergence. Thus, by (5.5)(ii), $\varphi \in C_c^1(A)$, and Hölder's inequality we obtain (5.8).

Step 3. We finally prove that, up to a subsequence, there holds

$$\lim_{k\to\infty} \int_{A\cap J_{u_k}} \big\langle G(u_k^+) - G(u_k^-), \nu_{u_k}\big\rangle\, \varphi\, \mathrm{d}\mathcal{H}^{d-1} = \int_{A\cap J_u} \big\langle G(u^+) - G(u^-), \nu_u\big\rangle\, \varphi\, \mathrm{d}\mathcal{H}^{d-1}. \tag{5.9}$$

As a preliminary step, we observe that, up to a subsequence,

$$\begin{aligned} &\lim_{k\to\infty} u_k^\pm(x) \to u^\pm(x) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in A.\\ &\lim_{k\to\infty} \nu_{u_k}(x) = \nu_u(x) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in A. \end{aligned} \tag{5.10}$$

In fact, by (5.5)(iv) we get that $u_k^\pm$ converges to $u^\pm$ in measure with respect to the measure $\mathcal{H}^{d-1}$, i.e.,

$$\lim_{k\to\infty} \mathcal{H}^{d-1}\big(\big\{x \in (J_{u_k}\cup J_u)\cap A\colon\ |u_k^\pm(x) - u^\pm(x)| > \varepsilon\big\}\big) = 0$$

for all $\varepsilon > 0$. Then, up to passing to a subsequence, we get that (5.10)(i) holds true. We further observe that (5.10)(ii) follows directly from (5.5)(iii). Now, by (5.10), dominated convergence, and the fact that $G \in C(\mathbb{R}^d;\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d;\mathbb{R}^d)$ as well as $\varphi \in C(A) \cap L^\infty(A)$ we obtain (5.9).

Finally, taking into account (5.7), (5.8), and (5.9), we can pass to the limit in (5.6). This concludes the proof. □

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. The proof is in the spirit of that of [2, Theorem 3.6]. The essential step is to show that for every every $A \in \mathcal{A}(\Omega)$ and for every conservative vector field $G \in C^1(\mathbb{R}^d;\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ there holds

$$\int_{J_u\cap A} \big\langle G(u^+) - G(u^-), \nu_u\big\rangle^+ \,\mathrm{d}\mathcal{H}^{d-1} \le \liminf_{k\to\infty} \int_{J_{u_k}\cap A} \big\langle G(u_k^+) - G(u_k^-), \nu_{u_k}\big\rangle^+ \,\mathrm{d}\mathcal{H}^{d-1}, \tag{5.11}$$

where $t^+ := \max\{t,0\}$ for $t \in \mathbb{R}$. For the moment, we assume that (5.11) holds and show the statement (Step 1). Afterwards, we will prove (5.11) (Step 2).

Step 1: Proof of the statement. Fix $\varepsilon > 0$. By Remark 3.2 and the fact that f is nonnegative there exist conservative vector fields $(g_h^\varepsilon)_h \subset C^1(\mathbb{R}^d;\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ such that

$$f(i,j,\nu) - \varepsilon \le \sup_{h\in\mathbb{N}} \langle g_h^\varepsilon(i) - g_h^\varepsilon(j),\, \nu\rangle^+ \le f(i,j,\nu) + \varepsilon \quad \text{for all } (i,j,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}, i \ne j. \tag{5.12}$$

For each $h \in \mathbb{N}$ and all $A \in \mathcal{A}(\Omega)$, we get by (5.11) and (5.12)

$$\int\limits_{J_u\cap A} \big\langle g_h^\varepsilon(u^+) - g_h^\varepsilon(u^-), \nu_u\big\rangle^+ \,\mathrm{d}\mathcal{H}^{d-1} \le \liminf_{k\to\infty} \int\limits_{J_{u_k}\cap A} f(u_k^+, u_k^-, \nu_{u_k})\,\mathrm{d}\mathcal{H}^{d-1} + \varepsilon \sup_k \mathcal{H}^{d-1}(J_{u_k}\cap A).$$

We set

$$\Lambda_\varepsilon(A) := \liminf_{k\to\infty} \int_{J_{u_k}\cap A} f(u_k^+, u_k^-, \nu_{u_k})\,\mathrm{d}\mathcal{H}^{d-1} + \varepsilon \sup_k \mathcal{H}^{d-1}(J_{u_k}\cap A) \tag{5.13}$$

and apply Lemma 2.7 to find

$$\int_{J_u\cap A} \sup_{h\in\mathbb{N}} \big\langle g_h^\varepsilon(u^+) - g_h^\varepsilon(u^-), \nu_u\big\rangle^+ \,\mathrm{d}\mathcal{H}^{d-1} \le \Lambda_\varepsilon(A)$$

for all $A \in \mathcal{A}(\Omega)$. Then, by (5.12) we get

$$\int_{J_u} f(u^+, u^-, \nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \le \Lambda_\varepsilon(\Omega) + \varepsilon\mathcal{H}^{d-1}(J_u).$$

We conclude the proof of (5.2) by passing to $\varepsilon \to 0$ and using (5.1) as well as (5.13).

Step 2: Proof of (5.11). We now show (5.11). By condition (5.1) we get that the sequence $(|e(u_k)|)_k$ is equiintegrable and so, for every $\varepsilon > 0$, we can find an open set $B \subset \Omega$ such that

$$\sup_{k\in\mathbb{N}} \int_B |e(u_k)|\,\mathrm{d}x + \int_B |e(u)|\,\mathrm{d}x \le \varepsilon, \qquad \text{and} \qquad \bigcup_{k\in\mathbb{N}} J_{u_k} \cup J_u \subset B. \tag{5.14}$$

Let $\Phi := \{\varphi \in C_c^1(B)\colon\ 0 \le \varphi \le 1\}$. Then, by (5.14), by dominated convergence, and by applying Lemma 5.3 twice, we get

$$\begin{aligned}
\int_{J_u\cap A} \big\langle G(u^+) - G(u^-), \nu_u\big\rangle^+ \,\mathrm{d}\mathcal{H}^{d-1} &= \sup_{\varphi\in\Phi} \left\{ \int_{J_u\cap A} \big\langle G(u^+) - G(u^-), \nu_u\big\rangle\, \varphi \,\mathrm{d}\mathcal{H}^{d-1} \right\} \\
&\le C\varepsilon + \sup_{\varphi\in\Phi} \left\{ -\int_A \langle G(u), \nabla\varphi\rangle \,\mathrm{d}x \right\} \\
&\le C\varepsilon + \liminf_{k\to\infty} \sup_{\varphi\in\Phi} \left\{ -\int_A \langle G(u_k), \nabla\varphi\rangle \,\mathrm{d}x \right\} \\
&\le 2C\varepsilon + \liminf_{k\to\infty} \int_{J_{u_k}\cap A} \big\langle G(u_k^+) - G(u_k^-), \nu_{u_k}\big\rangle^+ \,\mathrm{d}\mathcal{H}^{d-1},
\end{aligned}$$

where the constant $C > 0$ depends only on G. By the arbitrariness of $\varepsilon > 0$, the proof of (5.11) is concluded. □

Remark 5.5 (Symmetric biconvex functions)**.** While for general symmetric biconvex functions (see Subsection 4.3) the lower semicontinuity in $GSBD^p$ remains an open problem, we point out that the proof strategy devised in Theorem 5.1 allows to prove (5.2) for symmetric biconvex functions, under the additional assumption that the sequence of functions lies in $SBD(\Omega)$. In fact, in this case, the integration by parts formula (5.4) still holds for $G(u) = Zu$ with $Z \in \mathbb{R}^{d\times d}_{\rm sym}$, cf. (4.14), thanks to the approximation result in [24, Theorem 1.1]. This provides an alternative proof of the lower semicontinuity results in [10].

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Appendix A. Proof of Lemma 4.9

Proof of Lemma 4.9. The proof relies on slicing for $GSBD$ functions and we therefore first need to introduce some notation. For fixed $\xi \in \mathbb{S}^{d-1}$, we let

$$\Pi_\xi := \{y \in \mathbb{R} \colon \langle y, \xi\rangle = 0\}, \qquad D^\xi_y := \{t \in \mathbb{R} \colon y + t\xi \in D\} \quad \text{for any } y \in \mathbb{R}^d \text{ and } D \subset \mathbb{R}^d\,,$$

and for every function $v \colon D \to \mathbb{R}^d$ and $t \in D^\xi_y$ let

$$v^\xi_y(t) := v(y + t\xi), \qquad \widehat{v}^\xi_y(t) := \langle v^\xi_y(t), \xi\rangle.$$

Let $A := \{u = u_0\} \subset B$. As $B \setminus A \subset\subset B$, we find that

$$\mathcal{L}^1(B^\xi_y)/\mathcal{L}^1(A^\xi_y) \le C \qquad \text{for all } \xi \in \mathbb{S}^{d-1} \text{ and all } y \in \Pi_\xi \tag{A.1}$$

for a universal $C > 0$, whenever $\mathcal{L}^1(B^\xi_y) > 0$. Since $u \in PR(B)$, the function lies in $GSBD(B)$. Then [27, Definition 4.2, Theorem 8.1, Theorem 9.1] imply that for any $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi_\xi$, the function $\widehat{u}^\xi_y$ belongs to $SBV_{\rm loc}(B^\xi_y)$ and there holds

$$\langle e(u)(y + t\xi)\xi, \xi\rangle = (\widehat{u}^\xi_y)'(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in B^\xi_y, \tag{A.2}$$

as well as

$$(J^\xi_u)^\xi_y = J_{\widehat{u}^\xi_y} \ \text{ for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi, \ \text{ where } \ J^\xi_u := \{x \in J_u \colon \langle [u](x), \xi\rangle \neq 0\}\,, \tag{A.3a}$$

$$\langle u^\pm(y + t\xi), \xi\rangle = (\widehat{u}^\xi_y)^\pm \quad \text{for every } t \in (J_u)^\xi_y. \tag{A.3b}$$

As $u \in PR(B)$, we find $(\widehat{u}^\xi_y)' = 0$ by (A.2). Then, for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi_\xi$, we use the Poincaré inequality in BV to get $b^\xi_y \in \mathbb{R}$ such that

$$\int_{B^\xi_y} |\widehat{u}^\xi_y(t) - b^\xi_y|\,\mathrm{d}\mathcal{L}^1(t) \le C \int_{J_{\widehat{u}^\xi_y}} |[\widehat{u}^\xi_y]|\,\mathrm{d}\mathcal{H}^0,$$

where $C>0$ only depends on the diameter of B. Since $u=u_0$ on A for $u_0\in L^1(B)$, we get

$$\int_{A_y^\xi}\big|\langle u_0(y+t\xi),\xi\rangle - b_y^\xi\big|\,\mathrm{d}\mathcal{L}^1 \le C\int_{J_{\widehat{u}_y^\xi}}|[\widehat{u}_y^\xi]|\,\mathrm{d}\mathcal{H}^0$$

and therefore, combining the previous two estimates, we derive

$$\begin{aligned}\int_{B_y^\xi}|\widehat{u}_y^\xi|\,\mathrm{d}\mathcal{L}^1 &\le C\int_{J_{\widehat{u}_y^\xi}}|[\widehat{u}_y^\xi]|\,\mathrm{d}\mathcal{H}^0+\mathcal{L}^1(B_y^\xi)|b_y^\xi|\\ &\le C\Big(1+\frac{\mathcal{L}^1(B_y^\xi)}{\mathcal{L}^1(A_y^\xi)}\Big)\int_{J_{\widehat{u}_y^\xi}}|[\widehat{u}_y^\xi]|\,\mathrm{d}\mathcal{H}^0+C\frac{\mathcal{L}^1(B_y^\xi)}{\mathcal{L}^1(A_y^\xi)}\int_{A_y^\xi}\big|(u_0)_y^\xi\big|\,\mathrm{d}\mathcal{L}^1, \qquad (A.4)\end{aligned}$$

for $\mathcal{H}^{d-1}$-a.e. $y\in \Pi_\xi$. We note that by the area formula (cf. e.g. [37, (12.4)]) and (A.3) there holds

$$\int_{\Pi_\xi}\int_{J_{\widehat{u}_y^\xi}}|[\widehat{u}_y^\xi]|\,\mathrm{d}\mathcal{H}^0\,\mathrm{d}\mathcal{H}^{d-1}(y)=\int_{J_u}\big|[\langle u,\xi\rangle]\big|\,|\langle\xi,\nu_u\rangle|\,\mathrm{d}\mathcal{H}^{d-1}\le\int_{J_u}|[u]|\,\mathrm{d}\mathcal{H}^{d-1}<+\infty. \qquad (A.5)$$

Then, by (A.1), (A.4), (A.5), and by Fubini-Tonelli we find that $\langle u,\xi\rangle\in L^1(B)$ with

$$\int_B|\langle u,\xi\rangle|\,\mathrm{d}x=\int_{\Pi_\xi}\int_{B_y^\xi}|\widehat{u}_y^\xi|\,\mathrm{d}\mathcal{L}^1\mathrm{d}\mathcal{H}^{d-1}(y)\le C\int_{J_u}|[u]|\,\mathrm{d}\mathcal{H}^{d-1}+C\|u_0\|_{L^1(B)}<+\infty.$$

Since this holds for every $\xi\in\mathbb{S}^{d-1}$, we derive $u\in L^1(B;\mathbb{R}^d)$.

In a similar fashion, we get that for each $\xi\in\mathbb{S}^{d-1}$ the distributional derivative in ξ-direction of the the function $\langle u,\xi\rangle$ exists since for each $\varphi\in C_c^\infty(B)$ there holds in view of (A.5)

$$\begin{aligned}\Big|\int_B\langle u,\xi\rangle\,\partial_\xi\varphi\,\mathrm{d}x\Big| &= \Big|\int_{\Pi_\xi}\int_{B_y^\xi}\widehat{u}_y^\xi(t)\,(\varphi_y^\xi(t))'\,\mathrm{d}\mathcal{L}^1(t)\,\mathrm{d}\mathcal{H}^{d-1}(y)\Big| = \Big|-\int_{\Pi_\xi}\int_{J_{\widehat{u}_y^\xi}}\varphi_y^\xi\,[\widehat{u}_y^\xi]\,\mathrm{d}\mathcal{H}^0\,\mathrm{d}\mathcal{H}^{d-1}(y)\Big|\\ &\le\|\varphi\|_{L^\infty(B)}\int_{J_u}|[u]|\,\mathrm{d}\mathcal{H}^{d-1},\end{aligned}$$

where $\partial_\xi\varphi:=\langle\nabla\varphi,\xi\rangle$ and $\varphi_y^\xi(t):=\varphi(y+t\xi)$. This implies that the symmetric part of the distributional derivative of u is a finite Radon measure, see [6, Section 3]. Thus, $u\in BD(B)$. Finally, as the functions $\widehat{u}_y^\xi$ lie in $SBV(B_y^\xi)$ for all $\xi\in\mathbb{S}^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. $y\in\Pi_\xi$, we even get $u\in SBD(B)$, see [6, Proposition 4.7]. This concludes the proof. □

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